Quasi-Periodic Solutions for 1D Nonlinear Wave Equation with a General Nonlinearity *

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Abstract

In this paper, one–dimensional ($1D$) wave equation with a general nonlinearity

$$u_{tt}-u_{xx}+mu+f(u)=0,\ m>0$$

under Dirichlet boundary conditions is considered; the nonlinearity f is a real analytic, odd function and $f(u)=au^{2\bar{r}+1}+\sum\limits_{k\geq\bar{r}+1}f_{2k+1}u^{2k+1},\ a\neq 0$ and $\bar{r}\in\mathbb{N}$. It is proved that for almost all $m>0$ in Lebesgue measure sense, the above equation admits small-amplitude quasi-periodic solutions corresponding to finite dimensional invariant tori of an associated infinite dimensional dynamical system. The proof is based on infinite dimensional KAM theorem, partial normal form and scaling skills.

1 Statement of the main result

In this paper, we are going to study the nonlinear wave equation

$$u_{tt}-u_{xx}+mu+f(u)=0,\ m>0 \tag{1.1}$$

on the finite $x-$interval $[0,\pi]$ with Dirichlet boundary conditions

$$u(t,0)=0=u(t,\pi). \tag{1.2}$$

Here, $m>0$ is a real parameter, sometimes referred to as "mass", and f is a real analytic, odd function of u of the form

$$f(u)=au^{2\bar{r}+1}+\sum_{k\geq\bar{r}+1}f_{2k+1}u^{2k+1},\ a\neq 0 \text{ and } \bar{r}\in\mathbb{N}. \tag{1.3}$$

As [24], we study this equation (1.1) as an infinite dimensional hamiltonian system on $\mathcal{P}=H_0^1([0,\pi])\times L^2([0,\pi])$ with coordinates u and $v=u_t$. Let

$$\phi_j=\sqrt{\frac{2}{\pi}}sinjx,\ \lambda_j=\sqrt{j^2+m},\ j\geq 1$$

*The work was supported by the Special Funds for Major State Basic Research Projects of China (973 projects).

be the basic modes and frequencies of the linear equation $u_{tt}=u_{xx}-mu$ with Dirichlet boundary conditions. Then every solution of the linear equation is the superposition of their harmonic oscillations and of the form

$$u(t,x)=\sum_{j\geq 1} q_j(t)\phi_j(x),\ q_j(t)=I_j\cos(\lambda_j t+\phi_j^0)$$

with amplitudes $I_j\geq 0$ and initial phases ϕ_j^0. Their combined motion is periodic, quasi-periodic or almost periodic, respectively, depending on whether one, finitely many or infinitely many modes are excited. In particular, for every choice

$$\mathcal{J}=\{j_1<j_2<\cdots<j_b\}\subset\mathbb{N}$$

of finitely many modes there is an invariant $2b-$dimensional linear subspace $E_{\mathcal{J}}$ that is completely foliated into rotational tori with frequencies $\lambda_{j_1},\cdots,\lambda_{j_b}$:

$$E_{\mathcal{J}}=\{(u,v)=(q_1\phi_{j_1}+\cdots+q_b\phi_{j_b},p_1\phi_{j_1}+\cdots+p_b\phi_{j_b})\}=\bigcup_{I\in\overline{P^b}}\mathcal{T}_{\mathcal{J}}(I),$$

where $P^b=\{I\in\mathbb{R}^b: I_j>0 \text{ for } 1\leq j\leq b\}$ is the positive quadrant in $\mathbb{R}^n$ and

$$\mathcal{T}_{\mathcal{J}}(I)=\{(u,v): q_j^2+\lambda_j^{-2}p_j^2=I_j \text{ for } 1\leq j\leq b\},$$

using the above representation of u and v.

Upon restoration of the nonlinearity f, we show that there exists a Cantor set $\mathcal{C}\subset P^b$, a specially chosen index set $\mathcal{I}=\{n_1<n_2<\cdots<n_b\}\subset\mathbb{N}$ (see below) and a family of b-tori

$$\mathcal{T}_{\mathcal{I}}[\mathcal{C}]=\bigcup_{I\in\mathcal{C}}\mathcal{T}_{\mathcal{I}}(I)\subset E_{\mathcal{I}}$$

over $\mathcal{C}$, and a Whitney smooth embedding

$$\Phi:\mathcal{T}_{\mathcal{I}}[\mathcal{C}]\to\mathcal{E}_{\mathcal{I}}\subset\mathcal{P},$$

such that the restriction of Φ to each $\mathcal{T}_{\mathcal{I}}(I)$ in the family is an embedding of a rotational b-torus for the nonlinear equation. In [24], The image $\mathcal{E}_{\mathcal{I}}$ of $\mathcal{T}_{\mathcal{I}}[\mathcal{C}]$ is called a Cantor manifold of rotational b-tori.

Theorem 1 (Main Theorem) *For almost all $m>0$ and each index set $\mathcal{I}=\{n_1<\cdots<n_b\}$ with $b\geq 2$, satisfying*

$$sn_i\neq n_j \text{ for any } s=1,2,\cdots,\bar{r},\ i<j,\ i,j\in\{1,\cdots,b\}, \tag{1.4}$$

the wave equation (1.1) with (1.2) possesses a local, positive-measure, $2b$ dimensional invariant Cantor manifold $\mathcal{E}_{\mathcal{I}}$ given by a Whitney smooth embedding $\Phi:\mathcal{T}_{\mathcal{I}}[\mathcal{C}]\to\mathcal{E}_{\mathcal{I}}$, which is a higher order perturbation of the inclusion map $\Phi_0: E_{\mathcal{I}}\hookrightarrow\mathcal{P}$ restricted to $\mathcal{T}_{\mathcal{I}}[\mathcal{C}]$. Moreover, the Cantor manifold $\mathcal{E}_{\mathcal{I}}$ is foliated by real analytic, linearly stable, b dimensional invariant tori carrying quasiperiodic solutions.

Remark 1.1 *The size of Cantor manifold $\mathcal{E}_{\mathcal{I}}$ is not uniform, but depends on m, b, $\mathcal{I}$ and etc..*

Remark 1.2 *The frequencies of the diophantine tori are also under control. They are $\omega(I)=(\omega_{n_1}(I),\cdots,\omega_{n_b}(I))$, where $\omega_{n_j}(I)=\lambda_{n_j}+A_j+\mathcal{O}(||I||^{\bar{r}+\frac{1}{6}})$, $j=1,\cdots,b$,*

$$A_j=\frac{c_{\bar{r}}(2\bar{r}+1)}{\bar{r}+1}\frac{I_{n_j}^{\bar{r}}}{\lambda_{n_j}^{\bar{r}+1}}+\frac{c_{\bar{r}}(\bar{r}+1)}{\lambda_{n_j}}\sum_{i\neq j}\frac{I_{n_i}^{\bar{r}}}{\lambda_{n_i}^{\bar{r}}}+\sum_{\substack{0\le p_t<\bar{r},\ t=1,\cdots,b\\ p_1+\cdots+p_b=\bar{r}}}\mathcal{O}(I_{n_1}^{p_1}\cdots I_{n_b}^{p_b})$$

and

$$c_{\bar{r}}=C_{2\bar{r}+2}^{\bar{r}+1}\frac{(2\bar{r}-1)!!}{4\pi^{\bar{r}}(2\bar{r})!!}.$$

Remark 1.3 *The similar conclusion holds for 1D nonlinear wave equation*

$$u_{tt}-u_{xx}+mu+g(u)=0,\ m>0 \tag{1.5}$$

with periodic boundary condition

$$u(t,x+2\pi)=u(t,x), \tag{1.6}$$

where the nonlinearity $g(u)$ is real analytic and

$$g(u)=au^{2\bar{r}+1}+\mathcal{O}(u^{2\bar{r}+2}),\ a\neq 0 \text{ and } \bar{r}\in\mathbb{N},$$

if one uses the so-called "compactness property" observed in [16]. For $\bar{r}=1$, the result has been obtained in [22].

The rest of the paper is organized as follows: In section 2 we firstly discuss some known results on nonlinear wave equations, and then present the idea of the proof. Section 3 contains a concluding theorem about 1D nonlinear beam equation with a general nonlinearity under the hinged boundary conditions. In section 4, the hamiltonian function is written in infinitely many coordinates and then put into partial normal form in section 5. In section 6 we introduce an infinite dimensional KAM theorem and the measure estimates are given in section 7. Some lemmata are proved in the Appendix.

2 Discussion and idea of the proof

In this section, we will mainly discuss the relations of our results with previous results on 1D nonlinear wave equations with constant potentials. For $\bar{r}>1$, the known results are all about periodic solutions. The first result is due to Walter Craig and C. E. Wayne, who discuss φ^d-nonlinear Klein-Gordon equation

$$u_{tt}-u_{xx}+b^2u-u^{d-1}=0,\quad d>4 \tag{2.1}$$

in [15]. They obtain families of periodic solutions for open set of parameters b^2 of full measure when $d=2n(n>2)$ under either periodic or Dirichlet boundary conditions. For the existences of periodic solutions of 1D completely resonant wave equation($m=0$) with all kinds of nonlinearities, see [1, 3, 4, 5, 18, 23].

All the above cases are about the existences of periodic solutions. There are also many results on the existences of quasi-periodic solutions for $\bar{r}=1$. The first result in this direction is due to A.I.Bobenko and Kuksin [6]. They get the quasi-periodic solutions corresponding to finite dimensional invariant tori of (1.1)+(1.2). Their starting point is to take the equation (1.1) as a perturbed sine-Gordon equation. This result is regained by Pöschel [24] by the infinite KAM theory and normal form technique. The existence of quasi-periodic solutions of the equation (1.5)+(1.6) is firstly proved by Bourgain [10] with the famous Craig-Wayne-Bourgain's methods (see [7, 8, 9, 10, 11, 14]). Later Jean Bricmont, Antti Kupiainen and Alain Schenkel [12] give a new proof of this result based on a renormalization group procedure. For the existences of quasi-periodic solutions corresponding to finite dimensional invariant tori of (1.5)+(1.6), see [22].

The remained case is whether there exist quasi-periodic solutions corresponding to finite dimensional invariant tori of (1.1)+(1.2) when $\bar{r}>1$. Theorem 1 actually gives a positive answer towards this problem.

The proof follows the main steps of the infinite dimensional KAM theorem. However, as one will see, there exist some technical difficulties when considering 1D wave equation with a general nonlinearity. The first appears when we want to get a partial Birkhoff normal form. For this purpose, we mainly expect that the following inequality holds

$$|\lambda_{i_1}\pm\lambda_{i_2}\pm\cdots\pm\lambda_{i_{2\bar{r}}}+\lambda_i-\lambda_j|\geq c(m)>0, \tag{2.2}$$

where $i_1,\cdots,i_{2\bar{r}}\in\{n_1,\cdots,n_b\}$, $i\neq j$, and $n_1,\cdots,n_b$ are tangent sites while i, j are normal ones. For the 1D Schrödinger equations with higher order nonlinearities(see [20]), (2.2) is guaranteed by carefully choosing the tangent sites. But this method couldn't be applied to the wave equation as considered in this paper. Comparing with [20], one gets (2.2) from throwing a small set of m. The second technical difficulty lies in the measure estimates. Since the nonlinear term is of higher order, the measure estimate becomes much more complicated. Since it is very technical, the reader is deferred to section 7.

3 A Concluding Theorem

The same conclusion holds for 1D beam equation with a general nonlinearity

$$u_{tt}+u_{xxxx}+mu+f(u)=0,\ m>0 \tag{3.1}$$

under the hinged boundary conditions

$$u(0,t)=u_{xx}(0,t)=u(\pi,t)=u_{xx}(\pi,t)=0, \tag{3.2}$$

where f is a real analytic, odd function and

$$f(u)=au^{2\bar{r}+1}+\sum_{k\geq\bar{r}+1}f_{2k+1}u^{2k+1},\ a\neq 0 \text{ and } \bar{r}\in\mathbb{N}.$$

Here we give the result while omitting the proof.

Theorem 2 (Main Theorem) *For almost all $m>0$ and each index set $\mathcal{I}=\{n_1<\cdots<n_b\}$ with $b\geq 2$, satisfying*

$$sn_i \neq n_j \text{ for any } s=1,2,\cdots,\bar{r},\ i<j,\ i,j\in\{1,\cdots,b\}, \tag{3.3}$$

the beam equation (3.1) with (3.2) possesses a local, positive-measure, $2b$ dimensional invariant Cantor manifold $\mathcal{E}_{\mathcal{I}}$ given by a Whitney smooth embedding $\Phi : \mathcal{T}_{\mathcal{I}}[\mathcal{C}] \to \mathcal{E}_{\mathcal{I}}$, which is a higher order perturbation of the inclusion map $\Phi_0 : E_{\mathcal{I}} \hookrightarrow \mathcal{P}$ restricted to $\mathcal{T}_{\mathcal{I}}[\mathcal{C}]$. Moreover, the Cantor manifold $\mathcal{E}_{\mathcal{I}}$ is foliated by real analytic, linearly stable, b dimensional invariant tori carrying quasiperiodic solutions.

Remark 3.1 *When $\bar{r}=1$, for any index set $J=\{n_1<\cdots<n_b\}$ and almost all $m>0$, our conclusion holds while in [17] a set of positive measure of m is thrown.*

Remark 3.2 *For the completely resonant beam equation*

$$u_{tt}+u_{xxxx}+f(u)=0 \tag{3.4}$$

under the hinged boundary condition (3.2), the result is obtained in [21], where f is a real analytic, odd function and

$$f(u)=au^3+\sum_{k\geq 2} f_{2k+1}u^{2k+1},\ a\neq 0.$$

Remark 3.3 *The similar conclusions hold for 1D nonlinear beam equation*

$$u_{tt}+u_{xxxx}+mu+g(u)=0,\ m>0 \tag{3.5}$$

with periodic boundary condition

$$u(t,x+2\pi)=u(t,x), \tag{3.6}$$

where the nonlinearity $g(u)$ is real analytic and

$$g(u)=au^{2\bar{r}+1}+\mathcal{O}(u^{2\bar{r}+2}),\ a\neq 0 \text{ and } \bar{r}\in\mathbb{N}.$$

4 The hamiltonian setting of wave equations

Without losing generality, let $a=1$. In the following, we always suppose $m\in I=(0,M_*]$, where M_* is a fixed large number. Let us rewrite the wave equation (1.1) as follows

$$u_{tt}+Au=-f(u),\quad Au\equiv -u_{xx}+mu,\ x,\ t\in\mathbb{R}, \tag{4.1}$$

$$u(0,t)=0=u(\pi,t), \tag{4.2}$$

Equation (4.1) may be rewritten as

$$\dot{u}=v,\qquad \dot{v}+Au=-f(u), \tag{4.3}$$

which, as is well known, may be viewed as the (infinite dimensional) hamiltonian equations $\dot{u} = H_v$, $\dot{v} = -H_u$ associated to the hamiltonian

$$H = \frac{1}{2}\langle v, v\rangle + \frac{1}{2}\langle Au, u\rangle + \int_0^\pi g(u)\ dx, \tag{4.4}$$

where g is a primitive of $f(u)$ (with respect to the u variable) and $\langle \cdot\ ,\cdot\rangle$ denotes the scalar product in L^2.

As in [13], we introduce coordinates $q = (q_1, q_2, \cdots), p = (p_1, p_2, \cdots)$ through the relations

$$u(x) = \sum_{j\geq 1} \frac{q_j}{\sqrt{\lambda_j}}\phi_j(x), \quad v = \sum_{j\geq 1}\sqrt{\lambda_j}p_j\phi_j(x),$$

where $\phi_j = \sqrt{2/\pi}\sin jx$ for $j = 1, 2, \cdots$ are the orthonormal eigenfunctions of the operator A with eigenvalues $\lambda_j^2 = j^2 + m$. The coordinates are taken from some Hilbert space $\mathcal{H}^{a,\rho}$ of all real valued sequences $w = (w_1, w_2, \cdots)$ with finite norm

$$\|w\|_{a,\rho}^2 = \sum_{j\geq 1}|w_j|^2 j^{2a} e^{2j\rho}.$$

Below we will assume that $a \geq 0$ and $\rho > 0$. We formally obtain the hamiltonian

$$H = \Lambda + G = \frac{1}{2}\sum_{j\geq 1}\lambda_j(p_j^2 + q_j^2) + \int_0^\pi g(\sum_{j\geq 1}\frac{q_j}{\sqrt{\lambda_j}}\phi_j)\ dx \tag{4.5}$$

with the lattice hamiltonian equations

$$\dot{q}_j = \frac{\partial H}{\partial p_j} = \lambda_j p_j, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j} = -\lambda_j q_j - \frac{\partial G}{\partial q_j}, \tag{4.6}$$

Rather than discussing the above formal validity, we shall use the following elementary observation:

Lemma 4.1 *Let $\bar{I}$ be an interval and let*

$$t \in \bar{I} \to (q(t), p(t)) \equiv \Big(\{q_j(t)\}_{j\geq 1},\ \{p_j(t)\}_{j\geq 1}\Big)$$

be a real analytic solution of (4.6) for some $\rho > 0$. Then

$$u(t, x) \equiv \sum_{j\geq 1}\frac{q_j(t)}{\sqrt{\lambda_j}}\phi_j(x),$$

is classical solution of (4.1) that is real analytic on $\bar{I} \times [0, \pi]$.

For the proof, see [24].

Next we consider the regularity of the gradient of G. Following Pöschel [24], we have the following Lemma.

Lemma 4.2 *For $a>0$ and $\rho \geq 0$, the gradient G_q is real analytic as a map from some neighbourhood of the origin in $\mathcal{H}^{a,\rho}$ into $\mathcal{H}^{a+1,\rho}$, with*

$$\|G_q\|_{a+1,\rho} = \mathcal{O}(\|q\|_{a,\rho}^{2\bar{r}+1}).$$

For the nonlinearity $u^{2\bar{r}+1}$, we find

$$G = \frac{1}{2\bar{r}+2}\int_0^\pi u^{2\bar{r}+2}\, dx \;= \frac{1}{2\bar{r}+2}\sum_{i_1,\cdots,i_{2\bar{r}+2}} G_{i_1\cdots i_{2\bar{r}+2}} q_{i_1}\cdots q_{i_{2\bar{r}+2}} \tag{4.7}$$

with

$$G_{i_1\cdots i_{2\bar{r}+2}} = \frac{1}{\sqrt{\lambda_{i_1}\cdots\lambda_{i_{2\bar{r}+2}}}}\int_0^\pi \phi_{i_1}\phi_{i_2}\cdots\phi_{i_{2\bar{r}+2}}\, dx. \tag{4.8}$$

It is not difficult to verify that $G_{i_1\cdots i_{2\bar{r}+2}} = 0$ unless $i_1 \pm i_2 \pm \cdots \pm i_{2\bar{r}+2} = 0$, for some combination of plus and minus signs. For simplicity, write $G_{\underbrace{n_t\cdots n_t}_{2\bar{r}+2}} = G_{n_t}$ and $G_{\underbrace{n_j\cdots n_j}_{2\bar{r}} n_i n_i} = G_{n_j n_i}$, where $j \neq i, i,j \in \{1,\cdots,b\}$. Note (1.4), we have

$$G_{n_t} = \frac{2^{\bar{r}+1}(2\bar{r}+1)!!}{\lambda_{n_t}{}^{\bar{r}+1}\pi^{\bar{r}}(2\bar{r}+2)!!},\ G_{n_j n_i} = \frac{2^{\bar{r}}(2\bar{r}-1)!!}{\lambda_{n_i}\lambda_{n_j}^{\bar{r}}\pi^{\bar{r}}(2\bar{r})!!}. \tag{4.9}$$

by elementary calculation.

5 Partial Birkhoff normal form

Next we transform the hamiltonian (4.5) into some partial Birkhoff normal form so that it may serve for our aim. We first switch to complex coordinates

$$z_j = \frac{1}{\sqrt{2}}(q_j + ip_j), \quad \bar{z}_j = \frac{1}{\sqrt{2}}(q_j - ip_j),$$

then the hamiltonian (with respect to the symplectic structure $i\sum_j dz_j \wedge d\bar{z}_j$) is given by

$$\begin{aligned} H &= \sum_j \lambda_j |z_j|^2 + \int_0^\pi g\Big(\sum_j \frac{z_j+\bar{z}_j}{\sqrt{2\lambda_j}}\phi_j\Big)\, dx \\ &= \sum_j \lambda_j |z_j|^2 + \frac{1}{(\bar{r}+1)2^{\bar{r}+2}}\sum_{i_1\pm\cdots\pm i_{2\bar{r}+2}=0} G_{i_1\cdots i_{2\bar{r}+2}}(z_{i_1}+\bar{z}_{i_1})\cdots(z_{i_{2\bar{r}+2}}+\bar{z}_{i_{2\bar{r}+2}}) + \mathcal{O}(2\bar{r}+4). \end{aligned}$$

For simplicity, we will write $c = \frac{1}{(\bar{r}+1)2^{\bar{r}+2}}$. So we obtain the hamiltonian

$$\begin{aligned} H = &\sum_j \lambda_j |z_j|^2 + c\sum_{i_1\pm\cdots\pm i_{2\bar{r}+2}=0} G_{i_1\cdots i_{2\bar{r}+2}} z_{i_1}\cdots z_{i_{2\bar{r}+2}} \\ &+ cC^1_{2\bar{r}+2}\sum_{i_1\pm\cdots\pm i_{2\bar{r}+2}=0} G_{i_1\cdots i_{2\bar{r}+2}}\bar{z}_{i_1} z_{i_2}\cdots z_{i_{2\bar{r}+2}} + \cdots \\ &+ cC^k_{2\bar{r}+2}\sum_{i_1\pm\cdots\pm i_{2\bar{r}+2}=0} G_{i_1\cdots i_{2\bar{r}+2}}\bar{z}_{i_1}\cdots\bar{z}_{i_k} z_{i_{k+1}}\cdots z_{i_{2\bar{r}+2}} + \cdots \\ &+ cC^{2\bar{r}+2}_{2\bar{r}+2}\sum_{i_1\pm\cdots\pm i_{2\bar{r}+2}=0} G_{i_1\cdots i_{2\bar{r}+2}}\bar{z}_{i_1}\bar{z}_{i_2}\cdots\bar{z}_{i_{2\bar{r}+2}} + \mathcal{O}(2\bar{r}+4). \end{aligned} \tag{5.1}$$

We define the index sets Δ_*, $*=0,1,2$ and Δ_3. $\Delta_*(*=0,1,2)$ is the set of index $(i_1,\cdots,i_{2\bar{r}+1})$ such that there exist exactly $*$ components not in $\{n_1,n_2,\cdots,n_b\}$. Δ_3 is the set of the index $(i_1,\cdots,i_{2\bar{r}+1})$ such that there exist at least three components not in $\{n_1,n_2,\cdots,n_b\}$. Define the resonance sets

$$\mathcal{N}=\{(i_1,i_1,\cdots,i_{\bar{r}+1},i_{\bar{r}+1})\}\cap\Delta_0$$

and

$$\mathcal{M}=\{(i_1,i_1,\cdots,i_{\bar{r}+1},i_{\bar{r}+1})\}\cap\Delta_2.$$

We will transform the hamiltonian (5.1) into a partial Birkhoff form of $2\bar{r}+2$ order so that the infinite dimensional KAM Theorem, which will be given in section 6, can be applied. In the following, $A(\mathcal{H}^{a,\rho},\mathcal{H}^{a,\rho+1})$ denotes the class of all real analytic maps from some neighbourhood of the origin in $\mathcal{H}^{a,\rho}$ into $\mathcal{H}^{a,\rho+1}$. For conveniences, we also denote $J=I\setminus J_*$,where $J_*=J_1\bigcup Z_0$ and $Z_0=Z_0^1\bigcup Z_0^2\bigcup Z_0^3\bigcup Z_0^4\bigcup Z_0^5$. The relative constants τ, γ^* in J_1 are chosen suitably in section 7. We remark that[1] $|Z_0|=0$ and $|J_1|\le c\gamma^*$, where c depends on b, n_b, M_*. For the explanations for J_1, see Lemma 8.4 in the Appendix. For the explanations of Z_0^i, $i=1,\cdots,$ 5, see Lemma 5.2, Lemma 5.3, Lemma 5.4 and Lemma 7.8.

Lemma 5.1 *For $m\in J$ and any given index set $\{n_1<n_2<\cdots<n_b\}$, there exists a real analytic, symplectic change of coordinates X_F^1 in some neighborhood of the origin that takes the hamiltonian (5.1) into*

$$H\circ X_F^1=\Lambda+\overline{G}+\widehat{G}+K,$$

where $X_{\overline{G}}$, $X_{\widehat{G}}$, $X_K\in A(\mathcal{H}^{a,\rho},\mathcal{H}^{a,\rho+1})$,

$$\begin{aligned}\overline{G}=&cC_{2\bar{r}+2}^{\bar{r}+1}(\sum_{j=1}^{b}G_{n_j}|z_{n_j}|^{2\bar{r}+2})\\&+\sum_{k=2}^{b}\sum_{\substack{i_1,\cdots,i_k\in\{1,\cdots,b\}\\ i_1<\cdots<i_k}}\sum_{\substack{l_1,\cdots,l_k,\ 0<l_t<\bar{r}+1,t=1,\cdots,k,\\ l_1+\cdots+l_k=\bar{r}+1}}\mathcal{O}(|z_{n_{i_1}}|^{2l_1}\cdots|z_{n_{i_k}}|^{2l_k})\\&+c(\bar{r}+1)^2C_{2\bar{r}+2}^{\bar{r}+1}\sum_{i=1}^{b}\sum_{\substack{i\neq j\\ i,j\in\{1,\cdots,b\}}}G_{n_jn_i}|z_{n_j}|^{2\bar{r}}|z_{n_i}|^2\\&+c(\bar{r}+1)^2C_{2\bar{r}+2}^{\bar{r}+1}\sum_{t=1}^{b}\sum_{j\neq n_1,\cdots,n_b}G_{n_tj}|z_{n_t}|^{2\bar{r}}|z_j|^2\\&+\sum_{k=2}^{b}\sum_{\substack{i_1,\cdots,i_k\in\{1,\cdots,b\}\\ i_1<\cdots<i_k}}\sum_{\substack{l_1,\cdots,l_k,\ 0<l_t<\bar{r},\ t=1,\cdots,k\\ l_1+\cdots+l_k=\bar{r}}}\\&\sum_{j\neq n_1,\cdots,n_b}C_{i_1\cdots i_k}^{l_1\cdots l_k}G_{\underbrace{n_{i_1}\cdots n_{i_1}}_{2l_1}\cdots\underbrace{n_{i_k}\cdots n_{i_k}}_{2l_k}jj}|z_{n_{i_1}}|^{2l_1}\cdots|z_{n_{i_k}}|^{2l_k}|z_j|^2,\end{aligned}\tag{5.2}$$

[1]For a set, we denote by $|\cdot|$ the Lebesgue measure of the set.

$$\widehat{G}=c\sum_{\substack{(i_1,\cdots,i_{2\bar{r}+2})\in\Delta_3\\ i_1\pm\cdots\pm i_{2\bar{r}+2}=0}} G_{i_1\cdots i_{2\bar{r}+2}}(z_{i_1}+\bar{z}_{i_1})\cdots(z_{i_{2\bar{r}+2}}+\bar{z}_{i_{2\bar{r}+2}}),$$

$K=\mathcal{O}(||z||_{a,\rho}^{2\bar{r}+4})$ and the constant $C_{i_1\cdots i_k}^{l_1\cdots l_k}=\left(C_{\bar{r}+1}^{1}C_{\bar{r}}^{l_1}C_{\bar{r}-l_1}^{l_2}\cdots C_{\bar{r}-l_1-\cdots-l_{k-1}}^{l_k}\right)^2$ and c depends on $\bar{r}$, τ, n_b.

The proof is obvious from the following Lemmata.

Lemma 5.2 *For $m\in J$, if $(i_1,\cdots,i_{2\bar{r}+2})\in\Delta_0$ and $i_1\pm\cdots\pm i_{2\bar{r}+2}=0$, then $\lambda_{j_1}+\cdots+\lambda_{j_s}=\lambda_{j_{s+1}}+\cdots+\lambda_{j_{2\bar{r}+2}}$ if and only if $(j_1,\cdots,j_s)=(j_{s+1},\cdots,j_{2\bar{r}+2})$, where $(j_1,\cdots,j_{2\bar{r}+2})=(i_1,\cdots,i_{2\bar{r}+2})$. For the case $\lambda_{i_1}\pm\cdots\pm\lambda_{i_{2\bar{r}+2}}\neq 0$, one has*

$$|\lambda_{i_1}\pm\cdots\pm\lambda_{i_{2\bar{r}+2}}|>c(m)>0. \tag{5.3}$$

Proof: If $\lambda_{j_1}+\cdots+\lambda_{j_s}=\lambda_{j_{s+1}}+\cdots+\lambda_{j_{2\bar{r}+2}}$, from collecting terms, one obtains

$$l_1\lambda_{k_1}+l_2\lambda_{k_2}+\cdots+l_p\lambda_{k_p}=0, \tag{5.4}$$

where $0\leq|l_1|+\cdots+|l_p|\leq 2\bar{r}+2$ and $k_1,\cdots,\ k_p\in\{n_1,\cdots,\ n_b\}$ and $l_1l_2\cdots l_p\neq 0$ $\big($otherwise one obtains $(j_1,\cdots,\ j_s)=(j_{s+1},\cdots,\ j_{2\bar{r}+2})\big)$. Now we denote

$$Z_0^1=\left\{m\in I\left|\begin{matrix}l_1\lambda_{k_1}+\cdots+l_p\lambda_{k_p}=0,|l_1|,\cdots,|l_p|\in\{1,\cdots,2\bar{r}+2\},\\ \{k_1,k_2,\cdots,k_p\}\subseteq\{1,\cdots,n_b\},1\leqslant p\leqslant b,p\in Z\end{matrix}\right.\right\}.$$

From Lemma 8.5, we have $|Z_0^1|=0$. Since $m\in J$, then (5.4) can't hold unless $(l_1,\ l_2,\cdots,\ l_p)=(0,\ 0,\cdots,\ 0)$. (5.3) is obvious when $\lambda_{i_1}\pm\cdots\pm\lambda_{i_{2\bar{r}+2}}\neq 0$.

Lemma 5.3 *For $m\in J$, if $(i_1,\cdots,i_{2\bar{r}+2})\in\Delta_1$ and $i_1\pm\cdots\pm i_{2\bar{r}+2}=0$, then*

$$|\lambda_{i_1}\pm\cdots\pm\lambda_{i_{2\bar{r}+2}}|>c(m)>0.$$

Proof: We only give a sketch since it is similar with above. After collecting terms, one gets

$$l_1\lambda_{k_1}+\cdots+l_p\lambda_{k_p}+\lambda_i=0, \tag{5.5}$$

where i is the unique normal site. Since $i_1\pm\cdots\pm i_{2\bar{r}+2}=0$, we have $|i|\leq(2\bar{r}+1)n_b$. Denote

$$Z_0^2=\left\{m\in I\left|\begin{matrix}l_1\lambda_{k_1}+\cdots+l_p\lambda_{k_p}+\lambda_i=0,\ |l_1|,\cdots,|l_p|\in\{1,\cdots,2\bar{r}+1\},\\ \{k_1,k_2,\cdots,k_p,i\}\subseteq\{1,\cdots,(2\bar{r}+1)n_b\},\ 1\leqslant p\leqslant b,\ p\in Z\end{matrix}\right.\right\}.$$

Similarly, one has $|Z_0^2|=0$. Note $m\in J$, one easily gets the conclusion.

Lemma 5.4 *For $m\in J$, if $(i_1,\cdots,i_{2\bar{r}+2})\in\Delta_2$ and $i_1\pm\cdots\pm i_{2\bar{r}+2}=0$, then $\lambda_{j_1}+\cdots+\lambda_{j_s}=\lambda_{j_{s+1}}+\cdots+\lambda_{j_{2\bar{r}+2}}$ if and only if $(j_1,\cdots,j_s)=(j_{s+1},\cdots,j_{2\bar{r}+2})$, where $(j_1,\cdots,j_{2\bar{r}+2})=(i_1,\cdots,i_{2\bar{r}+2})$. For the case $\lambda_{i_1}\pm\cdots\pm\lambda_{i_{2\bar{r}+2}}\neq 0$, one has*

$$|\lambda_{i_1}\pm\cdots\pm\lambda_{i_{2\bar{r}+2}}|\geq c(m)>0. \tag{5.6}$$

Proof: We only need prove the necessary condition. After collecting terms, one obtains

$$l_1\lambda_{k_1}+l_2\lambda_{k_2}+\cdots+l_p\lambda_{k_p}\pm\lambda_i\pm\lambda_j=0,$$

where i, j are normal sites and the others are tangent ones. We only need prove the conclusions in the following two cases.
(i)

$$l_1\lambda_{k_1}+l_2\lambda_{k_2}+\cdots+l_p\lambda_{k_p}+\lambda_i+\lambda_j=0,\ i\geq j. \tag{5.7}$$

It is obvious that (5.7) can't hold When $i>2\bar{r}\sqrt{n_b^2+M_*}$. Hence, one has $i\leq 2\bar{r}\sqrt{n_b^2+M_*}$. Denote

$$Z_0^3=\left\{m\in I\middle|\begin{array}{l}l_1\lambda_{k_1}+\cdots+l_p\lambda_{k_p}+\lambda_i+\lambda_j=0,\ |l_1|,\cdots,|l_p|\in\{1,\cdots,2\bar{r}\},\\ \{k_1,k_2,\cdots,k_p,i,j\}\subseteq\left\{1,\cdots,[2\bar{r}\sqrt{n_b^2+M_*}]\right\},\ 1\leqslant p\leqslant b,\ p\in Z\end{array}\right\}.$$

From Lemma 8.5, one has $|Z_0^3|=0$, since $m\in J$, (5.7) can't hold.
(ii)

$$l_1\lambda_{k_1}+l_2\lambda_{k_2}+\cdots+l_p\lambda_{k_p}+\lambda_i-\lambda_j=0,\ i>j. \tag{5.8}$$

Write $p'=i-j$. We have

$$l_1\lambda_{k_1}+\cdots+l_p\lambda_{k_p}+p'+\mathcal{O}(\frac{1}{j})=0. \tag{5.9}$$

It is obvious that (5.8) can't hold when $p'>4\bar{r}n_bM_*$. Therefore

$$p'\leq 4\bar{r}n_bM_*. \tag{5.10}$$

From (5.8), one gets $(l_1,\cdots,l_p)\neq 0$. Combined with $m\in J$, it results in

$$|l_1\lambda_{k_1}+\cdots+l_p\lambda_{k_p}+p'|\geq\frac{2\gamma^{*b}}{(2\bar{r})^{\frac{\tau}{4\bar{r}}}}.$$

Hence, when $\frac{M_*}{j}\leq\frac{\gamma^{*b}}{(2\bar{r})^{\frac{\tau}{4\bar{r}}}}$, (5.8) can't hold. In other words, when $j\geq\frac{M_*(2\bar{r})^{\frac{\tau}{4\bar{r}}}}{\gamma^{*b}}$, (5.8) can't hold true. The left case is that $j\leq\frac{M_*(2\bar{r})^{\frac{\tau}{4\bar{r}}}}{\gamma^{*b}}$ and $i\leq\frac{M_*(2\bar{r})^{\frac{\tau}{4\bar{r}}}}{\gamma^{*b}}+4\bar{r}n_bM_*$. Similarly, denote

$$Z_0^4=\left\{m\in I\middle|\begin{array}{l}l_1\lambda_{k_1}+\cdots+l_p\lambda_{k_p}+\lambda_i-\lambda_j=0,\ |l_1|,\cdots,|l_p|\in\{1,\cdots,2\bar{r}\},\\ \{k_1,k_2,\cdots,k_p,i,j\}\subseteq\left\{1,\cdots,[\frac{M_*(2\bar{r})^{\frac{\tau}{4\bar{r}}}}{\gamma^{*b}}+4\bar{r}n_bM_*]\right\},\ 1\leqslant p\leqslant b,\ p\in Z\end{array}\right\}.$$

From Lemma 8.5, one has $|Z_0^4|=0$. Since $m\in J$, (5.8) can't hold.

The conclusion of (5.6) is obvious from the above proof. ∎

Now we introduce the symplectic polar and complex coordinates by setting

$$z_j=\begin{cases}\sqrt{(\xi_j+y_j)}e^{-\mathrm{i}x_j}, j=n_1,n_2,\cdots,n_b\\ w_j, \qquad\qquad\quad j\neq n_1,n_2,\cdots,n_b\end{cases}$$

depending on parameters $\xi\in\Pi=[0,1]^b$. The precise domain will be specified later. In order to simplify the expression, we substitute $\xi_{n_j}, j=1,2,\cdots,b$ by $\xi_j, j=1,2,\cdots,b$. Then one gets

$$i\sum_{j\geq 1}dz_j\wedge d\bar{z}_j=\sum_{j=n_1,n_2,\cdots,n_b}dx_j\wedge dy_j+i\sum_{j\neq n_1,n_2,\cdots,n_b}dw_j\wedge d\bar{w}_j.$$

Note (4.9), the new hamiltonian

$$H=\Lambda+\overline{G}+\widehat{G}+K=\langle\omega(\xi),y\rangle+\langle\Omega(\xi)w,\bar{w}\rangle+\tilde{G}+\widehat{G}+K$$

with frequencies $\omega(\xi)=\alpha'+A(\xi)$, $\Omega(\xi)=\beta'+B(\xi)$, where

$$\alpha'=(\lambda_{n_1},\lambda_{n_2},\cdots,\lambda_{n_b}),\ \beta'=(\lambda_i)_{i\neq n_1,\cdots,n_b},$$

$$A(\xi)=(A_j)_{j=1}^b=(A_1,\cdots,A_b),\ B(\xi)=(B_j)_{j\neq n_1,\cdots,n_b}$$

$$A_j=\frac{c_{\bar{r}}(2\bar{r}+1)}{\bar{r}+1}\frac{\xi_j^{\bar{r}}}{\lambda_{n_j}^{\bar{r}+1}}+\frac{c_{\bar{r}}(\bar{r}+1)}{\lambda_{n_j}}\sum_{i\neq j}\frac{\xi_i^{\bar{r}}}{\lambda_{n_i}^{\bar{r}}}+\sum_{\substack{0\leq p_t<\bar{r},\ t=1,\cdots,b\\ p_1+\cdots+p_b=\bar{r}}}\mathcal{O}(\xi_1^{p_1}\cdots\xi_b^{p_b}),$$

$$\begin{aligned}B_j=&c(\bar{r}+1)^2C_{2\bar{r}+2}^{\bar{r}+1}\sum_{t=1}^{b}G_{n_tj}\xi_t^{\bar{r}}\\ &+\sum_{k=2}^{b}\sum_{\substack{i_1,\cdots,i_k\in\{1,\cdots,b\}\\ i_1<\cdots<i_k}}\sum_{\substack{l_1,\cdots,l_k,\ 0<l_t<\bar{r},\ t=1,\cdots,k\\ l_1+\cdots+l_k=\bar{r}}}C_{i_1\cdots i_k}^{l_1\cdots l_k}G_{\underbrace{n_{i_1}\cdots n_{i_1}}_{2l_1}\cdots\underbrace{n_{i_k}\cdots n_{i_k}}_{2l_k}jj}\xi_{i_1}^{l_1}\cdots\xi_{i_k}^{l_k},\end{aligned}\tag{5.11}$$

$$c_{\bar{r}}=C_{2\bar{r}+2}^{\bar{r}+1}\frac{(2\bar{r}-1)!!}{4\pi^{\bar{r}}(2\bar{r})!!}.$$

We obtain

$$H=\langle\omega(\xi),y\rangle+\langle\Omega(\xi)w,\bar{w}\rangle+\mathcal{O}(|y|^2|\xi|^{r-1})+\mathcal{O}(|w|_{a,\rho}^2|y||\xi|^{\bar{r}-1})+\mathcal{O}(|w|_{a,\rho}^3|\xi|^{\frac{2\bar{r}-1}{2}})+\mathcal{O}(|\xi|^{\bar{r}+2}).$$

Rescaling ξ by $\epsilon^6\xi$,$w,\bar{w}$ by $\epsilon^4w,\epsilon^4\bar{w}$, and y by ϵ^8y, one obtains a hamiltonian given by the rescaled hamiltonian

$$\begin{aligned}\tilde{H}(x,y,w,\bar{w},\xi)&=\epsilon^{-(6\bar{r}+8)}H(x,\epsilon^8y,\epsilon^4w,\epsilon^4\bar{w},\epsilon^6\xi,\epsilon)\\ &=\langle\tilde{\omega}(\xi),y\rangle+\langle\tilde{\Omega}(\xi)w,\bar{w}\rangle+\epsilon\tilde{P}(x,y,w,\bar{w},\xi,\epsilon),\end{aligned}$$

where $\tilde{\omega}(\xi)=\epsilon^{-6\bar{r}}\alpha'+A(\xi)$, $\tilde{\Omega}=\epsilon^{-6\bar{r}}\beta'+B(\xi)$, $\xi\in[1,2]^b$. For simplicity, we rewrite $\tilde{H}$ by H, $\tilde{\omega}$ by ω, $\tilde{\Omega}$ by Ω and $\tilde{P}$ by P.

In the following, we will use the KAM iteration which involves infinite many steps of coordinate transformations to prove the existence of the KAM tori. To make this quantitative we introduce the following notations and spaces.

Define

$$D(r,s)=\{(x,y,z,\bar{z}):|Imx|<s,|y|<r^2,||z||_{a,\rho}<r,||\bar{z}||_{a,\rho}<r\}$$

a complex neighborhood of $\mathbb{T}^b\times\{y=0\}\times\{z=0\}\times\{\bar{z}=0\}$, where $|\cdot|$ denotes the sup-norm for complex vectors. For a p ($p\geq 1$) order Whitney smooth function $F(\xi)$, define

$$||F||^*=\max\Big\{\sup_{\xi\in\Pi}|F|,\cdots,\sup_{\xi\in\Pi}|\frac{\partial^p F}{\partial\xi^p}|\Big\},$$
$$||F||_*=\max\Big\{\sup_{\xi\in\Pi}|\frac{\partial F}{\partial\xi}|,\cdots,\sup_{\xi\in\Pi}|\frac{\partial^p F}{\partial\xi^p}|\Big\}.$$

If $F(\xi)$ is a vector function from ξ to $\mathcal{H}^{a,\bar{\rho}}(R^n)$ which is p order whitney smooth on ξ, define $||F||^*_{a,\bar{\rho}}=||(||F_i(\xi)||^*)_i||_{a,\bar{\rho}}\big(||F||^*_{R^n}=\max_i(||F_i(\xi)||^*)\big)$. If $F(\eta,\xi)$ is a vector function from $D\times\Pi$ to $\mathcal{H}^{a,\bar{\rho}}$, define $||F||^*_{a,\bar{\rho},D}=\sup_{\eta\in D}||F||^*_{a,\bar{\rho}}$. We usually omit D for brevity. To functions F, associate a hamiltonian vector field defined as $X_F=\{F_y,-F_x,iF_{\bar{z}},-iF_z\}$. Denote the weighted norm for X_F by letting

$$|X_F|^*_{r,D(r,s)}=||F_y||^*+\frac{1}{r^2}||F_x||^*+\frac{1}{r}||F_z||^*_{a,\bar{\rho}}+\frac{1}{r}||F_{\bar{z}}||^*_{a,\bar{\rho}}.$$

6 An Infinite Dimensional KAM Theorem

Theorem 1 is a direct result of the following Theorem 3 and measure estimates in section 7. Consider small perturbations of an infinite dimensional hamiltonian in the parameter dependent normal form

$$N=\langle\omega(\xi),y\rangle+\langle\Omega(\xi)z,\bar{z}\rangle$$

on a phase space

$$\mathcal{P}^{a,\rho}=\mathbb{T}^n\times\mathbb{R}^n\times\mathcal{H}^{a,\rho}\times\mathcal{H}^{a,\rho}\ni(x,y,z,\bar{z}),$$

where

$$\omega_j=\frac{j^d+\cdots}{\epsilon^t}+\mathcal{O}(\xi^p)^{\dagger},\ \Omega_j=\frac{j^d+\cdots}{\epsilon^t}+\mathcal{O}(\xi^p),$$

$t,\ p\in\mathbb{N},\ \rho>0,\ a\geq 0$. Suppose that $||\omega||_*\leq M_1,\ ||\Omega_j||_*\leq M_2 j^{\delta},\ M_1+M_2\geq 1$. Define $M=(M_1+M_2)^p$. The parameter set Π is $[1,2]^n$.

For the hamiltonian $H=N+P$, there exists n-dimensional, linearly stable torus $\mathcal{T}_0^n=\mathbb{T}^n\times\{0,0,0\}$ with frequencies $\omega(\xi)$ when $P=0$. Our aim is to prove the persistence of a large portion of this family of linearly stable rotational tori under small perturbations. Suppose that the perturbation P is real analytic in the space variables, C^p in ξ, and for each $\xi\in\Pi$ its hamiltonian vector field $X_P=(P_y,-P_x,iP_{\bar{z}},-iP_z)^T$ defines near $\mathcal{T}_0^n$ a real analytic map $X_P:\mathcal{P}^{a,\rho}\to\mathcal{P}^{a,\bar{\rho}}(\bar{\rho}\geq\rho)$. Without losing generality, suppose $\rho-\bar{\rho}\leq\delta<d-1$. Under the previous assumptions, we have the following theorem.

$^{\dagger}\mathcal{O}(\xi^p)$ means pth order terms in $\xi_1,\cdots,\xi_b$

Theorem 3 *Suppose that $H = N + P$ satisfies*

$$|X_P|^*_{r,D(s,r)} \leq \gamma s^{2(1+\mu)}, \tag{6.1}$$

where γ depends on n, p, τ and M, $\mu = (p+1)\tau + p + \frac{n}{2}$. Then there exists a Cantor set $\Pi_\epsilon \subset \Pi$, a Whitney smooth family of torus embeddings $\Phi : \mathbb{T}^n \times \Pi_\epsilon \to \mathcal{P}^{a,\bar{\rho}}$, and a Whitney smooth map $\omega_* : \Pi_\epsilon \to \mathbb{R}^n$, such that for each $\xi \in \Pi_\epsilon$, the map Φ restricted to $\mathbb{T}^n \times \{\xi\}$ is a real analytic embedding of a rotational torus with frequencies $\omega_*(\xi)$ for the hamiltonian H at ξ.

Each embedding is real analytic on $|\mathrm{Im}x| < \frac{s}{2}$, and

$$||\Phi - \Phi_0||^*_r \leq c\epsilon^{\frac{1}{2}},$$
$$||\omega_* - \omega||^* \leq c\epsilon,$$

uniformly on that domain and Π_ϵ, where Φ_0 is the trivial embedding $\mathbb{T}^n \times \Pi \to \mathcal{T}^n_0$. Moreover, there exist whitney smooth maps ω_m and Ω_m on Π for $m \geq 1$ satisfying $\omega_1 = \omega, \Omega_1 = \Omega$ and

$$||\omega_m - \omega||^* \leq c\epsilon, \tag{6.2}$$

$$||\Omega_m - \Omega||^*_{-\delta} \leq c\epsilon. \tag{6.3}$$

Remark 6.1 *Note that in the theorem, we didn't claim that the measure of Π_ϵ is positive. For positive measure, one needs further information of the frequencies $\omega(\xi)$ and $\Omega(\xi)$. We shall come back to this point in Section 7.*

Since the proof of Theorem 2 is essentially standard, we only state the main step of KAM iteration. The more detailed steps can be found in [25] and other papers.

6.1 Solving the Linearized Equations and KAM Step

At each step of KAM iteration, the symplectic coordinate change Φ is obtained as the the time 1-map $X^t_F|_{t=1}$ of the flow of hamiltonian vector field X_F. Its generating function F and some normal correction $\widehat{N}$ to the given normal form N are solutions of the linear equation

$$\{F, N\} + \widehat{N} = R, \tag{6.4}$$

where $R = \sum_{2m+|q+\bar{q}|\leq 2} R_{kmq\bar{q}} y^m z^q \bar{z}^{\bar{q}} e^{i\langle k,x\rangle}, R_{kmq\bar{q}} = P_{kmq\bar{q}}$, and the coefficients $R_{kmq\bar{q}}$ depend on ξ such that $X_R : P^{a,\rho} \to P^{a,\bar{\rho}}$ is real analytic and whitney smooth in ξ. Below we solve the linear equation and estimate the generating function F.

Lemma 6.1 *Suppose that uniformly on $\Pi_+ \subset \Pi$,*

$$|\langle k, \omega\rangle| \geq \frac{\epsilon^\beta}{A_k} \text{ for } k \neq 0, \tag{6.5}$$

$$|\langle k, \omega\rangle + \Omega_i| \geq \frac{\epsilon^\beta}{A_k}, \tag{6.6}$$

$$|\langle k, \omega\rangle + \Omega_i + \Omega_j| \geq \frac{\epsilon^\beta(|i-j|+1)}{A_k}, \tag{6.7}$$

$$|\langle k, \omega\rangle + \Omega_i - \Omega_j| \geq \frac{\epsilon^\beta(|i-j|+1)}{A_k}, \ i \neq j, \tag{6.8}$$

Then the linear equation has solution F and $\widehat{N}$, which satisfy $[F]=0$, $[\widehat{N}]=\widehat{N}$. Moreover,

$$|X_{\widehat{N}}|^*_{r,D(s,r)} \leq |X_R|^*_{r,D(s,r)}, |X_F|^*_{r,D(s-\sigma,r)} \leq \frac{cM}{\epsilon^{(p+1)\beta}\sigma^{\mu}}|X_R|^*_{r,D(s,r)}, \tag{6.9}$$

where $A_k = 1+|k|^{\tau}$, β will be denoted later.

For the proof refer to [25].

Lemma 6.2 *If $|X_F|^*_{r,D(s-\sigma,r)} \leq \sigma$, then for any $\xi \in \Pi_+$, the flow $X_F^t(\cdot\ ,\xi)$ exists on $D(s-2\sigma,\frac{r}{2})$ for $|t|\leq 1$ and maps $D(s-2\sigma,\frac{r}{2})$ into $D(s-\sigma,r)$. Moreover, for $|t|\leq 1$,*

$$|X_F^t - id|^*_{r,D(s-2\sigma,\frac{r}{2})}, \sigma||DX_F^t - Id||^*_{r,r,D(s-3\sigma,\frac{r}{4})} \leq c|X_F|^*_{r,D(s-\sigma,r)},$$

where D is the differentiation operator with respect to $(x,y,z,\bar{z})$, id and Id are identity mapping and unit matrix, and the operator norm

$$||A(\xi,\eta)||_{\bar{r},r,D(s,r)} = \sup_{\eta\in D(s,r)}\sup_{w\neq 0}\frac{||A(\xi,\eta)w||_{a,\bar{r}}}{||w||_{a,r}},$$

$$||A||^*_{r,r} = \max\{||A||_{r,r},\cdots,||\frac{\partial^p A}{\partial \xi^p}||_{r,r}\}.$$

For the proof see [26].

Below we consider the new perturbation under the sympletic transformation $\Phi = X_F^t|_{t=1}$. Let $|X_P|^*_{r,D(s,r)} \leq \epsilon$. From the above we have

$$R = \sum_{2|m|+|q+\bar{q}|\leq 2} R_{kmq\bar{q}}y^m z^q \bar{z}^{\bar{q}} e^{i\langle k,x\rangle}.$$

Thus $|X_R|^*_{r,D(s,r)} \leq \cdot |X_P|^*_{r,D(s,r)} \leq \cdot\epsilon$, and for $\eta \leq \frac{1}{8}$,

$$|X_{P-R}|^*_{\eta r,D(s,4\eta r)} \leq \cdot \eta |X_P|^*_{r,D(s,r)} \leq \cdot\eta\epsilon. \tag{6.10}$$

Since $\widehat{N} = \sum_{2|m|+|q+\bar{q}|\leq 2, q=\bar{q}} P_{0mq\bar{q}}y^m z^q \bar{z}^{\bar{q}} e^{i\langle k,x\rangle}$, the new normal form is

$$N_+ = N + \widehat{N} = \langle \omega_+, y\rangle + \langle \Omega_+ z, \bar{z}\rangle.$$

By Lemma 6.1, one has $|X_{\widehat{N}}|^*_{r,D(s,r)} \leq \cdot\epsilon$. Note that $\rho - \bar{\rho} \leq \delta$, therefore,

$$||\omega_+ - \omega||^*, ||\Omega_+ - \Omega||^*_{-\delta} \leq \cdot\epsilon, \tag{6.11}$$

where $||\Omega||^*_{-\delta} = \max_{j\geq 1}||\Omega_j||^* j^{-\delta}$. If $\frac{cM\epsilon^{1-\beta(p+1)}}{\sigma^{\mu+1}} \leq 1$, by Lemma 6.1 and Lemma 6.2, it follows that for $|t|\leq 1$,

$$\frac{1}{\sigma}|X_F^t - id|^*_{r,D(s-2\sigma,\frac{r}{2})}, ||DX_F^t - Id||^*_{r,r,D(s-3\sigma,\frac{r}{4})} \leq \frac{cM\epsilon^{1-(p+1)\beta}}{\sigma^{\mu+1}}. \tag{6.12}$$

Under the transformation $\Phi = X_F^1$, $(N+R)\circ\Phi = N_+ + R_+$, where $R_+ = \int_0^1 \left\{(1-t)\widehat{N} + tR, F\right\}\circ X_F^t$. Thus, $H\circ\Phi = N_+ + R_+ + (P-R)\circ\Phi = N_+ + P_+$, where the new perturbation

$$P_+ = R_+ + (P-R)\circ\Phi = (P-R)\circ\Phi + \int_0^1 \{\bar{R}(t), F\}\circ X_F^t dt$$

where $\bar{R}(t) = (1-t)\widehat{N} + tR$. Hence, the hamiltonian vector field of the new perturbation is $X_{P_+} = (X_F^1)^*(X_{P-R}) + \int_0^1 (X_F^t)^*[X_{\bar{R}(t)}, X_F]dt$. For the estimate of X_{P_+}, we need the following lemma.

Lemma 6.3 *If the hamiltonian vector field $W(\cdot,\xi)$ on $V = D(s-4\sigma, 2\eta r)$ depends on the parameter $\xi \in \Pi_+$ with $||W||_{r,V}^* < +\infty$, and $\Phi = X_F^t : U = D(s-5\sigma, \eta r) \to V$, then $\Phi^* W = D\Phi^{-1} W\circ\Phi$ and if $\frac{cM\epsilon^{1-(p+1)\beta}}{\eta^2\sigma^{\mu+1}} \le 1$, we have $||\Phi^* W||_{\eta r, U}^* \le c||W||_{\eta r, V}^*$.*

For the proof refer to [25].

Now we estimate X_{P_+}. By Lemma 6.3, if $\frac{cM\epsilon^{1-(p+1)\beta}}{\eta^2\sigma^{\mu+1}} \le 1$,

$$|X_{P_+}|_{\eta r, D(s-5\sigma,\eta r)}^* \le c|X_{P-R}|_{\eta r, D(s-4\sigma, 2\eta r)}^* + c\int_0^1 |[X_{\bar{R}(t)}, X_F]|_{\eta r, D(s-4\sigma, 2\eta r)}^* dt.$$

By Cauchy's inequality and Lemma 6.2, one obtains

$$\begin{aligned} |[X_{\bar{R}(t)}, X_F]|_{\eta r, D(s-4\sigma, 2\eta r)}^* &\le \frac{cM\epsilon^{2-(p+1)\beta}}{\eta^2\sigma^{\mu+1}} \\ &= cM\eta\epsilon, \end{aligned}$$

where one chooses $\eta^3 = \frac{\epsilon^{1-(p+1)\beta}}{\sigma^{\mu+1}}$. Combining (6.10) we have

$$|X_{P_+}|_{\eta r, D(s-5\sigma,\eta r)}^* \le cM\eta\epsilon.$$

6.2 Iteration and Proof of Theorem 2

To iterate the KAM step infinitely we must choose suitable sequences. For $m \ge 1$ set

$$\epsilon_{m+1} = \frac{cM(m)\epsilon_m^{\frac{4}{3}-\frac{1}{3}(p+1)\beta}}{\sigma_m^{\frac{1}{3}(1+\mu)}},\ \sigma_{m+1} = \frac{\sigma_m}{2},\ \eta_m^3 = \frac{\epsilon_m^{1-(p+1)\beta}}{\sigma_m},$$

where $\beta = \frac{1}{2(p+1)}$. Furthermore, $s_{m+1} = s_m - 5\sigma_m$, $r_{m+1} = \eta_m r_m$, $M(m) = (M_1 + M_2 + 2c(\epsilon_1 + \cdots + \epsilon_{m-1}))^p$, and $D_m = D(s_m, r_m)$. As initial value fix $\sigma_1 = \frac{s_1}{20} \le \frac{1}{2}$. Choose $\epsilon_1 \le \gamma_0\sigma_1^{2(1+\mu)}$, where $\gamma_0 \le \gamma_*$, γ_* is a constant depending on M, p, n, τ. Finally, let $K_m = K_1 2^{m-1}$ with $K_1 = \ln\frac{1}{\epsilon_1}$, $K_0 = 0$.

Lemma 6.4 *Suppose $H_m = N_m + P_m$ is given on $D_m \times \Pi_m$, where $N_m = \langle \omega_m(\xi), y\rangle + \langle \Omega_m, z\bar{z}\rangle$ is a normal form satisfying*

$$|\langle k, \omega_m\rangle| \geq \frac{\epsilon_m^\beta}{A_k} \text{ for } k \neq 0, \tag{6.13}$$

$$|\langle k, \omega_m\rangle + \Omega_{m,i}| \geq \frac{\epsilon_m^\beta}{A_k}, \tag{6.14}$$

$$|\langle k, \omega_m\rangle + \Omega_{m,i} + \Omega_{m,j}| \geq \frac{\epsilon_m^\beta(|i-j|+1)}{A_k}, \tag{6.15}$$

$$|\langle k, \omega_m\rangle + \Omega_{m,i} - \Omega_{m,j}| \geq \frac{\epsilon_m^\beta(|i-j|+1)}{A_k}, i \neq j, \tag{6.16}$$

for any $\xi \in \Pi_m$. and

$$|X_{P_m}|^*_{r_m, D_m} \leq \epsilon_m.$$

Then there exists a Whitney smooth family of real analytic symplectic coordinate transformations $\Phi_{m+1} : D_{m+1} \times \Pi_m \to D_m$ and a closed subset

$$\Pi_{m+1} = \Pi_m \setminus \bigcup_{|k| > K_m} R_{kl}^{m+1}(\epsilon_{m+1})$$

of Π_m, where

$$R_{kl}^{m+1}(\epsilon_{m+1}) = A_{k1}^{m+1} \cup A_{k2}^{m+1} \cup A_{k3}^{m+1} \cup A_{k4}^{m+1},$$

and

$$A_{k1}^{m+1} = \{\xi \in \Pi_m : |\langle k, \omega_{m+1}\rangle| < \frac{\epsilon_{m+1}^\beta}{A_k}\},$$

$$A_{k2}^{m+1} = \bigcup_i B_{ki}^{m+1,1} = \bigcup_i \{\xi \in \Pi_m : |\langle k, \omega_{m+1}\rangle + \Omega_{m+1,i}| < \frac{\epsilon_{m+1}^\beta}{A_k}\},$$

$$A_{k3}^{m+1} = \bigcup_{i,j} B_{kij}^{m+1,11} = \bigcup_{i,j} \{\xi \in \Pi_m : |\langle k, \omega_{m+1}\rangle + \Omega_{m+1,i} + \Omega_{m+1,j}| < \frac{\epsilon_{m+1}^\beta(|i-j|+1)}{A_k}\},$$

$$A_{k4}^{m+1} = \bigcup_{i \neq j} B_{kij}^{m+1,12} = \bigcup_{i \neq j} \{\xi \in \Pi_m : |\langle k, \omega_{m+1}\rangle + \Omega_{m+1,i} - \Omega_{m+1,j}| < \frac{\epsilon_{m+1}^\beta(|i-j|+1)}{A_k}\},$$

such that for $H_{m+1} = H_m \circ \Phi_{m+1} = N_{m+1} + P_{m+1}$ the same assumptions are satisfied with $m+1$ in place of m.

Proof: Note the value for p_1, ϵ_1, β and σ_1, one verifies that

$$\frac{M_{m+1}\epsilon_{m+1}^{1-(p+1)\beta}}{\sigma_{m+1}^{1+\mu}} \leq \frac{1}{2}\frac{M_m \epsilon_m^{1-(p+1)\beta}}{\sigma_m^{1+\mu}} \tag{6.17}$$

for all $m \geq 1$. So the smallness condition of the KAM step is satisfied. For the remained proof, see Iterative Lemma in [25].

With (6.11) and (6.12), we also obtain the following estimate.

Lemma 6.5 *For $m \geq 1$,*

$$\frac{1}{\sigma_m}||\Phi_{m+1}-id||^*_{r_m,D_{m+1}}, ||D\Phi_{m+1}-I||^*_{r_m,r_m,D_{m+1}} \leq \frac{cM(m)\epsilon_m^{1-(p+1)\beta}}{\sigma_m^{\mu+1}} \tag{6.18}$$

$$||\omega_{m+1}-\omega_m||^*_{\Pi_m}, ||\Omega_{m+1}-\Omega_m||^*_{-\delta,\Pi_m} \leq c\epsilon_m. \tag{6.19}$$

Proof of Theorem 3. The smallness condition is

$$\epsilon_1 \leq \frac{\gamma_0}{20^{2(1+\mu)}} s_1^{2(1+\mu)}. \tag{6.20}$$

To apply Lemma 6.4 with $m=1$, set $s_1 = s, r_1 = r, \cdots, N_1 = N, P_1 = P$,

$$\gamma = \frac{\gamma_0}{20^{2(1+\mu)}} \text{ and } \epsilon_1 = \gamma s_1^{2(1+\mu)}.$$

The smallness condition is satisfied, because

$$|X_{P_1}|^*_{r_1,D(s_1,r_1)} = |X_P|^*_{r,D(r,s)} \leq \gamma s^{2(1+\mu)} = \epsilon_1.$$

The small divisor conditions are satisfied by setting $\Pi_1 = \Pi \setminus \bigcup_{kl} R^1_{kl}(\epsilon)$, where $k \neq 0$ for A^1_{k1}, and $\Pi_0 = \Pi$. Then the Iterative Lemma applies. ∎

Remark 6.2 *For the rescaled hamiltonian H, we fix $r=1$. Then*

$$|X_{\epsilon P}|^*_{1,D(s,1)} \leq |X_{\epsilon P}|^*_{1,D(1,1)} \leq c\epsilon \leq \gamma s^{2(1+\mu)},$$

for ϵ small enough. If fix $\rho > 0$ and $a > 0$ arbitrarily, Theorem 3 can be applied to the rescaled hamiltonian.

7 Measure Estimates

The whole measure estimates for all the steps are given by the following Lemmata. From section 5, we have $p = \bar{r}$, $\bar{\rho} = \rho + 1$, $\beta = \frac{1}{2(\bar{r}+1)}$, $t = 6\bar{r}$. Note (4.8) and (5.11), we have $\delta = -1$. For our conveniences, we will extend ω_ν and Ω_ν defined in Π_ν to Π.

Before giving all the measure estimates, we outline the main idea of this section in the following. The main trouble lies in estimating the measure of the set $\bigcup_{|k|>K_{\nu-1}} \bigcup_{i\neq j} B^{\nu,12}_{kij}$. In fact, we have

$$\begin{aligned} |\bigcup_{|k|>K_{\nu-1}} \bigcup_{i\neq j} B^{\nu,12}_{kij}| &\leq \cdot |\bigcup_{|k|>K_{\nu-1}} \bigcup_{0<p\leq c|k|} \bigcup_{j} B^{\nu,12}_{k(j+p)j}| \\ &\leq \cdot \sum_{|k|>K_{\nu-1}} \sum_{0<p\leq c|k|} \sum_{0<j<j_0} |B^{\nu,12}_{k(j+p)j}| + \cdot \sum_{|k|>K_{\nu-1}} \sum_{0<p\leq c|k|} |Q^\nu_{kpj_0}| \\ &\leq \cdot \sum_{|k|>K_{\nu-1}} \frac{j_0 \epsilon_\nu^{\frac{\beta}{\bar{r}}}}{|k|^{\frac{\tau-\bar{r}-1}{\bar{r}}}} + \cdot \sum_{|k|>K_{\nu-1}} \left(\frac{\epsilon_\nu^\beta}{|k|^{\tau-1}} + \frac{1}{j_0} + \frac{|k|}{j_0^2 \epsilon^t}\right)^{\frac{1}{\bar{r}}} |k|. \end{aligned} \tag{7.1}$$

The trouble lies in $\frac{1}{\epsilon^t}$, where $\epsilon = \cdot \epsilon_0$. It's hard for us to choose a suitable j_0. Our method is to pick up a suitable t_0, which satisfies

$$\frac{1}{\epsilon^t} < |k|^{t_0}. \tag{7.2}$$

It is obvious that (7.2) holds when $\nu > N_* = \frac{t}{t_0 \ln 2} \ln \frac{1}{\epsilon} + 2$. Now we can choose a suitable j_0 at the cost of choosing a larger τ when $\nu > N_*$. In order to get the measure estimates for $\nu > N_*$, we further require

$$\frac{\tau}{2\bar{r}} \geq 1 + t_0. \tag{7.3}$$

At this time, we can't fix the value for t_0. In fact, we have to combine the constraint condition for t_0 from estimating the measure for $1 < \nu \leq N_*$. To obtain the measure estimates of these parts, we also require $m \in J$ and

$$t_0 > \frac{\tau}{4\bar{r}} + 1. \tag{7.4}$$

Now we fix $t_0 = \frac{\tau}{2\bar{r}} - 1$ which satisfies (7.3) and (7.4) clearly when $\tau > 8\bar{r}$. We remark that the choose of t_0 is not unique. Different choices for t_0 will correspond to different range for τ. The remained is the measure estimates for the first step. In order to get them, we also require $m \in J$.

We turn to the concrete measure estimates. A couple of Lemmata are needed. The following Lemma has been used many times in this section. For the proof see [27].

Lemma 7.1 *Suppose that $g(x)$ is an mth differentiable function on the closure $\bar{I}$ of I, where $I \subset \mathbb{R}$ is an interval. Let $I_h = \{x | g(x) < h\}$, $h > 0$. If for some constant $d > 0$, $|g^m(x)| \geq d$ for any $x \in I$, then $|I_h| \leq c h^{\frac{1}{m}}$, where I_h denotes the Lebesgue measure of I_h and $c = 2(2 + 3 + \cdots + m + d^{-1})$.*

Lemma 7.2 *For $|k| > c_* > 0$,*

$$|B^{\nu,12}_{kij}| \leq \cdot [\frac{(|i-j|+1)\epsilon_\nu^\beta}{A_k}]^{\frac{1}{\bar{r}}}.$$

Proof: For our conveniences, we write ω' and Ω' for ω_ν and Ω_ν. Define $v_1 = (1, 0, \cdots, 0)^T$, and $v_b = (0, 0, \cdots, 1)^T$. Define $S^{b-1} = \{(x_1, x_2, \cdots, x_b) \in \mathbb{R}^b : |x_1| + |x_2| + \cdots + |x_b| = 1\}$. Write $A(m) = (D^{\bar{r}}_{v_1}\omega, D^{\bar{r}}_{v_2}\omega, \cdots, D^{\bar{r}}_{v_b}\omega)^T$. It is easy to check that $|A(m)| = \frac{(-1)^{b-1} C}{(\lambda_{n_1}\lambda_{n_2}\cdots\lambda_{n_b})^{\bar{r}+1}} \neq 0$ for any $\xi \in \Pi$, where C is a positive constant depending on $\bar{r}$, b. For any $(\xi, v) \in \Pi \times S^{b-1}$,

$$|A(m)v|_1 \geq c_1 > 0. \tag{7.5}$$

Thus for any $(\xi, v) \in \Pi \times S^{b-1}$, there exists an open neighborhood S_v of v in S^{b-1}, such that for some i, $|\langle D^{\bar{r}}_{v_i}\omega, v' \rangle| \geq \frac{c_1}{2b}$, for any $(\xi, v') \in \Pi \times S_v$. Since $\{\Pi \times S_v\}$ covers the compact set $\Pi \times S^{b-1}$, there exist finite covers: $\Pi \times S_1, \cdots, \Pi \times S_{k_0}$ such that $\bigcup_{i=1}^{k_0} \Pi \times S_i \supset \Pi \times S^{b-1}$ and for any $(\xi, v) \in \Pi \times S_i$,

$$|\langle D^{\bar{r}}_{\bar{v}}\omega, v \rangle| \geq \frac{c_1}{2b},$$

where $\bar{v} \in \{v_1, v_2, \cdots, v_b\}$.

Now fix $k \neq 0$ and suppose $\frac{k}{|k|} \in S_i$. Then for any $\xi \in \Pi$,

$$|\langle D^{\bar{r}}_{\bar{v}}\omega, \frac{k}{|k|} \rangle| \geq \frac{c_1}{2b} > 0. \tag{7.6}$$

Define $f(\xi)=\langle k,\omega'\rangle+\Omega_i'-\Omega_j'$. Note

$$D_{\bar{v}}^{\bar{r}}\frac{f(\xi)}{|k|}=\langle\frac{k}{|k|},D_{\bar{v}}^{\bar{r}}(\omega)\rangle+\frac{D_{\bar{v}}^{\bar{r}}(\Omega_i-\Omega_j)}{|k|}+\frac{D_{\bar{v}}^{\bar{r}}(\Omega_i'-\Omega_i)}{|k|}+\frac{D_{\bar{v}}^{\bar{r}}(\Omega_j-\Omega_j')}{|k|}+\langle\frac{k}{|k|},D_{\bar{v}}^{\bar{r}}(\omega'-\omega)\rangle. \tag{7.7}$$

We estimate every term in (7.7). From (6.2) and (6.3), one obtains

$$|\langle\frac{k}{|k|},D_{\bar{v}}^{\bar{r}}(\omega'-\omega)\rangle|\le|D_{\bar{v}}^{\bar{r}}(\omega'-\omega)|\le c\epsilon, \tag{7.8}$$

$$\frac{|D_{\bar{v}}^{\bar{r}}(\Omega_i'-\Omega_i)|}{|k|}\le\frac{c\epsilon}{|k|}\le\frac{1}{|k|}, \tag{7.9}$$

$$\frac{|D_{\bar{v}}^{\bar{r}}(\Omega_j-\Omega_j')|}{|k|}\le\frac{c\epsilon}{|k|}\le\frac{1}{|k|}. \tag{7.10}$$

Note

$$\frac{|D_{\bar{v}}^{\bar{r}}(\Omega_i-\Omega_j)|}{|k|}\le\frac{c}{|k|} \tag{7.11}$$

and (7.8), (7.9), (7.10), (7.6), we arrive at $|D_{\bar{v}}^{\bar{r}}\frac{f(\xi)}{|k|}|\ge\frac{c_1}{4b}$ when $|k|\ge c_*>0$. It is obvious that $|D_{\bar{v}}^{\bar{r}}f(\xi)|\ge\frac{c_1}{4b}|k|>c>0$, when $|k|\ge c_*>0$. The result now follows with Lemma 7.1.

Remark 7.1 *If define $F(m,x)=|A(m)v|_1$, $(m,v)\in[0,M_*]\times S^{b-1}$, it is easy to check that F is continuous on $[0,M_*]\times S^{b-1}$. Therefore c_1 in (7.5) depends on $\bar{r}$, b, M_*. Further, we also know c_* depends on $\bar{r}$, b, M_*.*

In the following, we choose $t_0=\frac{\tau}{2\bar{r}}-1$, $N_*=\frac{t}{t_0\ln 2}\ln\frac{1}{\epsilon}+2$.

Lemma 7.3 *For $\tau>2(b+1)\bar{r}^2+2$, $\nu>N_*$*

$$|\bigcup_{|k|>K_{\nu-1}}\bigcup_{i\neq j}B_{kij}^{\nu,12}|\le\cdot\epsilon_\nu^{\frac{\beta}{2\bar{r}^2}}.$$

Proof: Write $p=i-j$. We firstly prove

$$|\bigcup_{|k|>K_{\nu-1}}\bigcup_{p\ge c|k|}\bigcup_{j}B_{k(j+p)j}^{\nu,12}|=\emptyset. \tag{7.12}$$

In fact, we have

$$\begin{aligned}\frac{|\langle k,\omega_\nu\rangle+\Omega_{\nu,i}-\Omega_{\nu,j}|}{p+1}&\ge\frac{|\langle k,\omega\rangle+\Omega_i-\Omega_j|}{p+1}-\frac{c|k|}{p+1}\\&\ge\frac{|p+\mathcal{O}(\frac{1}{j})+\langle k,\lambda\rangle|}{(p+1)\epsilon^t}-\frac{c|k|}{p+1}\\&\ge\cdot\frac{1}{\epsilon^t}>>1\end{aligned}$$

From above, it is clear that (7.12) holds. From (7.12), one has

$$\begin{aligned}|\bigcup_{|k|>K_{\nu-1}}\bigcup_{i\neq j}B^{\nu,12}_{kij}|&=\cdot|\bigcup_{|k|>K_{\nu-1}}\bigcup_{i>j}B^{\nu,12}_{kij}|\\&=\cdot|\bigcup_{|k|>K_{\nu-1}}\bigcup_{j,p}B^{\nu,12}_{k(j+p)j}|\\&\leq\cdot|\bigcup_{|k|>K_{\nu-1}}\bigcup_{0<p\leq c|k|}\bigcup_{j}B^{\nu,12}_{k(j+p)j}|\end{aligned}\tag{7.13}$$

In order to get the estimate for (7.13), noticing the following inequality

$$\begin{aligned}|\langle k,\omega_\nu\rangle+\frac{p}{\epsilon^t}|&\leq|\langle k,\omega_\nu\rangle+\Omega_{\nu,i}-\Omega_{\nu,j}|+|\Omega_{\nu,i}-\Omega_i|\\&\quad+|\Omega_{\nu,j}-\Omega_j|+|\Omega_i-\Omega_j-\frac{p}{\epsilon^t}|\\&\leq\frac{c\epsilon_\nu^\beta p}{A_k}+\frac{c}{j}+\frac{cp}{j^2\epsilon^t},\end{aligned}$$

we define

$$Q^\nu_{kpj}=\{\xi\in\Pi:|\langle k,\omega_\nu\rangle+\frac{p}{\epsilon^t}|\leq\frac{c\epsilon_\nu^\beta p}{A_k}+\frac{c}{j}+\frac{cp}{j^2\epsilon^t}\}.$$

It is obvious that $B^{\nu,12}_{k(j+p)j}\subseteq Q^\nu_{kpj}(p>0)$ and $Q^\nu_{kpj}\subseteq Q^\nu_{kpj_0}$ for $j\geq j_0$. Therefore

$$\begin{aligned}(7.13)&\leq|\bigcup_{|k|>K_{\nu-1}}\bigcup_{0<p\leq c|k|}[(\bigcup_{0<j<j_0}B^{\nu,12}_{k(j+p)j})\bigcup Q^\nu_{kpj_0}]|\\&\leq\sum_{|k|>K_{\nu-1}}\sum_{0<p\leq c|k|}\sum_{0<j<j_0}|B^{\nu,12}_{k(j+p)j}|+\sum_{|k|>K_{\nu-1}}\sum_{0<p\leq c|k|}|Q^\nu_{kpj_0}|\\&\leq\cdot\sum_{|k|>K_{\nu-1}}\frac{j_0\epsilon_\nu^{\frac{\beta}{\bar{r}}}}{|k|^{\frac{\tau-\bar{r}-1}{\bar{r}}}}+\cdot\sum_{|k|>K_{\nu-1}}(\frac{\epsilon_\nu^\beta}{|k|^{\tau-1}}+\frac{1}{j_0}+\frac{|k|}{j_0^2\epsilon^t})^{\frac{1}{\bar{r}}}|k|.\end{aligned}\tag{7.14}$$

Note $\nu>N_*$, it is easy to check that

$$\frac{1}{\epsilon^t}<|k|^{t_0}.$$

Therefore

$$(7.14)\leq\cdot\sum_{|k|>K_{\nu-1}}\frac{j_0\epsilon_\nu^{\frac{\beta}{\bar{r}}}}{|k|^{\frac{\tau-\bar{r}-1}{\bar{r}}}}+\cdot\sum_{|k|>K_{\nu-1}}(\frac{\epsilon_\nu^\beta}{|k|^{\tau-1}}+\frac{1}{j_0}+\frac{|k|^{1+t_0}}{j_0^2})^{\frac{1}{\bar{r}}}|k|\tag{7.15}$$

Choosing $j_0 = \frac{|k|^{\frac{\tau}{2\bar{r}}}}{\epsilon_\nu^{\frac{\beta}{2\bar{r}}}}$, note $\tau > 2(b+1)\bar{r}^2 + 2$, then

$$\begin{aligned}(7.15) &\leq \cdot \sum_{|k|>K_{\nu-1}} \frac{\epsilon_\nu^{\frac{\beta}{2\bar{r}}}}{|k|^{\frac{\tau-2}{2\bar{r}}-1}} + \cdot \sum_{|k|>K_{\nu-1}} \Big(\frac{\epsilon_\nu^{\beta}}{|k|^{\tau-1}} + \frac{\epsilon_\nu^{\frac{\beta}{2\bar{r}}}}{|k|^{\frac{\tau}{2\bar{r}}}} + \frac{\epsilon_\nu^{\frac{\beta}{\bar{r}}}}{|k|^{\frac{\tau}{2\bar{r}}}}\Big)^{\frac{1}{\bar{r}}} |k| \\ &\leq \cdot \sum_{|k|>K_{\nu-1}} \frac{\epsilon_\nu^{\frac{\beta}{2\bar{r}}}}{|k|^{\frac{\tau-2}{2\bar{r}}-1}} + \cdot \sum_{|k|>K_{\nu-1}} \frac{\epsilon_\nu^{\frac{\beta}{2\bar{r}^2}}}{|k|^{\frac{\tau}{2\bar{r}^2}-1}} \\ &\leq \cdot \epsilon_\nu^{\frac{\beta}{2\bar{r}^2}}.\end{aligned} \tag{7.16}$$

Lemma 7.4 *For* $1 \leq \nu \leq N_*,\ \tau > 2(b+1)\bar{r}^2 + 2$,

$$|\bigcup_{|k|>\epsilon^{-\frac{t}{t_0}}} \bigcup_{i\neq j} B_{kij}^{\nu,12}| \leq \cdot \epsilon_\nu^{\frac{\beta}{2\bar{r}^2}}.$$

The proof is similar to Lemma 7.3, we omit it.

Lemma 7.5 *For* $2 \leq \nu \leq N_*,\ m \in J,\ \tau > \max\{8\bar{r},\ \frac{4}{3}[\bar{r}(b+1)+1]\}$,

$$|\bigcup_{\epsilon^{-\frac{t}{t_0}} \geq |k| > K_{\nu-1}} \bigcup_{i\neq j} B_{kij}^{\nu,12}| \leq \cdot \epsilon_\nu^{\frac{\beta}{\bar{r}}}.$$

Proof: We write $p = i - j$. One easily has

$$\begin{aligned}|\langle k, \omega_\nu\rangle + \Omega_{\nu,i} - \Omega_{\nu,j}| &\geq |\langle k, \omega_\nu\rangle + \Omega_i - \Omega_j| - |\Omega_{\nu,i} - \Omega_i| - |\Omega_{\nu,j} - \Omega_j| \\ &\geq \frac{|\langle k, \lambda\rangle + p| - \frac{c}{j}}{\epsilon^t} - c|k|.\end{aligned} \tag{7.17}$$

Note $m \in J$, we get

$$|\langle k, \lambda\rangle + p| \geq \frac{2\gamma^{*b}}{|k|^{\frac{\tau}{4\bar{r}}}},\ k \neq 0.$$

For $j > \frac{c|k|^{\frac{\tau}{4\bar{r}}}}{\gamma^{*b}}$, one then obtains

$$(7.17) \geq \frac{\gamma^{*b}}{\epsilon^t |k|^{\frac{\tau}{4\bar{r}}}} - c|k|. \tag{7.18}$$

In fact

$$\frac{\gamma^{*b}}{\epsilon^t |k|^{\frac{\tau}{4\bar{r}}}} - c|k| > 1. \tag{7.19}$$

Firstly, from $t > \frac{t}{t_0}(\frac{\tau}{4\bar{r}} + 1)$ and $|k| \leq \epsilon^{-\frac{t}{t_0}}$, we get

$$\epsilon^{-t} > \cdot \epsilon^{-\frac{t}{t_0}(\frac{\tau}{4\bar{r}}+1)} \geq \cdot |k|^{\frac{\tau}{4\bar{r}}+1}.$$

Now it is easy to get (7.19) when ϵ is small enough(also depending the constant γ^*). On the other hand, we have

$$\begin{aligned}\frac{\epsilon_\nu^\beta(|i-j|+1)}{A_k} &\leq\cdot\frac{\epsilon_\nu^\beta}{|k|^{\tau-1}}\\ &\leq\cdot\epsilon_\nu^\beta<<1,\end{aligned}$$

when $p=|i-j|\leq c|k|$. Hence, one gets

$$\begin{aligned}|\bigcup_{\epsilon^{-\frac{t}{t_0}}\geq|k|>K_{\nu-1}}\bigcup_{i\neq j}B_{kij}^{\nu,12}| &\leq\cdot|\bigcup_{\epsilon^{-\frac{t}{t_0}}\geq|k|>K_{\nu-1}}\bigcup_{0<p\leq c|k|}\bigcup_{j\leq\cdot\frac{|k|^{\frac{\tau}{4\bar{r}}}}{\gamma^{*b+2}}}B_{kij}^{\nu,12}|\\ &\leq\cdot\sum_{|k|>K_{\nu-1}}(\frac{\epsilon_\nu^\beta|k|}{A_k})^{\frac{1}{\bar{r}}}|k|^{1+\frac{\tau}{4\bar{r}}}\\ &\leq\cdot\epsilon_\nu^{\frac{\beta}{\bar{r}}}.\end{aligned}$$

From Lemma 7.4 and Lemma 7.5, we obtain the following Lemma.

Lemma 7.6 $2\leq\nu\leq N_*,\ m\in J,\ \tau>\tau_*=\max\{8\bar{r},\ \frac{4}{3}[\bar{r}(b+1)+1],\ 2(b+1)\bar{r}^2+2\}$,

$$|\bigcup_{|k|>K_{\nu-1}}\bigcup_{i\neq j}B_{kij}^{\nu,12}|\leq\cdot\epsilon_\nu^{\frac{\beta}{2\bar{r}^2}}.$$

The remained is the measure estimate for the first step. We will consider two cases to get it. From Lemma 7.4 and the same method from Lemma 7.5, we obtain the following Lemma.

Lemma 7.7 *For* $m\in J,\ \tau>\tau_*$,

$$|\bigcup_{|k|>c_*}\bigcup_{i\neq j}B_{kij}^{1,12}|\leq\cdot\epsilon_1^{\frac{\beta}{2\bar{r}^2}},$$

where τ_* *as above.*

Lemma 7.8 *For* $m\in J$, *then*

$$|\bigcup_{0\leq|k|\leq c_*}\bigcup_{i\neq j}B_{kij}^{1,12}|=\emptyset.$$

Proof:

$$\begin{aligned}|\bigcup_{0<|k|\leq c_*}\bigcup_{i\neq j}B_{kij}^{1,12}| &\leq\cdot|\bigcup_{0<|k|\leq c_*}\bigcup_{i>j}B_{kij}^{1,12}|\\ &\leq\cdot|\bigcup_{0<|k|\leq c_*}\bigcup_{0<p\leq c|k|}\bigcup_{j}B_{kij}^{1,12}|\end{aligned}\tag{7.20}$$

If $j > \frac{c|k|^{\frac{\tau}{4\bar{r}}}}{\gamma^{*b}}$, as Lemma 7.5, we obtain

$$|\bigcup_{0<|k|\leq c_*}\bigcup_{0<p\leq c|k|}\bigcup_{j>\frac{c|k|^{\frac{\tau}{4\bar{r}}}}{\gamma^{*b}}} B_{kij}^{1,12}| = \emptyset. \tag{7.21}$$

Note Remark 7.1, if $j \leq \frac{c|k|^{\frac{\tau}{4\bar{r}}}}{\gamma^{*b}} \leq \frac{C(M_*)}{\gamma^{*b}}$, one has $i = j+p \leq \frac{C(M_*)}{\gamma^{*b}}$. In this case we will prove that for $m \in J$,

$$|\langle k,\lambda\rangle + \lambda_i - \lambda_j| \geq c'' > 0. \tag{7.22}$$

Denote

$$Z_0^5 = \left\{ m \in I \middle| \begin{array}{l} l_1\lambda_{k_1} + \cdots + l_p\lambda_{k_p} + \lambda_i - \lambda_j = 0, |l_1|, \cdots, |l_p| \in \{1, \cdots, [C(M_*)]\}, \\ \{k_1, k_2, \cdots, k_p, i, j\} \subseteq \left\{1, \cdots, [\frac{C(M_*)}{\gamma^{*b}}]\right\}, 1 \leqslant p \leqslant b, p \in Z \end{array} \right\}.$$

From Lemma 8.5, we have $|Z_0^5| = 0$. Since $m \in J$, then $\langle k,\lambda\rangle + \lambda_i - \lambda_j \neq 0$. It is obvious that (7.22) holds. This follows that

$$|\bigcup_{0<|k|\leq c_*}\bigcup_{0<p\leq c|k|}\bigcup_{j\leq\frac{c|k|^{\frac{\tau}{4\bar{r}}}}{\gamma^{*b}}} B_{kij}^{1,12}| = \emptyset, \tag{7.23}$$

when ϵ is small enough. From the direct computation, one gets

$$|\bigcup_{k=0}\bigcup_{i\neq j} B_{kij}^{1,12}| = \emptyset. \tag{7.24}$$

Note (7.21), (7.23) and (7.24), we complete the proof.

From the above two Lemmata, we arrive at the following Lemma.

Lemma 7.9 *For $m \in J$ and $\tau > \tau_*$, then*

$$|\bigcup_{|k|\geq 0}\bigcup_{i\neq j} B_{kij}^{1,12}| \leq \cdot \epsilon_1^{\frac{\beta}{2\bar{r}^2}}.$$

Since the measure estimates for the remained are obvious, we only give the Lemmata and omit the proofs.

Lemma 7.10 *For $\nu > 1$ and $\tau > (b+2)\bar{r}+1$, then*

$$|\bigcup_{|k|>K_{\nu-1}} A_{k3}^{\nu}| = |\bigcup_{|k|>K_{\nu-1}}\bigcup_{i,j} B_{kij}^{\nu,11}| \leq \cdot \epsilon_\nu^{\frac{\beta}{\bar{r}}}.$$

Lemma 7.11 *For $\nu > 1$ and $\tau > (b+1)\bar{r}$, then*

$$|\bigcup_{|k|>K_{\nu-1}} A_{k2}^{\nu}| = |\bigcup_{|k|>K_{\nu-1}}\bigcup_{i} B_{ki}^{\nu,1}| \leq \cdot \epsilon_\nu^{\frac{\beta}{\bar{r}}}.$$

Lemma 7.12 *For $\nu > 1$ and $\tau > b\bar{r}$, then*

$$|\bigcup_{|k|>K_{\nu-1}} A^{\nu}_{k1}| \leq \cdot \epsilon_{\nu}^{\frac{\beta}{\bar{r}}}.$$

Lemma 7.13 *If $\tau > (b+2)\bar{r}+1$, then*

$$|\bigcup_{|k|\geq 0} A^1_{k3}| = |\bigcup_{|k|\geq 0} \bigcup_{i,j} B^{1,11}_{kij}| \leq \cdot \epsilon_1^{\frac{\beta}{\bar{r}}}.$$

Lemma 7.14 *If $\tau > (b+1)\bar{r}$, then*

$$|\bigcup_{|k|\geq 0} A^1_{k2}| = |\bigcup_{|k|\geq 0} \bigcup_{i} B^{1,1}_{ki}| \leq \cdot \epsilon_1^{\frac{\beta}{\bar{r}}}.$$

Lemma 7.15 *If $\tau > b\bar{r}$, then*

$$|\bigcup_{|k|\geq 0} A^1_{k1}| \leq \cdot \epsilon_1^{\frac{\beta}{\bar{r}}}.$$

Combining with all the Lemmata in this section, we easily get the total measure of the parameter sets Π_ϵ, which are thrown in all the steps. Before that, one has to fix some constants. We choose $\tau > \tau_{**} = \max\{\tau_*, 4\bar{r}b(b+1)\}$. Then, for $m \in J$,

$$|\Pi_\epsilon| \leq \cdot \epsilon_1^{\frac{\beta}{2\bar{r}^2}} \leq \cdot \epsilon^{\frac{\beta}{2\bar{r}^2}} = \cdot \epsilon^{\frac{1}{4\bar{r}^2(\bar{r}+1)}},$$

where $|J_*| \leq c\gamma^*$ and c depends on M_*, b, n_b and ϵ is small enough and depends on all the constants including m. If we choose a series of $\gamma^*_{(n)} = \frac{1}{c2^n}$, $n = 1, \cdots$, then one gets a series of $J_*^{(n)}$ correspondingly and $|J_*^{(n)}| \leq 2^{-n}$, $n = 1, 2, \cdots$. Write $J_n = I \setminus J_*^{(n)}$. Now, we choose $\widehat{J} = \bigcup\limits_n J_n$. It is obvious that $|I \setminus \widehat{J}| = 0$. Since M_* is arbitrary, Theorem 1 is proved for almost all $m > 0$. ∎

8 Appendix

The following two lemmata are born from Lemma 6.8 and Lemma 6.9 of Bambusi [2].

Lemma 8.1 *Let $m \in I = (0, M_*]$, if write*

$$\bar{A} = \begin{pmatrix} \frac{d\lambda_{n_1}}{dm} & \frac{d\lambda_{n_2}}{dm} & \cdots & \frac{d\lambda_{n_b}}{dm} \\ \frac{d^2\lambda_{n_1}}{dm^2} & \frac{d^2\lambda_{n_2}}{dm^2} & \cdots & \frac{d^2\lambda_{n_b}}{dm^2} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{d^b\lambda_{n_1}}{dm^b} & \frac{d^b\lambda_{n_2}}{dm^b} & \cdots & \frac{d^b\lambda_{n_b}}{dm^b} \end{pmatrix} \tag{8.1}$$

we have $|det(\bar{A})| \geq c > 0$, where c depends on b, M_*, n_b.

Lemma 8.2 *Let $u^{(1)}, \cdots, u^{(K)}$ be K independent vectors with $||u^{(i)}||_1 \le 1$. Let $\omega \in \mathbb{R}^K$ be an arbitrary vector, then there exists $i \in [1, \cdots, K]$ such that*

$$|u^{(i)} \cdot \omega| \ge \frac{||\omega||_1 det(u^{(i)})}{K^{\frac{3}{2}}}$$

where $det(u^{(i)})$ is the determinant of the matrix formed by the components of the vectors $u^{(i)}$.

From the above two Lemmata and the proof of Corollary 6.10 in Bambusi [2], we get the following Lemma.

Lemma 8.3 *Denote $f(m) = \langle k, \lambda\rangle (k \neq 0)$, then there exists $i_0 \in \{1, \cdots, b\}$, so that $|f^{(i_0)}(m)| \ge c_0 > 0$, where $\lambda = (\lambda_{n_1}, \cdots, \lambda_{n_b})$ and c_0 depends on b, n_b, M_*.*

From Lemma 8.3, one gets the following lemma easily.

Lemma 8.4 *For $\tau > 4\bar{r}b(b+1)$,*

$$|J_1| = |\bigcup_{k\neq 0, p\in\mathbb{Z}} \{m \in I : |\langle k, \lambda\rangle + p| < \frac{2\gamma^{*b}}{|k|^{\frac{\tau}{4\bar{r}}}}\}| \le c\gamma^*, \tag{8.2}$$

where $\lambda = (\lambda_{n_1}, \cdots, \lambda_{n_b})$ and c depends on b, n_b, M_*.

Lemma 8.5 *If $f(m) = k_1\lambda_{i_1} + \cdots + k_s\lambda_{i_s}$, where $0 \le |k_1| + \cdots + |k_s| \le ß$, $k_1, k_2, \cdots, k_s \in \mathbb{Z}$, $k_1k_2\cdots k_s \neq 0$ and $i_1, \cdots, i_s$ are different with each other and $i_l = \min\{i_1, \cdots, i_s\}$, we have*

$$\text{Number of } \{m \in I | f(m) = 0\} < \frac{3}{2} + \frac{\ln ß + 2}{\ln \frac{(i_l+1)^2 + M_*}{i_l^2 + M_*}}.$$

Proof: For our convenience, write $a_i = i^2 + m$. We have

$$f^{(n)}(m) = c_n(k_1 a_{i_1}^{\frac{1}{2}-n} + k_2 a_{i_2}^{\frac{1}{2}-n} + \cdots + k_s a_{i_s}^{\frac{1}{2}-n}), \ n \ge 2,$$

where $c_n = \frac{(-1)^{n+1}(2n-3)!}{2^{n-2}(n-2)!2^n}$. It is easy to check that

$$|k_l a_{i_l}^{\frac{1}{2}-n}| \ge |k_1 a_{i_1}^{\frac{1}{2}-n} + \cdots + k_{l-1} a_{i_{l-1}}^{\frac{1}{2}-n} + k_{l+1} a_{i_{l+1}}^{\frac{1}{2}-n} + \cdots + k_s a_{i_s}^{\frac{1}{2}-n}|$$

when $n > N_0 = \frac{\ln(ß)}{\ln \frac{(i_l+1)^2 + M_*}{i_l^2 + M_*}} + \frac{1}{2}$. Therefore, there exists positive integer n_0 such that $|f^{(n_0)}(m)| > 0$. For example, we choose $n_0 = [N_0] + 1$. This results in at most n_0 $m's$ such that $f(m) = 0$. The proof is similar with Lemma 2.1 in [27]. We omit it.

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