Stationary flow past a semi-infinite flat plate: analytical and numerical evidence for a symmetry-breaking solution

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Abstract

We consider the incompressible stationary flow of a fluid past a semi-infinite flat plate. This is a very old and well studied problem and is discussed in most introductory texts on fluid mechanics. Indeed, an easy scaling argument shows that far downstream the flow should be to leading order described by the so called Blasius solution, and this has been confirmed to good precision by experiments. However, there still exists no mathematical proof of the existence of a solution of the Navier-Stokes equations for this situation. Here, we do not prove existence of a solution either, but rather show that the problem might be even more complicated than hitherto thought, by providing solid arguments that a solution with broken symmetry should exist. Namely, by using techniques from dynamical system theory we analyze in detail the vorticity equation for this problem, and show that a symmetry-breaking term fits naturally into a downstream asymptotic expansion of a solution. This new term replaces the symmetric second order logarithmic term found in the literature. In contrast to all earlier work our expansion produces order by order smooth divergence free vector fields satisfying all the boundary conditions. To check that our asymptotic expressions can be completed to a solution of the Navier-Stokes equations we also solve the problem numerically, by using our results to prescribe artificial boundary conditions for a sequence of truncated domains. The results of these numerical computations are clearly compatible with the existence of a symmetry-breaking solution.

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1 Introduction

The study of the stationary Navier-Stokes flow of an incompressible fluid past a semi-infinite flat plate that is aligned with the flow at infinity has a long history [3], [22], [13], [24], [20]. The so called Blasius solution [3], is discussed in essentially any introductory textbook on fluid dynamics [2], [17], [19], [7]. It is one of the cornerstones of boundary layer theory [19] and serves as an input to the so called triple deck solution at the trailing end of a flat plate of finite length [18], [6]. Quantities that are routinely used by engineers, like the "displacement thickness" for instance, are based on the intuition that we have gained through this analysis [19], [3], [17], [7]. The semi-infinite flat plate flow also plays a role in the Orr-Sommerfeld stability theory [19] which was developed to predict the point at which the boundary layer on a wing becomes unstable. A complete mathematical understanding of this problem is therefore very desirable, would put a whole body of work on a solid footing, and might in addition shed light on a related important open question [10].

Given its practical importance, it is astonishing how little is known about this problem on a mathematical level. Indeed, there still exists no proof that the Navier-Stokes equations admit a stationary solution in the corresponding domain. In order to gain some insight into the structure of such a solution various authors have constructed higher order terms of a downstream asymptotic expansion which has as its leading order term (order zero) the solution of the Blasius equation. A first very nice paper on this

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subject was written by H. L. Alden [1]. It was however rapidly pointed out by other authors [22] that the second order term found by Alden could not be correct, since it predicted a vorticity that was not decaying exponentially fast transverse to the flow, in contradiction with experimental observation. This problem was then discussed by S. Goldstein [13] and later by M. Van Dyke [24]. In his very interesting article [13] Goldstein showed the impossibility to cure the problem by the introduction of a first order term, and reluctantly concluded that the only possibility for correcting the problem encountered by Alden was the introduction of a second order term containing logarithms. This theory has been recently reviewed in [20]. Interestingly enough it appears to have passed unnoticed in the literature that the logarithmic term violates the boundary conditions at infinity. Such difficulties with boundary conditions are not new and are considered not to be fundamental by many, but it should not be forgotten that a similar problem with the boundary conditions for the Blasius solution was the very reason for which higher order terms and matched asymptotic expansions were introduced in the first place (see Section 3).

Be this as it may, an asymptotic expansion satisfying term by term divergence freeness and all the boundary conditions in a natural way is evidently satisfactory. Similar expansions for the case of laminar flows around an obstacle of finite size have recently been discussed in [15], [4], [5]. There, such well-behaved expansions were used for prescribing artificial boundary conditions when solving the corresponding problem numerically by truncating the infinite domain to a finite computational domain. Here, we will use similar techniques in order to verify numerically that our asymptotic expressions can be completed to a solution of the Navier-Stokes equations.

As in [5] our construction of an asymptotic expansion follows basically the old ideas of Alden [1], Goldstein [13] and Van Dyke [24], supplemented with the more recent ideas from dynamical systems theory. It is these new ideas which allow us to properly address questions related to the boundary conditions. As mentioned above, Goldstein introduced his second order logarithmic correction term in order to resolve the problem with the slowly decaying vorticity term found by Alden. Our new asymmetric first order term achieves the same and makes logarithmic terms superfluous. The reason for which Goldstein could not find such a first order term is that he was restricting himself (implicitly) to symmetric flows. More historic details can be found in Section 4 and Section 5.

To summarize, the goal of this paper is two-fold: First, by providing solid evidence that a solution with broken symmetry should exist we show that the mathematical problem concerning the existence of solutions may be more complicated than hitherto thought. Second, by formulating our result as a detailed conjecture we provide an explicit framework for further research. Such an approach has already proved fruitful in fluid dynamics in the past. An example is the introduction of the concept of physically reasonable solutions by Finn [9], a framework which helped to solve the questions concerning regularity and uniqueness of stationary solutions for the case of flows for exterior domains with a smooth finite boundary.

So, consider a semi-infinite flat plate that is put into a uniform stream of a homogeneous incompressible fluid filling up all of $\mathbf{R}^2$, aligned such that the fluid flows at infinity parallel to the plate. The same problem can be posed in $\mathbf{R}^3$, but reduces to the problem in $\mathbf{R}^2$ if we restrict ourselves to solutions that are independent of the third coordinate. The situation under consideration is therefore in both cases modeled by the stationary Navier-Stokes equations

$$-\rho\left(\tilde{\mathbf{u}}\cdot\nabla\right)\tilde{\mathbf{u}}+\mu\Delta\tilde{\mathbf{u}}-\nabla\tilde{p}=0\ , \tag{1}$$

$$\nabla\cdot\tilde{\mathbf{u}}=0\ , \tag{2}$$

in $\Omega=\mathbf{R}^2\setminus\mathbf{B}$, with $\mathbf{B}=[0,\infty)$, subject to the boundary conditions

$$\tilde{\mathbf{u}}|_{\mathbf{B}}=0\ , \tag{3}$$

$$\lim_{\substack{\rho\to\infty\\ \varphi\in(0,2\pi)\\ x\in\mathbf{R}}}\tilde{\mathbf{u}}((x,0)+\rho\mathbf{e}(\varphi))=\tilde{\mathbf{u}}_\infty\ . \tag{4}$$

Here, $\tilde{\mathbf{u}}$ is the velocity field, $\tilde{p}$ is the pressure, $\tilde{\mathbf{u}}_\infty=u_\infty\mathbf{e}_1$ with $\mathbf{e}_1=(1,0)$ and $u_\infty>0$, and $\mathbf{e}(\varphi)=(\cos(\varphi),\sin(\varphi))$. The notation in the limit in (4) means that ρ goes to plus infinity for arbitrary but fixed $\varphi\in(0,2\pi)$ and $x\in\mathbf{R}$. The density ρ and the viscosity μ are arbitrary positive constants. The boundary condition (4) may not look very natural at first. It can however not be replaced by the limit where the argument of $\mathbf{u}$ goes to infinity in an arbitrary way since, because of (3), one expects that $\lim_{x\to\infty}\tilde{\mathbf{u}}(x,y)=0$ for fixed $y\in\mathbf{R}$. In directions transversal to the flow the vector field $\tilde{\mathbf{u}}$ should however converge to $\tilde{\mathbf{u}}_\infty$, and the formulation in (4) in particular ensures that $\lim_{y\to\pm\infty}\mathbf{u}(x,y) = \tilde{\mathbf{u}}_\infty$ for arbitrary fixed $x \in \mathbf{R}$. From μ, ρ and u_∞ we can form the length ℓ,

$$\ell = \frac{\mu}{\rho u_\infty} \ , \tag{5}$$

the so called viscous length of the problem. Usually, for an exterior problem with a domain of diameter A, we can compute the Reynolds number $\mathrm{Re} = A/\ell$. The geometry of the present problem is however invariant under rescaling (*i.e.*, $\mathrm{Re} = \infty$) so that we can assume without restriction of generality that $\mu = \rho = 1$. Namely, if we define dimensionless coordinates $\mathbf{x} = \tilde{\mathbf{x}}/\ell$, and introduce a dimensionless vector field $\mathbf{u}$ and a dimensionless pressure p through the definitions

$$\tilde{\mathbf{u}}(\tilde{\mathbf{x}}) = u_\infty \mathbf{u}(\mathbf{x}) \ , \tag{6}$$

$$\tilde{p}(\tilde{\mathbf{x}}) = (\rho u_\infty^2) p(\mathbf{x}) \ , \tag{7}$$

then in the new coordinates we get instead of (1)-(4) the equations

$$-(\mathbf{u}\cdot\nabla)\,\mathbf{u} + \Delta\mathbf{u} - \nabla p = 0 \ , \tag{8}$$

$$\nabla\cdot\mathbf{u} = 0 \ , \tag{9}$$

in the same domain $\Omega = \mathbf{R}^2 \setminus \mathbf{B}$, subject to the boundary conditions

$$\mathbf{u}|_{\mathbf{B}} = 0 \ , \tag{10}$$

$$\lim_{\substack{\rho\to\infty \\ \varphi\in(0,2\pi) \\ x\in\mathbf{R}}} \mathbf{u}((x,0) + \rho\mathbf{e}(\varphi)) = \mathbf{e}_1 \ . \tag{11}$$

In (8)-(9) all derivatives are with respect to the new coordinates.

The following conjecture is our main result.

Conjecture 1 *There exists a vector field $\mathbf{u} = (u,v)$ and a function p satisfying the Navier-Stokes equations (8), (9) in $\Omega = \mathbf{R}^2 \setminus [0,\infty)$, subject to the boundary conditions (10), (11), with the following properties:*
(i) there exists a sequence of divergence free vector fields $\mathbf{u}_N = \sum_{n=0}^{N}(u_n, v_n)$, $N = 0,1,2$, defined in Ω, such that

$$\lim_{x\to\infty} x^{N/2} \sup_{y\in\mathbf{R}} \left| u(x,y) - \sum_{n=0}^{N} u_n(x,y) \right| = 0 \ , \tag{12}$$

$$\lim_{x\to\infty} x^{(N+1)/2} \sup_{y\in\mathbf{R}} \left| v(x,y) - \sum_{n=0}^{N} v_n(x,y) \right| = 0 \ , \tag{13}$$

and

$$\lim_{\substack{\rho\to\infty \\ \varphi\in(0,2\pi) \\ x\in\mathbf{R}}} \rho^{[N/2]+1/2}\,(\mathbf{u} - \mathbf{u}_N)\,((x,0) + \rho\mathbf{e}(\varphi)) = 0 \ . \tag{14}$$

Here, [ ] means integer part (i.e., $[N/2] = N/2$ for N even and $(N-1)/2$ for N odd), and $\mathbf{e}(\varphi) = (\cos(\varphi), \sin(\varphi))$, and the notation in the limit in (14) means that ρ goes to plus infinity for arbitrary but fixed $\varphi \in (0,2\pi)$ and $x \in \mathbf{R}$.

(ii) the functions ω_n, $\omega_n(x,y) = -\partial_y u_n(x,y) + \partial_x v_n(x,y)$ are rapidly decaying functions of y for fixed x, in the sense that $\lim_{y\to\pm\infty} e^{C|y|}\omega_n(x,y) = 0$ for all $C > 0$, $x \in \mathbf{R}$, and $n = 0,1,2$.

(iii) the vector fields (u_0, v_0) and (u_2, v_2) are mirror symmetric with respect to the x-axis, but (u_1, v_1), and therefore $\mathbf{u}$, are not.

Below we give explicit expressions for the vector fields $\mathbf{u}_N$. The rest of the paper is organized as follows. In Section 2 we reformulate the problem in terms of the vorticity equation and give an outline of our method. In Section 3 we recall the Blasius' scaling ansatz and show that it leads, when applied correctly, to an approximate solution satisfying all the boundary conditions. In Section 4 we compute higher order terms for the case of a solution with broken symmetry. These computations involve limits of certain functions. All these limits, as well as all solutions of ordinary differential equations involved, have been calculated using the computer algebra system Maple (Maple V, Release 4, and Maple 9.51). For comparison with the literature we recall in Section 5 the symmetric expansion with Goldstein's logarithmic corrections. In Section 6 we discuss the stress tensor and give an expansion for the drag. Section 7 contains the numerical results. The corresponding computer programs are written in ADA 95 and were executed on various PC's. In Appendix I we give details concerning the the Blasius equation, the computation of the drag, and discuss the Green's function of the Laplacean for our domain. Appendix II contains all the computational details related to the asymptotic expansion.

2 The vorticity equation

Let $\mathbf{u} = (u, v)$, and let

$$\omega(x, y) = -\partial_y u(x, y) + \partial_x v(x, y) \ . \tag{15}$$

The function ω is the vorticity of the fluid. To solve (8) and (9) we can first solve (9) together with the equation that we get by taking the curl of (8),

$$W(u, v, \omega) \equiv -(\mathbf{u} \cdot \nabla)\,\omega + \Delta\omega = 0 \ . \tag{16}$$

Once (9), (15) and (16) are solved for $\mathbf{u}$ and ω, the pressure p can be constructed by solving the equation that we get by taking the divergence of (8) subject to the appropriate boundary conditions.

As we will see below, Conjecture 1 follows from a detailed analysis of the vorticity equation (16). So assume a solution $(\mathbf{u}, \omega)$ to the above problem exists. Then, in analogy with recent results [26], [28], [15], [27], [25],[12], [11], we expect the existence of functions $\omega_n \colon \Omega \to \mathbf{R}$ and a nonnegative integer $N_{\max} > 0$ (possibly infinity), such that

$$\lim_{x\to\infty} x^{(1+N)/2} \sup_{y\in\mathbf{R}} \left| \omega(x, y) - \sum_{n=0}^{N} \omega_n(x, y) \right| = 0 \ , \tag{17}$$

for $0 \leq N \leq N_{\max}$. More precisely, let $0 < \varepsilon < 1/4$, and let $\mathcal{W}$ be the Banach space of continuous functions from Ω to $\mathbf{R}$ for which the norm $\| \ \|_{\mathcal{W}}$,

$$\|\tilde{\omega}\|_{\mathcal{W}} = \sup_{(x,y)\in\Omega} \left| \tilde{\omega}(x, y\,|x|^{1/2}) \right| e^{|y|}\, |x|^{3/2+\varepsilon} \left(1 + e^{-x}\right) \ ,$$

is finite. Then we expect that

$$\omega = \sum_{n=0}^{2} \omega_n + \tilde{\omega} \ , \tag{18}$$

and for the symmetry breaking case of Conjecture 1 the functions ω_n are conjectured to be of the form

$$\omega_n(x, y) = \theta(x) x^{-(n+1)/2} \varphi_n''\left(\frac{y}{\sqrt{x}}\right) \ , \tag{19}$$

with φ_n certain smooth functions with derivatives φ_n', φ_n'' decaying at infinity faster than exponential, with φ_0 and φ_2 odd and with φ_1 even (symmetry-breaking), with θ the Heaviside function (*i.e.*, $\theta(x) = 1$ for $x > 1$ and $\theta(x) = 0$ for $x \leq 0$), and with $\tilde{\omega} \in \mathcal{W}$. From the representation (18) the decomposition of the vector field $\mathbf{u}$ in Conjecture 1 is obtained by solving equation (9) and (15).

In this paper we stay on a formal level and explain the construction of the functions φ_n by asymptotic expansion techniques, using equation (16) as a starting point. The main problem with (16) is that it involves in addition to the vorticity ω also the velocity $\mathbf{u}$. For this reason, the traditional approach for constructing an asymptotic expansions is to use an ad hoc ansatz for the stream function ψ from which one then computes expansions for u and v and ω via

$$u(x, y) = \partial_y \psi(x, y) \ , \qquad v(x, y) = -\partial_x \psi(x, y) \ , \tag{20}$$

and

$$\omega(x,y) = -\Delta\psi(x,y) \ , \tag{21}$$

and these expansions are then plugged into (16) and solved order by order. The stream function has however a more complicated structure than the vorticity and in spite of the efforts of various authors the matched asymptotic expansion for ψ of the traditional ansatz is plagued with inconsistencies concerning the boundary condition (4). Here, we solve this problem by avoiding this ad hoc ansatz. The basic observation is that from the vorticity ω and its downstream asymptotic expansion (18) an expansion of the stream function can be obtained simply by using the definitions.

Namely, let ω be given. Then, the stream function ψ has to satisfy (21) in Ω, subject to the boundary conditions

$$\psi|_{\mathbf{B}} = 0 \ , \tag{22}$$

$$\partial_{\mathbf{n}}\psi|_{\mathbf{B}} = 0 \ , \tag{23}$$

$$\lim_{\substack{\rho\to\infty \\ \varphi\in(0,2\pi) \\ x_0\in\mathbf{R}}} (\partial_y\psi, -\partial_x\psi)((x_0,0)+\rho\mathbf{e}(\varphi)) = (1,0) \ . \tag{24}$$

Equations (22) and (23) are equivalent to (10), and (24) is equivalent to (11). Note that the system of equations (21)-(24) is a priori over-determined, since for a problem of the form (21) only (22) (Dirichlet problem) or (23) (Neumann problem) can be imposed[1]. The assumption that the Navier-Stokes problem (8)-(11) has a solution therefore has the important implication that the vorticity ω has to be such that (22) and (23) are equivalent, *i.e.*, lead to the same solution ψ, and it is therefore essential to construct an asymptotic expansion which is compatible with this requirement.

Definition 2 *A function $\omega\colon \Omega \to \mathbf{R}$ is called admissible, if there exists a unique solution ψ of equation (21) subject to the boundary conditions (22) and (24) which satisfies (23).*

The functions $\sum_{n=0}^{N} \omega_n$ constructed below for $N = 0, 1, 2$ will be shown to be admissible.

In practice we simply first solve (21) by using the Dirichlet boundary condition (22) and verify then in a second step (23). So let ω be given, and define for $(x, y) \in \Omega$ the functions r and r_- by the equations

$$r(x,y) = \sqrt{x^2+y^2} \ , \qquad r_-(x,y) = \sqrt{2r(x,y)-2x} \ . \tag{25}$$

Then, the general solution of (21) satisfying the boundary conditions (22) and (24) is (see Appendix I for details),

$$\psi(x,y) = y + \alpha r_-(x,y) + \psi_\omega(x,y) \ , \tag{26}$$

with $\alpha \in \mathbf{R}$ arbitrary, and with $\psi_\omega = L_G(\omega)$, where

$$L_G(\omega)(x,y) = -\int_\Omega G(x,y;x_0,y_0)\ \omega(x_0,y_0)\ dx_0 dy_0 \ , \tag{27}$$

with G the Green's function of the Laplacean in Ω with Dirichlet boundary conditions on $[0,\infty)$ and at infinity. Namely,

$$G(x,y;x_0,y_0) = \tilde{G}(y/r_-(x,y),\ r_-(x,y)/2;\ y_0/r_-(x_0,y_0),\ r_-(x_0,y_0)/2) \ , \tag{28}$$

where

$$\begin{aligned}\tilde{G}(\xi,\eta;\xi_0,\eta_0) = {} & \frac{1}{4\pi}\log((\xi-\xi_0)^2+(\eta-\eta_0)^2) \\ & -\frac{1}{4\pi}\log((\xi-\xi_0)^2+(\eta+\eta_0)^2) \ . \end{aligned} \tag{29}$$

Note that $\tilde{G}$ is nothing else than the Green's function of the Laplacean in the upper half plane $H = \{z \in \mathbf{C} \mid \mathrm{Im}(z) > 0\}$ with Dirichlet boundary conditions on the real axis, and the arguments in the definition

[1]For the (singular) domain Ω at hand the solution of the Dirichlet or Neumann problem is determined by the above boundary conditions only up to a multiple of a certain harmonic function, since the boundary condition (24) at infinity is not sufficient to ensure uniqueness.

(28) are obtained from the inverse of the conformal mapping $H \to \Omega$, $z \mapsto z^2$. Here we have interpreted Ω as a subset of the complex plane. Let $\psi_{0,\infty}(x,y) = y$, $\psi_{1,\infty}(x,y) = \alpha r_-(x,y)$ and $\psi_{n,\infty} = 0$ for $n \geq 2$. For the function ψ we will then use below for $0 \leq N \leq 2$ the decomposition

$$\psi = \sum_{n=0}^{N} \psi_n + R_N \ , \tag{30}$$

where,

$$\psi_n = \psi_{n,\infty} - L_G(\omega_n) \ , \tag{31}$$

$$R_N = \sum_{n=N}^{\infty} \psi_{n,\infty} - L_G(\omega - \sum_{n=0}^{N} \omega_n) \ , \tag{32}$$

and we will show formally that there are functions ω_n such that

$$\lim_{x,y\to\infty} r^{3/2}\partial_x R_2(x,y) = \lim_{x,y\to\infty} r^{3/2}\partial_y R_2(x,y) = 0 \ , \tag{33}$$

provided the solution ω is indeed as conjectured in (18) with $\tilde{\omega} \in \mathcal{W}$.

Basically, the idea is now to use the functions $\sum_{n=0}^{N} \psi_n$ as an approximation to ψ in order to compute approximations for $\mathbf{u} = (u,v)$ using (20). These approximations are then plugged together with the approximation $\sum_{n=0}^{N} \omega_n$ for ω into (16) in order to obtain recursively equations for the functions ω_n. This way, by construction, all vector fields are smooth in Ω and satisfy the boundary conditions (22) and (24) and a posteriori also (23), since the functions $\sum_{n=0}^{N} \omega_n$ turn out to be admissible in the sense of Definition in 2. This solves the above mentioned consistency problems of the ad hoc procedures found in the literature at the price of introducing non-local expressions for ψ_n due to the integration in the definition (31). Such non-local expressions are not manipulated easily when trying to solve the resulting equations for ω_n, and for $0 \leq N \leq 2$ we have therefore analyzed the functions ψ_n in detail. It turns out that, modulo terms obeying the same bounds as R_2 in (33), local approximations $\psi_{n,\mathrm{loc}}$ for ψ_n can be constructed, such that if we use these approximations instead of ψ_n to compute the approximations $\mathbf{u}_N = \sum_{n=0}^{N}(u_n, v_n)$ for $\mathbf{u}$, the vector fields $\mathbf{u}_N$ nevertheless satisfy all the boundary conditions.

3 Blasius equation and beyond

In order to motivate the mathematical analysis in subsequent sections we recall here briefly the Blasius' theory [3], [18]. This also allows us to give the reader a first glimpse at our method. Let $x, y > 0$ and set $\psi(x,y) = \psi_B(x,y) \equiv \sqrt{x} f(y/\sqrt{x})$, with f the solution of the Blasius equation, defined for $z \geq 0$ by,

$$f'''(z) + \frac{1}{2} f(z) f''(z) = 0 \ , \qquad f(0) = f'(0) = 0 \ , \qquad \lim_{z\to\infty} f'(z) = 1 \ . \tag{34}$$

See equation (40) below and Appendix I for details concerning the equation. We have that

$$f''(0) = a_2 = 0.332057\ldots \ , \tag{35}$$

$$\lim_{z\to\infty}(f(z) - z) = a = -1.72078\ldots \ , \tag{36}$$

and the function $z \mapsto f(z) - z - a$ and all its derivatives decay at infinity faster than exponential. See Figure 3 for a graph of f', f'' and $z \mapsto f(z) - z - a$. The idea behind the above ansatz for the stream function is the experimental observation that a boundary layer of width $\sqrt{x}$ forms along the plate (see for example [19]), and ψ_B is supposed to describe the flow in this boundary layer to leading order of an expansion for large x and fixed ratio $y/\sqrt{x} > 0$. From ψ_B we find with (20)

$$u_B(x,y) = \partial_y \psi_B(x,y) = f'(\frac{y}{\sqrt{x}}) \ , \tag{37}$$

$$v_B(x,y) = -\partial_x \psi_B(x,y) = -\frac{1}{2}\frac{1}{\sqrt{x}}\left(f(\frac{y}{\sqrt{x}}) - \frac{y}{\sqrt{x}} f'(\frac{y}{\sqrt{x}})\right) \ , \tag{38}$$

and from (21) we find, neglecting terms of order $1/x^{3/2}$,

$$\omega_B(x,y) = -\frac{1}{\sqrt{x}} f''(\frac{y}{\sqrt{x}}) \ . \tag{39}$$

By construction the vector field $(u_B,\ v_B)$ is divergence free. We now substitute (37)-(39) into (16) and compute the limit as $x \to \infty$, keeping $z = y/\sqrt{x} > 0$ fixed. We find (by hand, or using a computer algebra system) that

$$\lim_{x\to\infty} x^{3/2} W(u_B, v_B, \omega_B)(x, z\sqrt{x}) = -(\frac{1}{2} f f'' + f''')'(z) \ , \tag{40}$$

and the right hand side in (40) equals zero since f solves the Blasius equation (34). Therefore, in the sense of the limit in (40), (37)-(39) provide a solution of equation (16) to leading order. Note that the boundary conditions on f in (34) imply that $u_B(x,0) = v_B(x,0) = 0$ and that $\lim_{y\to\infty} u_B(x,y) = 1$ for $x \geq 0$. Therefore the boundary condition (10) is satisfied, but because of (36) we find that for $x > 0$

$$\lim_{y\to\infty} (u_B, v_B)(x,y) = (1, -\frac{a}{2\sqrt{x}}) \neq (1,0) \ , \tag{41}$$

i.e., the vector field (u_B, v_B) does not satisfy the boundary condition (11). At this point one usually resolves to some sort of hand-waving and explains that the Blasius' theory was after all only meant to describe the flow within the boundary layer. The following proposition shows that, by following the procedure outlined in the preceding section, the Blasius' ansatz naturally leads to a vector field satisfying all the boundary conditions:

Proposition 3 *Let f be the solution of the Blasius equation (34) and define the function $\omega_0 \colon \Omega \to \mathbf{R}$ by the equation*

$$\omega_0(x,y) = -\mathrm{sign}(y) \frac{\theta(x)}{\sqrt{x}} f''(\frac{|y|}{\sqrt{x}}) \ , \tag{42}$$

with θ the Heaviside function (i.e., $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x \leq 0$). Then ω_0 is admissible in the sense of Definition 2.

A proof of this proposition is given in Appendix II.

From Proposition 3 it follows that there is a unique solution ψ_0 of $\Delta\psi_0 = -\omega_0$ in Ω, such that the vector field $(\partial_y\psi_0, -\partial_x\psi_0)$ satisfies the boundary conditions (10), (11). In Appendix II we moreover extract from ψ_0 a local approximation $\psi_{0,\mathrm{loc}}$,

$$\psi_{0,\mathrm{loc}}(x,y) = y + a \frac{y}{\sqrt{2\sqrt{x^2+y^2} - 2x}} + \theta(x)\mathrm{sign}(y)\sqrt{x}(f(\frac{|y|}{\sqrt{x}}) - \frac{|y|}{\sqrt{x}} - a) \ . \tag{43}$$

Note that since $\lim_{y\to 0} y/r_-(x,y) = \sqrt{x}\ \mathrm{sign}(y)$ for $x > 0$, we find that that $\psi_{0,\mathrm{loc}}(x,0) = 0$ for $x > 0$. From (43) one finds the vector field $\mathbf{u}_0 = (u_0, v_0) = (\partial_y\psi_{0,\mathrm{loc}}, -\partial_x\psi_{0,\mathrm{loc}})$,

$$u_0(x,y) = u_{0,E}(x,y) + \theta(x)(f'(\frac{|y|}{\sqrt{x}}) - 1) \ , \tag{44}$$

$$v_0(x,y) = v_{0,E}(x,y) - \theta(x)\mathrm{sign}(y)\frac{1}{2\sqrt{x}}(f(\frac{|y|}{\sqrt{x}}) - \frac{|y|}{\sqrt{x}} f'(\frac{|y|}{\sqrt{x}}) - a) \ , \tag{45}$$

with

$$u_{0,E}(x,y) = 1 + \frac{a}{4}\frac{r_-(x,y)}{r(x,y)} \ , \qquad v_{0,E}(x,y) = -\frac{a}{2}\frac{y}{r_-(x,y)\ r(x,y)} \ , \tag{46}$$

and r and r_- as defined in (25). It is easily checked that the vector field $\mathbf{u}_0$ is smooth in Ω. Note that

$$u_0(x,y) = u_{0,E}(x,y) + \theta(x)(u_B(x,|y|) - 1) \ ,$$
$$v_0(x,y) = v_{0,E}(x,y) + \theta(x)\mathrm{sign}(y)(v_B(x,|y|) + \frac{a}{2\sqrt{x}}) \ ,$$

and therefore we see using (41) that the boundary conditions (10) and (11) are satisfied. Moreover we find (see Appendix II) that

$$\lim_{x\to\infty} x^{3/2}W(\partial_y\psi_0, -\partial_x\psi_0, \omega_0)(x, z\sqrt{x}) = \lim_{x\to\infty} x^{3/2}W(u_0, v_0, \omega_0)(x, z\sqrt{x}) \;, \tag{47}$$

and as in (40), that for $z \in \mathbf{R}$

$$\lim_{x\to\infty} x^{3/2}W(u_0, v_0, \omega_0)(x, z\sqrt{x}) = -\mathrm{sign}(z)(\frac{1}{2}ff'' + f''')'(|z|) \;, \tag{48}$$

with the right hand side of (48) being equal to zero because f solves the Blasius equation (34). This means that the theoretical prediction for the leading order asymptotic shape of the flow in the boundary layer is not affected by the replacement of (u_B, v_B, ω_B) by (u_0, v_0, ω_0). This is what we should expect, since the correctness of the Blasius velocity profile has been experimentally checked to good precision [19].

We conclude that, at least on the level of the Blasius ansatz, there is in fact no problem with boundary conditions, provided we interpret it as an ansatz for the vorticity and not for the stream function. Last but not least we note that the vector field $\mathbf{u}_0$ is similar to the leading order term of the matched asymptotic expansion that Goldstein proposed in order to cure the problem with the limit (41). See [13].

3.1 Pressure

In Section 8 we need an approximate expression for the pressure. Let $\mathbf{u} = (u, v) = (\partial_y\psi, -\partial_x\psi)$. From (8) we find for p the equation

$$\Delta p = 2(\partial_x u\partial_y v - \partial_x v\partial_y u) = 2\; J(\psi) \;, \tag{49}$$

where $J(\psi)$ is the Jacobian of ψ,

$$J(\psi) = \det\begin{pmatrix} \partial_x^2\psi & \partial_x\partial_y\psi \\ \partial_x\partial_y\psi & \partial_y^2\psi \end{pmatrix} \;.$$

Furthermore we get from (10), using (8) and (15) for $x \geq 0$ the boundary condition

$$\lim_{y\to\pm 0} \partial_y p(x, y) = \lim_{y\to\pm 0} \partial_x\omega(x, y) \;. \tag{50}$$

By hand, or using a computer algebra system, we find that

$$\lim_{x\to\infty} x^2 J(\psi_{0,\mathrm{loc}})(x, zx^{1/2}) = \rho_0''(|z|) \;, \tag{51}$$

where

$$\rho_0(z) = -\frac{1}{4}f(z)^2 + \frac{1}{4}zf(z)f'(z) + \frac{1}{2}zf''(z) + \frac{a}{4}z + \frac{a^2}{4} \;.$$

Note that $\lim_{z\to\infty}\rho_0(z) = 0$. From (50) and (42) we get that an approximation p_0 to the pressure has to satisfy the boundary condition

$$\lim_{y\to\pm 0} \partial_y p_0(x, y) = \lim_{y\to\pm 0} \partial_x\omega_0(x, y) = \frac{a_2}{2}\frac{\mathrm{sign}(y)}{x^{3/2}} \;. \tag{52}$$

Since $\Delta\rho_0(y/x^{1/2}) \approx \partial_y^2\rho_0(y/x^{1/2})$ in the sense of limit (48), we conclude from (51) that the function ρ_0 determines the pressure to leading order, modulo a harmonic function which has to be chosen such that the boundary condition (52) is satisfied. We therefore get that $p \approx p_0$, where

$$p_0(x, y) = \frac{\theta(x)}{x}\rho_0(\frac{|y|}{\sqrt{x}}) - \frac{a}{4}\frac{\sqrt{2\sqrt{x^2+y^2} - 2x}}{\sqrt{x^2+y^2}} \;. \tag{53}$$

4 The symmetry-breaking case

When analyzing equation (16) to leading order in the sense of limit (48) we found the Blasius equation, which is a nonlinear third order ordinary differential equation. Similarly, when discussing the higher order term of order $n \geq 1$, one finds the equation

$$L_n g_n = j_n \ ,$$

for certain functions j_n depending on the solution up to order $n-1$, and with L_n the third order linear ordinary differential operator defined for $n \geq 1$ and $z \geq 0$ by the equation

$$(L_n g)(z) = g'''(z) + \frac{1}{2} f(z) g''(z) + \frac{n}{2} f'(z) g'(z) - \frac{1}{2}(n-1) f''(z) g(z) \ , \tag{54}$$

where f is the solution of the Blasius equation. The operators L_n have been analyzed in some detail by Alden [1], and then by Goldstein [13]. It is easily verified that the multiples of the function f' are in the kernel of L_n for all $n \geq 1$. The kernel of L_1 contains in addition the constant functions and the kernel of L_2 the multiples of the function $f_{2,0}$,

$$f_{2,0}(z) = (f(z) - z f'(z))/a \ , \tag{55}$$

with a as defined in (36). With this normalization $\lim_{z\to\infty} f_{2,0}(z) = 1$. See Figure 3 for a graph of $f_{2,0}$.

Alden, in the paper mentioned in the introduction [1], tried to get higher order corrections by an ad hoc ansatz for the stream function which corresponds to keeping only terms with n even in (17). He found the equation $L_2 g_2 = j_{2,0}$ for a certain function $j_{2,0}$ given below. The problem with this equation is that the function $j_{2,0}$ is not in the image of L_2 of a function with derivatives of rapid decrease. This is related to the fact that the function $f_{2,0}$ is in the kernel of L_2. The equation still has a solution though and this is the solution that Alden constructed, but its derivatives decay only algebraically at infinity, and as explained above this is in contradiction with experimental observations. For this reason Goldstein made an ansatz which corresponds to also keeping terms with n odd in (17) which, on the basis of more recent mathematical results [25], is indeed expected to be the correct ansatz for the problem.

4.1 The first order term

Goldstein's hope was that through the introduction of a term with $n = 1$ one would be able to adjust the right hand side of the second order equation in order to obtain a solution with derivatives of rapid decrease. By restricting himself implicitly to vector fields that are symmetric with respect to the x-axis, Goldstein found for $n = 1$ the homogeneous equation $L_1 g_1 = 0$. The only solution of this equation satisfying the "natural" boundary conditions $g_1(0) = g_1'(0) = 0$ is $g_1 \equiv 0$, and one therefore again finds for $n = 2$ the solution of Alden. Goldstein then also tried to use the boundary conditions $g_1(0) = 1$, $g_1'(0) = 0$ instead, for which $g_1 \equiv 1$. He then correctly concluded that this leads to a vector field violating the boundary conditions. He therefore reluctantly put $g_1 \equiv 0$ and instead introduced the logarithmic correction term, which for comparison with the literature is discussed in Section 5.

The discovery here is that Goldstein's original idea actually works. Namely, if we use the boundary conditions $g_1(0) = 0$, $g_1'(0) = 1$ for which the solution of the homogeneous equation $L_1 g_1$ is $g_1 = f'$ and furthermore give up the mirror-symmetry of the vector field with respect to the x-axis, then we can construct a solution satisfying the boundary conditions. More precisely we have the following proposition:

Proposition 4 *Let ω_0 be as defined in (42). Let $f_1 \colon \mathbf{R}_+ \to \mathbf{R}$ be the solution of the equation*

$$f_1''(z) + \frac{1}{2} f(z) f_1'(z) = \frac{1}{2}(f(z) - z - a), \qquad f_1(0) = 0,\ f_1'(0) = 1 \ , \tag{56}$$

and define $\omega_1 \colon \Omega \to \mathbf{R}$ by the equation

$$\omega_1(x,y) = -\frac{b}{2}\ \theta(x) \frac{1}{x} f_1''\left(\frac{|y|}{\sqrt{x}}\right) \ , \tag{57}$$

for $b \in \mathbf{R}$. Then, the function $\omega_0 + \omega_1$ is admissible in the sense of Definition 2.

A proof of this proposition is given in Appendix II.

Note that, in contrast to the order zero term (42), the function ω_1 is even in y (otherwise $\omega_0+\omega_1$ would not be admissible), and the corresponding vector field is therefore not mirror symmetric with respect to the x-axis. Taking the derivative of equation (56) we get that $(L_1 f_1)(z) = (f'(z) - 1)/2$. What comes as a surprise is the fact that the function f_1 does not have to solve the homogeneous equation $L_1 f_1 = 0$ as one would have expected on a heuristic level, but the inhomogeneous equation (56). The right hand side in this equation is produced through a nonlinear coupling between the stream functions of order zero and order one. We take the fact that this equation has a nontrivial solution with the desired properties as an indication in favor of a symmetry breaking solution. The equation (56) can be solved explicitly. One finds

$$f_1(z) = \frac{1}{2}\int_0^z d\zeta\ f''(\zeta)\int_0^\zeta \frac{f(\eta)-\eta-a}{f''(\eta)}\ d\eta + f'(z)/a_2\ , \tag{58}$$

with f the solution of the Blasius equation and a_2 as in (35). See Figure 4 for a graph of f_1. The derivatives of f_1 decay faster than exponential at infinity.

From Proposition 4 it follows that there is a unique solution ψ_1 of $\Delta\psi_1 = -\omega_1$ in Ω, such that the vector field $\mathbf{u}_0 + (\partial_y\psi_1, -\partial_x\psi_1)$, with $\mathbf{u}_0$ as defined in (44)-(46), satisfies the boundary conditions (10), (11). In Appendix II we also extract from ψ_1 a local approximation $\psi_{1,\mathrm{loc}}$,

$$\psi_{1,\mathrm{loc}}(x,y) = -\frac{b}{2}\sqrt{2\sqrt{x^2+y^2}-2x} - \frac{b}{2}c_1 + \frac{b}{2}\theta(x)(f_1(\frac{|y|}{\sqrt{x}}) - c_1)\ , \tag{59}$$

where

$$c_1 = \lim_{z\to\infty} f_1(z) = 5.353\ldots\ .$$

We use the function $\psi_{1,\mathrm{loc}}$ to define the vector field $(u_1, v_1) = (\partial_y\psi_{1,\mathrm{loc}}, -\partial_x\psi_{1,\mathrm{loc}})$,

$$u_1(x,y) = u_{1,E}(x,y) + \theta(x)\frac{b}{2}\frac{\mathrm{sign}(y)}{\sqrt{x}}f_1'(\frac{|y|}{\sqrt{x}})\ , \tag{60}$$

$$v_1(x,y) = v_{1,E}(x,y) + \theta(x)\frac{b}{4}\frac{1}{x}\frac{|y|}{\sqrt{x}}f_1'(\frac{|y|}{\sqrt{x}})\ , \tag{61}$$

where

$$u_{1,E}(x,y) = -\frac{b}{2}\frac{y}{r_-(x,y)\ r(x,y)}\ , \qquad v_{1,E}(x,y) = -\frac{b}{4}\frac{r_-(x,y)}{r(x,y)}\ , \tag{62}$$

with r and r_- as defined in (25). It is easily checked that the vector field $\mathbf{u}_1 = \mathbf{u}_0 + (u_1, v_1)$ is smooth in Ω and satisfies the boundary conditions (10), (11). Equation (56) is obtained from (16) in the limit (computed with a computer algebra system)

$$\begin{aligned}
&\lim_{x\to\infty} x^2 W(\partial_y\psi_0 + \partial_y\psi_1, -\partial_x\psi_0 - \partial_x\psi_1, \omega_0+\omega_1)(x, z\sqrt{x}) \\
&\quad = \lim_{x\to\infty} x^2 W(u_0+u_1, v_0+v_1, \omega_0+\omega_1)(x, z\sqrt{x}) \\
&\quad = \frac{b}{4}f''(|z|) - \frac{b}{2}(\frac{1}{2}ff_1' + f_1'')''(|z|)\ ,
\end{aligned} \tag{63}$$

and the right hand side of (63) is equal to zero because f_1 solves equation (56). Note the presence of the term $bf''/4$ on the right hand side of equation (63), which as explained above comes as a surprise when compared with naïve perturbation theory. The constant b in (63) remains undetermined at this stage. It will be determined from the computation to second order. Note that, since $\lim_{y\to 0}\psi_{1,\mathrm{loc}}(x,y) = -bc_1$ for $x > 0$, there is for $b > 0$ a finite amount more of the fluid passing above the plate than below, and vice versa for $b < 0$.

4.2 The second order term

As mentioned above the source of all difficulties in the construction of an asymptotic expansion is the equation $L_2 g_2 = j_2$, which is obtained when studying (16) to second order. Without the contribution coming from a nonzero term of order one (or logarithmic corrections, see Section 5), the right hand side in this equation is not in the image of L_2 of functions with derivatives of rapid decrease. With our first order term we get to second order the equation

$$(L_2 f_2)(z) = j_{2,0}(z) + b^2 j_{2,1}(z), \qquad f_2(0) = f_2'(0) = f_2''(0) = 0 \ , \tag{64}$$

with

$$\begin{aligned} j_{2,0}(z) = {} & \frac{a}{16} z^2 f''(z) - \frac{1}{8} z f(z) f'(z) + \frac{1}{8} z^2 f''(z) f(z) \\ & - \frac{3}{2} z f''(z) - \frac{1}{8} f'(z)^2 z^2 - \frac{a}{8} f(z) + \frac{1}{4} f(z)^2 \\ & - \frac{1}{4} a z f'(z) - \frac{1}{8} a^2 \ , \end{aligned} \tag{65}$$

which is (modulo normalization) the function already obtained by Alden in [1], and

$$j_{2,1}(z) = \frac{1}{4} f_1'(z)(1 - \frac{1}{2} f_1'(z)) \ . \tag{66}$$

The (real) number b in (64) has to be chosen such that

$$\int_0^\infty f(z)(j_{2,0}(z) + b^2 j_{2,1}(z)) \ dz = 0 \ . \tag{67}$$

The condition (67) ensures that the right hand side is the image of L_2 of functions with derivatives of rapid decrease at infinity. Namely (see Alden [1]), the function f is an integrating factor for L_2, *i.e.*,

$$f L_2(g) = (f g'' + (\frac{1}{2} f^2 - f') g' + f'' g)' \ , \tag{68}$$

and therefore (64) is equivalent to the equation

$$(f f_2'' + (\frac{1}{2} f^2 - f') f_2' + f'' f_2)(z) = \int_0^z f(\xi)(j_{2,0}(\xi) + b^2 j_{2,1}(\xi)) \ d\xi \ . \tag{69}$$

Equation (69) can again be solved explicitly in terms of quadratures (see Alden [1]), and by virtue of (67) the derivatives of the solution f_2 are functions of rapid decrease at infinity. Note that the functions $j_{2,0}$ and $j_{2,1}$ decay at infinity also faster than exponential so that the integral in (67) is well defined. See Figure 4 for a graph of $j_{2,0}$ and $j_{2,1}$. A priori it is however not clear that the signs in equation (67) are such that the resulting equation can be solved for $b \in \mathbf{R}$. We take the fact that this is indeed the case as a further indication for the existence of a solution with broken symmetry. Numerically we find that

$$b = \pm 1.2378\ldots \ , \tag{70}$$

and we use from now on f_2 to mean the solution of equation (64) obtained with this value of b. See Figure 4 for a graph of f_2.

After these preparatory remarks we can now formulate the results concerning the expansion to second order:

Proposition 5 *Let f be the Blasius function and let f_1 be as defined in (56). Let*

$$\tilde{f}_2(z) = f_2(z) + c_{2,0} f_{2,0}(z) \ , \tag{71}$$

with f_2 the solution of equation (64), with $f_{2,0}$ as defined in (55) and with $c_{2,0}$ an arbitrary real constant. Let furthermore

$$\tilde{f}_0(z) = \frac{1}{4} z^2 f(z) - \frac{3}{4} z \ F_1(z) + \frac{3}{4} F_2(z) + \frac{a z^2}{8} + \frac{3}{4} \lambda_1 \ , \tag{72}$$

with

$$F_1(z) = \int_0^z f(\xi) \ d\xi \ , \qquad F_2(z) = \int_0^z F_1(\xi) \ d\xi \ ,$$

and $\lambda_1 = \int_0^\infty d\xi \int_\xi^\infty (f(\eta) - \eta - a) \ d\eta$. Let $\omega_2 = \tilde{\omega}_0 + \tilde{\omega}_2$, with $\tilde{\omega}_i \colon \Omega \to \mathbf{R}$, $i \in \{0, 2\}$ defined by the equation

$$\tilde{\omega}_i(x, y) = -\mathrm{sign}(y) \theta(x) \frac{1}{x^{3/2}} \tilde{f}_i''(\frac{|y|}{\sqrt{x}}) \ . \tag{73}$$

Then, the function $\sum_{n=0}^2 \omega_n$ is admissible in the sense of Definition 2.

A proof of this proposition is given in Appendix II.

From Proposition 5 it follows that there exists a function ψ_2 that solves the equation $\Delta\psi_2 = -\omega_2$ in Ω, such that the vector field $\mathbf{u}_1 + (\partial_y\psi_2, -\partial_x\psi_2)$ satisfies the boundary conditions (10) and (11). The reason for introducing ω_2 as the sum of two terms is that $\psi_0 - \psi_{0,\mathrm{loc}}$ is of the same order as ψ_2. In fact (see Appendix II), we have that $\Delta(\psi_0 - \psi_{0,\mathrm{loc}}) = -\tilde{\omega}_0$, so that is is sufficient to compute a solution $\tilde{\psi}_2$ of the equation $\Delta\tilde{\psi}_2 = -\tilde{\omega}_2$. In Appendix II we extract from $\tilde{\psi}_2$ a local approximations $\tilde{\psi}_{2,\mathrm{loc}}$,

$$\tilde{\psi}_{2,\mathrm{loc}}(x,y) = \tilde{c}_2 \frac{y}{r(x,y)\; r_-(x,y)} + \theta(x)\mathrm{sign}(y)\frac{1}{\sqrt{x}}(\tilde{f}_2(\frac{|y|}{\sqrt{x}}) - \tilde{c}_2) \;, \tag{74}$$

with r and r_- as defined in (25), and with $\tilde{c}_2 = c_2 + c_{2,0}$, where

$$c_2 = \lim_{z\to\infty} f_2(z) = -3.777\ldots \;. \tag{75}$$

Note that $\lim_{y\to\pm 0}\tilde{\psi}_{2,\mathrm{loc}}(x,y) = 0$ for $x > 0$. We use $\tilde{\psi}_{2,\mathrm{loc}}$ to define the vector field $(u_2, v_2) = (\partial_y\tilde{\psi}_{2,\mathrm{loc}}, -\partial_x\tilde{\psi}_{2,\mathrm{loc}})$,

$$\begin{aligned} u_2(x,y) &= u_{2,E}(x,y) + \theta(x)\frac{1}{x}\tilde{f}_2'(\frac{|y|}{\sqrt{x}}) \;, \\ v_2(x,y) &= v_{2,E}(x,y) + \theta(x)\frac{\mathrm{sign}(y)}{2x^{3/2}}(\tilde{f}_2(\frac{|y|}{\sqrt{x}}) + \frac{|y|}{\sqrt{x}}\tilde{f}_2'(\frac{|y|}{\sqrt{x}}) - \tilde{c}_2) \;, \end{aligned} \tag{76}$$

where

$$u_{2,E}(x,y) = -\frac{\tilde{c}_2}{4}\frac{r_-}{r^2}(1 + \frac{2x}{r}) \;, \qquad v_{2,E}(x,y) = -\frac{\tilde{c}_2}{2}\frac{y}{r_-\; r^2}(1 - \frac{2x}{r}) \;. \tag{77}$$

The vector field $\mathbf{u}_2 = \mathbf{u}_1 + (u_2, v_2)$ is smooth in Ω and satisfies the boundary conditions (10), (11). Finally, equation (64) is obtained from (16) by the limit (computed with a computer algebra system),

$$\begin{aligned} &\lim_{x\to\infty} x^{5/2} W(u_0 + u_1 + u_2, v_0 + v_1 + v_2, \omega_0 + \omega_1 + \omega_2)(x, z\sqrt{x}) \\ &\qquad = -\mathrm{sign}(z)(L_2\tilde{f}_2 - j_{2,0} - b^2 j_{2,1})'(|z|) \;, \end{aligned} \tag{78}$$

and the right hand side of (78) is equal to zero because f_2 solves equation (64) and $f_{2,0}$ is in the kernel of L_2.

5 The symmetric case

For comparison with the literature we recall in this section some facts about the symmetric expansion involving logarithmic corrections proposed by Goldstein [13]. For this case we still expect (17), but the functions ω_n are more complicated than in (19). Namely, Goldstein proposed that there should be functions $\varphi_{n,m}$, with derivatives decaying rapidly at infinity, such that

$$\omega_n(x,y) = \sum_{m=0}^{n} \rho_{n,m}(x)\varphi_{n,m}''(\frac{y}{\sqrt{x}})$$

with

$$\rho_{n,m}(x) = \theta(x)\frac{\log(x)^{n-m}}{x^{(n+1)/2}} \;.$$

See [24] for a motivation concerning the logarithmic terms. To leading order one finds as before the vector field (44)-(46) and (48), the first order term is identically zero, and for the second order terms one makes the ansatz

$$\omega_2(x,y) = \omega_{2,1}(x,y) + \omega_{2,2}(x,y) \tag{79}$$

with

$$\omega_{2,1}(x,y) = -b_s\;\mathrm{sign}(y)\theta(x)\frac{\log(x)}{x^{3/2}} f_{2,0}''(\frac{|y|}{\sqrt{x}}) \;, \tag{80}$$

with $f_{2,0}$ as defined in (55), and with $\omega_{2,2} = \tilde{\omega}_0 + \tilde{\omega}_{2,2}$, with $\tilde{\omega}_0$ as defined in (73), and where

$$\tilde{\omega}_{2,2}(x,y) = -\text{sign}(y)\theta(x)\frac{1}{x^{3/2}}\tilde{g}_2''(\frac{|y|}{\sqrt{x}}) \ , \tag{81}$$

with $\tilde{g}_2 = g_2 + c_{2,0}f_{2,0}$, with $f_{2,0}$ as defined in (55) and $c_{2,0}$ an arbitrary real constant, and with g_2 the solution of the equation

$$(L_2 g_2)(z) = j_{2,0}(z) + b_s\ j_{s,2,1}(z), \qquad g_2(0) = g_2'(0) = g_2''(0) = 0 \ , \tag{82}$$

with $j_{2,0}$ as defined in (65) and with

$$j_{s,2,1}(z) = -\frac{1}{a}f''(z)f(z) \ , \tag{83}$$

where b_s has to be chosen such that

$$\int_0^\infty f(z)\left(j_{2,0}(z) + b_s\ j_{s,2,1}(z)\right)\ dz = 0 \ . \tag{84}$$

Numerically we find that

$$b_s = 1.427\ldots \ , \tag{85}$$

and we use from now on g_2 to mean the solution of equation (82) obtained with this value of b_s.

The problem with the above ansatz is that the function $\omega_0 + \omega_{2,1}$ is not admissible in the sense of Definition 2. More precisely, there is no solution $\psi_{2,1}$ to $\Delta\psi_{2,1} = -\omega_{2,1}$ such that the vector field $\mathbf{u}_0 + (\partial_y\psi_{2,1}, -\partial_x\psi_{2,1})$ satisfies both of the boundary conditions (10) and (11). Here, in order to circumvent this problem for numerical purposes and for comparison with the literature, we have added to the local approximation obtained from $\psi_{2,1}$ as defined by Dirichlet boundary conditions an ad hoc term of higher order, in the spirit of our results in [5]. This produces a modified local approximation $\psi_{2,1,\text{loc}}$ such that the vector field $\mathbf{u}_{2,1} = \mathbf{u}_0 + (\partial_y\psi_{2,1,\text{loc}}, -\partial_x\psi_{2,1,\text{loc}})$ satisfies both of the boundary conditions (10) and (11). Explicitly we have

$$\begin{aligned}\psi_{2,1,\text{loc}}(x,y) &= b_s y\frac{\log(r)}{r\ r_-} + \frac{b_s}{2}\frac{r_-}{r}(\arctan(\frac{y}{x}) - \pi\theta(x)\text{sign}(y)) \\ &\quad + b_s\text{sign}(y)\theta(x)\frac{\log(x)}{x^{1/2}}(f_{2,0}(\frac{|y|}{\sqrt{x}}) - 1) + \lambda\frac{b_s\pi}{2a_2^2}\frac{1}{x}f'(\frac{|y|}{\sqrt{x}})f''(\frac{|y|}{\sqrt{x}}) \ ,\end{aligned} \tag{86}$$

and the term proportional to λ is the just mentioned ad hoc term, chosen such that for $\lambda = 1$, $\lim_{y\to\pm 0}$ $\partial_y\psi_{2,1,\text{loc}}(x,y) = 0$. With $(u_{2,1}, v_{2,1}) = (\partial_y\psi_{2,1,\text{loc}}, -\partial_x\psi_{2,1,\text{loc}})$ we get (using a computer algebra system) that

$$\lim_{x\to\infty} x^{5/2}/\log(x)W(u_0 + u_{2,1}, v_0 + v_{2,1}, \omega_0 + \omega_{2,1})(x, z\sqrt{x}) = -\frac{b_s}{2}\text{sign}(z)\ (L_2 f_{2,0})'(|z|) \ , \tag{87}$$

with the right hand side being equal to zero because $f_{2,0}$ is in the kernel of L_2. Finally, a local approximation to the solution $\tilde{\psi}_{2,2}$ of the equation $\Delta\tilde{\psi}_{2,2} = -\tilde{\omega}_{2,2}$ is $\tilde{\psi}_{2,2,\text{loc}}$,

$$\tilde{\psi}_{2,2,\text{loc}}(x,y) = \tilde{c}_{2,2}\frac{y}{r\ r_-} + \theta(x)\text{sign}(y)\frac{1}{\sqrt{x}}(\tilde{g}_2(\frac{|y|}{\sqrt{x}}) - \tilde{c}_{2,2}) \ , \tag{88}$$

with r and r_- as defined in (25), and with $\tilde{c}_{2,2} = c_{2,2} + c_{2,0}$, where

$$c_{2,2} = \lim_{z\to\infty} g_2(z) = -4.436\ldots \ . \tag{89}$$

We use $\tilde{\psi}_{2,2,\text{loc}}$ to define the vector field $(u_{2,2}, v_{2,2}) = (\partial_y\tilde{\psi}_{2,2,\text{loc}}, -\partial_x\tilde{\psi}_{2,2,\text{loc}})$. The vector field $\mathbf{u}_{2,1} + (u_{2,2}, v_{2,2})$ is smooth in Ω and satisfies the boundary conditions (10), (11). Finally, equation (82) is obtained from (16) by the limit (computed with a computer algebra system),

$$\begin{aligned}&\lim_{x\to\infty} x^{5/2}W(u_0 + u_{2,1} + u_{2,2}, v_0 + v_{2,1} + v_{2,2}, \omega_0 + \omega_1 + \omega_2)(x, z\sqrt{x}) \\ &\qquad = -\text{sign}(z)\ (L_2\tilde{g}_2 - j_{2,0} - b_s j_{s,2,1})'(|z|) \ ,\end{aligned} \tag{90}$$

and the right hand side of (90) is equal to zero because g_2 solves equation (64) and $f_{2,0}$ is in the kernel of L_2.

6 The stress tensor

Using that $u(x,0) = v(x,0) = 0$ for $x \geq 0$, the stress tensor Σ of our problem evaluated on $\partial\Omega = [0,\infty)$ is

$$\Sigma(x,\pm 0) = \lim_{y\to\pm 0} \begin{pmatrix} -p(x,y) & \partial_y u(x,y) \\ \partial_y u(x,y) & -p(x,y) \end{pmatrix} . \tag{91}$$

For $x \geq 0$ we set

$$\tau_\pm(x) = \pm \lim_{y\to\pm 0} \partial_y u(x,y) = \mp \lim_{y\to\pm 0} \omega(x,y) . \tag{92}$$

From (91) we get for the average drag $\bar{D}$ exerted on the interval $[0,x]$ of the plate

$$\bar{D}(x) = \frac{1}{x}\int_0^x \left(\tau_+(s) + \tau_-(s)\right)\, ds .$$

For the symmetry breaking case we get from the asymptotic expansion (42), (57), (73), and using that $f_1''(0) = -a/2$ (see (56)) and that $\tilde{f}''(0) = a/4$ (see (72)), that $\tau_\pm = \tau_{a,\pm}$,

$$\tau_{a,\pm}(x) = \frac{a_2}{\sqrt{x}} \mp \frac{ab}{2}\frac{1}{x} + \frac{a}{4}\frac{1}{x^{3/2}} + \frac{c_{2,0}}{x^{3/2}} + \ldots , \tag{93}$$

with a_2 as in (35) , with a as in (36), and b as in (70), and $c_{2,0}$ an arbitrary real constant. Similarly, the theory with the second order logarithmic term predicts that $\tau_\pm = \tau$, where

$$\tau(x) = \frac{a_2}{\sqrt{x}} - \frac{b_s a_2}{a}\frac{\log(x)}{x^{3/2}} + \frac{a}{4}\frac{1}{x^{3/2}} + \frac{c_{2,0}}{x^{3/2}} + \ldots . \tag{94}$$

In the asymmetric case we therefore have that

$$\frac{1}{2}(\tau_{a,+} + \tau_{a,-})(x) = \frac{a_2}{\sqrt{x}} + \frac{\text{const.}}{x^{3/2}} + \ldots . \tag{95}$$

Note that the terms proportional to b which are not integrable at $x = 0$ and $x = \infty$ cancel out. In the asymmetric case we therefore get for the average drag $\bar{D}(x)$ acting on $[0,x]$ that $\bar{D}(x) = \bar{D}_a(x)$, where

$$\begin{aligned} \bar{D}_a(x) &= \frac{2}{x}\int_0^x \frac{a_2}{\sqrt{s}}ds + \frac{2}{x}\int_0^\infty \left(\frac{1}{2}(\tau_{a,+} + \tau_{a,-})(s) - \frac{a_2}{\sqrt{s}}\right) ds - \frac{2}{x}\int_x^\infty \left(\frac{\text{const.}}{s^{3/2}} + \ldots\right) ds \\ &= \frac{4a_2}{\sqrt{x}} + \frac{C_0}{x} + \frac{\text{const.}}{x^{3/2}} + \ldots . \end{aligned} \tag{96}$$

Here we have used the fact that we expect $(\tau_{a,+} + \tau_{a,-})(s)$ to be integrable at $s = 0$ (otherwise the tip of the plate produces an infinite amount of drag), to absorb all our lack of knowledge on $\tau_{a,\pm}$ for small values of x into the constant C_0. Namely, if $(\tau_{a,+} + \tau_{a,-})(s)$ is integrable at $s = 0$, then because of (95) the function $(\tau_{a,+} + \tau_{a,-})(s) - 2a_2/\sqrt{s}$ is integrable at zero and infinity. Astonishingly enough, the constant C_0 can be determined from the asymptotic expansion to leading order by using the integral form of the momentum equations. This fact has been first pointed out by Imai [16], [24]. One finds (see Appendix I),

$$C_0 = \frac{a^2\pi}{4} = 2.3256\ldots , \tag{97}$$

and therefore

$$\bar{D}_a(x) = \frac{1.328\ldots}{\sqrt{x}} + \frac{2.3256\ldots}{x} + \frac{\text{const.}}{x^{3/2}} + \ldots . \tag{98}$$

Similarly, we find for the symmetric case that $\bar{D}(x) = \bar{D}_s(x)$, where

$$\begin{aligned} \bar{D}_s(x) &= \frac{2}{x}\int_0^x \frac{a_2}{\sqrt{s}}ds + \frac{2}{x}\int_0^\infty \left(\tau(s) - \frac{a_2}{\sqrt{s}}\right) ds - \frac{2}{x}\int_x^\infty \left(-b_s\frac{a_2}{a}\frac{\log(s)}{s^{3/2}} + \frac{\text{const.}}{s^{3/2}} + \ldots\right) ds \\ &= \frac{4a_2}{\sqrt{x}} + \frac{C_0}{x} + \frac{2b_s a_2}{a}\frac{2\log(x) + 4}{x^{3/2}} + \frac{\text{const.}}{x^{3/2}} + \ldots , \end{aligned} \tag{99}$$

and therefore

$$D_s(x) = \frac{1.328\ldots}{\sqrt{x}} + \frac{2.3256\ldots}{x} - 1.1018\ldots\frac{\log(x)}{x^{3/2}} + \frac{\text{const.}}{x^{3/2}} . \tag{100}$$

For comparison with the literature see [24] equation (7.46) page 140.

7 Numerical solution

In order to check that the asymptotic expressions obtained in Section 4 can be completed to a solution of the Navier-Stokes equations we solve the problem (8)-(11) numerically by restricting the equations from the exterior infinite domain Ω to a sequence of bounded domains $\mathbf{D}_L = \{(x,y) \in \mathbf{R}^2 \mid \max\{|x|, |y|\} \leq L\} \subset \Omega$. This leads to the problem of finding appropriate boundary conditions on the surface $\Gamma_L = \partial\mathbf{D}_L \setminus \partial\boldsymbol{\Omega}$ of the truncated domain. In a recent paper [4], [5] we have introduced for the case of the flow around an obstacle of finite size a novel scheme that uses on the boundary the vector field obtained from an asymptotic analysis of the problem to second order [15]. Here, we use similar techniques and use on Γ_L Dirichlet boundary conditions obtained from the vector fields calculated in the previous sections through our asymptotic analysis. In contrast to the work in [4], [5] the boundary $\mathbf{B}$ of the original domain also gets truncated in the present case, and forms a corner of ninety degrees with the artificial boundary Γ_L. This fact is numerically somewhat delicate and we have therefore chosen to use a very straightforward, unsophisticated but robust numerical implementation of the problem. See for example [14], [21], [8]. Namely, we use after truncation to a finite domain $\mathbf{D}_L$ a simple first order finite difference scheme on staggered lattices and solve then the time dependent Navier-Stokes equation

$$\partial_t \mathbf{u} = -(\mathbf{u} \cdot \nabla)\mathbf{u} + \Delta\mathbf{u} - \nabla p$$

by iterating a first order discretization in time with a sufficiently small time step until convergence to a stationary solution, on each of a sequence of nested lattices (see [8]). The pressure is computed at each time step with high precision in order to keep the vector fields divergence free. This method is numerically robust, but convergence is slow and many weeks of computer time on a PC equipped with a Pentium 4, 2.8GHz processor were necessary to obtain the results that we discuss now.

Let $L = 125$, 250, 500, 1000. Then, on each of the corresponding domains $\mathbf{D}_L$, with we have solved (8)-(11) on a sequence of nested lattices using:

A the symmetric vector field $\mathbf{u}_0$ obtained from perturbation theory to leading order (see (44)-(46)),

B the symmetric vector field $\mathbf{u}_0 + (\partial_y \psi_{1,2,\mathrm{loc}}, -\partial_x \psi_{1,2,\mathrm{loc}})$ obtained from perturbation theory with logarithmic corrections (see (86)),

C the asymmetric vector field $\mathbf{u}_1$ obtained from perturbation theory to second order (see (60)-(62)).

Some care has to be taken when discretizing these vector field in order to ensure that numerically the total flux through the surface of the truncated domain is zero, since otherwise the equation for the pressure cannot be solved. In a finite domain the boundary conditions determine the flux, and since the boundary conditions **A** and **B** are mirror symmetric with respect to the x-axis the flux above and below the plate has to be the same. It turns out that the vector field converges in these cases to a symmetric vector field, even when starting from asymmetric initial conditions. Similarly, the vector field **C** forces the flux to be asymmetric with respect to the plate and in this case the vector field converges to an asymmetric solution. Let $(u_{L,X}, v_{L,X})$ be the numerical solution of the problem obtained in the domain $\mathbf{D}_L$ with Dirichlet boundary conditions X being either of the vector fields described in **A**, **B** and **C**. For the symmetric cases we have computed upon convergence to a stationary solution the function $\tau_{L,X}(x) = \lim_{y\to+0} \partial_y u_{L,X}(x,y)$, and in the asymmetric case the functions $\tau_{\pm,L,X}(x) = \pm \lim_{y\to\pm 0} \partial_y u_{L,X}(x,y)$. The results are summarized in Figure 1 for the symmetric case, and in Figure 2 for the asymmetric case. We expect that, for a given type of boundary conditions, the functions $\tau_{L,X}$ and $\tau_{\pm,L,X}$ converge as a function of L uniformly on compact sets to the corresponding limiting function. This limit should be the same for the two symmetric boundary conditions. This is indeed what the figures suggest. In particular the convergence to a limit appears to be faster when one includes the term with logarithmic corrections in the symmetric case, and the results are close to the numerical solution found previously by other groups [23]. Taking the good convergence of the procedure in the symmetric case (Figure 1) as a confirmation for the validity of our method, we conclude from Figure 2 that there is good evidence for the existence of an asymmetric stationary solution to the problem (8)-(11).

8 Appendix I

In this appendix we discuss in more detail the Blasius equation (34), recall the computation of drag (and lift) through surface integrals and give some more details concerning the Green's function for the

Laplacean in Ω.

8.1 Blasius equation

Let f be the solution of the Blasius equation (34). In order to find this function numerically one usually uses the following scaling property, which is a consequence of the scale-invariance of the domain Ω. Namely, define for $\beta > 0$ the function f_β by the equation $f(z) = \beta f_\beta(\beta z)$. Then f_β satisfies the same equation as f and $f_\beta(0) = f'_\beta(0) = 0$, but

$$\lim_{z\to\infty} f'_\beta(z) = 1/\beta^2 \ . \tag{101}$$

Since furthermore $f''(0) = \beta^3 f''_\beta(0)$, we can first solve the equation (34) with the additional boundary condition at zero $f''_\beta(0) = 1$, and use (101) to determine β. The boundary condition $\lim_{z\to\infty} f'(z) = 1$ is therefore equivalent to setting

$$f''(0) = a_2 = \beta^3 \ .$$

Numerically we find $\beta = 0.69247\ldots$ and therefore $a_2 = 0.33205\ldots$. Furthermore one finds numerically that

$$\lim_{z\to\infty} f(z) - z = a = -1.7207\ldots \ .$$

Note that the functions $z \mapsto f(z) - z - a$, $z \mapsto f'(z) - 1$, $z \mapsto f(z) - zf'(z)$ and f'' all decay faster than exponential at infinity. For graphs of these functions see Figure 3. Additional details can be found in many textbooks. See for example [2]. For convenience later on we also define the functions F_1,

$$\begin{aligned} F_1(z) &= \int_0^z f(\zeta)d\zeta = \frac{z^2}{2} + az + \int_0^z (f(\zeta) - \zeta - a)\ d\zeta \\ &= \frac{z^2}{2} + az + \lambda_0 - \int_z^\infty (f(\zeta) - \zeta - a)\ d\zeta \ , \end{aligned} \tag{102}$$

where $\lambda_0 = \int_0^\infty (f(\zeta) - \zeta - a)\ d\zeta = 2.182\ldots$. The function $z \mapsto F_1(z) - z^2/2 - az - \lambda_0$ decays faster than exponential at infinity. There is also an explicit expression for F_1 in terms of f. Namely, using the equation for f (see (34)) we find that

$$F_1(z) = -2\int_0^z \frac{f''(\zeta)}{f'''(\zeta)} d\zeta = -2\log(f''(z)/a_2) \ .$$

We also need the function F_2,

$$\begin{aligned} F_2(z) &= \int_0^z F_1(\zeta)d\zeta = \frac{z^3}{6} + a\frac{z^2}{2} + \lambda_0 z - \int_0^z d\zeta \int_\zeta^\infty f_0(\eta)d\eta \\ &= \frac{z^3}{6} + a\frac{z^2}{2} + \lambda_0 z + \lambda_1 + \int_z^\infty d\zeta \int_\zeta^\infty f_0(\eta)d\eta \ . \end{aligned} \tag{103}$$

with $\lambda_1 = \int_0^\infty d\zeta \int_\zeta^\infty (f(\eta) - \eta - a)\ d\eta$. The function $z \mapsto F_2(z) - z^3/6 - az^2/2 - \lambda_0 z - \lambda_1$ also decays faster than exponential at infinity. The functions F_1 and F_2 are used in Section 4.2.

8.2 Computation of Drag

Let $\mathbf{u}$, p be a solution of the Navier-Stokes equations (8), (9) subject to the boundary conditions (10), (11), and let $\mathbf{e}$ be some arbitrary unit vector in $\mathbf{R}^2$. Multiplying (8) with $\mathbf{e}$ leads to

$$-(\mathbf{u}\cdot\nabla)(\mathbf{u}\cdot\mathbf{e}) + \Delta(\mathbf{u}\cdot\mathbf{e}) - \nabla\cdot(p\mathbf{e}) = 0 \ . \tag{104}$$

Since

$$\begin{aligned} \nabla\cdot((\mathbf{u}\cdot\mathbf{e})\,\mathbf{u}) &= \mathbf{u}\cdot(\nabla(\mathbf{u}\cdot\mathbf{e})) + (\mathbf{u}\cdot\mathbf{e})(\nabla\cdot\mathbf{u}) = (\mathbf{u}\cdot\nabla)(\mathbf{u}\cdot\mathbf{e}) \ , \\ \Delta(\mathbf{u}\cdot\mathbf{e}) &= \nabla\cdot(\ [\nabla\mathbf{u} + (\nabla\mathbf{u})^T]\cdot\mathbf{e}) \ , \end{aligned}$$

equation (104) can be written as $\nabla \cdot \mathbf{P}(\mathbf{e}) = 0$, where

$$\mathbf{P}(\mathbf{e}) = -\left(\mathbf{u}\cdot\mathbf{e}\right)\mathbf{u} + [\nabla\mathbf{u} + (\nabla\mathbf{u})^T]\cdot\mathbf{e} - p\mathbf{e} \ , \tag{105}$$

i.e., the vector field $\mathbf{P}(\mathbf{e})$ is divergence free. Therefore, applying Gauss's theorem to the region $\Omega_S = [-x, x] \times [-s, s]$ for $x,\ s > 0$, we find (with inward normal vectors on $\partial\Omega$ and outward normal vectors on S) that

$$\int_{\partial\Omega} \mathbf{P}(\mathbf{e})\cdot\mathbf{n}\ d\sigma = \int_S \mathbf{P}(\mathbf{e})\cdot\mathbf{n}\ d\sigma \ . \tag{106}$$

We have that $\mathbf{P}(\tilde{\mathbf{e}})\cdot\mathbf{e} = \mathbf{P}(\mathbf{e})\cdot\tilde{\mathbf{e}}$ for any two unit vectors $\mathbf{e}$ and $\tilde{\mathbf{e}}$, and therefore it follows from (106), since $\mathbf{e}$ is arbitrary, that

$$\int_{\partial\Omega} \mathbf{P}(\mathbf{n})\ d\sigma = \int_S \mathbf{P}(\mathbf{n})\ d\sigma \ . \tag{107}$$

Since $\mathbf{u} = 0$ on $\partial\Omega$, we finally get from (107) and (105) that the total force the fluid exerts on the body is

$$\mathbf{F} = \int_{\partial\Omega} \Sigma(\mathbf{u}, p)\mathbf{n}\ d\sigma = \int_S \Big(-\left(\mathbf{u}\cdot\mathbf{n}\right)\mathbf{u} + [\nabla\mathbf{u} + (\nabla\mathbf{u})^T]\mathbf{n} - p\mathbf{n}\Big)\ d\sigma \ , \tag{108}$$

with $\Sigma(\mathbf{u}, p) = \nabla\mathbf{u} + (\nabla\mathbf{u})^T - p$ the stress tensor. The force $\mathbf{F}$ is traditionally decomposed into a component D parallel to the flow at infinity called drag and a component L perpendicular to the flow at infinity called lift. We compute here the drag only. Since $\lim_{|y|\to\infty}\mathbf{u}(x, y) = (1, 0)$ and since p can be chosen such that $\lim_{|y|\to\infty} p(x, y) = 0$ for all $x \in \mathbf{R}$, we can take the limit $s \to \infty$ and replace S by two vertical lines, one at $-x < 0$ and one at $x > 0$. To leading order we therefore get that $D \approx D_0$, where

$$D_0(x) = \int_{\mathbf{R}} \mu_0(x, y)\ dy \ , \tag{109}$$

with

$$\mu_0(x, y) = -u_0(x, y)^2 - p_0(x, y) + u_0(-x, y)^2 + p_0(-x, y) \ ,$$

with u_0 as defined in (44) and p_0 as defined in (53). On the scale $y \sim x^{1/2}$ we have

$$\nu_0(z) = \lim_{x\to\infty} \mu_0(x, zx^{1/2}) = 1 - f'(|z|)^2 \ ,$$

whereas on the scale $y \sim x$ we get that

$$\nu_1(z) = \lim_{x\to\infty} x^{1/2}\mu_0(x, zx) = \frac{a}{4}\frac{r_-(-1, z) - r_-(1, z)}{r(1, z)} \ , \tag{110}$$

and that

$$\nu_2(z) = \lim_{x\to\infty} x\left(\mu_0(x, zx) - \frac{1}{\sqrt{x}}\nu_1(z)\right) = \frac{a^2}{4}\frac{1}{r(1, z)^2} \ . \tag{111}$$

Therefore, since

$$\begin{aligned} -\int_0^\infty (f'(z)^2 - 1)\ dz &= -\left[f(z)f'(z) - z\right]_{z=0}^{z=\infty} + \int_0^\infty f(z)f''(z)\ dz \\ &= -a - 2\int_0^\infty f'''(z)\ dz = 2a_2 - a \ , \end{aligned} \tag{112}$$

with a_2 as defined in (15), we find that

$$\begin{aligned} D(x) \approx D_0(x) &\approx -2\sqrt{x}\int_0^\infty (f'(z)^2 - 1)\ dz + 2\sqrt{x}\int_0^\infty \nu_1(z)\ dz + 2\int_0^\infty \nu_2(z)\ dz \\ &= 4a_2\sqrt{x} - 2a\sqrt{x} + 2a\sqrt{x} + \frac{a^2\pi}{4} = 4a_2\sqrt{x} + \frac{a^2\pi}{4} \ , \end{aligned} \tag{113}$$

from which, after division by x, (96) and (99) follow with C_0 as defined in (97). It is tedious but straightforward to verify that all the neglected terms are smaller than the ones computed here.

8.3 Green's function

In this section we derive the Green's function for the Laplacean in Ω with Dirichlet boundary conditions on $[0,\infty)$, *i.e.*, a function $G\colon \Omega\times\Omega \to \mathbf{R}$, such that

$$f(x,y) = \int_\Omega G(x,y;x_0,y_0)\ g(x_0,y_0)\ dx_0\ dy_0 \tag{114}$$

solves the equation $\Delta f = g$ in Ω with $f(x,0)=0$ for $x\geq 0$. We use complex notation, *i.e.*, $\Omega = \mathbf{C}\backslash[0,\infty)$. Let $H=\{z\in\mathbf{C}\,|\,\mathrm{Im}(z)>0\}$ be the upper half plane. The map $z\mapsto z^2$ maps H conformally onto Ω. Let $z=\xi+i\eta\in H$. Then $z^2 = x+iy$ with

$$x = \xi^2-\eta^2\ , \tag{115}$$

$$y = 2\xi\eta\ . \tag{116}$$

The inverse of (115), (116) is $\xi = y/r_-(x,y)$, $\eta = r_-(x,y)/2$, with r_- as defined in (25).

The following observation concerning the limit towards the boundary will be useful below: Let $\eta\to 0$ for fixed $\xi\neq 0$, then $x\to\xi^2>0$, and y converges to zero from above or below depending on the sign of ξ. In other words, the limit when $y\to\pm 0$ for $x>0$ corresponds to taking the limit $\eta\to 0$ (from above) for fixed $\xi=\pm\sqrt{x}$. The differential version of the change of coordinates (115), (116) is $(dx,dy) = A(d\xi,d\eta)$ with

$$A = \left(\begin{array}{cc} 2\xi & -2\eta \\ 2\eta & 2\xi \end{array}\right)\ .$$

We have that $\det(A) = 4(\xi^2+\eta^2)$ and the inverse infinitesimal change of coordinates is therefore given by $(d\xi,d\eta) = B(dx,dy)$, with

$$B = A^{-1} = \frac{1}{4}\frac{1}{(\xi^2+\eta^2)}\left(\begin{array}{cc} 2\xi & 2\eta \\ -2\eta & 2\xi \end{array}\right)\ .$$

Define now, for given functions f and g the functions $\tilde f$ and $\tilde g$ by the equation $\tilde f(\xi,\eta) = f(x,y)$, and $\tilde g(\xi,\eta) = g(x,y)$, with x, y given by (115), (116). Then, we find by direct calculation that

$$(\Delta f)\,(x,y) = \frac{1}{4}\frac{1}{(\xi^2+\eta^2)}(\Delta\tilde f)(\xi,\eta)\ ,$$

and therefore we get from (114) by the change of variables $x_0 = \xi_0^2-\eta_0^2$, $y_0 = 2\xi_0\eta_0$ with inverse $\xi_0 = y_0/r_-(x_0,y_0)$, $\eta_0 = r_-(x_0,y_0)/2$, the identity

$$(\Delta f)\,(x,y) = \frac{1}{(\xi^2+\eta^2)}\int_H(\Delta\tilde G)(\xi,\eta;\xi_0,\eta_0)\ \tilde g(\xi_0,\eta_0)\ (\xi_0^2+\eta_0^2)\ d\xi_0\ d\eta_0\ , \tag{117}$$

where $\tilde G(\xi,\eta;\xi_0,\eta_0) = G(x,y;x_0,y_0)$. It is now easy to see that the Green's function $\tilde G$ of our problem is given by

$$\tilde G(\xi,\eta;\xi_0,\eta_0) = \frac{1}{4\pi}\left[\log((\xi-\xi_0)^2+(\eta-\eta_0)^2) - \log((\xi-\xi_0)^2+(\eta+\eta_0)^2)\right]\ . \tag{118}$$

Namely, by definition of $\tilde G$ we have that $(\Delta\tilde G)(\xi,\eta;\xi_0,\eta_0) = \delta(\xi-\xi_0)\delta(\eta-\eta_0)$, for $(\xi,\eta)\in H$, and therefore we find that $\Delta f = g$ in Ω. Furthermore $\lim_{\eta\to+0}\tilde G(\xi,\eta;\xi_0,\eta_0) = 0$, and therefore $G(x,\pm 0;x_0,y_0) = 0$, for $x>0$. This implies that $f(x,0)=0$ for $x>0$ as required. From the above it follows that $(\partial_x^2+\partial_y^2)G(x,y;x_0,y_0) = \delta(x-y)\delta(y-y_0)$, and similarly one can show that

$$(\partial_{x_0}^2+\partial_{y_0}^2)G(x,y;x_0,y_0) = \delta(x-y)\delta(y-y_0)\ . \tag{119}$$

Next, we note that

$$\lim_{y_0\to\pm 0} G(x,y;x_0,y_0) = \lim_{\eta_0\to+0}\tilde G(\xi,\eta;\pm\sqrt{x_0},\eta_0) = 0\ , \tag{120}$$

and an explicit computation shows that

$$\begin{aligned}\lim_{y\to\pm 0}\partial_y G(x,y;x_0,y_0) &= \lim_{\eta\to 0}\left(\partial_\xi\tilde G(\xi,\eta;\xi_0,\eta_0)\frac{\partial\xi}{\partial y} + \partial_\eta\tilde G(\xi,\eta;\xi_0,\eta_0)\frac{\partial\eta}{\partial y}\right)\\ &= -\frac{1}{2\pi}\frac{1}{\xi}\frac{\eta_0}{(\xi-\xi_0)^2+\eta_0^2}\ ,\end{aligned} \tag{121}$$

where the right hand side has to be evaluated at $\xi = \text{sign}(y)\sqrt{x}$ and at $\xi_0 = y_0/r_-(x_0, y_0)$, $\eta_0 = r_-(x_0, y_0)/2$. Similarly we have that

$$\lim_{y_0\to -0} \partial_{y_0} G(x, y; x_0, y_0) + \lim_{y_0\to +0} \partial_{y_0} G(x, y; x_0, y_0) = -\frac{2}{\pi} \frac{\xi\eta}{((\xi - \xi_0)^2 + \eta^2)\,((\xi + \xi_0)^2 + \eta^2)} \;, \tag{122}$$

$$\lim_{y_0\to -0} \partial_{y_0} G(x, y; x_0, y_0) - \lim_{y_0\to +0} \partial_{y_0} G(x, y; x_0, y_0) = -\frac{1}{\pi} \frac{\eta(\xi^2 + \eta^2 + \xi_0^2)}{((\xi - \xi_0)^2 + \eta^2)\,((\xi + \xi_0)^2 + \eta^2)} \;, \tag{123}$$

where $\xi_0 = \sqrt{x_0}$ and where $\xi = y/r_-(x, y)$, $\eta = r_-(x, y)/2$. Finally we have that

$$\begin{aligned} -\frac{2}{\pi} \int_0^\infty & \frac{\xi\eta}{((\xi - \xi_0)^2 + \eta^2)\,((\xi + \xi_0)^2 + \eta^2)} \sqrt{x_0}\; dx_0 \\ &= -\frac{4}{\pi}\xi\eta \int_0^\infty \frac{\xi_0^2}{((\xi - \xi_0)^2 + \eta^2)\,((\xi + \xi_0)^2 + \eta^2)}\; d\xi_0 = -\xi \;, \end{aligned} \tag{124}$$

and similarly that

$$-\frac{1}{\pi} \int_0^\infty \frac{\eta(\xi^2 + \eta^2 + \xi_0^2)}{((\xi - \xi_0)^2 + \eta^2)\,((\xi + \xi_0)^2 + \eta^2)}\; dx_0 = -1 \;. \tag{125}$$

9 Appendix II

This appendix contains the details concerning the asymptotic expansion.

9.1 Proof of Proposition 3

Let ψ_0 be as defined in (31), *i.e.*,

$$\psi_0(x, y) = y - \int_\Omega G(x, y; x_0, y_0)\; \omega_0(x_0, y_0)\; dx_0 dy_0 \;. \tag{126}$$

To check that ω_0 is admissible we first show that $\lim_{y\to\pm 0} \partial_y \psi_0(x, y) = 0$ for $x \geq 0$. Using (121) we find that

$$\begin{aligned} \lim_{y\to\pm 0} \partial_y \psi_0(x, y) &= 1 - \int_\Omega \lim_{y\to\pm 0} \partial_y G(x, y; x_0, y_0)\; \omega_0(x_0, y_0)\; dx_0 dy_0 \\ &= 1 + \frac{1}{2\pi}\frac{1}{\xi} \int_\Omega \frac{\eta_0}{(\xi - \xi_0)^2 + \eta_0^2}\; \omega_0(x_0, y_0)\; dx_0 dy_0 \;, \end{aligned} \tag{127}$$

where $\xi = \text{sign}(y)\; \sqrt{x}$ and where $\xi_0 = y_0/r_-(x_0, y_0)$, $\eta_0 = r_-(x_0, y_0)/2$, with r_- as defined in (25). Next, using the definition of ω_0 we get

$$\begin{aligned} \lim_{y\to\pm 0} \partial_y \psi_0(x, y) &= 1 - \frac{1}{2\pi}\frac{1}{\xi} \int_0^\infty dx_0 \int_{-\infty}^0 dy_0 \frac{\eta_0}{(\xi - \xi_0)^2 + \eta_0^2}\; \frac{1}{\sqrt{x_0}} f''(\frac{-y_0}{\sqrt{x_0}}) \\ &\quad + \frac{1}{2\pi}\frac{1}{\xi} \int_0^\infty dx_0 \int_0^\infty dy_0 \frac{\eta_0}{(\xi - \xi_0)^2 + \eta_0^2}\; \frac{1}{\sqrt{x_0}} f''(\frac{y_0}{\sqrt{x_0}}) \\ &= 1 - \frac{1}{2\pi}\frac{1}{\xi} \int_0^\infty dx_0 \int_0^\infty dy_0 \left(\frac{\eta_0}{(\xi - \xi_0)^2 + \eta_0^2} - \frac{\eta_0}{(\xi + \xi_0)^2 + \eta_0^2} \right) \frac{1}{\sqrt{x_0}} f''(\frac{y_0}{\sqrt{x_0}}) \\ &= 1 - \frac{2}{\pi} \int_0^\infty dx_0 \int_0^\infty dy_0 \frac{\xi_0\eta_0}{((\xi - \xi_0)^2 + \eta_0^2)\,((\xi + \xi_0)^2 + \eta_0^2)} \frac{1}{\sqrt{x_0}} f''(\frac{y_0}{\sqrt{x_0}}) \;. \end{aligned} \tag{128}$$

We change coordinates by setting $y_0 = \sqrt{x_0} z$. We then get

$$\lim_{y\to\pm 0} \partial_y \psi_0(x, y) = 1 - \frac{2}{\pi} \int_0^\infty dx_0 \int_0^\infty dz \frac{\tilde{\xi}_0\tilde{\eta}_0}{((\xi - \tilde{\xi}_0)^2 + \tilde{\eta}_0^2)((\xi + \tilde{\xi}_0)^2 + \tilde{\eta}_0^2)} f''(z) \;, \tag{129}$$

where $\tilde{\xi}_0 = z\sqrt{x_0}/r_-(x_0, z\sqrt{x_0})$, $\tilde{\eta}_0 = r_-(x_0, z\sqrt{x_0})/2$. Next we exchange the integrals and change then coordinates by setting $x_0 = z^2 s$. We get

$$\lim_{y\to\pm 0} \partial_y \psi_0(x,y) = 1 - \frac{2}{\pi}\int_0^\infty dz\ f''(z)\int_0^\infty ds \frac{\tilde{\tilde{\xi}}_0 \tilde{\tilde{\eta}}_0}{((\xi/z - \tilde{\tilde{\xi}}_0)^2 + \tilde{\tilde{\eta}}_0^2)((\xi/z + \tilde{\tilde{\xi}}_0)^2 + \tilde{\tilde{\eta}}_0^2)}\ , \tag{130}$$

where $\tilde{\tilde{\xi}}_0 = \sqrt{s}/r_-(s,\sqrt{s})$, $\tilde{\tilde{\eta}}_0 = r_-(s,\sqrt{s})/2$. The integral over s can be computed explicitly to be equal to $\pi/2$, independent of ξ, and therefore since $\lim_{z\to\infty} f'(z) = 1$ and $f'(0) = 0$ we find that $\lim_{y\to\pm 0}\partial_y\psi_0(x,y) = 1 - 1 = 0$ as required. Finally, since $\lim_{y\to\pm 0}\partial_y r_-(x,y) \neq 0$ for $x > 0$, (126) is the only solution satisfying all the boundary conditions, and therefore ω_0 is admissible in the sense of Definition 2. This completes the proof of Proposition 3.

9.2 Local approximation for ψ_0

We now extract a local approximation from ψ_0. From (126) we find

$$\begin{aligned}\psi_0(x,y) = y &- \int_0^\infty dx_0 \int_{-\infty}^0 dy_0\ G(x,y;x_0,y_0)\frac{1}{\sqrt{x_0}} f''(\frac{-y_0}{\sqrt{x_0}}) \\ &+ \int_0^\infty dx_0 \int_0^\infty dy_0\ G(x,y;x_0,y_0)\frac{1}{\sqrt{x_0}} f''(\frac{y_0}{\sqrt{x_0}})\ .\end{aligned} \tag{131}$$

Integrating by parts once we find

$$\begin{aligned}\psi_0(x,y) = y &- \int_0^\infty dx_0 \left[G(x,y;x_0,y_0)\left(-f'(\frac{-y_0}{\sqrt{x_0}}) + 1\right)\right]_{y_0=-\infty}^{y_0=0} \\ &+ \int_0^\infty dx_0 \left[G(x,y;x_0,y_0)\left(f'(\frac{y_0}{\sqrt{x_0}}) - 1\right)\right]_{y_0=0}^{y_0=\infty} \\ &+ \int_0^\infty dx_0 \int_{-\infty}^0 dy_0\ \partial_{y_0} G(x,y;x_0,y_0)\left(-f'(\frac{-y_0}{\sqrt{x_0}}) + 1\right) \\ &- \int_0^\infty dx_0 \int_0^\infty dy_0\ \partial_{y_0} G(x,y;x_0,y_0)\left(f'(\frac{y_0}{\sqrt{x_0}}) - 1\right)\ .\end{aligned} \tag{132}$$

The boundary terms compensate each other. Therefore we get, integrating by parts a second time,

$$\begin{aligned}\psi_0(x,y) = y &+ \int_0^\infty dx_0 \left[\partial_{y_0} G(x,y;x_0,y_0)\left(\sqrt{x_0} f(\frac{-y_0}{\sqrt{x_0}}) + y_0 - a\sqrt{x_0}\right)\right]_{y_0=-\infty}^{y_0=0} \\ &- \int_0^\infty dx_0 \left[\partial_{y_0} G(x,y;x_0,y_0)\left(\sqrt{x_0} f(\frac{y_0}{\sqrt{x_0}}) - y_0 - a\sqrt{x_0}\right)\right]_{y_0=0}^{y_0=\infty} \\ &- \int_0^\infty dx_0 \int_{-\infty}^0 dy_0\ \partial_{y_0}^2 G(x,y;x_0,y_0)\left(\sqrt{x_0} f(\frac{-y_0}{\sqrt{x_0}}) + y_0 - a\sqrt{x_0}\right) \\ &+ \int_0^\infty dx_0 \int_0^\infty dy_0\ \partial_{y_0}^2 G(x,y;x_0,y_0)\left(\sqrt{x_0} f(\frac{y_0}{\sqrt{x_0}}) - y_0 - a\sqrt{x_0}\right)\ ,\end{aligned} \tag{133}$$

and therefore

$$\begin{aligned}\psi_0(x,y) = y &- a\int_0^\infty \left(\lim_{y_0\to -0}\partial_{y_0} G(x,y;x_0,y_0) + \lim_{y_0\to +0}\partial_{y_0} G(x,y;x_0,y_0)\right)\ \sqrt{x_0}\ dx_0 \\ &+ \int_\Omega \partial_{y_0}^2 G(x,y;x_0,y_0)\mathrm{sign}(y_0)\theta(x_0)\sqrt{x_0}\left(f(\frac{|y_0|}{\sqrt{x_0}}) - \frac{|y_0|}{\sqrt{x_0}} - a\right)\ dx_0 dy_0\ .\end{aligned} \tag{134}$$

From (134) we find for $(x,y)\in\Omega$ using (119), (122) and (124) the decomposition $\psi_0 = \psi_{0,\mathrm{loc}} + \psi_{0,\mathrm{nonloc}}$, with $\psi_{0,\mathrm{loc}}$ as defined in (43) and with

$$\psi_{0,\mathrm{nonloc}}(x,y) = -\int_\Omega \partial_{x_0}^2 G(x,y;x_0,y_0)\mathrm{sign}(y_0)\theta(x_0)\sqrt{x_0}\left(f(\frac{|y_0|}{\sqrt{x_0}}) - \frac{|y_0|}{\sqrt{x_0}} - a\right)\ dx_0 dy_0\ . \tag{135}$$

9.3 The remainder $\psi_{0,\text{nonloc}}$

The remainder $\psi_{0,\text{nonloc}}$ does not contribute to the limit (63), since as we now show it is of the same size as the second order term and therefore contributes to the limit (78). Namely, we have that

$$\begin{aligned}\psi_{0,\text{nonloc}}(x,y) = &-\int_0^\infty dy_0 \int_0^\infty dx_0\ \partial_{x_0}^2 G(x,y;x_0,y_0)\ \sqrt{x_0}\ (f(\frac{|y_0|}{\sqrt{x_0}}) - \frac{|y_0|}{\sqrt{x_0}} - a) \\ &+ \int_{-\infty}^0 dy_0 \int_0^\infty dx_0\ \partial_{x_0}^2 G(x,y;x_0,y_0)\ \sqrt{x_0}\ (f(\frac{|y_0|}{\sqrt{x_0}}) - \frac{|y_0|}{\sqrt{x_0}} - a)\ .\end{aligned} \tag{136}$$

Integrating twice by parts with respect to x_0 we find that

$$\begin{aligned}\psi_{0,\text{nonloc}}(x,y) = &-\int_0^\infty dy_0 \int_0^\infty dx_0\ G(x,y;x_0,y_0)\ \partial_{x_0}^2 \left(\sqrt{x_0}\ (f(\frac{|y_0|}{\sqrt{x_0}}) - \frac{|y_0|}{\sqrt{x_0}} - a)\right) \\ &+ \int_{-\infty}^0 dy_0 \int_0^\infty dx_0\ G(x,y;x_0,y_0)\ \partial_{x_0}^2 \left(\sqrt{x_0}\ (f(\frac{|y_0|}{\sqrt{x_0}}) - \frac{|y_0|}{\sqrt{x_0}} - a)\right)\ .\end{aligned} \tag{137}$$

We have that

$$\partial_{x_0}^2 \left(\sqrt{x_0}\ (f(\frac{|y_0|}{\sqrt{x_0}}) - \frac{|y_0|}{\sqrt{x_0}} - a)\right) = \frac{1}{x_0^{3/2}} \tilde{f}_0''(\frac{|y_0|}{\sqrt{x_0}})\ , \tag{138}$$

where

$$\tilde{f}_0''(z) = \frac{1}{4}(z^2 f''(z) - (f(z) - zf'(z) - a))\ . \tag{139}$$

Equation (139) can be integrated explicitly to yield $\tilde{f}_0$ as given in (72). In terms of $\tilde{f}_0$ we find for (137)

$$\psi_{0,\text{nonloc}}(x,y) = \int_\Omega G(x,y;x_0,y_0)\ \tilde{\omega}_0(x,y)\ dx_0 dy_0\ , \tag{140}$$

with $\tilde{\omega}_0$ as defined in (73). As we will see below the function $\psi_{0,\text{nonloc}}$ has the same form as the second order term, and is therefore discussed in the corresponding subsection below.

9.4 Proof of Proposition 4

Let ψ_1 be as defined in (31) with $\alpha = b/2$, *i.e.*,

$$\psi_1(x,y) = \frac{b}{2} r_-(x,y) - \int_\Omega G(x,y;x_0,y_0)\ \omega_1(x_0,y_0)\ dx_0 dy_0\ , \tag{141}$$

with r_- as defined in (25). We now check that $\omega_0+\omega_1$ is admissible. First we show that $\lim_{y\to\pm 0} \partial_y \psi_1(x,y) = 0$ for $x \geq 0$. Using (121) we find that

$$\begin{aligned}\lim_{y\to\pm 0} \partial_y \psi_1(x,y) &= \frac{b}{2} \lim_{y\to\pm 0} \partial_y r_-(x,y) - \int_\Omega \lim_{y\to\pm 0} \partial_y G(x,y;x_0,y_0)\ \omega_1(x_0,y_0)\ dx_0 dy_0 \\ &= \frac{b}{2\xi} + \frac{1}{2\pi} \frac{1}{\xi} \int_\Omega \frac{\eta_0}{(\xi-\xi_0)^2 + \eta_0^2}\ \omega_1(x_0,y_0)\ dx_0 dy_0\ ,\end{aligned} \tag{142}$$

where $\xi = \text{sign}(y)\ \sqrt{x}$ and where $\xi_0 = y_0/r_-(x_0,y_0)$, $\eta_0 = r_-(x_0,y_0)/2$. Next, using the definition (17) of ω_1 we get

$$\begin{aligned}\lim_{y\to\pm 0} \partial_y \psi_1(x,y) =& \frac{b}{2\xi} - \frac{b}{4\pi} \frac{1}{\xi} \int_0^\infty dx_0 \int_{-\infty}^0 dy_0 \frac{\eta_0}{(\xi-\xi_0)^2+\eta_0^2}\ \frac{1}{x_0} f_1''(\frac{-y_0}{\sqrt{x_0}}) \\ & \frac{b}{2\xi} - \frac{b}{4\pi} \frac{1}{\xi} \int_0^\infty dx_0 \int_0^\infty dy_0 \frac{\eta_0}{(\xi-\xi_0)^2+\eta_0^2}\ \frac{1}{x_0} f_1''(\frac{y_0}{\sqrt{x_0}}) \\ =& \frac{b}{2\xi} - \frac{b}{4\pi} \frac{1}{\xi} \int_0^\infty dx_0 \int_0^\infty dy_0 \left(\frac{\eta_0}{(\xi-\xi_0)^2+\eta_0^2} + \frac{\eta_0}{(\xi+\xi_0)^2+\eta_0^2}\right) \frac{1}{x_0} f_1''(\frac{y_0}{\sqrt{x_0}}) \\ =& \frac{b}{2\xi} - \frac{b}{2\pi} \frac{1}{\xi} \int_0^\infty dx_0 \int_0^\infty dy_0 \frac{(\xi^2+\xi_0^2+\eta_0^2)\ \eta_0}{((\xi-\xi_0)^2+\eta_0^2)\ ((\xi+\xi_0)^2+\eta_0^2)} \frac{1}{x_0} f_1''(\frac{y_0}{\sqrt{x_0}})\ .\end{aligned} \tag{143}$$

We change coordinates by setting $y_0 = \sqrt{x_0}z$. We get

$$\lim_{y\to\pm 0} \partial_y \psi_1(x,y) = \frac{b}{2\xi} - \frac{b}{2\pi}\frac{1}{\xi}\int_0^\infty dx_0 \int_0^\infty dz \frac{\left(\xi^2 + \tilde{\xi}_0^2 + \tilde{\eta}_0^2\right)\tilde{\eta}_0}{((\xi - \tilde{\xi}_0)^2 + \tilde{\eta}_0^2)((\xi + \tilde{\xi}_0)^2 + \tilde{\eta}_0^2)} \frac{1}{\sqrt{x_0}} f_1''(z) \ , \tag{144}$$

where $\tilde{\xi}_0 = z\sqrt{x_0}/r_-(x_0, z\sqrt{x_0})$, $\tilde{\eta}_0 = r_-(x_0, z\sqrt{x_0})/2$. Next we exchange the integrals and then change coordinates by setting $x_0 = z^2 s$. We get

$$\lim_{y\to\pm 0} \partial_y \psi_1(x,y) = \frac{b}{2\xi} - \frac{b}{2\pi}\frac{1}{\xi}\int_0^\infty dz\ f_1''(z) \int_0^\infty \frac{ds}{\sqrt{s}} \frac{((\xi/z)^2 + \tilde{\tilde{\xi}}_0^2 + \tilde{\tilde{\eta}}_0^2)\tilde{\tilde{\eta}}_0}{((\xi/z - \tilde{\tilde{\xi}}_0)^2 + \tilde{\tilde{\eta}}_0^2)((\xi/z + \tilde{\tilde{\xi}}_0)^2 + \tilde{\tilde{\eta}}_0^2)} \ , \tag{145}$$

where $\tilde{\tilde{\xi}}_0 = \sqrt{s}/r_-(s, \sqrt{s})$, $\tilde{\tilde{\eta}}_0 = r_-(s, \sqrt{s})/2$. The integral over s can be computed explicitly and is equal to π, independent of ξ and therefore, since $\lim_{z\to\infty} f_1'(z) = 0$ and $f_1'(0) = 1$, we find that for $x > 0$

$$\lim_{y\to\pm 0} \partial_y \psi_1(x,y) = \frac{b}{2\xi} - \frac{b}{2\xi}\int_0^\infty dz\ f_1''(z) = 0 \ ,$$

as required. Finally, since $\lim_{y\to\pm 0} \partial_y r_-(x,y) \neq 0$ for $x > 0$, (141) is the only solution such that $\psi_0 + \psi_1$ satisfies all the boundary conditions, and therefore $\omega_0 + \omega_1$ is admissible in the sense of Definition 2. This completes the proof of Proposition 4.

9.5 Local approximation for ψ_1

We now extract a local approximation from ψ_1. From (141) we find

$$\begin{aligned}\psi_1(x,y) = \frac{b}{2} r_-(x,y) &+ \frac{b}{2}\int_0^\infty dx_0 \int_{-\infty}^0 dy_0\ G(x,y;x_0,y_0)\frac{1}{x_0} f_1''(\frac{-y_0}{\sqrt{x_0}}) \\ &+ \frac{b}{2}\int_0^\infty dx_0 \int_0^\infty dy_0\ G(x,y;x_0,y_0)\frac{1}{x_0} f_1''(\frac{y_0}{\sqrt{x_0}}) \ .\end{aligned} \tag{146}$$

Integrating by parts once we find

$$\begin{aligned}\psi_1(x,y) = \frac{b}{2} r_-(x,y) &+ \frac{b}{2}\int_0^\infty dx_0 \left[G(x,y;x_0,y_0)\frac{1}{\sqrt{x_0}}\left(-f_1'(\frac{-y_0}{\sqrt{x_0}})\right)\right]_{y_0=-\infty}^{y_0=0} \\ &+ \frac{b}{2}\int_0^\infty dx_0 \left[G(x,y;x_0,y_0)\frac{1}{\sqrt{x_0}} f_1'(\frac{y_0}{\sqrt{x_0}})\right]_{y_0=0}^{y_0=\infty} \\ &- \frac{b}{2}\int_0^\infty dx_0 \int_{-\infty}^0 dy_0\ \partial_{y_0} G(x,y;x_0,y_0)\frac{1}{\sqrt{x_0}}\left(-f_1'(\frac{-y_0}{\sqrt{x_0}})\right) \\ &- \frac{b}{2}\int_0^\infty dx_0 \int_0^\infty dy_0\ \partial_{y_0} G(x,y;x_0,y_0)\frac{1}{\sqrt{x_0}}\left(f_1'(\frac{y_0}{\sqrt{x_0}})\right) \ ,\end{aligned} \tag{147}$$

and therefore, since $f_1'(0) = 1$,

$$\begin{aligned}\psi_1(x,y) = \frac{b}{2} r_-(x,y) &- \frac{b}{2}\int_0^\infty \left(\lim_{y_0\to -0} G(x,y;x_0,y_0) + \lim_{y_0\to +0} G(x,y;x_0,y_0)\right)\frac{dx_0}{\sqrt{x_0}} \\ &+ \frac{b}{2}\int_0^\infty dx_0 \int_{-\infty}^0 dy_0\ \partial_{y_0} G(x,y;x_0,y_0)\frac{1}{\sqrt{x_0}} f_1'(\frac{-y_0}{\sqrt{x_0}}) \\ &- \frac{b}{2}\int_0^\infty dx_0 \int_0^\infty dy_0\ \partial_{y_0} G(x,y;x_0,y_0)\frac{1}{\sqrt{x_0}}\left(f_1'(\frac{y_0}{\sqrt{x_0}})\right) \ .\end{aligned} \tag{148}$$

The second term on the right hand side is equal to zero by (120). Therefore we get, integrating by parts again,

$$\begin{aligned}\psi_1(x,y) = \frac{b}{2}r_-(x,y) &+ \frac{b}{2}\int_0^\infty dx_0 \left[\partial_{y_0}G(x,y;x_0,y_0)\left(-f_1(\frac{-y_0}{\sqrt{x_0}})+c_1\right)\right]_{y_0=-\infty}^{y_0=0} \\ &- \frac{b}{2}\int_0^\infty dx_0 \left[\partial_{y_0}G(x,y;x_0,y_0)\left(f_1(\frac{y_0}{\sqrt{x_0}})-c_1\right)\right]_{y_0=0}^{y_0=\infty} \\ &- \frac{b}{2}\int_0^\infty dx_0 \int_{-\infty}^0 dy_0\ \partial_{y_0}^2 G(x,y;x_0,y_0)\left(-f_1(\frac{-y_0}{\sqrt{x_0}})+c_1\right) \\ &+ \frac{b}{2}\int_0^\infty dx_0 \int_0^\infty dy_0\ \partial_{y_0}^2 G(x,y;x_0,y_0)\left(f_1(\frac{y_0}{\sqrt{x_0}})-c_1\right)\ ,\end{aligned} \tag{149}$$

and therefore

$$\begin{aligned}\psi_1(x,y) = \frac{b}{2}r_-(x,y) &+ \frac{b}{2}c_1\int_0^\infty dx_0 \left(\lim_{y_0\to -0}\partial_{y_0}G(x,y;x_0,y_0) - \lim_{y_0\to +0}\partial_{y_0}G(x,y;x_0,y_0)\right) \\ &- \frac{b}{2}\int_\Omega \partial_{y_0}^2 G(x,y;x_0,y_0)\theta(x_0)\left(f_1(\frac{|y_0|}{\sqrt{x_0}})-c_1\right)\ dx_0dy_0\ .\end{aligned} \tag{150}$$

From (150) we find for $(x,y)\in\Omega$ using (119), (122) and (125) the decomposition $\psi_1 = \psi_{1,\mathrm{loc}} + \psi_{1,\mathrm{nonloc}}$, with $\psi_{1,\mathrm{loc}}$ as defined in (59) and with

$$\psi_{1,\mathrm{nonloc}}(x,y) = \frac{b}{2}\int_\Omega \partial_{x_0}^2 G(x,y;x_0,y_0)\ \theta(x_0)\left(f_1(\frac{|y_0|}{\sqrt{x_0}})-c_1\right) dx_0dy_0\ . \tag{151}$$

A careful analysis shows that

$$\lim_{x,y\to\infty} r^{3/2}\partial_x\psi_{1,\mathrm{nonloc}}(x,y) = \lim_{x,y\to\infty} r^{3/2}\partial_y\psi_{1,\mathrm{nonloc}}(x,y) = 0\ ,$$

and therefore $\psi_{1,\mathrm{nonloc}}$ does not contribute to the limits (63) and (78).

9.6 Proof of Proposition 5

By definition $\omega_2 = \tilde{\omega}_0 + \tilde{\omega}_2$, with $\tilde{\omega}_0 = \Delta\psi_{0,\mathrm{nonloc}}$, see (140). Therefore, $\psi_2 = -\psi_{0,\mathrm{nonloc}} + \tilde{\psi}_2$, where

$$\tilde{\psi}_2(x,y) = -\int_\Omega G(x,y;x_0,y_0)\ \tilde{\omega}_2(x_0,y_0)\ dx_0dy_0\ . \tag{152}$$

Since ψ_0 and $\psi_{0,\mathrm{loc}}$ both satisfy all the boundary conditions, it follows that $\omega_0+\omega_1+\tilde{\omega}_0$ is admissible, and it therefore suffices to show that $\lim_{y\to\pm 0}\partial_y\tilde{\psi}_2(x,y) = 0$ for $x\geq 0$ in order to prove that $\sum_{n=0}^2 \omega_n$ is admissible. Using (121) we find that

$$\begin{aligned}\lim_{y\to\pm 0}\partial_y\tilde{\psi}_2(x,y) &= -\int_\Omega \lim_{y\to\pm 0}\partial_y G(x,y;x_0,y_0)\ \tilde{\omega}_2(x_0,y_0)\ dx_0dy_0 \\ &= \frac{1}{2\pi}\frac{1}{\xi}\int_0^\infty dx_0 \int_{\mathbf{R}} dy_0\ \frac{\eta_0}{(\xi-\xi_0)^2+\eta_0^2}\ \tilde{\omega}_2(x_0,y_0)\ ,\end{aligned} \tag{153}$$

where $\xi = \mathrm{sign}(y)\ \sqrt{x}$ and where $\xi_0 = y_0/r_-(x_0,y_0)$, $\eta_0 = r_-(x_0,y_0)/2$. Next, using the definition of $\tilde{\omega}_2$ in (73) we get,

$$\begin{aligned}\lim_{y\to\pm 0}\partial_y\tilde{\psi}_2(x,y) =& \frac{1}{2\pi}\frac{1}{\xi}\int_0^\infty dx_0\int_{-\infty}^0 dy_0 \frac{\eta_0}{(\xi-\xi_0)^2+\eta_0^2}\ \frac{1}{x_0^{3/2}}\tilde{f}_2''(\frac{-y_0}{\sqrt{x_0}}) \\ &-\frac{1}{2\pi}\frac{1}{\xi}\int_0^\infty dx_0\int_0^\infty dy_0 \frac{\eta_0}{(\xi-\xi_0)^2+\eta_0^2}\ \frac{1}{x_0^{3/2}}\tilde{f}_2''(\frac{y_0}{\sqrt{x_0}}) \\ =& \frac{1}{2\pi}\frac{1}{\xi}\int_0^\infty dx_0\int_0^\infty dy_0 \left(\frac{\eta_0}{(\xi-\xi_0)^2+\eta_0^2} - \frac{\eta_0}{(\xi+\xi_0)^2+\eta_0^2}\right)\frac{1}{x_0^{3/2}}\tilde{f}_2''(\frac{y_0}{\sqrt{x_0}}) \\ =& \frac{2}{\pi}\int_0^\infty dx_0\int_0^\infty dy_0 \frac{\xi_0\eta_0}{((\xi-\xi_0)^2+\eta_0^2)\,((\xi+\xi_0)^2+\eta_0^2)}\frac{1}{x_0^{3/2}}\tilde{f}_2''(\frac{y_0}{\sqrt{x_0}})\ .\end{aligned} \tag{154}$$

We change coordinates by setting $y_0 = \sqrt{x_0} z$. We get

$$\lim_{y\to\pm 0} \partial_y \psi_2(x,y) = \frac{2}{\pi}\int_0^\infty dx_0 \int_0^\infty dz \frac{\tilde{\xi}_0 \tilde{\eta}_0}{((\xi-\tilde{\xi}_0)^2+\tilde{\eta}_0^2)((\xi+\tilde{\xi}_0)^2+\tilde{\eta}_0^2)} \frac{1}{x_0} \tilde{f}_2''(z) \ ,$$

where $\tilde{\xi}_0 = z\sqrt{x_0}/r_-(x_0, z\sqrt{x_0})$, $\tilde{\eta}_0 = r_-(x_0, z\sqrt{x_0})/2$. Next we exchange the integrals and then change coordinates by setting $x_0 = z^2 s$. We get

$$\lim_{y\to\pm 0} \partial_y \psi_2(x,y) = \frac{2}{\pi}\int_0^\infty dz \frac{1}{z^2} \tilde{f}_2''(z) \int_0^\infty \frac{ds}{s} \frac{\tilde{\tilde{\eta}}_0 \tilde{\tilde{\xi}}_0}{((\xi/z-\tilde{\tilde{\xi}}_0)^2+\tilde{\tilde{\eta}}_0^2)((\xi/z+\tilde{\tilde{\xi}}_0)^2+\tilde{\tilde{\eta}}_0^2)} \ ,$$

where $\tilde{\tilde{\xi}}_0 = \sqrt{s}/r_-(s,\sqrt{s})$, $\tilde{\tilde{\eta}}_0 = r_-(s,\sqrt{s})/2$. The integral over s can be computed explicitly and is equal to $(\pi/2)(z/\xi)^2$, and therefore since $\lim_{z\to\infty} \tilde{f}_2'(z) = \tilde{f}_2'(0) = 0$ we find that $\lim_{y\to\pm 0} \partial_y \tilde{\psi}_2(x,y) = 0$ for $x > 0$ as required. Finally, since $\lim_{y\to\pm 0} \partial_y r_-(x,y) \neq 0$ for $x > 0$, (152) is the only solution such that $\sum_{n=0}^2 \psi_n$ satisfies all the boundary conditions, and therefore $\sum_{n=0}^2 \omega_n$ is admissible in the sense of Definition 2. This completes the proof of Proposition 5.

9.7 Local approximation for ψ_2 and higher order terms

We now extract a local approximation from $\tilde{\psi}_2$. From (152) we find

$$\begin{aligned}\tilde{\psi}_2(x,y) = &-\int_0^\infty dx_0 \int_{-\infty}^0 dy_0 \ G(x,y;x_0,y_0) \frac{1}{x_0^{3/2}} \tilde{f}_2''(\frac{-y_0}{\sqrt{x_0}}) \\ &+ \int_0^\infty dx_0 \int_0^\infty dy_0 \ G(x,y;x_0,y_0) \frac{1}{x_0^{3/2}} \tilde{f}_2''(\frac{y_0}{\sqrt{x_0}}) \ ,\end{aligned} \tag{155}$$

Integrating by parts once we find

$$\begin{aligned}\tilde{\psi}_2(x,y) = &-\int_0^\infty dx_0 \left[G(x,y;x_0,y_0)\frac{1}{x_0}\left(-\tilde{f}_2'(\frac{-y_0}{\sqrt{x_0}})\right)\right]_{y_0=-\infty}^{y_0=0} \\ &+ \int_0^\infty dx_0 \left[G(x,y;x_0,y_0)\frac{1}{x_0}\tilde{f}_2'(\frac{y_0}{\sqrt{x_0}})\right]_{y_0=0}^{y_0=\infty} \\ &+ \int_0^\infty dx_0 \int_{-\infty}^0 dy_0 \ \partial_{y_0} G(x,y;x_0,y_0)\frac{1}{x_0}\left(-\tilde{f}_2'(\frac{-y_0}{\sqrt{x_0}})\right) \\ &- \int_0^\infty dx_0 \int_0^\infty dy_0 \ \partial_{y_0} G(x,y;x_0,y_0)\frac{1}{x_0}\left(\tilde{f}_2'(\frac{y_0}{\sqrt{x_0}})\right) \ .\end{aligned} \tag{156}$$

Since $\lim_{z\to\infty} \tilde{f}_2'(z) = \tilde{f}_2'(0) = 0$ the boundary terms are both equal to zero. Integrating by parts again,

$$\begin{aligned}\tilde{\psi}_2(x,y) = &\int_0^\infty dx_0 \left[\partial_{y_0} G(x,y;x_0,y_0)\frac{1}{\sqrt{x_0}}\left(\tilde{f}_2(\frac{-y_0}{\sqrt{x_0}}) - \tilde{c}_2\right)\right]_{y_0=-\infty}^{y_0=0} \\ &- \int_0^\infty dx_0 \left[\partial_{y_0} G(x,y;x_0,y_0)\frac{1}{\sqrt{x_0}}\left(\tilde{f}_2(\frac{y_0}{\sqrt{x_0}}) - \tilde{c}_2\right)\right]_{y_0=0}^{y_0=\infty} \\ &- \int_0^\infty dx_0 \int_{-\infty}^0 dy_0 \ \partial_{y_0}^2 G(x,y;x_0,y_0)\frac{1}{\sqrt{x_0}}\left(\tilde{f}_2(\frac{-y_0}{\sqrt{x_0}}) - \tilde{c}_2\right) \\ &+ \int_0^\infty dx_0 \int_0^\infty dy_0 \ \partial_{y_0}^2 G(x,y;x_0,y_0)\frac{1}{\sqrt{x_0}}\left(\tilde{f}_2(\frac{y_0}{\sqrt{x_0}}) - \tilde{c}_2\right) \ ,\end{aligned} \tag{157}$$

and therefore

$$\begin{aligned}\tilde{\psi}_2(x,y) = &-\tilde{c}_2 \int_0^\infty \frac{dx_0}{\sqrt{x_0}} \left(\lim_{y_0\to -0} \partial_{y_0} G(x,y;x_0,y_0) + \lim_{y_0\to +0} \partial_{y_0} G(x,y;x_0,y_0)\right) \\ &+ \int_\Omega \partial_{y_0}^2 G(x,y;x_0,y_0) \ \theta(x_0) \ \mathrm{sign}(y_0) \frac{1}{\sqrt{x_0}} \left(\tilde{f}_2(\frac{|y_0|}{\sqrt{x_0}}) - \tilde{c}_2\right) \ dx_0 dy_0 \ .\end{aligned} \tag{158}$$

From (158) we find for $(x,y) \in \Omega$ using (119), (122) and (124) the decomposition $\tilde{\psi}_2 = \tilde{\psi}_{2,\mathrm{loc}} + \tilde{\psi}_{2,\mathrm{nonloc}}$, with $\tilde{\psi}_{2,\mathrm{loc}}$ as defined in (74) and with

$$\tilde{\psi}_{2,\mathrm{nonloc}}(x,y) = -\int_{\Omega} \partial_{x_0}^2 G(x,y;x_0,y_0)\ \theta(x_0)\ \mathrm{sign}(y_0)\frac{1}{\sqrt{x_0}}\left(\tilde{f}_2(\frac{|y_0|}{\sqrt{x_0}}) - \tilde{c}_2\right)\ dx_0 dy_0\ .$$

A careful analysis shows that

$$\lim_{x,y\to\infty} r^{3/2}\partial_x \psi_{2,\mathrm{nonloc}}(x,y) = \lim_{x,y\to\infty} r^{3/2}\partial_y \psi_{2,\mathrm{nonloc}}(x,y) = 0\ ,$$

and therefore $\psi_{2,\mathrm{nonloc}}$ does not contribute to the limits (63) and (78). Finally, using the same techniques and assuming that ω is of the form (18) with $\tilde{\omega} \in \mathcal{W}$ one shows the bounds (33).

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References

[1] H. L. Alden, *Second approximation to the laminar boundary layer flow over a flat plate*, Journal of Mathematics and Physics **XVII** (1948), 91–104.

[2] G. K. Batchelor, *An introduction to fluid dynamics*, Cambridge University Press, 1967.

[3] H. Blasius, Zeitschrift für Mathematik und Physik **56** (1908), 4–13.

[4] S. Bönisch, V. Heuveline, and P. Wittwer, *Adaptive boundary conditions for exterior flow problems*, Journal of Mathematical Fluid Mechanics, to appear (2003).

[5] ______, *Second order adaptive boundary conditions for exterior flow problems: non-symmetric stationary flows in two dimensions*, Geneva/Heidelberg preprint (2004).

[6] P. Bradshaw, *Topics in applied physics: Turbulence*, vol. 12, Springer-Verlag.

[7] J. F. Douglas, J. M. Gasiorek, and J. A. Swaffield, *Fluid mechanics*, Pearson, Prentice Hall.

[8] J. H. Ferziger and M. Péric, *Computational methods in fluid dynamics*, Springer-Verlag.

[9] R. Finn, *An energy theorem for viscous fluid motions*, Arch. Rational Mech. Anal. **6** (1960), 371–381.

[10] G.P. Galdi, *On the existence of symmetric steady-state solutions to the plane exterior Navier-Stokes problem for a large Reynolds number*, Advances in Fluid Dynamics, Quaderni Di Matematica della II Universita' di Napoli **4** (1999), 1–25.

[11] Th. Gallay and C.E. Wayne, *Invariant manifolds and the long-time asymptotics of the Navier-Stokes and vorticity equations on R^2*, Arch. Rat. Mech. Anal. **163** (2002), 209–258.

[12] ______, *Long-time asymptotics of the Navier-Stokes and vorticity equations on R^3*, Phil. Trans. Roy. Soc. Lond. **360** (2002), 2155–2188.

[13] S. Goldstein, *Lectures on fluid mechanics*, Interscience Publishers, Ltd., London, 1957.

[14] M. Griebel, T. Dornseifer, and T. Neunhoeffer, *Numerical simulation in fluid dynamics. A practical introduction*, SIAM.

[15] F. Haldi and P. Wittwer, *Leading order down-stream asymptotics of non-symmetric stationary Navier-Stokes flows in two dimensions*, Journal of Mathematical Fluid Mechanics, to appear (2003).

[16] I. Imai, *Second approximation to the laminar boundary-layer flow over a flat plate*, Journal Aeronaut. Sci. **24** (1957), 155–156.

[17] L. Landau and E. Lifchitz, *Physique théorique, tome 6, mécanique des fluides*, Editions MIR, 1989.

[18] H. Ockendon and J. R. Ockendon, *Viscous flow*, Cabridge texts in applied mathematics.

[19] H. Schlichting and K. Gersten, *Boundary layer theory*, Springer, 1999.

[20] I. J. Sobey, *Introduction to interactive boundary layer theory*, Oxford University Press, 2000.

[21] K. Stüben and U. Trottenberg, *Multigrid methods: Fundamental algorithms, model problem analysis and applications*, vol. 960, Springer-Verlag.

[22] K. Stewartson, *On asymptotic expansions in the theory of boundary layers*, J. Maths Physics **36** (1957), 173–191.

[23] A. I. van de Vooren and D. Dijkstra, *The Navier-Stokes solution for laminar flow past a semi-infinite flat plate*, Journal of Engineering Mathematics **4** (1970), 9–27.

[24] M. van Dyke, *Perturbation methods in fluid mechanics*, The parabolic press, Stanford, California, 1975.

[25] E. Wayne, *Invariant manifolds for parabolic partial differential equations on unbounded domains*, Arch. Rational Mech. Anal. **138** (1997), 279–306.

[26] P. Wittwer, *On the structure of Stationary Solutions of the Navier-Stokes equations*, Commun. Math. Phys. **226** (2002), 455–474.

[27] ———, *Leading order down-stream asymptotics of non-symmetric stationary Navier-Stokes flows in three dimensions*, Geneva preprint (2003).

[28] ———, *Supplement: On the structure of stationary solutions of the Navier-Stokes equations*, Commun. Math. Phys. **234** (2003).

10 Figures

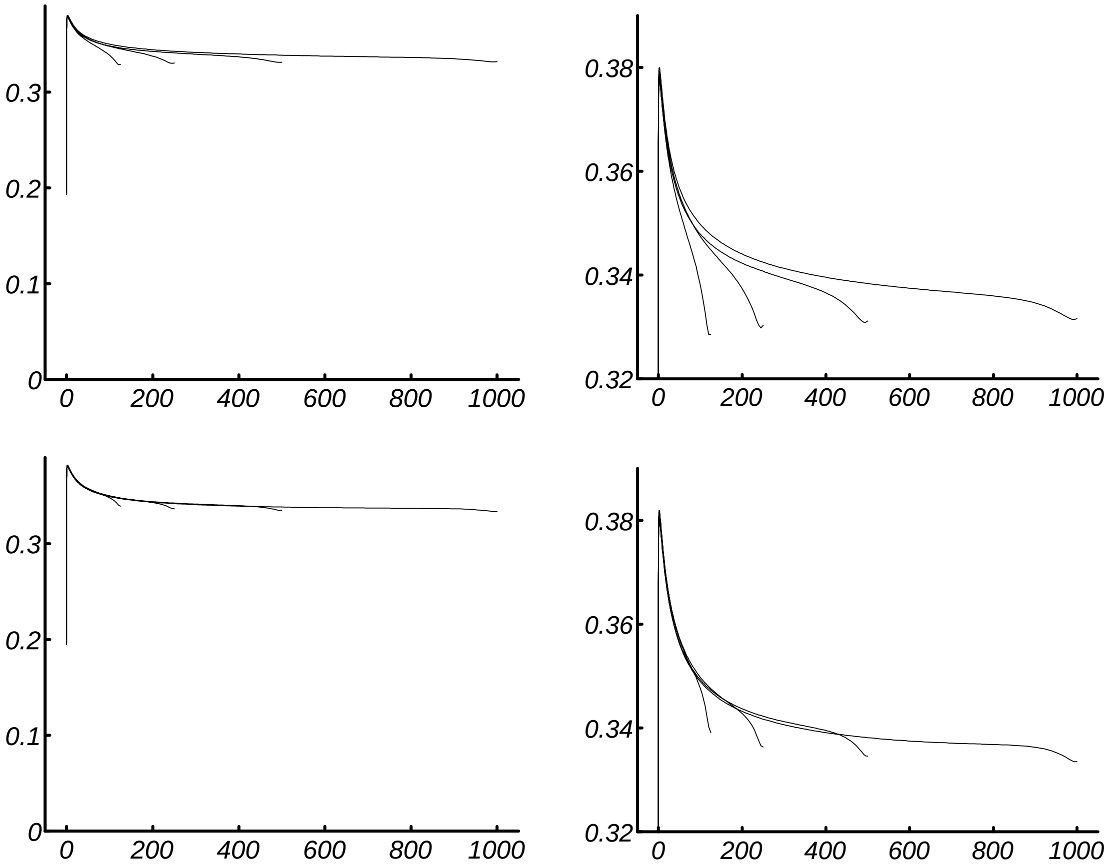

Figure 1. Plot of the function $x \mapsto x^{1/2}\tau(x)$ as a function of domain size L with artificial boundary conditions computed from first order symmetric perturbation theory (top left) and zoom on the same quantity (top right). Bottom: same results for artificial boundary conditions obtained from second order logarithmic symmetric perturbation theory.

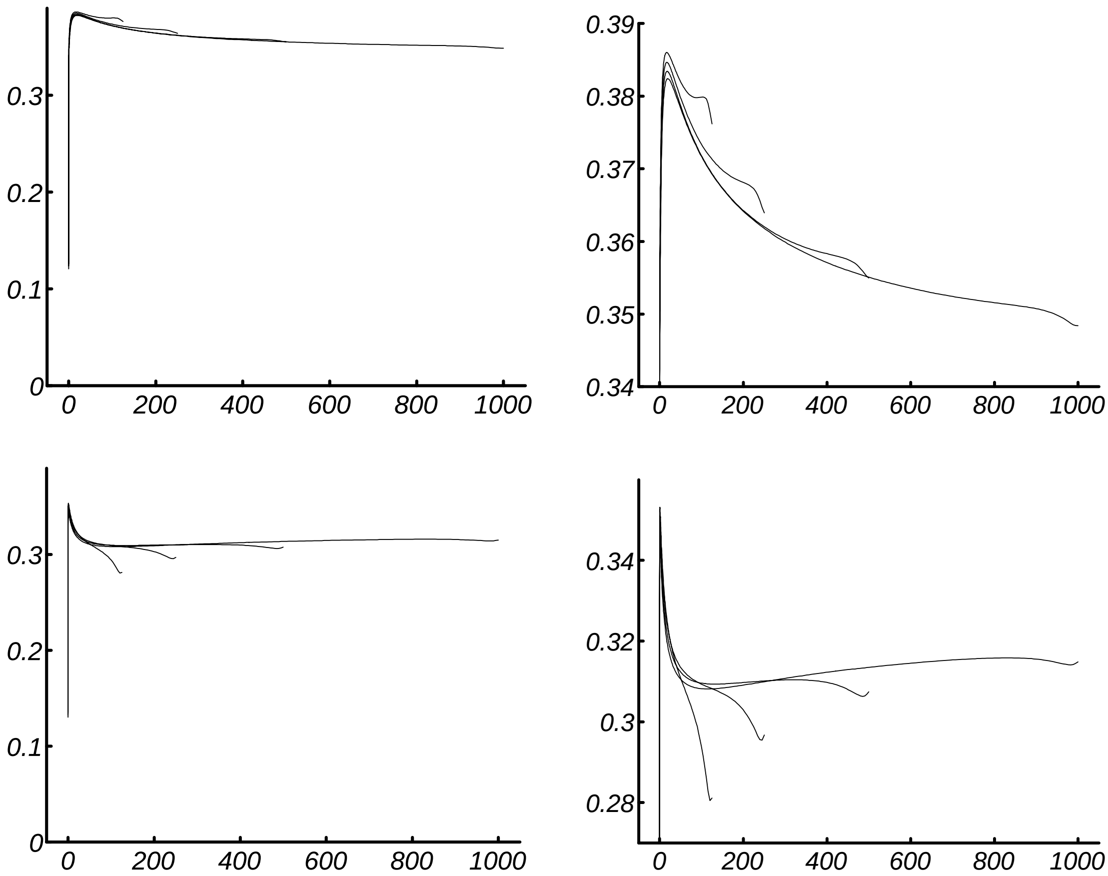

Figure 2. Plot of the function $x \mapsto x^{1/2}\tau_+(x)$ as a function of domain size L with artificial boundary conditions obtained from first order asymmetric perturbation theory (top left) and zoom on the same quantity (top right). Bottom: same results for the function $x \mapsto x^{1/2}\tau_-(x)$.

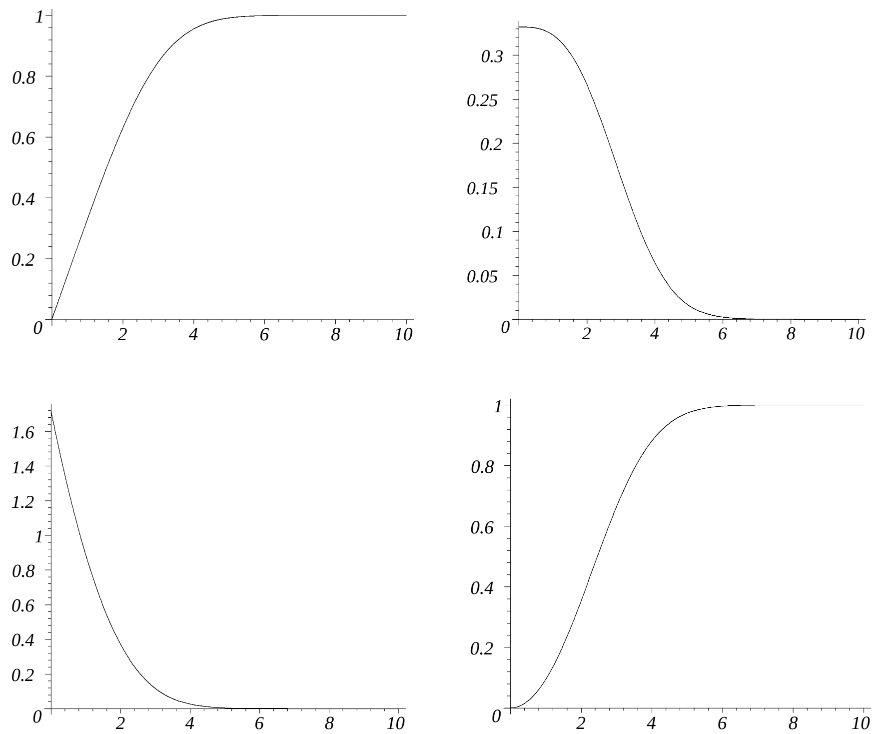

Figure 3. From left to right, top to bottom: graph of the function f', f'', $z \mapsto f(z) - z - a$, and $z \mapsto f_{2,0}(z) = (f(z) - zf'(z))/a$

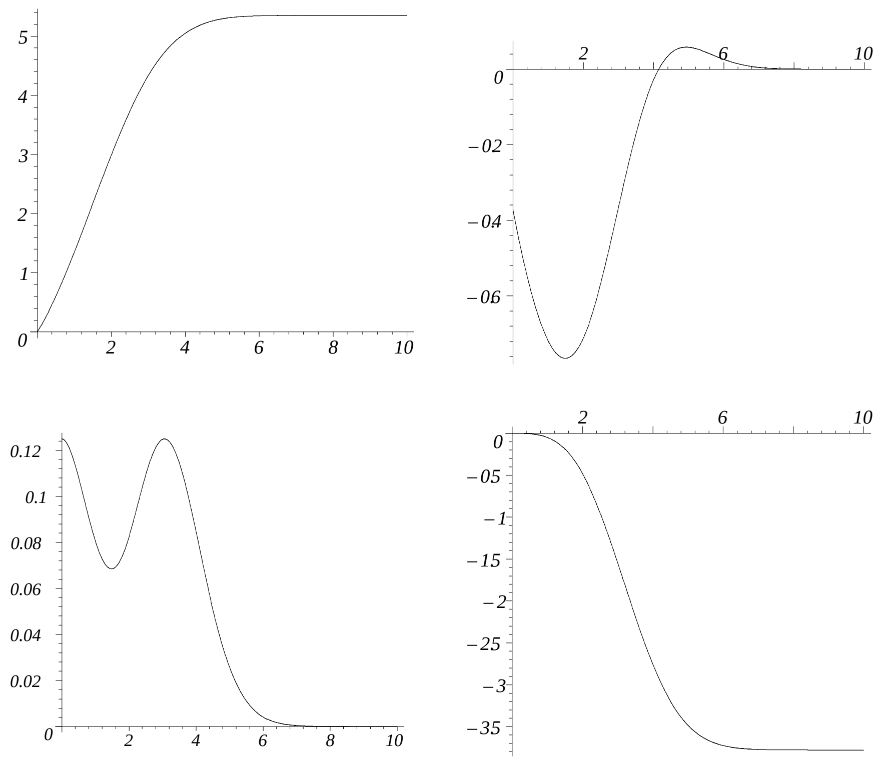

Figure 4. From left to right, top to bottom: graph of the function f_1, $j_{2,0}$, $j_{2,1}$, and f_2.