Long Time Motion of NLS Solitary Waves in a Confining Potential

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Abstract

We study the motion of solitary-wave solutions of a family of focusing generalized nonlinear Schrödinger equations with a confining, slowly varying external potential, $V(x)$.

A Lyapunov-Schmidt decomposition of the solution combined with energy estimates allows us to control the motion of the solitary wave over a long, but finite, time interval.

We show that the center of mass of the solitary wave follows a trajectory close to that of a Newtonian point particle in the external potential $V(x)$ over a long time interval.

1 Introduction

We consider a family of generalized nonlinear Schrödinger and Hartree equations with a focusing nonlinearity. These equations have solitary wave solutions, and, in this paper, we study the effective dynamics of such solitary waves. The equations have the form:

$$i\partial_t \psi(x,t) = -\Delta \psi(x,t) + V(x)\psi(x,t) - f(\psi)(x,t), \qquad (1.1)$$

where $t \in \mathbb{R}$ is time, $x \in \mathbb{R}^d$ denotes a point in physical space, $\psi : \mathbb{R}^d \times \mathbb{R} \mapsto \mathbb{C}$ is a (one-particle) wave function, V is the external potential, which is a real-valued, confining, and slowly varying function on $\mathbb{R}^d$, and $f(\psi)$ describes a nonlinear self-interaction with the properties that $f(\psi)$ is "differentiable" in ψ, $f(0) = 0$, and $f(\bar{\psi}) = \overline{f(\psi)}$. Precise assumptions on V and f are formulated in Section 2.

The family of nonlinearities of interest to us includes local nonlinearities, such as

$$f(\psi) = \lambda |\psi|^s \psi, \ 0 < s < \frac{4}{d}, \ \lambda > 0, \tag{1.2}$$

and Hartree nonlinearities

$$f(\psi) = \lambda(\Phi * |\psi|^2)\psi, \ \lambda > 0, \tag{1.3}$$

where the (two-body) potential Φ is real-valued, of positive type, continuous, spherically symmetric, and tends to 0 as $|x| \to \infty$. Here $\Phi * g := \int \Phi(x - y)g(y)\,\mathrm{d}^d y$ denotes convolution. Such equations are encountered in the theory of Bose gases (BEC), in nonlinear optics, in the theory of water waves and in other areas of physics.

It is well known that Eq. (1.1) has solitary wave solutions when $V \equiv 0$. Let $\eta_\mu \in \mathrm{L}^2$ be a spherically symmetric, positive solution of the nonlinear eigenvalue problem

$$-\Delta\eta + \mu\eta - f(\eta) = 0. \tag{1.4}$$

The function η_μ is called a "solitary wave profile". Among the solitary wave solutions of (1.1) are Galilei transformations of η_μ,

$$\psi_{\mathrm{sol}} := \mathcal{S}_{a(t)p(t)\gamma(t)}\eta_{\mu(t)}, \tag{1.5}$$

where $\mathcal{S}_{ap\gamma}$ is defined by

$$(\mathcal{S}_{ap\gamma}\psi)(x) := \mathrm{e}^{\mathrm{i}p \cdot (x-a) + \mathrm{i}\gamma}\psi(x - a). \tag{1.6}$$

Let $\sigma := \{a, p, \gamma, \mu\}$, where μ is as in Eq. (1.4). For ψ_{sol} to be a solution to (1.1) with $V \equiv 0$ the modulation parameters, σ, must satisfy the equations of motion

$$a(t) = 2pt + a, \ p(t) = p, \ \gamma(t) = \mu t + p^2 t + \gamma, \ \mu(t) = \mu \tag{1.7}$$

with $\gamma \in \mathbb{S}^1$, $a, p \in \mathbb{R}^d$, $\mu \in \mathbb{R}_+$. In other words, σ satisfy (1.7), then

$$\psi_{\mathrm{sol}}(x, t) = (\mathcal{S}_{a(t)p(t)\gamma(t)}\eta_{\mu(t)})(x) \tag{1.8}$$

solves Eq. (1.1) with $V \equiv 0$. Thus (1.5), with $a(t), p(t), \gamma(t), \mu(t)$ as above, describes a $2d+2$-dimensional family solutions of Eq. (1.1) with $V \equiv 0$. Let the **soliton manifold**, $\mathrm{M_s}$ be defined by

$$\mathrm{M_s} := \left\{ \mathcal{S}_{ap\gamma}\eta_\mu : \{a, p, \gamma, \mu\} \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^1 \times I \right\}, \tag{1.9}$$

where I is a bounded interval in $\mathbb{R}_+$.

Solutions to (1.4) behave roughly like $e^{-\sqrt{\mu}|x|}$, as $|x| \to \infty$. So $\sqrt{\mu}$ is a reciprocal length scale that indicates the "size" of the solitary wave.

We consider the Cauchy problem for Eq. (1.1), with initial condition ψ_0 in a weighted Sobolev space. For Hartree nonlinearities, global wellposedness is known [20]. For local nonlinearities, the situation is more delicate; see Condition 1 and Remark 2.1 in Section 2. Let the initial condition ψ_0 be "close" to M_s. Then, we will show, the corresponding solution ψ will remain "close" to M_s, over a long time interval. A certain "symplectically orthogonal" projection of ψ onto M_s is then well defined and traces out a unique curve on M_s. We denote this curve by $\eta_{\sigma(t)}$, see Figure 1.1.
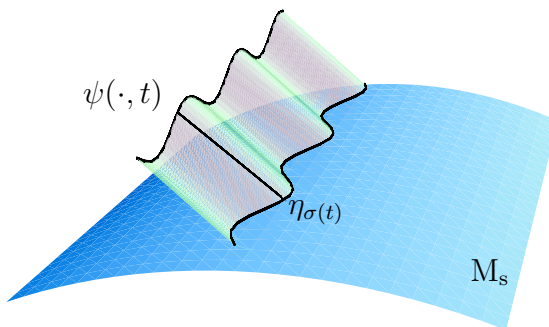

PSfrag replacements

Figure 1.1: The trajectory $\psi(\cdot, t)$ over the soliton Manifold M_s.

An essential part of this paper is to determine the leading order behavior of $\sigma(t) = \{a(t), p(t), \gamma(t), \mu(t)\}$ and to estimate error terms. To this end, let W be a smooth, positive, polynomially bounded function, and define

$$V(x) = W(\epsilon_V x) \qquad (1.10)$$

where ϵ_V is a small parameter. Furthermore, let ψ_0 be an initial condition "ϵ_0–close" to $\eta_{\sigma_0} \in M_s$, for some σ_0. Roughly speaking, this initial condition has length scale $1/\sqrt{\mu}_0$. We will consider external potentials, V, as in (1.10), for a scaling parameter ϵ_V satisfying

$$\epsilon_V \ll \sqrt{\mu_0}, \qquad (1.11)$$

i.e., we assume that the external potential varies very little over the length scale of ψ_0. For simplicity, we choose $\mu = 1$ and $\epsilon_V \ll 1$, at the price of re-scaling the nonlinearity.

We decompose the solution ψ of (1.1) into a part which is a solitary wave and a small part, a "perturbation", w. That is, we write ψ as

$$\psi = \mathcal{S}_{ap\gamma}(\eta_\mu + w). \qquad (1.12)$$

This does not define a unique decomposition, unless $2d + 2$ additional conditions are imposed. These conditions say that the perturbation w is 'symplectically orthogonal' to the soliton manifold M_s.

The main idea used to control the perturbation w is to derive differential equations in time for the modulation parameters, σ, which depend on the external potential. These equations appear naturally when one projects solutions of (1.1) onto the soliton manifold. To control the motions of σ and w, we make use of conserved quantities: the energy

$$\mathcal{H}_V(\psi) := \frac{1}{2} \int (|\nabla\psi|^2 + V|\psi|^2)\, \mathrm{d}^d x - F(\psi), \tag{1.13}$$

where $F'(\psi) = f(\psi)$ (this is a variational derivative), the mass (or charge)

$$\mathcal{N}(\psi) := \frac{1}{2} \int |\psi|^2\, \mathrm{d}^d x, \tag{1.14}$$

and the "almost conserved" momentum

$$\mathcal{P}(\psi) := \frac{1}{4} \int (\bar{\psi}\nabla\psi - \psi\nabla\bar{\psi})\, \mathrm{d}^d x. \tag{1.15}$$

To achieve control over the perturbation w, we introduce a 'Lyapunov functional'

$$\Lambda(\psi, t) := K_\sigma(\psi) - K_\sigma(\mathcal{S}_{ap\gamma}\eta_\mu), \tag{1.16}$$

where $\sigma = \sigma(t) = \{a(t), p(t), \gamma(t), \mu(t)\}$, and where

$$\begin{aligned} K_\sigma(\psi) := \mathcal{H}_V(\psi) + (p^2 + \mu)\mathcal{N}(\psi) - 2p \cdot \mathcal{P}(\psi) \\ - \frac{1}{2} \int \left(V(a) + \nabla V(a) \cdot (x - a)\right)|\psi|^2\, \mathrm{d}^d x, \end{aligned} \tag{1.17}$$

i.e., K_σ is essentially a linear combination of the conserved and almost conserved quantities. Using the linear transformation $u := \mathcal{S}_{ap\gamma}^{-1}\psi$, we change questions about the size of fluctuations around $\mathcal{S}_{ap\gamma}\eta_\mu$ to ones about the size of fluctuations around the solitary wave profile $\eta_{\mu(t)}$. In this "moving frame", the $K_\sigma(\psi)$ terms in the Lyapunov functional introduced above take the form

$$K_\sigma(\mathcal{S}_{ap\gamma}u) = \mathcal{E}_\mu(u) + \frac{1}{2} \int \mathcal{R}_V |u|^2\, \mathrm{d}^d x, \tag{1.18}$$

where

$$\mathcal{R}_V(x) := V(x + a) - V(a) - \nabla V(a) \cdot (x - a) \tag{1.19}$$

and

$$\mathcal{E}_\mu(u) := \mathcal{H}_{V=0}(u) + \mu\mathcal{N}(u). \tag{1.20}$$

In the moving frame the Lyapunov functional depends on the parameters μ and a, but not on p and γ. Furthermore, η_μ is a critical point of $\mathcal{E}_\mu(\eta_\mu)$, i.e., $\mathcal{E}'_\mu(\eta_\mu) = 0$. The change of frame discussed above simplifies the analysis leading to our main result.

Simply stated, our main theorem shows that, for initial conditions ψ_0 ϵ_0-close to M_s, the perturbation w is of order $\epsilon := \epsilon_V + \epsilon_0$, for all times smaller than $C\epsilon^{-1}$. Furthermore, the center of mass of the solitary wave, a, and the center of mass momentum p satisfy the following equations

$$\dot{a} = 2p + \mathcal{O}(\epsilon^2), \qquad \dot{p} = -\nabla V(a) + \mathcal{O}(\epsilon^2). \tag{1.21}$$

The remaining modulation parameters μ and γ satisfy

$$\dot{\mu} = \mathcal{O}(\epsilon^2), \qquad \dot{\gamma} = \mu - V(a) + p^2 + \mathcal{O}(\epsilon^2). \tag{1.22}$$

A precise statement is found in the next section.

This is the first result of its type covering *confining* external potentials. Indeed, we can exploit the confining nature of the potential to obtain a *stronger* result than that of [12] (and that stated above) for a certain class of initial conditions which we now describe. Consider the classical Hamiltonian function:

$$h(a, p) := \big(p^2 + V(a)\big)/2. \tag{1.23}$$

Given an initial condition ψ_0 ϵ_0–close to $\eta_{\sigma_0} \in M_s$, where $\sigma_0 = \{a_0, p_0, \gamma_0, \mu_0\}$, we require the initial position a_0 and momentum p_0 to satisfy

$$h(a_0, p_0) - \min_a h(a, 0) \le \epsilon_h, \tag{1.24}$$

with $\epsilon_V \le C\epsilon_h \le 1$, for some constant C. For this class of initial conditions, our main result shows that the perturbation w remains $\mathcal{O}(\epsilon)$ for longer times:

$$t < \frac{C}{\epsilon_V \sqrt{\epsilon_h} + \epsilon^2}. \tag{1.25}$$

This improvement is non-trivial. For example, it means that we can control the perturbation of a solitary wave which undergoes many oscillations near the bottom of a potential well.

Remark: We can also extend our analysis to a class of slowly time-dependent external potentials without much additional work. We introduce a scale parameter, τ, in time: $V(x, t) := W(\epsilon_V x, \tau t)$. To determine the size of τ heuristically we consider

$$\frac{d}{dt}h(a, p, t) = p\big(\dot{p} + \nabla V(a, t)\big) + \frac{1}{2}(\dot{a} - 2p) \cdot \nabla V(a, t) + \partial_t V(a, t). \tag{1.26}$$

We want the last two terms to have the same size. The second but last term is of size $\epsilon^2 \epsilon_V$, since $\dot{a}$ satisfy the classical equations of motion to order ϵ^2. The last term is of size τ. Thus if τ is chosen to be $\tau = \mathcal{O}(\epsilon_V^3)$ all our estimates will survive.

The following example suggests that accelerating solitary wave solutions of Eq. (1.1) in a confining external potential can, in fact, survive for arbitrarily long times. Choose $V(x) := x \cdot Ax + d \cdot x + c \geq 0$ and $A > 0$ (positive matrix). Then (1.1) has the following solution:

$$\psi(x, t) = e^{ip(t) \cdot (x - a(t)) + i\gamma(t)} \tilde{\eta}_\mu(x - a(t)) \tag{1.27}$$

with

$$\dot{p} = -\nabla V(a), \ \dot{a} = 2p, \ \dot{\gamma} = p^2 + \mu - V(a), \tag{1.28}$$

where $\tilde{\eta}_\mu$ solves the equation

$$-\Delta \eta + \mu \eta - f(\eta) + (x \cdot Ax)\eta = 0. \tag{1.29}$$

Thus, given a solution of the equations of motion (1.28), a family of solitary wave solutions is given by (1.27), for arbitrary times t. For details see Appendix D.

The first results of the above type, for bounded, time-independent potentials were proved in [13, 14] for the Hartree equation under a spectral assumption. This result was later extended to a general class of nonlinearities in [12]. Neither of these works deals with a confining external potential. In particular, their results do not extend to the longer time interval (1.25) described above.

For local pure-power nonlinearities and a small parameter ϵ_V, it has been shown in [4] that if an initial condition is of the form $\mathcal{S}_{a_0 p_0 \gamma_0} \eta_{\mu_0}$, then the solution $\psi(x, t)$ of Eq. (1.1) satisfies

$$\epsilon_V^{-d} |\psi(\frac{x}{\epsilon_V}, \frac{t}{\epsilon_V})|^2 \to \|\eta_\mu\|_{L^2}^2 \delta_{a(t)} \tag{1.30}$$

in the C^{1*} topology (dual to C^1), provided $a(t)$ satisfies the equation $\frac{1}{2}\ddot{a} = \nabla W(a)$, where $V(x) = W(\epsilon_V x)$. This result was strengthened in [19] for a bounded external potential and in [8] for a potential given by a quadratic polynomial in x.

There have been many recent works on asymptotic properties for generalized nonlinear Schrödinger equations. Asymptotic stability, scattering and asymptotic completeness of solitary waves for bounded external potential tending to 0 at ∞ has been shown under various assumptions. See for example, [26, 27, 28, 5, 6, 10, 11, 7, 30, 31, 32, 24, 29, 17, 15, 22].

Though these are all-time results, where ours is long (but finite)-time, our approach has some advantages: we can handle confining potentials (for which the above-described results are meaningless); we require a much less stringent (and verifiable) spectral condition; we track the finite-dimensional soliton dynamics (Newton equations); and our methods are comparatively elementary.

Our paper is organized as follows. In Section 2, we state our hypotheses and the main result. In Section 3, we recall the Hamiltonian nature of Eq. (1.1) and describe symmetries of (1.1) for $V \equiv 0$. We give a precise definition of the soliton manifold M_s and its tangent space. In Section 4, we introduce a convenient parametrization of functions in a small neighborhood of M_s in phase space, and we derive equations for the

modulation parameters $\sigma = \{a, p, \gamma, \mu\}$ and the perturbation w around a solitary wave $\eta_\sigma = \mathcal{S}_{ap\gamma}\eta_\mu$. In this parametrization, the perturbation w is symplectically orthogonal to the tangent space $\mathrm{T}_{\eta_\sigma}\mathrm{M}_\mathrm{s}$ to M_s at η_σ. In Section 5, we similarly decompose the initial condition ψ_0 deriving in this way the initial conditions, σ_0 and w_0, for σ and w, and estimating w_0. In Section 6, we derive bounds on the solitary wave position, a, and the momentum, p, by using the fact that the Hamiltonian, $h(a, p)$ is almost conserved in time. In Section 7, we construct the Lyapunov functional, $\Lambda(\psi, t)$, and compute its time derivative. This computation is used in Section 8 in order to obtain an upper bound on $\Lambda(\psi, t)$. This bound, together with the more difficult lower bound derived in Section 9, is used in Section 10 in order to estimate the perturbation w and complete the proof of our main result, Theorem 2.1. Some basic inequalities are collected in Appendices A–C. In Appendix D, we construct a family of time-dependent solutions with parameters exactly satisfying the classical equations of motion.

2   Notation, assumptions and main result

Let L^s denote the usual Lebesgue space of functions, C^s the space of functions with s continuous derivatives, and H_s the Sobolev space of order s. Abbreviate $\langle x \rangle^2 := 1 + |x|^2$.

Assumptions on the external potential. Let $W(x)$ be a C^3 function, and let $\min_x W(x) = 0$. Let $\beta \in \mathbb{Z}^d$ with $\beta_j \geq 0 \ \forall j = 1, \ldots, d$ be a multi-index. Given a number $r \geq 1$ let W be such that

$$|\partial_x^\beta W(x)| \leq C_{\max V} \langle x \rangle^{r - |\beta|} \text{ for } |\beta| \leq 3, \tag{2.1}$$

$$\mathrm{Hess}\, W(x) \geq \rho_1 \langle x \rangle^{r-2}, \tag{2.2}$$

and

$$W(x) \geq c_V |x|^r, \text{ for } |x| \geq c_L \tag{2.3}$$

for some positive constants $C_{\max V}$, ρ_1, c_V, c_L.

The number r is called the growth rate of the external potential. Here $\mathrm{Hess}\, W$ is the Hessian of W with respect to spatial variables. Define $V(x) := W(\epsilon_V x)$. Then, for $r \geq 1$,

$$|\partial_x^\beta V(x)| \leq C_V \epsilon_V^{|\beta|} \langle \epsilon_V x \rangle^{r - |\beta|}, \text{ for } |\beta| \leq 3, \tag{2.4}$$

$$\mathrm{Hess}\, V(x) \geq \rho_1 \epsilon_V^2 \langle \epsilon_V x \rangle^{r-2}, \tag{2.5}$$

and

$$V(x) \geq c_V (\epsilon_V |x|)^r, \text{ for } \epsilon_V |x| \geq c_L. \tag{2.6}$$

Assumptions on the initial condition ψ_0. The energy space, $\mathscr{H}_{1,r}$, for a given growth rate r of the external potential, is defined as

$$\mathscr{H}_{1,r} := \{\psi \in \mathrm{H}_1 : \langle x \rangle^{r/2} \psi \in \mathrm{L}^2\}. \tag{2.7}$$

Let $\mathscr{H}'_{1,r}$ denote the dual space of $\mathscr{H}_{1,r}$. The energy norm is defined as

$$\|\psi\|^2_{\mathscr{H}_{1,r}} := \|\psi\|^2_{\mathrm{H}_1} + \|\langle \epsilon_V x \rangle^{r/2} \psi\|^2_{\mathrm{L}^2} \tag{2.8}$$

We require $\psi_0 \in \mathscr{H}_{1,r}$.

In what follows, we identify complex functions with real two-component functions via

$$\mathbb{C} \ni \psi(x) = \psi_1(x) + \mathrm{i}\psi_2(x) \longleftrightarrow \vec{\psi}(x) = (\psi_1(x), \psi_2(x)) \in \mathbb{R}^2.$$

Consider a real function $F(\vec{\psi})$ on a space of real two-component functions, and let $F'(\vec{\psi})$ denote its L^2-gradient. We identify this gradient with a complex function denoted by $F'(\psi)$. Then

$$F'(\bar{\psi}) = \overline{F'(\psi)} \longleftrightarrow F(\sigma\vec{\psi}) = F(\vec{\psi}),$$

where $\sigma := \mathrm{diag}(1, -1)$, since the latter property is equivalent to $F'(\vec{\psi}) = \sigma F'(\sigma\vec{\psi})$.

Assumptions on the nonlinearity f.

1. (GWP [9, 34, 35, 20]) Equation (1.1) is globally well-posed in the space $\mathrm{C}(\mathbb{R}, \mathscr{H}_{1,r}) \cap \mathrm{C}^1(\mathbb{R}, \mathscr{H}'_{1,r})$. See Remark 2.1 below.

2. The nonlinearity f maps from H_1 to H_{-1}, with $f(0) = 0$. $f(\psi) = F'(\psi)$ is the L^2-gradient of a C^3 functional $F : H_1 \to \mathbb{R}$ defined on the space of real-valued, two-component functions, satisfying the following conditions:

   (a) (Bounds)

   $$\sup_{\|u\|_{\mathrm{H}_1} \leq M} \|F''(u)\|_{\mathrm{B}(\mathrm{H}_1, \mathrm{H}_{-1})} < \infty, \quad \sup_{\|u\|_{\mathrm{H}_1} \leq M} \|F'''(u)\|_{\mathrm{H}_1 \mapsto \mathrm{B}(\mathrm{H}_1, \mathrm{H}_{-1})} < \infty, \quad (2.9)$$

   where $\mathrm{B}(X, Y)$ denotes the space of bounded linear operators from X to Y.

   (b) (Symmetries [12]) $F(\mathcal{T}\psi) = F(\psi)$ where $\mathcal{T}$ is either translation $\psi(x) \mapsto \psi(x+a) \ \forall a \in \mathbb{R}^d$, or spatial rotation $\psi(x) \mapsto \psi(R^{-1}x)$, $\forall R \in \mathrm{SO}(d)$, or boosts $\mathcal{T}^{\mathrm{b}}_p : u(x) \mapsto \mathrm{e}^{\mathrm{i}p \cdot x} u(x), \forall p \in \mathbb{R}^d$, or gauge transformations $\psi \mapsto \mathrm{e}^{\mathrm{i}\gamma}\psi, \forall \gamma \in \mathbb{S}^1$, or complex conjugation $\psi \mapsto \bar{\psi}$.

3. (Solitary waves) There exists a bounded open interval $\tilde{I}$ on the positive real axis such that for all $\mu \in \tilde{I}$:

(a) (Ground state [3, 1, 2, 21]) The equation

$$-\Delta\psi + \mu\psi - f(\psi) = 0. \tag{2.10}$$

has a spherically symmetric, positive $\mathrm{L}^2 \cap \mathrm{C}^2$ solution, $\eta = \eta_\mu$.

(b) (Stability: see *e.g.*, [16]) This solution, η, satisfies

$$\partial_\mu \int \eta_\mu^2 \, \mathrm{d}^d x > 0. \tag{2.11}$$

(c) (Null space condition: see *e.g.*, [12]) Let $\mathcal{L}_\eta$ be the linear operator

$$\mathcal{L}_\eta := \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix} \tag{2.12}$$

where $L_1 := -\Delta + \mu - f^{(1)}(\eta)$, and $L_2 := -\Delta + \mu - f^{(2)}(\eta)$, with $f^{(1)} := \left(\partial_{\mathrm{Re}\,\psi} \big(\mathrm{Re}(f) \big) \right)(\eta)$, and $f^{(2)} := \left(\partial_{\mathrm{Im}\,\psi} \big(\mathrm{Im}(f) \big) \right)(\eta)$. We require that

$$\mathrm{N}(\mathcal{L}_\eta) = \mathrm{span}\{ \begin{pmatrix} 0 \\ \eta \end{pmatrix}, \begin{pmatrix} \partial_{x_j}\eta \\ 0 \end{pmatrix}, \ j = 1, \ldots, d\}. \tag{2.13}$$

Conditions 2–3 on the nonlinearity are discussed in [12], where further references can be found. Examples of nonlinearities that satisfy the above requirements are local nonlinearities

$$f(\psi) = \beta|\psi|^{s_1}\psi + \lambda|\psi|^{s_2}\psi, \ 0 < s_1 < s_2 < \frac{4}{d}, \ \beta \in \mathbb{R}, \ \lambda > 0, \tag{2.14}$$

and Hartree nonlinearities

$$f(\psi) = \lambda(\Phi * |\psi|^2)\psi, \ \lambda > 0, \tag{2.15}$$

where Φ is of positive type, continuous and spherically symmetric and tends to 0, as $|x| \to \infty$. Of course, λ can be scaled out by rescaling ψ. For precise conditions on Φ we refer to [9, 20].

Remark 2.1. *For Hartree nonlinearities global well-posedness is known for potentials $0 \le V \in \mathrm{L}^1_{loc}$ [20]. For local nonlinearities, the situation is more delicate. Global well-posedness and energy conservation is known for potentials with growth-rate $r \le 2$ [9]. For $r > 2$ and local nonlinearities, local well-posedness has been shown in the energy space [34, 35]. For local nonlinearities, a proof of the energy conservation needed for global well-posedness, and the application of this theory to our results, is missing.*

For $V \equiv 0$, Eq. (1.1) is the usual generalized nonlinear Schrödinger (or Hartree) equation. For self-focusing nonlinearities as in examples (2.14) and (2.15), it has stable solitary wave solutions of the form

$$\eta_{\sigma(t)}(x) := \mathrm{e}^{\mathrm{i}p(t)\cdot(x-a(t))+\mathrm{i}\gamma(t)}\eta_{\mu(t)}(x - a(t)), \tag{2.16}$$

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where $\sigma(t) := \{a(t), p(t), \gamma(t), \mu(t)\}$, and

$$a(t) = 2pt + a, \ \gamma(t) = \mu t + p^2 t + \gamma, \ p(t) = p, \ \mu(t) = \mu, \tag{2.17}$$

with $\gamma \in \mathbb{S}^1$, $a, p \in \mathbb{R}^d$ and $\mu \in \mathbb{R}^+$, and where η_μ is the spherically symmetric, positive solution of the nonlinear eigenvalue problem

$$-\Delta \eta + \mu \eta - f(\eta) = 0. \tag{2.18}$$

Recall from (1.6) that the linear map $\mathcal{S}_{ap\gamma}$ is defined as

$$(\mathcal{S}_{ap\gamma} g)(x) := e^{ip \cdot (x-a) + i\gamma} g(x - a). \tag{2.19}$$

In analyzing solitary wave solutions to (1.1) we encounter two length scales: the size $\propto \mu^{-1/2}$ of the support of the function η_μ, which is determined by our choice of initial condition ψ_0, and a length scale determined by the potential, V, measured by the small parameter ϵ_V. We consider the regime,

$$\frac{\epsilon_V}{\sqrt{\mu}} \ll 1. \tag{2.20}$$

We claim in the introduction that if ψ_0 is close to η_σ, for some σ then we retain control for times $\propto \epsilon^{-1}$. Restricting the initial condition to a smaller class of η_σ, with small initial energy, we retain control for longer times. In our main theorem, which proves this claim, we wish to treat both cases uniformly. To this end, let ϵ_h and K be positive numbers such that $\epsilon_h \in K[\epsilon_V, \min_{\mu \in I} \sqrt{\mu}]$ and assume

$$h(a_0, p_0) := \frac{1}{2}\big(p_0^2 + V(a)\big) \le \epsilon_h \tag{2.21}$$

(recall $\min_a V(a) = 0$). The lower bound for ϵ_h corresponds to our restricted class of initial data, the upper bound to the larger class of data. In particular, $\epsilon_h \ge K\epsilon_V$.

We are now ready to state our main result. Fix an open proper sub-interval $I \subset \tilde{I}$.

Theorem 2.1. *Let f and V satisfy the conditions listed above. There exists $T > 0$ such that for $\epsilon := \epsilon_V + \epsilon_0$ sufficiently small, and $\epsilon_h \ge K\epsilon_V$, if the initial condition ψ_0 satisfies*

$$\|\psi_0 - \mathcal{S}_{a_0 p_0 \gamma_0} \eta_{\mu_0}\|_{\mathrm{H}_1} + \|\langle \epsilon_V x \rangle^{r/2} (\psi_0 - \mathcal{S}_{a_0 p_0 \gamma_0} \eta_{\mu_0})\|_{\mathrm{L}^2} \le \epsilon_0 \tag{2.22}$$

for some $\sigma_0 := \{a_0, p_0, \gamma_0, \mu_0\} \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^1 \times I$ such that

$$h(a_0, p_0) \le \epsilon_h, \tag{2.23}$$

then for times $0 \le t \le T(\epsilon_V \sqrt{\epsilon_h} + \epsilon^2)^{-1}$, the solution to Eq. (1.1) with this initial condition is of the form

$$\psi(x, t) = \mathcal{S}_{a(t)p(t)\gamma(t)}\big(\eta_{\mu(t)}(x) + w(x, t)\big), \tag{2.24}$$

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where $\|w\|_{\mathrm{H}_1} + \left\|\langle\epsilon_V x\rangle^{r/2} w\right\|_{\mathrm{L}^2} \leq C\epsilon$. The modulation parameters a, p, γ and μ satisfy the differential equations

$$\dot{p} = -(\nabla V)(a) + \mathcal{O}(\epsilon^2), \tag{2.25}$$

$$\dot{a} = 2p + \mathcal{O}(\epsilon^2), \tag{2.26}$$

$$\dot{\gamma} = \mu - V(a) + p^2 + \mathcal{O}(\epsilon^2), \tag{2.27}$$

$$\dot{\mu} = \mathcal{O}(\epsilon^2). \tag{2.28}$$

Remark 2.2 (Remark about notation). *Fréchet derivatives are always understood to be defined on real spaces. They are denoted by primes. C and c denote various constants that often change between consecutive lines and which do not depend on ϵ_V, ϵ_0 or ϵ.*

3   Soliton manifold

In this section we recall the Hamiltonian nature of Eq. (1.1) and some of its symmetries. We also define the soliton manifold and its tangent space.

An important part in our approach is played by the variational character of (1.1). More precisely, the nonlinear Schrödinger equation (1.1) is a Hamiltonian system with Hamiltonian

$$\mathcal{H}_V(\psi) := \frac{1}{2} \int \left(|\nabla\psi|^2 + V|\psi|^2\right) \mathrm{d}^d x - F(\psi). \tag{3.1}$$

The Hamiltonian $\mathcal{H}_V$ is conserved *i.e.*,

$$\mathcal{H}_V(\psi) = \mathcal{H}_V(\psi_0). \tag{3.2}$$

A proof of this can be found, for local nonlinearities and $r \leq 2$, in *e.g.*, Cazenave [9], and for Hartree nonlinearities in [20]. An important role is played by the mass

$$\mathcal{N}(\psi) := \int |\psi|^2 \, \mathrm{d}^d x, \tag{3.3}$$

which also is conserved,

$$\mathcal{N}(\psi(t)) = \mathcal{N}(\psi_0). \tag{3.4}$$

We often identify complex spaces, such as the Sobolev space $\mathrm{H}_1(\mathbb{R}^d, \mathbb{C})$, with real spaces; *e.g.*, $\mathrm{H}_1(\mathbb{R}^d, \mathbb{R}^2)$, using the identification $\psi = \psi_1 + \mathrm{i}\psi_2 \leftrightarrow (\psi_1, \psi_2) =: \vec{\psi}$. With this identification, the complex structure i^{-1} corresponds to the operator

$$J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.5}$$

The real L^2-inner product in the real notation is

$$\langle \vec{u}, \vec{w} \rangle := \int (u_1 w_1 + u_2 w_2) \, \mathrm{d}^d x, \tag{3.6}$$

where $\vec{u} := (u_1, u_2)$. In the complex notation it becomes

$$\langle u, w \rangle := \text{Re} \int u\bar{w}\, \mathrm{d}^d x. \tag{3.7}$$

We henceforth abuse notation and drop the arrows. The symplectic form is

$$\omega(u, w) := \text{Im} \int u\bar{w}\, \mathrm{d}^d x. \tag{3.8}$$

We note that $\omega(u, w) = \langle u, J^{-1}v \rangle$ in the real notation.

Equation (1.1) with $V \equiv 0$ is invariant under spatial translations, $\mathcal{T}_a^{\text{tr}}$, gauge transformations, $\mathcal{T}_\gamma^{\text{g}}$, and boost transformations, $\mathcal{T}_p^{\text{boost}}$, where

$$\mathcal{T}_a^{\text{tr}} : \psi(x,t) \mapsto \psi(x-a, t)\,,\ \ \mathcal{T}_\gamma^{\text{g}} : \psi(x,t) \mapsto \mathrm{e}^{\mathrm{i}\gamma}\psi(x,t)\,, \tag{3.9}$$

$$\mathcal{T}_p^{\text{boost}} : \psi(x,t) \mapsto \mathrm{e}^{\mathrm{i}(p \cdot x - p^2 t)}\psi(x - 2pt, t)\,. \tag{3.10}$$

The transformations (3.9)–(3.10) map solutions of eq. (1.1) with $V \equiv 0$ into solutions of (1.1) with $V \equiv 0$.

Let $\mathcal{T}_p^{\text{b}} : \psi(x) \mapsto \mathrm{e}^{\mathrm{i}p \cdot x}\psi(x)$ be the $t = 0$ slice of the boost transform. The combined symmetry transformations $\mathcal{S}_{ap\gamma}$ introduced in (2.19) can be expressed as

$$\mathcal{S}_{ap\gamma}\eta = \mathcal{T}_a^{\text{tr}}\mathcal{T}_p^{\text{b}}\mathcal{T}_\gamma^{\text{g}}\eta_\mu(x) = \mathrm{e}^{\mathrm{i}(p \cdot (x-a) + \gamma)}\eta_\mu(x - a). \tag{3.11}$$

We define the soliton manifold as

$$\mathrm{M_s} := \{\mathcal{S}_{ap\gamma}\eta_\mu : \{a, p, \gamma, \mu\} \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^1 \times I\}\,. \tag{3.12}$$

The tangent space to this manifold at the solitary wave profile $\eta_\mu \in \mathrm{M_s}$ is given by

$$\mathrm{T}_{\eta_\mu}\mathrm{M_s} = \text{span}(z_\text{t}, z_\text{g}, z_\text{b}, z_\text{s})\,, \tag{3.13}$$

where

$$z_\text{t} := \nabla_a \mathcal{T}_a^{\text{tr}}\eta_\mu\big|_{a=0} = \begin{pmatrix} -\nabla\eta_\mu \\ 0 \end{pmatrix}\,, \qquad z_\text{g} := \frac{\partial}{\partial\gamma}\mathcal{T}_\gamma^{\text{g}}\eta_\mu\Big|_{\gamma=0} = \begin{pmatrix} 0 \\ \eta_\mu \end{pmatrix}\,, \tag{3.14}$$

$$z_\text{b} := \nabla_p \mathcal{T}_p^{\text{boost}}\eta_\mu\big|_{p=0,t=0} = \begin{pmatrix} 0 \\ x\eta_\mu \end{pmatrix}\,, \qquad z_\text{s} := \begin{pmatrix} \partial_\mu\eta_\mu \\ 0 \end{pmatrix}\,. \tag{3.15}$$

Above, we have explicitly written the basis of tangent vectors in the real space.

Recall that the equation (2.10) can be written as $\mathcal{E}'_\mu(\eta_\mu) = 0$ where

$$\mathcal{E}_\mu(\psi) = \mathcal{H}_{V\equiv 0}(\psi) + \frac{\mu}{2}\mathcal{N}(\psi).$$

Then the tangent vectors listed above are generalized zero modes of the operator $\mathcal{L}_\mu := \mathcal{E}''_\mu(\eta_\mu)$. That is, $(J\mathcal{L}_\mu)^2 z = 0$ for each tangent vector z above. To see this fact for z_g, for example, recall that $\mathcal{E}'_\mu(\psi)$ is gauge-invariant. Hence $\mathcal{E}'_\mu(\mathcal{T}_\gamma^{\text{g}}\eta_\mu) = 0$. Taking the derivative with respect to the parameter γ at $\gamma = 0$ gives $\mathcal{L}_\eta z_\text{g} = 0$. The other relations are derived analogously (see [33]).

4 Symplectically orthogonal decomposition

In this section we make a change of coordinates for the Hamiltonian system $\psi \mapsto (\sigma, w)$, where $\sigma := (a, p, \gamma, \mu)$. We also give the equations in this new set of coordinates.

Let

$$m(\mu) := \frac{1}{2} \int \eta_\mu^2(x) \, \mathrm{d}^d x. \tag{4.1}$$

Let

$$C_I := \max_{\substack{z \in \{x\eta_\mu, \eta_\mu, \nabla_\mu, \partial_\mu \eta_\mu\} \\ \mu \in \tilde{I}}} (\|z\|_{\mathrm{H}_1}, \|\langle \epsilon_V x \rangle^{r/2} z\|_{\mathrm{L}^2}, \|\mathcal{K}z\|_{\mathrm{L}^2}). \tag{4.2}$$

When it will not cause confusion, for $\sigma = \{a, p, \gamma, \mu\}$ we will abbreviate

$$\eta_\sigma := \mathcal{S}_{ap\gamma} \eta_\mu.$$

Now define the neighborhood of $\mathrm{M_s}$:

$$U_\delta := \{\psi \in L^2 : \inf_{\sigma \in \Sigma} \|\psi - \eta_\sigma\|_{\mathrm{L}^2} \leq \delta\}, \tag{4.3}$$

where $\Sigma := \{a, p, \gamma, \mu : a \in \mathbb{R}^d, p \in \mathbb{R}^d, \gamma \in \mathbb{S}^1, \mu \in I\}$. Our goal is to decompose a given function $\psi \in U_\delta$ into a solitary wave and a perturbation:

$$\psi = \mathcal{S}_{ap\gamma}(\eta_\mu + w). \tag{4.4}$$

We do this according to the following theorem. Let $\tilde{\Sigma} := \{a, p, \gamma, \mu : a \in \mathbb{R}^d, p \in \mathbb{R}^d, \gamma \in \mathbb{S}^1, \mu \in \tilde{I}\}$.

Theorem 4.1. *There exists $\delta > 0$ and a unique map $\varsigma \in \mathrm{C}^1(U_\delta, \tilde{\Sigma})$ such that (i)*

$$\langle \psi - \eta_{\varsigma(\psi)}, J^{-1} z \rangle = 0, \quad \forall z \in \mathrm{T}_{\eta_{\varsigma(\psi)}} \mathrm{M_s}, \quad \forall \psi \in U_\delta \tag{4.5}$$

and (ii) if, in addition, $\delta \ll (2C_I)^{-1} \min(m(\mu), m'(\mu))$ then there exists a constant c_I independent of δ such that

$$\sup_{\psi \in U_\delta} \|\varsigma'(\psi)\|_{\mathrm{L}^2} \leq c_I. \tag{4.6}$$

Proof. Part (i): Let the map $G : L^2 \times \tilde{\Sigma} \mapsto \mathbb{R}^{2d+2}$ be defined by

$$G_j(\psi, \varsigma) := \langle \psi - \eta_\varsigma, J^{-1} z_{\varsigma, j} \rangle, \quad \forall j = 1, \ldots 2d + 2. \tag{4.7}$$

Part (i) is proved by applying the implicit function theorem to the equation $G(\psi, \varsigma) = 0$, around a point (η_σ, σ). For details we refer to Proposition 5.1 in [12].

Part (ii): Abbreviate:

$$\Omega_{jk} := \langle \partial_{\varsigma_j} \eta_\varsigma, J^{-1} z_{\varsigma, k} \rangle, \tag{4.8}$$

where $z_{\varsigma,k}$ is the k:th element of $\mathcal{S}_{ap\gamma}\{z_{\mathrm{t}}, z_{\mathrm{g}}, z_{\mathrm{b}}, z_{\mathrm{s}}\}$. By explicitly inserting the tangent vectors, we find that $\|\Omega\|_{\mathrm{L}^2} \geq \inf_{\mu\in I}(m(\mu), m'(\mu))$. Thus, Ω is invertible by Condition 3b in Section 2.

From a variation of ψ in $G(\psi, \varsigma(\psi)) = 0$ we find

$$\varsigma'_k(\psi) = \sum_{j=1}^{2d+2} (J^{-1}z_\varsigma)_j (\tilde{\Omega}^{-1})_{jk}. \tag{4.9}$$

where

$$\tilde{\Omega}_{jk} := \Omega_{jk} + \langle \psi - \eta_{\varsigma(\psi)}, J^{-1}\partial_{\varsigma_j} z_{\varsigma,k}\rangle \tag{4.10}$$

Using the upper bound of δ, and the definition of C_I above, we find

$$\sup_{\psi\in U_\delta} \|\varsigma'(\psi)\|_{\mathrm{L}^2} \leq \frac{2C_I}{\inf_{\mu\in I}(m(\mu), m'(\mu))} =: c_I. \tag{4.11}$$

$$\square$$

We now assume $\psi(t) \in U_\delta \cap \mathscr{H}_{1,r}$, and set $\sigma(t) := \varsigma(\psi(t))$ as defined by Theorem 4.1. Write

$$u := \mathcal{S}_{ap\gamma}^{-1}\psi = \eta_\mu + w \tag{4.12}$$

so that w satisfies

$$\langle w, J^{-1}z\rangle = 0, \ \ \forall z \in \mathrm{T}_{\eta_\mu}\mathrm{M}_{\mathrm{s}}. \tag{4.13}$$

Here u is the solution in a moving frame.

Denote the anti-self-adjoint infinitesimal generators of symmetries as

$$\mathcal{K}_j = \partial_{x_j}, \ \ \mathcal{K}_{d+j} = \mathrm{i}x_j, \ \ \mathcal{K}_{2d+1} = \mathrm{i}, \ \ \mathcal{K}_{2d+2} = \partial_\mu, \ \ j = 1, ..., d \tag{4.14}$$

and define corresponding coefficients

$$\alpha_j = \dot{a}_j - 2p_j, \ \ \alpha_{d+j} = -\dot{p}_j - \partial_{x_j}V(a), \ \ j = 1, ..., d, \tag{4.15}$$

$$\alpha_{2d+1} = \mu - p^2 + \dot{a}\cdot p - V(a) - \dot{\gamma}, \ \ \alpha_{2d+2} = -\dot{\mu}. \tag{4.16}$$

Denote

$$\underline{\alpha} \cdot \underline{\mathcal{K}} := \sum_{j=1}^{2d+1} \alpha_j \mathcal{K}_j, \ \ \text{and} \ \ \alpha \cdot \mathcal{K} := \underline{\alpha} \cdot \underline{\mathcal{K}} + \alpha_{2d+2}\partial_\mu. \tag{4.17}$$

Substituting $\psi = \mathcal{S}_{ap\gamma}u$ into (1.1) we obtain

$$\mathrm{i}\dot{u} = \mathcal{E}'_\mu(u) + \mathcal{R}_V u + \mathrm{i}\underline{\alpha} \cdot \underline{\mathcal{K}}u, \tag{4.18}$$

where

$$\mathcal{R}_V(x) = V(x+a) - V(a) - \nabla V(a) \cdot x. \tag{4.19}$$

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To obtain the equations for (σ, w) we project Eqn. (4.18) onto $\mathrm{T}_\eta \mathrm{M}_\mathrm{s}$ and $(J\mathrm{T}_\eta \mathrm{M}_\mathrm{s})^\perp$ and use (4.12). We illustrate this method of deriving the equations for σ, for the projection of (4.18) along $\mathrm{i}\eta$:

$$\langle \eta, \dot{\mu}\partial_\mu \eta + \dot{w}\rangle = \langle \mathrm{i}\eta, \mathcal{L}_\eta w + N_\eta(w) + \mathcal{R}_V(\eta + w) + \mathrm{i}\underline{\alpha} \cdot \underline{\mathcal{K}}(\eta + w)\rangle. \tag{4.20}$$

where we have used $u = \eta + w$ and $\mathcal{E}'_\mu(u) = \mathcal{L}_\eta w + N_\eta(w)$ where $\mathcal{L}_\eta := \mathcal{E}''_\mu(\eta)$ is given explicitly as

$$\mathcal{L}_\eta w = -\Delta w + \mu w - f'(\eta)w. \tag{4.21}$$

In particular, for local nonlinearities of the form $g(|\psi|^2)\psi$, we have in the complex notation, since $\eta(x) \in \mathbb{R}$,

$$\mathcal{L}_\eta w := -\Delta w + \mu w - g(\eta^2)w - 2\eta g'(\eta^2)\,\mathrm{Re}\,w. \tag{4.22}$$

Here

$$N_\eta(w) := -f(\eta + w) + f(\eta) + f'(\eta)w. \tag{4.23}$$

We find the equation for $\dot{\mu}$ once we note that $\partial_t \langle \eta, w\rangle = 0$, $\mathcal{L}_\eta \mathrm{i}\eta = 0$, $\langle \mathrm{i}\eta, \mathcal{R}_V \eta\rangle = 0$, $\langle \eta, \underline{\mathcal{K}}\eta\rangle = 0$ and $\underline{\mathcal{K}}^* = -\underline{\mathcal{K}}$. Inserting this into (4.20) gives

$$\dot{\mu}m'(\mu) = \langle \mathrm{i}\eta, N_\eta(w) + \mathcal{R}_V w\rangle - \alpha \cdot \langle \mathcal{K}\eta, w\rangle. \tag{4.24}$$

The projection along the other directions works the same way: we use the fact that these directions are the generalized zero modes of $\mathcal{L}_\eta$, and furthermore that they are orthogonal to Jw. The calculations are worked out in detail in [12] (See Eqns. (6.20)–(6.22) in [12].) We give the result:

$$\dot{\gamma} = \mu - p^2 + \dot{a} \cdot p - V(a) - (m'(\mu))^{-1}\left(\langle \partial_\mu \eta, N_\eta(w) + \mathcal{R}_V w\rangle \right. \tag{4.25}$$
$$\left. -\alpha \cdot \langle \mathcal{K}\partial_\mu \eta, \mathrm{i}w\rangle + \langle \partial_\mu \eta, \mathcal{R}_V \eta\rangle\right),$$

$$\dot{\mu} = \left(m'(\mu)\right)^{-1}\left(\langle \mathrm{i}\eta, N_\eta(w) + \mathcal{R}_V w\rangle - \alpha \cdot \langle \mathcal{K}\eta, w\rangle\right), \tag{4.26}$$

$$\dot{a}_k = 2p_k + \left(m(\mu)^{-1}\right)\left(\langle \mathrm{i}x_k\eta, N_\eta(w) + \mathcal{R}_V w\rangle - \alpha \cdot \langle \mathcal{K}x_k\eta, w\rangle\right), \tag{4.27}$$

$$\dot{p}_k = -\partial_{a_k}V(a) + (m(\mu))^{-1}\left(-\frac{1}{2}\langle(\partial_{x_k}\mathcal{R}_V)\eta, \eta\rangle + \langle \partial_k\eta, N_\eta(w) + \mathcal{R}_V w\rangle\right.$$
$$\left. - \alpha \cdot \langle \mathcal{K}\partial_k\eta, \mathrm{i}w\rangle\right), \tag{4.28}$$

and

$$\mathrm{i}\dot{w} = \mathcal{L}_\eta w + N(w) + \mathcal{R}_V(\eta + w) + \mathrm{i}\underline{\alpha} \cdot \underline{\mathcal{K}}(\eta + w) - \mathrm{i}\dot{\mu}\partial_\mu \eta. \tag{4.29}$$

Note that the first two terms on the right-hand side of Eqn. (4.28) can be written as $-\partial_{a_k} V_{\text{eff}}(a, \mu)$, where

$$V_{\text{eff}}(a, \mu) := \|\eta_\mu\|_{\text{L}^2}^{-2} \int V(a+x)|\eta_\mu(x)|^2 \, \mathrm{d}^d x. \qquad (4.30)$$

Hence,

$$\dot{p}_k = -\nabla_a V_{\text{eff}}(a, \mu) + (m(\mu)^{-1}\langle \partial_{x_k} \eta_\mu, N_\eta(w)\rangle + \mathcal{O}(\|w\|_{\text{L}^2}(\epsilon_V^2 + |\alpha|)), \qquad (4.31)$$

where $|\alpha|^2 = \sum |\alpha_j|^2$.

Thus we have obtained the dynamical equations for (σ, w).

Remark 4.1. *The transformation*

$$\sigma := (a, p, \gamma, \mu) \mapsto \hat{\sigma} := (a, P, \gamma, m) \qquad (4.32)$$

with $P := \frac{1}{2}p\|\eta_\mu\|_{\text{L}^2}^2$ *and* $m := \frac{1}{2}\|\eta_\mu\|_{\text{L}^2}^2$ *gives a canonical symplectic structure and Darboux coordinates on* $\mathrm{M_s}$, *i.e., for* $w = 0$

$$\dot{P} = -\partial_a \mathcal{H}_V(\mathcal{S}_{ap\gamma}\eta_\mu), \qquad\qquad \dot{a} = \partial_P \mathcal{H}_V(\mathcal{S}_{ap\gamma}\eta_\mu), \qquad (4.33)$$
$$\dot{m} = \partial_\gamma \mathcal{H}_V(\mathcal{S}_{ap\gamma}\eta_\mu), \qquad\qquad \dot{\gamma} = -\partial_m \mathcal{H}_V(\mathcal{S}_{ap\gamma}\eta_\mu). \qquad (4.34)$$

Here $\nabla_{\hat{\sigma}} \mathcal{H}_V(\mathcal{S}_{ap\gamma}\eta_\mu) = (m\nabla_a V_{\text{eff}}, 2P/m, 0, -P^2/m^2 + V(a) - \mu)$.

5   Initial conditions $\tilde{\sigma}_0$, w_0.

In this section we use Theorem 4.1 in order to decompose the initial condition ψ_0 as (see Figure 5.1)

$$\psi_0 = \mathcal{S}_{\tilde{a}_0, \tilde{p}_0, \tilde{\gamma}_0}(\eta_{\tilde{\mu}_0} + w_0) \qquad (5.1)$$

so that $w_0 \perp J^{-1}\mathrm{T}_{\eta_{\tilde{\mu}_0}} \mathrm{M_s}$. This decomposition provides the initial conditions $\tilde{\sigma}_0$ and w_0, for the parameters, σ, and fluctuation, w (determined for later times by Theorem 4.1). The main work here goes into estimating w_0. Let $\varsigma : U_\delta \mapsto \tilde{\Sigma}$ be the map established in Theorem 4.1. Then $\tilde{\sigma}_0 = \{\tilde{a}_0, \tilde{p}_0, \tilde{\gamma}_0, \tilde{\mu}_0\}$ and w_0 are given as $\tilde{\sigma}_0 := \varsigma(\psi_0)$ and

$$w_0 := \mathcal{S}_{\tilde{a}_0 \tilde{p}_0 \tilde{\gamma}_0}^{-1}(\psi_0 - \eta_{\tilde{\sigma}_0}), \quad w_0 \perp J\mathrm{T}_{\eta_{\tilde{\mu}_0}} \mathrm{M_s}. \qquad (5.2)$$

Recall the definitions of $\mathcal{K}$ (4.14), and C_I (4.2). Theorem 4.1 states $\sup_{\psi \in U_\delta} \|\varsigma'(\psi)\|_{\text{L}^2} \le c_I$.

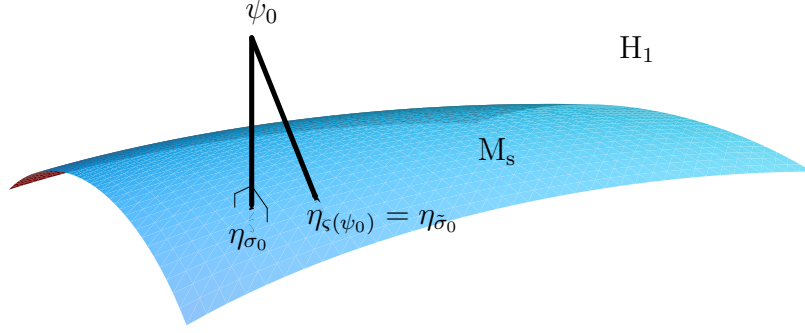Bounds for w_0 and $\tilde{\sigma}_0$ are stated in the following proposition

Figure 5.1: Orthogonal decomposition versus skew-orthogonal decomposition.

Proposition 5.1. *Let w_0 be defined as above. Let $\sigma_0 := \{a_0, p_0, \gamma_0, \mu_0\}$ and let ψ_0 satisfy $\|\psi_0 - \eta_{\sigma_0}\|_{L^2} \le \delta$ (where δ is from Theorem 4.1), and let $\psi_0 \in \mathscr{H}_{1,r}$. Then there exists positive constants C_1, C_2, such that*

$$|\tilde{\sigma}_0 - \sigma_0| \le c_I \|\psi_0 - \eta_{\sigma_0}\|_{L^2}, \tag{5.3}$$

$$\|w_0\|_{H_1} \le C_1(1 + p_0^4 + \|\psi_0 - \eta_{\sigma_0}\|_{L^2}^4)\|\psi_0 - \eta_{\sigma_0}\|_{H_1} \tag{5.4}$$

and

$$\|\langle \epsilon_V x \rangle^{r/2} w_0\|_{L^2} \le 3^{r/2} \|\langle \epsilon_V x \rangle^{r/2} (\psi_0 - \eta_{\sigma_0})\|_{L^2}$$
$$+ C_2(1 + |p_0|^2 + \epsilon_V^r |a_0|^r + \|\psi_0 - \eta_{\sigma_0}\|_{L^2}^2 + \epsilon_V^r \|\psi_0 - \eta_{\sigma_0}\|_{L^2}^r)\|\psi_0 - \eta_{\sigma_0}\|_{L^2}. \tag{5.5}$$

where C_1 and C_2 depend only on C_I, c_I and r, where C_I is defined in (4.2) and c_I in Theorem 4.1.

Proof. First we consider inequality (5.3). Abbreviate $\tilde{\sigma}_0 := \varsigma(\psi_0)$ and analogously for the components a, p, γ, μ of ς. Let $|\varsigma|^2 := \sum_{j=1}^{2d+2} |\varsigma_j|^2$. From Theorem 4.1 we know that $\varsigma(\psi)$ is a C^1-map. Thus, for $j \in 1, ..., 2d+2$ and some $\theta_1 \in [0,1]$

$$(\tilde{\sigma}_0 - \sigma_0)_j = \langle \varsigma_j'(\theta_1 \psi_0 + (1-\theta_1)\eta_{\sigma_0}), (\psi_0 - \eta_{\sigma_0}) \rangle. \tag{5.6}$$

Since $\sup_{\psi \in U_\delta} \|\varsigma'(\psi)\| \le c_I$ the inequality (5.3) follows.

Consider inequality (5.4) and rewrite $w(\cdot, 0) =: w_0$ from (5.2) as

$$w_0 = \mathcal{S}_{\tilde{a}_0 \tilde{p}_0 \tilde{\gamma}_0}^{-1}(\psi_0 - \eta_{\sigma_0}) + \mathcal{S}_{\tilde{a}_0 \tilde{p}_0 \tilde{\gamma}_0}^{-1}(\eta_{\sigma_0} - \eta_{\tilde{\sigma}_0}). \tag{5.7}$$

To estimate this, we first estimate the linear operator $\mathcal{S}_{ap\gamma}^{-1}$:

$$\|\mathcal{S}_{ap\gamma}^{-1}\psi\|_{H_1} \le 2(1 + |p|^2)^{1/2}\|\psi\|_{H_1}. \tag{5.8}$$

The first term in (5.7) is in the appropriate form, for the second term we recall that η is a C^1-map. Thus for some $\theta_2 \in [0,1]$

$$\eta_{\tilde{\sigma}_0} - \eta_{\sigma_0} = \sum_{j=1}^{2d+2} (\tilde{\sigma}_0 - \sigma_0)_j \partial_{\sigma_j} \eta_\sigma \big|_{\sigma = \theta_1 \tilde{\sigma}_0 + (1-\theta_2)\sigma_0}. \tag{5.9}$$

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To calculate the norm of this expression, note that

$$\partial_\sigma \eta_\sigma = \mathcal{S}_{ap\gamma} z_{\mu,p}, \text{ where } z_{\mu,p} := \{ip\eta_\mu + \nabla \eta_\mu, ix\eta_\mu, i\eta_\mu, \partial_\mu \eta_\mu\} \tag{5.10}$$

and $\|z_{\mu,p}\|_{H_1} \le \sqrt{5}C_I(1 + |p|^2)^{1/2}$. Let $n(\sigma, \sigma_0) := (\sigma - \sigma_0)\theta_2 + \sigma_0$, and define $g^2 := 1 + |\tilde{p}_0 - p_0|^2 + p_0^2$. The H_1-norm of (5.9), using (5.8) and (5.10) is

$$\begin{aligned}
\|\eta_{\tilde{\sigma}_0} - \eta_{\sigma_0}\|_{H_1} &\le |\tilde{\sigma}_0 - \sigma_0| \|\partial_\sigma \eta_\sigma\|_{H_1}\big|_{\sigma = n(\tilde{\sigma}_0, \sigma_0)} \\
&\le 2\sqrt{5}C_I(1 + |p|^2)\big|_{p = n(\tilde{p}_0, p_0)} |\tilde{\sigma}_0 - \sigma_0| \le 9C_I g^2 |\tilde{\sigma}_0 - \sigma_0|.
\end{aligned} \tag{5.11}$$

We now calculate the H_1 norm of w_0 (see (5.7)) using (5.3), (5.8) with momentum $p = \tilde{p}_0 - p_0 + p_0$ and (5.11). We find

$$\begin{aligned}
\|w_0\|_{H_1} &\le 2g(\|\psi_0 - \eta_{\sigma_0}\|_{H_1} + \|\eta_{\tilde{\sigma}_0} - \eta_{\sigma_0}\|_{H_1}) \\
&\le 2g\big(1 + 9C_I c_I g^2\big)\|\psi_0 - \eta_{\sigma_0}\|_{H_1}.
\end{aligned} \tag{5.12}$$

The coefficient above is less then $cg^4 + C$, and $g^4 \le 3(1 + c_I^4 \|\psi_0 - \eta_{\sigma_0}\|_{L^2}^4 + |p_0|^4)$. Inserting and simplifying gives the inequality (5.4).

The quantity appearing in the third and last inequality (5.5), can be rewritten as

$$\langle \epsilon_V x \rangle^{r/2} w_0 = \langle \epsilon_V x \rangle^{r/2} \mathcal{S}_{\tilde{a}_0 \tilde{p}_0 \tilde{\gamma}_0}^{-1} \big((\psi_0 - \eta_{\sigma_0}) + (\eta_{\sigma_0} - \eta_{\tilde{\sigma}_0})\big). \tag{5.13}$$

We begin our calculation of the norm of (5.13) by considering the linear operator $\langle \epsilon_V x \rangle^{r/2} \mathcal{S}_{ap\gamma}$. We have

$$\langle \epsilon_V x \rangle^{r/2} \mathcal{S}_{ap\gamma} \psi = \mathcal{S}_{ap\gamma} \langle \epsilon_V (x - a) \rangle^{r/2} \psi \tag{5.14}$$

and $\|\mathcal{S}_{ap\gamma} \psi\|_{L^2} = \|\psi\|_{L^2}$. From Lemma A.4 we obtain

$$\begin{aligned}
\|\langle \epsilon_V x \rangle^{r/2} \mathcal{S}_{ap\gamma} \psi\|_{L^2} &\le \|\langle \epsilon_V (x - (a - a_0) - a_0) \rangle^{r/2} \psi\|_{L^2} \\
&\le 3^{\max(r/2, r-1)} \big(\|\langle \epsilon_V x \rangle^{r/2} \psi\|_{L^2} + g_2 \|\psi\|_{L^2}\big),
\end{aligned} \tag{5.15}$$

where $g_2 := (\epsilon_V |a - a_0|)^{r/2} + (\epsilon_V |a_0|)^{r/2}$. Using this we find the L^2-norm of (5.13) to be

$$\begin{aligned}
\|\langle \epsilon_V x \rangle^{r/2} w_0\|_{L^2} \le C\big(&\|\langle \epsilon_V x \rangle^{r/2} (\psi_0 - \eta_{\sigma_0})\|_{L^2} + g_2 \|\psi_0 - \eta_{\sigma_0}\|_{L^2} \\
&+ \|\langle \epsilon_V x \rangle^{r/2} (\eta_{\tilde{\sigma}_0} - \eta_{\sigma_0})\|_{L^2} + g_2 \|\eta_{\tilde{\sigma}_0} - \eta_{\sigma_0}\|_{L^2}\big).
\end{aligned} \tag{5.16}$$

The first and second term of the above expression is in an appropriate form. We bound the third term by using (5.9), (5.10) and (5.14) to get

$$\begin{aligned}
\|\langle \epsilon_V x \rangle^{r/2} (\eta_{\tilde{\sigma}_0} - \eta_{\sigma_0})\|_{L^2} &\le |\tilde{\sigma}_0 - \sigma_0| \, \|\langle \epsilon_V (x - a) \rangle^{r/2} z_{p,\mu}\|_{L^2}\big|_{\sigma = n(\tilde{\sigma}_0, \sigma_0)} \\
&\le 3^{\max(r/2, r-1)} \sqrt{5} C_I g(1 + g_2) |\tilde{\sigma}_0 - \sigma_0|.
\end{aligned} \tag{5.17}$$

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The last term of (5.16) is straight forward to bound:

$$
\begin{aligned}
\|\eta_{\tilde{\sigma}_0} - \eta_{\sigma_0}\|_{\mathrm{L}^2} &\leq |\tilde{\sigma}_0 - \sigma_0|\ \|\partial_\sigma \eta_\sigma\|_{\mathrm{L}^2}\big|_{\sigma = n(\tilde{\sigma}_0, \sigma_0)} \\
&\leq |\tilde{\sigma}_0 - \sigma_0|\|z_{p,\mu}\|_{\mathrm{L}^2}\big|_{p = n(\tilde{p}_0, p_0)\mu = n(\tilde{\mu}_0, \mu_0)} \leq \sqrt{5} C_I g |\tilde{\sigma}_0 - \sigma_0|.
\end{aligned}
\tag{5.18}
$$

Inserting (5.17) and (5.18) into (5.16) gives

$$
\begin{aligned}
\|\langle \epsilon_V x \rangle^{r/2} w_0\|_{\mathrm{L}^2} \leq C\Big(\|\langle \epsilon_V x \rangle^{r/2}(\psi_0 - \eta_{\sigma_0})\|_{\mathrm{L}^2} \\
+ \big(g_2 + g(1 + 2g_2)\big)\|\psi_0 - \eta_{\sigma_0}\|_{\mathrm{L}^2}\Big),
\end{aligned}
\tag{5.19}
$$

where C depend only on C_I, c_I and r. We simplify this, by repeatedly using Cauchy's inequality and (5.3) on the expression in front of the $\|\psi_0 - \eta_{\sigma_0}\|_{\mathrm{L}^2}$-term, to obtain

$$
\begin{aligned}
\|\langle \epsilon_V x \rangle^{r/2} w_0\|_{\mathrm{L}^2} \leq C\Big(\|\langle \epsilon_V x \rangle^{r/2}(\psi_0 - \eta_{\sigma_0})\|_{\mathrm{L}^2} + \big(1 + \|\psi_0 - \eta_{\sigma_0}\|_{\mathrm{L}^2}^2 \\
+ \epsilon_V^r \|\psi_0 - \eta_{\sigma_0}\|_{\mathrm{L}^2}^r + |p_2|^2 + (\epsilon_V|a_0|)^r\big)\|\psi_0 - \eta_{\sigma_0}\|_{\mathrm{L}^2}\Big).
\end{aligned}
\tag{5.20}
$$

This gives the third inequality of the proposition. □

Recall the initial energy bound (2.22)

$$
\|\psi_0 - \eta_{\sigma_0}\|_{\mathrm{H}_1} + \|\langle \epsilon_V x \rangle^{r/2}(\psi_0 - \eta_{\sigma_0})\|_{\mathrm{L}^2} \leq \epsilon_0,
\tag{5.21}
$$

and the bound on the initial kinetic and potential energy for the solitary wave (2.23)

$$
\frac{1}{2}(p_0^2 + V(a_0)) \leq \epsilon_h.
\tag{5.22}
$$

We have the corollary

Corollary 5.2. *Let (2.22), (2.23) and (2.4)–(2.6) hold with $\epsilon_0 < \delta$. Then*

$$
|\tilde{\sigma}_0 - \sigma_0| \leq c_I \epsilon_0, \quad \|w_0\|_{\mathrm{H}_1} \leq C_1 \epsilon_0,
\tag{5.23}
$$

$$
\|\langle \epsilon_V x \rangle^{r/2} w_0\|_{\mathrm{L}^2} \leq C_2 \epsilon_0
\tag{5.24}
$$

and

$$
h(\tilde{a}_0, \tilde{p}_0) \leq C_3(\epsilon_h + \epsilon_0^2 + \epsilon_V \epsilon_0),
\tag{5.25}
$$

where C_1, C_2 and C_3 depend only on c_L, c_V (Eq. (2.6)), $C_E := \max(\epsilon_V, \epsilon_0, \epsilon_h)$ and the constants in Proposition 5.1.

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Proof. Starting from Proposition 5.1 the first three inequalities follow directly through the energy bounds (2.22), (2.23) together with the observation that either $\epsilon_V|a_0| \le c_L$ or $c_V(\epsilon_V|a_0|)^r \le V(a_0) \le 2\epsilon_h$. We also use that ϵ_h, ϵ_0 and ϵ_V are all bounded by a constant C_E.

The last inequality follows from the fact that $h(a,p) := (p^2 + V(a))/2$ is a C^1 function. For some $\theta \in [0,1]$

$$
\begin{aligned}
h(a,p) - h(a_0,p_0) &= ((p - p_0)\theta + p_0) \cdot (p - p_0) \\
&\quad + \frac{1}{2}(a - a_0) \cdot \nabla V((a - a_0)\theta + a_0).
\end{aligned}
\tag{5.26}
$$

Thus, using (2.4), and $\langle x + y \rangle^{r-1} \le 3^{\max(0,(r-3)/2)}\big(1 + 2^{(r-1)/2}(|x|^{r-1} + |y|^{r-1})\big)$ gives

$$
\begin{aligned}
|h(a,p) - h(a_0,p_0)| \le C\Big(&|p - p_0|^2 + |p_0|^2 + \\
&\epsilon_V^2|a - a_0|\big(1 + |\epsilon_V(a - a_0)|^{r-1} + |\epsilon_V a_0|^{r-1}\big)\Big).
\end{aligned}
\tag{5.27}
$$

With $p = \tilde{p}_0$ and $a = \tilde{a}_0$ above, and $|\tilde{\sigma}_0 - \sigma_0| \le c_I \epsilon_0$, $h(a_0,p_0) \le \epsilon_h$, (2.23) and (2.6) we have have shown (5.25). $\square$

6 Bounds on soliton position and momentum

In this section we use the bounded initial soliton energy, Corollary 5.2, to find upper bounds on position and momentum of the solitary wave. We express the norms first in terms of $h(\tilde{a}_0, \tilde{p}_0)$ and the small parameters. In Corollary 6.2 we state the final result, where the bounds are just constants times the small parameters ϵ_0, ϵ_h and ϵ_V.

Recall (see (2.4) and (2.6)) that the potential V is non-negative and satisfies the following upper and lower bounds:

$$
|\partial_x^\beta V| \le C_V \epsilon_V \langle \epsilon_V a \rangle^{r-1}, \text{ for } |\beta| = 1,
\tag{6.1}
$$

and, if $\epsilon_V|a| \ge c_L$ then

$$
V(a) \ge c_V(\epsilon_V|a|)^r.
\tag{6.2}
$$

To obtain the desired estimates on a and p we will use the fact that the soliton energy,

$$
h(a,p) := \frac{1}{2}\big(p^2 + V(a)\big),
\tag{6.3}
$$

is essentially conserved. We abbreviate $\alpha := \{\alpha^{\mathrm{tr}}, \alpha^{\mathrm{b}}, \alpha_{2d+1}, \alpha_{2d+2}\}$. The size of α is measured by $|\alpha|^2 := \sum_j |\alpha_j|^2$ and $|\alpha|_\infty := \sup_{s \le t}|\alpha(s)|$. We have the following:

Proposition 6.1. *Let V satisfy conditions (6.1) and (6.2). Let $h_0 := h(\tilde{a}_0, \tilde{p}_0)$, and set*

$$\tilde{T}_1 := \frac{C_{T_1}}{(\epsilon_V^2 + |\alpha|_\infty)(1 + \epsilon_V + h_0)}, \quad C_{\tilde{T}_1} := \frac{c_V}{2^{\max(2,r-1)/2} C_V d}, \qquad (6.4)$$

where the constants C_V and c_V are related to the growth rate of the potential (see (2.4) and (2.6)). Then for times $t \leq \tilde{T}_1$:

$$|p| \leq C_{\tilde{p}}(\sqrt{h_0} + |\alpha|_\infty t + \epsilon_V) \text{ and } \epsilon_V |a| \leq C_a, \qquad (6.5)$$

where C_a and $C_{\tilde{p}}$ depend only on c_L, c_V, $C_{\tilde{T}_1}$, r, d, C_3 and $C_E = \max(\epsilon_V, \epsilon_0, \epsilon_h)$. C_3 is the constant in Corollary 5.2 and

Proof. First we estimate p in terms of a, using the almost conservation of $h(a, p)$

$$\frac{d}{dt} h(a, p) = \frac{1}{2} \left(2p \cdot (\dot{p} + \nabla V(a)) + \nabla V(a) \cdot (\dot{a} - 2p)\right). \qquad (6.6)$$

Now recall the definitions $\alpha^{\mathrm{b}} := -\dot{p} - \nabla V(a)$ and $\alpha^{\mathrm{tr}} := \dot{a} - 2p$ together with the upper bound (6.1) of the potential $|\nabla V| \leq d^{1/2} C_V \epsilon_V \langle \epsilon_V a \rangle^{r-1}$ to obtain

$$|\mathrm{d}_t h(a, p)| \leq |\alpha||p| + \frac{1}{2} C_V d^{1/2} \epsilon_V |\alpha| \langle \epsilon_V a \rangle^{r-1}. \qquad (6.7)$$

Integration in time and simplification gives

$$h(a(t), p(t)) \leq h_0 + t(|\alpha|_\infty) \left(|p|_\infty + 2^{-1} d^{1/2} C_V \epsilon_V \langle \epsilon_V |a|_\infty \rangle^{r-1}\right). \qquad (6.8)$$

Recall that $h = 2^{-1}(p^2 + V(a))$ and that $V \geq 0$, thus $|p|^2 \leq 2h$. Solving the resulting quadratic inequality for $|p|_\infty > 0$ we find that

$$|p|_\infty \leq \sqrt{2h_0} + 3t|\alpha|_\infty + 2^{-1} d^{1/2} C_V \epsilon_V \langle \epsilon_V |a|_\infty \rangle^{r-1}. \qquad (6.9)$$

The Eqn. (6.8) also implies

$$\sup_{s \leq t} V(a(s)) \leq 2h_0 + 2t|\alpha|_\infty \left(|p|_\infty + 2^{-1} d^{1/2} C_V \epsilon_V \langle \epsilon_V |a|_\infty \rangle^{r-1}\right). \qquad (6.10)$$

As can be seen in (6.9) we need to consider the possibility of large $\epsilon_V |a|$. Let $\epsilon_V |a| \geq c_L$, with c_L as in (6.2) then $V(a) \geq c_V(\epsilon_V |a|)^r$. Inserting this lower bound and (6.9) into (6.10) we obtain

$$c_V(\epsilon_V |a|_\infty)^r \leq 2h_0 + 2t|\alpha|_\infty \left(\sqrt{2h_0} + 3t|\alpha|_\infty + d^{1/2} C_V \epsilon_V \langle \epsilon_V |a|_\infty \rangle^{r-1}\right). \qquad (6.11)$$

Lemma A.4 shows $\langle \epsilon_V |a|_\infty \rangle^{r-1} \leq 2^{\max(0,r-3)/2}(1 + (\epsilon_V |a|_\infty)^{r-1})$ for $r \geq 1$. If the maximal time satisfies the inequality $t \leq \tilde{T}_1$ (see (6.4)), then the above inequality implies

$$\epsilon_V |a|_\infty \leq (\frac{2}{c_V}(C_4 + 2C_{\tilde{T}_1} + 6C_{\tilde{T}_1}^2 + \frac{1}{2} c_V)^{1/r} =: \tilde{C}_a, \qquad (6.12)$$

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where we have used that h_0 is bounded by the constant C_E. Thus, either $\epsilon_V |a| \leq c_L$ holds or, for the given time interval, (6.12) holds. In both cases $\epsilon_V |a| \leq C_a$, where the constant only depends on $C_4 = C_3 C_E$, $C_{\tilde{T}_1}$, c_V, c_L and r. We insert this upper bound on $\epsilon_V |a|$ into (6.9) and for times $t \leq \tilde{T}_1$ we find

$$|p|_\infty \leq C_{\tilde{p}}(\sqrt{h_0} + |\alpha|_\infty t + \epsilon_V), \tag{6.13}$$

where $C_{\tilde{p}} := 3 + d^{1/2} C_V C_{\tilde{a}}^{r-1}$. $\qquad\square$

Using the Corollary 5.2 we express the above proposition in terms of ϵ_h rather than h_0. Recall the requirement on δ from Theorem 4.1

Corollary 6.2. *Let V satisfy (2.4)–(2.6) and let $\psi_0 \in U_\delta \cap \mathscr{H}_{1,r}$. Furthermore, let ψ_0 satisfy the ϵ_0-energy bound (2.22) for η_{σ_0} with $\sigma_0 = \{a_0, p_0, \gamma_0, \mu_0\}$, and let $h(a_0, p_0) \leq \epsilon_h$ (i.e., (2.23)). Let*

$$T_1 := \frac{C_{T_1}}{(\epsilon_V^2 + |\alpha|_\infty)(1 + \epsilon_V + \epsilon_h + \epsilon_V)}, \quad T_2 := \frac{\sqrt{\epsilon_h}}{|\alpha|_\infty + \epsilon_V^2}, \tag{6.14}$$

where

$$C_{T_1} := \frac{C_{\tilde{T}_1}}{(1 + C_3)(1 + C_E^2)}. \tag{6.15}$$

Then for times $t \leq \min(T_1, T_2)$:

$$|p| \leq C_p(\sqrt{\epsilon_h} + \epsilon_0 + \epsilon_V) \text{ and } \epsilon_V |a| \leq C_a, \tag{6.16}$$

where C_p depends on $C_E = \max(\epsilon_V, \epsilon_0, \epsilon_h)$, C_V, d, r and C_a. C_3 is defined in Corollary 5.2 and C_a in Proposition 6.1. The constant C_V is defined in (2.4).

Proof. Under the assumptions of the corollary we have that Corollary 5.2 holds and hence

$$h(\tilde{a}_0, \tilde{p}_0) \leq C_3(\epsilon_h + \epsilon_0^2 + \epsilon_V \epsilon_0). \tag{6.17}$$

We now modify the constants and estimates of Proposition 6.1 to take the upper bound of h_0 into account. The new, maximal time derived from $\tilde{T}_1$ becomes $T_1 \leq \tilde{T}_1$. For times shorter than this time, $t \leq T_1$, the bound on $\epsilon_V |a|$ remains the same. Using this estimate for $\epsilon_V |a|$, we simplify the $|p|$ estimate. Note first that $\sqrt{h_0} + \epsilon_V \leq (\sqrt{\epsilon_h} + \epsilon_0 + \epsilon_V)(1 + 2\sqrt{C_3})$, inserted into (6.5) gives

$$|p| \leq \frac{1}{2} C_p(\sqrt{\epsilon_h} + \epsilon_V + \epsilon_0 + |\alpha|t), \tag{6.18}$$

where C_p depends on C_3, C_E, C_a and d and r. With the choice of time interval T_2 such that $t \leq T_2$, where T_2 is given in (6.14), we obtain $|p| \leq C_p(\sqrt{\epsilon_h} + \epsilon_0 + \epsilon_V)$. $\qquad\square$

7 Lyapunov functional

In this section we define the Lyapunov functional and calculate its time derivative in the moving frame. Recall the definition of $\mathcal{E}_\mu(\psi)$ in (1.20) together with decomposition (4.4): $\psi = \mathcal{S}_{ap\gamma}(\eta_\mu + w)$, with $w \perp J\mathrm{T}_\eta \mathrm{M}_{\mathrm{s}}$. Define the Lyapunov functional, Λ, as

$$\Lambda := \mathcal{E}_\mu(\eta_\mu + w) + \frac{1}{2}\langle \mathcal{R}_V(\eta_\mu + w), \eta_\mu + w \rangle - \mathcal{E}_\mu(\eta_\mu) - \frac{1}{2}\langle \mathcal{R}_V \eta_\mu, \eta_\mu \rangle. \tag{7.1}$$

Here we show that the Lyapunov functional Λ is an almost conserved quantity. We begin by computing its time derivative. Let $\alpha^{\mathrm{b}} := -\dot{p} - \nabla V(a)$ and $\alpha^{\mathrm{tr}} := \dot{a} - 2p$ (boost and translation coefficients). We have the following proposition

Proposition 7.1. *Given a solution $\psi \in \mathscr{H}_{1,r} \cap U_\delta$ to (1.1), define η_μ and w as above. Then*

$$\frac{d}{dt}\Lambda = p \cdot \langle \nabla_a \mathcal{R}_V w, w \rangle - \alpha^{\mathrm{tr}} \cdot \mathrm{D}^2 V(a) \cdot \langle xw, w \rangle + R, \tag{7.2}$$

where

$$R := \alpha^{\mathrm{b}} \cdot \langle \mathrm{i}w, \nabla w \rangle + 2p \cdot \langle \nabla_a \mathcal{R}_V \eta_\mu, w \rangle - \frac{1}{2}\alpha^{\mathrm{tr}} \cdot \langle \nabla_a \mathcal{R}_V \eta_\mu, \eta_\mu \rangle$$
$$+ \frac{\dot\mu}{2}\|w\|_{\mathrm{L}^2}^2 - \dot\mu \langle \mathcal{R}_V \eta_\mu, \partial_\mu \eta_\mu \rangle. \tag{7.3}$$

Before proceeding to the proof, we recall the definition of the moving frame solution u defined by

$$u(x,t) := \mathrm{e}^{-\mathrm{i}p \cdot x - \mathrm{i}\gamma}\psi(x + a, t). \tag{7.4}$$

Here a, p and γ depend on time, in a way determined by the splitting of Section 4, and the function ψ is a solution of the nonlinear Schrödinger equation (1.1). In the moving frame the Lyapunov functional Λ takes the form

$$\Lambda = \mathcal{E}_\mu(u) + \frac{1}{2}\langle \mathcal{R}_V u, u \rangle - \mathcal{E}_\mu(\eta_\mu) - \frac{1}{2}\langle \mathcal{R}_V \eta_\mu, \eta_\mu \rangle. \tag{7.5}$$

We begin with some auxiliary lemmas.

Lemma 7.2. *Let $\psi \in \mathscr{H}_{1,r}$ be a solution to (1.1). Then*

$$\partial_t \langle \psi, -\mathrm{i}\nabla\psi \rangle = -\langle (\nabla V)\psi, \psi \rangle \;\; and \;\; \partial_t \langle x\psi, \psi \rangle = 2\langle \psi, -\mathrm{i}\nabla\psi \rangle. \tag{7.6}$$

Proof. The first part of this lemma was proved in [12]. To prove the second part we use the equation

$$\partial_t(x_k|\psi|^2) = \mathrm{i}\nabla \cdot (x_k \bar\psi \nabla\psi - x_k \psi \nabla\bar\psi) - \mathrm{i}(\bar\psi \partial_k \psi - \psi \partial_k \bar\psi), \tag{7.7}$$

understood in a weak sense, which follows from the nonlinear Schrödinger equation (1.1). Formally, integrating this equation and using that the divergence term vanishes

gives the second equation in (7.6). To do this rigorously, let χ be a C^1 function such that $|\nabla \chi(x)| \leq C$ and

$$\chi(x) := \begin{cases} 1 & |x| \leq 1, \\ 0 & |x| > 2, \end{cases} \tag{7.8}$$

and let $\chi_R(x) := \chi(\frac{x}{R})$. Abbreviate $j_k := (x_k \bar{\psi} \nabla \psi - x_k \psi \nabla \bar{\psi})$ and let $R > 1$. We multiply the divergence term by χ_R. Integration by parts gives

$$\left| \int (\nabla \cdot j_k) \chi_R \, \mathrm{d}^d x \right| = \left| \int j_k \cdot \nabla \chi_R(x) \, \mathrm{d}^d x \right| \leq \frac{C}{R} \int |j_k| \, \mathrm{d}^d x. \tag{7.9}$$

We note that $j_k \in \mathrm{L}^1$ for all k, and is independent of R, thus as $R \to \infty$, this term vanishes. The remaining terms give in the limit $R \to \infty$ the second equation in (7.6). $\quad \square$

Lemma 7.3. *Let $\psi \in \mathscr{H}_{1,r}$ be a solution to (1.1), and let u be defined as above. Then*

$$\frac{d}{dt}\left(\mathcal{E}_\mu(u) + \frac{1}{2}\langle \mathcal{R}_V u, u \rangle\right) = p \cdot \langle \nabla_a \mathcal{R}_V u, u \rangle - \frac{1}{2}\alpha^{\mathrm{tr}} \cdot \mathrm{D}^2 V(a) \cdot \langle xu, u \rangle$$

$$+ \frac{1}{2}\dot{\mu}\|u\|_{\mathrm{L}^2}^2 + \alpha^{\mathrm{b}} \cdot \langle iu, \nabla u \rangle, \tag{7.10}$$

where $\alpha^{\mathrm{tr}} := \dot{a} - 2p$ and $\alpha^{\mathrm{b}} = -\dot{p} - \nabla V(a)$.

Proof. The functional $\mathcal{E}_\mu(u) + \frac{1}{2}\langle \mathcal{R}_V u, u \rangle$, is related to the Hamiltonian functional by

$$\mathcal{E}_\mu(u) + \frac{1}{2}\langle \mathcal{R}_V u, u \rangle = \mathcal{H}_V(\psi) + \frac{1}{2}(p^2 + \mu)\|\psi\|_{\mathrm{L}^2}^2 - p \cdot \langle i\psi, \nabla \psi \rangle$$

$$- \frac{1}{2}\int (V(a) + \nabla V(a) \cdot (x - a))|\psi|^2 \, \mathrm{d}^d x, \tag{7.11}$$

which is obtained by substituting (7.4) into $\mathcal{E}_\mu(u) + \frac{1}{2}\langle \mathcal{R}_V u, u \rangle$. Using the facts that the mass $\|\psi\|_{\mathrm{L}^2}^2$ and Hamiltonian $\mathcal{H}_V(\psi)$ are time independent, together with the Ehrenfest relations, Lemma 7.2, we obtain

$$\frac{d}{dt}\left(\mathcal{E}_\mu(u) + \frac{1}{2}\langle \mathcal{R}_V u, u \rangle\right) = (\frac{\dot{\mu}}{2} + p \cdot \dot{p})\|\psi\|_{\mathrm{L}^2}^2 - \dot{p} \cdot \langle i\psi, \nabla \psi \rangle + p \cdot \langle (\nabla V)\psi, \psi \rangle$$

$$- \frac{\dot{a}}{2} \cdot \mathrm{D}^2 V(a) \cdot \int (x - a)|\psi|^2 \, \mathrm{d}^d x - \nabla V(a) \cdot \langle i\psi, \nabla \psi \rangle.$$

Collecting $p \cdot \dot{p}$ and $p \cdot \nabla V$ together, and combining $\dot{p}$ and $\nabla V(a)$ gives

$$\frac{d}{dt}\left(\mathcal{E}_\mu(u) + \frac{1}{2}\langle \mathcal{R}_V u, u \rangle\right) = \frac{\dot{\mu}}{2}\|\psi\|_{\mathrm{L}^2}^2 + p \cdot \langle (\dot{p} + \nabla V)\psi, \psi \rangle$$

$$- (\dot{p} + \nabla V(a)) \cdot \langle i\psi, \nabla \psi \rangle - \frac{1}{2}\dot{a} \cdot \mathrm{D}^2 V(a) \cdot \int (x - a)|\psi|^2 \, \mathrm{d}^d x. \tag{7.12}$$

From the definition of u, (7.4), the following relations hold

$$\|\psi\|_{\mathrm{L}^2} = \|u\|_{\mathrm{L}^2}, \qquad \langle \mathrm{i}\psi, \nabla\psi \rangle = p\|u\|_{\mathrm{L}^2}^2 + \langle \mathrm{i}u, \nabla u \rangle, \tag{7.13}$$

$$\langle (\nabla V)\psi, \psi \rangle = \langle (\nabla V_a)u, u \rangle, \qquad \langle (x-a)\psi, \psi \rangle = \langle xu, u \rangle. \tag{7.14}$$

Substitution of (7.13)–(7.14) into (7.12) gives, after cancellation of the $p \cdot \dot{p}$ terms,

$$\frac{d}{dt}\big(\mathcal{E}_\mu(u) + \tfrac{1}{2}\langle \mathcal{R}_V u, u \rangle\big) = \frac{\dot{\mu}}{2}\|u\|_{\mathrm{L}^2}^2 + p \cdot \langle (\nabla V_a - \nabla V(a))u, u \rangle$$

$$- (\dot{p} + \nabla V(a)) \cdot \langle \mathrm{i}u, \nabla u \rangle - \frac{1}{2}\dot{a} \cdot \mathrm{D}^2 V(a) \cdot \int x|u|^2 \, \mathrm{d}^d x. \tag{7.15}$$

The last remaining step is to rewrite the second last term as $\dot{a} - 2p + 2p$ and combine its p term with the difference of the potentials, recalling the definition of $\mathcal{R}_V$, to obtain

$$\frac{d}{dt}\big(\mathcal{E}_\mu(u) + \tfrac{1}{2}\langle \mathcal{R}_V u, u \rangle\big) = \frac{\dot{\mu}}{2}\|u\|_{\mathrm{L}^2}^2 + p \cdot \langle (\nabla_a \mathcal{R}_V)u, u \rangle$$

$$- (\dot{p} + \nabla V(a)) \cdot \langle \mathrm{i}u, \nabla u \rangle + \frac{1}{2}(2p - \dot{a}) \cdot \mathrm{D}^2 V(a) \cdot \int x|u|^2 \, \mathrm{d}^d x. \tag{7.16}$$

Identification of the boost coefficient $\alpha^{\mathrm{b}} := -\dot{p} - \nabla V(a)$ and the translation coefficient $\alpha^{\mathrm{tr}} := \dot{a} - 2p$ gives the lemma. $\qquad\square$

The time derivative of the second part of the Lyapunov functional (7.5) is computed in the next lemma.

Lemma 7.4. *Let η_μ be the solution of* (2.18), *and let μ depend on t. Then*

$$\frac{d}{dt}\big(\mathcal{E}_\mu(\eta_\mu) + \tfrac{1}{2}\langle \mathcal{R}_V \eta_\mu, \eta_\mu \rangle\big) =$$

$$\frac{\dot{\mu}}{2}\|\eta_\mu\|_{\mathrm{L}^2}^2 + (p + \tfrac{1}{2}\alpha^{\mathrm{tr}}) \cdot \langle \nabla_a \mathcal{R}_V \eta_\mu, \eta_\mu \rangle + \dot{\mu}\langle \mathcal{R}_V \eta_\mu, \partial_\mu \eta_\mu \rangle, \tag{7.17}$$

where $\alpha^{\mathrm{tr}} := \dot{a} - 2p$.

Proof. The result follows directly, upon recalling that $\mathcal{E}'_\mu(\eta_\mu) = 0$ and $\tfrac{1}{2}\alpha^{\mathrm{tr}} + p = \tfrac{\dot{a}}{2}$. $\quad\square$

To proceed to the proof of Proposition 7.1, we restate our condition for unique decomposition of the solution to the nonlinear Schrödinger equation, $\psi \in U_\delta \cap \mathscr{H}_{1,r}$, in terms of u:

$$u = \eta_\mu + w \quad \text{and} \quad w \perp J\mathrm{T}_\eta \mathrm{M}_{\mathrm{s}}. \tag{7.18}$$

Given Lemma 7.3 and Lemma 7.4, Proposition 7.1 follows directly.

Proof of Proposition 7.1. Lemma 7.3 states

$$\frac{d}{dt}\big(\mathcal{E}_\mu(u) + \frac{1}{2}\langle \mathcal{R}_V u, u\rangle\big) = p \cdot \langle \nabla_a \mathcal{R}_V u, u\rangle - \frac{1}{2}\alpha^{\mathrm{tr}} \cdot \mathrm{D}^2 V(a) \cdot \langle xu, u\rangle$$
$$+ \frac{1}{2}\dot{\mu}\|u\|_{\mathrm{L}^2}^2 + \alpha^{\mathrm{b}} \cdot \langle \mathrm{i}u, \nabla u\rangle. \tag{7.19}$$

Insert $u = \eta_\mu + w$ above, and use $w \perp \{\eta_\mu, \mathrm{i}\nabla\eta_\mu, x\eta_\mu\}$. Recall that η_μ is a real valued symmetric function, hence $\langle x\eta_\mu, \eta_\mu\rangle = 0$ as well as $\langle \mathrm{i}\eta_\mu, \nabla\eta_\mu\rangle = 0$. We obtain

$$\frac{d}{dt}\big(\mathcal{E}_\mu(u) + \frac{1}{2}\langle \mathcal{R}_V u, u\rangle\big) =$$
$$p \cdot \big(\langle \nabla_a \mathcal{R}_V w, w\rangle + 2\langle \nabla_a \mathcal{R}_V \eta_\mu, w\rangle + \langle \nabla_a \mathcal{R}_V \eta_\mu, \eta_\mu\rangle\big)$$
$$- \frac{1}{2}\alpha^{\mathrm{tr}} \cdot \mathrm{D}^2 V(a) \cdot \langle xw, w\rangle + \frac{1}{2}\dot{\mu}(\|w\|_{\mathrm{L}^2}^2 + \|\eta_\mu\|_{\mathrm{L}^2}^2) + \alpha^{\mathrm{b}} \cdot \langle \mathrm{i}w, \nabla w\rangle \tag{7.20}$$

Subtracting the result of Lemma 7.4 we find

$$\frac{d}{dt}\Lambda = p \cdot \langle (\nabla_a \mathcal{R}_V)w, w\rangle - \frac{1}{2}\alpha^{\mathrm{tr}} \cdot \mathrm{D}^2 V(a) \cdot \langle xw, w\rangle$$
$$+ \alpha^{\mathrm{b}} \cdot \langle \mathrm{i}w, \nabla w\rangle + 2p \cdot \langle \nabla_a \mathcal{R}_V \eta_\mu, w\rangle - \frac{1}{2}\alpha^{\mathrm{tr}} \cdot \langle \nabla_a \mathcal{R}_V \eta_\mu, \eta_\mu\rangle + \frac{\dot{\mu}}{2}\|w\|_{\mathrm{L}^2}^2$$
$$- \dot{\mu}\langle \mathcal{R}_V \eta_\mu, \partial_\mu \eta\rangle. \tag{7.21}$$

Note that the terms on the second and third line are at least fourth order in the small parameters. The last two lines is the definition of R in the proposition. $\square$

8 Upper bound on Λ

This section we estimate Λ from above using Corollary 6.2 in Proposition 7.1. Taylor expansion of $\mathcal{E}_\mu\big(\eta(t) + w(x, t)\big)$ around η at $t = 0$, gives

$$|\mathcal{E}_{\mu(t)}(u(x, t)) - \mathcal{E}_\mu(\eta_{\mu(t)}(x))|_{t=0} \le C\|w_0\|_{\mathrm{H}_1}^2. \tag{8.1}$$

The remaining terms in the Lyapunov functional are estimated using the inequality $\mathrm{Hess}\, V(x) \le C\epsilon_V^2 |x|^2 \langle \epsilon_V x\rangle^{r-2}$ together with Taylor's formula and Lemma B.3. Furthermore, we use from Corollary 6.2. that $|\epsilon_V \tilde{a}_0| \le C$. We obtain for a $\theta \in [0, 1]$

$$|\langle \mathcal{R}_V u, u\rangle\rangle - \langle \mathcal{R}_V \eta, \eta\rangle|_{t=0} = |\langle \mathcal{R}_V w, w\rangle\rangle + 2\langle \mathcal{R}_V \eta, w\rangle|_{t=0}$$
$$= \epsilon_V^2 |\langle x \cdot \mathrm{Hess}\, V(x\theta + \tilde{a}_0) \cdot x, 2\eta_{\mu_0} \mathrm{Re}(w_0)\rangle + |\langle \mathcal{R}_V w_0, w_0\rangle|$$
$$\le C(\epsilon_V^2 \|w_0\|_{\mathrm{L}^2} + \|w_0\|_{\mathrm{L}^2}^2 + \big\|\epsilon_V x\langle \epsilon_V x\rangle^{(r-2)/2} w_0\big\|_{\mathrm{L}^2}^2). \tag{8.2}$$

We now use Corollary 5.2 and Lemma C.1 in (8.2) and (8.1) to obtain

$$|\langle \mathcal{R}_V u, u\rangle\rangle - \langle \mathcal{R}_V \eta, \eta\rangle|_{t=0} \le C(\epsilon_V^2 \epsilon_0 + \epsilon_0^2) \tag{8.3}$$

and

$$|\mathcal{E}_{\mu(t)}(u(x,t)) - \mathcal{E}_{\mu}(\eta_{\mu(t)}(x))|_{t=0} \le C\epsilon_0^2. \tag{8.4}$$

Thus, finally

$$|\Lambda|_{t=0} \le C(\epsilon_0^2 + \epsilon_V^2 \epsilon_0). \tag{8.5}$$

Proposition 8.1. *Let $\psi \in U_\delta \cap \mathscr{H}_{1,r}$, and let Λ, w and α be defined as above, and δ as defined in Theorem 4.1. Then*

$$|\frac{d}{dt}\Lambda| \le C\Big((\epsilon_V + \epsilon_0 + \sqrt{\epsilon_h})\epsilon_V \|\epsilon_V x \langle \epsilon_V x \rangle^{(r-2)/2} w\|_{\mathrm{L}^2}^2 + |\alpha|\epsilon_V \|(\epsilon_V |x|)^{1/2} w\|_{\mathrm{L}^2}^2$$
$$+ \big((\epsilon_V + \epsilon_0 + \sqrt{\epsilon_h})\epsilon_V^2 + |\alpha|\big)(\|w\|_{\mathrm{H}_1}^2 + \epsilon_V^2)\Big), \quad (8.6)$$

for times $0 \le t \le \min(T_1, T_2)$, where T_1 and T_2 are defined in Corollary 6.2.

Proof. Proposition 7.1 implies

$$|\frac{d}{dt}\Lambda| \le C\big(|p||\langle \nabla_a \mathcal{R}_V w, w\rangle| + |\alpha^{\mathrm{tr}}||\operatorname{Hess} V(a)||\langle xw, w\rangle|$$
$$+ |\alpha^{\mathrm{b}}|\|w\|_{\mathrm{L}^2}\|\nabla w\|_{\mathrm{L}^2} + |p|\epsilon_V^3\|w\|_{\mathrm{L}^2} + |\alpha^{\mathrm{tr}}|\epsilon_V^3 + |\dot{\mu}|\|w\|_{\mathrm{L}^2}^2 + |\dot{\mu}|\epsilon_V^2\big). \quad (8.7)$$

An alternative form of Eqn. (8.7) is

$$|\frac{d}{dt}\Lambda| \le C\big(|p||\langle \nabla_a \mathcal{R}_V w, w\rangle| + |\alpha||\operatorname{Hess} V(a)||\langle xw, w\rangle|$$
$$+ (|p|\epsilon_V^2 + |\alpha|)(\|w\|_{\mathrm{H}_1}^2 + \epsilon_V^2)\big), \quad (8.8)$$

where we have used $\epsilon_V < C$ and $|\alpha_j| \le |\alpha|$, $\forall j$.

Using Corollary B.2 we estimate the $\mathcal{R}_V$ terms to obtain

$$|\frac{d}{dt}\Lambda| \le C\big(|p|\epsilon_V \|\epsilon_V |x| \langle \epsilon_V x \rangle^{(r-2)/2} w\|_{\mathrm{L}^2}^2 + |\alpha|\epsilon_V \langle \epsilon_V |a|_\infty \rangle^{r-2} |\langle \epsilon_V xw, w\rangle|$$
$$+ (|p|\epsilon_V^2 + |\alpha|)(\|w\|_{\mathrm{H}_1}^2 + \epsilon_V^2)\big). \quad (8.9)$$

The proposition now follows upon using $\epsilon_V |a| \le C_a$ and $|p| \le C(\epsilon_V + \epsilon_0 + \sqrt{\epsilon_h})$ for $t \le \min(T_1, T_2)$ from Corollary 6.2 and the inequality:

$$\langle \epsilon_V xw, w\rangle \le \|(\epsilon_V |x|)^{1/2} w\|_{\mathrm{L}^2}^2. \tag{8.10}$$

$\square$

Equation (8.5) and Proposition 8.1 yield an upper bound on Λ:

$$|\Lambda| \le C\epsilon_0^2 + C\epsilon_V^2 \epsilon_0 + t \sup_{s \le t} |\frac{d}{dt}\Lambda|. \tag{8.11}$$

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9 Lower bound on Λ

In this section we estimate the Lyapunov-functional Λ from below. Recall the definition (7.1) of Λ:

$$\Lambda := \mathcal{E}_\mu(\eta + w) - \mathcal{E}_\mu(\eta) + \frac{1}{2}\langle \mathcal{R}_V(\eta + w), \eta + w \rangle - \frac{1}{2}\langle \mathcal{R}_V \eta, \eta \rangle. \tag{9.1}$$

We have the following result.

Proposition 9.1. *Let Λ and w be defined as above. Then for a positive constant C,*

$$\Lambda \geq \frac{1}{2}\rho_2\|w\|_{\mathrm{H}_1}^2 + C_0\rho_1\|\epsilon_V|x|\langle\epsilon_V x\rangle^{(r-2)/2}w\|_{\mathrm{L}^2}^2 - C\|w\|_{\mathrm{H}_1}^3 - C\epsilon_V^2\|w\|_{\mathrm{L}^2}. \tag{9.2}$$

where r and $\rho_1 > 0$ are defined in (2.4), C_0 is the positive constant defined in Lemma A.1 and $\rho_2 > 0$ is a positive number. The constant C_0 depends on the constant C_a defined in Corollary 6.2 bounding the size of $\epsilon_V|a|$.

Proof. By Taylor expansion we have

$$\mathcal{E}_\mu(\eta + w) - \mathcal{E}_\mu(\eta) = \frac{1}{2}\langle \mathcal{L}_\eta w, w \rangle + R_\eta^{(3)}(w), \tag{9.3}$$

where $\mathcal{L}_\eta := (\mathrm{Hess}\,\mathcal{E}_\mu)(\eta)$ and by Condition 2a, $|R_\eta^{(3)}(w)| \leq C\|w\|_{\mathrm{H}_1}^3$. The coercivity of $\mathcal{L}_\eta$ for $w \perp J\mathrm{T}_\eta \mathrm{M_s}$ is proved in Proposition D.1 of [12] under Conditions 1–3 on the nonlinearity (in Section 2). Thus

$$\langle \mathcal{L}_\eta w, w \rangle \geq \rho_2\|w\|_{\mathrm{H}_1}^2 \text{ for } w \perp J\mathrm{T}_\eta \mathrm{M_s}. \tag{9.4}$$

The remaining terms of Λ can be rewritten as

$$\langle \mathcal{R}_V(\eta + w), \eta + w \rangle - \langle \mathcal{R}_V \eta, \eta \rangle = \langle \mathcal{R}_V w, w \rangle + 2\langle \mathcal{R}_V \eta, w \rangle. \tag{9.5}$$

In Lemma A.1 we show that

$$\mathcal{R}_V \geq C_0\rho_1(\epsilon_V|x|)^2\langle\epsilon_V x\rangle^{r-2} \text{ for } r \geq 1. \tag{9.6}$$

Using Lemma A.1, (9.4), (9.6) and the fact that $\langle \mathcal{R}_V \eta, w \rangle \leq C\epsilon_V^2\|w\|_{\mathrm{L}^2}$ we obtain the lower bound on Λ. $\qquad\square$

10 Proof of Theorem 2.1

The upper bound (8.11) together with the bound from below in Proposition 9.1 yield the inequality

$$\frac{1}{2}\rho_2\|w\|_{\mathrm{H}_1}^2 + C_0\rho_1\big\|\epsilon_V x\langle\epsilon_V x\rangle^{(r-2)/2}w\big\|_{\mathrm{L}^2}^2 - C\|w\|_{\mathrm{H}_1}^3 - C\epsilon_V^2\|w\|_{\mathrm{L}^2} \leq C\epsilon_0^2 + C\epsilon_V^2\epsilon_0$$

$$+ tC\sup_{s \leq t}\Big((\epsilon + \sqrt{\epsilon_h})\epsilon_V\|\epsilon_V x\langle\epsilon_V x\rangle^{(r-2)/2}w\|_{\mathrm{L}^2}^2 + |\alpha|\epsilon_V\|(\epsilon_V|x|)^{1/2}w\|_{\mathrm{L}^2}^2$$

$$+ ((\epsilon + \sqrt{\epsilon_h})\epsilon_V^2 + |\alpha|)(\|w\|_{\mathrm{H}_1}^2 + \epsilon_V^2)\Big), \tag{10.1}$$

for $0 \leq t \leq \min(T_1, T_2)$, where T_1 and T_2 are defined in Corollary 6.2 and $\epsilon := \epsilon_V + \epsilon_0$. The right-hand side is independent of the operator $t \mapsto s$, $\sup_{s \leq t}$ in the given time interval, we can therefore apply this to both sides of (10.1). To simplify, let

$$\rho := \min(\frac{\rho_2}{8}, \frac{C_0 \rho_1}{3}). \tag{10.2}$$

We absorb higher order terms into lower order ones. Furthermore, we assume

$$t \leq \min(T_1, T_2, T_3), \text{ where } T_3 := \frac{\rho}{C(|\alpha|_\infty + \epsilon_V(\epsilon + \sqrt{\epsilon_h}))(1 + \epsilon_V)}, \tag{10.3}$$

in agreement with Corollary 6.2. Both ρ and C above depend on I, clarifying the need for $\epsilon \ll C(I)$. Note that

$$T_3 C(\epsilon + \sqrt{\epsilon_h})\epsilon_V \leq \rho, \ T_3 C |\alpha|_\infty \epsilon_V \leq \rho, \text{ and } T_3 C((\epsilon + \sqrt{\epsilon_h})\epsilon_V^2 + |\alpha|_\infty \leq 2\rho. \tag{10.4}$$

We obtain

$$\rho \sup_{s \leq t} \left(4\|w\|_{H_1}^2 + 3\|\epsilon_V x \langle \epsilon_V x \rangle^{(r-2)/2} w\|_{L^2}^2 \right)$$

$$\leq C\left(\sup_{s \leq t}(\|w\|_{H_1}^3 + \epsilon_V^2 \|w\|_{L^2}) + \epsilon_0^2 + \epsilon_V^2 \epsilon_0 \right)$$

$$+ \rho \sup_{s \leq t} \left(\|\epsilon_V x \langle \epsilon_V x \rangle^{(r-2)/2} w\|_{L^2}^2 + \||\epsilon_V x|^{1/2} w\|_{L^2}^2 + 2\epsilon_V^2 + 2\|w\|_{H_1}^2 \right). \tag{10.5}$$

Note that $g(y) := |y| - y^2 \langle y \rangle^{-1} \leq 2^{-1}$, $y \in \mathbb{R}$. Indeed $g(-y) = g(y)$ and g is continuously differentiable on $(0, \infty)$, $g(y) \geq 0$ since $|y| \geq y^2 \langle y \rangle^{-1}$ with $g(0) = g(\infty) = 0$. The function $g(y)$ has one critical point on $(0, \infty)$ at $y = (2^{-1}(\sqrt{5} - 1))^{-1/2}$ with value $\max g = (3 - \sqrt{5})(2(\sqrt{5} - 1))^{-1/2} \leq 2^{-1}$. This proves the claim. We now use this intermediate function $g(x)$ to estimate the term above with $|x|^{1/2}$. We have

$$\epsilon_V |x| - (\epsilon_V |x|)^2 \langle \epsilon_V x \rangle^{r-2} \leq g(\epsilon_V |x|) \leq \frac{1}{2}. \tag{10.6}$$

We also have the inequalities

$$C\|w\|_{H_1}^3 \leq \rho^{-1} C^2 \|w\|_{H_1}^4 + 4^{-1} \rho \|w\|_{H_1}^2, \quad C\epsilon_V^2 \|w\|_{H_1} \leq C^2 \rho^{-1} \epsilon_V^4 + 4^{-1} \rho \|w\|_{H_1}^2. \tag{10.7}$$

Thus we have $3\rho \|w\|_{H_1}^2$ on the right-hand side and 2ρ of terms containing $\langle \epsilon_V x \rangle$. Moving those to the left-hand side of (10.5) using the above inequalities and simplifying we obtain

$$\sup_{s \leq t} \left(\|w\|_{H_1}^2 + \|\epsilon_V x \langle \epsilon_V x \rangle^{(r-2)/2} w\|_{L^2}^2 \right) \leq C' \epsilon^2 + C^2 \rho^{-2} (\sup_{s \leq t} \|w\|_{H_1}^4). \tag{10.8}$$

Abbreviate $\kappa := C' \epsilon^2$. Let

$$X := \sup_{s \leq t} \left(\|w\|_{H_1}^2 + \|\epsilon_V |x| \langle \epsilon_V x \rangle \epsilon_V x^{(r-2)/2} w\|_{L^2}^2 \right). \tag{10.9}$$

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Equation (10.8) implies

$$X \leq C^2 \rho^{-2} X^2 + \kappa. \tag{10.10}$$

Solving this inequality, we find

$$X \leq 2\kappa, \text{ provided } \kappa \leq \frac{\rho^2}{4C^2}. \tag{10.11}$$

The definition of X and κ implies

$$\|w\|_{H_1} \leq c'\epsilon, \text{ and } \left\|\epsilon_V x \langle \epsilon_V x \rangle \epsilon_V x^{(r-2)/2} w\right\|_{L^2} \leq c'\epsilon. \tag{10.12}$$

Lemma C.1 allow us to rewrite (10.12) as $\|w\|_{\mathscr{H}_{1,r}} \leq c'\epsilon$. Inserting (10.12) into the expressions for our modulation parameters, the estimate of the α_j-terms in (4.25)–(4.28) gives us $|\alpha| \leq c\epsilon^2$ and time interval $t \leq T'$, where

$$T' := c \min(\epsilon^{-2}, \frac{\sqrt{\epsilon_h}}{\epsilon^2}, \frac{1}{\epsilon^2 + \epsilon_V \sqrt{\epsilon_h}}) \tag{10.13}$$

Using $\epsilon_h \geq K\epsilon_V$ (that is, ϵ_h is not an order of magnitude smaller then ϵ_V), we can shorten the time-interval to have an upper limit of

$$T'' := C(\epsilon^2 + \epsilon_V \sqrt{\epsilon_h})^{-1}. \tag{10.14}$$

We now choose ϵ such that (10.11) holds and $c'\epsilon \leq \frac{1}{2}\delta$, where δ is defined in Theorem 4.1. Then there is a maximum T_0 such that the solution ψ of (1.1) is in U_δ for $t \leq T_0$. Thus the decomposition (4.4) is valid and the above upper bounds for $\|w\|_{H_1}$ and α are valid for $t \leq \min(T_0, C(\epsilon^2 + \epsilon_V \sqrt{\epsilon_h})^{-1})$. Thus there exists a constant C_T such that $0 < C_T \leq C$, such that for $t \leq C_T(\epsilon^2 + \epsilon_V \sqrt{\epsilon_h})^{-1}$ the theorem holds. This concludes the proof of Theorem 2.1. $\square$

A Lower bound on $\mathcal{R}_V$

In this appendix we estimate $\mathcal{R}_V$ from below. Recall that

$$\mathcal{R}_V(x) := V(x+a) - V(a) - \nabla V(a) \cdot x \tag{A.1}$$

and

$$\text{Hess } V \geq \rho_1 \epsilon_V^2 \langle \epsilon_V x \rangle^{(r-2)/2}, \tag{A.2}$$

where ρ_1 is a positive constant. We have the following result:

Lemma A.1. *Let $a, x \in \mathbb{R}^d$ and $0 < \epsilon_V \in \mathbb{R}$. Then (i) if $r \geq 2$ or (ii) if $r \leq 2$ and $\epsilon_V |a| \leq C_a$:*

$$\mathcal{R}_V(x) \geq C_0 \rho_1 \epsilon_V^2 |x|^2 \langle \epsilon_V x \rangle^{r-2}, \tag{A.3}$$

where

$$C_0 := \begin{cases} \frac{1}{2^{r-2+\max(0,\frac{r-4}{2})}r(r-1)} & \text{in case (i)} \\ \frac{1}{2(2(1+C_a^2))^{(2-r)/2}} & \text{in case (ii)}. \end{cases} \tag{A.4}$$

Proof. Consider the case $x = 0$, from the definition of $\mathcal{R}_V$ it follows that $\mathcal{R}_V(0) = 0$, thus the estimate holds. Let $x \neq 0$, the function $\mathcal{R}_V(x)$ is the Taylor expansion of $V(x + a)$ around a to first order. The Taylor series remainder

$$\int_0^1 (1 - \theta)x \cdot \mathrm{Hess}\, V(a + \theta x) \cdot x \, \mathrm{d}\theta, \tag{A.5}$$

integrated by parts, gives the identity

$$\int_0^1 (1 - \theta)x \cdot \mathrm{Hess}\, V(a + \theta x) \cdot x \, \mathrm{d}\theta =$$

$$(1 - \theta)x \cdot \nabla V(a + \theta x)|_0^1 + \int_0^1 \nabla V(a + \theta x) \cdot x \, \mathrm{d}\theta = \mathcal{R}_V(x, t). \tag{A.6}$$

Inserting the lower bound of the $\mathrm{Hess}\, V$ into the Taylor remainder gives the inequality

$$\mathcal{R}_V(x) \geq \rho_1 \epsilon_V^2 |x|^2 \int_0^1 (1 - \theta)(1 + \epsilon_V^2 |a + x\theta|^2)^{(r-2)/2} \, \mathrm{d}\theta = \rho_1 \epsilon_V^2 |x|^2 I. \tag{A.7}$$

To estimate I, we first consider case (a), with $r \geq 2$. The integrand of I is estimated by the following lemma.

Lemma A.2. *Let $y \geq 0$ and $b \geq 0$ then*

$$\frac{1}{2^{\max(0, \frac{2-b}{2})}} \leq \frac{(1 + y^2)^{b/2}}{1 + |y|^b} \leq 2^{\max(0, \frac{b-2}{2})}. \tag{A.8}$$

This lemma is proved at the end of this appendix. For $x \neq 0$ denote $\hat{x} = x/|x|$, $a_\parallel = a \cdot \hat{x}$, and $a_\perp = a - a_\parallel \hat{x}$, and abbreviate $b := r - 2$. Lemma A.2 then implies

$$\left(1 + \epsilon_V^2 |a + x\theta|^2\right)^{b/2} = \left(1 + \epsilon_V^2 |a_\perp|^2\right)^{b/2} \left(1 + \frac{\epsilon_V^2}{1 + \epsilon_V^2 |a_\perp|^2} \left(a_\parallel + |x|\theta\right)^2\right)^{b/2}$$

$$\geq \frac{1}{2^{\max(0, \frac{2-b}{2})}} \left(\left(1 + \epsilon_V^2 |a_\perp|^2\right)^{b/2} + \epsilon_V^b \left|a_\parallel + |x|\theta\right|^b\right)$$

$$\geq \frac{1}{2} + \frac{1}{2}\epsilon_V^b \left|a_\parallel + |x|\theta\right|^b, \tag{A.9}$$

where we used $2^{\max(0, \frac{2-b}{2})} \leq 2$. Thus

$$I \geq \frac{1}{4} + \frac{1}{2}\epsilon_V^{r-2} \int_0^1 (1 - \theta)\left|a_\parallel + |x|\theta\right|^{r-2} \mathrm{d}\theta = \frac{1}{4} + \frac{1}{2}\epsilon_V^{r-2} I_2. \tag{A.10}$$

The integral I_2 evaluates to

$$I_2 = \frac{1}{r(r-1)|x|^2} \left(\left|a_\parallel + |x|\right|^r - |a_\parallel|^r - r|x|\, \mathrm{sign}(a_\parallel)|a_\parallel|^{r-1}\right), \tag{A.11}$$

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which is the remainder of a Taylor series of $\left|a_\parallel + |x|\right|^r$ around $a_\parallel$, as expected. At $a_\parallel = 0$ we have

$$I_2 \geq \frac{|x|^{r-2}}{r(r-1)}. \tag{A.12}$$

To estimate I_2 for $a_\parallel \neq 0$, we use the following lemma

Lemma A.3. *Let $y \in \mathbb{R}$ and $r \geq 2$. Then*

$$|1+y|^r - 1 - ry \geq \frac{1}{2^{r-2}}|y|^r. \tag{A.13}$$

The lemma is proved at the end of this appendix. Since $a_\parallel \neq 0$ we can pull it out of I_2, and use $y := |x|/a_\parallel$ in the Lemma to obtain

$$I_2 \geq \frac{|x|^{r-2}}{2^{r-2}r(r-1)}. \tag{A.14}$$

Note that the above result is a lower limit than we obtained in (A.12), so we can use (A.14) for all $a_\parallel$. Now, inserting this inequality into (A.10) and the result into (A.7) to obtain

$$\mathcal{R}_V(x) \geq \rho_1 \epsilon_V^2 |x|^2 I \geq \rho_1 \left(\frac{\epsilon_V^2 |x|^2}{4} + \frac{\epsilon_V^r |x|^r}{2^{r-1}r(r-1)} \right). \tag{A.15}$$

Once again using Lemma A.2 gives

$$\mathcal{R}_V(x) \geq C_1 \rho_1 \epsilon_V^2 |x|^2 \langle \epsilon_V x \rangle^{r-2}, \tag{A.16}$$

where

$$C_1 := \frac{1}{2^{\max(0, \frac{r-4}{2})} 2^{r-2} r(r-1)}, \quad r \geq 2. \tag{A.17}$$

This concludes part (i) of Lemma A.1, except for the proofs of Lemma A.2 and A.3 which is done below.

Now, we estimate the integral I for the case (ii), with $r \leq 2$ and $\epsilon_V|a| \leq C_a$. Introduce the change of variables $p = (1-\theta)^2$. The integral takes the form

$$I = \frac{1}{2} \int_0^1 \frac{1}{g(p)^{(2-r)/2}} \, dp, \quad \text{where } g(p) := 1 + \epsilon_V^2 |a + x(1 + \sqrt{p})|^2. \tag{A.18}$$

The triangle inequality together with $0 \leq p \leq 1$ gives

$$g(p) \leq 1 + 2\epsilon_V^2 |a|^2 + 2\epsilon_V^2 |x|^2 (1 - \sqrt{p}) \leq \left(1 + \max(2\epsilon_V^2 |a|^2, 1)\right) \langle \epsilon_V x \rangle^2. \tag{A.19}$$

The upper bound $\epsilon_V|a| \leq C_a$ and the estimate $1 + \max(2\epsilon_V^2 |a|^2, 1) \leq 2 + 2C_a^2$ together with a trivial integral gives that I is bounded from below as

$$I \geq C_2 \langle \epsilon_V x \rangle^{r-2}, \quad \text{where } C_2 := \frac{1}{2\left(2(1+C_a^2)\right)^{(2-r)/2}}. \tag{A.20}$$

Inserting this result into (A.7) concludes the Lemma. $\qquad\square$

Now consider Lemma A.2. It is a combination of the inequalities Theorem 13 and Theorem 19 in [18]

Lemma A.4. *Let $y \in \mathbb{R}$ and $b \geq 0$ then*

$$\frac{1}{2^{\max(0, \frac{2-b}{2})}} \leq \frac{(1+y^2)^{b/2}}{1+|y|^b} \leq 2^{\max(0, \frac{b-2}{2})}. \tag{A.21}$$

Proof. Denote

$$f(y, b) := \frac{(1+y^2)^{b/2}}{1+|y|^b}. \tag{A.22}$$

We first note that $f(y, b) = f(-y, b)$, thus we can restrict our attention to $y \geq 0$. At $y = 0$ we have $f(0, b) = 1$ and at $y = \infty$, $f(\infty, b) = 1$. The function is differentiable for $y > 0$, the only critical point for $y > 0$ is at $y = 1$, where the function takes the value

$$f(1, b) = 2^{b/2-1}. \tag{A.23}$$

If $b > 2$ its a maximum, and if $b < 2$ its a minimum, the lemma follows. $\square$

To prove Lemma A.3 we begin by stating it again.

Lemma. *Let $y \in \mathbb{R}$ and $r \geq 2$. Then*

$$|1+y|^r - 1 - ry \geq \frac{1}{2^{r-2}}|y|^r. \tag{A.24}$$

Proof. Denote

$$f(y\,; r) := |1+y|^r - 1 - ry - \frac{1}{2^{r-2}}|y|^r. \tag{A.25}$$

The lemma is equivalent to $f \geq 0$, for $r \geq 2$. We note that f is twice differentiable at all points except $y = -1$ and $y = 0$. We observe that the inequality is satisfied for both of these points since we have $f(0\,; r) = 0$ and $f(-1, r) = r - 1 - \frac{1}{2^{r-1}} > 0$ for $r \geq 2$. Consider the derivative of f with respect to y:

$$\partial_y f(y\,; r) = r \left(\text{sign}(1+y)|1+y|^{r-1} - \frac{1}{2^{r-2}} \text{sign}(y)|y|^{r-1} - 1 \right). \tag{A.26}$$

We wish to show that f decays monotonically on $y < 0$ and hence, that $\partial_y f \leq 0$ for $-1 < y < 0$ and $y < -1$. We also wish to show that f increases monotonically for $y > 0$ with $\partial_y f \geq 0$. To show this, consider first the case $y > 0$: using that $(1+y)^{r-1} \geq 1+y^{r-1}$, we have $\partial_y f > 0$ for $y > 0$.

For the interval $-1 < y < 0$, use that $b^r < b$ for any $b \in (0, 1)$, to obtain

$$\partial_y f(y\,; r) = (1 - |y|)^{r-1} + 2^{2-r}|y|^{r-1} - 1 < -|y| + 2^{2-r}|y|$$
$$= -\left(1 - 2^{2-r}\right)|y| \leq 0, \ r \geq 2. \tag{A.27}$$

For the last interval $y < -1$ we re-write (A.26) as

$$\partial_y f(y\,;r) = -r\left((|y|-1)^{r-1} - 2^{2-r}|y|^{r-1} + 1\right).\tag{A.28}$$

Upon calculating $\partial_y^2 f$, and solving $\partial_y^2 f = 0$ for y in this interval we find that $\partial_y f$ has a maximum at $|y| = 2$ with value $\partial_y f(2\,;r) = 0$. Hence, $\partial_y f \leq 0$ for $y \leq -1$ and $-1 < y < 0$. We have showed that f decays monotonically on $y < 0$ and increases monotonically on $y > 0$, and $f(0\,;r) = 0$. Hence $f \geq 0$ for all $y \in \mathbb{R}$, which proves the lemma. $\qquad\square$

B Upper bound on $\mathcal{R}_V$ and $\nabla_a \mathcal{R}_V$

In this appendix we estimate $\mathcal{R}_V$ and derivatives of $\mathcal{R}_V$ from above. From the proof of Lemma A.1 we have the following identity

$$\mathcal{R}_V(x) = \int_0^1 (1-\theta)x \cdot \mathrm{Hess}\,V(a+\theta x) \cdot x \,\mathrm{d}\theta.\tag{B.1}$$

Furthermore, in (2.4) we made the assumptions that, for $\beta \in \mathbb{Z}^d$ and $|\beta| \leq 3$,

$$|\partial_x^\beta V(x)| \leq C_V \epsilon_V^{|\beta|} \langle \epsilon_V x \rangle^{r-|\beta|}.\tag{B.2}$$

We begin with the following result for derivatives of $\mathcal{R}_V$.

Lemma B.1. *Let $a, x \in \mathbb{R}^d$, $0 < \epsilon_V \in \mathbb{R}$ and $\epsilon_V |a| \leq C_a$, as in Corollary 6.2. Furthermore let $\beta \in \mathbb{Z}^d$, with $0 \leq \beta_j \leq 1\ \forall j = 1, ..., d$ and $|\beta| = 1$ Then, (i) if $r \geq 2$:*

$$|\partial_a^\beta \mathcal{R}_V| \leq C_1 \epsilon_V^3 |x|^2 \langle \epsilon_V x \rangle^{\max(r-3,0)},\tag{B.3}$$

or (ii) if $1 \leq r < 2$:

$$|\partial_a^\beta \mathcal{R}_V| \leq C_2 \epsilon_V^3 |x|^2 \langle \epsilon_V x \rangle^{r-2},\tag{B.4}$$

where

$$C_1 := 2^{-1} C_V d(2(1+C_a^2))^{\max(r-3,0)/2}, \quad C_2 := C_V d\big(6\sqrt{2} + \ln(1+C_a)\big).\tag{B.5}$$

Here C_V is the constant in (2.4).

Corollary B.2. *Under the same conditions as in Lemma B.1 we have*

$$|\partial_a^\beta \mathcal{R}_V| \leq C \epsilon_V^3 |x|^2 \langle \epsilon_V x \rangle^{r-2},\tag{B.6}$$

where C depends on C_1 and C_2 above.

Proof. Use that $\langle \epsilon_V x \rangle^{\max(r-3,0)} \leq \langle \epsilon_V x \rangle^{r-2}$ in Lemma B.1. $\qquad\square$

Proof of Lemma B.1. For the case $x = 0$, $(\partial_a^\beta \mathcal{R}_V)(0) = 0$ and thus the Lemma is satisfied. For $x \neq 0$, and since $V \in C^3$, we have from (B.1) that

$$\partial_a^\beta \mathcal{R}_V(x,t) = \sum_{k,l=1}^d x_l x_k \int_0^1 (1-\theta)(\partial_x^\beta \partial_{x_k} \partial_{x_l} V)(a+\theta x)\, d\theta. \tag{B.7}$$

The upper bound on the potential, (B.2), gives

$$|\partial_{a_j} \mathcal{R}_V(x)| \leq \epsilon_V^3 C_V \sum_{k,l=1}^d |x_l x_k| \int_0^1 (1-\theta)(1+\epsilon_V^2|a+\theta x|^2)^{(r-3)/2}\, d\theta, \tag{B.8}$$

here C_V is the constant in (2.4). To estimate $|\partial_a^\beta \mathcal{R}_V|$, we use the inequality

$$\sum_{k,l=1}^d |x_k x_l| \leq d|x|^2, \tag{B.9}$$

to obtain

$$|\partial_a^\beta \mathcal{R}_V(x)| \leq C_V d\epsilon_V^3 |x|^2 \int_0^1 (1-\theta)(1+\epsilon_V^2|a+\theta x|^2)^{(r-3)/2} = dC_V \epsilon_V^3 |x|^2 I. \tag{B.10}$$

To estimate the integrand, we consider first case (i), with $r > 3$, $\epsilon_V|a| \leq C_a$. Before we estimate the integral I, we estimate part of its integrand with the triangle inequality together with $\epsilon_V|a| \leq C_a$ and $\theta \leq 1$ to obtain

$$1 + \epsilon_V^2|a+\theta x|^2 \leq 1 + 2C_a^2 + 2|x|^2 \leq (1+\max(2C_a^2,1))\langle \epsilon_V x \rangle^2. \tag{B.11}$$

Thus

$$|\partial_a^\beta \mathcal{R}_V(x,t)| \leq \tilde{C}_1 \epsilon_V^3 |x|^2 \langle \epsilon_V x \rangle^{r-3}, \tag{B.12}$$

where

$$\tilde{C}_1 := 2^{-1} C_V d(2(1+C_a^2))^{(r-3)/2}. \tag{B.13}$$

To extend this case to include $r \geq 2$, we note that for $r \in [2,3]$ the exponent in the integrand of I, $r - 3 \leq 0$, and that $1 + \epsilon_V^2|a+\theta x|^2 \geq 1$ to obtain

$$I \leq 2^{-1}. \tag{B.14}$$

We conclude that for $r \in [2,3]$ $|\partial_a^\beta \mathcal{R}_V| \leq 2^{-1} dC_V \epsilon_V^3 |x|^2$. Thus upon changing (B.12) and (B.13) into

$$|\partial_a^\beta \mathcal{R}_V(x)| \leq C_1 \epsilon_V^3 |x|^2 \langle \epsilon_V x \rangle^{\max(r-3,0)}, \tag{B.15}$$

where

$$C_1 := 2^{-1} C_V d(2(1+C_a^2))^{\max(r-3,0)/2}, \tag{B.16}$$

part (i) is proved.

For the case (ii), with $r < 2$ we need a more precise estimate that the case of $r \in [2, 3]$. To obtain this, recall that $x \neq 0$ and let us introduce the notations $\hat{x} := x/|x|$, $a_\| := a \cdot \hat{x}$ and $a_\perp := a - a_\|$. Then $|a + \theta x|^2 = a_\perp^2 + |a_\| + \theta|x||^2$. By the change of variables $y = \epsilon_V(a_\| + \theta|x|)$ and that $1 + a_\perp^2 \geq 1$, we find

$$I \leq \frac{1}{\epsilon_V |x|} \int_{\epsilon_V a_\|}^{\epsilon_V(a_\| + |x|)} \frac{dy}{(1 + y^2)^{(3-r)/2}} =: \frac{1}{\epsilon_V |x|} I_2. \tag{B.17}$$

The goal is to show that $\langle \epsilon_V x \rangle^{2-r} I$ is bounded by a constant independent of ϵ_V. To show this, we consider two intervals for $|x|$ first $\epsilon_V |x| \leq 1$. For this interval $\langle \epsilon_V x \rangle^{2-r} \leq 2^{2-r}$ and $I_2 \leq \epsilon_V |x|$, thus $\langle \epsilon_V x \rangle^{2-r} I \leq 2$.

For the intervals $\epsilon_V |x| \geq 1$ and $1 \leq r \leq 1.5$. We show that I_2 is bounded by a constant. Indeed, regardless of the values of $a_\|$ and $|x|$ we have

$$I_2 \leq 2 \int_0^\infty \frac{dy}{(1 + y^2)^{(3-r)/2}} = \sqrt{\pi} \frac{\Gamma(1 - \frac{r}{2})}{\Gamma(\frac{r-3}{2})} \leq 6 \tag{B.18}$$

for $1 \leq r \leq 1.5$. Thus

$$\langle \epsilon_V x \rangle^{2-r} I \leq 6 \frac{\langle \epsilon_V x \rangle^{2-r}}{\epsilon_V |x|} \leq 6\sqrt{2}. \tag{B.19}$$

For $r \in [1.5, 2]$ we need a better estimate, we use that $(1 + y^2)^{(3-r)/2} \geq (1 + y^2)^{1/2}$, thus

$$I_2 \leq \int_{\epsilon_V a_\|}^{\epsilon_V(a_\| + |x|)} \frac{dy}{\sqrt{1 + y^2}} = \ln\left(\frac{\epsilon_V a_\| + \epsilon_V |x| + \sqrt{1 + \epsilon_V^2 |a + x|^2}}{\epsilon_V a_\| + \sqrt{1 + \epsilon_V^2 a_\|^2}} \right). \tag{B.20}$$

To estimate this, we consider four different regions, For $a_\| > 0$ and $|x| > |a|$ it is bounded by $\ln(1 + 4\epsilon_V |x|)$. For $a_\| > 0$ and $|x| \leq |a|$ it is bounded by $\ln 2$. For $a_\| < 0$ and $|x| < |a|$ it is bounded by $\ln(1 + 2C_a)$. For $a_\| < 0$ and $|x| > |a|$ it is bounded by $\ln\left((1 + 2C_a)(1 + 4\epsilon_V |x|)\right)$. Thus

$$I_2 \leq \ln(2 + 2C_a) + \ln(1 + 4\epsilon_V |x|) \leq (\epsilon_V |x|)^{1/2} \ln\left(10(1 + C_a)\right), \text{ for } \epsilon_V |x| \geq 1. \tag{B.21}$$

where we have used that for $\epsilon_V |x| \geq 1$, $q \geq 0$ we have $q + \ln(1 + 4\epsilon_V |x|) \leq (q + \ln 5)(\epsilon_V |x|)^{1/2}$. Thus

$$\langle \epsilon_V x \rangle^{2-r} I \leq \ln(10(1 + C_a)) \frac{\langle \epsilon_V x \rangle^{2-r}}{(\epsilon_V |x|)^{1/2}} \leq 2^{1/4} \ln(10(1 + C_a)). \tag{B.22}$$

Comparing the constants above for the I estimate we find that

$$C_2 := C_V d(6\sqrt{2} + \ln(1 + C_a)), \tag{B.23}$$

is sufficient. This concludes the proof of the lemma. $\qquad \square$

To bound $\mathcal{R}_V$ from above we could use the same methods as above, but the upper bound will be to large to fit into the energy space. But we have the following

Lemma B.3. *For $r \geq 1$ and $\epsilon_V |a| \leq C_a$*

$$\mathcal{R}_V \leq C_1 (1 + \epsilon_V^2 |x|^2 \langle \epsilon_V x \rangle^{r-2}), \tag{B.24}$$

where

$$C_1 := 2C_V (2 + 2C_a^2)^{(r-1)/2}. \tag{B.25}$$

Proof. For $r \geq 2$, we there exists a $\theta \in [0,1]$ such that

$$\mathcal{R}_V \leq C_V \epsilon_V^2 |x|^2 \langle x\theta + a \rangle^{r-2}. \tag{B.26}$$

Since $r - 2 \geq 0$ we estimate $\epsilon_V |x\theta + a| \leq \epsilon_V |x| + C_a$ and $(1 + 2a|x|^2 + 2C_a^2) \leq (2 + 2C_a^2)\langle \epsilon_V x \rangle^2$ we obtain the lemma for $r \geq 2$ as

$$\mathcal{R}_V \leq C_V (2 + 2C_a^2)^{(r-2)/2} |\epsilon_V|^2 |x|^2 \langle \epsilon_V x \rangle^{r-2}. \tag{B.27}$$

For $r \in [1,2)$ we use that there exists a $\theta \in [0,1]$ such that

$$\mathcal{R}_V = (\nabla V(x\theta + a, t) - \nabla V(a)) \cdot x \leq C_V \epsilon_V |x| (\langle \epsilon_V (x\theta + a) \rangle^{r-1} + \langle C_a \rangle^{r-1}). \tag{B.28}$$

Once again $\epsilon_V |x\theta + a| \leq \epsilon_V |x| + C_a$ and we obtain

$$\mathcal{R}_V \leq C_V \epsilon_V |x| ((2 + 2C_a^2)^{(r-1)/2} \langle \epsilon_V x \rangle^{r-1} + \langle C_a \rangle^{r-1}). \tag{B.29}$$

To estimate the second term, recall (10.6), to get

$$\epsilon_V |x| \leq 2^{-1} + (\epsilon_V |x|)^2 \langle \epsilon_V x \rangle^{r-2}. \tag{B.30}$$

To estimate the first term in (B.29), let $y = \epsilon_V |x| \geq 0$, $r \in [1,2]$ and calculate

$$y\langle y \rangle^{r-1} - y^2 \langle y \rangle^{r-2} = y\langle y \rangle^{r-1}(1 - \frac{y}{\langle y \rangle}) = \frac{y\langle y \rangle^{r-1}}{(y + \langle y \rangle)\langle y \rangle} \leq \frac{y}{(y + \langle y \rangle)} \leq \frac{1}{2}. \tag{B.31}$$

Thus

$$\epsilon_V |x| \langle \epsilon_V x \rangle^{r-1} \leq 2^{-1} + (\epsilon_V |x|)^2 \langle \epsilon_V x \rangle^{r-2}. \tag{B.32}$$

Collecting the above two terms gives

$$\mathcal{R}_V \leq 2C_V (2 + 2C_a^2)^{(r-1)/2}(1 + (\epsilon_V |x|)^2 \langle \epsilon_V x \rangle^{r-2}). \tag{B.33}$$

Since (B.33) for $r \geq 2$ is larger than (B.27) we have proved the lemma. $\square$

C   Bound in energy-space

In Eq. (10.12) we show that

$$\|w\|_0 := \|w\|_{\mathrm{H}_1} + \left\||\epsilon_V|x|\langle \epsilon_V x\rangle^{(r-2)/2}w\right\|_{\mathrm{L}^2} \le C\epsilon. \tag{C.1}$$

We want to show that $\|w\|_{\mathscr{H}_{1,r}} \le C_r\|w\|_0$. This result follows from the following lemma:

Lemma C.1. *For $r \in (0, \infty)$ there exists a constant c_r such that*

$$0 < 1 + \min(0, c_r) \le \frac{1 + y^2\langle y\rangle^{r-2}}{\langle y\rangle^r} \le 1 + \max(0, c_r) < 2, \tag{C.2}$$

where

$$c_r := \frac{2-r}{2}\left(\frac{2}{r}\right)^{2/(r-2)}, \quad r \ne 2, \tag{C.3}$$

and for $r = 2$, $c_r = 0$.

Proof. Denote

$$f(z) = \frac{1 + (z^2 - 1)z^{r-2}}{z^r} = 1 + z^{-r} - z^{-2}, \ z \ge 1. \tag{C.4}$$

Note that for $z^2 = 1 + y^2$, f is the function we want to estimate for the lemma. For $r = 2$, $f = 1$, thus $c_r = 0$. The function f is at least C^1 for $z \ge 1$. Now consider $r \ne 2$. We note that $f(1) = 1$ and $f \to 1$ as $z \to \infty$. The critical point on $[1, \infty)$ of f is at $z_c := (r/2)^{1/(r-2)} > 1$, where the function take the value

$$f(z_c) = 1 + \frac{2-r}{r}\left(\frac{2}{r}\right)^{2/(r-2)}. \tag{C.5}$$

A maximum(minimum) for $r < 2(r > 2)$. This concludes the proof. $\qquad\square$

D   A family of time-dependent solutions

In this appendix, we construct a family of solutions to the nonlinear Schrödinger equation with a quadratic, time-independent potential.

Let $\psi(x, t)$ have the form

$$\psi(x, t) = \mathrm{e}^{\mathrm{i}p(t)\cdot(x-a(t))+\mathrm{i}\gamma(t)}\tilde{\eta}_\mu(x - a(t)), \tag{D.1}$$

where $\tilde{\eta}_\mu$ is a real-valued function, not yet determined. We substitute this function into (1.1), and let $y := x - a$ to obtain

$$0 = \dot{p} \cdot y\tilde{\eta}_\mu + (\dot{\gamma} + p^2 - \dot{a}\cdot p)\tilde{\eta}_\mu + \mathrm{i}\nabla\tilde{\eta}_\mu \cdot (\dot{a} - 2p) - \Delta\tilde{\eta}_\mu - f(\tilde{\eta}_\mu) + V(y + a)\tilde{\eta}_\mu. \tag{D.2}$$

By adding and subtracting the terms $(\mu + V(a))\tilde{\eta}_\mu$ and $\nabla V(a) \cdot y\tilde{\eta}_\mu$ and as usual defining $\mathcal{R}_V := V(y+a) - V(a) - \nabla V(a) \cdot y$ we find

$$0 = (\dot{p} + \nabla V(a)) \cdot y\tilde{\eta}_\mu + (\dot{\gamma} + p^2 - \dot{a} \cdot p + V(a) - \mu)\tilde{\eta}_\mu + i\nabla\tilde{\eta}_\mu \cdot (\dot{a} - 2p) \\ + (-\Delta + \mu)\tilde{\eta}_\mu - f(\tilde{\eta}_\mu) + \mathcal{R}_V\tilde{\eta}_\mu. \quad \text{(D.3)}$$

If we choose

$$\dot{p} = -\nabla V(a), \ \dot{a} = 2p, \ \dot{\gamma} = p^2 + \mu - V(a), \quad \text{(D.4)}$$

then the Eqn. D.3 reduces to

$$0 = -\Delta\tilde{\eta}_\mu + \mu\tilde{\eta}_\mu - f(\tilde{\eta}_\mu) + \mathcal{R}_V\tilde{\eta}_\mu, \quad \text{(D.5)}$$

where $\tilde{\eta}_\mu = \tilde{\eta}_\mu(y)$, and $\Delta = \sum \partial_{y_j}^2$. In general this equation is time-dependent due to the appearance of a in $\mathcal{R}_V$, but for potentials of the form $V(x) = x \cdot A \cdot x + v \cdot x + d$, with constant matrix A, vector v and scalar d, we have

$$\mathcal{R}_V = (y+a) \cdot A \cdot (y+a) + v \cdot (y+a) + d - (a \cdot A \cdot a + v \cdot a + d) \\ - (a \cdot A \cdot y + y \cdot A \cdot a + v \cdot y) = y \cdot A \cdot y. \quad \text{(D.6)}$$

The right-hand side is independent of a, and hence of time. Equation (D.3) reduces to

$$0 = -\Delta\tilde{\eta}_\mu + \mu\tilde{\eta}_\mu - f(\tilde{\eta}_\mu) + y \cdot A \cdot y\tilde{\eta}_\mu. \quad \text{(D.7)}$$

Thus, if there exists nontrivial solutions to this equation, we have constructed a family of solutions $e^{ip(t)\cdot(x-a)+i\gamma}\tilde{\eta}_\mu(x-a)$, where $\tilde{\eta}_\mu$ solves (D.7). Existence of solutions to a general class of equations that includes (D.7) under some restrictions on $b := \mu + y \cdot A \cdot y$ and with a class of local nonlinearities is shown by Rabinowitz [23] and extended to more general potentials by Sikarov [25]. Sikarov require the following potential conditions: $b > -c$, where $|c| < \infty$,

$$\inf_{u \in H_1, \|u\|_{L^2}=1} \|\nabla u\|_{L^2}^2 + \langle bu, u \rangle > 0 \quad \text{(D.8)}$$

and that b grows to infinity in almost all directions as $|y| \to \infty$.

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