An estimate of smoothing and composition with applications to conjugation problems

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October 20th 2005

Abstract

An estimation of the difference between smoothing a composition of two functions and composing their smoothings is given. The smoothing operator considered here is defined by the convolution operator with a holomorphic kernel. It is also shown, by means of an example, how apply the estimate given here to obtain finite differentiable versions of theorems on conjugation of maps in non-perturbative settings.

Keywords: *smoothing, composition, differentiable functions, conjugation of torus maps, KAM theory.*

1 Introduction

It is known that finite differentiable functions can be approximated by analytic ones. One way of obtaining approximating analytic functions is to use a *smoothing operator* [Kra83, Mos70, Ste70, Zeh75]. Here we consider a smoothing operator defined by the convolution operator with a holomorphic kernel. The main result of this note is an estimate for the norm of the difference between smoothing a composition of two functions and composing their smoothings. We also show how such an estimate can be used in some conjugation problems when the involved functions are only finite differentiable.

The classification – under conjugacy – and the study of the existence of invariant objects for a given dynamical system are important to understand the dynamics of the complete system. Invariant objects and conjugating functions are often found solving functional equations that involve the composition operator and whose unknowns are homeomorphisms. Conjugation problems can be formulated in an abstract form as *generalised implicit function theorems* [Mos66b, Zeh75, Ham82, Van02].

Frequently, in conjugation problems – or in problems concerning the existence of invariant objects – the so-called *'small divisors'* appear in the infinitesimal equations.

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These small divisors yield a reduction of the analyticity domain if the considered functions are analytic and a loss of derivatives in the finite differentiable setting.

It turns out that, in certain cases, it is preferable to deal with reduction of the analyticity domain than loss of differentiability. In particular, it is easier to prove the existence of a solution of the functional equation when the involved functions are analytic. Then, using an idea of Moser [Mos66b], one proves the existence of a solution in the finite differentiable case by constructing a double sequence of approximate solutions: the finite differentiable functions are first approximated by analytic ones, and then the result for the analytic case is applied to the approximating functions.

We call attention to Chapter IV in [Her83] where the author describes a technique to conjugate finite differentiable circle maps with constant type rotation number to the rotation given by the rotation number. The Herman's technique – different from Moser's – enables him to lose only one derivative.

It seems not straightforward a generalisation of the Herman's technique to dimension greater than one.

Since the composition operator does not commute with the convolution operator, in the passage from the analytic to the finite differentiable case, is sometimes necessary to estimate a norm of the difference between smoothing a composition of two functions and composing their smoothings. This happens in certain non-perturbative conjugation results where the main hypothesis is that an approximate solution of the problem is known. Roughly speaking the proof of such results goes as follows: the problem is written as finding zeros of a suitable functional equation defined in appropriate spaces of functions. Assuming that we are given an approximate solution of the functional equation and that the infinitesimal equations are approximately solvable – maybe with some loss of differentiability. Then a modified Newton method is constructed to find a true solution. It turns out that the method is convergent if both the initial error is 'sufficiently' small and the error on resolving the infinitesimal equations is 'quadratic'. We will refer to such proofs as 'polishing'.[1] Some results within this context are [SZ89, CC97, dlLGJV05, dlL05, HdlL04b].

In this work we give an estimate of the supremum norm, computed on complex strips, of the difference between smoothing a composition of two functions and composing their smoothings. The estimation is given in terms of the order of differentiability of the two functions (Theorem 1). We also show how this estimate can be used to obtain a finite differentiable version, from an analytic one, of a non-perturbative result on the conjugation of torus diffeomorphism (see Theorem 2 and Theorem 3 in Section 5 for the formulation of, respectively, the finite differentiable and the analytic version of this result).[2]

The procedure we use to obtain a finite differentiable version of Theorem 3 is essentially the same used in the proof of Theorem 7.1.1 in [Van02] (which is a generalisation of Theorem 2.1 in [Zeh75]), and it consists of the following steps (see Section 5 for more details):

[1]This type of proofs provides the basis for algorithms to compute invariant objects c.f. [HdlL04a] and at the same time they can be used to perform computed assisted proofs c.f. [CC03].

[2]We will not give here any proof of Theorem 3, a polishing proof of it will appear in a future paper [GE05] .

Step 0: Write the conjugacy problem as a functional equation in appropriate function spaces.

Step 1: Smooth both the torus map and the approximate conjugation (which are assumed to be finite differentiable). Apply the estimate in Theorem 1 to obtain an analytic approximate solution of the functional equation.

Step 2: Construct a double approximation following the proof of Theorem 2.1 in [Zeh75] (see also [Mos62, Jac72]).

For readers who are familiar with [Zeh75] we remark that Theorem 1, (which enables us to perform Step 1) proves that the composition operator satisfies the hypothesis F.S4 of Theorem 7.1.1 in [Van02] which improves the condition in Theorem 2.1 in [Zeh75]: the initial solution has to be in the biggest spaces $X_1 \times Y_1$, see Theorem 2.1 in [Zeh75] and Remark 4.3.8 in [Van02].

Conjugation of torus maps to rigid rotations is a special case of problems involving the functional equation

$$F(f, g, \varphi) = 0\,, \tag{1}$$

where

$$F(f, g, \varphi) \stackrel{\text{def}}{=} f \circ g - g \circ \varphi\,. \tag{2}$$

The main idea in Step 1 of the above procedure is the following. Assume that f_0, g_0 and φ_0 are finite differentiable functions which satisfies (1) approximately. Then using estimate given in Theorem 1 one obtains an analytic approximate solution of (1).

We emphasise that in the case that φ in (2) is a fixed rigid rotation on the torus, if g_0 is polynomial, then it is not necessary to use an estimate like that in Theorem 1 to obtain an analytic approximate solution of (1) from f_0, and g_0. For example, perturbative results on conjugacy problems [Arn65, Mos62, Zeh76] satisfy this property, because one can assume g_0 to be the identity. Although, in certain non-perturbative cases it is possible to assume that g_0 is the identity map e.g. [SZ89], in general imposing such condition weakens the obtained results. For example, the conjugation problem considered in Section 5 and that considered in [dlLGE05] where the authors use the scheme described above to give a finite differentiable version of a non-perturbative result on the existence of maximal dimensional invariant tori for exact symplectic maps. [3]

This note is organised as follows. In Section 2 we define the smoothing operator and state Theorem 1 which gives an estimate the norm of the difference between smoothing a composition of two functions and composing their smoothings. Section 3 contains some quantitative properties of the smoothing operator. In Section 4 we apply the results of Section 3, to prove Theorem 1. Finally, in Section 5 we discuss an application of the estimate given in Theorem 1 to conjugations of torus maps to rigid rotations (Theorem 2).

[3] The analytic version of this result can be found in [dlLGJV05].

2 Smoothing and composition

We begin by defining the function spaces we work with. Let $\mathbb{Z}_+$ denote the set of non-negative integers.

Definition 1. *$C^0(\mathbb{R}^d)$ denotes the space of functions $f:\mathbb{R}^d \to \mathbb{R}$ such that*

$$|f|_{C^0} \stackrel{\text{def}}{=} \sup_{x\in\mathbb{R}^d} |f(x)| < \infty\,.$$

Let $\ell = p+\alpha$, with $p\in\mathbb{Z}_+$ and $0<\alpha<1$. Define the Hölder space $C^\ell(\mathbb{R}^d)$ to be the set of all functions $f:\mathbb{R}^d\to\mathbb{R}$ with continuous derivatives up to order p for which the norm

$$|f|_{C^\ell(\mathbb{R}^d)} \stackrel{\text{def}}{=} \sup_{\substack{x\in\mathbb{R}^d\\|k|\le p}} \left\{|D^k f(x)|\right\} + \sup_{\substack{x,y\in\mathbb{R}^d, x\neq y\\|k|=p}} \left\{\frac{|D^k f(x) - D^k f(y)|}{|x-y|^\alpha}\right\} < \infty\,.$$

Following [Mos66b, Zeh75, Sal04] we consider a linear smoothing operator taking functions in $C^\ell(\mathbb{R}^d,\mathbb{R})$ into real entire functions which are bounded in complex strips. For $\rho>0$, let $\mathbb{T}^d_\rho$ denote the complex strip

$$\mathbb{T}^d_\rho = \{x+iy\in\mathbb{C}^d : |y_j|\le\rho, 1\le j\le d\}.$$

Definition 2. *Let $\ell=p+\alpha$, with $p\in\mathbb{Z}_+$ and $0\le\alpha<1$. Define the Banach space $A(\rho, C^\ell, d)$ to be the set of all holomorphic functions $f:\mathbb{T}^d_\rho\subset\mathbb{C}^d\to\mathbb{C}$ which are real valued on $\mathbb{R}^d$ (i.e. $\bar{f(x)} = f(\bar{x})$ for all $x\in\mathbb{R}^d$) and such that the norm*

$$|f|_{\rho,C^\ell} \stackrel{\text{def}}{=} \sup_{\substack{x\in U_\rho\\|k|\le p}} \left\{|D^k f(x)|\right\} + \sup_{\substack{x,y\in\mathbb{T}^d_\rho, x\neq y\\|k|=p}} \left\{\frac{|D^k f(x) - D^k f(y)|}{|x-y|^\alpha}\right\} < \infty\,.$$

We denote by $|\cdot|_\rho$ the norm of $A(\rho, C^0, d)$.

For a matrix or vector valued-function G with components $G_{i,j}$ in either $C^\ell(\mathbb{R}^d)$ or in $A(\rho,C^\ell,d)$ we use the norm $|G|_{C^\ell(\mathbb{R}^d)} \stackrel{\text{def}}{=} \max_{i,j} |G_{i,j}|_{C^\ell(\mathbb{R}^d)}$ or $|G|_{\rho,C^\ell} \stackrel{\text{def}}{=} \max_{i,j}|G_{i,j}|_{\rho,C^\ell}$, respectively. The space of functions $g=(g_1,\dots,g_d):\mathbb{R}^n\to\mathbb{R}^d$ such that $g_i\in C^\ell(\mathbb{R}^n)$, for $i=1,\dots,d$, is denoted by $C^\ell(\mathbb{R}^n,\mathbb{R}^d)$. Now we define the smoothing operator S_t.

Definition 3. *Let d be a natural number and $0<\mathbf{s}\le 1$. Choose $u:\mathbb{R}^d\to\mathbb{R}$ to be C^∞, even, identically equal to 1 in a neighbourhood of the origin and with support contained in the ball with centre in the origin and radius $\mathbf{s}$. Let $\hat{u}:\mathbb{R}^d\to\mathbb{R}$ be the Fourier transform of u and denote by s the holomorphic continuation of $\hat{u}$. Define linear operator S_t as*

$$S_t[f](z) = t^d\int_{\mathbb{R}^d} s(t(y-z))f(y)dy \qquad \text{for} \quad f\in C^0(\mathbb{R}^d). \tag{3}$$

Applying obvious modifications Definition 3 can be extended to a linear operator on $C^0(\mathbb{R}^n, \mathbb{R}^d)$ and on $C^0(\mathbb{R}^n)$. In the sequel these operators are denoted by the same symbol S_t

In Section 3 we state some properties of S_t. From Definition 3 one obtains the following remark.

Remark 1.

1. *S_t takes functions in $C^0(\mathbb{R}^d)$ to the space of entire functions on $\mathbb{C}^d$.*
2. *Using the change of variables $\xi = t\,\mathrm{Re}\,(y-z) = ty - t\,\mathrm{Re}\,(z)$, one has*
$$S_t[f](z) = \int_{\mathbb{R}^d} s(\xi - it\,\mathrm{Im}\,(z)) f(\mathrm{Re}\,(z) + \xi/t) d\xi. \tag{4}$$
3. *S_t maps periodic functions to periodic functions.*
4. *S_t commutes with constant coefficients differential operators.*
5. *The definition of S_t can be extended to functions defined on subsets of $\mathbb{R}^d$ which are connected and have sufficiently regular boundary.*
6. *The definition of the linear operator S_t, given in (3), depends on d. However, we do not include this dependence in the notation.*
7. *$S_t[f]$ is the convolution $s_t * f$ where $s_t(z) = t^d s(tz)$.*

The following theorem provides an estimate for the norm of the difference between the composition of the smoothing and the smoothing of the composition.

Theorem 1. *Let $\ell_1, \ell_2 > 1$ with $\ell_1, \ell_2 \notin \mathbb{N}$. For $f \in C^{\ell_1}(\mathbb{R}^d)$ and $g \in C^{\ell_2}(\mathbb{R}^n, \mathbb{R}^d)$, then the composition $f \circ g$ belongs to $C^{\ell}(\mathbb{R}^n)$ with $\ell = \min(\ell_1, \ell_2)$.*

Moreover, given two real numbers $\beta > 0$, $0 \le \mu < \ell - 1$, there exist two positive constants $\kappa = \kappa(n, d, \ell_1, \ell_2, \beta, \mu)$ and $t^* = t^*(n, \ell_2, \beta)$ such that for every $f \in C^{\ell_1}(\mathbb{R}^d)$, and $g \in C^{\ell_2}(\mathbb{R}^n, \mathbb{R}^d)$, with $|Dg|_{C^0(\mathbb{R}^n)} < \beta$, the following holds [4]
$$|\, S_t[f] \circ S_t[g] - S_t[f \circ g] \,|_{t^{-1}} \le \kappa \, |f|_{C^{\ell_1}(\mathbb{R}^d)} \; t^{-\mu} , \qquad \forall\, t \ge t^* . \tag{5}$$

Remark 2. *Theorem 1 is stated only for Hölder functions, with ℓ_1, ℓ_2 not integer. A natural question is whether it is possible to give an estimate, similar to that given in (5), for Λ_ℓ-*functions *– briefly, Λ_ℓ-*functions *coincide with Hölder functions when $\ell \notin \mathbb{Z}$ and with the space of Zygmund functions when $\ell \in \mathbb{Z}$.*[5] *However the composition of Zygmund functions will not always result in another Zygmund function, for example $f(x) = x \log(|x|) \in \Lambda_1$ but $f^2 \notin \Lambda_1$.*

Remark 3. *Let $\mathbb{T} \stackrel{\text{def}}{=} \mathbb{R}/\mathbb{Z}$ and let $C^{\ell}(\mathbb{T}^d)$ be the subset of functions in $C^{\ell}(\mathbb{R}^d)$ which are $\mathbb{Z}^d$-periodic. Denote by $C^{\ell}(\mathbb{T}^d, \mathbb{R}^d)$ the space of d-dimensional vector functions with coordinates in $C^{\ell}(\mathbb{T}^d)$. We remark that Theorem 1 also holds for the space functions $C^{\ell}(\mathbb{T}^d, \mathbb{R}^d)$. See sections 3 and 5.*

[4] $|\cdot|_{t^{-1}}$ denotes the supremum norm on the complex strip of width t^{-1}, see Definition 2.

[5] For a definition of Λ_ℓ-functions see, for example, Section 4 of Chapter V of [Ste70] or [Kra83].

3 Some properties of S_t

In the present section we give some properties of the operator S_t, defined in (3), that will be the key to establish the necessary estimates to prove Theorem 1. Throughout this section ℓ denotes a positive non integer number. Let us start with some remarks on S_t.

Remark 4.

1. *S_t acts as the identity on polynomials: for any polynomial $P : \mathbb{R}^d \to \mathbb{R}$*

$$S_t[P] = P.$$

2. *Assume that $f \in C^0(\mathbb{R}^d)$ admits a Fourier series expansion*

$$f(x) = \sum_{k \in \mathbb{Z}^d} f_k \, e^{-2i\pi k \cdot x} \,.$$

Then

$$S_t[f](z) = \sum_{k \in \mathbb{Z}^d} u(k/t) f_k \, e^{2\pi i k \cdot z} \,,$$

where u is as in Definition 3.

The following proposition ensures that the operator S_t is an analytic smoothing operator, for a proof see Lemma 2.1 in [Zeh75].

Proposition 5. *There exists a constant $\kappa_1 = \kappa_1(d, \ell)$ such that*

1. $|(S_t - 1)[f]|_{C^0(\mathbb{R}^d)} \leq \kappa_1 \, \|f\|_{C^\ell(\mathbb{R}^d)} \; t^{-\ell}$ *, for $f \in C^\ell(\mathbb{R}^d)$.*
2. $|S_t[f]\,|_{t^{-1}} \leq \kappa_1 \, |f|_{C^0(\mathbb{R}^d)}$*, for $f \in C^0(\mathbb{R}^d)$.*
3. $|(S_\tau - S_t)[f]\,|_{\tau^{-1}} \leq \kappa_1 \; |f|_{C^\ell(\mathbb{R}^d)} \; t^{-\ell}$*, for $f \in C^\ell(\mathbb{R}^d)$, with $\tau \geq t \geq 1$*

Using the fact that S_t commutes with constant coefficients differential operators one proves the following (see Lemma 3 in [Sal04]).

Proposition 6. *Assume that $f \in C^\ell(\mathbb{R}^d)$. Then there exists a constant $\kappa_2 = \kappa_2(d, \ell)$ such that for any $\alpha \in \mathbb{Z}^d_+$, $|\alpha|_1 \leq \ell$, and $|\,\mathrm{Im}\,(z)\,| \leq t^{-1}$ the following inequality holds*

$$\left| D^\alpha \; (\, S_t[f])\,(z) \; - \sum_{|k|_1 \leq \ell - |\alpha|_1} \frac{1}{k\,!} \, D^{k+\alpha} \, f(\mathrm{Re}\,(z))\,(i\,\mathrm{Im}\,(z))^k \right| \leq \; \kappa_2 \; |f|_{C^\ell(\mathbb{R}^d)} \;\; t^{-(\ell - |\alpha\,|_1)} \,, \tag{6}$$

and for $\tau \geq t \geq 1$

$$|D^\alpha \, S_t[f] - D^\alpha \, S_\tau[f]|_{\tau^{-1}} \leq \kappa_2 \; |f|_{C^\ell(\mathbb{R}^d)} \;\; t^{-(\ell - |\alpha\,|_1)} \,.$$

In particular, for analytic functions we have the following estimate.

Proposition 7. *Assume that* $f \in A(t^{-1}, C^\ell, d)$. *Then there exists a constant* $\kappa_3 = \kappa_3(d,\ell)$ *depending on* ℓ *such that*

$$|(S_t - 1)[f]\,|_{t^{-1}} \le \kappa_3\ |f|_{t^{-1},C^\ell}\ t^{-\ell}.$$

Proof. From inequality (6) in Proposition 6, one has for $|\,\mathrm{Im}\,(z)\,| \le t^{-1}$

$$|\,S_t[f](z)\ -\ P_\ell(\mathrm{Re}\,(z)\,, i\,\mathrm{Im}\,(z))\,|\ \le \kappa_2\ |f|_{C^\ell(\mathbb{R}^d)}\ t^{-\ell}\,,$$

where

$$P_\ell(\mathrm{Re}\,(z)\,, i\,\mathrm{Im}\,(z)) = \sum_{|k|_1 \le \ell} \frac{1}{k\,!}\,D^k f(\mathrm{Re}\,(z))\,(i\,\mathrm{Im}\,(z))^k.$$

Moreover, from the Taylor Theorem we have

$$|\,f(z)\ -\ P_\ell(\mathrm{Re}\,(z)\,, i\,\mathrm{Im}\,(z))\,|\ \le c\ |f|_{t^{-1},C^\ell}\ \ |\,\mathrm{Im}\,(z)\,|^\ell\,,$$

for some constant c. Therefore,

$$|(S_t - 1)[f]|_{t^{-1}} \le \kappa_3\ |f|_{t^{-1},C^\ell}\ t^{-\ell}\,.$$

□

One important property of the smoothing operator S_t is that we know an estimate for the norm of $S_t[f]$ on the complex strip $\mathbb{T}^d_{t^{-1}}$ (this is given by part 2 of Proposition 5). However, we need more accurate estimates which we provide in the Lemma 8. Similar estimates for a different smoothing operator are given in the proof of Lemma 2 in [Mos66b].

Lemma 8. *Let* $\mathbf{s}$ *be as in Definition 3. Given nonnegative constants* r, C *and* $\ell > 1$ *there exist positive constants* $\kappa_4 = \kappa_4(d,\ell,C)$, $\kappa_5 = \kappa_5(d,\ell)$, *and* $\kappa_6 = \kappa_6(d,\ell,C,r)$, *such that for all* $t \ge 1$ *with*

$$0 <\ \ \rho(t) \overset{\mathrm{def}}{=} (\mathbf{s}\,t)^{-1}(C + r\log(t)) \le 1 \tag{7}$$

the following holds

1. $|S_{2t}[f] - S_t[f]\,|_{\rho(t)} \le\ \kappa_4\ |f|_{C^\ell(\mathbb{R}^d)}\ t^{-\ell+2\,r}, \quad f \in C^\ell(\mathbb{R}^d)$
2. *If* $f \in C^\ell(\mathbb{R}^d)$ *and* k *is such that* $2^k \le t < 2^{k+1}$, *then*
$$|S_t[f] - S_{2^k}[f]\,|_{\rho(t)} \le \kappa_4\ |f|_{C^\ell(\mathbb{R}^d)}\ 2^{-k(\ell-r)+r}\,.$$
3. $|\,(1 - S_t)\ [\,S_t[f]\,]\,|_{\rho(t)} \le \kappa_5\ |f|_{C^\ell(\mathbb{R}^d)}\ t^{-\ell+r}\,, \quad f \in C^\ell(\mathbb{R}^d)$
4. $|\,S_t[f]\,|_{\rho(t)} \le\ \kappa_4\ |f|_{C^0(\mathbb{R}^d)}\ t^r, \quad f \in C^0(\mathbb{R}^d)$
5. *If* $f \in C^\ell(\mathbb{R}^d)$, $\ell > 2\,r$, *and* t *is sufficiently large then*
$$|\,S_t[f]\,|_{\rho(t)} \le\ \kappa_6\ |f|_{C^\ell(\mathbb{R}^d)}\ .$$

In the proof Lemma 8 the following well known estimate is used.

Lemma 9. *Let s and $\mathbf{s}$ be as in Definition 3. For any m and N there exists a constant $c = c(m, N) > 0$ such that if $|\alpha|_1 \leq m$ then*

$$|D^\alpha s(z)| \leq c\ (1 + |\operatorname{Re}(z)|)^{-N}\, e^{\mathbf{s}\,|\operatorname{Im}(z)|}\,.$$

Proof of Lemma 8 . We follow the ideas given in the proof of Lemma 2.1 in [Zeh75] (see also [Sal04, Mos66b]). Expand the function $f \in C^\ell(\mathbb{R}^d)$ in Taylor series

$$f(x+\eta) = P_p(x,\eta) + R(x,\eta),$$

where

$$P_p(x,\eta) = \sum_{\substack{k\in\mathbb{Z}^m_+\\ |k|_1\leq p}} \frac{1}{k\,!}\, D^k\, f(x)\, \eta^k\,.$$

and R is the integral remainder. Applying the Taylor series to $f(\operatorname{Re}(z) + \xi/t)$ in (4) and using that S_t acts as the identity on polynomials one obtains

$$S_t[f](z) = P_p(\operatorname{Re}(z)\,, i\ \operatorname{Im}(z)) + \hat{R}_f(z,t), \tag{8}$$

where

$$\hat{R}_f(z,t) = \int s\,(\xi - i\,t\operatorname{Im}(z))\ R(\operatorname{Re}(z)\,,\ \xi/t)\,d\,\xi. \tag{9}$$

From the Taylor Theorem, one has

$$|R(\operatorname{Re}(z)\,,\ \xi/t)| \leq c\ |f|_{C^\ell(\mathbb{R}^d)}\ |\xi/t|^\ell\,,$$

where $c = c(d, \ell)$. Then from (9) we have

$$|\hat{R}_f(z,t)| \leq\ t^{-\ell}\ |f|_{C^\ell(\mathbb{R}^d)}\ \phi_\ell(t\ \operatorname{Im}(z))\,, \tag{10}$$

where

$$\phi_\ell(\eta) \stackrel{\text{def}}{=} c\int_{\mathbb{R}^d} |s(\xi - i\eta)|\,|\xi|^\ell\,d\xi\,, \tag{11}$$

The key point is to bound $\phi_\ell(t\ \operatorname{Im}(z))$ in (10). This is achieved by Lemma 9. Indeed, fixing $N > 0$ and $\alpha = 0$, Lemma 9 implies

$$\begin{aligned}\phi_\ell(t\ \operatorname{Im}(z)) &= c\int_{\mathbb{R}^d} |s(\xi - i\,t\ \operatorname{Im}(z))|\,|\xi|^{\,\ell}\,d\xi\\ &\leq c\,e^{\mathbf{s}\,|t\ \operatorname{Im}(z)|}\int_{\mathbb{R}^d}\frac{|\xi|^{\,\ell}}{(1+|\xi|)^N}d\xi.\end{aligned}$$

Hence

$$|\phi_\ell(t\ \operatorname{Im}(z))| \leq c\,e^{\mathbf{s}\,t\,|\ \operatorname{Im}(z)|}\int_{\mathbb{R}^d}\frac{|\xi|^\ell}{(1+|\xi|)^N}\,d\xi. \tag{12}$$

We first prove part 1 of Lemma 8. Notice that

$$S_{2\,t}[f](z) = P_p(\mathrm{Re}\,(z)\,, i\ \mathrm{Im}\,(z)) + \hat{R}_f(z, 2\,t)\,.$$

Therefore, if $|\,\mathrm{Im}\,(z)\,| \leq \rho(t)$, using (10) and (12) one obtains

$$\begin{aligned}
|(S_{2\,t}[f] - S_t[f]\,)(z)| &\leq |\hat{R}_f(z, 2t)| + |\hat{R}_f(z, t)| \\
&\leq t^{-\ell}\ |f|_{C^\ell(\mathbb{R}^d)}\,[\,\phi_\ell(2t\ \mathrm{Im}\,(z))\ +\ \phi_\ell(t\ \mathrm{Im}\,(z))\,] \\
&\leq \hat{\kappa}_4\ t^{-\ell+2r}\ \ |f|_{C^\ell(\mathbb{R}^d)}\ ,
\end{aligned}$$

with $\hat{\kappa}_4 \stackrel{\mathrm{def}}{=} 2c\,e^{2\,C}\int_{\mathbb{R}^d}\frac{|\xi|^\ell}{(1+|\xi|)^N}\,d\xi$. This proves part 1.

Let us prove part 2 of Lemma 8 Let k be a positive integer such that $2^k \leq t < 2^{k+1}$. From equalities (8), (10), and (12) we have for $|\,\mathrm{Im}\,(z)\,| < \rho(t)$

$$\begin{aligned}
|S_t[f](z) - S_{2^k}[f](z)| &= |\hat{R}_f(z, t) + \hat{R}_f(z, 2^k)\,| \\
&\leq |f|_{C^\ell(\mathbb{R}^d)}\ \left(t^{-\ell} + 2^{-k\ell}\right)\ \phi(t\,\mathrm{Im}\,(z)) \\
&\leq |f|_{C^\ell(\mathbb{R}^d)}\ 2^{-k\,\ell}\,\hat{\kappa}_4\,t^r \\
&\leq \hat{\kappa}_4\ |f|_{C^\ell(\mathbb{R}^d)}\ 2^{-k(\ell-r)+r}\,.
\end{aligned}$$

This proves 2.

Now we prove part 3 of Lemma 8. From equality (4) we have

$$(1 - S_t)[\,S_t[f]\,](z) = \int_{\mathbb{R}^d} s(\xi - it\,\mathrm{Im}\,(z))\ (f(\mathrm{Re}\,(z) + \xi/t) - S_t[f](\mathrm{Re}\,(z) + \xi/t))\ d\xi\,. \tag{13}$$

From equality (8) we have for $x \in \mathbb{R}^d$

$$S_t[f](x) = P_p(x, 0) + \hat{R}_f(x, t) = f(x) + \hat{R}_f(x, t)$$

Then using (13), (10) and (12) one obtains for $|\,\mathrm{Im}\,(z)\,| < \rho(t)$

$$\begin{aligned}
|(1 - S_t)[\,S_t[f]\,](z)| &= \left|\int_{\mathbb{R}^d} s(\xi - it\,\mathrm{Im}\,(z))\,\hat{R}_f(\mathrm{Re}\,(z) + \xi/t, t)d\xi\right| \\
&\leq t^{-\ell}\ |f|_{C^\ell(\mathbb{R}^d)}\ \phi_\ell(0)\,\phi_0(t\ \mathrm{Im}\,(z)) \\
&\leq \kappa_5\,|f|_{C^\ell(\mathbb{R}^d)}\ t^{-\ell+r},
\end{aligned}$$

and this proves part 3.

Next we consider part 4 of Lemma 8. From (4) and (11) we have for $f \in C^0(\mathbb{R}^d)$ and $|\,\mathrm{Im}\,(z)\,| < \rho(t)$

$$\begin{aligned}
|S_t[f](z)| &\leq |f|_{C^0(\mathbb{R}^d)}\ c^{-1}\phi_0(t\ \mathrm{Im}\,(z)) \\
&\leq |f|_{C^0(\mathbb{R}^d)}\ t^r\ \left(e^C\int_{\mathbb{R}^d}\frac{1}{(1 + |\xi|)^N}\ d\xi\right) \\
&\leq \kappa_4\,|f|_{C^0(\mathbb{R}^d)}\,t^r\,.
\end{aligned}$$

Finally, we prove part 5 of Lemma 8. Let k be such that $2^k \le t < 2^{k+1}$, since $\rho(t)$ goes to zero as t goes to infinity, we can assume that t is is sufficiently large so that $\rho(2^i) \ge \rho(t)$ for all $i = 0, \ldots, k$. Then, using part 1 and part 2 of Lemma 8 one has

$$\begin{aligned}
|S_t[f]|_{\rho(t)} &\le |S_t[f] - S_{2^k}[f]|_{\rho(t)} + \sum_{j=2}^{k} |S_{2^j}[f] - S_{2^{j-1}}[f]\,|_{\rho(2^{j-1})} + |S_1[f]\,|_{\rho(1)} \\
&\le \kappa_4\ |f|_{C^\ell(\mathbb{R}^d)}\, 2^{-k(\ell-r)+r} + \kappa_4\ |f|_{C^\ell(\mathbb{R}^d)} \sum_{j=0}^{k-1} 2^{-j(\ell-2r)} + |S_1[f]\,|_{\rho(1)} \\
&\le\ 2^r\, \kappa_4\ |f|_{C^\ell(\mathbb{R}^d)} \sum_{j=0}^{\infty} 2^{-j(\ell-2r)} + |S_1[f]\,|_{\rho(1)} \\
&\le (\hat{\kappa}_4 + \kappa_1)\, |f|_{C^\ell(\mathbb{R}^d)}\ ,
\end{aligned}$$

where we have used that $\ell > 2r$. This proves part 5, and finishes the proof of Lemma 8. □

Proposition 10. *Let C be non negative constant, $r > 0$, and let κ_4 be as in Lemma 8. For each $f \in C^\ell(\mathbb{R}^d)$ and $0 \le \mu < \ell$, there exists a constant and $\kappa_7 = \kappa_7(d, \ell, \mu, r, C)$ such that if $t \ge e^{1/(\mathbf{s}\, r)}$, and $t^{-1}(C + r\log(t)) \le 1$ then the following hold*

1. $|S_{2t}[f] - S_t[f]\,|_{(\mathbf{s}\, t)^{-1}C,\, C^\mu} \le\ \kappa_4\ |f|_{C^\ell(\mathbb{R}^d)}\ \ t^{-\ell+\mu+2\, r}\,.$
2. *If k is such that $2^k \le t < 2^{k+1}$, then*
$$|S_t[f] - S_{2^k}[f]\,|_{(\mathbf{s}\, t)^{-1}C,\, C^\mu} \le \kappa_4\ |f|_{C^\ell(\mathbb{R}^d)}\ 2^{-k(\ell-\mu-r)+r+\mu}.$$
3. *If $\ell > 2\, r\ + \mu$, then $|\, S_t[f]\,|_{(\mathbf{s}t)^{-1}C,\, C^\mu} \le \kappa_7\, |f|_{X_0^\ell}$.*

Proof. This is a consequence of Cauchy's inequalities and Lemma 8. Indeed, if $t \ge e^{1/(\mathbf{s}\, r)}$, then we have

$$\begin{aligned}
|S_{2t}[f] - S_t[f]\,|_{\mathbf{s}\, t^{-1}C,\, C^\mu} &\le (\mathbf{s}\, t^{-1} r\ \log t)^{-\mu}\ |S_{2t}[f] - S_t[f]\,|_{\mathbf{s}\, t^{-1}(C + r\log(t))} \\
&\le\ \kappa_4\ |f|_{C^\ell(\mathbb{R}^d)}\ \ t^{\mu-\ell+2\, r}\,.
\end{aligned}$$

Similarly, if k is such that $2^k \le t < 2^{k+1}$, then

$$\begin{aligned}
|S_t[f] - S_{2^k}[f]\,|_{t^{-1}\, C,\, C^\mu} &\le (\mathbf{s}\, t^{-1} r\ \log t)^{-\mu}\, |S_t[f] - S_{2^k}[f]\,|_{\mathbf{s}\, t^{-1}(C + r\log(t))} \\
&\le 2^{r+\mu}\, \kappa_4\ |f|_{C^\ell(\mathbb{R}^d)}\ 2^{-k(\ell-\mu-r)}\,.
\end{aligned}$$

Finally, assume that $\ell > 2\, r + \mu$ and let k be such that $2^k \le t < 2^{k+1}$, then Cauchy's inequalities and parts 1 and 2 of Proposition 10 imply

$$
\begin{aligned}
|S_t[f]|_{t^{-1}C,\,C^\mu} &\le |S_t[f]-S_{2^k}[f]|_{t^{-1}C,\,C^\mu} + \sum_{j=2}^{k} |S_{2^j}[f]-S_{2^{j-1}}[f]\,|_{2^{-j}C,\,C^\mu} + |S_1[f]\,|_{t^{-1}C,\,C^\mu} \\
&\le \tilde{\kappa}_6 \; |f|_{C^\ell(\mathbb{R}^d)} \sum_{j=0}^{k} 2^{-j(\ell-2\,r-\mu)} + \left(C - Ct^{-1}\right)^{-\mu} \, |S_1[f]\,|_C \\
&\le \tilde{\kappa}_6 \; |f|_{C^\ell(\mathbb{R}^d)} \sum_{j=0}^{\infty} 2^{-j(\ell-2\,r-\mu)} + C^{-\mu} \left(1-t^{-1}\right)^{\mu} \, |S_1[f]\,|_C \\
&\le \kappa_6 \;\; |f|_{C^\ell(\mathbb{R}^d)} \;.
\end{aligned}
$$

□

Remark 11. *Take $r \in (0,1)$ in Proposition 10. Then part 3 in Proposition 10 establishes that for any $C > 0$, $0 < \mu < \ell - 2\,r$, and t satisfying $(\mathbf{s}\,t)^{-1}\,(C + log(t^r)) \le 1$ and $t \ge e^{1/(\mathbf{s}\,r)}$ the analytic function $S_t[f]$ belongs to $A(C(\mathbf{s}\,t)^{-1}, C^\mu, d)$, and moreover*

$$
|\,S_t[f]\,|_{(\mathbf{s}\,t)^{-1}C,\,C^\mu} \le \kappa_7 \; |f|_{X_0^\ell} \;,
$$

for some constant κ_7 depending on d, ℓ, μ, r, and C.

Notice that for any $f \in C^\ell(\mathbb{R}^d)$, $S_t[f]$ is an entire function. For any $f \in C^\ell(\mathbb{R}^d)$, and that we know bounds of $S_t[f]$ on complex strips (see Proposition 5 and Remark 11). This enables us to bound the norm of the imaginary part of $S_t[f]$ in the complex strip $\mathbb{T}^d_{t^{-1}}$ as follows.

Lemma 12. *Assume $f \in C^\ell(\mathbb{R}^d)$, with $\ell > 1$. Then for any $t \ge 1$*

$$
|\mathrm{Im}\,(S_t[f]\,)|_{t^{-1}} \le \; t^{-1} \; |DS_t[f]\,|_{t^{-1}} \;.
$$

Proof. For any function satisfying $\overline{g(z)} \;=\; g(\bar{z})$, we have

$$
\mathrm{Im}\,(g(z)) \;=\; \frac{1}{2\,i}\,(g(\xi(1,z)) - g(\xi(0,z)))\,.
$$

were $\xi(t,z) = \bar{z} + t(z-\bar{z})$. Then, applying the Mean Value Theorem we have

$$
|\,\mathrm{Im}\,(g(z))\;| \le \; \frac{1}{2}\Big|\int_0^1 \frac{d}{dt}\,(g(\xi(t,z)))\;dt\Big| \le \;|\,\mathrm{Im}\,(z)\,|\,|Dg(z)|.
$$

□

4 Proof of Theorem 1

Throughout this section we assume that $\ell_1, \ell_2 > 1$ are positive non integer numbers and $\ell \stackrel{\mathrm{def}}{=} \min(\ell_1, \ell_2)$. Let us start with some properties of the composition operator.

Proposition 13.

1. *If $f \in C^{\ell_1}(\mathbb{R}^d)$ and $g \in C^{\ell_2}(\mathbb{R}^n, \mathbb{R}^d)$, the composition $f \circ g$ is an element of $C^{\ell}(\mathbb{R}^n)$ with $\ell = \min(\ell_1, \ell_2)$.*

2. *Given $C \geq 1$, $f \in A(\rho', C^0, d)$, and ρ such that $\rho C \leq \rho'$, then for any $g \in \left[A(\rho, C^0, n)\right]^d$ with $|Dg|_\rho \leq C$ the following holds*
$$|f \circ g|_\rho \leq |f|_{C\,\rho} \leq |f|_{\rho'} \; .$$

3. *Given $C \geq 1$, $f \in A(\rho', C^\mu, d)$, with $\mu > 1$, and ρ such that $\rho C \leq \rho'$, then for any $g \in [A(\rho, C^\mu, n)]^d$ with $|Dg|_\rho \leq C$ the following holds*
$$\begin{aligned} |f \circ g|_{\rho, C^\mu} &\leq \kappa_8 \; |f|_{C\,\rho, C^\mu} \, (1 + |g|^\mu_{\rho, C^\mu}) \\ &\leq \kappa_8 \; |f|_{\rho', C^\mu} \, (1 + |g|^\mu_{\rho, C^\mu}) \, , \end{aligned} \tag{14}$$
for some constant $\kappa_8 = \kappa_8(d, \mu)$.

4. *Given $0 < \alpha < \beta < 1$, and $f : \mathbb{R}^d \to \mathbb{R} \in C^{1+\beta}$, for every $g_1 : \mathbb{R}^n \to \mathbb{R}^d \in C^{1+\alpha}$ there exists a $\delta > 0$ such that if $g_2 : \mathbb{R}^n \to \mathbb{R}^d \in C^{1+\alpha}$ with $|g_1 - g_2|_{C^0(\mathbb{R}^n)} < \delta$, then*
$$|f \circ g_1 - f \circ g_2|_{C^0(\mathbb{R}^n)} \leq \kappa_9 \; |f|_{C^1(\mathbb{R}^d)} \; |g_1 - g_2|_{C^0(\mathbb{R}^n)} \; , \tag{15}$$
where κ_9 is a constant.

Proof. 2 is straightforward. A proof of 1, 3, and 4 can be found in [dlLO99]. We just remark that 3 follows from Theorem 4.3 in [dlLO99] because complex strips are compensated domains. $\square$

We now prove Theorem 1. Let $f \in C^{\ell_1}(\mathbb{R}^d)$ and $g \in C^{\ell_2}(\mathbb{R}^n, \mathbb{R}^d)$. In order to prove inequality (5) we decompose $S_t[f] \circ S_t[g] - S_t[f \circ g]$ as follows
$$\begin{aligned} S_t[f] \circ S_t[g] - S_t[f \circ g] = S_t[f] \circ S_t[g] &- S_t[\; S_t[f] \circ S_t[g] \;] + \\ &+ S_t\,[\, S_t[f] \circ S_t[g] - S_t[f] \circ g \,] + \\ &+ S_t\,[\; S_t[f] \circ g - f \circ g \;] \, . \end{aligned} \tag{16}$$
It is easy to estimate the second and the third terms on the right hand side of (16). Indeed, from part 1 of Proposition 5 one has
$$|(S_t - 1)[g]|_{C^0(\mathbb{R}^n)} \leq \kappa_1(n, \ell_2) \; |g|_{C^{\ell_2}(\mathbb{R}^n)} \;\; t^{-\ell_2} \, .$$
Then there exists $t^* = t^*(n, \ell_2, |g|_{C^{\ell_2}(\mathbb{R}^n)})$ such that for $t \geq t^*$
$$\begin{aligned} |S_t\,[\; S_t[f] \circ S_t[g] - S_t[f] \circ g \;]|_{t^{-1}} &\leq \kappa_1(n, \ell) \; |S_t[f] \circ S_t[g] - S_t[f] \circ g|_{C^0(\mathbb{R}^n)} \\ &\leq \kappa_1(n, \ell) \, \kappa_9 \; |S_t[f]|_{C^1(\mathbb{R}^n)} \;\; |(S_t - 1)[g]|_{C^0(\mathbb{R}^n)} \\ &\leq \; \kappa_{10} \; |f|_{C^{\ell_1}(\mathbb{R}^d)} \; |g|_{C^{\ell_2}(\mathbb{R}^n)} \;\; t^{-\ell_2} \, , \end{aligned}$$
where the first inequality follows from part 2 of Proposition 5, the second from (15), and in the third inequality we have used that S_t commutes with the derivate operator to bound $|S_t[f]|_{C^1}$.

Similarly, for the fourth term on the right hand side of (16) we have the following estimate

$$\begin{aligned}
|S_t\,[\,S_t[f]\circ g - f\circ g\,]\,|_{t^{-1}} &\leq\ \kappa_1(n,\ell)\ |S_t[f]\circ g - f\circ g\,|_{C^0(\mathbb{R}^n)} \\
&\leq \kappa_1(n,\ell)\ |(S_t-1)[f]\,|_{C^0(\mathbb{R}^d)} \\
&\leq \kappa_{11}\ |f|_{C^{\ell_1}(\mathbb{R}^d)}\ \ t^{-\ell_1}\,.
\end{aligned}$$

Therefore, for $t \geq t^* = t^*(n,\ell_2,|g|_{C^{\ell_2}(\mathbb{R}^n)})$

$$|S_t\,[\,S_t[f]\circ S_t[g] - S_t[f]\circ g\,]|_{t^{-1}} \leq \kappa_{10}\ |f|_{C^{\ell_1}(\mathbb{R}^d)}\ |g|_{C^{\ell_2}(\mathbb{R}^n)}\ t^{-\ell}\,, \tag{17}$$

and

$$|S_t\,[\,S_t[f]\circ g - f\circ g\,]\,|_{t^{-1}} \leq \kappa_{11}\ |f|_{C^{\ell_1}(\mathbb{R}^d)}\ t^{-\ell}\,, \tag{18}$$

where $\ell = \min(\ell_1,\ell_2)$, and $\kappa_{10} = \kappa_{10}(d,n,\ell_1,\ell_2)$, $\kappa_{11} = \kappa_{11}(d,n,\ell_1,\ell_2)$ are constants.

Hence in order to prove inequality (5) it is enough to bound the first two terms on the right hand side of (16). That is

$$S_t[f]\circ S_t[g] - S_t[\ S_t[f]\circ S_t[g]\ ] = (1-S_t)\,(S_t[f]\circ S_t[g])\ ,$$

This is achieved by the following two lemmas.

Lemma 14. *Given $\beta > 0$ and $0 \leq \mu < \ell - 1$ there exist two constants $\hat{t} = \hat{t}(n,\ell_2,\beta) > 1$ and $\kappa_{12} = \kappa_{12}(n,d,\ell_1,\ell_2,\beta,\mu)$ such that for any $f \in C^{\ell_1}(\mathbb{R}^d)$, $g \in C^{\ell_2}(\mathbb{R}^n,\mathbb{R}^d)$, with $|\,Dg|_{C^0(\mathbb{R}^n)} < \beta$, and $t \geq \hat{t}$ the composition $S_t[f]\circ S_t[g]$ belongs to $A(t^{-1},C^\mu,n)$ and*

$$|S_t[f]\circ S_t[g]\,|_{t^{-1},\,C^\mu} \leq \kappa_{12}\ |f|_{C^{\ell_1}(\mathbb{R}^d)}\,.$$

Proof. From Lemma 12 and part 2 of Proposition 5 we have that there exists a constant $c = c(n,\ell_2)$ such that

$$\begin{aligned}
|\mathrm{Im}\,(S_t[g])|_{t^{-1}} &\leq\ |\,DS_t[g]\,|_{t^{-1}}\ t^{-1} \\
&\leq\ |\,S_t[Dg]\,|_{t^{-1}}\ t^{-1} \\
&\leq\ \kappa_1(n,\ell_1-1)\,|Dg\,|_{C^0(\mathbb{R}^n)}\ t^{-1} \\
&\leq c\,\beta\,t^{-1}\,.
\end{aligned}$$

Therefore, if $C \stackrel{\text{def}}{=} c\,\beta\mathbf{s}$, then from Remark 11 we have that if $\hat{t}$ satisfies

$$\hat{t} \geq e^{2/\mathbf{s}}\,, \qquad (\mathbf{s}\,\hat{t}\,)^{-1}\left(C + \log\left(\hat{t}^{1/2}\right)\right) \leq 1$$

and $0 \leq \mu < \ell - 1$ then $S_t[g] \in \left[\,A(t^{-1},C^\mu,n)\,\right]^d$ and $S_t[f] \in A(C\,(\mathbf{s}\,t)^{-1},C^\mu,d)$, for any $t \geq \hat{t}$. Thus from (14) one obtains

$$\begin{aligned}
|S_t[f]\circ S_t[g]\,|_{t^{-1},\,C^\mu} &\leq \kappa_8\ \left(1 + |S_t[g]\,|^{\mu}_{t^{-1},\,C^\mu}\right)\ |S_t[f]\,|_{|\mathrm{Im}(S_t[g])|_{t^{-1}}\,,C^\mu} \\
&\leq \kappa_8\ \left(1 + |S_t[g]\,|^{\mu}_{t^{-1},\,C^\mu}\right)|S_t[f]\,|_{t^{-1}\,C\,,C^\mu} \\
&\leq \kappa_8\,\kappa_7\ \left[1 + \left(\kappa_7\ |g|_{C^{\ell_2}(\mathbb{R}^n)}\right)^{\mu}\right]\ |f|_{C^{\ell_1}(\mathbb{R}^d)}
\end{aligned}$$

where we have used Remark 11 to obtain the last inequality. ∎

Lemma 15. *Given $\beta > 0$ and $0 \le \mu < \ell - 1$ there exist two constants $\hat{t} = \hat{t}(n, \ell_2, \beta) > 1$ and $\kappa_{13} = \kappa_{13}(n, d, \ell_1, \ell_2, \beta, \mu)$ such that for any $f \in C^{\ell_1}(\mathbb{R}^d)$, $g \in C^{\ell_2}(\mathbb{R}^n, \mathbb{R}^d)$, with $|\, Dg|_{C^0(\mathbb{R}^n)} < \beta$, the following holds*

$$|\,(1 - S_t)\,(S_t[f] \circ S_t[g])\,|_{t^{-1}} \le \kappa_{13}\, |f|_{C^\ell(\mathbb{R}^d)}\; t^{-\mu}\,, \qquad \forall \quad t \ge t^*.$$

Proof. This is an immediate consequence of Lemma 14 and Proposition 7. Indeed, let $0 \le \mu < \ell - 1$, form Lemma 14 we have $S_t[f] \circ S_t[g] \in A(t^{-1}, C^\mu, n)$, then Proposition 7 implies

$$\begin{aligned} |(1 - S_t)\,(S_t[f] \circ S_t[g])\,|_{t^{-1}} &\le \kappa_3\; |S_t[f] \circ S_t[g]\,|_{t^{-1}, C^\mu}\;\; t^{-\mu} \\ &\le \kappa_3\, \kappa_{12}\; |f|_{C^{\ell_1}(\mathbb{R}^d)}\;\; t^{-\mu}. \end{aligned}$$

□

Theorem 1 follows from inequalities (17), (18), and Lemma 15.

5 An Application: torus maps

In this section, as an application to Theorem 1, we consider the problem of conjugation of finite differentiable torus maps to rigid rotations: $T_\omega : \theta \to \theta + \omega$, $\omega \in \mathbb{T}^d$. More precisely, given a finite differentiable torus diffeomorphism f and a Diophantine[6] frequency vector ω, we consider the solvability of the non-linear functional equation

$$f \circ h = h \circ T_\omega + \lambda\,, \tag{19}$$

where h and λ are the unknowns. The main assumption we do is the existence of an approximate solution (h_0, λ_0) of (19), where h is finite differentiable and satisfies a non-degeneracy condition (see Theorem 2).

We emphasise that we do not assume that f is a perturbation of a rotation. In fact, as we mentioned in the introduction of this work, if f is a perturbation of a rotation map, to obtain a finite differentiable conjugation result from the analytic one – by using the Moser's method – we do not need an estimate of the type given in Theorem 1 [Mos62, Mos66a, Zeh76, Sal04].

5.1 Smoothing torus maps

Given a continuous torus map $f : \mathbb{T}^d \to \mathbb{T}^d$, a *lift* of f to $\mathbb{R}^d$ (the universal cover of $\mathbb{T}^d$) is a continuous map $\hat{f} : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\pi \circ \hat{f} = f \circ \pi\,, \tag{20}$$

where π is the covering map

$$\pi : \mathbb{R}^d \to \mathbb{T}^d, \qquad \pi(x) = x \mod \mathbb{Z}^d. \tag{21}$$

[6]See Definition 4 in Section 5.2.

Proposition 16. *Given a continuous torus map $f : \mathbb{T}^d \to \mathbb{T}^d$ any lift $\hat{f} : \mathbb{R}^d \to \mathbb{R}^d$ has the form*

$$\hat{f}(x) = A\,x + u(x) \tag{22}$$

where $A \in \mathcal{M}_{d\times d}(\mathbb{Z}^d)$ and $u \in C^0\left(\mathbb{T}^d, \mathbb{R}^d\right)$ (here $\mathcal{M}_{d\times d}(\mathbb{Z}^d)$ is the set $d \times d$ integer valued matrices and $C^0\left(\mathbb{T}^d, \mathbb{R}^d\right)$ denotes the set functions in $C^0\left(\mathbb{R}^d, \mathbb{R}^d\right)$ which are Z^d-periodic). Moreover, if f has additional regularity the corresponding periodic function u has the same regularity.

Let $\mathrm{Diff}^{\,0}(\mathbb{R}^d)$ and $\mathrm{Diff}^{\,0}(\mathbb{T}^d)$ denote the group of homeomorphisms of $\mathbb{R}^d$ and $\mathbb{T}^d$, respectively. For $r \geq 0$, let $\mathrm{Diff}^{\,r}(\mathbb{R}^d)$ and $\mathrm{Diff}^{\,r}(\mathbb{T}^d)$ denote the set of $C^{\,r}$-diffeomorphisms of $\mathbb{R}^d$ and $\mathbb{T}^d$, respectively. For $r \in [1, \infty) \cup \{0\}$, $D^{\,r}(\mathbb{R}^d)$ denotes the subset of diffeomorphism in $\mathrm{Diff}^{\,r}(\mathbb{R}^d)$ that can be written in the form (22). And for $0 < r < 1$, $D^{\,r}(\mathbb{R}^d)$ denotes the set of $C^{\,r}$-diffeomorphisms $\tilde{f} \in \mathrm{Diff}^{\,0}(\mathbb{R}^d)$ such that $\tilde{f}$ and $\tilde{f}^{-1}$ can be written in the form (22).

Remark 17. *Notice that any diffeomorphism $\hat{f} \in D^{\,r}(\mathbb{R}^d)$ define a torus diffeomorphism $f \in \mathrm{Diff}^r(\mathbb{T}^d)$ such that $f \circ \pi = \pi \circ \hat{f}$, with π defined in (21). Moreover, even though lifts of continuous torus diffeomorphism are not unique, they differ from a constant vector in $\mathbb{Z}^d$. This enables us to work with lifts of torus maps. For notational reasons we will the same letter to denote the torus map and a lift of it.*

Let $f \in D^{\,r}(\mathbb{R}^d)$ with $f(x) = Ax + u(x)$ and let S_t be as in Definition 3. Notice that (because S_t acts as the identity on polynomials)

$$S_t[f](z) = A\,z + S_t[u](z)\,. \tag{23}$$

Let $\mathcal{P}(\rho, C^\ell)$ the subset of functions in $A(\rho, C^\ell, d)$ (see Definition 2) which are $\mathbb{Z}^d$-periodic. Since S_t takes periodic functions in periodic functions one has that if $u \in C^\ell(\mathbb{T}^d)$ then $S_t[u] \in \mathcal{P}(t^{-1}, C^0)$. Moreover, all the properties of S_t given in Section 3 as well as Theorem 1 hold for functions in $C^\ell(\mathbb{T}^d)$. Hence one has the following result.

Corollary 18. *Let $\ell_1, \ell_2 > 1$ with $\ell_1, \ell_2 \notin \mathbb{N}$. For $f \in D^{\ell_1}(\mathbb{T}^d)$, $g \in D^{\ell_2}(\mathbb{T}^d)$ the composition $f \circ g$ belongs to $D^\ell(\mathbb{T}^d)$ with $\ell = \min(\,\ell_1, \ell_2)$.*

Moreover, let κ_1 be as in Proposition 5, given two real numbers $\beta > 0$, $0 \leq \mu < \ell - 1$, there exist two positive constants $\kappa = \kappa(n, d, \ell_1, \ell_2, \beta, \mu)$ and $t^* = t^*(n, \ell_2, \beta)$ such that for every $f = A + u \in D^{\ell_1}(\mathbb{T}^d)$ and $g = B + v \in D^{\ell_2}(\mathbb{T}^d)$, with $\kappa_1\, |Dv|_{C^0(\mathbb{T}^d)} < \beta - |B|$, the following holds

$$|\, S_t[f] \circ S_t[g] - S_t[f \circ g]\, |_{t^{-1}} \leq\, \kappa\; |u|_{C^{\ell_1}(\mathbb{T}^d)}\; t^{-\mu}\,, \qquad \forall\, t\, \geq t^*\,.$$

Proof. Notice that with the hypothesis of Corollary 18 one has

$$S_t[f] \circ S_t[g] - S_t[f \circ g] \,= S_t[u] \circ S_t[g] - S_t[u \circ g] \in C^\ell(\mathbb{T}^d)\,,$$

with $\ell = \min(\ell_1, \ell_2)$. Moreover, for any z with $|\mathrm{Im}\,(z)| \leq t^{-1}$ one has

$$\begin{aligned} |\,\mathrm{Im}\,(S_t[g])| &= |\; B\;\mathrm{Im}\,(z) + \mathrm{Im}\,(S_t[v](z))\;| \\ &\leq |B|\; t^{-1} + \kappa_1\; |Dv|_{C^0(\mathbb{T}^d)}\; t^{-1} \\ &\leq \beta\, t^{-1}\,. \end{aligned}$$

The proof is finished following the same lines of the proof to Theorem 1 given in Section 4. $\square$

5.2 A non-perturbative conjugation theorem

Definition 4. *Given $\gamma > 0$ and $\sigma \geq d$, we define $D(\gamma, \sigma)$ as the set of frequency vectors $\omega \in \mathbb{R}^d$ satisfying the Diophantine condition:*

$$| k \cdot \omega - m| \geq \gamma |k|_1^{-\sigma} \qquad \forall \ell \in \mathbb{Z}^d \setminus \{0\}, m \in \mathbb{Z},$$

where $|k|_1 = |k_1| + \cdots + |k_d|$.

Given a map $f \in D^0(\mathbb{T}^d)$ we use the following notation

$$\operatorname{avg}\{f\}_\theta \stackrel{\text{def}}{=} \int_{[0,1]^d} f(x)dx \,.$$

Taking coordinate-function the above notation is extended to matrix or vector valued-functions G with components $G_{i,j} \in D^0(\mathbb{T}^d)$.

We will prove the following non-perturbative conjugation result.

Theorem 2. *Let $\omega \in D(\gamma, \sigma)$, for some $\gamma > 0$ and $\sigma \geq d$, let ℓ, μ and q be such that $2(\sigma + 1) \leq q < \mu < \ell - 1$. and such that ℓ and $\ell - \sigma$ are not integer.*

Let $f = A + u \in D^\ell(\mathbb{T}^d)$, $H_0 = B + v_0 \in D^\ell(\mathbb{T}^d)$, and $\lambda_0 \in \mathbb{R}^d$ be fixed. Define the error function

$$E_0 \stackrel{\text{def}}{=} f \circ H_0 - H_0 \circ T_\omega - \lambda_0 \,.$$

Assume that the following hypothesis hold

1. *$E_0 \in C^\ell(\mathbb{T}^d)$.*
2. *$\kappa_1 \, |Dv_0|_{C^0(\mathbb{T}^d)} < \beta - |B|$, for some $\beta > |B|$, where κ_1 is as in Proposition 5.*
3. *The matrix $\Phi \stackrel{\text{def}}{=} \operatorname{avg}\Big\{ (DH_0)^{-1} \Big\}_\theta$ is invertible and there are two positive numbers η and $\tilde{\eta}$ such that*

$$\Big| (DH_0)^{-1} \Big|_{C^0(\mathbb{R}^d)} \leq \tilde{\eta} \,, \qquad \big| \Phi^{-1} \big| \leq \eta \,.$$

Then there exists a constant $C > 1$, depending on d, ℓ, μ, q, σ, γ^{-2}, η, $\tilde{\eta}$, β, , $|B| + |v_0|_{C^\ell(\mathbb{T}^d)}$, $|v_0|_{C^\ell}$ and $|u|_{C^\ell(\mathbb{T}^d)}$, such that if

$$C \, |E_0|_{C^0(\mathbb{T}^d)} < \min(1, \beta) \,,$$

then there exists constant vector $\lambda^* \in \mathbb{R}^d$ and diffeomorphism $H^* \in D^{\ell - \sigma}(\mathbb{T}^d)$ such that

$$f \circ H^* = H^* \circ T_\omega + \lambda^* \,.$$

Moreover, $H^* - H_0 \in C^{\ell - \sigma}(\mathbb{T}^n)$ and the following holds

$$|H^* - H_0|_{C^\alpha(\mathbb{T}^d)} \leq C \, | E_0|_{C^0(\mathbb{T}^d)}^{1/\mu} \,,$$

and

$$|\lambda^* - \lambda_0| \leq C \, | E_0|_{C^0(\mathbb{T}^d)}^{1/\mu} \,.$$

Let us briefly explain the hypotheses of Theorem 2. Condition 1 ensures that the error function is periodic so that the linear part of the conjugation map has not to be changed. Condition 2 enables us to control the size of the imaginary part of $S_t[H_0](\mathbb{T}^d_{t^{-1}})$. So that Corollary 18 applies. Finally condition 3 is a non-degenerate condition on the approximate solution H_0.

The proof to Theorem 2 we present here uses the estimate given in Corollary 18 and the analytic version of Theorem 2 formulated in Theorem 3. We will not give here any proof of Theorem 3, a proof using a 'polishing' method[7] will appear in a future paper [GE05].

Theorem 3. *Let $\omega \in D(\gamma,\sigma)$, for some $\gamma > 0$ and $\sigma \geq d$. Assume that $\beta,\rho > 0$, $\lambda_0 \in \mathbb{R}^d$, and $f = A + u$ and h_0 are given and define the error function*

$$e_0 \stackrel{\text{def}}{=} f \circ h_0 - h_0 \circ T_\omega - \lambda_0 \,.$$

Assume that the following hypothesis hold

1. *$u \in \mathcal{P}(2\,\beta\,\rho, C^2)$ and $|\,\mathrm{Im}\,(h_0)|_\rho < \beta\,\rho$.*
2. *$e_0 \in \mathcal{P}(\rho, C^0)$.*
3. *The matrix $Dh_0(z)$ is invertible for any $z \in \mathbb{T}^d_\rho$. The matrix $\Phi \stackrel{\text{def}}{=} \mathrm{avg}\left\{\, Dh_0^{-1} \,\right\}_\theta$ is invertible. Moreover, there exist two positive numbers η and $\tilde\eta$ such that*

$$\left|\,\Phi^{-1}\,\right| \leq \eta\,, \qquad \left|\,(Dh_0)^{-1}\,\right|_\rho \leq \tilde\eta\,.$$

Then there exists a constant $M > 0$, depending on d, σ, γ^{-2}, β, η, $\tilde\eta$, $|B| + |Dv_0|_\rho$, and $|u|_{2\,\beta\rho, C^2}$, such that if $q \geq 2(\sigma+1)$ and

$$M\,\rho^{\,q}\,|e_0|_\rho < \min(\,1,\beta)$$

then there exists a constant vector $\lambda^* \in \mathbb{R}^d$ and a diffeomorphism $h^* \in D^0(\mathbb{T}^d)$ such that $(h^* - h_0) \in \mathcal{P}(\rho/2, C^0)$, and such that

$$f \circ h^* = h^* \circ T_\omega + \lambda^*\,.$$

Moreover, the following inequalities hold

$$\begin{aligned} |h^* - h_0|_{\rho/2} &\leq M\,\rho^{-\sigma}\ \left|\,e_0\,\right|_\rho\ , \\ |Dh^* - Dh_0|_{\rho/2} &\leq M\,\rho^{-(\sigma+1)}\ \left|\,e_0\,\right|_\rho\ , \\ |\lambda^* - \lambda_0| &\leq M\ \left|\,e_0\,\right|_\rho\,. \end{aligned}$$

Remark 19. *It turns out [GE05] that the constant M in Theorem 3 is increasing with respect to the initial data β, η, $\tilde\eta$, $|Dv_0|_\rho$, and $|u|_{2\,\beta\rho, C^2}$.*

[7] See Section 1.

5.2.1 Proof of Theorem 2

Throughout this section we assume that the hypotheses of Theorem 2 hold, that S_t is defined as in Definition 3, and that in Definition 3 $\mathbf{s} = 1$.

Following the scheme explained in Section 1, we prove Theorem 2 in several lemmas. The procedure of the proof consists of three main steps:

Step 0 Write our conjugation problem in a functional form. Consider the functional

$$F(f, h, \lambda) \stackrel{\text{def}}{=} f \circ h - h \circ T_\omega \; - \lambda. \tag{24}$$

Then (f, H_0, λ_0) is an approximate solution of the problem

$$F(f, h, \lambda) = 0. \tag{25}$$

Step 1 Find an analytic approximate solution of (25) which satisfies the hypotheses of Theorem 3. This is done in Lemmas 20 and 21.
Step 2 Construct a double sequence of approximate solution of (25) (see lemmas 22 and 24) which, under the hypotheses of Theorem 2, converge (see Lemma 25).

We remark that lemmas 20 and 21 make the difference between the procedure we use to proof Theorem 2 and that described in the proof of Theorem 1.2 in [Zeh75].

Lemma 20. *Let $2\sigma < \mu < \ell - 1$, and $\beta > 0$ be fixed, and let E_0 is as Theorem 2. Define $t_0 \geq 1$ by*

$$t_0^{-\mu} \stackrel{\text{def}}{=} |\, E_0|_{C^0(\mathbb{T}^d)} \; .$$

Let t^* be as in Corollary 18, assume that $|\, E_0|_{C^0(\mathbb{T}^d)}$ is sufficiently small such that $t_0 \geq t^*$. There exists a constant C_0 depending on d, ℓ, β, μ, and $|u|_{C^\ell(\mathbb{T}^d)}$ such that for any $t \geq t_0$, one has $F\,(S_t[f], S_t[H_0], \lambda_0) \in \mathcal{P}(t^{-1}, C^0)$ and

$$|\, F(S_t[f], S_t[H_0], \lambda_0)\,|_{t^{-1}} \leq \; C_0 \; |\, E_0|_{C^0(\mathbb{T}^d)} \; .$$

Proof. First of all notice that, since $S_t[T_\omega] = T_\omega$, one has

$$S_t[\, H_0 \circ T_\omega\,] = S_t[H_0] \; \circ T_\omega \, ,$$

hence, performing some simple computations, one obtains

$$\begin{aligned} F\,(S_t[f], S_t[H_0], \lambda_0) &= (F(S_t[f], S_t[H_0], \lambda_0) - S_t[E_0]) \; + \; S_t[E_0] \\ &= \{\, S_t[f] \circ S_t[H_0] - S_t[f \circ H_0]\,\} \; + \; S_t[E_0]\,. \end{aligned}$$

Let κ_1 be as in Proposition 5, let t^* and κ be as in Corollary 18. Assume that $|E_0|_{C^0}$ is sufficiently small such that $t_0 \geq t^*$, then Corollary 18 and Proposition 5 imply for all $t \geq t_0$

$$|\, F\,(S_t[f], S_t[H_0], \lambda_0)\,|_{t^{-1}} \leq \kappa \; |u|_{D^\ell(\mathbb{T}^d)} \; t^{-\mu} + \kappa_1 \; |E_0|_{C^0(\mathbb{T}^d)} \leq C_0 \; |E_0|_{C^0(\mathbb{T}^d)}$$

□

The following lemma ensures that if $|E_0|_{C^0}$ is sufficiently small there is an analytic approximate solution of (25) which satisfies the hypothesis of Theorem 3.

Lemma 21. *Let β be as in Theorem 2 and let μ and t_0 be as in Lemma 20. Define*

$$\rho_0 \stackrel{\text{def}}{=} t_0^{-1}\,, \qquad f_0 \stackrel{\text{def}}{=} S_{t_0}[f]\,, \qquad h_0 \stackrel{\text{def}}{=} S_{t_0}[H_0]\,, \qquad e_0 \stackrel{\text{def}}{=} F(f_0, h_0, \lambda_0)\,,$$

and let u_0 and φ_0 denote the periodic parts of f_0 and h_0, respectively.

There exists a positive constant C_1, depending on d, ℓ, $\tilde{\eta}$, η, and $|v_0|_{C^\ell(\mathbb{T}^d)}$, such tat if

$$C_1\,|E_0| < 1\,, \qquad\qquad \rho_0\,(2\,\beta)\,\leq\,1, \tag{26}$$

then the following hold

1. $|Dh_0|_{\rho_0} < \beta$.
2. $u_0 \in \mathcal{P}(2\,\beta\,\rho_0, C^2\,)$*, and the following estimates hold*

$$|u_0|_{2\,\beta\,\rho_0} \leq \kappa_6\;|u|_{C^\ell(\mathbb{T}^d)}\;, \qquad |u_0|_{2\,\beta\,\rho_0,C^2} \leq \kappa_7\;|u|_{C^\ell(\mathbb{T}^d)}\;, \tag{27}$$

   where κ_6 is as in Lemma 8 taking $C = 2\,\beta$, and $r = 0$ and κ_7 is as in Proposition 10 taking $C = 2\,\beta$, and $r = 1/2$.
3. $e_0 \in \mathcal{P}(\rho_0, C^0)$.
4. *$Dh_0(z)$ is invertible for each $z \in \mathbb{T}^d_{\rho_0}$ and the matrix $\Phi_0 \stackrel{\text{def}}{=} \operatorname{avg}\{\,Dh_0^{-1}\,\}_\theta$ is also invertible. Moreover,*

$$\left|\,(Dh_0)^{-1}\,\right|_{\rho_0} \leq \frac{3}{2}\,\tilde{\eta}\,, \qquad \left|\,\Phi_0^{-1}\,\right| \leq \frac{3}{2}\,\eta\,. \tag{28}$$

Proof. From Proposition 5 and assumption 2 of Theorem 2 one has

$$|Dh_0|_{\rho_0} \;=\; \sup_{z\in\mathbb{T}^d_{\rho_0}}\;|\,B + S_{t_0}[Dv_0](z)\,| \leq\,|B| +\,\kappa_1\;|Dv_0|_{C^0} < \beta\,.$$

Now assume that $|E_0|_{C^0(\mathbb{T}^d)}$ is sufficiently small such that the second inequality in (26) holds, then Part 5 of Lemma 8 and Part 3 of Proposition 10 imply that $u_0 = S_{t_0}[u] \in \mathcal{P}(\,2\,\beta\,,\rho_0, C^2)$ and that the estimates in (27) hold. This proves part 1 and 2 of Lemma 21, and since we are assuming that $E_0 \in C^\ell(\mathbb{T}^d)$, then from (23) one has part 3 of Lemma 21.

In order to prove Part 4 of Lemma 21, we first notice that

$$|\,Dh_0 - DH_0\,|_{C^0(\mathbb{R}^d)} = |\,S_{t_0}[Dv_0] - Dv_0\,|_{C^0(\mathbb{T}^d)} \leq (\kappa_1\;|\,Dv_0\,|_{C^{\ell-1}}\,)\;t_0^{-\ell+1}\,,$$

hence if t_0 is sufficiently big (equivalently $|E_0|_{C^0}$ sufficiently small) such that

$$C_2\,t_0^{-\ell+1}\,\leq 1\,,$$

with

$$C_2 \stackrel{\text{def}}{=} 2^3\,\max\,(\tilde{\eta}\,,1)\;\kappa_1\;|Dv_0|_{C^{\ell-1}}\,\leq 1\,,$$

then we have that $Dh_0(x)$ is invertible for all $x \in \mathbb{R}^d$, and

$$\left| (Dh_0)^{-1} \right|_{C^0(\mathbb{R}^d)} \leq \tilde{\eta} + 2\,\tilde{\eta}^2 \, |Dh_0 - DH_0\,|_{C^0(\mathbb{R}^d)} \leq \tilde{\eta} + \tilde{\eta}/4 \tag{29}$$

Now, let $z \in \mathbb{T}^d_{\rho_0}$, then

$$Dh_0(z) = Dh_0(\mathrm{Re}\,(z)) + [\,D\varphi_0(z) - D\varphi_0(\mathrm{Re}\,(z))\,]\ ,$$

from Remark 11 we have that

$$|\,D\varphi_0(z) - D\varphi_0(\mathrm{Re}\,(z))\,| \leq \ |\mathrm{Im}\,(z)|\ |\,D\varphi_0\,|_{\rho,\, C^1(\mathbb{T}^d)} \ \leq \kappa_7\ |\,Dv_0|_{C^{\ell-1}(\mathbb{T}^d)}\ t_0^{-1}$$

Therefore, if

$$C_3 \stackrel{\rm def}{=} \max\left(C_2, 2^3 \max(2\,\tilde{\eta}, 1)\,\kappa_7\ |Dv_0|_{C^{\ell-1}(\mathbb{T}^d)}\right)$$

and

$$C_3\, t_0^{-1} \ \leq\ 1\,,$$

one has that $Dh_0(z)$ for any $z \in \mathbb{T}^d_{\rho_0}$ is invertible and using (29) one has

$$\left|\,(\,Dh_0)^{-1}\right|_{\rho_0} \ \leq\ \left|\,(\,Dh_0)^{-1}\right|_{C^0(\mathbb{R}^d)} (1 + 1/4) \leq 3\,\tilde{\eta}/2$$

Similarly, there exists a constant $C_1 \geq C_3$, depending on d, ℓ, η, and $|v_0|_{C^\ell(\mathbb{T}^d)}$, such tat if

$$C_1\ |E_0| < 1\,,$$

then Φ_0 is invertible and the second estimate in (28) holds. □

Now we have the necessary conditions to construct a sequence of analytic solutions of (25) by using the method described in the proof to Theorem 1.2. in [Zeh75].

Lemma 22. *Assume that for $n \geq 0$ fixed there exists an approximate solution (f_n, h_n, λ_n) of (25), with error function $e_n \stackrel{\rm def}{=} F(f_n, h_n, \lambda_n)$. Let u_n and φ_n be the periodic parts of f_n and h_n, respectively. Assume that the following conditions hold*

C1(n) $|\,Dh_n|_{\rho_n} <\ \beta_n$ *for β_n, $\rho_n > 0$ such that*

$$2\,\rho_n\,\beta_n < 1\,, \quad \beta_n \geq\ 2^{-(n+1)}\,\beta. \tag{30}$$

C2(n) $u_n \in \mathcal{P}(2\,\beta_n\,\rho_n, C^{\,2}\,)$ *and the following estimates hold*

$$|u_n|_{2\,\beta\,\rho_0} \leq \kappa_6\ |u|_{C^\ell(\mathbb{T}^d)}\ , \qquad |u_n|_{2\,\beta\,\rho_0, C^2} \leq \kappa_7\ |u|_{C^\ell(\mathbb{T}^d)}\ , \tag{31}$$

where κ_6 and κ_7 are as in Lemma 21.

C3(n) $e_n \in \mathcal{P}(\rho_n, C^0)$.

C4(n) $Dh_n(z)$ *is invertible for each $z \in \mathbb{T}^d_{\rho_n}$ and the matrix $\Phi_n \stackrel{\rm def}{=} \mathrm{avg}\,\{\,Dh_n^{-1}\,\}_\theta$ is also invertible. Moreover, there exists two positive numbers η_n and $\tilde{\eta}_n$ such that*

$$\left|(Dh_n)^{-1}\right|_{\rho_n} \leq \tilde{\eta}_n\,, \qquad \left|\,\Phi_n^{-1}\,\right| \leq \eta_n\,. \tag{32}$$

there exists a constant $\tilde M_n$, depending on d, σ, γ^{-2}, β_n, η_n, $\tilde\eta_n$, $|B| + |\varphi_n|_\rho$, and $|u_n|_{2\,\beta_n\,\rho_n, C^2}$, such that if

$$\tilde M_n\,\rho_n^{-\,q}\,|e_n|_{\rho_n} < \min(1, \beta_n)\,, \tag{33}$$

then there exists a constant vector $\lambda_{n+1} \in \mathbb{R}^d$ and a diffeomorphism h_{n+1} such that $(h_{n+1} - h_n) \in \mathcal{P}(\rho_n/2, C^0)$, and such that

$$F(f_n, h_{n+1}, \lambda_{n+1}) = 0\,. \tag{34}$$

and such that the following estimates hold

$$\begin{aligned} |h_{n+1} - h_n|_{\rho_{n+1}} &\le \tilde M_n\,\rho_n^{-\sigma}\,|\,e_n\,|_{\rho_n}\,, \\ |Dh_{n+1} - Dh_n|_{\rho_{n+1}} &\le \tilde M_n\,\rho_n^{-(\sigma+1)}\,|\,e_n\,|_{\rho_n}\,, \\ |\lambda_{n+1} - \lambda_n| &\le \tilde M_n\,|\,e_n\,|_{\rho_{n+1}}\,. \end{aligned} \tag{35}$$

Define

$$\rho_{n+1} \stackrel{\text{def}}{=} \rho_n/2\,, \qquad t_{n+1} = 2\,t_n\,, \tag{36}$$

and

$$f_{n+1} \stackrel{\text{def}}{=} S_{t_{n+1}}[f]\,, \qquad e_{n+1} \stackrel{\text{def}}{=} F(f_{n+1}, h_{n+1}, \lambda_{n+1})\,,$$

and let $\eta_0, \tilde\eta_0$ be given by (28). There exists a constant $\hat M_n$, depending on $\tilde M_n$, η_n, and $\tilde\eta_n$ such that if

$$\hat M_n\,2^{(n+1)}\,\rho_n^{-(\sigma+1)}\,|e_n|_{\rho_n} \le \min(\,\beta, 1) \tag{37}$$

then h_{n+1} and f_{n+1} satisfy properties **C1(n+1)**, **C2(n+1)**, *and* **C3(n+1)** *with*

$$\begin{aligned} \tilde\eta_{n+1} &\stackrel{\text{def}}{=} \tilde\eta_n + \tilde\eta_n\,2^{-(n+1)}\,, \\ \eta_{n+1} &\stackrel{\text{def}}{=} \eta_n + \eta_n\,2^{-(n+1)} \\ \beta_{n+1} &\stackrel{\text{def}}{=} \beta_n + \beta\,2^{-(n+1)}\,. \end{aligned} \tag{38}$$

Moreover the following estimate holds

$$|\,e_{n+1}|_{\rho_{n+1}} \le \kappa_4\,|u|_{C^\ell(\mathbb{T}^d)}\,\rho_n^\ell\,, \tag{39}$$

where κ_4 is as in part 1 of Proposition 10.

Proof. The existence of h_{n+1}, λ_{n+1}, such that (34) and (35) hold is ensured by Theorem 3. Indeed, notice that Lemma 12 and **C1(n)** imply

$$|\mathrm{Im}\,(h_n)|_{\rho_n} \le \rho_n\,|Dh_n|_{\rho_n} < \beta_n\,\rho_n\,.$$

Hence, the hypotheses of Theorem 3 are met.

Let us first verify that **C1(n+1)** holds. From the second estimate in (35) one has

$$|\,Dh_{n+1}(z)|_{\rho_{n+1}} \le |\,Dh_n(z)|_{\rho_{n+1}} + \tilde M_n\,\rho_n^{-(\sigma+1)}\,|e_n|_n\,. \tag{40}$$

Moreover, since by assumption (30) holds, one has that if ρ_{n+1} and β_{n+1} are defined by (36) and (38), respectively, then (30) also holds for $(n+1)$. Now assume that (37) holds with

$$\hat{M}_n \geq \tilde{M}_n,$$

then (40) implies that **C1(n+1)** holds with β_{n+1} defined in (38) and ρ_{n+1} defined in (36).

From the fact that the first inequality in (30) holds for $(n+1)$, Part 5 of Lemma 8 and Part 3 of Proposition 10 imply that $u_{n+1} = S_{t_{n+1}}[u] \in \mathcal{P}(2\,\beta_n\,, \rho_n, C^2)$ and that the estimates in (31) hold for $(n+1)$. Hence, **C2(n+1)** holds.

Notice that **C3(n+1)** is a consequence of **C3(n)**, **C1(n+1)** and the fact that $h_{n+1} - h_n \in \mathcal{P}(\rho_{n+1}, C^0)$.

Now we prove that **C4(n+1)** holds. From (40) we have that if

$$\hat{M} \overset{\text{def}}{=} \max\Big(2\,\max(\tilde{\eta}_n, 1)\,\tilde{M}_n,\ 2\,\max(\eta_n, 1)\,\tilde{M}_n,\ \tilde{M}_n\Big)$$

and if inequality (37) holds, then

$$\tilde{M}\,\rho_n^{-(\sigma+1)}\ \left|e_n\right|_{\rho_n} \leq 1/2$$

this and (40) imply that, $Dh_{n+1}(z)$ is invertible for any $z \in \mathbb{T}^d_{\rho_{n+1}}$ and using (37) one has

$$\Big|\,(Dh_{n+1}(z))^{-1}\,\Big|_{\rho_{n+1}} \leq \tilde{\eta}_n\ +\ 2\,\tilde{\eta}_n^2\,\tilde{M}_n\,\rho_n^{-(\sigma+1)}\ \left|e_n\right|_{\rho_n}\ \leq \tilde{\eta}_n\ +\ \tilde{\eta}_n\,2^{-(n+1)}$$

Therefore, the first inequality in (32) holds for $n+1$ That the second inequality in (32) holds for $n+1$ is proved similarly.

We now prove estimate (39). From Lemma 12 and since **C1(n+1)** holds we have

$$\left|\mathrm{Im}\,(h_{n+1})\right|_{\rho_{n+1}} \leq \rho_{n+1}\,\beta_{n+1}\,.$$

Then applying Part 1 of Lemma 8 (because the first inequality in (30) holds) one obtains

$$\begin{aligned}
\left|e_{n+1}\right|_{\rho_{n+1}} &= \left|\,f_{n+1}\circ h_n - f_n\circ h_n\right|_{\rho_{n+1}}\\
&\leq \left|\,(S_{2\,t_n}[u] - S_{t_n}[u])\ \circ h_n\,\right|_{\rho_{n+1}}\\
&\leq \left|\,(S_{2\,t_n} - S_{t_n})\,[u]\,\right|_{\beta_{n+1}\rho_{n+1}}\\
&\leq \kappa_4\ |u|_{C^\ell(\mathbb{T}^d)}\ \rho_n^\ell,
\end{aligned}$$

where κ_4 is as in Lemma 8 taking $r=0$, and $C = 2\,\beta_{n+1}$. $\square$

Remark 23. *Let $\tilde{M}_n$ and $\hat{M}_n$ be as in Lemma 22. For $n \geq 0$ define*

$$M_n \overset{\text{def}}{=} \max\left(\hat{M}_n, \hat{M}_n\right)\ .$$

From Remark 19 we know that $\tilde{M}_n$ is increasing with respect o to β_n, η_n, $\tilde{\eta}_n$, $|D\varphi_n|_{\rho_n}$, and $|u_n|_{2\beta_n\rho_n, C^2}$. And it follows from the proof to Lemma 22 that $\tilde{M}_n$ is also increasing with respecto to the same quantities. Let us write explicitly dependence on these variables as follows:

$$M_n = \Omega\left(\beta_n, \eta_n, \tilde{\eta}_n, |D\varphi_n|_{\rho_n}, |u_n|_{2\beta_n\rho_n, C^2}\right).$$

Define

$$M_\infty = \Omega\left(2\,\beta, 4\,\eta, 4\,\tilde{\eta}, \kappa_7\, |u|_{C^\ell(\mathbb{T}^d)}\right).$$

We will show in Lemma 24 that if $|E_0|_{C^0}$ is sufficiently small, then $M_n \le M_\infty$ for all $n \ge 0$.

Lemma 24. *Assume that the hypothesis of Theorem 2 hold. Let t_0, ρ_0 be as in Lemma 21. Consider the sequences of numbers $\{\rho_n\}_{n\ge0}$, $\{\beta\}_{n\ge0}$, $\{\eta\}_{n\ge0}$, and $\{\tilde{\eta}\}_{n\ge0}$ defined in (36) and (38) for $n \ge 1$ and*

$$\beta_0 \stackrel{\rm def}{=} \beta, \qquad \eta_0 \stackrel{\rm def}{=} \frac{3\,\eta}{2}, \qquad \tilde{\eta}_0 \stackrel{\rm def}{=} \frac{3\,\tilde{\eta}}{2}, \qquad .$$

There exists a constant M, depending on d, σ, γ^{-2}, β, η, $\tilde{\eta}$, $|B| + |Dv_0|_\rho$, $|v_0|_{C^\ell(\mathbb{T}^d)}$, and $|u|_{2\beta\rho, C^2}$, such that if

$$M\,\rho_0^{(\mu-q)} < \min(1,\beta), \qquad \textit{and} \qquad 2\,\beta\,\rho_0 < 1, \tag{41}$$

then there exists a sequence of numbers $\{\lambda_n\}_{n\in\mathbb{N}}$ and two sequences of functions $\{h_n\}_{n\ge0} \subset D^0(\mathbb{T}^d)$ and $\{f_n\}_{n\ge0} \subset D^0(\mathbb{T}^d)$ satisfying conditions **C1(n)**-**C4(n)** *and (34). Moreover, for each $n \ge 0$ $(h_{n+1} - h_n) \in \mathcal{P}(\rho_n, C^0)$ and the following estimates hold*

$$\begin{aligned} |h_{n+1} - h_n|_{\rho_{n+1}} &\le M\,\rho_n^{\ell-\sigma} \\ |Dh_{n+1} - Dh_n|_{\rho_{n+1}} &\le M\,\rho_n^{\ell-(\sigma+1)} \\ |\lambda_{n+1} - \lambda_n| &\le M\,\rho_n^\ell\,. \end{aligned} \tag{42}$$

Proof. Let C_1 be as in Lemma 21 and let M be a constant greater than C_1, then if (41) holds then properties **C1(0)-C4(0)** in Lemma 22 hold.

Let M_∞ be as in Remark 23, C_0 as in Lemma 20, and κ_4 as in Lemma 22 and assume that (41) holds with

$$M \ge \max\left(C_1,\, 2\,M_\infty\,C_0\,,\, 2^{\ell+1}\,\kappa_4\,M_\infty\,|u|_{C^\ell(\mathbb{T}^d)}\right) \tag{43}$$

Then, (33) and (37) hold for $n = 0$. Indeed, from Lemma 20 we have

$$\tilde{M}_0\,\rho_0^{-q}\,|e_0|_{\rho_0} \le M_\infty\,C_0\,\rho_0^{\mu-q} \le M\,\rho_0^{\mu-q} < \min(1,\beta)\,,$$

and

$$2\,\hat{M}_0\,\rho_0^{-(\sigma+1)}\,|e_0|_{\rho_0} \le 2\,M_\infty\,C_0\,\rho_0^{\mu-(\sigma+1)} \le M\,\rho_0^{\mu-q} < \min(1,\beta)\,,$$

where we have used that

$$\rho_0^\mu = |E_0|_{C^0(\mathbb{T}^d)}\,, \qquad \text{and} \qquad 2(\sigma+1) \le q < \mu < \ell - 1.$$

Hence Lemma 22 implies the existence of h_1 and λ_1 such that (34), (35), and (39) hold for $n = 0$, and such that $f_1 = S_{t_1}[u]$ and h_1 satisfy **C1(1)-C4(1)**. Moreover, estimate (35) and Lemma 20 imply (42) for $n = 0$.

Now we iterate the above procedure. Assume that for $k \geq 1$ there exist h_k and λ_k such that (34), (35), (39), and (42) hold for $(k-1)$, and such that $f_k = S_{t_k}[u]$ and h_k satisfy **C1(k)-C4(k)**. To obtain h_{k+1} and λ_{k+1} satisfying the same conditions for k we only have to verify that estimates (33) and (37) in Lemma 22 hold for k.
From (38) one obtains

$$\beta_k = \beta \sum_{j=0}^{k} 2^{-j} < 2\,\beta\,, \quad \tilde{\eta}_k = \tilde{\eta}_0 \prod_{j=1}^{k} \left(1 + 2^{-j}\right) < 4\,\tilde{\eta}\,, \quad \eta_k = \eta_0 \prod_{j=1}^{k} \left(1 + 2^{-j}\right) < 4\,\eta$$

Therefore, if M_k and M_∞ are as in Remark 23, then

$$M_k \leq M_\infty\,.$$

This and (39) imply

$$\tilde{M}_k\, \rho_k^{-q}\, |e_k|_{\rho_k} \leq \left(2^\ell\, \kappa_4\, M_\infty\, |u|_{C^\ell(\mathbb{T}^d)}\right) \rho_k^{\ell - q} \leq M\, \rho_0^{\mu - q} < \min(1,\, \beta)\,,$$

and

$$\hat{M}_k\, 2^{(k+1)}\, \rho_k^{-(\sigma+1)}\, |e_k|_{\rho_k} \leq M\, \rho_0^{\ell - q} \leq M\, \rho_0^{\mu - q} < \min(1,\, \beta)\,,$$

where we have used $\rho_k = 2^{-k}\, \rho_0$, and $\ell - (\sigma + 2) \geq \ell - q > \mu - q$.

Hence, if M satisfies (41) and (43), then (33) and (37) hold, and Lemma 22 applies yielding h_{k+1} and λ_{k+1} such that (34) and (35) hold. Moreover, $f_{k+1} = S_{t_{k+1}}[u]$ and h_{k+1} satisfy **C1(k+1)-C4(k+1)** and from (35) and (34) one has that (42) holds. $\square$

Lemma 25. *Assume that the hypotheses of Lemma 24 hold. There exist a constant vector $\lambda^* \in \mathbb{R}^d$ and a function $H^* \in D^{\ell-\sigma}$ such that $H^* - H_0 \in C^{\ell-\sigma}(\mathbb{T}^n)$ and*

$$\begin{aligned} |H^* - H_0|_{C^\alpha(\mathbb{T}^d)} &\leq \tilde{M}\, \rho_0\,, \\ |\, \lambda^* - \lambda_0| &\leq \tilde{M}\, \rho_0\,, \end{aligned} \tag{44}$$

for some constant $\tilde{M}$, depending on M, $|v_0|_{C^\ell(\mathbb{T}^d)}$, ℓ, σ, and α. Moreover (f, H^*, λ^*) satisfies (25).

Proof. First of all notice that (42) implies that the sequence $\{\lambda_n\}_{n\geq 0}$ of Lemma 24 converges to some vector $\lambda^* \in \mathbb{R}^d$.

To prove the existence of H^* define $w_n \stackrel{\text{def}}{=} h_n - h_0$ with $\{\, h_n\}_{n\geq 0}$ as in Lemma 24. Then from Lemma 24 we have that $\{w_n\}_{n\geq 0} \subset \mathcal{P}(\rho_n, C^0)$ and

$$\sup_{n\geq 1} \left(\rho_n^{(\ell-\sigma)}\, |w_n - w_{n+1}|_{\rho_n}\right) < M\,,$$

where M and ρ_n are as in Lemma 24. Moreover, if $\alpha < \ell - \sigma$ we have

$$|w_n|_{C^\alpha(\mathbb{T}^d)} \leq \sum_{k=0}^{n-1} |\, h_{k+1} - h_k|_{C^\alpha(\mathbb{T}^d)} \leq C_4 \, \rho_0 \,,$$

and

$$|w_n - w_{n+1}|_{C^\alpha(\mathbb{T}^d)} \leq \, \rho_{n+1}^{-\alpha} \, |h_n - h_{n+1}|_{\rho_{n+1}} \leq \, C_4 \, \rho_{n+1}^{\ell-\sigma-\alpha} \,,$$

for some constant C_4 depending on M, ℓ, σ, and α.

Hence, if $\ell - \sigma$ is not an integer there exists a function[8] $w \in C^{\ell-\sigma}(\mathbb{T}^d)$ with

$$|\, w \,|_{C^{\ell-\sigma}} \, \leq \, C_5 \,,$$

for some constant C_5 depending on the same variables as C_4 and such that such that

$$\lim_{n\to\infty} |w - w_n|_{C^\alpha(\mathbb{T}^d)} = 0 \,,$$

for any $\alpha < \ell - \sigma$.

Lemma 25 follows by taking $H^* \stackrel{\text{def}}{=} h_0 + w$. Indeed, $H^* - H_0 = h_0 - H_0 + w \in C^{\ell-\sigma}(\mathbb{T}^d)$ and

$$\begin{aligned} |\, H^* - H_0|_{C^\alpha(\mathbb{T}^d)} &\leq |\, H^* - h_n|_{C^0(\mathbb{T}^d)} \, + \, |\, h_0 - H_0|_{C^0(\mathbb{T}^d)} \, + \, |\, h_n - h_0|_{C^0(\mathbb{T}^d)} \\ &= |\, w - w_n|_{C^\alpha(\mathbb{T}^d)} \, + \, |\, (S_{t_0} - 1)[v_0] \,|_{C^\alpha(\mathbb{T}^d)} \, + \, |\, w_n|_{C^\alpha(\mathbb{T}^d)} \\ &\leq |\, w - w_n|_{C^\alpha(\mathbb{T}^d)} \, + \, \kappa_2 \, |v_0|_{C^\ell(\mathbb{T}^d)} \, \rho_0^{\ell-\alpha} \, + \, C_4 \, \rho_0 \end{aligned}$$

and using (42)

$$|\, \lambda^* - \lambda_0| \, \leq |\, \lambda^* - \lambda_n| \, + \sum_{k=0}^{n-1} |\lambda_k - \lambda_{k-1}| \leq C_4 \, \rho_0 \,.$$

This proves (44). Moreover, from (34) and the continuity of the operator F defined in (24) one has $F(f, H^*, \lambda^*) = 0$.

□

Acknowledgement

Authors are deeply grateful to Prof. R. de la Llave for his help, encouragement and many useful suggestions in the preparation of this note. Thanks also to the Mathematics department of The University of Texas at Austin, where part of this work was done. This work has been supported by the DMS 0099399 grant.

[8] See Lemma 2.2 in [Zeh75] or Lemma 6.14 in [BHS96].

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