Lipschitz stability of a non-standard problem for the non-stationary transport equation via Carleman estimate

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Abstract

The Lipschitz stability estimate for the non-stationary single-speed transport equation with the lateral boundary data is obtained. The method of Carleman estimates is used. Uniqueness of the solution follows.

1. Introduction

This paper addresses the question of the Lipschitz stability for the non-stationary single-speed transport equation with the lateral boundary data. This is a non-standard problem for such equation. The author plans to use this estimate for establishing a stability estimate for a coefficient inverse problem for this equation. The author is not aware about other publications with similar results for the non-stationary transport equation. Reviews of uniqueness and stability results for non-standard Cauchy problems for partial differential equations (PDEs) can be found in, e. g., books of Lavrentev, Romanov, Shishatskii [10], Ames and Straughan [1], Isakov [6], and Klibanov and Timonov [9]. A derivation of the transport equation for non-stationary case can be found, for example, in the book of Case and Zweifel [4].

The proof of the main result of this paper is based on a new Carleman estimate. Traditionally, Carleman estimates have been used for proof of stability and uniqueness results for non-standard Cauchy problems for PDEs. They were first introduced by Carleman in 1939 [3], also see Hörmander [5], and [10], [6], [9]. Since the work of Bukhgeim and Klibanov [2] Carleman estimates are also used for proofs of uniqueness and stability results for coefficient inverse problems and, most recently, for construction of numerical methods, see the book of Klibanov and Timonov [9] for details and more references. Works of Klibanov and Malinsky [8] and Kazemi and Klibanov [7] were the first ones, where Carleman estimates were used for proofs of the Lipschitz stability estimates for hyperbolic equations with lateral Cauchy data; also see [9]. The method of this paper is similar with one of [9], [7] and [8]. In Section 2 the statements of the results are given; in Section 3 the proofs of these results are provided.

2. Statements of results

Denote

$$\Omega = \{x \in \mathbb{R}^n : |x| < R\}, \quad S^n = \{\nu \in \mathbb{R}^n : |\nu| = 1\},$$
$$H = \Omega \times S^n \times (-T,T), \quad \Gamma = \partial\Omega \times S^n \times (-T,T).$$

The transport equation in H has the form [4]

$$u_t + (\nu,\nabla u) + a(x,t,\nu)u + \int_{S^n} g(x,t,\nu,\mu)u(x,t,\mu)d\sigma_\mu = F(x,t,\nu), \tag{2.1}$$

where $\nu \in S^n$ is a unit vector of particle velocity, $u(x,t,\nu)$ is a density of particle flow, $a(x,t,\nu)$ is an absorption coefficient, $F(x,t,\nu)$ is an angular density of sources, $g(x,t,\nu,\mu)$ is a scattering indicatrix and $(\nu,\nabla u)$ represents a scalar product of two vectors.

Consider the following boundary condition

$$u|_\Gamma = p(x,t,\nu), \text{ where } (x,t,\nu) \in \partial\Omega \times [-T,T] \times S^n \text{ and } (n,\nu) < 0. \tag{2.2}$$

Here (n,ν) is the scalar product of the outer normal vector n to the surface $\partial\Omega$ and the direction of the velocity ν. So, only incoming radiation is given at the boundary in this case.

Equation (2.1) with the boundary condition (2.2) and the initial condition at $t = -T$

$$u(x,-T,\nu) = q(x,\nu), \text{ where } (x,\nu) \in \Omega \times S^n, \tag{2.3}$$

form the classical forward problem for the transport equation. Standard uniqueness, existence and stability results for this problem are well known, see, e. g., Prilepko and Ivankov [11].

Suppose now that the initial data $q(x,\nu)$ is unknown, but the following additional boundary condition is given:

$$u|_\Gamma = p(x,t,\nu), \text{ where } (x,t,\nu) \in \partial\Omega \times [0,T] \times S^n \text{ and } (n,\nu) > 0. \tag{2.4}$$

Hence, the function $p(x,t,\nu)$ describes the outgoing radiation on the boundary. Thus, we obtain a non-standard Cauchy problem for the non-stationary transport equation:

Problem 1: Given the lateral data (2.2), (2.4), determine the solution $u(x,t,\nu)$ of the equation (2.1).

Theorem 1. [Lipschitz stability] *Let functions $a(x,t,\nu)$ and $g(x,t,\nu,\mu)$ be bounded, i.e. $|a(x,t,\nu)| < a_1 \ \forall(x,t,\nu) \in H$ and $|g(x,t,\nu,\mu)| < g_1 \ \forall(x,t,\nu,\mu) \in H \times S^n$, where a_1 and g_1 are positive constants. Let functions $p(x,t,\nu), F(x,t,\nu) \in L_2(H)$ and let $T > R$. Suppose that the function $u \in C^1(\overline{\Omega} \times [-T,T]) \times C(S^n)$ satisfies the conditions* (2.1), (2.2), (2.4). *Then the following Lipschitz stability estimate holds*:

$$\|u\|_{L_2(H)} \leq K \bullet [\|p\|_{L_2(\Gamma)} + \|F\|_{L_2(H)}], \tag{2.5}$$

where $K = K(g,\Omega,a,T)$ is the positive constant independent on functions u, p and F.

Corollary. [Uniqueness] *Suppose that conditions of Theorem* 1 *are fulfilled. Then the Problem* 1 *has at most one solution.*

Below conditions of Theorem 1 are assumed to be satisfied. The proof of Theorem 1 is based on the Carleman estimate formulated in Lemma 1.

Let

$$L_0 u = u_t + (\nu, \nabla u) = u_t + \sum_{i=1}^{n} \nu_i u_i, \tag{2.6}$$

where $u_i \equiv \partial u / \partial x_i$. Introduce the function

$$\psi(x,t) = |x|^2 - \eta t^2, \quad \eta = const \in (0,1). \tag{2.7}$$

Let $c = const \in (0,R)$. Denote

$$G_c = \{(x,t) : \psi(x,t) > c \quad and \quad |x| < R\}. \tag{2.8}$$

Introduce the Carleman Weight Function (CWF) as

$$\mathsf{C}(x,t) = \exp[\lambda \psi(x,t)]. \tag{2.9}$$

Lemma 1. *Choose the number* η *such that* $\eta \in (0,1)$ *and* $T > R/\sqrt{\eta}$. *Also, choose the constant* $c \in (0,R)$ *such that* $G_c \subset \Omega \times (-T,T)$. *Then there exist positive constants* $\lambda_0 = \lambda_0(G_c)$ *and* $M = M(G_c)$, *depending only on the domain* G_c, *such that the following pointwise Carleman estimate holds in* $G_c \times S^n$ *for all functions* $u(x,t,\nu) \in C^1(\overline{G}_c) \times C(S^n)$ *and for all* $\lambda \geq \lambda_0(G_c)$:

$$(L_0 u)^2 \mathsf{C}^2 \geq 2\lambda(1-\eta) u^2 \mathsf{C}^2 + \nabla \bullet U + V_t, \tag{2.10}$$

where the vector function (U,V) *satisfies the estimate*

$$|(U,V)| \leq M \lambda u^2 \mathsf{C}^2. \tag{2.11}$$

3. Proofs of results

Proof of Lemma 1.

Denote $\upsilon = u \bullet \mathsf{C}$, where C is the CWF. Hence,

$$u = \upsilon \bullet \mathsf{C}^{-1} = \upsilon \bullet exp[\lambda(\eta t^2 - |x|^2)],$$

$$u_t = (\upsilon_t + 2\lambda \eta t \upsilon)\mathsf{C}^{-1}, \quad u_i = (\upsilon_i - 2\lambda x_i \upsilon)\mathsf{C}^{-1}. \tag{3.1}$$

Then

$$L_o u = \left[\upsilon_t + \sum_{i=1}^{n} \nu_i \upsilon_i + 2\lambda \left(\eta t - \sum_{j=1}^{n} \nu_j x_j \right) \upsilon \right] \mathsf{C}^{-1}. \tag{3.2}$$

Hence,

$$(L_o u)^2 \mathrm{C}^2 \geq 4\lambda\left(\eta t - \sum_{j=1}^{n} \nu_j x_j\right) \bullet \upsilon \bullet \left(\upsilon_t + \sum_{i=1}^{n} \nu_i \upsilon_i\right) =$$

$$\left[2\lambda\left(\eta t - \sum_{j=1}^{n} \nu_j x_j\right)\upsilon^2\right]_t - 2\lambda\eta\upsilon^2 +$$

$$+\sum_{i=1}^{n}\left[2\lambda\left(\eta t - \sum_{j=1}^{n} \nu_j x_j\right)\nu_i\upsilon^2\right]_i + 2\lambda\left(\sum_{i=1}^{n} \nu_i^2\right)\upsilon^2. \tag{3.3}$$

Since

$$\sum_{i=1}^{n} \nu_i^2 = 1 \quad \text{and} \quad \upsilon^2 = u^2 \mathrm{C}^2, \tag{3.4}$$

then

$$(L_o u)^2 \mathrm{C}^2 \geq 2\lambda(1-\eta)u^2\mathrm{C}^2 + \nabla \bullet U + V_t, \tag{3.5}$$

where

$$|(U,V)| \leq \widetilde{K}\lambda u^2 \mathrm{C}^2. \tag{3.6}$$

□

Proof of Theorem 1:
Choose the number $\eta \in (0,1)$ such that $T > R/\sqrt{\eta}$. Also, choose the constant c such that $c \in (0, R/12)$ and $G_c \subset \Omega \times (-T,T)$, and choose sufficiently small $\delta > 0$ such that $G_{c+3\delta} \cap \{\Omega \times (-T,T)\} \neq \varnothing$. Consider the sets $G_{c+3\delta} \subset G_{c+2\delta} \subset G_{c+\delta} \subset G_c$. (See fig.1 for a schematic representation in the 1 - D case)

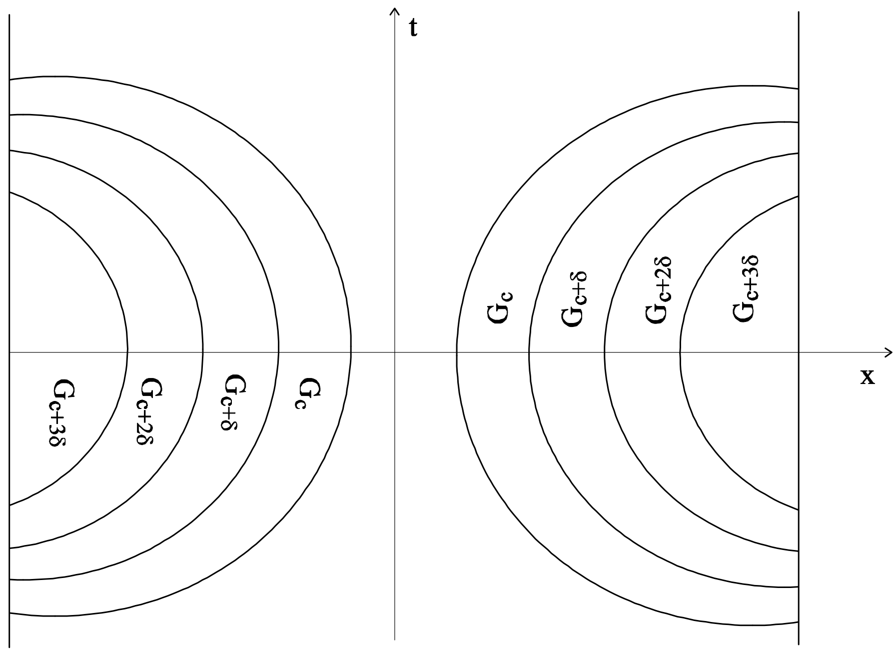

Fig.1. Sets $G_{c+3\delta} \subset G_{c+2\delta} \subset G_{c+\delta} \subset G_c$.

Also, consider the function $\chi(x,t) \in C^1(\overline{\{\Omega \times (-T,T)\}})$, such that

$$\chi(x,t) = \begin{cases} 1 & in \;\; G_{c+2\delta}, \\ 0 & in \;\; \{\Omega \times (-T,T)\} \backslash G_{c+\delta}, \\ between \;\; 0 \;\; and \;\; 1 & otherwise. \end{cases} \tag{3.7}$$

The equation (2.1) implies that

$$|u_t + (\nu, \nabla u)| \leq K \left[a_1 |u| + \int_{S^n} |u| d\sigma_\mu + |F| \right]. \tag{3.8}$$

Here and below K denotes different positive constants depending on $\Omega,\ T,\ a_1,\ g_1,$ but independent on λ and u.

Let $w(x,t,\nu) = u(x,t,\nu) \bullet \chi(x,t)$. Then

$$w_t + \sum_{i=1}^{n} \nu_i w_i = \chi \left(u_t + \sum_{i=1}^{n} \nu_i u_i \right) + u \left(\chi_t + \sum_{i=1}^{n} \nu_i \chi_i \right). \tag{3.9}$$

Derivatives $\chi_t, \chi_i, i = 1,...,n$ are not equal to zero only in $G_{c+\delta} \backslash G_{c+2\delta}$ and are bounded. So, using inequality (3.8), we obtain

$$|w_t + \sum_{i=1}^{n} v_i w_i| \leq K \bullet \left[\chi\left(|u| + \int_{S^n} |u| d\sigma_\mu + |F| \right) + (1-\chi) \bullet |u| \right]. \tag{3.10}$$

Thus

$$|w_t + \sum_{i=1}^{n} v_i w_i| \leq K\left[\left(|w| + \int_{S^n} |w| d\sigma_\mu + |F| \right) + (1-\chi) \bullet |u| \right]. \tag{3.11}$$

Multiplying (3.11) by the CWF and squaring both sides, we obtain

$$|w_t + \sum_{i=1}^{n} v_i w_i|^2 \mathrm{C}^2 \leq K\left[\left(|w|^2 + \int_{S^n} w^2 d\sigma_\mu + |F|^2 \right)\mathrm{C}^2 + (1-\chi) \bullet |u|^2 \mathrm{C}^2 \right]. \tag{3.12}$$

The Carleman estimate (2.10) leads to

$$2\lambda(1-\eta) w^2 \mathrm{C}^2 + \nabla \bullet U + V_t \leq K\left[\left(|w|^2 + \int_{S^n} w^2 d\sigma_\mu + |F|^2 \right)\mathrm{C}^2 + (1-\chi) \bullet |u|^2 \mathrm{C}^2 \right], \tag{3.13}$$

where $(x,t,v) \in H_c,\ H_c = G_c \times S^n$. Integrating over H_c and applying the Gauss' formula, we obtain

$$2\lambda(1-\eta) \int_{H_c} w^2 \mathrm{C}^2 dh \leq \tag{3.14}$$

$$\leq K\left[\int_{H_c} \left(|w|^2 + \int_{S^n} w^2 d\sigma_\mu + |F|^2 \right)\mathrm{C}^2 dh + \int_{H_c} (1-\chi) u^2 \mathrm{C}^2 dh \right] + \int_{M_c} |(U,V)| dS,$$

where $dh \equiv dxdvdt$ and $M_c = \partial G_c \times S^n$. Noticing that

$$\int_{H_c} \left(\int_{S^n} w^2 d\sigma_\mu \right)\mathrm{C}^2 dh = A \bullet \int_{H_c} w^2 \mathrm{C}^2 dh, \tag{3.15}$$

where A is the area of the unit sphere S^n, we remove the inner integral over S^n in (3.14). So, (3.14) becomes

$$2\lambda(1-\eta) \int_{H_c} w^2 \mathrm{C}^2 dh \leq \tag{3.16}$$

$$\leq K\left(\int_{H_c} w^2 C^2 dh + \int_{H_c} |F|^2 C^2 dh + (1-\chi)\bullet \int_{H_c} u^2 C^2 dh \right) + \int_{M_c} |(U,V)| dS.$$

Choose λ_0 such that $K/(2\lambda_0(1-\eta)) < 1/2$. Then for all $\lambda > \lambda_0$ we have

$$\lambda \int_{H_c} w^2 C^2 dh \leq K\left(\int_{H_c} |F|^2 C^2 dh + \int_{H_c} (1-\chi) u^2 C^2 dh \right) + \int_{M_c} |(U,V)| dS. \tag{3.17}$$

From the Carleman estimate (2.10) we have

$$\int_{M_c} |(U,V)| dS \leq K\lambda \int_{M_c} w^2 C^2 dS, \tag{3.18}$$

and since $w = 0$ on the part of the boundary M_c where $|x|^2 - \eta t^2 = c$,

$$\int_{M_c} |(U,V)| dS \leq K\lambda \int_{M_c} w^2 C^2 dS = K\lambda \int_{M'_c} p^2 C^2 dS. \tag{3.19}$$

Here $M'_c = M_c \cap \{(x,t) : |x| = R\} \times S^n$.

Estimate both sides of the inequality (3.17). Note that since $w = u$ in $H_{c+2\delta}$ and $H_{c+3\delta} \subset H_c$, then

$$\lambda \int_{H_c} w^2 C^2 dh \geq \lambda \int_{H_{c+3\delta}} w^2 C^2 dh \geq \lambda e^{2\lambda(c+3\delta)} \int_{H_{c+3\delta}} u^2 dh. \tag{3.20}$$

Also, since $1 - \chi(x,t) = 0$ in $G_{c+2\delta}$, then

$$\sup_{H_c} (1-\chi) C^2 = e^{2\lambda(c+2\delta)}. \tag{3.21}$$

Hence,

$$\int_{H_c} (1-\chi) u^2 C^2 dh \leq e^{2\lambda(c+2\delta)} \int_{H_c} u^2 dh. \tag{3.22}$$

Therefore (3.17) leads to

$$\lambda e^{2\lambda(c+3\delta)} \int_{H_{c+3\delta}} u^2 dh \leq K\left(\int_{H_c} |F|^2 C^2 dh + e^{2\lambda(c+2\delta)} \bullet \int_{H_c} u^2 dh + \lambda \int_{M'_c} p^2 C^2 dS \right). \tag{3.23}$$

Let $m = \sup_{G_c} (|x|^2 - \eta t^2)$. Then

$$\lambda e^{2\lambda(c+3\delta)} \|u\|^2_{L_2(H_{c+3\delta})} \leq K\left(e^{2\lambda m} \|F\|^2_{L_2(H_c)} + e^{2\lambda(c+2\delta)} \|u\|^2_{L_2(H_c)} + \lambda e^{2\lambda m} \|p\|^2_{L_2(M'_c)} \right). \tag{3.24}$$

Dividing this inequality by $\lambda exp(2\lambda(c+3\delta))$, we obtain

$$\|u\|^2_{L_2(H_{c+3\delta})} \leq K\left(\frac{e^{2\lambda m}}{\lambda} \|F\|^2_{L_2(H_c)} + \frac{e^{-2\lambda\delta}}{\lambda} \|u\|^2_{L_2(H_c)} + e^{2\lambda m} \|p\|^2_{L_2(M'_c)} \right). \tag{3.25}$$

Choose an x_0 such that $|x_0| = R/4$. Consider domain $G_c(x_0) = \{(x,t) : |x - x_0|^2 - \eta t^2 > c\}$, which is obtained by a shift of the domain G_c. (See fig.2).

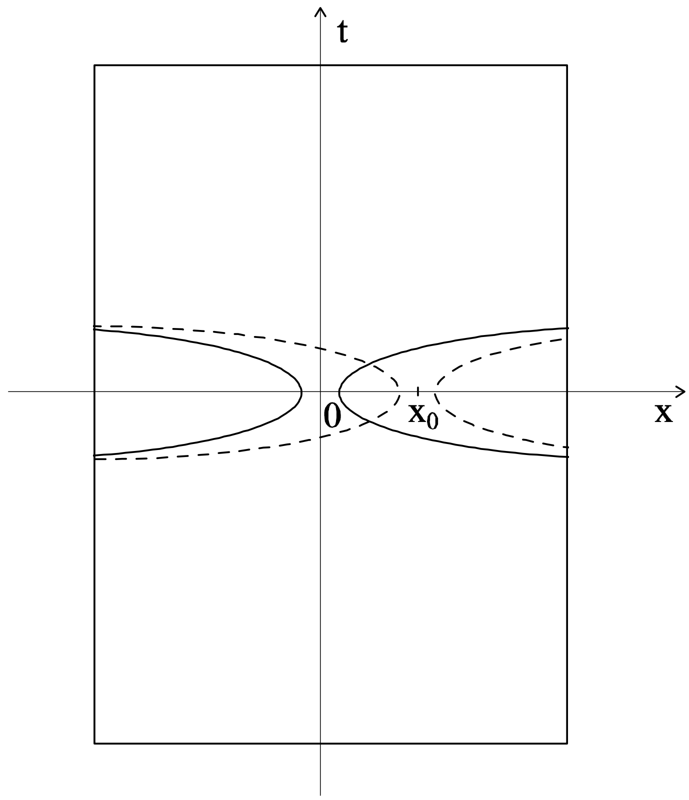

Fig.2. G_c – Solid line, $G_c(x_0)$ – Dashed line.

The Carleman estimate (2.10)-(2.11) is valid for $G_c(x_0)$. So, we can obtain an estimate similar to (3.25)

$$\|u\|^2_{L_2(H_{c+3\delta}(x_0))} \leq K\Big(\frac{e^{2\lambda m}}{\lambda}\|F\|^2_{L_2(H_c(x_0))} + \frac{e^{-2\lambda\delta}}{\lambda}\|u\|^2_{L_2(H_c(x_0))} + e^{2\lambda m}\|p\|^2_{L_2(M'_c(x_0))}\Big), \quad (3.26)$$

where $H_c(x_0) = G_c(x_0) \times S^n$ and $M'_c(x_0) = \partial G_c(x_0) \cap \{(x,t) : |x| = R\}$.

One can see from fig.2 that our choice of c and x_0 provides us with a layer

$$E_{\delta_1} = \{(x,t) : x \in \Omega, |t| < \delta_1\} \subset (G_c \cap G_c(x_0)) \quad (3.27)$$

for some sufficiently small δ_1. Estimates (3.25) and (3.26) lead to the following estimate in $E_{\delta_1} \times S^n$:

$$\|u\|^2_{L_2(E_{\delta_1}\times S^n)} \leq K\Big(\frac{e^{2\lambda m}}{\lambda}\|F\|^2_{L_2(H)} + \frac{e^{-2\lambda\delta}}{\lambda}\|u\|^2_{L_2(H)} + e^{2\lambda m}\|p\|^2_{L_2(\Gamma)}\Big). \quad (3.28)$$

Since there exists $t_1 \in (-\delta_1, \delta_1)$ such that

$$\int\limits_{S^n}\int\limits_{\Omega} u^2(x,t_1,v)dvdx \le \frac{1}{2\delta_1}\|u\|^2_{L_2(E_{\delta_1}\times S^n)}, \tag{3.29}$$

then by (3.28) we obtain

$$\int\limits_{S^n}\int\limits_{\Omega} u^2(x,t_1,v)dvdx \le N, \tag{3.30}$$

where

$$N = K\left(\frac{e^{2\lambda m}}{\lambda}\|F\|^2_{L_2(H)} + \frac{e^{-2\lambda\delta}}{\lambda}\|u\|^2_{L_2(H)} + e^{2\lambda m}\|p\|^2_{L_2(\Gamma)}\right). \tag{3.31}$$

Denote

$$Y(x,t,v) = u_t + \sum_{i=1}^{n} v_i u_i, \tag{3.32}$$

$$u(x,t_1,v) = u_0(x,v), \tag{3.33}$$

$$u|_{S^+(t_1)} = p(x,t,v), \tag{3.34}$$

where $S^+(t_1) = \partial\Omega\times(t_1,T)\times S^n$. Let $H^+(t_1) = \Omega\times(t_1,T)\times S^n$.

Estimate the $L_2(H^+(t_1))$ norm of u. Multiplying (3.32) by $2u$ and integrating over $\Omega\times S^n\times(t_1,t)$, where $t\in(t_1,T)$, we obtain

$$\int\limits_{t_1}^{t}\int\limits_{S^n}\int\limits_{\Omega} \frac{\partial}{\partial\tau}(u^2)dxdvd\tau + \int\limits_{t_1}^{t}\int\limits_{S^n}\int\limits_{\Omega}\sum_{i=1}^{n}(v_iu^2)_i dxdvd\tau = \int\limits_{t_1}^{t}\int\limits_{S^n}\int\limits_{\Omega} 2uYdxdvd\tau. \tag{3.35}$$

Consider the vector function $\vec{B} = (v_1u^2, v_2u^2, \dots, v_nu^2)$. Then

$$\sum_{i=1}^{n}(v_iu^2)_i = div\vec{B}, \tag{3.36}$$

so (3.35) becomes

$$\int\limits_{S^n}\int\limits_{\Omega} u^2(x,t,v)dxdv - \int\limits_{S^n}\int\limits_{\Omega} u^2(x,t_1,v)dxdv + \int\limits_{t_1}^{t}\int\limits_{S^n}\int\limits_{\partial\Omega}(\vec{B},\vec{n})dSdvd\tau \le \tag{3.37}$$

$$\le K\left(\int\limits_{t_1}^{t}\int\limits_{S^n}\int\limits_{\Omega} u^2dxdvd\tau + \int\limits_{t_1}^{t}\int\limits_{S^n}\int\limits_{\Omega} Y^2dxdvd\tau\right).$$

Here $(\vec{B},\vec{n})$ denotes the scalar product of vector $\vec{B}$ with vector $\vec{n}$, where $\vec{n}$ is the outward normal vector on $\partial\Omega$.

Noticing that $\vec{B} = \vec{v}\bullet u^2$, where $|\vec{v}| = 1$, we obtain

$$\int\limits_{S^n}\int\limits_{\Omega} u^2(x,t,v)dxdv \le \int\limits_{S^n}\int\limits_{\Omega} u^2\ (x,t_1,v)dxdv + \int\limits_{t_1}^{t}\int\limits_{S^n}\int\limits_{\partial\Omega} u^2 dSdvd\tau + \tag{3.38}$$

$$+ K\left(\int\limits_{t_1}^{t}\int\limits_{S^n}\int\limits_{\Omega} u^2 dxdvd\tau + \int\limits_{t_1}^{t}\int\limits_{S^n}\int\limits_{\Omega} Y^2 dxdvd\tau \right),$$

Estimate $|Y|$ using (3.8) and (3.32)

$$|Y| \le K\left[a_1|u| + \int\limits_{S^n} |u|d\sigma_\mu + |F| \right]. \tag{3.39}$$

Estimates (3.38) and (3.39) lead to

$$\int\limits_{S^n}\int\limits_{\Omega} u^2(x,t,v)dxdv \le \int\limits_{S^n}\int\limits_{\Omega} u^2\ (x,t_1,v)dxdv + \int\limits_{t_1}^{t}\int\limits_{S^n}\int\limits_{\partial\Omega} p^2 dSdvd\tau + \tag{3.40}$$

$$+ K\left(\int\limits_{t_1}^{t}\int\limits_{S^n}\int\limits_{\Omega} u^2 dxdvd\tau + \int\limits_{t_1}^{t}\int\limits_{S^n}\int\limits_{\Omega} F^2 dxdvd\tau \right).$$

Using the Gronwall's inequality, we obtain

$$\int\limits_{S^n}\int\limits_{\Omega} u^2(x,t,v)dxdv \le K\left(\int\limits_{S^n}\int\limits_{\Omega} u^2\ (x,t_1,v)dxdv + \int\limits_{t_1}^{t}\int\limits_{S^n}\int\limits_{\partial\Omega} p^2 dSdvd\tau + \int\limits_{t_1}^{t}\int\limits_{S^n}\int\limits_{\Omega} F^2 dxdvd\tau \right), \tag{3.41}$$

and by using (3.30) and (3.31) to estimate norm of $u(x,t_1,v)$, we get

$$\int\limits_{S^n}\int\limits_{\Omega} u^2(x,t,v)dxdv \le K\left(N + \int\limits_{t_1}^{t}\int\limits_{S^n}\int\limits_{\partial\Omega} p^2 dSdvd\tau + \int\limits_{t_1}^{t}\int\limits_{S^n}\int\limits_{\Omega} F^2 dxdvd\tau \right) =$$

$$= K\left(\frac{e^{2\lambda m}}{\lambda}\|F\|^2_{L_2(H)} + \frac{e^{-2\lambda\delta}}{\lambda}\|u\|^2_{L_2(H)} + e^{2\lambda m}\|p\|^2_{L_2(\Gamma)} + \int\limits_{t_1}^{t}\int\limits_{S^n}\int\limits_{\partial\Omega} p^2 dSdvd\tau + \int\limits_{t_1}^{t}\int\limits_{S^n}\int\limits_{\Omega} F^2 dxdvd\tau \right) \le \tag{3.42}$$

$$\le K\left(\frac{e^{2\lambda m}}{\lambda}\|F\|^2_{L_2(H)} + \frac{e^{-2\lambda\delta}}{\lambda}\|u\|^2_{L_2(H)} + e^{2\lambda m}\|p\|^2_{L_2(\Gamma)} \right).$$

So,

$$\|u\|^2_{L_2(H^+(t_1))} \leq K\left(\frac{e^{2\lambda m}}{\lambda}\|F\|^2_{L_2(H)} + \frac{e^{-2\lambda\delta}}{\lambda}\|u\|^2_{L_2(H)} + e^{2\lambda m}\|p\|^2_{L_2(\Gamma)}\right). \qquad (3.43)$$

One can obtain similar estimate for $\|u\|^2_{L_2(H^-(t_1))}$, where $H^-(t_1) = \Omega \times (-T, t_1) \times S^n$.

Summing up that estimate with (3.43), we obtain

$$\|u\|^2_{L_2(H)} \leq K\left(\frac{e^{2\lambda m}}{\lambda}\|F\|^2_{L_2(H)} + \frac{e^{-2\lambda\delta}}{\lambda}\|u\|^2_{L_2(H)} + e^{2\lambda m}\|p\|^2_{L_2(\Gamma)}\right). \qquad (3.44)$$

Choosing $\lambda \geq \lambda_1$ such that

$$K\frac{e^{-2\lambda\delta}}{\lambda} < \frac{1}{2}, \qquad (3.45)$$

we obtain

$$\|u\|^2_{L_2(H)} \leq K\left(\frac{e^{2\lambda m}}{\lambda}\|F\|^2_{L_2(H)} + e^{2\lambda m}\|p\|^2_{L_2(\Gamma)}\right), \qquad (3.46)$$

which implies the desired estimate (2.5). □

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