KAM tori for N–body problems
(a brief history)

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Abstract. We review analytical (rigorous) results about the existence of invariant tori for planetary many–body problems.

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1. Introduction

In this paper we review analytical results concerning the existence of KAM tori (smooth invariant tori, for a nearly–integrable Hamiltonian system, on which the flow is quasi–periodic with Diophantine frequencies) in the context of the planetary many body problem.

The main body of the paper is divided in two sections and two appendices.

In § 2 general existence theorems for the planetary $(1+n)$–body problem are discussed. In particular, after a brief reminder about the Hamiltonian setting for the many–body problem (§ 2.1) and about classical KAM theory (§ 2.2), it is shown how Kolmogorov's 1954 Theorem yields easily the existence of KAM tori in the special non–degenerate case of the restricted, planar, circular three–body problem (§ 2.2.2). Kolmogorov's Theorem, on the other hand, does not apply to the general case because of the proper degeneracy of the $(1+n)$–body problem, when $n \geq 2$. In this context, Arnold, in 1963, stated a general result, which he proved only in the planar three–body case; Arnold's theorem was proven in 2004 by Fejóz, who completed Herman's work on the matter (§ 2.3).

In § 3 rigorous "computer–assisted" results about the existence of KAM tori for Hamiltonian models of Solar subsystems are reviewed. In particular, in § 3.2, the KAM stability of the subsystem Sun–Jupiter–Victoria, modelled by a truncated restricted, planar, circular three–body problem, obtained recently by the authors, is discussed.

In Appendix A several sets of symplectic variables relevant for analytical investigations of the many–body problem are reviewed.

In Appendix B a numerical comparison between the dynamics of the truncated model considered in § 3.2 and the non truncated model is discussed.

2. KAM tori for general many–body problems

2.1. Hamiltonian models for planetary many–body problems

2.1.1. *Newton's equations*

Newton's equations for $n+1$ bodies (point masses), interacting only through gravitational attraction, are given by

$$\ddot{u}^{(i)} = \sum_{\substack{0\leq j\leq n \\ j\neq i}} m_j \frac{u^{(j)}-u^{(i)}}{|u^{(i)}-u^{(j)}|^3} \ , \qquad i=0,1,...,n \ , \tag{2.1}$$

where $u^{(i)} = (u_1^{(i)}, u_2^{(i)}, u_3^{(i)}) \in \mathbb{R}^3$ are cartesian coordinates of the $i^{\rm th}$ body of mass m_i, $|u| = \sqrt{u\cdot u} = \sqrt{\sum_i u_i^2}$ is the standard Euclidean norm, "dot" denotes time derivative, and the gravitational constant has been normalized to one (by rescaling time t). Equations (2.1) are invariant by change of inertial frames, i.e., by change of variables of the form $u^{(i)} \to u^{(i)} - (a+ct)$ with fixed $a, c \in \mathbb{R}^3$. This allows to restrict the attention to the manifold of initial data given by[1]

$$\sum_{i=0}^n m_i u^{(i)}(0) = 0 \ , \qquad \sum_{i=0}^n m_i \dot{u}^{(i)}(0) = 0 \ . \tag{2.2}$$

The total linear momentum $\mathrm{M}_{\rm tot} := \sum_{i=0}^n m_i \dot{u}^{(i)}$ does not change along the flow of (2.1), i.e., $\dot{\mathrm{M}}_{\rm tot} = 0$ along trajectories; therefore, by (2.2), $\mathrm{M}_{\rm tot}(t)$ vanishes for all t. But, then, also the position of the barycenter $B(t) := \sum_{i=0}^n m_i u^{(i)}(t)$ is constant ($\dot{B} = 0$) and, again by (2.2), $B(t) \equiv 0$. In other words, the manifold of initial data (2.2) is invariant under the flow (2.1).

2.1.2. *Hamiltonian point of view*

Equations (2.1) are the Hamiltonian equations generated by the Hamiltonian function

$$\widehat{\mathcal{H}}_{\rm New} := \sum_{i=0}^n \frac{|U^{(i)}|^2}{2m_i} - \sum_{0\leq i<j\leq n} \frac{m_i m_j}{|u^{(i)}-u^{(j)}|} \ , \tag{2.3}$$

where $(U^{(i)}, u^{(i)})$ are standard symplectic variables ($U^{(i)} = m_i \dot{u}^{(i)}$ is the momentum conjugated to $u^{(i)}$) and the phase space is the "collisionless" open domain in $\mathbb{R}^{6(n+1)}$ given by

$$\widehat{\mathcal{M}} := \{U^{(i)}, u^{(i)} \in \mathbb{R}^3 : \ u^{(i)} \neq u^{(j)} \ , \ 0 \leq i \neq j \leq n\}$$

endowed with the standard symplectic form

$$\sum_{i=0}^n dU^{(i)} \wedge du^{(i)} := \sum_{\substack{0\leq i\leq n \\ 1\leq k\leq 3}} dU_k^{(i)} \wedge du_k^{(i)} \ . \tag{2.4}$$

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[1] Replace the coordinates $u^{(i)}$ by $u^{(i)} - (a+ct)$ with

$$a := \mathrm{m}_{\rm tot}^{-1} \sum_{i=0}^n m_i u^{(i)}(0) \qquad \text{and} \qquad c := \mathrm{m}_{\rm tot}^{-1} \sum_{i=0}^n m_i \dot{u}^{(i)}(0) \ , \qquad \mathrm{m}_{\rm tot} := \sum_{i=0}^n m_i \ .$$

As explained above, the physically relevant motions governed by (2.3) lie on

$$\widehat{\mathcal{M}}_0 := \{(U,u) \in \widehat{\mathcal{M}} : \sum_{i=0}^n m_i u^{(i)} = 0 = \sum_{i=0}^n U^{(i)}\}$$

(which corresponds to the manifold described in (2.2)). The submanifold $\widehat{\mathcal{M}}_0$ is symplectic (i.e., the restriction of the form (2.4) to $\widehat{\mathcal{M}}_0$ is again a symplectic form) and the $\widehat{\mathcal{H}}_{\rm New}$–flow on it is best described in terms of heliocentric coordinates. Let $\phi_{\rm hel} : (R,r) \to (U,u)$ be the linear symplectic transformation given by

$$\phi_{\rm hel} : \begin{cases} u^{(0)} = r^{(0)} , & u^{(i)} = r^{(0)} + r^{(i)} , \quad (i=1,...,n) \\ U^{(0)} = R^{(0)} - \sum_{i=1}^n R^{(i)} , & U^{(i)} = R^{(i)} , \quad (i=1,...,n) . \end{cases} \tag{2.5}$$

In such variables $\widehat{\mathcal{M}}_0$ reads

$$\Big\{(R,r) \in \mathbb{R}^{6(n+1)} : \ R^{(0)} = 0 , \ \ r^{(0)} = -{\rm m}_{\rm tot}^{-1} \sum_{i=1}^n m_i r^{(i)} \quad \text{and} \quad 0 \neq r^{(i)} \neq r^{(j)} \ \forall \ 1 \le i \neq j \le n\Big\} ;$$

the restriction of the 2–form (2.4) on $\widehat{\mathcal{M}}_0$ is simply $\sum_{i=1}^n dR^{(i)} \wedge dr^{(i)}$ and

$$(\widehat{\mathcal{H}}_{\rm New} \circ \phi_{\rm hel})|_{\mathcal{M}_0} = \sum_{i=1}^n \Big(\frac{|R^{(i)}|^2}{2\frac{m_0 m_i}{m_0+m_i}} - \frac{m_0 m_i}{|r^{(i)}|}\Big) + \sum_{1\le i<j\le n} \Big(\frac{R^{(i)}\cdot R^{(j)}}{m_0} - \frac{m_i m_j}{|r^{(i)} - r^{(j)}|}\Big) =: \ \mathcal{H}_{\rm New} .$$

Thus, the dynamics generated by $\widehat{\mathcal{H}}_{\rm New}$ on $\widehat{\mathcal{M}}_0$ is equivalent to the dynamics generated by the Hamiltonian $(R,r) \in \mathbb{R}^{6n} \to \mathcal{H}_{\rm New}(R,r)$ on

$$\mathcal{M}_0 := \Big\{(R,r) = (R^{(1)},...,R^{(n)},r^{(1)},...,r^{(n)}) \in \mathbb{R}^{6n} : \ \ 0 \neq r^{(i)} \neq r^{(j)} \ \forall \ 1 \le i \neq j \le n\Big\}$$

with respect to the standard symplectic form $\sum_{i=1}^n dR^{(i)} \wedge dr^{(i)}$; to recover the full dynamics on $\widehat{\mathcal{M}}_0$ from the dynamics on $\mathcal{M}_0$ one will simply set $R^{(0)}(t) \equiv 0$ and $r^{(0)}(t) := -{\rm m}_{\rm tot}^{-1} \sum_{i=1}^n m_i r^{(i)}(t)$.

Motivated by the planetary case, let us perform the trivial rescaling by a small positive parameter ε:

$$\bar m_0 := m_0 \ , \ m_i = \varepsilon \bar m_i \ (i \ge 1) \ , \quad X^{(i)} := \frac{R^{(i)}}{\varepsilon} \ , \ x^{(i)} := r^{(i)} \ , \quad \mathcal{H}_{\rm plt}(X,x) := \frac{1}{\varepsilon}\mathcal{H}_{\rm New}(\varepsilon X, x) \ ,$$

which leaves unchanged Hamilton's equations. Explicitly, if $\mu_i := \frac{\bar m_0 \bar m_i}{\bar m_0 + \varepsilon \bar m_i}$ and $M_i := \bar m_0 + \varepsilon \bar m_i$, then

$$\begin{aligned} \mathcal{H}_{\rm plt}(X,x) \ &:= \ \sum_{i=1}^n \left(\frac{|X^{(i)}|^2}{2\mu_i} - \frac{\mu_i M_i}{|x^{(i)}|}\right) + \varepsilon \sum_{1\le i<j\le n} \left(\frac{X^{(i)}\cdot X^{(j)}}{\bar m_0} - \frac{\bar m_i \bar m_j}{|x^{(i)} - x^{(j)}|}\right) \\ &=: \ \mathcal{H}_{\rm plt}^{(0)}(X,x) + \varepsilon \mathcal{H}_{\rm plt}^{(1)}(X,x) \ , \end{aligned} \tag{2.6}$$

the phase space being

$$\mathcal{M} := \left\{ (X,x) = (X^{(1)},...,X^{(n)},x^{(1)},...,x^{(n)}) \in \mathbb{R}^{6n} \ : \ \ 0 \neq x^{(i)} \neq x^{(j)} \ \forall \ 1 \leq i \neq j \leq n \right\} ,$$

endowed with the standard symplectic form $\sum_{i=1}^n dX^{(i)} \wedge dx^{(i)}$.

Remarks (i) The reduction of the general dynamics to the $\mathcal{H}_{\rm plt}$–dynamics on $\mathcal{M}$ is sometimes referred to as the "reduction of the linear momentum". Notice that a reflection of such reduction is that there is no more "conservation of the total linear momentum", as $\sum_{i=1}^n X^{(i)}$ is obviously *not* an integral[2] for $\mathcal{H}_{\rm plt}$. On the other hand, the transformation (2.5) does preserve the total *angular momentum* $\sum_{i=0}^n U^{(i)} \times u^{(i)}$, where " $\times$ " denotes the standard vector product in $\mathbb{R}^3$. Thus, the Hamiltonian $\mathcal{H}_{\rm plt}$ admits, besides the energy, three more integrals, which are the three components of the total angular momentum

$$C = (C_1, C_2, C_3) := \sum_{i=1}^n X^{(i)} \times x^{(i)} \ . \tag{2.7}$$

Such integrals do not commute (i.e., their Poisson brackets do not vanish):

$$\{C_1, C_2\} = C_3 \ , \qquad \{C_2, C_3\} = C_1 \ , \qquad \{C_3, C_1\} = C_2 \ ,$$

but, for example, $|C|^2$ and C_3 are two commuting, independent integrals.

(ii) The two–body case (corresponding to $n = 1$ and no $\mathcal{H}_{\rm plt}^{(1)}$ term) is integrable for any $\varepsilon > 0$ (Kepler). Therefore also the term $\mathcal{H}_{\rm plt}^{(0)}$ in the planetary Hamiltonian (2.6) is integrable, being the sum of n decoupled two–body problems. In Delaunay action–angle variables $((L, G, \Theta), (\ell, g, \theta))$ defined on the phase space[3]

$$\mathcal{M}_{\rm plt} := \left\{ (L, G, \Theta) \in \mathbb{R}^{3n} \ : \ L_i > G_i > \Theta_i > 0 \ , \quad \frac{L_i}{\mu_i \sqrt{M_i}} \neq \frac{L_j}{\mu_j \sqrt{M_j}} \ , \ \forall \ i \neq j \right\} \times \mathbb{T}^{3n} \ , \tag{2.8}$$

the Hamiltonian $\mathcal{H}_{\rm plt}^{(0)}$ takes the form

$$\mathcal{H}_{\rm plt}^{(0)} = - \sum_{i=1}^n \frac{\mu_i^3 M_i^2}{2 L_i^2} \ ; \tag{2.9}$$

the phase space $\mathcal{M}_{\rm plt}$, which corresponds to an open subset of $\mathcal{M}$ in (2.6), is endowed with the standard symplectic form

$$\sum_{i=1}^n dL_i \wedge d\ell_i + dG_i \wedge dg_i + d\Theta_i \wedge d\theta_i = \sum_{i=1}^n \sum_{j=1}^3 dX_j^{(i)} \wedge dx_j^{(i)} \ ;$$

for more information on Delaunay variables, see § A.1.

[2] We recall that $F(X,x)$ is an integral for $\mathcal{H}(X,x)$ if $\{F, \mathcal{H}\} = 0$ where $\{F, G\} = F_X \cdot G_x - F_x \cdot G_X$ denotes the (standard) Poisson bracket.

[3] $\mathbb{T}^n$ denotes the standard flat torus $\mathbb{T}^n := \mathbb{R}^n / (2\pi \mathbb{Z}^n)$.

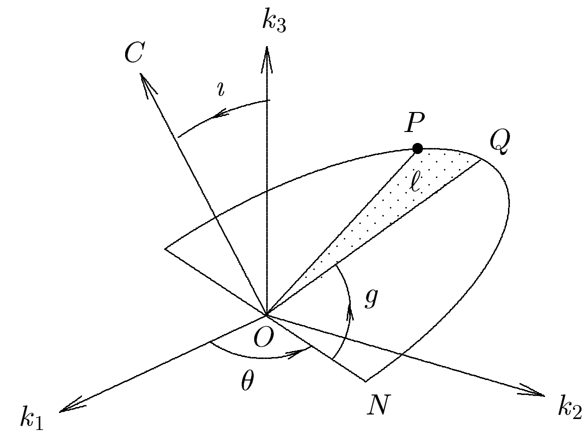

Figure 1. Spatial Delaunay angle variables.

Notice that the $6n$–dimensional phase space $\mathcal{M}_{\rm plt}$ is foliated by $3n$–dimensional $\mathcal{H}^{(0)}_{\rm plt}$–invariant tori $\{L, G, \Theta\} \times \mathbb{T}^3$, which, in turn, are foliated by n–dimensional tori $\{L\} \times \mathbb{T}^n$, expressing geometrically the degeneracy of the integrable Keplerian limit of the $(1 + n)$–body problem.

2.2. KAM theory

The perturbative approach to the many–body problem is based on the modern theory of conservative dynamical systems as developed, mainly, by Poincaré, Birkhoff, Siegel, Kolmogorov, Arnold, Moser and Herman. We recall here, briefly, some classical results.

2.2.1. *Quasi–periodic motions and KAM tori*

Consider a smooth Hamiltonian $(p, q) \in \mathcal{M} \to H(p, q)$ on a $(2d)$–dimensional phase space $\mathcal{M}$ endowed with standard symplectic coordinates (p, q). A (maximal) *KAM torus* for H is a d–dimensional H–invariant torus, on which the H–flow is conjugated to $\theta \in \mathbb{T}^d \to \theta + \omega t$ with $\omega \in \mathbb{R}^d$ Diophantine. We recall that ω is Diophantine if there exist positive constants γ and τ such that

$$|\omega \cdot k| \geq \frac{\gamma}{|k|^\tau} \ , \qquad \forall\ k \in \mathbb{Z}^d \backslash \{0\} \ . \tag{2.10}$$

In particular, the motion on a KAM torus is quasi–periodic with frequencies $\omega_1, ..., \omega_d$.

2.2.2. *Kolmogorov's 1954 Theorem*

In (Kolmogorov, 1954) Kolmogorov stated (and gave a beautiful albeit sketchy proof) of his famous theorem on the persistence of invariant tori, which may be formulated as follows.

Consider a "nearly–integrable" Hamiltonian system with phase space $\mathcal{M} := V \times \mathbb{T}^d$, V being an open bounded region in $\mathbb{R}^d$, and with Hamiltonian function given by

$$H_\varepsilon(I, \varphi) := h(I) + \varepsilon f(I, \varphi) \tag{2.11}$$

with real–analytic functions h, f and ε a small real parameter. The variables (I,φ) are standard symplectic "action–angle" variables, the symplectic form being $dI \wedge d\varphi := \sum_{i=1}^d dI_i \wedge d\varphi_i$.

THEOREM 2.1 (Kolmogorov, 1954). *In any neighborhood of any torus* $\{I_0\} \times \mathbb{T}^d \subset \mathcal{M}$ *such that*

$$\det h''(I_0) := \det \left(\frac{\partial^2 h}{\partial I_i \partial I_j}(I_0) \right)_{i,j=1,\dots,d} \neq 0 \ , \tag{2.12}$$

there exists a positive measure set of phase points belonging to analytic KAM tori for H_ε*, provided* ε *is small enough.*

A simple variation of the proof of Kolmogorov's theorem leads to the "iso–energetic" version of Theorem 2.1, namely:

THEOREM 2.2. *Let* I_0 *be such that*[4]

$$\det \begin{pmatrix} h''(I_0) & h'(I_0) \\ h'(I_0) & 0 \end{pmatrix} \neq 0 \ ; \tag{2.13}$$

let $\mathcal{M}_0 := \{(I,\varphi) \in \mathcal{M} : \ H_\varepsilon(I,\varphi) = h(I_0)\}$ *be the energy level corresponding to the "unperturbed" energy* $h(I_0)$*. Then, there exists on* $\mathcal{M}_0$ *a positive measure set of phase points belonging to analytic KAM tori for* H_ε*, provided* ε *is small enough.*

Clearly, the measure referred to in Theorem 2.1 is the $2d$–dimensional Liouville measure in phase space, while the measure referred to in Theorem 2.2 is the restriction of the Liouville measure on the energy level $\mathcal{M}_0$.

2.2.3. *Proper degeneracies*

A nearly–integrable system with Hamiltonian (2.11) for which h does not depend upon all the actions I_1,...,I_d is called *properly degenerated*. This is the case of the many–body problem since $\mathcal{H}^{(0)}_{\rm plt}$ in (2.9) depends only on the actions L's.

For properly degenerate systems neither condition (2.12) nor (2.13) holds and KAM tori may not exist at all[5]. To establish the existence of KAM tori in properly degenerate systems it is necessary to have more information on the perturbation f. In (Arnold, 1963), Arnold proved the following theorem, which he intended (and partially succeeded) to apply to the planetary many–body problem.

Let $\mathcal{M}$ denote the phase space

$$\mathcal{M} := \Big\{ (I,\varphi,p,q) : \ (I,\varphi) \in V \times \mathbb{T}^d \text{ and } (p,q) \in B \Big\} \ ,$$

[4] The matrix in (2.13) is a $(d+1)\times(d+1)$–matrix and the gradient $h'(I_0) := (\partial_{I_1} h(I_0), ..., \partial_{I_d} h(I_0))$ has to be thought of as a column in the upper right corner and as a row in lower left corner.

[5] Trivially, any unperturbed properly–degenerate system on a $2d$ dimensional phase space with $d \geq 2$ will have motions with frequencies not rationally independent over $\mathbb{Z}^d$.

where V is an open bounded region in $\mathbb{R}^d$ and B is a ball around the origin in $\mathbb{R}^{2m}$; $\mathcal{M}$ is equipped with the standard symplectic form

$$dI \wedge d\varphi + dp \wedge dq = \sum_{i=1}^{d} dI_i \wedge d\varphi_i + \sum_{i=1}^{m} dp_i \wedge dq_i \ .$$

Let, also, H_ε be a real analytic Hamiltonian on $\mathcal{M}$ of the form

$$H_\varepsilon(I,\varphi,p,q) := h(I) + \varepsilon f(I,\varphi,p,q) \ , \tag{2.14}$$

and denote by $\bar{f}$ the average of f over the "fast angles" φ:

$$\bar{f}(I,p,q) := \int_{\mathbb{T}^d} f(I,\varphi,p,q) \frac{d\varphi}{(2\pi)^d} \ . \tag{2.15}$$

THEOREM 2.3 (Arnold, 1963). *Assume that* $\bar{f}$ *is of the form*

$$\bar{f} = f_0(I) + \sum_{j=1}^{m} \Omega_j(I) J_j + \frac{1}{2} A(I) J \cdot J + o_4 \ , \qquad J_j := \frac{p_j^2 + q_j^2}{2} \ , \tag{2.16}$$

where A *is a symmetric* $(m \times m)$*–matrix and* $\lim_{(p,q)\to 0} |o_4|/|(p,q)|^4 = 0$. *Assume, also, that* $I_0 \in V$ *is such that*

$$\det h''(I_0) \neq 0 \ , \tag{2.17}$$

$$\sum_{j=1}^{m} \Omega_j(I_0) k_j \neq 0 \ , \qquad \forall\ k \in \mathbb{Z}^m \ \text{ with } \ 0 < \sum_{j=1}^{m} |k_j| \leq 6 \ , \tag{2.18}$$

$$\det A(I_0) \neq 0 \ . \tag{2.19}$$

Then, in any neighborhood of $\{I_0\} \times \mathbb{T}^d \times \{(0,0)\} \subset \mathcal{M}$ *there exists a positive measure set of phase points belonging to analytic KAM tori for* H_ε*, provided* ε *is small enough.*

This theorem has been generalized by Herman (Herman, 1998), as we shall, now, briefly explain. To formulate the non–degeneracy assumption of Herman's theorem, we need the notion of *non planar map* introduced by Pyartli (Pyartli, 1969). A smooth curve $u \in U \subset \mathbb{R} \to \omega(u) \in \mathbb{R}^n$, U open non empty interval, is called non planar at $u_0 \in U$ if all the u–derivatives up to order $(n-1)$ at u_0, $\omega(u_0)$, $\omega'(u_0)$,..., $\omega^{(n-1)}(u_0)$ are linearly independent over $\mathbb{R}^n$; a smooth map $u \in U \subset \mathbb{R}^d \to \omega(u) \in \mathbb{R}^n$, $d \leq n$, is called non planar at $u_0 \in U$ if there exists a smooth curve $\alpha : \hat{U} \subset \mathbb{R} \to U$ such that $\omega \circ \alpha$ is non planar at $t_0 \in \hat{U}$ with $\alpha(t_0) = u_0$.

THEOREM 2.4 (Herman, 1998). *Let* H_ε *and* $\bar{f}$ *be* C^∞ *functions as in* (2.14) *and* (2.15). *Assume that* $\bar{f}$ *is of the form*

$$\bar{f} = f_0(I) + \sum_{j=1}^{m} \Omega_j(I) J_j + o_2 \ , \qquad J_j := \frac{p_j^2 + q_j^2}{2} \ ,$$

where $\lim_{(p,q)\to 0} |o_2|/|(p,q)|^2 = 0$. *Assume, also, that* $I_0 \in V$ *is such that the "frequency map"*

$$I \in V \to \Big(h'(I), \Omega_1(I), ..., \Omega_m(I)\Big) \in \mathbb{R}^{d+m} \tag{2.20}$$

is non planar at I_0. *Then, in any neighborhood of* $\{I_0\} \times \mathbb{T}^d \times \{(0,0)\} \subset \mathcal{M}$ *there exists a positive measure set of phase points belonging to* C^∞ *KAM tori for* H_ε, *provided* ε *is small enough.*

This theorem is based on a C^∞ local inversion theorem on "tame" Frechet spaces due to F. Sergeraert and R. Hamilton (which, in turn, is related to the Nash–Moser implicit function theorem; see (Bost, 1984/1985)). A non–properly–degenerate version of Theorem 2.4 was established by Rüssmann in (Rüssmann, 2001). A proof of Herman's Theorem 2.4 can be found in (Féjoz, 2004).

2.3. "Arnold's Theorem" on planetary motions

The main question, longly studied by astronomers and mathematicians, which Arnold addressed in his 1963 paper is the following (Arnold, 1963, Chapter III, p. 125):

"*Do there exist, in the* n*–body problem, a set of initial conditions having positive measure such that, if the initial position and velocities of the bodies belong to this set, then the distances of the bodies from each other will remain perpetually bounded?*"

Indeed, in a (very) special case, Kolmogorov's Theorem yields immediately a positive answer to such a question: it is the case of the restricted, planar, circular three–body problem (RPC3BP, for short).

The RPC3BP, largely investigated by Poincaré, consists in studying the motion of a "zero mass" asteroid moving on the plane containing the trajectories of two unperturbed major bodies (say, Sun and Jupiter) revolving on a Keplerian circle. The mathematical model for the restricted three body problem is obtained by taking $n = 2$ and setting $m_2 = 0$ in (2.1): the equations for the two major bodies ($i = 0, 1$) decouple from the equation for the asteroid ($i = 2$) and form an integrable two body–system; the problem consists, then, in studying the evolution of the asteroid $u^{(2)}(t)$. In the circular, planar case the motion of the two primaries is assumed to be circular and the motion of the asteroid is assumed to take place on the plane containing the motion of the two primaries; in fact (to avoid collisions) one considers either inner or outer (with respect to the circle described by the relative motion of the primaries) asteroid motions. Using "rotating" planar Delaunay variables (see § A.2)

$$\Big((L, G), (\ell, g)\Big) \in \{(L, G) \in \mathbb{R}^2 : L > G > 0\} \times \mathbb{T}^2 \ ,$$

the Hamiltonian $\mathcal{H}_{\rm rcp}$ governing the motion of the RCP3BP problem, in suitably normalized units, is given by

$$\mathcal{H}_{\rm rcp}(L, G, \ell, g; \varepsilon) := -\frac{1}{2L^2} - G + \varepsilon \mathcal{H}_1(L, G, \ell, g; \varepsilon) \ , \tag{2.21}$$

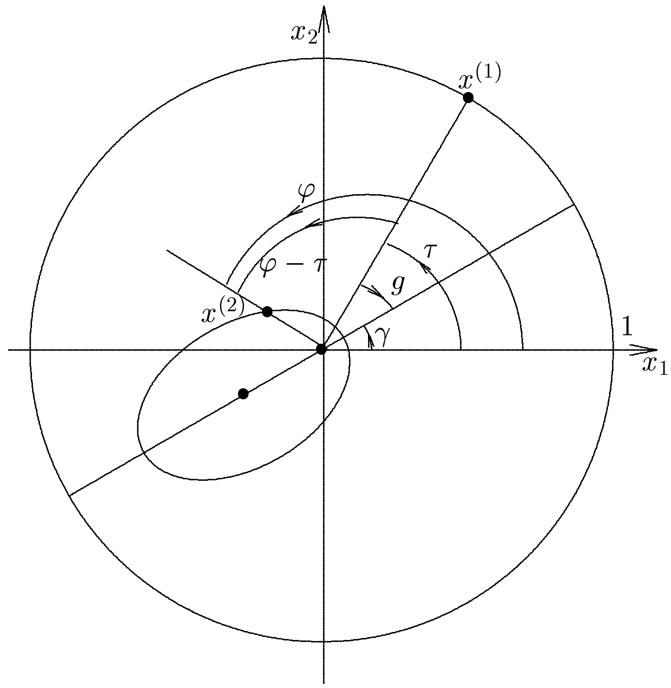

Figure 2. Planar Delaunay angle variables.

where the perturbation is given by

$$\mathcal{H}_1 := x^{(2)} \cdot x^{(1)} - \frac{1}{|x^{(2)} - x^{(1)}|} \tag{2.22}$$

expressed in the above Delaunay variables, $x^{(2)}$ being the heliocentric coordinate of the asteroid and $x^{(1)}$ that of the planet (Jupiter); the parameter ε represents essentially the mass ratio of the two main bodies; see Appendix A.2 for more information.

The integrable limit Hamiltonian $\mathcal{H}_{\rm rcp}|_{\varepsilon=0} = -\frac{1}{2L^2} - G$ satisfies (2.13) in a neighborhood of any point of the phase space (the determinant in (2.13) being equal, in the present case, to $3/L^4$) and, therefore, Theorem 2.2 yields the existence of a positive measure set of inital data, in each energy level $\mathcal{M}_0 := \{\mathcal{H}_{\rm rcp} = -\frac{1}{2L_0^2} - G_0\}$, that belong to KAM tori for $H_{\rm rcp}$, provided ε is small enough. In particular, the distance between the asteroid and the Keplerian circle described by the major bodies remains forever bounded.

REMARK 2.1. Indeed, in this very special case, much more is true: since two–dimensional KAM tori separate the three dimensional energy levels, also *all* trajectories starting between two KAM tori remain forever trapped in the region bounded by such two tori; compare Fig. 4 below.

As for the general planetary many body problem, Arnold in (Arnold, 1963) stated the following

THEOREM 2.5 (Arnold's Theorem on planetary motions). *Let $n \geq 2$. Then if ε is small enough, the Hamiltonian $\mathcal{H}_{\rm plt}$ in (2.6) admits a positive measure set of phase points, in a neighborhood of circular and coplanar Keplerian motions, leading to quasi–periodic motions with $3n-1$ frequencies.*

This statement is taken from (Féjoz, 2004), where a proof of Arnold's Theorem, in this generality, appeared for the first time. Actually, in (Arnold, 1963) a somewhat stronger result was announced[6], but the proof was given only for the planar three–body case[7]. A brief history of the proof of Arnold's Theorem is the following.

1. In (Arnold, 1963) Arnold gave a complete proof for the case of three coplanar bodies: $n=2$ and $(X,x) \in \mathbb{R}^2 \times \mathbb{R}^2$ in (2.6). In such a case, the word "coplanar" in Theorem 2.5, is redundant and $3n-1$ has to be replaced by 4. Arnold's proof is based upon his KAM Theorem 2.3: first, by means of planar Poincaré variables (see § A.5 with $n=2$), the Hamiltonian $\mathcal{H}_{\rm plt}$ is put in the form (2.14), (2.16) (with $d=m=2$); then conditions (2.18) and (2.19) ((2.17) is trivial) are checked by means of Leverrier's tables in the aymptotic regime $a_1/a_2 \to 0$ (a_i being the semimajor axis of the osculating Keplerian ellipse of the $i^{\rm th}$ planet).

2. The spatial three–body case was proven in (Laskar and Robutel, 1995) and (Robutel, 1995). The strategy is similar to that of Arnold and, in particular, it is again based upon Theorem 2.5: first, by means of spatial "osculating" Poincaré variables, Jacobi's "reduction of the nodes" (see, e.g., § A.4) and Birkhoff theory of normal form (see, e.g., (Siegel and Moser, 1971)), the Hamiltonian $\mathcal{H}_{\rm plt}$ is put in the form (2.14), (2.16) (again, $d=m=2$); then, the nondegeneracy conditions (2.18) and (2.19) are numerically checked, with the aid of computers, in a relatively large region of semiaxes.

3. The full proof of Theorem 2.5, as mentioned, was published in 2004 by Féjoz in (Féjoz, 2004), where Herman's work[8] on the subject was presented for the first time in a complete manner. The first step is to introduce Poincaré variables (see § A.3) and, in view of the conservation of the total angular momentum (2.7), to restrict the attention to the symplectic manifold of vertical total angular momentum, $\mathcal{M}_{\rm vert} := \{C_1 = 0 = C_2\}$. The idea is then to use the KAM Theorem 2.4 and hence to check the nonplanarity of the frequency map (2.20). However, this strategy fails for $\mathcal{H}_{\rm plt}$ (expressed in Poincaré variables and restricted to $\mathcal{M}_{\rm vert}$); the reason being the presence of an extra resonance ("Herman's resonance"). To overcome this problem, following Poincaré, Féjoz considers the modified

[6] "*If the masses, eccentricities and inclinations of the planets are sufficiently small, then for the majority of initial conditions the true motion is conditionally periodic and differs little from Lagrangian motion with suitable initial conditions throughout an infinite interval of time* $-\infty < t < \infty$"; (Arnold, 1963, Chapter III, p. 127).

[7] In fact, Arnold gave indications on how to generalize his approach to the general case, but, apparently, nobody has succeeded in implementing Arnold's indications.

[8] Michel Herman worked for long time on the planetary problem and gave several lectures and seminars on it in the mid nineties but his untimely death (2nd of November, 2000) did not allow him to publish the complete results of his researches. Herman's work on the planetary problem was, then, taken up by friends and colleagues in Paris and completed in (Féjoz, 2004).

Hamiltonian $\mathcal{H}^{\delta}_{\rm plt} := \mathcal{H}_{\rm plt} + \delta C_3^2$. For such Hamiltonian the nonplanarity condition of the frequency map is satisfied; but since the Hamiltonian $\mathcal{H}^{\delta}_{\rm plt}$ and $\mathcal{H}_{\rm plt}$ commute they have the same Lagrangian tori and hence the result is established also for $\mathcal{H}_{\rm plt}$.

3. KAM tori in Solar subsystems

3.1. Results

Certainly the main motivation for KAM theory was the existence of regular (relatively bounded) motions in the Solar system. In fact, as soon as the first KAM theorems were established, astronomers tried to apply them to astronomical models. However, such direct applications lead to very poor "practical" results, the restriction on ε (i.e., the size of the mass ratios) being far too strong to allow for applications to the Solar system (or Solar subsystems). At this regard, in a 1966 paper (Hénon, 1966), Hénon concludes: "Ainsi, ces théorèmes, bien que d'un très grand intérêt théorique, ne semblent pas pouvoir en leur état actuel être appliqués à des problèmes pratiques[9]."

A major breakthrough towards applications of KAM theory to physical models came from the interaction between KAM theory and techniques for *computer–assisted proofs*. Such techniques, which are based upon the so–called *interval arithmetic*[10], allow to perform long computations on computers keeping rigorously track of the rounding errors introduced by the machine.

For more information about computer–aided proofs and computer–assisted KAM theory applied to model problems (such as the standard map or a simple forced pendulum), see, e.g., (Celletti and Chierchia 1987a, Celletti et al 1987b, Celletti and Chierchia 1988a, Celletti and Giorgilli 1988b, Celletti and Chierchia 1995, Celletti et al 2000, Llave and Rana 1990, Rana 1987), and references therein.

[9] (Hénon, 1966), p. 64: "Les théorèmes d'Arnold et Moser ne s'appliquent qu'à des problèmes qui diffèrent d'un problème intégrable par une perturbation extrêmement petite. [⋯] Par exemple, dans la démonstration d'Arnold (1963, *Russian math. Surveys*, 18, 9, p. 16) on a: [⋯] Dans le cas du problème restreint, on a: $n = 2$. D'autre part, le cas intégrable est représenté par $\mu = 0$; on retrouve alors le problème des deux corps. Pour $\mu \neq 0$, la perturbation est proportionelle à la masse μ du second corps. M et μ sont donc du même ordre de grandeur. Des íneqalités ci–dessus, on tire:

$$M < 10^{-333} . \tag{14}$$

Une estimation du même genre peut être faite dans la démonstration de Moser (1962, *Nach. Akad. Wiss. Göttingen*, Math. Phys. Kl., 1); on aboutit à:

$$M < 10^{-48} . \tag{15}$$

Ainsi, ces théorèmes, bien que d'un très grand intérêt théorique, ne semblent pas pouvoir en leur état actuel être appliqués à des problèmes pratiques, où les perturbations sont toujours beaucoup plus grandes que les limites (14) ou (15)."

[10] Roughly speaking, computers work with special classes of rational numbers ("representable numbers"). In general, an elementary operation $(+, -, *, \div)$ between two representable numbers is no more a representable number, since the result is affected by rounding–off and propagation errors. Therefore one needs to provide the result as an interval, whose endpoints are representable numbers and which yield lower and upper bounds on the result of the elementary operation.

Computer–aided existence of KAM tori for three–body problems with mass ratios within at most three orders of magnitude of the observed values have been (rigorously) established in the following three papers.

1. In (Celletti and Chierchia, 1997) the Sun–Jupiter–Ceres problem has been investigated in the context of the RPC3BP using rotating planar Delaunay variables. The observed average frequency of Ceres is about $\Omega_C \simeq 2.577107$, while $e_C \simeq 0.0766$ is the observed eccentricity. The perturbing function has been expanded in Fourier–Taylor series, retaining only the terms whose size is bigger than the gravitational influence due to Saturn and the Jupiter/Sun mass ratio (which is about 10^{-3}) has been replaced by ε. Implementing computer–assisted KAM estimates, existence of quasi–periodic tori with Diophantine frequencies close to Ω_C has been established for any mass–ratio $\varepsilon \leq 10^{-6}$.

2. In (Locatelli and Giorgilli, 2000) the planetary problem formed by the Sun, Jupiter and Saturn has been considered. After Jacobi's reduction of the nodes (see, e.g., § A.4), one obtains a Hamiltonian function with 4 degrees of freedom. Such Hamiltonian is expanded up to the second order in the masses and averaged over the fast angles $(\lambda_1^*, \lambda_2^*)$ (in the notation of § A.4). In this way a two degree–of–freedom Hamiltonian is obtained, which nearly gives the slow motion of the parameters characterizing the Keplerian approximation (e.g., the eccentricities). Looking for invariant tori in the proximity of an equilibrium elliptic point, the perturbation, written in Poincaré variables, is expanded up to the order 6 in the eccentricities. Then, a Birkhoff normal form, combined with a computer–assisted implementation of a KAM theorem, provides the existence of two invariant tori bounding the *secular* motions of Jupiter and Saturn for the observed values of the parameters[11].

3. In (Celletti and Chierchia, 2005), which is extensively reviewed in the next section 3.2, a truncated RPC3BP model for Sun, Jupiter and Asteroid 12 Victoria is investigated. On a *fixed* energy level[12], invariant KAM tori trapping the motion of Victoria have been established for the astronomical value of the Jupiter–Sun mass–ratio.

For other computer–aided KAM results of interest for Celestial Mechanics, see (Celletti 1990a, 1990b, Celletti and Falcolini, 1992) (spin-orbit problem) and (Celletti, 1993) (librational tori).

3.2. KAM stability of the Sun–Jupiter–Victoria system modelled by a truncated RPC3BP

Here, we describe with some details the results in (Celletti and Chierchia, 2005) mentioned in item 3 above. Let us begin by describing precisely the mathematical model. The framework is that of RPC3BP as described in § 2.3 and § A.2; see in particular (2.21), (2.22) and Figure 2. As main bodies we take the Sun (P_0) and Jupiter (P_1), which are therefore assumed to revolve

[11] For interesting numerical results related to (Locatelli and Giorgilli, 2000), see (Locatelli and Giorgilli 2005a, 2005b).

[12] In comparing this result with (Celletti and Chierchia, 1997), keep in mind that there the energy level is not a priori fixed as it is done here.

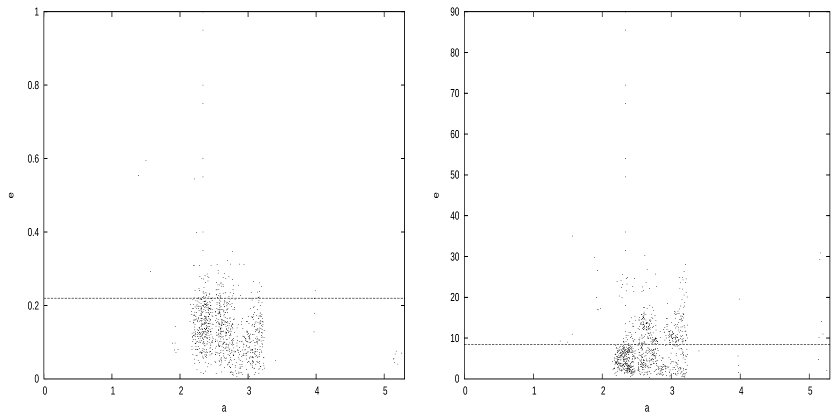

Figure 3. Orbital elements of the numbered asteroids: semimajor axis versus eccentricity (left panel), semimajor axis versus inclination (right panel). The internal lines locate the position of the asteroid 12 Victoria.

on a circle of radius one. In such a case the perturbative parameter ε is the Jupiter/Sun mass ratio, which amounts to

$$\varepsilon = \varepsilon_J := 0.954 \cdot 10^{-3} \tag{3.23}$$

(the normalizations are described in § A.2; see, in particular, Eq. (A.3) and (A.4)). We, then, proceed to select a minor ("zero mass") body, P_2, within the asteroidal belt; in order to avoid the introduction of another small parameter, we privileged those asteroids whose eccentricity is not too small (which also happen to be quite common in the asteroidal belt, as we shall shortly explain). We pick Asteroid 12 Victoria, whose orbital elements are:

$$\begin{array}{lll} a_{\rm V} \simeq 2.334\ AU & e_{\rm V} \simeq 0.220 & \imath_{\rm V} \simeq 8.363^o \\ \hat{g} \simeq 69.717^o & \Omega \simeq 235.548^o & M \simeq 135.908^o\ , \end{array}$$

where $\imath_{\rm V}$ is the inclination with respect to the ecliptic, $\hat{g}$ is the argument of perihelion, Ω is the longitude of the ascending node and M is the mean anomaly referred to the epoch MJD 53400.

In order to explore the peculiarity of this choice, we report in Figure 3 the elements of the numbered asteroids[13]. The majority of the asteroids lie within the region $1.8 \le a \le 3.5$, while the eccentricity is typically confined to $0 \le e \le 0.35$ and, as Figure 3 shows, the orbital elements of Victoria (which are located by the internal lines) appear to be rather typical in the nearly planar, non–too eccentric region of the orbital elements of the numbered asteroids.

In our model we disregarded the eccentricity of Jupiter, the mutual inclinations, the gravitational effects of the other bodies (notably those of Mars and Saturn), any dissipative phenomena like tides, solar winds, Yarkovsky effect, etc. As empirical criterion, we decide to expand the perturbation in the eccentricity and semimajor axes ratio, disregarding the contributions smaller than the most important term we have neglected in our model, which is actually due

[13] The elements of the numbered asteroids are provided by the JPL's DASTCOM database at http://ssd.jpl.nasa.gov/sb_elem.html

to the eccentricity of the orbit of Jupiter. Moreover, in order to balance the fact that lower harmonics are physically more relevant than higher ones, we reintroduce in the lowest order harmonics the first discarded term. We are thus led to consider the one–parameter family of Hamiltonians

$$\begin{aligned} H_{\rm SJV}(\ell,g,L,G;\varepsilon) &:= -\frac{1}{2L^2} - G - \varepsilon P_{\rm SJV}(\ell,g,L,G) \\ &=: H_0(L,G) + \varepsilon H_1(\ell,g,L,G) \ , \end{aligned} \tag{3.24}$$

where: $0 < G < L$ (being $e = \sqrt{1-\frac{G^2}{L^2}}$; see Eq. (A.2), § A.1) and, setting $a_0 := L^2$, the perturbing function is given by

$$\begin{aligned} P_{\rm SJV}(\ell,g,L,G) := 1 \ &+ \ \frac{{a_0}^2}{4} + \frac{9}{64}\,{a_0}^4 + \frac{3}{8}\,{a_0}^2 e^2 - \Big(\frac{1}{2} + \frac{9}{16}{a_0}^2\Big)\,{a_0}^2 e\ \cos\ell \\ &+ \ \Big(\frac{3}{8}\,{a_0}^3 + \frac{15}{64}{a_0}^5\Big)\cos(\ell+g) - \Big(\frac{9}{4} + \frac{5}{4}{a_0}^2\Big)\,{a_0}^2 e\ \cos(\ell+2g) \\ &+ \ \Big(\frac{3}{4}\,{a_0}^2 + \frac{5}{16}\,{a_0}^4\Big)\cos(2\,\ell+2\,g) + \frac{3}{4}\,{a_0}^2 e\cos(3\,\ell+2\,g) \\ &+ \ \Big(\frac{5}{8}\,{a_0}^3 + \frac{35}{128}\,{a_0}^5\Big)\cos(3\,\ell+3\,g) + \frac{35}{64}\,{a_0}^4\,\cos(4\,\ell+4\,g) \\ &+ \ \frac{63}{128}{a_0}^5\cos(5\ell+5g)\ . \end{aligned} \tag{3.25}$$

Fixing the perturbing parameter $\varepsilon = \varepsilon_J$ as in (3.23), we obtain the *Sun-Jupiter-Victoria Hamiltonian*:

$$\begin{aligned} H^*_{\rm SJV}(\ell,g,L,G) &:= -\frac{1}{2L^2} - G - \varepsilon_J P_{\rm SJV}(\ell,g,L,G) \\ &= H_0(L,G) + \varepsilon_J H_1(\ell,g,L,G)\ . \end{aligned}$$

We next fix the energy level. To this end, we remark that the observed values of the Delaunay's action variables are $\sqrt{a_{\rm V}} \simeq 0.670 =: L_{\rm V}$ and $L_{\rm V}\sqrt{1-e^2_{\rm V}} \simeq 0.654 =: G_{\rm V}$. Let

$$E^{(0)}_{\rm V} := -\frac{1}{2L^2_{\rm V}} - G_{\rm V} \simeq -1.768\ , \quad E^{(1)}_{\rm V} := \langle H_1(\cdot, L_{\rm V}, G_{\rm V})\rangle \simeq -1.060\ , \quad E_{\rm V}(\varepsilon) := E^{(0)}_{\rm V} + \varepsilon E^{(1)}_{\rm V}.$$

We define the osculating energy level of the Sun–Jupiter–Victoria model as

$$E^*_{\rm V} := E_{\rm V}(\varepsilon_J) = E^{(0)}_{\rm V} + \varepsilon_J E^{(1)}_{\rm V} \simeq -1.769\ . \tag{3.26}$$

On $\mathcal{S}^*_{\rm SJV} := (H^*_{\rm SJV})^{-1}(E^*_{\rm V})$ we want to prove the existence of two invariant tori, bounding from above and below the observed values $L_{\rm V}$ and $G_{\rm V}$. More precisely, if $\tilde L_\pm = L_V \pm 0.001$ we consider the frequencies

$$\tilde\omega_\pm := \Big(\frac{\partial H_0}{\partial L}, \frac{\partial H_0}{\partial G}\Big) = \Big(\frac{1}{\tilde L^3_\pm}, -1\Big) =: (\tilde\alpha_\pm, -1)\ .$$

In order to obtain two bounding *Diophantine* frequencies we compute the continued fraction expansion up to the order 5 of $\tilde{\alpha}_\pm$ and we add a tail of one's to obtain the following Diophantine numbers:

$$\alpha_- := [3;3,4,2,1^\infty] = 3.30976937631389... \ , \quad \alpha_+ := [3;2,1,17,5,1^\infty] = 3.33955990647860... \ .$$

Finally, we define

$$\omega_\pm := (\alpha_\pm, -1) \ ,$$

which satisfy the Diophantine condition (2.10) with constants

$$\tau_\pm := \tau = 1 \ , \qquad \gamma_- := 7.224496 \cdot 10^{-3} \ , \qquad \gamma_+ := 3.324329 \cdot 10^{-2} \ .$$

The stability of the asteroid Victoria is an immediate consequence of the following theorem, which yields the existence of the KAM continuations of the unperturbed tori $\mathcal{T}_0^\pm := \{(L_\pm, G_\pm)\} \times \mathbb{T}^2$.

THEOREM 3.1. *For $|\varepsilon| \le \varepsilon_* := 10^{-3}$ the unperturbed tori $\mathcal{T}_0^\pm$ can be analytically continued into invariant KAM tori $\mathcal{T}_\varepsilon^\pm$ on the energy level $\mathcal{S}_\varepsilon := H_{\rm SJV}^{-1}(E_{\rm V}(\varepsilon))$ keeping fixed the ratio of the frequencies.*

As a consequence (recall Remark 2.1), *the orbital elements corresponding to the semimajor axis and to the eccentricity* (which are simply related to the Delaunay's variables L and G) *stay forever ε–close to their unperturbed values.*

The idea of the proof relies on the combination of a new KAM iso–energetic theorem with accurate computer–assisted construction of approximate solutions. First, one observes that the parametric representation $\theta \in \mathbb{T}^2 \to (x,y) = (u(\theta), v(\theta))$ of a KAM torus lying with Diophantine frequencies (ω_1, ω_2), on the energy level E satisfies the following semi–linear PDE

$$\begin{aligned} &Du = \frac{\partial H}{\partial y}(u,v) \ , \qquad Dv = -\frac{\partial H}{\partial x}(u,v) \ , \\ &H(u(0), v(0)) = E \ , \end{aligned} \tag{3.27}$$

where D denotes the vector field $\left(\omega_1 \frac{\partial}{\partial\theta_1} + \omega_2 \frac{\partial}{\partial\theta_2}\right)$. Then, the system (3.27) is solved by a "hard implicit function theorem" *à la* Nash–Moser (compensating the effect of the small divisors with a quadratic scheme). To apply effectively this implicit function theorem, we first compute explicitly an "approximate solution", say $z^{(1)}$, and, then, we prove that close to it there exists a much better approximate solution, $z^{(2)}$, to which the stringent smallness condition dictated by the KAM implicit function theorem applies. In fact, $z^{(1)}$ is a Fourier–Taylor polynomial function (depicted in Figure 4 and Figure 5), while $z^{(2)}$ is obtained via iteration of a certain nonlinear operator and can only be controlled by estimating suitable norms. The construction of $z^{(1)}$ is based on an algorithm for computing iso–energetic Lindstedt series[14].

[14] Lindstedt series – already known at the times of Poincaré – are formal Fourier–Taylor series expansions of the solution of system (3.27).

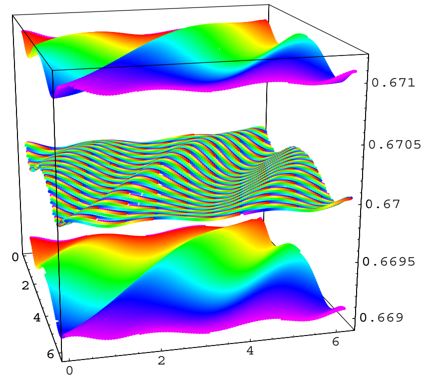

Figure 4. The upper and lower surfaces are the graphs on the 3–dimensional energy level $(H^*_{\rm SJV})^{-1}(E^*_V)$ of the approximate solutions $z^{(1)}$ described in the text for the two frequency vectors $\omega_+ = (3.30976937631389..., -1)$ and $\omega_- = (3.33955990647860..., -1)$; the intermediate surface is obtained integrating numerically a Sun–Jupiter–Victoria sample motion on the same energy level. The coordinates used are the (rotating) Delaunay angles $(\ell, g) \in \mathbb{T}^2$ in abscissa and the action $L > 0$ in ordinates; the perturbing parameter is set equal to the actual Jupiter–Sun mass ratio $\varepsilon_{\rm J} = 0.954 \cdot 10^{-3}$.

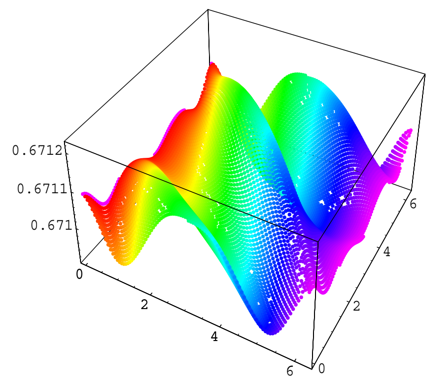

Figure 5. The upper bounding surface, on a different scale, showing the oscillatory structure of the KAM trapping tori.

REMARK 3.1. From the mathematical point of view, the Furier–truncation introduced in this model is rather unsatisfactory. However, we believe that a similar strategy to that leading to Theorem 3.1, could be applied to the full RPC3BP. From the physical point of view, instead, the truncation does not seem to affect much the dynamics. In fact, numerical studies suggest that for the frequencies and parameter values considered in Theorem 3.1, the truncated Hamiltonian (3.24)–(3.25) provides results very close to those obtained using the complete perturbing function; see (Celletti et al, 2004), briefly reviewed in Appendix B.

Appendix

A. Appendix: Symplectic variables for many–body problems

A.1. Delaunay variables

We begin by briefly describing the Delaunay variables for the Keplerian two–body problem. Let $\mathcal{H}_{\rm Kep} = \frac{|X|^2}{2\mu} - \frac{\mu M}{|x|}$ denote the (reduced) two–body Hamiltonian with $(X, x) \in \mathbb{R}^3 \times \mathbb{R}^3 \backslash \{0\}$, where M denotes the total mass of the two bodies and μ is a free rescaling parameter, and consider negative energies $\mathcal{H}_{\rm Kep} < 0$. In such a case, if $(X(t), x(t))$ denotes the $\mathcal{H}_{\rm Kep}$–flow, then $x(t)$ describes an ellipse lying in the plane π_C orthogonal to $C := X \times x$, with focus in the origin and fixed symmetry axes. Assume that the angular momentum C is not vertical and that the ellipse is not a circle. Introduce the following notations:

a is the semi–major axis of the ellipse spanned by x;

ι (the inclination) is the angle between the x_3–axis and C;

$G = |C| = \sqrt{C_1^2 + C_2^2 + C_3^2}$;

$\Theta = G \cos \iota = C_3$;

$L = \mu\sqrt{Ma}$;

ℓ is the mean anomaly of x (:=2π times the normalized area spanned by x measured from the perihelion Q, which is the point of the ellipse closest to the origin);

θ is the angle between the x_1–axis and the node line N (i.e. the intersection of the (x_1, x_2)–plane with π_C);

g is the argument of the perihelion (:= the angle between N and (O, Q)).

Then

$$\Big((L, G, \Theta), (\ell, g, \theta)\Big) \in \mathcal{M}_{\rm Kep} := \{L > G > \Theta > 0\} \times \mathbb{T}^3 \tag{A.1}$$

are conjugated symplectic coordinates (i.e., $dL \wedge d\ell + dG \wedge dg + d\Theta \wedge d\theta = \sum_{i=1}^3 dX_i \wedge dx_i$) and if $\phi_{\rm Del}$ is the corresponding symplectic map, then

$$\mathcal{H}_{\rm Kep} \circ \phi_{\rm Del} = -\frac{\mu^3 M^2}{2L^2}\ .$$

The eccentricity e of the Keplerian ellipse with energy $-\frac{\mu^3 M^2}{2L^2}$ and absolute value of angular momentum G is, then, given by

$$e = \sqrt{1 - \frac{G^2}{L^2}}\ . \tag{A.2}$$

Thus, the inequalities in (A.1) are seen to correspond to regions in phase space of non–degenerate elliptical motions (i.e., ellipses with $0 < e < 1$) taking place on the plane transversal with and not perpendicular to the (x_1, x_2)–plane.

In expressing the planetary $(1 + n)$–problem in Delaunay action–angle variables one considers Delaunay variables $(L_i, G_i, \Theta_i), (\ell_i, g_i, \theta_i)$ associated to the limiting two–body problem formed by the Sun ($i = 0$) and the $i^{\rm th}$ planet ($1 \le i \le n$). The (clearly sympletic) variables $(L_i, G_i, \Theta_i), (\ell_i, g_i, \theta_i)$ are well defined in the Cartesian product of the Keplerian phase spaces

$$\prod_{1\le i\le n} \{L_i > G_i > \Theta_i > 0\} \times \mathbb{T}^{3n} \ ,$$

and the relations

$$a_i = \frac{L_i}{\mu_i\sqrt{M_i}} \neq a_j = \frac{L_j}{\mu_j\sqrt{M_j}} \ , \qquad\qquad \forall\, 1 \le i \neq j \le n \ ,$$

avoid collisions; this accounts for the definition of $\mathcal{M}_{\rm plt}$ given in (2.8).

Complete details may be found, e.g., in (Biasco et al, 2003, § C.1, p. 117–119) and (Celletti and Chierchia, 2005, § 3.2).

A.2. Planar Delaunay variables and the RPC3BP Hamiltonian

We start by describing planar Delaunay variables $((L, G), (\ell, \hat{g}))$ and then describe the "rotating" planar Delaunay variables $((L, G), (\ell, g))$. Consider a planar two–body problem with $(X, x) \in \mathbb{R}^2 \times \mathbb{R}^2 \backslash \{0\}$ and $\mathcal{H}_{\rm Kep,pl} = \frac{|X|^2}{2\mu} - \frac{\mu M}{|x|}$, M being the total mass of the two body and μ a free rescaling parameter. Introduce the following notations:

- a is the semi–major axis of the ellipse spanned by x;
- $L = \mu\sqrt{Ma}$;
- e is the eccentricity of the ellipse spanned by x and $G = L\sqrt{1-e^2}$;
- ℓ is the mean anomaly of x;
- $\hat{g}$ is the argument of the perihelion.

Then

$$\Big((L, G), (\ell, \hat{g})\Big) \in \mathcal{M}_{\rm Kep,pl} := \{L > G > 0\} \times \mathbb{T}^2$$

are conjugated symplectic coordinates (i.e., $dL \wedge d\ell + dG \wedge d\hat{g} = \sum_{i=1}^2 dX_i \wedge dx_i$) and if $\phi_{\rm Del,pl}$ is the corresponding symplectic map, then

$$\mathcal{H}_{\rm Kep,pl} \circ \phi_{\rm Del,pl} = -\frac{\mu^3 M^2}{2L^2} \ .$$

The rotating planar Delaunay variables for the RPC3BP for P_0 (main body), P_1 (planet) and P_2 (zero–mass asteroid) are, then, given by

$$\Big((L, G), (\ell, g)\Big) \in \mathcal{M}_{\rm Kep,pl} := \{L > G > 0\} \times \mathbb{T}^2 \ , \qquad g := \hat{g} - \tau \ ,$$

τ being the longitude of P_1 (i.e., the angle between the x_1–axis and $x^{(1)}(\tau)$, which denotes the relative position P_1–P_0). The units are choosen so that:

$$m_0 + m_1 = 1 \ , \qquad\qquad |x^{(1)}(\tau)| = 1 \ , \qquad\qquad \text{(A.3)}$$

where m_i denote the masses of P_i. With such normalization the period of the P_0–P_1 motion is 2π (so that $\tau \in \mathbb{T}$).

Now, if we also set

$$\mu := \frac{1}{m_0^{2/3}} \ , \qquad \varepsilon := \frac{m_1}{m_0^{2/3}} = \frac{m_1}{(1-m_1)^{2/3}} \ , \tag{A.4}$$

then the Hamiltonian of the RCP3BP, in rotating planar Delaunay variables, takes the form (2.21) with

$$\mathcal{H}_1(L,G,\ell,g;\varepsilon) := x^{(2)} \cdot x^{(1)}_{\rm circ}(\tau) - \frac{1}{|x^{(2)} - x^{(1)}_{\rm circ}(\tau)|} \ , \qquad x^{(1)}_{\rm circ}(\tau) := (\cos\tau, \sin\tau) \ ,$$

where, of course, $x^{(2)}$ (the heliocentric position of the asteroid) has to be expressed in term of the rotating planar Delaunay variables.

Complete details may be found, e.g., in (Celletti and Chierchia, 2005, § 3.2 and § 3.3).

A.3. Poincaré variables

The spatial Poincaré variables for the planetary $(1+n)$ body–problem is a set of symplectic variables for an open (physically relevant) subset of the phase space $\mathcal{M}_{\rm plt}$; in particular such variables are well defined (and analytic) in a neighborhood of circular and co–planar motions. For $1 \le i \le n$, let $((L_i, G_i, \Theta_i), (\ell_i, g_i, \theta_i))$ denote the Delaunay variables associated to the two–body system Sun–$i^{\rm th}$ planet. The (spatial) Poincaré variables are given by $((\Lambda_i, \lambda_i), (\eta_i, \xi_i), (p_i, q_i))$ where

$$\Lambda_i = L_i \ , \qquad \lambda_i = \ell_i + g_i + \theta_i \ ,$$

and

$$\begin{cases} \eta_i = \sqrt{2(L_i - G_i)}\ \cos(g_i + \theta_i) \\ \xi_i = -\sqrt{2(L_i - G_i)}\ \sin(g_i + \theta_i) \end{cases} \qquad \begin{cases} p_i = \sqrt{2(G_i - \Theta_i)}\ \cos\theta_i \\ q_i = -\sqrt{2(G_i - \Theta_i)}\ \sin\theta_i \ . \end{cases}$$

Then, for any $\Lambda_+ > \Lambda_- > 0$ there exists $r > 0$ such that the Poincaré variables are symplectic and analytic on the domain

$$\begin{array}{ll} \Lambda_- < \Lambda_i < \Lambda_+ \quad \text{for } 0 \le i \le n \ , & (\lambda_i, .., \lambda_n) \in \mathbb{T}^n \ , \\ \eta_i^2 + \xi_i^2 < r^2 \quad \text{and} & p_i^2 + q_i^2 < r^2 \quad \text{for } 0 \le i \le n \ . \end{array}$$

If e_i, $C^{(i)}$ and ι_i denote, respectively, the eccentricity, angular momentum and inclination of the (instantaneous or osculating) two–body system Sun–$i^{\rm th}$ planet, then the following relations hold

$$\begin{aligned} \frac{\eta_i^2 + \xi_i^2}{2} &= \Lambda_i\,(1 - \sqrt{1 - e_i^2}) \ , \\ |C^{(i)}| &= \Lambda_i \sqrt{1 - e_i^2} \ , \\ \frac{p_i^2 + q_i^2}{2} &= |C^{(i)}|(1 - \cos\iota_i) \ . \end{aligned}$$

For details, see, e.g., (Biasco et al, 2003, § C.1).

A.4. Osculating Poincaré variables and Jacobi's reduction of the nodes

Poincaré introduced another set of symplectic variables, particularly suited to describe the classical Jacobi's reduction of the nodes, which allows to give a representation of the spatial three–body in terms of a four–degree–of–freedom Hamiltonian system[15].

Let, for $i = 1, 2$, $\big((L_i, G_i, \Theta_i), (\ell_i, g_i, \theta_i)\big)$ denote the Delaunay variables introduced in § A.1. Then the variables

$$\big((\Lambda_i^*, \lambda_i^*), (\eta_i^*, \xi_i^*), (\Theta_i, \theta_i)\big) \tag{A.5}$$

defined by

$$\begin{cases} \Lambda_i^* = L_i \\ \lambda_i^* = \ell_i + g_i \end{cases} \qquad \begin{cases} \eta_i^* = \sqrt{2(L_i - G_i)}\cos g_i \\ \xi_i^* = -\sqrt{2(L_i - G_i)}\sin g_i \end{cases}$$

are symplectic and analytic near circular, *non* co–planar motions; for details, see, e.g., (Biasco et al, 2003). Denote by

$$\mathcal{H}_{3\text{bp}} := \mathcal{H}^{(0)}(\Lambda^*) + \varepsilon \mathcal{H}^{(1)}(\Lambda^*, \lambda^*, \eta^*, \xi^*, \Theta, \theta)$$

the Hamiltonian (2.6) (with $n = 2$) expressed in terms of the symplectic variables (A.5), $\Lambda^* = (\Lambda_1^*, \Lambda_2^*)$, etc. Then, $\Theta_1 + \Theta_2$ is the vertical component, $C_3 = C \cdot k_3$, of the total argument $C = C^{(1)} + C^{(2)}$. Introduce, now, the symplectic variables

$$(\Lambda^*, \lambda^*, \eta^*, \xi^*, \Psi, \psi) = \phi(\Lambda^*, \lambda^*, \eta^*, \xi^*, \Theta, \theta)$$

where $(\Psi_1, \Psi_2, \psi_1, \psi_2) := (\Theta_1, \Theta_1 + \Theta_2, \theta_1 - \theta_2, \theta_2)$ and let $\mathcal{H}_{3\text{bp}}^* := \mathcal{H}_{3\text{bp}} \circ \phi^{-1}$ denote the Hamiltonian of the spatial three–body problem in these symplectic variables. Since the Poisson bracket of $\Psi_2 = \Theta_1 + \Theta_2$ and $\mathcal{H}_{3\text{bp}}^*$ vanishes (C_3 being an integral for the $\mathcal{H}_{3\text{bp}}$–flow), the conjugate angle ψ_2 is cyclic for $\mathcal{H}_{3\text{bp}}^*$, i.e.,

$$\mathcal{H}_{3\text{bp}}^* = \mathcal{H}_{3\text{bp}}^*(\Lambda^*, \lambda^*, \eta^*, \xi^*, \Psi_1, \Psi_2, \psi_1)\ .$$

Because the total angular momentum C is preserved, one can restrict the attention to the 10-dimensional invariant (and symplectic) submanifold $\mathcal{M}_{\text{ver}}$ defined by fixing the total angular momentum to be vertical. Such submanifold, in terms of Delaunay variables, is given by

$$\theta_1 - \theta_2 = \pi \qquad \text{and} \qquad G_1^2 - \Theta_1^2 = G_2^2 - \Theta_2^2\ ,$$

so that $\mathcal{M}_{\text{ver}}^* := \phi(\mathcal{M}_{\text{ver}}) = \Big\{\psi_1 = \pi\ ,\quad \Psi_1 = \widehat{\Psi}_1(\Lambda^*, \eta^*, \xi^*; \Psi_2)\Big\}$ with

$$\widehat{\Psi}_1 := \frac{\Psi_2}{2} + \frac{(\Lambda_1^* - H_1^*)^2 - (\Lambda_2^* - H_2^*)^2}{2\Psi_2}\ , \qquad H_i^* := \frac{{\eta_i^*}^2 + {\xi_i^*}^2}{2}\ .$$

[15] This description is borrowed from (Chierchia, 2005).

Since $\mathcal{M}^*_{\rm ver}$ is invariant for the flow ϕ^t_* of $\mathcal{H}^*_{\rm 3bp}$, $\psi_1(t) := \pi$ and $\dot\psi_1 := 0$ for motions starting on $\mathcal{M}^*_{\rm ver}$, which implies that $(\partial_{\Psi_1}\mathcal{H}^*_{\rm 3bp})|_{\mathcal{M}^*_{\rm ver}} = 0$. This fact allows to introduce, for fixed values of the vertical angular momentum $\Psi_2 = c \neq 0$, the following reduced Hamiltonian:

$$\mathcal{H}^c_{\rm red}(\Lambda^*, \lambda^*, \eta^*, \xi^*) := \mathcal{H}^*_{\rm 3bp}(\Lambda^*, \lambda^*, \eta^*, \xi^*, \widehat{\Psi}_1(\Lambda^*, \eta^*, \xi^*; c), c, \pi)$$

on the 8–dimensional phase space $\mathcal{M}_{\rm red} := \{\Lambda^*_i > 0, \lambda \in \mathbb{T}^2, (\eta^*, \xi^*) \in B^4\}$ endowed with the standard symplectic form $d\Lambda^* \wedge d\lambda^* + d\eta^* \wedge d\xi^*$ (B^4 being a ball aroud the origin in $\mathbb{R}^4$). In fact, *the (standard) Hamilton's equations for $\mathcal{H}^c_{\rm red}$ are immediately recognized to be a subsystem of the full (standard) Hamilton's equations for $\mathcal{H}_{\rm 3bp}$ when the initial data are restricted on $\mathcal{M}^*_{\rm ver}$ and the constant value of Ψ_2 is chosen to be c.*

A.5. Planar Poincaré variables and the planar $(1+n)$–body problem

The planar Poincaré variables for $(1+n)$ co–planar bodies are defined as follows. For $0 \le i \le n$, let $((L_i, G_i), (\ell_i, \hat g_i))$ be the planar Delaunay variables (as defined in § A.2 above) associated to the two–body system Sun–$i^{\rm th}$ planet and let

$$\begin{cases} \Lambda_i = L_i \\ \lambda_i = \ell_i + \hat g_i \ , \end{cases} \qquad \begin{cases} \eta_i = \sqrt{2(L_i - G_i)}\ \cos \hat g_i \\ \xi_i = -\sqrt{2(L_i - G_i)}\ \sin \hat g_i \ . \end{cases}$$

Then, for any $\Lambda_+ > \Lambda_- > 0$ there exists $r > 0$ such that the planar Poincaré variables are symplectic and analytic on the domain

$$\Lambda_- < \Lambda_i < \Lambda_+ \quad \text{for } 0 \le i \le n \ , \qquad (\lambda_i, .., \lambda_n) \in \mathbb{T}^n \ ,$$
$$\eta_i^2 + \xi_i^2 < r^2 \ \ \text{for } 0 \le i \le n \ .$$

If e_i denotes the eccentricity of the (instantaneous or osculating) two–body system Sun–$i^{\rm th}$ planet then

$$\frac{\eta_i^2 + \xi_i^2}{2} = \Lambda_i \,(1 - \sqrt{1 - e_i^2}) \ .$$

For complete details, see, e.g., (Biasco et al, 2005, Appendix A).

B. Appendix: Numerical investigation of the RPC3BP

A complementary numerical study of the stability of the asteroid Victoria has been performed in (Celletti et al, 2004) using frequency analysis as introduced in (Laskar et al 1992), (Laskar, 1993). The dynamical system described by (3.24)–(3.25) has been compared to the system where no truncation of the perturbing function has been performed[16]. If (ω_L, ω_G) are the fundamental frequencies, we denote by $\gamma := |\frac{\omega_L}{\omega_G}|$ the frequency ratio.

[16] In (Celletti et al, 2004), also more realistic models, like those in which Jupiter moves on an eccentric orbit or where the relative inclination of Jupiter and of the asteroid is not neglected, have been considered.

In practice one can proceed as follows. Fix $E = E_0$ and $\varepsilon = \varepsilon_0$; set the initial data as $L = L_0$, $\ell = 0$, $g = g_0$, where L_0, g_0 vary over a grid (which corresponds to consider a slice projection by fixing $\ell = 0$). Find G_0 by solving the relation

$$E_0 = -\frac{1}{2L_0^2} - G + \varepsilon_0 R(L_0, G, 0, g_0) \ .$$

Using the solution of the equations of motion, frequency analysis is implemented to compute (ω_L, ω_G). We remark that according to a standard criterion (see Laskar et al, 1992), the dynamics is discriminated on the basis of the graph of γ versus the initial conditions L_0, g_0. More precisely: a region of invariant tori is characterized by a regular (i.e., monotonically increasing or decreasing) behavior of the frequency–map; no variation of the frequency ratio corresponds to a *resonant regime; a chaotic region* is characterized by consecutive sudden jumps of the frequency map.

Having fixed the energy level according to (3.26), let ε_c be the critical value of the perturbing parameter at which the transition from stability to instability occurs. The results are shown in Table 1, where we provide an interval, say $\varepsilon_c \in [\varepsilon_-, \varepsilon_+]$ such that if $\varepsilon_c \leq \varepsilon_-$, then both lower and upper bounding tori (with frequencies $\omega_\pm$) exist; whenever $\varepsilon_c \geq \varepsilon_+$ we have numerical evidence of the disappearance of both tori. Due to the topology of the model (compare Remark 2.1), for $\varepsilon_c \leq \varepsilon_-$ the motion of the asteroid is confined on the given energy level between the two bounding tori. We also provide an intermediate value at which one of the two tori still survives. The results provided in Table 1 suggest that the truncated model provides a good approximation of the complete model, at least as far as the above energy level and frequencies are considered.

Table 1.

	Truncated	Complete
$\varepsilon_c \in$	$[0.07, 0.09]$	0.08
Intermediate value	$[0.08, 0.1]$	0.09

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