Bogoliubov Hamiltonians and one-parameter groups of Bogoliubov transformations

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Abstract

On the bosonic Fock space, a family of Bogoliubov transformations corresponding to a strongly continuous one-parameter group of symplectic maps $(R(t))_{t\in\mathbb{R}}$ is considered. Under suitable assumptions on the generator A of this group, which guarantee that the induced representations of CCR are unitarily equivalent for all time t, it is known that the unitary operator $U_{\rm nat}(t)$ which implement this transformation gives a projective unitary representation of $R(t)$. Under rather general assumptions on the generator A, we prove that the corresponding Bogoliubov transformations can be implemented by a one-parameter group $U(t)$ of unitary operators. The generator of $U(t)$ will be called a Bogoliubov Hamiltonian. We will introduce two kinds of Bogoliubov Hamiltonians (type I and II) and give conditions so that they are well defined.

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1 Introduction

Given a real symplectic space $\mathcal{Y}$ (with a symplectic form σ) and a complex Hilbert space $\mathcal{H}$, a representation of the Canonical Commutation Relations (CCR) over $\mathcal{Y}$ in $\mathcal{H}$ is a map $\mathcal{Y} \ni y \mapsto W(y) \in U(\mathcal{H})$ (the unitary operators on $\mathcal{H}$) such that

$$W(y)W(y') = \mathrm{e}^{-\frac{\mathrm{i}}{2}\sigma(y,y')}W(y+y'), \quad \forall y, y' \in \mathcal{Y}. \tag{1.1}$$

These representations arise naturally in the study of bosonic systems (*e.g.* a free Bose field).

For any symplectic map R on $\mathcal{Y}$ (*i.e.* $\sigma(Ry, Ry') = \sigma(y, y')$ for all y, y'), the map

$$\mathcal{Y} \ni y \mapsto W_R(y) := W(Ry)$$

is also a representation of CCR. The tranformation of $W(y)$ into $W_R(y)$ is often called the Bogoliubov transformation. The question is whether these two representations (W and W_R) are unitarily equivalent, i.e. is there a unitary operator U on $\mathcal{H}$ such that for all y in $\mathcal{Y}$, $UW(y)U^{-1} = W_R(y)$.

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In our paper we consider the so-called Fock representation, *i.e.* the Hilbert space $\mathcal{H}$ is a bosonic Fock space $\Gamma_{\mathrm{s}}(\mathfrak{h})$ and the operators $W(y)$ are the usual Weyl operators. In this case the symplectic spaces is $\mathcal{Y} := \{(f,\bar{f}), f \in \mathfrak{h}\}$, where $f \mapsto \bar{f}$ is a conjugation (i.e. an antilinear involution) on the Hilbert space $\mathfrak{h}$. The symplectic form on $\mathcal{Y}$ will be the imaginary part of the scalar product on $\mathfrak{h}$ (*i.e.* $\sigma((f,\bar{f}),(g,\bar{g})) = \mathrm{Im}\langle f|g\rangle$).

Consider a real symplectic map R on $\mathcal{Y}$. One can write it as a 2×2 matrix, $R = \begin{pmatrix} P & \bar{Q} \\ Q & \bar{P} \end{pmatrix}$, where P and Q are bounded operators on $\mathfrak{h}$ which satisfy some conditions (see (3.1)). We also introduce $J = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}$. Finally, let $W(f)$ denote the Weyl operators on the bosonic Fock space $\Gamma_{\mathrm{s}}(\mathfrak{h})$. In that setting, the question of unitarily equivalence has been solved by Shale [Sh]: the representations of W and W_R are unitarily equivalent if and only if the operator $[R,J]$ is a Hilbert-Schmidt operator on $\mathfrak{h}\oplus\mathfrak{h}$, which is equivalent to say that the operator Q is Hilbert-Schmidt on $\mathfrak{h}$. Moreover, if this condition holds, then the operator, which we will call the natural Bogoliubov implementer,

$$U_{\mathrm{nat}} := \det(1-K^*K)^{1/4}\mathrm{e}^{-\frac{1}{2}a^*(K)}\Gamma((P^{-1})^*)\mathrm{e}^{-\frac{1}{2}a(L)},$$

with $K = \overline{QP^{-1}}$ and $L = -P^{-1}\bar{Q}$, extends to a unitary operator on $\Gamma_{\mathrm{s}}(\mathfrak{h})$ and satisfies [IH, Ru1, Ru2]: $\forall f \in \mathfrak{h}$, $W_R(f) = U_{\mathrm{nat}}W(f)U_{\mathrm{nat}}^*$. Here, $a(L)$ and $a^*(K)$ denote the "quadratic" annihilation and creation operators associated to the Hilbert-Schmidt operators K and L via the natural identification between Hilbert-Schmidt operators on $\mathfrak{h}$ and vectors in $\mathfrak{h}\otimes\mathfrak{h}$ (Section 2.2).

Suppose $(R(t))_{t\in\mathbb{R}}$ is a group of symplectic maps such that, for all t, $[R(t),J]$ is Hilbert-Schmidt. We can then define the operators $U_{\mathrm{nat}}(t)$ for all t. In general $(U_{\mathrm{nat}}(t))_{t\in\mathbb{R}}$ will not be a one-parameter group as well. Since $(R(t))_{t\in\mathbb{R}}$ is a one-parameter group and the Weyl representation W is irreducible we know that

$$U_{\mathrm{nat}}(t)U_{\mathrm{nat}}(s) = \mathrm{e}^{\mathrm{i}\rho(t,s)}U_{\mathrm{nat}}(t+s).$$

Clearly, for any $\theta(t)\in\mathbb{R}$, $U(t) := \mathrm{e}^{\mathrm{i}\theta(t)}U_{\mathrm{nat}}(t)$ also intertwines W and $W_{R(t)}$. A suitable choice of the phase $\theta(t)$ may give rise to a strongly continuous unitary group $U(t)$. When such a unitary group exists, $R(t)$ will be called unitarily implementable and its selfadjoint generator H a Bogoliubov Hamiltonian. Note that the irreducibility of W guarantees that all the Bogoliubov Hamiltonians associated to a given symplectic group are equal up to a constant.

There are at least two natural choices for this constant, corresponding to two distinguished classes of Bogoliubov Hamiltonians, which we call type I and type II. The type I Bogoliubov Hamiltonian is such that its expectation value on the Fock vacuum vanishes. The type II Bogoliubov Hamiltonian is such that its infimum is zero. We will see that such choices are not always possible, i.e. one of these distinguished Bogoliubov Hamiltonians (or both) may not exist, even if $R(t)$ is unitarily implementable.

Let $A = \mathrm{i}\begin{pmatrix} h & -v \\ \bar{v} & -\bar{h} \end{pmatrix}$ denote the generator of the symplectic group $R(t)$. $R(t)$ is symplectic for all t if and only if h is selfadjoint and $v^* = \bar{v}$. We will see that, at least formally, the Bogoliubov Hamiltonians associated to $R(t)$ are given by

$$H := \mathrm{d}\Gamma(h) + \frac{1}{2}(a^*(v)+a(v)) + c. \tag{1.2}$$

Here c is a constant, which may be infinite – this means that one may have to perform an apprrorpriate renormalization (see Section 4 for a concrete example).

It is easy to see that the constant c_I corresponding to the type I Bogoliubov Hamiltonian is zero. The constant c_{II} (at least in the case where $\mathfrak{h}$ is finite dimensional) is given by

$$c_{II} = -\frac{1}{4}\mathrm{Tr}\left[\begin{pmatrix} \bar{h}^2 - \bar{v}v & \bar{h}\bar{v} - \bar{v}h \\ hv - v\bar{h} & h^2 - v\bar{v} \end{pmatrix}^{1/2} - \begin{pmatrix} \bar{h} & 0 \\ 0 & h \end{pmatrix}\right].$$

The unitary group generated by the type I Hamiltonian can be written explicitly:

$$U_I(t) := \mathrm{e}^{\mathrm{i}tH_I} = \det(\bar{P}(t)\mathrm{e}^{\mathrm{i}t\bar{h}})^{-1/2}\mathrm{e}^{-\frac{1}{2}a^*(K(t))}\Gamma((P^{-1}(t))^*)\mathrm{e}^{-\frac{1}{2}a(L(t))}. \tag{1.3}$$

In our paper we give conditions on the generator A of $R(t)$ guaranteeing so that $R(t)$ is unitarily implementable and conditions which ensure that Bogoliubov Hamiltonians of type I, resp. type II, are well defined.

We prove that, if for all t the operator

$$v(t) := \int_0^t \mathrm{e}^{\mathrm{i}\tau h} v \mathrm{e}^{\mathrm{i}\tau\bar{h}} \mathrm{d}\tau \tag{1.4}$$

is Hilbert-Schmidt such that its Hilbert-Schmidt norm is locally integrable and continuous at $t = 0$, then $R(t)$ is unitarily implementable.

To guarantee the existence of type I, we need some additional assumption on A, namely: the operator $\bar{v}v(t)$ is trace class, and its trace norm is locally integrable and continuous at $t = 0$. Under this condition, we prove that the operators $U(t)$ defined by (1.3) with $c = 0$ form a strongly continuous unitary group. Note that this does note require v to be Hilbert-Schmidt and allows to give a meaning to the formal operator (1.2) in a more general situation. On the other hand, if v is Hilbert-Schmidt, then the above assumptions are satisfied and hence the Bogoliubov Hamiltonian of type I exists. Moreover, we then prove that the a priori formal expression (1.2) indeed defines an essentially selfadjoint operator.

The selfadjointness of H is not obvious, and to our knowledge there is no (rigorous) proof of it. We would like to emphasize the fact that v is naturally associated to an element of $\Gamma_\mathrm{s}^2(\mathfrak{h})$ and not $\Gamma_\mathrm{s}^1(\mathfrak{h}) = \mathfrak{h}$, so that the "perturbation" $a^*(v) + a(v)$ has really to be thought as an operator quadratic, and not linear, in a and a^*.

In order to study Bogoliubov Hamiltonians of type II, one needs to compute the infimum of operators H of the form (1.2). In particular, the Bogoliuobov Hamiltonian of type II is well defined if and only if $R(t)$ has bounded from below Bogoliubov Hamiltonians.

If $\mathfrak{h}$ has finite dimension, we will prove that H is bounded from below (and compute its infimum) if and only if, for all $f \in \mathfrak{h}$,

$$\langle f|hf\rangle + \langle \bar{f}|\bar{h}\bar{f}\rangle + \langle f|v\bar{f}\rangle + \langle \bar{f}|\bar{v}f\rangle \geq 0.$$

When h is positive, we also give a condition under which the "perturbation" $a^*(v) + a(v)$ is relatively bounded with respect to $\mathrm{d}\Gamma(h)$, and we give an upper bound on this relative bound. We thus get another class of symplectic groups for which both type I and type II Bogoliubov Hamiltonians exist.

Finally, we study completely the simple (non trivial) following situation: $\mathfrak{h} = L^2(\mathbb{N})$ and the operators h and v are both diagonal with respect to the canonical basis of $L^2(\mathbb{N})$, i.e. $h = \sum h_n|e_n\rangle\langle e_n|$ and $v = \sum v_n|e_n\rangle\langle e_n|$. We prove that $R(t)$ is unitarily implementable if and only if $\sum \frac{|v_n|^2}{1+h_n^2} < +\infty$. Then we prove that the Bogoliubov Hamiltonian of type I, resp. type II, is well defined if and only if $\sum \frac{|v_n|^2}{1+|h_n|} < +\infty$, resp. $\sum_{|h_n|\leq 1} \frac{|v_n|^2}{|h_n|} < +\infty$. In particular one can see that all kinds of situation can occur: neither type I nor type II exist, type I exists but not type II, etc.

We end this introduction with a few comments on the related results which exist in the litterature. In [Be], under the same conditions on $v(t)$ (see (1.4) and below), it is proved that the operators $U_I(t)$ are well defined, and that they form a one-parameter group of unitary operators whose generator is given by (1.2) provided the latter makes sense as an essentially selfadjoint operator. The author also proves this essential selfadjointness when v is Hilbert-Schmidt. However, the proofs at some places are not quite complete (the proof of essential selfadjointness for instance is not completely rigorous). Similar results are also obtained in [Ne] but the author considers only the case where v is Hilbert-Schmidt and partially relies on the results of [Be], such as for the essential selfadjointness of Bogoliubov Hamiltonians.

More recently, in [IH] the authors have also considered the question of finding a unitary group $\mathrm{e}^{\mathrm{i}tH}$ which intertwines W and $W_{R(t)}$ but only for norm continuous symplectic groups.

Finally, we would like to mention that similar "quadratic operators" as (1.2) have been studied in *e.g.* [A, AY]. However, the authors use the field operator $\phi(f) = a(f) + a^*(f)$ instead of the annihilation/creation operators. Namely, if $\Gamma_{\mathrm{s}}^2(\mathfrak{h}) \ni v = \sum \lambda_n e_n \otimes e_n$ where $(e_n)_n$ is an orthonormal basis of $\mathfrak{h}$ and λ_n are positive numbers, then the operator $\sum \lambda_n \phi(e_n)\phi(e_n)$ is considered, while we use operators of the form $\sum \lambda_n a^*(e_n) a^*(e_n)$. In particular, the use of the field operators in the previous sum leads to quadratic expressions which are not normal ordered. Therefore, in order to make them well defined, one has to impose that v is actually trace class.

2 Fock spaces and representation of the CCR

2.1 Generalities on the Fock space

Let $\mathfrak{h}$ be a Hilbert space. We denote by $\Gamma_{\mathrm{s}}(\mathfrak{h})$ the bosonic Fock space over the one-particle space $\mathfrak{h}$,

$$\Gamma_{\mathrm{s}}(\mathfrak{h}) := \bigoplus_{n=0}^{\infty} \Gamma_{\mathrm{s}}^n(\mathfrak{h}),$$

where $\Gamma_{\mathrm{s}}^n(\mathfrak{h}) := \otimes_{\mathrm{s}}^n \mathfrak{h}$ denotes the symmetric $n-$fold tensor product of $\mathfrak{h}$ with the convention $\otimes_{\mathrm{s}}^0 \mathfrak{h} = \mathbb{C}$. $\Omega := (1, 0, \cdots)$ will denote the vacuum vector and

$$\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathfrak{h}) := \{\Psi = (\Psi^{(0)}, \cdots, \Psi^{(n)}, \cdots) \in \Gamma_{\mathrm{s}}(\mathfrak{h}) \mid \Psi^{(n)} = 0 \text{ for all but a finite number of } n\},$$

the finite particle space. Note that $\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathfrak{h})$ is dense in $\Gamma_{\mathrm{s}}(\mathfrak{h})$.

For any $f \in \mathfrak{h}$, $a(f)$ and $a^*(f)$ denote the usual annihilation/creation operators on $\Gamma_{\mathrm{s}}(\mathfrak{h})$. They satisfy

$$[a(f_1), a(f_2)] = [a^*(f_1), a^*(f_2)] = 0, \qquad [a(f_1), a^*(f_2)] = \langle f_1 | f_2 \rangle. \tag{2.1}$$

We also denote denote by $\phi(f) := \frac{1}{\sqrt{2}}(a(f) + a^*(f))$ the field operators and by $W(f) := \mathrm{e}^{\mathrm{i}\phi(f)}$ the Weyl operators. The Weyl operators are unitary and satisfy the following version of the CCR:

$$W(f)W(g) = \mathrm{e}^{-\frac{\mathrm{i}}{2}\mathrm{Im}\langle f|g\rangle} W(f+g). \tag{2.2}$$

If h is an operator on $\mathfrak{h}$, $\mathrm{d}\Gamma(h)$ will denote the second quantization of h :

$$\mathrm{d}\Gamma(h)\lceil_{\otimes_{\mathrm{s}}^n \mathfrak{h}} := \sum_{j=1}^{n} \underbrace{1 \otimes \cdots \otimes 1}_{j-1} \otimes h \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n-j}.$$

The operator $N := \mathrm{d}\Gamma(1)$ is the number operator. The following estimates are well known and sometimes called $N_\tau-$estimates [Ar, BFS, DJ, GJ].

Proposition 2.1. *Let h be a positive selfadjoint operator on $\mathfrak{h}$, and $f \in \mathfrak{h}$. Then, for all $\Psi \in \mathrm{Dom}(\mathrm{d}\Gamma(h)^{1/2})$,*

$$\begin{aligned} \|a(f)\Psi\| &\leq \|h^{-1/2} f\| \|\mathrm{d}\Gamma(h)^{1/2}\Psi\|, \\ \|a^*(f)\Psi\| &\leq \|h^{-1/2} f\| \|(1 + \mathrm{d}\Gamma(h))^{1/2}\Psi\|. \end{aligned}$$

Finally, if q is a bounded operator on $\mathfrak{h}$, we define $\Gamma(q) : \Gamma_{\mathrm{s}}(\mathfrak{h}) \to \Gamma_{\mathrm{s}}(\mathfrak{h})$ by $\Gamma(q)\lceil_{\otimes_{\mathrm{s}}^n \mathcal{H}} := q \otimes \cdots \otimes q$.

2.2 Quadratic annihilation and creation operators

Let $v \in \Gamma_s^2(\mathfrak{h})$. We define the annihilation and creation operators associated to v as follows:

$$a^*(v)\Psi := \sqrt{n+2}\sqrt{n+1}v \otimes_s \Psi, \qquad \Psi \in \Gamma_s^n(\mathfrak{h}),$$

$$a(v)\Psi := \sqrt{n+2}\sqrt{n+1}\left(\langle v| \otimes 1^{\otimes n}\right)\Psi, \qquad \Psi \in \Gamma_s^{n+2}(\mathfrak{h}),$$

where $\langle v| \otimes 1^{\otimes n} : \Gamma_s^{n+2}(\mathfrak{h}) \ni f_1 \otimes \cdots \otimes f_{n+2} \mapsto \langle v|f_1 \otimes f_2\rangle f_3 \otimes \cdots \otimes f_{n+2} \in \Gamma_s^n(\mathfrak{h})$. These operators are well defined on $\Gamma_s^{\mathrm{fin}}(\mathfrak{h})$ and can be extended to $\mathrm{Dom}(N)$.

Proposition 2.2. *Let $\Psi \in \Gamma_s^{\mathrm{fin}}(\mathfrak{h})$, then*

(i) $\|a(v)\Psi\| \leq \|v\|\|N\Psi\|$, (2.3)

(ii) $\|a^*(v)\Psi\| \leq \|v\|\|(N+2)^{1/2}(N+1)^{1/2}\Psi\|$. (2.4)

This result will be a particular case of Propositions 3.23 and 3.24 (Section 3.8).

Note also that if we write $v = \sum \lambda_n\ \phi_n \otimes \psi_n$, where $(\phi_n)_{n\in\mathbb{N}}, (\psi_n)_{n\in\mathbb{N}}$ are two orthonormal bases of $\mathfrak{h}$ and $(\lambda_n)_{n\in\mathbb{N}}$ is a sequence of positive numbers (with $\sum \lambda_n^2 = \|v\|^2_{\Gamma_s(\mathfrak{h})} < +\infty$), then we have

$$a(v) = \sum \lambda_n a(\phi_n)a(\psi_n), \quad a^*(v) = \sum \lambda_n a^*(\phi_n)a^*(\psi_n), \tag{2.5}$$

where on the right-hand side a and a^* denote the usual annihilation/creation operators.

Before going further, we would like to make the link between elements of the 2-particle space and real symmetric Hilbert-Schmidt operators on $\mathfrak{h}$, which will play an important role in the sequel. Let us fix a conjugation $f \mapsto \bar{f}$ on $\mathfrak{h}$. We denote by $B^2(\mathfrak{h})$ the set of all Hilbert-Schmidt operators and by $B_s^2(\mathfrak{h})$ the set of real symmetric (i.e. $\bar{v} = v^*$) Hilbert-Schmidt operators. It is well known that $\mathfrak{h} \otimes \mathfrak{h}$ and $B^2(\mathfrak{h})$ are isomorphic (the map $T : \mathfrak{h} \otimes \mathfrak{h} \ni \phi \otimes \psi \mapsto |\phi\rangle\langle\bar{\psi}| \in B^2(\mathfrak{h})$ extends by linearity and defines an isometry). It is easy to see that $T(\Gamma_s^2(\mathfrak{h})) = B_s^2(\mathfrak{h})$. We will thus make no difference between a symmetric Hilbert-Schmidt operator and the corresponding element of $\Gamma_s^2(\mathfrak{h})$.

Using (2.1), one then easily gets the following commutation relations:

Proposition 2.3. *For all $v, v' \in \Gamma_s^2(\mathfrak{h})$, $f \in \mathfrak{h}$ and h selfadjoint operator on $\mathfrak{h}$,*

$$[a^*(v), a(f)] = -2a^*(v\bar{f}), \quad [a(v), a^*(f)] = 2a(v\bar{f}), \tag{2.6}$$

$$[a(v), a^*(v')] = 4\mathrm{d}\Gamma(v'v^*) + 2\mathrm{Tr}(v^*v'). \tag{2.7}$$

$$[\mathrm{d}\Gamma(h), a^*(v)] = a^*(hv + v\bar{h}), \quad [\mathrm{d}\Gamma(h), a(v)] = -a(hv + v\bar{h}) \tag{2.8}$$

To end this section, we would like to introduce the exponential of the operators $a(v)$ and $a^*(v)$, which will be used to define the unitary operators U_{nat} (Section 3.1).

Proposition 2.4. *Let $v \in B_s^2(\mathfrak{h})$.*

1) For all $\Psi \in \Gamma_s^{\mathrm{fin}}(\mathfrak{h})$, there exists $\mathrm{s}-\lim\limits_{n\to+\infty} \sum_{k=0}^{n} \frac{1}{k!}\left(\frac{1}{2}a(v)\right)^k \Psi =: \mathrm{e}^{\frac{1}{2}a(v)}\Psi$, *and* $\mathrm{e}^{\frac{1}{2}a(v)}\Psi \in \Gamma_s^{\mathrm{fin}}(\mathfrak{h})$.

2) If $\|v\|_{B(\mathfrak{h})} < 1$, then for all $\Psi \in \Gamma_s^{\mathrm{fin}}(\mathfrak{h})$, there exists $\mathrm{s}-\lim\limits_{n\to+\infty} \sum_{k=0}^{n} \frac{1}{k!}\left(\frac{1}{2}a^*(v)\right)^k \Psi =: \mathrm{e}^{\frac{1}{2}a^*(v)}\Psi$.

Proof. The proof of 1) is obvious since the right-hand side reduces to a finite sum. Now, part 2) follows from the fact that the function

$$f(z) := \sum_{n=0}^{\infty} \left(\frac{1}{2^n n!} \right)^2 \|a^*(v)^n \Omega\|^2 z^n$$

converges for all $|z| < \frac{1}{\|v\|^2_{B(\mathfrak{h})}}$ (see [Ru2]). Indeed, it is sufficient to prove the proposition for vectors of the form $\Psi = a^*(f_1) \cdots a^*(f_m)\Omega$, where $f_1, \cdots f_m \in \mathfrak{h}$. Now, we have, for all $n \in \mathbb{N}$,

$$\begin{aligned}
\| \sum_{k=0}^{n} \frac{1}{k!} \left(\frac{1}{2} a^*(v) \right)^k \Psi \|^2 &= \sum_{k=0}^{n} \| \frac{1}{k!} \left(\frac{1}{2} a^*(v) \right)^k \Psi \|^2 \\
&\leq \|f_1\|^2 \cdots \|f_m\|^2 \sum_{k=0}^{n} \left(\frac{1}{2^k k!} \right)^2 (2k+m+1)(2k+m) \cdots (2k+1) \|a^*(v)^k \Omega\|^2 \\
&\leq \|f_1\|^2 \cdots \|f_m\|^2 \sum_{k=0}^{+\infty} \frac{(2k+m+1)!}{(2k)!} \left(\frac{1}{2^k k!} \right)^2 \|a^*(v)^k \Omega\|^2 =: C_m \|\Psi\|^2 < +\infty.
\end{aligned}$$

□

Finally, we have the following

Proposition 2.5. *Let $(v_l)_{l \in \mathbb{N}}$ be a sequence in $\Gamma^2_{\mathrm{s}}(\mathfrak{h})$ such that $\|v_l\|_{B(\mathfrak{h})} < 1$ for all $l \in \mathbb{N}$ and $\lim_{l \to \infty} \|v_l\| = 0$. Then the operators $\mathrm{e}^{\frac{1}{2} a^*(v_l)}$ strongly converge to the identity on $\Gamma^{\mathrm{fin}}_{\mathrm{s}}(\mathfrak{h})$.*

Proof. The result follows by the same computation as in the proof of the previous proposition. Indeed, let $\Psi = a^*(f_1) \cdots a^*(f_m)\Omega$, then for all $n \in \mathbb{N}$, and using (2.4),

$$\| \sum_{k=0}^{n} \frac{1}{k!} \left(\frac{1}{2} a^*(v_l) \right)^k \Psi - \Psi \|^2 \leq \|f_1\|^2 \cdots \|f_m\|^2 \sum_{k=1}^{+\infty} \frac{(2k+m+1)!}{(2k)!} \left(\frac{1}{2^k k!} \right)^2 \|a^*(v_l)^k \Omega\|^2 \leq C_m \|v_l\| \|\Psi\|^2,$$

where $C_m < +\infty$. Hence $\|\mathrm{e}^{\frac{1}{2} a^*(v_l)} \Psi - \Psi\| \leq C_m \|v_l\| \|\Psi\|^2$ and the result follows. □

2.3 Fock representations of CCR

In this paper we are interested in Fock representations of CCR, *i.e.* $\mathcal{H} = \Gamma_{\mathrm{s}}(\mathfrak{h})$ where $\mathfrak{h}$ is a given complex Hilbert space. From now on, we assume that the real symplectic space $\mathcal{Y}$ is of the form $\mathcal{Y} = \{(f, \bar{f}) \in \mathfrak{h} \oplus \mathfrak{h} | f \in \mathfrak{h}\}$ and that the symplectic form $\sigma((f, \bar{f}), (g, \bar{g})) = \mathrm{Im}\langle f|g\rangle$, where $\langle \cdot|\cdot \rangle$ denotes the scalar product in $\mathfrak{h}$.

We consider the map $\mathcal{Y} \ni (f, \bar{f}) \mapsto W(f) \in U(\Gamma_{\mathrm{s}}(\mathfrak{h}))$ where $W(f)$ is the Weyl operator defined in Section 2.1. Using (2.2), we can see that this map is a representation of CCR. Moreover, it is well known that this representation is regular and irreducible [BR].

Finally, we define the following operator on $\mathcal{Y}$: $J = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}$. This operator is an antiinvolution ($J^2 = -1$) which preserves the symplectic form σ.

3 Bogoliubov transformations and Bogoliubov Hamiltonians

3.1 Bogoliubov implementer

A bounded real map R on $\mathcal{Y} = \{(f, \bar{f}) | f \in \mathfrak{h}\}$ will be written as $R = \begin{pmatrix} P & \bar{Q} \\ Q & \bar{P} \end{pmatrix}$, where P and Q are bounded linear maps on $\mathfrak{h}$, and $\bar{P} f := \overline{P \bar{f}}$ (and similarly for $\bar{Q}$). It is easy to see that a map R is symplectic if and only if $RJR^* = R^*JR = J$ which is equivalent to

$$\left\{\begin{array}{rclrcl} P^*P - Q^*Q &=& 1, & PP^* - \overline{QQ^*} &=& 1, \\ \bar{P}^*Q - \bar{Q}^*P &=& 0, & QP^* - \overline{PQ^*} &=& 0. \end{array}\right. \tag{3.1}$$

In particular, if R is symplectic, then $P^*P \geq 1$ and therefore P is invertible.

The following natural identification will be sometimes useful

$$I : \mathfrak{h} \ni f \mapsto (f, \bar{f}) \in \mathcal{Y}. \tag{3.2}$$

Given a symplectic map R, we define $W_R(f) := W(I^{-1}R(f, \bar{f}))$. The map $(f, \bar{f}) \mapsto W_R(f)$ is also a representation of CCR over $\mathcal{Y}$ in $\Gamma_{\mathrm{s}}(\mathfrak{h})$.

Definition 3.1. *A symplectic map R is called unitarily implementable if and only if there exists a unitary operator U on $\Gamma_{\mathrm{s}}(\mathfrak{h})$ such that $UW(f)U^{-1} = W_R(f)$, $\forall f \in \mathfrak{h}$. If it exists, U is called a Bogoliubov implementer of R.*

Assumption 3.A. *(Shale condition): $Q \in B^2(\mathfrak{h})$ ($\Leftrightarrow [R, J] \in B^2(\mathfrak{h} \oplus \mathfrak{h})$).*

We define the operators K and L as follows

$$K := \overline{QP^{-1}}, \qquad L := -P^{-1}\bar{Q}. \tag{3.3}$$

The following result is well known (see [Be, Ru1, Ru2, Sh]).

Theorem 3.2. *R is unitarily implementable if and only if the Shale condition is satisfied. If it is satisfied, then*

(i) the operators K and L belong to $B^2_{\mathrm{s}}(\mathfrak{h})$ and $\|K\| < 1$,

(ii) the operator
$$U_{\mathrm{nat}} := \det(1 - K^*K)^{1/4} \mathrm{e}^{-\frac{1}{2}a^*(K)} \Gamma((P^{-1})^*) \mathrm{e}^{-\frac{1}{2}a(L)} \tag{3.4}$$
is well defined on $\Gamma^{\mathrm{fin}}_{\mathrm{s}}(\mathfrak{h})$, extends to a unitary operator on $\Gamma_{\mathrm{s}}(\mathfrak{h})$, and implements R.

We call U_{nat} the natural Bogoliubov implementer of R. Since the Weyl representation is irreducible, if R is unitarily implementable, then the Bogoliubov implementer is unique up to a phase factor. U_{nat} has the particular feature that its expectation value on the vacuum is positive: $\langle \Omega | U_{\mathrm{nat}} \Omega \rangle = \det(1 - K^*K)^{1/4} > 0$.

3.2 Bogoliubov dynamics and Bogoliubov Hamiltonians

Suppose $t \mapsto R(t) = \begin{pmatrix} P(t) & \bar{Q}(t) \\ Q(t) & \bar{P}(t) \end{pmatrix}$ is a strongly continuous one parameter group of symplectic maps. We denote by $K(t)$ and $L(t)$ the operators defined in (3.3) associated to $R(t)$.

Definition 3.3. *A one parameter symplectic group $R(t)$ is called unitarily implementable if and only if there exists a strongly continuous unitary group $U(t)$ such that, for all t, $U(t)$ is a Bogoliubov implementer of $R(t)$. If $R(t)$ is unitarily implementable, we call a Bogoliubov dynamics implementing $R(t)$ the unitary group $U(t)$ and a Bogoliubov Hamiltonian (associated to $R(t)$) its selfadjoint generator.*

Since the Bogoliubov implementer of a symplectic map R is unique up to a phase, if $R(t)$ is unitarily implementable, then there exists $c(t) \in \mathbb{C}$, $|c(t)| = 1$, such that $U(t) = c(t)U_{\mathrm{nat}}(t)$, and where $U_{\mathrm{nat}}(t)$ is the natural Bogoliubov implementer of $R(t)$. $c(t)$ will be called the natural cocycle for $U(t)$.

One can actually prove that $R(t)$ is unitarily implementable under very weak assumptions.

Theorem 3.4. *Suppose $R(t)$ is a strongly continuous one-parameter symplectic group. Then $R(t)$ is unitarily implementable if and only if the Shale condition is satisfied for all time t and $\lim_{t\to 0}\|K(t)\|_2 = 0$.*

Proof. Suppose $R(t)$ is unitarily implementable. Using Theorem 3.2, we immediately get that the Shale condition is satisfied for all t. It remains to prove that $\|K(t)\|_2 \to 0$ as t goes to zero. Let $U(t)$ be a strongly continuous unitary group implementing $R(t)$ and let

$$\alpha_t : \mathcal{B}(\Gamma_{\mathrm{s}}(\mathfrak{h})) \ni B \mapsto U(t)BU(t)^* \in \mathcal{B}(\Gamma_{\mathrm{s}}(\mathfrak{h})).$$

Clearly α_t is a weak$*$ continuous one parameter group of $*-$automorphisms, and $\alpha_t(B) = U_{\mathrm{nat}}(t)BU_{\mathrm{nat}}(t)^*$ since we have $U(t) = c(t)U_{\mathrm{nat}}(t)$ where $c(t)$ is the natural cocycle for $U(t)$. Therefore the map

$$\mathbb{R} \ni t \mapsto \mathrm{Tr}(|\Omega\rangle\langle\Omega|\alpha_t(|\Omega\rangle\langle\Omega|)) = \det(1 - K(t)^*K(t))^{1/2}$$

is continuous. Since $\|K(t)\| < 1$, $\det(1 - K(t)^*K(t)) = \mathrm{e}^{\mathrm{Tr}(\log(1-K(t)^*K(t)))}$. Moreover $K(0) = 0$, so

$$\lim_{t\to 0} \mathrm{Tr}(\log(1 - K(t)^*K(t))) = 0,$$

from which the result follows using

$$\|K(t)\|_2^2 = \mathrm{Tr}(K(t)^*K(t)) \leq |\mathrm{Tr}(\log(1 - K(t)^*K(t)))|.$$

Suppose now that Shale condition is satisfied for all t. Hence, for all t, we can construct $U_{\mathrm{nat}}(t)$ the natural implementer associated to $R(t)$. Let us define the map

$$\alpha_t : \mathcal{B}(\Gamma_{\mathrm{s}}(\mathfrak{h})) \ni B \mapsto U_{\mathrm{nat}}(t)BU_{\mathrm{nat}}(t)^* \in \mathcal{B}(\Gamma_{\mathrm{s}}(\mathfrak{h})).$$

Obviously, for all t, α_t is a weak$*$ continuous $*-$automorphism of $\mathcal{B}(\Gamma_{\mathrm{s}}(\mathfrak{h}))$. Moreover, for all $t, s \in \mathbb{R}$,

$$\alpha_t(\alpha_s(W(f))) = \alpha_{t+s}(W(f)) = W_{R(t+s)}(f), \quad \forall f \in \mathfrak{h}.$$

Since the $*-$algebra generated by the Weyl operators is weak$*$ dense in $\mathcal{B}(\Gamma_{\mathrm{s}}(\mathfrak{h}))$, this proves that α_t forms a one-parameter group of $*-$automorphisms of $\mathcal{B}(\Gamma_{\mathrm{s}}(\mathfrak{h}))$.

In order to prove that it can be implemented by a selfadjoint operator H, it remains to show that this one parameter group is weak$*$ continuous with respect to t ([BR], Ex 3.2.35). Moreover, using the group property it suffices to prove that it is weak$*$ continuous at $t = 0$. For that purpose, we shall prove that $t \mapsto U_{\mathrm{nat}}(t)$ is strongly continuous at $t = 0$.

The map $t \mapsto K(t)$ is continuous at $t = 0$ in the Hilbert-Schmidt norm by assumption (recall that $K(0) = 0$). This together with Proposition 2.5 proves that $t \mapsto U_{\mathrm{nat}}(t)\Omega$ is continuous at $t = 0$.

Now, for any $f \in \mathfrak{h}$ one has $U_{\mathrm{nat}}(t)W(f)\Omega = W_{R(t)}(f)U_{\mathrm{nat}}(t)\Omega$. Hence

$$\begin{aligned}\|U_{\mathrm{nat}}(t)W(f)\Omega - W(f)\Omega\| &\leq \|W_{R(t)}(f)(U_{\mathrm{nat}}(t)\Omega - \Omega)\| + \|(W_{R(t)}(f) - W(f))\Omega\| \\ &\leq \|U_{\mathrm{nat}}(t)\Omega - \Omega\| + \|(W_{R(t)}(f) - W(f))\Omega\|.\end{aligned}$$

The first term of the second line goes to zero as t goes to zero, and the second one as well since $t \mapsto R(t)$ is strongly continuous and

$$\lim_{n\to+\infty} \|f_n - f\| = 0 \Longrightarrow \mathrm{s}-\lim_{n\to+\infty} W(f_n) = W(f).$$

Thus, we have proven that $U_{\mathrm{nat}}(t)$ is strongly continuous at $t = 0$ on $\mathrm{Span}\{W(f)\Omega,\, f \in \mathfrak{h}\}$. Since this subspace is dense in $\Gamma_{\mathrm{s}}(\mathfrak{h})$ and the $U_{\mathrm{nat}}(t)$ are unitary, this ends the proof. $\square$

3.3 Generator of unitarily implementable symplectic groups

In this section, we look for conditions on the generator A of $R(t)$ which make it unitarily implementable. The basic assumption on the generator A will be the following.

Assumption 3.B. *A can be written as* $A = \mathrm{i}\begin{pmatrix} h & -v \\ \bar{v} & -\bar{h} \end{pmatrix}$*, where h is a selfadjoint operator with domain* $\mathrm{Dom}(h)$*, v is a bounded operator such that $v^* = \bar{v}$, and* $\mathrm{Dom}(A) = \mathrm{Dom}(h) \oplus \mathrm{Dom}(\bar{h})$.

Proposition 3.5. *Suppose A satisfies Assumption 3.B, then A generates a strongly continuous one-parameter group $(R(t))_{t\in\mathbb{R}}$ of symplectic maps.*

Proof. h is selfadjoint, therefore the operator $A_0 := \mathrm{i}\begin{pmatrix} h & 0 \\ 0 & -\bar{h} \end{pmatrix}$ generates a one-parameter group of unitary maps $R_0(t) = \mathrm{e}^{tA_0}$. Moreover, one can see that $R_0(t)$ is symplectic. Let us also write $V := \mathrm{i}\begin{pmatrix} 0 & -v \\ \bar{v} & 0 \end{pmatrix}$. Then $A = A_0 + V$ where A_0 is the generator of a strongly continuous one-parameter group and V is a bounded operator. Hence, A generates a one parameter strongly continuous group $R(t)$.

Since $R(t)$ and J leave $\mathrm{Dom}(A)$ invariant, for all $f \in \mathrm{Dom}(A)$, $t \mapsto R(t)JR(t)^* f$ is differentiable, and

$$\frac{\mathrm{d}}{\mathrm{d}t} R(t)JR(t)^* f = R(t)(JA^* + AJ)R(t)f.$$

But, using $h^* = h$ and $v^* = \bar{v}$, one gets $(JA^* + AJ)f = 0$ for all $f \in \mathrm{Dom}(A)$. Hence, $R(t)JR(t)^* = J$ on $\mathrm{Dom}(A)$. Since they are both bounded operators and $\mathrm{Dom}(A)$ is dense, this proves that $R(t)JR(t)^* = J$ on $\mathfrak{h} \oplus \mathfrak{h}$. We prove similarly that $R(t)^* JR(t) = J$, so that $R(t)$ is symplectic. □

From now on, we will always assume that Assumption 3.B is satisfied. Let us define

$$v(t) := \int_0^t \mathrm{e}^{\mathrm{i}\tau h} v \mathrm{e}^{\mathrm{i}\tau \bar{h}} \mathrm{d}\tau. \tag{3.5}$$

Assumption 3.C. *For all t, $v(t) \in B^2(\mathfrak{h})$, the function $t \mapsto \|v(t)\|_2$ is locally integrable on $\mathbb{R}$ and continuous at $t = 0$.*

This assumption was already used in [Be].

Theorem 3.6. *Suppose A satisfies Assumption 3.C. Then $R(t)$ is unitarily implementable.*

Proof. We define $V(t) := R_0(t)VR_0(-t)$ and $\tilde{R}(t) := R(t)R_0(-t)$. Since V is bounded, we have

$$\tilde{R}(t) = 1 + \int_0^t \tilde{R}(\tau)V(\tau)\mathrm{d}\tau. \tag{3.6}$$

We introduce the following sequence of bounded operators

$$\tilde{R}_0(t) = 1, \qquad \tilde{R}_{n+1}(t) = \int_0^t \tilde{R}_n(\tau)V(\tau)\mathrm{d}\tau.$$

In particular we have

$$\tilde{R}_1(t) = \int_0^t V(\tau)\mathrm{d}\tau = \mathrm{i}\begin{pmatrix} 0 & -v(t) \\ \overline{v(t)} & 0 \end{pmatrix}.$$

Hence, $\tilde{R}_1(t)$ is Hilbert-Schmidt and $\|\tilde{R}_1(t)\|_2 = \sqrt{2}\|v(t)\|_2$. Then, using $\|V(\tau)\| = 2\|v\|$ for all τ, we get

$$\begin{aligned} \|\tilde{R}_{n+1}(t)\|_2 &\le 2\|v\| \int_0^t \|\tilde{R}_n(\tau)\|_2 \mathrm{d}\tau \\ &\le (2\|v\|)^n \int_0^t \frac{(t-\tau)^{n-1}}{(n-1)!} \sqrt{2}\|v(\tau)\|_2 \mathrm{d}\tau, \qquad \forall n \ge 1. \end{aligned}$$

Moreover, we have $\tilde{R}(t) - 1 = \sum_{n\ge 1} \tilde{R}_n(t)$, hence

$$\|\tilde{R}(t) - 1\|_2 \le \sqrt{2}\|v(t)\|_2 + 2\sqrt{2}\|v\| \int_0^t \mathrm{e}^{2(t-\tau)\|v\|} \|v(\tau)\|_2 \mathrm{d}\tau < +\infty. \tag{3.7}$$

Since $R(t)R_0(-t)-1$ is Hilbert-Schmidt, so is $R(t)-R_0(t)$. Now, $Q_0(t) = 0$ hence $Q(t)$ is Hilbert-Schmidt.

It remains to prove that $\lim_{t\to 0} \|K(t)\|_2 = 0$. Using (3.7) and the continuity of $\|v(t)\|_2$ at $t = 0$, we get $\lim_{t\to 0} \|\tilde{R}(t) - 1\|_2 = 0$. Thus we have $\lim_{t\to 0} \|\bar{Q}(t)\mathrm{e}^{\mathrm{i}t\bar{h}}\|_2 = 0$. Finally, by definition of $K(t)$ we have

$$\|K(t)\|_2 = \|\overline{Q(t)P(t)^{-1}}\|_2 \le \|\bar{Q}(t)\mathrm{e}^{\mathrm{i}t\bar{h}}\|_2 \|\mathrm{e}^{-\mathrm{i}t\bar{h}}\overline{P(t)^{-1}}\| \le \|\bar{Q}(t)\mathrm{e}^{\mathrm{i}t\bar{h}}\|_2. \qquad \square$$

3.4 Bogoliubov dynamics of type I

As mentioned in the introduction, there are natural choices for the Bogoliubov dynamics implementing $R(t)$, one of them being type I. However, it is not always possible to define it and one has to impose some additional assumption on $R(t)$. We will denote by $B^1(\mathfrak{h})$ the set of trace class operators on $\mathfrak{h}$ and by $\|\cdot\|_1$ the trace norm.

Definition 3.7. *Let $t \mapsto R(t)$ be a unitarily implementable symplectic group, with generator A. We say that it is type I if and only if, for all $t \in \mathbb{R}$, $P(t)\mathrm{e}^{-\mathrm{i}th} - 1 \in B^1(\mathfrak{h})$ and $\lim_{t\to 0} \|P(t)\mathrm{e}^{-\mathrm{i}th} - 1\|_1 = 0$.*

Theorem 3.8. *Suppose $R(t)$ is a type I symplectic group. Then, the operators*

$$U_I(t) := \det(\bar{P}(t)\mathrm{e}^{\mathrm{i}t\bar{h}})^{-1/2} \mathrm{e}^{-\frac{1}{2}a^*(K(t))} \Gamma((P(t)^{-1})^*) \mathrm{e}^{-\frac{1}{2}a(L(t))} \tag{3.8}$$

form a Bogoliubov dynamics implementing $R(t)$. Their natural cocycle is given by

$$c_I(t) = \det(\bar{P}(t)\mathrm{e}^{\mathrm{i}t\bar{h}})^{-1/2} \det(1 - K(t)^*K(t))^{-1/4}. \tag{3.9}$$

Definition 3.9. *The operator $H_I = \frac{1}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}t} U_I(t)\lceil_{t=0}$ is called a Bogoliubov Hamiltonian of type I.*

In the proof of Theorem 3.8, we will make use of the following lemmas.

Lemma 3.10. *Let B be a bounded operator and V a unitary operator such that $BV - 1$ is trace class. Then $VB - 1$ is trace class and* $\det(BV) = \det(VB)$.

Lemma 3.11. *Let $K, L \in B^2(\mathfrak{h})$ such that $\bar{K} = K^*, \bar{L} = L^*$ and $\|K\| < 1, \|L\| < 1$. Then*

$$\langle \mathrm{e}^{-\frac{1}{2}a^*(L)\Omega} | \mathrm{e}^{-\frac{1}{2}a^*(K)}\Omega\rangle = \det(1 - L^*K)^{-1/2}.$$

Proof. Since K is Hilbert-Schmidt and $\bar{K} = K^*$, there exist an orthonormal basis of $\mathfrak{h}$, $(f_n)_n$, and a sequence λ_n such that $K = \sum \lambda_n |f_n\rangle\langle \bar{f}_n|$. Similarly, we can write $L = \sum \mu_m |g_m\rangle\langle \bar{g}_m|$. Therefore, we have

$$\begin{aligned} \langle \mathrm{e}^{-\frac{1}{2}a^*(L)}\Omega | \mathrm{e}^{-\frac{1}{2}a^*(K)}\Omega\rangle &= \prod_{m,n} \langle \mathrm{e}^{-\frac{1}{2}\mu_m a^*(g_m)^2}\Omega | \mathrm{e}^{-\frac{1}{2}\lambda_n a^*(f_n)^2}\Omega\rangle \\ &= \prod_{m,n} \sum_j \left(-\frac{1}{2}\right)^{2j} \frac{\bar{\mu}_m^j \lambda_n^j (2j)!}{(j!)^2} \langle g_m | f_n\rangle^{2j} \\ &= \prod_{m,n} (1 - \bar{\mu}_m \lambda_n \langle g_m | f_n\rangle^2)^{-1/2}. \end{aligned}$$

Now, it suffices to see that $L^*K = \sum_{m,n} \bar{\mu}_m \lambda_n \langle g_m | f_n\rangle |\bar{g}_m\rangle\langle \bar{f}_n|$. □

Proof of Theorem 3.8 Since $U_{\rm nat}(t)$ is unitary and $U_I(t) = c_I(t)U_{\rm nat}(t)$, to prove that $U_I(t)$ is a Bogoliubov implementer, it suffices to show that $|c_I(t)| = 1$. Using (3.1), we have $1 - K(t)^*K(t) = (\overline{P(t)P(t)^*})^{-1}$. Now

$$\begin{aligned} |\det(\bar{P}(t)\mathrm{e}^{\mathrm{i}t\bar{h}})^{-1/2}| = \left|\det(\bar{P}(t)\mathrm{e}^{\mathrm{i}t\bar{h}})\overline{\det((\bar{P}(t)\mathrm{e}^{\mathrm{i}t\bar{h}})^*)}\right|^{-1/4} &= \left|\det(\bar{P}(t)\mathrm{e}^{\mathrm{i}t\bar{h}})\det(\mathrm{e}^{-\mathrm{i}t\bar{h}}P\bar{(t)}^*)\right|^{-1/4} \\ &= |\det(\overline{P(t)P(t)^*})|^{-1/4}. \end{aligned}$$

We now prove that the operators $U_I(t)$ form a one-parameter group. As for $U_{\rm nat}(t)$, for all s and t there exists $\alpha(t,s) \in \mathbb{R}$ such that $U_I(t)U_I(s) = \mathrm{e}^{\mathrm{i}\alpha(t,s)}U_I(t+s)$. Using Lemmas 3.10 and 3.11 we have

$$\begin{aligned} \langle\Omega|U_I(t)U_I(s)\Omega\rangle &= \det(\bar{P}(t)\mathrm{e}^{\mathrm{i}t\bar{h}})^{-1/2}\det(\bar{P}(s)\mathrm{e}^{\mathrm{i}s\bar{h}})^{-1/2}\langle \mathrm{e}^{-\frac{1}{2}a^*(L(t))}\Omega|\mathrm{e}^{-\frac{1}{2}a^*(K(s))}\Omega\rangle \\ &= \det(\bar{P}(t)\mathrm{e}^{\mathrm{i}t\bar{h}})^{-1/2}\det(\bar{P}(s)\mathrm{e}^{\mathrm{i}s\bar{h}})^{-1/2}\det(1 - L(t)^*K(s))^{-1/2} \\ &= \left(\det(\mathrm{e}^{\mathrm{i}t\bar{h}}\bar{P}(t))\det(1 + \bar{P}(t)^{-1}Q(t)\bar{Q}(s)\bar{P}(s)^{-1})\det(\bar{P}(s)\mathrm{e}^{\mathrm{i}s\bar{h}})\right)^{-1/2} \\ &= \det(\mathrm{e}^{\mathrm{i}t\bar{h}}(\bar{P}(t)\bar{P}(s) + Q(t)\bar{Q}(s))\mathrm{e}^{\mathrm{i}s\bar{h}})^{-1/2} \\ &= \det(\bar{P}(t+s)\mathrm{e}^{\mathrm{i}(t+s)\bar{h}})^{-1/2} = \langle\Omega|U_I(t+s)\Omega\rangle. \end{aligned}$$

Therefore $\mathrm{e}^{\mathrm{i}\alpha(t,s)} = 1$ and $U_I(t)$ is a one-parameter group.

Finally we have to prove that $U_I(t)$ is strongly continuous. Using the group property together with the same argument as in Theorem 3.4, it suffices to prove that $t \mapsto U_I(t)\Omega$ is continuous at $t = 0$. Now, $t \mapsto K(t)$ is continuous in the Hilbert-Schmidt norm since $R(t)$ is unitarily implementable (Theorem 3.4), and, by assumption, $t \mapsto P(t)\mathrm{e}^{-\mathrm{i}th}$ is continuous in the trace norm at $t = 0$, thus so is the map $t \mapsto \det(P(t)\mathrm{e}^{-\mathrm{i}th})$, which ends the proof. □

3.5 Generator of type I Bogoliubov dynamics

We would like in this section to give some sufficient conditions on the generator A of a symplectic group $R(t)$ so that it is of type I.

Assumption 3.D. *For all t, the operator $\bar{v}v(t)$ is trace class and the function $t \mapsto \|\bar{v}v(t)\|_1$ is locally integrable on $\mathbb{R}$ and continuous at $t = 0$.*

This condition was also used in [Be].

Assumption 3.E. *v is a Hilbert-Schmidt operator on $\mathfrak{h}$.*

Theorem 3.12. *(i) If Assumptions 3.C and 3.D are satisfied, then $R(t)$ is of type I.*

(ii) If Assumption 3.E is satisfied, then $R(t)$ is of type I and we have

$$U_I(t) = \mathrm{e}^{\frac{\mathrm{i}}{2}\mathrm{Tr}(\int_0^t Q(s)v\bar{P}(s)^{-1}\mathrm{d}s)}\mathrm{e}^{-\frac{1}{2}a^*(K(t))}\Gamma((P(t)^{-1})^*)\mathrm{e}^{-\frac{1}{2}a(L(t))}, \tag{3.10}$$

$$c_I(t) = \mathrm{e}^{\frac{\mathrm{i}}{2}\mathrm{Re}(\mathrm{Tr}(\int_0^t Q(s)v\bar{P}(s)^{-1}\mathrm{d}s))}. \tag{3.11}$$

Proof. We will use the notation introduced in the proof of Theorem 3.6. Let

$$\mathcal{V} := \left\{ R = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{B}(\mathfrak{h}\oplus\mathfrak{h}) \mid A, D \in B^1(\mathfrak{h}), B, C \in B^2(\mathfrak{h}) \right\},$$

with $\|R\|_{\mathcal{V}} := \|A\|_1 + \|D\|_1 + \|B\|_2 + \|C\|_2$. Note that if R and R' are in $\mathcal{V}$, then so is RR'.

(i) Suppose Assumption 3.D is satisfied. We have $\tilde{R}_1(t) \in \mathcal{V}$. Then,

$$\tilde{R}_2(t) = \int_0^t \tilde{R}_1(\tau)V(\tau)\mathrm{d}\tau = \begin{pmatrix} \int_0^t v(\tau)\mathrm{e}^{-\mathrm{i}\tau\bar{h}}\bar{v}\mathrm{e}^{-\mathrm{i}\tau h}\mathrm{d}\tau & 0 \\ 0 & \int_0^t \overline{v(\tau)}\mathrm{e}^{\mathrm{i}\tau h}v\mathrm{e}^{\mathrm{i}\tau\bar{h}}\mathrm{d}\tau \end{pmatrix}.$$

Using Assumption 3.D, one has

$$\| \int_0^t v(\tau)\mathrm{e}^{-\mathrm{i}\tau\bar{h}}\bar{v}\mathrm{e}^{-\mathrm{i}\tau h}\mathrm{d}\tau\|_1 \le \int_0^t \|\bar{v}v(\tau)\|_1\mathrm{d}\tau =: \rho(t) < +\infty$$

Therefore $\tilde{R}_2(t)$ is trace class and $\|\tilde{R}_2(t)\|_1 \le 2\rho(t)$. In the same way as in the proof of Theorem 3.6, we have that $\tilde{R}(t) - 1 - \tilde{R}_1(t)$ is trace class, and hence is in $\mathcal{V}$. Thus so is $\tilde{R}(t) - 1$. In particular, $P(t)\mathrm{e}^{-\mathrm{i}th} - 1$ is trace class.

Finally, the continuity of $\|\bar{v}v(t)\|_1$ at $t = 0$ implies the one of $\|P(t)\mathrm{e}^{-\mathrm{i}th} - 1\|_1$ in a similar way as in Theorem 3.6.

(ii) Suppose now that Assumption 3.E is satisfied. First note that it implies Assumptions 3.C and 3.D, so that $R(t)$ is of type I. According to the definition of $U_I(t)$, we have to prove that, for all t,

$$\det(\bar{P}(t)\mathrm{e}^{\mathrm{i}t\bar{h}}) = \mathrm{e}^{-\mathrm{i}\int_0^t \mathrm{Tr}(Q(s)v\bar{P}(s)^{-1})\mathrm{d}s}. \tag{3.12}$$

For all t, $V(t) \in \mathcal{V}$, and, using (3.6), we have as an identity in $\mathcal{V}$

$$\tilde{R}(t) - 1 = \int_0^t V(\tau)\mathrm{d}\tau + \int_0^t (\tilde{R}(\tau) - 1)V(\tau)\mathrm{d}\tau.$$

We have $V(t) = \mathrm{i}\begin{pmatrix} 0 & -\mathrm{e}^{\mathrm{i}th}v\mathrm{e}^{\mathrm{i}t\bar{h}} \\ \mathrm{e}^{-\mathrm{i}t\bar{h}}\bar{v}\mathrm{e}^{-\mathrm{i}th} & 0 \end{pmatrix}$. It is clear that $t \mapsto \mathrm{e}^{\mathrm{i}th}v\mathrm{e}^{\mathrm{i}t\bar{h}}$ is continuous in the weak operator topology, and therefore in the weak sense in $B^2(\mathfrak{h})$ considered as a Hilbert space ($i.e.$ for all $K \in B^2(\mathfrak{h})$, $t \mapsto \mathrm{Tr}(K\mathrm{e}^{\mathrm{i}th}v\mathrm{e}^{\mathrm{i}t\bar{h}})$ is continuous). Moreover, since $\mathrm{e}^{\mathrm{i}th}$ is unitary, we have $\|\mathrm{e}^{\mathrm{i}th}v\mathrm{e}^{\mathrm{i}t\bar{h}}\|_2 = \|v\|_2$. But in a Hilbert space, a function which takes values on a sphere and which is weakly continuous is actually norm continuous. Hence $\mathrm{e}^{\mathrm{i}th}v\mathrm{e}^{\mathrm{i}t\bar{h}}$ is continuous in the Hilbert-Schmidt norm. So $V(t)$ is continuous in $\mathcal{V}$ and thus $R(t)R_0(-t) - 1$ is differentiable in $\mathcal{V}$. In particular, $\bar{P}(t)\mathrm{e}^{\mathrm{i}t\bar{h}} - 1$ is differentiable in the trace class topology. Hence, $\det(\bar{P}(t)\mathrm{e}^{\mathrm{i}t\bar{h}})$ is differentiable and

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\det(\bar{P}(t)\mathrm{e}^{\mathrm{i}t\bar{h}}) &= \mathrm{Tr}\left(\frac{\mathrm{d}}{\mathrm{d}t}(\bar{P}(t)\mathrm{e}^{\mathrm{i}t\bar{h}}) \times (\bar{P}(t)\mathrm{e}^{\mathrm{i}t\bar{h}})^{-1}\right) \times \det(\bar{P}(t)\mathrm{e}^{\mathrm{i}t\bar{h}}) \\ &= -\mathrm{i}\mathrm{Tr}(Q(t)v\bar{P}(t)^{-1}) \times \det(\bar{P}(t)\mathrm{e}^{\mathrm{i}t\bar{h}}), \end{aligned}$$

which proves (3.12), and where we used (3.6) in the second line.

The proof of (3.11) follows from (3.9), (3.12) and the fact that $\det(1 - K(t)^*K(t))$ is positive. □

3.6 Essential selfadjointness of type I Bogoliubov Hamiltonians

Formally, it is easy to see that the Bogoliubov Hamiltonian of type I is given by (1.2) with $c = 0$. We can make this precise when v is Hilbert-Schmidt.

Theorem 3.13. *Suppose Assumption 3.E is satisfied, then the operator* $H_I = \mathrm{d}\Gamma(h) + \frac{1}{2}(a^*(v) + a(v))$ *is essentially selfadjoint on* $\mathcal{D} := \Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathfrak{h}) \cap \mathrm{Dom}(\mathrm{d}\Gamma(h))$ *and* $\mathrm{e}^{\mathrm{i}tH_I} = U_I(t)$.

Note that, since $\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathfrak{h}) \subset \mathrm{Dom}(a^*(v)+a(v))$, the operator H is therefore essentially selfadjoint on $\mathrm{Dom}(\mathrm{d}\Gamma(h)) \cap \mathrm{Dom}(a^*(v)+a(v))$. The strategy of the proof for the essential selfadjointness comes from [Be] and goes back to Carleman [Ca]. However, as we mentioned in the introduction, the proof in [Be] is not completely rigorous. In the case where h is bounded, a similar result has also been proven in [IH].

Note also that $\Omega \in \mathcal{D}(H_I)$ and that H_I has the particular feature that $\langle \Omega | H_I \Omega \rangle = 0$.

Recall that $L(t) = -P(t)^{-1}\bar{Q}(t)$. When v is Hilbert-Schmidt, the operator $v\bar{L}(t)$ is trace class and $\mathrm{Tr}(Q(s)v\bar{P}(s)^{-1}) = -\mathrm{Tr}(v\bar{L}(s))$. Therefore,

$$c_I(t) = \mathrm{e}^{-\frac{\mathrm{i}}{2}\mathrm{Re}(\int_0^t \mathrm{Tr}(v\bar{L}(s))\mathrm{d}s)}. \tag{3.13}$$

Lemma 3.14. *Suppose Assumption 3.E is satisfied. Then the map $t \mapsto L(t)$ is differentiable in the Hilbert-Schmidt topology.*

Proof. In the same way as in Theorem 3.6, we can prove that $R_0(-t)R(t) - 1$ is differentiable in $\mathcal{V}$. Hence, $\mathrm{e}^{-\mathrm{i}th}P(t) - 1$ is differentiable in the trace class norm, thus $\mathrm{e}^{-\mathrm{i}th}P(t)$ is norm differentiable and hence so is $P(t)^{-1}\mathrm{e}^{\mathrm{i}th}$. Moreover $\mathrm{e}^{-\mathrm{i}th}\bar{Q}(t)$ is differentiable in the Hilbert-Schmidt norm, so that $L(t) = -P(t)^{-1}\bar{Q}(t) = -(\mathrm{e}^{-\mathrm{i}th}P(t))^{-1}\mathrm{e}^{-\mathrm{i}th}\bar{Q}(t)$ is differentiable in the Hilbert-Schmidt norm. □

Lemma 3.15. *Suppose Assumption 3.E is satisfied, then $\langle \Omega | U_{\mathrm{nat}}(t)\Omega \rangle$ is continuously differentiable.*

Proof. Using (3.9) and (3.11) we have $\langle \Omega | U_{\mathrm{nat}}(t)\Omega \rangle = \mathrm{e}^{\frac{1}{2}\mathrm{Im}(\int_0^t \mathrm{Tr}(v\bar{L}(s))\mathrm{d}s)}$. The differentiability then follows from Lemma 3.14. □

Proof of Theorem 3.13 We first prove that H_I is essentially selfadjoint on $\mathcal{D}$. For that purpose, we consider the symmetric operator H defined as H_I on the domain $\mathcal{D}$ and we prove that for all $z \in \mathbb{C}, z \notin \mathbb{R}$, $\mathrm{Ker}(H^* - z) = \{0\}$.

We denote by P_n the orthogonal projection onto $\Gamma_{\mathrm{s}}^n(\mathfrak{h})$. In particular, for any vector Ψ, $P_n\Psi \in \Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathfrak{h})$. We also define, for all $\epsilon \in \mathbb{R}$,

$$\Psi_\epsilon := (1 - \mathrm{i}\epsilon \mathrm{d}\Gamma(h))^{-1}\,\Psi.$$

For any $\epsilon \neq 0$, $\Psi_\epsilon \in \mathrm{Dom}(\mathrm{d}\Gamma(h))$ and $\lim_{\epsilon\to 0}\Psi_\epsilon = \Psi$. Moreover, since the operator $\mathrm{d}\Gamma(h)$ leaves the subspace $\Gamma_{\mathrm{s}}^n(\mathfrak{h})$ invariant, we have $P_n\Psi_\epsilon = (P_n\Psi)_\epsilon \in \mathcal{D}$ for all n and $\epsilon \neq 0$.

Let us now fix $z \notin \mathbb{R}$ and let $\Phi \in \mathrm{Ker}(H^* - z)$. For all n we have

$$z\|P_n\Phi\|^2 = z\langle P_n\Phi|\Phi\rangle = \lim_{\epsilon\to 0} z\langle P_n\Phi_\epsilon|\Phi\rangle = \lim_{\epsilon\to 0}\langle P_n\Phi_\epsilon|H^*\Phi\rangle = \lim_{\epsilon\to 0}\langle HP_n\Phi_\epsilon|\Phi\rangle,$$

where in the last equality we have used the fact that $P_n\Phi_\epsilon \in \mathcal{D}$. Similarly, we have $\bar{z}\|P_n\Phi\|^2 = \lim_{\epsilon\to 0}\langle \Phi|HP_n\Phi_{-\epsilon}\rangle$. Therefore,

$$\begin{aligned} 2\mathrm{i}\mathrm{Im}z\|P_n\Phi\|^2 &= \lim_{\epsilon\to 0}(\langle \mathrm{d}\Gamma(h)P_n\Phi_\epsilon|\Phi\rangle - \langle \Phi|\mathrm{d}\Gamma(h)P_n\Phi_{-\epsilon}\rangle + \frac{1}{2}\langle (a(v)+a^*(v))P_n\Phi_\epsilon|\Phi\rangle \\ &\qquad -\frac{1}{2}\langle \Phi|(a(v)+a^*(v))P_n\Phi_{-\epsilon}\rangle). \end{aligned}$$

Since P_n commutes with $\mathrm{d}\Gamma(h)$, the two first terms of the right hand side cancel. Moreover, the operator $(a(v)+a^*(v))P_n$ is bounded. So finally we get, with the convention $P_{-1} = P_{-2} = 0$,

$$\begin{aligned} 4\mathrm{i}\mathrm{Im}z\|P_n\Phi\|^2 &= \langle (a(v)+a^*(v))P_n\Phi|\Phi\rangle - \langle \Phi|(a(v)+a^*(v))P_n\Phi\rangle \\ &= \langle a(v)P_n\Phi|P_{n-2}\Phi\rangle + \langle a^*(v)P_n\Phi|P_{n+2}\Phi\rangle - \langle a(v)P_{n+2}\Phi|P_n\Phi\rangle - \langle a^*(v)P_{n-2}\Phi|P_n\Phi\rangle. \end{aligned}$$

We now sum the previous identity for $0 \le n \le N$, which gives

$$\begin{aligned} 4\mathrm{i}\mathrm{Im}z\sum_{n=0}^N \|P_n\Phi\|^2 &= \langle a^*(v)P_N\Phi|P_{N+2}\Phi\rangle + \langle a^*(v)P_{N-1}\Phi|P_{N+1}\Phi\rangle - \langle a(v)P_{N+2}\Phi|P_N\Phi\rangle \\ &\qquad -\langle a(v)P_{N+1}\Phi|P_{N-1}\Phi\rangle. \end{aligned}$$

Therefore, for all $N \in \mathbb{N}$, and using Proposition 2.2, we have

$$\begin{aligned} 4|\mathrm{Im}z| \sum_{n=0}^{N} \|P_n\Phi\|^2 &\leq \|v\|_2 \left(2(N+2)\|P_N\Phi\|\|P_{N+2}\Phi\| + 2(N+1)\|P_{N-1}\Phi\|\|P_{N+1}\Phi\|\right) \\ &\leq (N+2)\|v\|_2 \left(\|P_{N-1}\Phi\|^2 + \|P_N\Phi\|^2 + \|P_{N+1}\Phi\|^2 + \|P_{N+2}\Phi\|^2\right). \end{aligned}$$

Suppose $\Phi \neq 0$. Hence there exists N_0 such that $\sum_{n=0}^{N_0} \|P_n\Phi\|^2 = C > 0$, and for all $N \geq N_0$, $\sum_{n=0}^{N} \|P_n\Phi\|^2 \geq C$. So we have, for all $N \geq N_0$,

$$\frac{4|\mathrm{Im}z|C}{N+2} \leq \|v\|_2 \sum_{j=N-1}^{N+2} \|P_j\Phi\|^2.$$

If now we sum over N this inequality, the right hand side converges (and is less that $4\|v\|_2\|\Phi\|^2$), while the left hand side diverges. Hence $\Phi = 0$ and H_I is essentially selfadjoint on $\mathcal{D}$.

It remains to prove that $\mathrm{e}^{\mathrm{i}tH_I} = U_I(t)$. For that purpose, we prove that $\mathrm{e}^{\mathrm{i}tH_I}$ is a Bogoliubov dynamics implementing $R(t)$ (first provided h is bounded and then for a general h) so that it equals $U_I(t)$ up to a phase factor. And then we prove that this phase is 1.

Given two operators B and C, let $\mathrm{ad}_B^0 C := C$ and $\mathrm{ad}_B^k C := [B, \mathrm{ad}_B^{k-1} C]$. Recall also that $\phi(f)$ stand for the field operators on $\Gamma_{\mathrm{s}}(\mathfrak{h})$.

Suppose h, and hence A, is bounded. Then, for all $f \in \mathfrak{h}$, and in the sense of quadratic forms on $\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathfrak{h})$,

$$\mathrm{ad}_{\mathrm{i}H_I}^k \phi(f) = \phi(I^{-1}A^k(f, \bar{f})), \tag{3.14}$$

where I was defined in (3.2). Indeed, as quadratic forms on $\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathfrak{h})$, we have

$$[\mathrm{i}H_I, a(f)] = \mathrm{i}[\mathrm{d}\Gamma(h), a(f)] + \frac{\mathrm{i}}{2}[a^*(v), a(f)] = a(\mathrm{i}hf) + a^*(-\mathrm{i}v\bar{f}).$$

In the same way, one proves that $[\mathrm{i}H_I, a^*(f)] = a^*(\mathrm{i}hf) + a(-\mathrm{i}v\bar{f})$. Hence, one has $[\mathrm{i}H_I, \phi(f)] = \phi(I^{-1}A(f, \bar{f}))$, and since A is bounded (3.14) follows easily.

Let now $\Phi, \Psi \in \Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathfrak{h})$. For $z \in \mathbb{C}$, we define

$$F_1(z) := \langle \Phi, \mathrm{e}^{\mathrm{i}zH_I} \phi(f) \Psi \rangle \quad \text{and} \quad F_2(z) := \langle \phi(I^{-1}\mathrm{e}^{zA}(f, \bar{f}))\Phi, \mathrm{e}^{\mathrm{i}zH_I}\Psi \rangle.$$

Using Proposition 2.2, it is easy to see that Φ and Ψ are analytic for H_I. Since moreover A is bounded, this proves that F_1 and F_2 are analytic functions in some neighborhood of 0. Moreover it is well known that $B^n C = \sum_{k=0}^{n} \binom{n}{k} \mathrm{ad}_B^k C\, B^{n-k}$. Thus, for all n we have

$$\begin{aligned} \frac{\mathrm{d}^n F_1(z)}{\mathrm{d}z^n}\lceil_{z=0} = \langle \Phi, (\mathrm{i}H_I)^n \phi(f)\Psi \rangle &= \sum_{k=0}^{n} \binom{n}{k} \langle \Phi, \mathrm{ad}_{\mathrm{i}H_I}^k \phi(f)\, (\mathrm{i}H_I)^{n-k}\Psi \rangle \\ &= \sum_{k=0}^{n} \binom{n}{k} \langle \Phi, \phi(I^{-1}A^k(f, \bar{f}))(\mathrm{i}H_I)^{n-k}\Psi \rangle \\ &= \frac{\mathrm{d}^n F_2(z)}{\mathrm{d}z^n}\lceil_{z=0}, \end{aligned}$$

where we have used (3.14) in the second line. Therefore, for all z in some neighborhood of 0, $F_1(z) = F_2(z)$, which implies that

$$\mathrm{e}^{\mathrm{i}tH_I} \phi(f) \mathrm{e}^{-\mathrm{i}tH_I} \Phi = \phi(I^{-1}\mathrm{e}^{tA}(f, \bar{f}))\Phi,$$

and hence
$$\mathrm{e}^{\mathrm{i}tH_I}W(f)\mathrm{e}^{-\mathrm{i}tH_I}\Phi = W_{R(t)}(f)\Phi,$$
for all $\Phi \in \Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathfrak{h})$, and all $t \in \mathbb{R}$ by the group property. Since $\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathfrak{h})$ is dense in $\Gamma_{\mathrm{s}}(\mathfrak{h})$, this proves that $\mathrm{e}^{\mathrm{i}tH_I}$ intertwines W and $W_{R(t)}$ when h is bounded.

We consider now the general case. Let us write $A = A_0 + V$ as in the proof of Proposition 3.5. Both A_0 and V are generators of a one-parameter group of symplectic maps. Moreover, V is bounded, therefore we can apply the first part of the proof and, for all $t \in \mathbb{R}$, we have
$$\mathrm{e}^{\frac{\mathrm{i}}{2}t(a^*(v)+a(v))}W(f)\mathrm{e}^{-\frac{\mathrm{i}}{2}t(a^*(v)+a(v))} = W(I^{-1}\mathrm{e}^{tV}(f,\bar{f})).$$
But, since h is selfadjoint, it is well known (see *e.g.* [DG]) that
$$\mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma(h)}W(f)\mathrm{e}^{-\mathrm{i}t\mathrm{d}\Gamma(h)} = W(\mathrm{e}^{\mathrm{i}th}f) = W(I^{-1}\mathrm{e}^{tA_0}(f,\bar{f})).$$
Thus, using the Trotter product formula,
$$\mathrm{e}^{\mathrm{i}tH_I} = \mathrm{s}-\lim_{n\to\infty}\left(\mathrm{e}^{\frac{\mathrm{i}t\mathrm{d}\Gamma(h)}{n}}\mathrm{e}^{\frac{\mathrm{i}t(a^*(v)+a(v))}{2n}}\right)^n = \mathrm{s}-\lim_{n\to\infty}\left(\mathrm{e}^{\frac{\mathrm{i}t(a^*(v)+a(v))}{2n}}\mathrm{e}^{\frac{\mathrm{i}t\mathrm{d}\Gamma(h)}{n}}\right)^n.$$
In the same way, we have $R(t) = \mathrm{s}-\lim_{n\to\infty}\left(\mathrm{e}^{\frac{tA_0}{n}}\mathrm{e}^{\frac{tV}{n}}\right)^n$. Hence,
$$\begin{aligned}
\mathrm{e}^{\mathrm{i}tH_I}W(f)\mathrm{e}^{-\mathrm{i}tH_I} &= \mathrm{s}-\lim_{n\to\infty}\left(\mathrm{e}^{\frac{\mathrm{i}t\mathrm{d}\Gamma(h)}{n}}\mathrm{e}^{\frac{\mathrm{i}t(a^*(v)+a(v))}{2n}}\right)^n W(f)\left(\mathrm{e}^{\frac{-\mathrm{i}t(a^*(v)+a(v))}{2n}}\mathrm{e}^{\frac{-\mathrm{i}t\mathrm{d}\Gamma(h)}{n}}\right)^n \\
&= \mathrm{s}-\lim_{n\to\infty} W\left(I^{-1}\left(\mathrm{e}^{\frac{tA_0}{n}}\mathrm{e}^{\frac{tV}{n}}\right)^n(f,\bar{f})\right) = W_{R(t)}(f).
\end{aligned}$$
This proves that $\mathrm{e}^{\mathrm{i}tH_I}$ is a Bogoliubov dynamics implementing $R(t)$. And hence $\mathrm{e}^{\mathrm{i}tH_I}$ and $U_I(t)$ are equal up to a phase factor. In order to prove that this phase is one, we will show that they have the same natural cocycle. By (3.13), we know that
$$U_I(t) = \mathrm{e}^{-\frac{\mathrm{i}}{2}\mathrm{Re}(\int_0^t \mathrm{Tr}(v\bar{L}(s))\mathrm{d}s)}U_{\mathrm{nat}}(t).$$
Let now $\rho(t) \in \mathbb{R}$ be such that $U_{\mathrm{nat}}(t) = \mathrm{e}^{\mathrm{i}\rho(t)}\mathrm{e}^{\mathrm{i}tH_I}$. Note that $\Omega \in \mathcal{D}$, hence,
$$\mathrm{e}^{\mathrm{i}\rho(t)} = \frac{\langle\Omega|U_{\mathrm{nat}}(t)\Omega\rangle}{\langle\Omega|\mathrm{e}^{\mathrm{i}tH_I}\Omega\rangle}$$
is continuously differentiable by Lemma 3.15. Moreover, for all t, $\langle\Omega|U_{\mathrm{nat}}(t)\Omega\rangle \in \mathbb{R}$, thus
$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\Omega|U_{\mathrm{nat}}(t)\Omega\rangle = \mathrm{i}\rho'(t)\langle\Omega|U_{\mathrm{nat}}(t)\Omega\rangle + \langle U_{\mathrm{nat}}(t)^*\Omega|\mathrm{i}H_I\Omega\rangle \ \in \mathbb{R},$$
and hence
$$\rho'(t)\langle\Omega|U_{\mathrm{nat}}(t)\Omega\rangle = -\mathrm{Im}\langle U_{\mathrm{nat}}(t)^*\Omega|\mathrm{i}H_I\Omega\rangle = -\frac{1}{2}\mathrm{Im}\langle U_{\mathrm{nat}}(t)^*\Omega|\mathrm{i}a^*(v)\Omega\rangle. \tag{3.15}$$
Therefore
$$\rho'(t) = -\mathrm{Im}\frac{\langle U_{\mathrm{nat}}(t)^*\Omega|\mathrm{i}a^*(v)\Omega\rangle}{2\langle\Omega|U_{\mathrm{nat}}(t)\Omega\rangle} = -\frac{1}{2}\mathrm{Im}\langle e^{-\frac{1}{2}a^*(L(t))}\Omega|\mathrm{i}a^*(v)\Omega\rangle = \frac{1}{4}\mathrm{Im}\langle a^*(L(t))\Omega|\mathrm{i}a^*(v)\Omega\rangle.$$
Now, using (2.7), we have
$$\langle\Omega|[a(L(t)),a^*(v)]\Omega\rangle = 2\mathrm{Tr}(L(t)^*v).$$
But $L(t)^* = \bar{L}(t)$, therefore
$$\rho(t) = \frac{1}{2}\int_0^t \mathrm{Re}\mathrm{Tr}(v\bar{L}(s))\mathrm{d}s, \tag{3.16}$$
and
$$\mathrm{e}^{\mathrm{i}tH_I} = \mathrm{e}^{-\frac{\mathrm{i}}{2}\mathrm{Re}(\int_0^t \mathrm{Tr}(v\bar{L}(s))\mathrm{d}s)}U_{\mathrm{nat}}(t).$$
□

3.7 Infimum of Bogoliubov Hamiltonians

In this section, we introduce another (natural) distinguished class of Bogoliubov Hamiltonians, those whose infimum is zero.

Definition 3.16. *A unitarily implementable symplectic group is of type II if and only if it has a bounded from below Bogoliubov Hamiltonian (and hence all its Bogoliubov Hamiltonians are bounded from below).*

Definition 3.17. *If $R(t)$ is a symplectic group of type II, we define the Bogoliubov Hamiltonian of type II to be the unique associated Bogoliubov Hamiltonian whose infimum of spectrum is 0. We denote it by H_{II}. The corresponding Bogoliubov unitary group is denoted $U_{II}(t) = \mathrm{e}^{\mathrm{i}tH_{II}}$.*

We denote by $\mathcal{Y}^{\#}$ the dual space of $\mathcal{Y}$.

Definition 3.18. *The classical symbol associated to a one parameter symplectic group $R(t) = \exp t \begin{pmatrix} \mathrm{i}h & -\mathrm{i}v \\ \mathrm{i}\bar{v} & -\mathrm{i}\bar{h} \end{pmatrix}$ is the bilinear symmetric form defined on $\mathcal{Y}^{\#}$ as*

$$\mathcal{Y}^{\#} \times \mathcal{Y}^{\#} \ni ((\bar{f}, f), (\bar{g}, g)) \mapsto \frac{1}{2}(\langle f|hg\rangle + \langle \bar{f}|\bar{h}\bar{g}\rangle + \langle f|v\bar{g}\rangle + \langle \bar{f}|\bar{v}g\rangle).$$

Theorem 3.19. *Suppose $\mathfrak{h}$ is finite dimensional. Then every symplectic group with a positive classical symbol is both of type I and II. Moreover, the Bogoliubov Hamiltonians H_I and H_{II} satisfy*

$$H_{II} = H_I - \frac{1}{4}\mathrm{Tr}\left[\begin{pmatrix} \bar{h}^2 - \bar{v}v & \bar{h}\bar{v} - \bar{v}h \\ hv - v\bar{h} & h^2 - v\bar{v} \end{pmatrix}^{1/2} - \begin{pmatrix} \bar{h} & 0 \\ 0 & h \end{pmatrix}\right]. \tag{3.17}$$

Proof. The operator v is Hilbert-Schmidt (we are in finite dimension), hence by Theorem 3.13 $R(t)$ is of type I and $H_I = \mathrm{d}\Gamma(h) + \frac{1}{2}(a(v) + a^*(v))$.

Let d denote the (complex) dimension of $\mathfrak{h}$, hence $\mathcal{Y}$ is a real symplectic space of dimension $2d$. We define, on $\mathcal{Y}^{\#}$, the operator

$$\beta(h, v) := \frac{1}{2}\begin{pmatrix} v & h \\ \bar{h} & \bar{v} \end{pmatrix}.$$

The operator $\beta(h, v)$ is real symmetric and hence induces a real quadratic form on $\mathcal{Y}^{\#}$

$$\mathcal{Y}^{\#} \ni (\bar{f}, f) \mapsto \langle (f, \bar{f})|\beta(h, v)(\bar{f}, f)\rangle,$$

which is nothing else but the classical symbol of $R(t)$. Its Weyl quantization, denoted $\mathrm{Op}(\beta)$, is then

$$\mathrm{Op}(\beta) = \mathrm{d}\Gamma(h) + \frac{1}{2}a^*(v) + \frac{1}{2}a(v) + \frac{1}{2}\mathrm{Tr}(h). \tag{3.18}$$

We also denote

$$\sigma = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}.$$

The map $\mathcal{Y} \times \mathcal{Y} \ni (y, y') \mapsto \frac{1}{2}\langle \bar{y}|\sigma y'\rangle = \sigma(y, y')$ is the symplectic form on $\mathcal{Y}$ introduced in Section 2.3.

Since β is positive real symmetric and σ is real antisymmetric, we can diagonalize them simultaneously, i.e. there is a basis $(y_1, \cdots, y_{2d})$ of $\mathcal{Y}$ and positive real numbers $\lambda_1, \cdots, \lambda_{2d}$ such that

$$\beta\bar{y}_j = \lambda_j y_j, \tag{3.19}$$

$$\sigma y_{2j-1} = \bar{y}_{2j}, \quad \sigma y_{2j} = -\bar{y}_{2j-1}, \tag{3.20}$$

where, if $y = (f, \bar{f})$, $\bar{y} = (\bar{f}, f)$. Let $f_k \in \mathfrak{h}$ be such that $y_k = (f_k, \bar{f}_k)$, and let $h_k = |f_k\rangle\langle f_k|$ and $v_k = |f_k\rangle\langle \bar{f}_k|$. Finally, we denote $\beta_k = \beta(h_k, v_k)$. One then gets, using (3.19), $\beta(h,v) = \sum_{j=1}^{2d} \lambda_j \beta_j$. Hence

$$\mathrm{Op}(\beta(h,v)) = \sum_{j=1}^{2d} \lambda_j \mathrm{Op}(\beta_j),$$

with

$$\mathrm{Op}(\beta_j) = \mathrm{d}\Gamma(|f_j\rangle\langle f_j|) + \frac{1}{2}(a(f_j \otimes f_j) + a^*(f_j \otimes f_j)) + \frac{1}{2} = \phi(f_j)^2,$$

and where $\phi(f)$ denotes the field operators (Section 2.1), so that

$$\mathrm{Op}(\beta(h,v)) = \sum_{j=1}^{d} (\lambda_{2j-1}\phi(f_{2j-1})^2 + \lambda_{2j}\phi(f_{2j})^2). \tag{3.21}$$

Now, since $(y_1, \cdots, y_{2d})$ diagonalizes σ, we have

$$\sigma(y_{2j}, y_{2k}) = \sigma(y_{2j-1}, y_{2k-1}) = 0, \quad \text{and} \quad \sigma(y_{2j}, y_{2k-1}) = \delta_{jk}.$$

And hence, $[\phi(f_{2j-1}), \phi(f_{2j})] = \mathrm{i}$ for all $j \in \{1, \ldots, d\}$ while the other field operators commute. Therefore, by (3.21), and the properties of the harmonic oscillator,

$$\inf \mathrm{Op}(\beta(h,v)) = \sum_{j=1}^{d} \sqrt{\lambda_{2j-1}\lambda_{2j}}.$$

On the other hand, using (3.19)-(3.20), one gets

$$-(\sigma\beta)^2 \bar{y}_{2j-1} = \lambda_{2j-1}\lambda_{2j}\bar{y}_{2j-1}, \quad -(\sigma\beta)^2 \bar{y}_{2j} = \lambda_{2j-1}\lambda_{2j}\bar{y}_{2j}.$$

Therefore we have

$$\inf \mathrm{Op}(\beta(h,v)) = \sum_{j=1}^{d} \sqrt{\lambda_{2j-1}\lambda_{2j}} = \frac{1}{2}\mathrm{Tr}\sqrt{-(\sigma\beta)^2}.$$

Finally, a simple computation gives $-(\sigma\beta)^2 = \frac{1}{4}\begin{pmatrix} \bar{h}^2 - \bar{v}v & \bar{h}\bar{v} - \bar{v}h \\ hv - v\bar{h} & h^2 - v\bar{v} \end{pmatrix}$, from which (3.17) follows since $H_I = \mathrm{Op}(\beta(h,v)) - \frac{1}{2}\mathrm{Tr}(h)$. $\square$

3.8 Relative boundedness of quadratic annihilation and creation operators

In this section, we consider $a^*(v) + a(v)$ as a perturbation of $\mathrm{d}\Gamma(h)$ and derive a condition so that it is relatively bounded with respect to it.

Theorem 3.20. *If h is a positive selfadjoint operator on $\mathfrak{h}$ and $v \in \mathrm{Dom}(h^{-1/2} \otimes h^{-1/2}) \cap \mathrm{Dom}(h^{-1/2} \otimes 1 + 1 \otimes h^{-1/2})$, then $a(v) + a^*(v)$ is $\mathrm{d}\Gamma(h)$ bounded with relative bound less than $2\|(h^{-1/2} \otimes h^{-1/2})v\|$.*

Using the Kato-Rellich Theorem ([RS2], Theorem X.39), one then immediately gets

Corollary 3.21. *Under the same assumption, if moreover $\|(h^{-1/2} \otimes h^{-1/2})v\| < 1$ then the Bogoliubov Hamiltonian $H := \mathrm{d}\Gamma(h) + \frac{1}{2}(a(v) + a^*(v))$ is selfadjoint on $\mathrm{Dom}(\mathrm{d}\Gamma(h))$ and bounded from below. In particular, the associated symplectic group $R(t)$ is both of type I and type II.*

The above condition should not be so surprising. Indeed, it closely resembles the condition one can find for the Van-Hove and the Pauli-Fierz Hamiltonians (see *e.g.* [De, DJ]), where a perturbation linear in the annihilation and creation operators is involved.

Lemma 3.22. *Suppose that* $v \in \mathrm{Dom}(h^{-1/2}\otimes h^{-1/2})$*. Then, there exist orthonormal bases of* $\mathfrak{h}$ $(\xi_n)_n, (\chi_n)_n$ *and positive numbers* μ_n *such that* $(h^{-1/2}\otimes h^{-1/2})v = \sum_n \mu_n \; \xi_n \otimes \chi_n$*. Moreover, for all* $\Psi \in \Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathfrak{h})$*,*

$$a(v)\Psi = \sum_n \mu_n a(h^{1/2}\xi_n) a(h^{1/2}\chi_n)\Psi.$$

Proof. It follows from (2.5) and the fact that $v = \sum \mu_n \; h^{1/2}\xi_n \otimes h^{1/2}\chi_n$. □

We now prove bounds on $a(v)$ and $a^*(v)$ which generalise the ones obtained in Proposition 2.2, and which are in the spirit of the $N_\tau-$estimate of Proposition 2.1.

Proposition 3.23. *Suppose* $v \in \mathrm{Dom}(h^{-1/2}\otimes h^{-1/2})$*. Then for all* $\Psi \in \mathrm{Dom}(\mathrm{d}\Gamma(h))$*,*

$$\|a(v)\Psi\| \le \|(h^{-1/2}\otimes h^{-1/2})v\| \|\mathrm{d}\Gamma(h)\Psi\|.$$

Proof. Using Lemma 3.22, we have

$$\|a(v)\Psi\|^2 = \|\sum_n \mu_n a(h^{1/2}\xi_n)a(h^{1/2}\chi_n)\Psi\|^2 \le \sum_n \mu_n^2 \; \sum_n \|a(h^{1/2}\xi_n)a(h^{1/2}\chi_n)\Psi\|^2.$$

Hence, using $\sum_n \mu_n^2 = \|(h^{-1/2}\otimes h^{-1/2})v\|^2$ and Proposition 2.1, we get

$$\begin{aligned}
\|a(v)\Psi\|^2 &\le \|(h^{-1/2}\otimes h^{-1/2})v\|^2 \sum_n \langle a(h^{1/2}\chi_n)\Psi | \mathrm{d}\Gamma(h) a(h^{1/2}\chi_n)\Psi\rangle \\
&= \|(h^{-1/2}\otimes h^{-1/2})v\|^2 \sum_n \Big(\langle a(h^{1/2}\chi_n)\Psi | a(h^{1/2}\chi_n)\mathrm{d}\Gamma(h)\Psi\rangle - \langle a(h^{1/2}\chi_n)\Psi | a(h^{3/2}\chi_n)\Psi\rangle \Big) \\
&= \|(h^{-1/2}\otimes h^{-1/2})v\|^2 \left(\|\mathrm{d}\Gamma(h)\Psi\|^2 - \langle \Psi | \mathrm{d}\Gamma(h^2)\Psi\rangle \right),
\end{aligned}$$

where in the last line we used the following identities

$$\sum a^*(h^{1/2}\chi_n)a(h^{1/2}\chi_n) = \mathrm{d}\Gamma(h), \quad \text{and} \quad \sum a^*(h^{1/2}\chi_n)a(h^{3/2}\chi_n) = \mathrm{d}\Gamma(h^2).$$ □

Proposition 3.24. *Suppose* $v \in \mathrm{Dom}(h^{-1/2}\otimes h^{-1/2}) \cap \mathrm{Dom}(h^{-1/2}\otimes 1 + 1 \otimes h^{-1/2})$*. For any* $\epsilon > 0$*, there exists* $C_\epsilon > 0$*, such that for all* $\Psi \in \mathrm{Dom}(\mathrm{d}\Gamma(h))$*,*

$$\|a^*(v)\Psi\|^2 \le \Big(\|(h^{-1/2}\otimes h^{-1/2})v\|^2 + \epsilon \Big) \|\mathrm{d}\Gamma(h)\Psi\|^2 + C_\epsilon \|\Psi\|^2.$$

In order to prove this estimate, we will need the following lemma which follows directly from (2.7).

Lemma 3.25. *Let* $v \in \Gamma_{\mathrm{s}}^2(\mathfrak{h})$*, then for all* $\Psi \in \mathrm{Dom}(N)$*,*

$$\|a^*(v)\Psi\|^2 = \|a(v)\Psi\|^2 + 4\langle \Psi | \mathrm{d}\Gamma(vv^*)\Psi\rangle + 2\|v\|^2 \|\Psi\|^2.$$

Proof of Proposition 3.24 Using Proposition 3.23 and Lemma 3.25, it suffices to show that

$$\langle \Psi, \mathrm{d}\Gamma(vv^*)\Psi\rangle \le \epsilon \|\mathrm{d}\Gamma(h)\Psi\|^2 + C'_\epsilon \|\Psi\|^2$$

for some C'_ϵ. One can write $vv^* = h^{1/2}(h^{-1/2}v)(h^{-1/2}v)^* h^{1/2}$. Now, $h^{-1/2}v$ is bounded. It is actually Hilbert-Schmidt since $v \in \mathrm{Dom}(h^{-1/2}\otimes 1 + 1 \otimes h^{-1/2})$. Thus $vv^* \le \|h^{-1/2}v\|^2 \, h$, and so

$$\langle \Psi | \mathrm{d}\Gamma(vv^*)\Psi\rangle \le \|h^{-1/2}v\|^2 \|\mathrm{d}\Gamma(h)^{1/2}\Psi\|^2,$$

which ends the proof. □

Proof of Theorem 3.20 It follows directly from Propositions 3.23 and 3.24. □

4 A concrete example: the diagonal case

The case where the one particle space $\mathfrak{h}$ is finite dimensional is completely understood: all symplectic groups are of type I and we have a necessary and sufficient condition on its generator to determine wether it is of type II or not (Section 3.7). In this section we consider the simplest "infinite dimensional" case. Namely, $\mathfrak{h} := L^2(\mathbb{N})$ with its canonical basis $(e_n)_{n\in\mathbb{N}}$ and h and v are both diagonal, i.e.

$$h := \sum_n h_n |e_n\rangle\langle e_n|, \qquad v := \sum_n v_n |e_n\rangle\langle e_n|, \tag{4.1}$$

and where the h_n are real numbers so that h is selfadjoint. Our goal is to describe, in this simple situation, what are the one parameter symplectic groups $R(t)$ which are unitarily implementable, which are those of type I, and those of type II. In the case where $R(t)$ is not of type I, we will also achieve the "phase renormalization" we have mentioned in the introduction. More precisely, we will prove the following

Theorem 4.1. *Consider on $L^2(\mathbb{N})$ the operators h and v defined by (4.1).*

(i) $R(t)$ defines a strongly continuous one parameter group of symplectic maps if and only if v is h-bounded with relative bound strictly less than one, i.e. there exists $a \in [0,1[$ and $b \geq 0$ such that for all $n \in \mathbb{N}$, $|v_n| \leq a|h_n| + b$.

(ii) $R(t)$ is unitarily implementable if and only if $\sum \frac{|v_n|^2}{1+h_n^2} < +\infty$. If it is unitarily implementable, the operators

$$U_{\mathrm{ren}}(t) := \mathrm{e}^{\mathrm{i}\mathrm{Tr}(\frac{1}{2}\mathrm{Re}\int_0^t Q_\tau v \bar{P}_\tau^{-1}\mathrm{d}\tau + \Lambda_{\mathrm{ren}} t)} U_{\mathrm{nat}}(t),$$

where $\Lambda_{\mathrm{ren}} = \sum_{|h_n|>1} \frac{|v_n|^2}{4h_n} |e_n\rangle\langle e_n|$, form a Bogoliubov dynamics implementing $R(t)$.

(iii) A unitarily implementable symplectic group $R(t)$ is of type I if and only if $\sum \frac{|v_n|^2}{1+|h_n|} < +\infty$.

(iv) A unitarily implementable symplectic group $R(t)$ is of type II if and only if $h_n \geq |v_n|$ for all n and $\sum_{|h_n|\leq 1} \frac{|v_n|^2}{|h_n|} < +\infty$.

Suppose now that $(O_1, \cdots, O_M)$ is a partition of $\mathbb{N}$, then we have $\mathfrak{h} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_M$ where $\mathfrak{h}_j = \mathrm{Span}\{e_n, n \in O_j\}$. But, since h and v are both diagonal they leave the $\mathfrak{h}_j$ invariant, and we can split the problem with respect to the above decomposition of $\mathfrak{h}$, i.e. the symplectic group $R(t)$ can be written as $R(t) = R_1(t) \oplus \cdots \oplus R_M(t)$ where for all $j = 1, \cdots, M$ $R_j(t)$ is a symplectic group on $\mathcal{Y}_j = \{(f, \bar{f}), f \in \mathfrak{h}_j\}$, and we can consider separately the M so obtained reduced problems. It is then easy to see that $R(t)$ is unitarily implementable if and only the $R_j(t)$ are all unitarily implementable, and that the same statement holds for the type I (resp. type II) character of $R(t)$. For that reason, as a first step we will consider the case of a single degree of freedom, i.e. the case $\mathfrak{h} = \mathbb{C}$.

4.1 Bogoliubov transformations of a single degree of freedom

Let $\mathfrak{h} = \mathbb{C}$ and $A = \mathrm{i}\begin{pmatrix} h & -v \\ \bar{v} & -h \end{pmatrix}$, where $h \in \mathbb{R}$ and $v \in \mathbb{C}$. One can compute explicitly the operators $P(t)$ and $Q(t)$:

- if $|h| < |v|$

$$P(t) = \cosh(t\sqrt{|v|^2 - h^2}) + \mathrm{i}h\frac{\sinh(t\sqrt{|v|^2 - h^2})}{\sqrt{|v|^2 - h^2}} \quad \text{and} \quad Q(t) = \mathrm{i}\bar{v}\frac{\sinh(t\sqrt{|v|^2 - h^2})}{\sqrt{|v|^2 - h^2}}, \tag{4.2}$$

- if $|h| = |v|$
$$P(t) = 1 + \mathrm{i}th \quad \text{and} \quad Q(t) = \mathrm{i}t\bar{v}, \tag{4.3}$$

- if $|h| > |v|$
$$P(t) = \cos(t\sqrt{h^2 - |v|^2}) + \mathrm{i}h\frac{\sin(t\sqrt{h^2 - |v|^2})}{\sqrt{h^2 - |v|^2}} \quad \text{and} \quad Q(t) = \mathrm{i}\bar{v}\frac{\sin(t\sqrt{h^2 - |v|^2})}{\sqrt{h^2 - |v|^2}}. \tag{4.4}$$

From Theorem 3.19 we know that $R(t)$ is always of type I with $H_I = \mathrm{d}\Gamma(h) + \frac{1}{2}(a^*(v) + a(v))$. Moreover, it is of type II if and only if its classical symbol is positive i.e. $\forall z \in \mathbb{C}, \mathrm{Re}(h|z|^2 + vz^2) \geq 0$, which is equivalent to $h \geq |v|$. The Bogoliubov Hamiltonian of type II then writes, according to (3.17),

$$H_{II} = H_I - \frac{1}{2}(\sqrt{h^2 - |v|^2} - h). \tag{4.5}$$

4.2 Proof of Theorem 4.1

We now turn back to the general situation (4.1). In view of (4.2)-(4.3)-(4.4), we consider the following partition of $\mathbb{N} := \mathcal{N}_< \cup \mathcal{N}_= \cup \mathcal{N}_>$, where $\mathcal{N}_< := \{n \in \mathbb{N}, |h_n| < |v_n|\}$, $\mathcal{N}_= := \{n \in \mathbb{N}, |h_n| = |v_n|\}$ and $\mathcal{N}_> := \{n \in \mathbb{N}, |h_n| > |v_n|\}$ and split the analysis with respect to this partition. It will also be convenient to split again the case $|h| > |v|$ in two cases, namely $|h_n|^2 - |v_n|^2 \leq \frac{1}{2}$ and $|h_n|^2 - |v_n|^2 > \frac{1}{2}$ (the choice of the value $\frac{1}{2}$ is purely arbitrary and could be replaced by any strictly positive number).

4.2.1 The case $|h| < |v|$

Throughout this section we assume that, for all n, $|h_n| < |v_n|$.

Proposition 4.2. *(i) $R(t)$ defines a strongly continuous symplectic group if and only if v is bounded.*

(ii) $R(t)$ is unitarily implementable if and only if v is Hilbert-Schmidt.

(iii) All unitary implementable symplectic groups are of type I.

(iv) A unitarily implementable symplectic group is never of type II.

Proof. (i) If the operator v is bounded then the result follows from Proposition 3.5. Suppose now that $R(t)$ defines a strongly continuous group. Then there exist two constants M and ω strictly positive such that, for all t, $\|R(t)\| \leq M\mathrm{e}^{\omega|t|}$. Then, using (4.2), one easily gets that the operators $\sqrt{|v|^2 - h^2}$ and $\frac{v}{\sqrt{|v|^2 - h^2}}$ have to be bounded which implies that v is bounded.

(ii) Once again the sufficient condition follows from the general theory (Theorem 3.6). Suppose now that $R(t)$ is unitarily implementable. Then, by Theorem 3.4, $Q(t)$ is Hilbert-Schmidt for all t, i.e.

$$\sum \left| |v_n|^2 \frac{\sinh^2(t\sqrt{|v_n|^2 - h_n^2})}{|v_n|^2 - h_n^2} \right|^2 < \infty.$$

The result follows immediately since for all x, $\sinh^2(x) \geq x^2$.

(iii) This follows from Theorem 3.12 (ii).

(iv) Since $|h_n| < |v_n|$, the result follows from the properties of "one degree of freedom" case. $\square$

4.2.2 The case $|h| = |v|$

We suppose in this section that for all n, $|h_n| = |v_n|$. This situation is very close to the previous one.

Proposition 4.3. *(i) $R(t)$ defines a strongly continuous symplectic group if and only if v is bounded.*

(ii) $R(t)$ is unitarily implementable if and only if v is Hilbert-Schmidt.

(iii) All unitarily implementable symplectic groups are of type I.

(iv) A unitarily implementable symplectic group is of type II if and only if $h \geq 0$ and is trace class. If it is of type II then $H_{II} = H_I + \frac{\mathrm{Tr}(h)}{2}$.

Proof. The proofs of (i)-(ii)-(iii) are the same as in the previous section. It remains to prove (iv). Since v is Hilbert-Schmidt, we know that $R(t)$ is of type I with Bogoliubov Hamiltonian H_I given by (1.2) with $c = 0$. Thus $R(t)$ is of type II if and only if H_I is bounded from below.

First assume that h is positive and trace class. Formally, H_I is given by $H_I = \sum_n H_n$, where $H_n = \mathrm{d}\Gamma(h_n|e_n\rangle\langle e_n|) + \frac{1}{2}a^*(v_n|e_n\rangle\langle e_n|) + \frac{1}{2}a(v_n|e_n\rangle\langle e_n|)$. Note that the H_n commute with one another. We shall prove that $H(N) := \sum_{n\leq N} H_n$ converges to H_I in the strong resolvent sense when N goes to infinity. If this holds, we have then ([RS1], Theorem VIII.24)

$$\inf H_I \geq \lim_{N\to\infty} \inf H(N).$$

On the other hand, $H(N)$ is bounded from below for all N (Section 4.1) and

$$\inf H(N) = -\frac{1}{2}\sum_{n=0}^{N} h_n - (h_n^2 - |v_n|^2)^{1/2} = -\frac{1}{2}\sum_{n=0}^{N} h_n. \tag{4.6}$$

Since h is positive and trace class this proves that H_I is bounded from below and

$$\inf H_I \geq -\frac{\mathrm{Tr}(h)}{2}. \tag{4.7}$$

Since $H(N)$ is selfadjoint for all N (Section 3.6), to prove that $H(N)$ converges to H_I in the strong resolvent sense it suffices to prove the strong convergence of the unitary groups, i.e. $U_N(t) := \mathrm{e}^{\mathrm{i}tH(N)}$ converges strongly to $U_I(t)$ for all t, which is equivalent to prove that $\tilde{U}_N(t) := U_N(t)^{-1}U_I(t)$ strongly converges to the identity. Moreover it is clearly sufficient to prove strong convergence on the dense set $\Gamma_{\mathrm{s}}^{\mathrm{fin}}(C_c(\mathbb{N}))$ where $C_c(\mathbb{N})$ denotes the set of sequences which have compact support (since $C_c(\mathbb{N})$ is not a Hilbert space, $\Gamma_{\mathrm{s}}^{\mathrm{fin}}(C_c(\mathbb{N}))$ denotes here, with an abuse of notation, the algebraic Fock space over $C_c(\mathbb{N})$).

Let

$$h(N) := \sum_{n>N} h_n|e_n\rangle\langle e_n| \quad \text{and} \quad v(N) := \sum_{n>N} v_n|e_n\rangle\langle e_n|.$$

We also denote by $R(t, N)$ the corresponding symplectic group and similarly for $P(t, N)\ldots$ One then easily gets

$$\tilde{U}_N(t) = \mathrm{e}^{-\frac{\mathrm{i}}{2}\mathrm{Tr}\left(\int_0^t Q(s,N)v(N)\bar{P}(s,N)^{-1}\right)}\mathrm{e}^{-\frac{1}{2}a^*(K(t,N))}\Gamma((P(t,N)^{-1})^*)\mathrm{e}^{-\frac{1}{2}a(L(t,N))}.$$

Since $\int_0^t Q(s)v\bar{P}(s)^{-1}\mathrm{d}s$ is trace class by Theorem 3.12, we have

$$\lim_{N\to+\infty} \mathrm{e}^{\frac{1}{2}\mathrm{Tr}\left(\int_0^t Q(s,N)v(N)\bar{P}^{-1}(s,N)\mathrm{d}s\right)} = 1. \tag{4.8}$$

Moreover, let $\Phi \in \Gamma_{\mathrm{s}}^{\mathrm{fin}}(C_c(\mathbb{N}))$, then for N large enough one has

$$\Gamma((P(t,N)^{-1})^*)\mathrm{e}^{-\frac{1}{2}a(L(t,N))}\Phi = \Phi. \tag{4.9}$$

Finally, since $K(t)$ is Hilbert-Schmidt, the sequence of operators $K(t,N)$ goes to zero in the Hilbert-Schmidt norm. This together with Proposition 2.5 proves that

$$\lim_{N\to+\infty} \mathrm{e}^{-\frac{1}{2}a^*(K(t,N))}\Phi = \Phi. \tag{4.10}$$

The strong convergence of $\tilde{U}_N(t)$ to the identity on $\Gamma_{\mathrm{s}}^{\mathrm{fin}}(C_c(\mathbb{N}))$ follows from (4.8)-(4.9)-(4.10).

We now suppose that H_I is bounded from below. Let $\mathfrak{h}_N := \mathrm{Span}\{e_n, n = 0, \cdots N\}$. $\Gamma_{\mathrm{s}}(\mathfrak{h})$ is isomorphic to $\Gamma_{\mathrm{s}}(\mathfrak{h}_N)\otimes\Gamma_{\mathrm{s}}(\mathfrak{h}_N^\perp)$ and via this identification, and with a slight abuse of notation, we have

$$H(N) = H(N)\otimes 1 \quad \text{and} \quad H_I = H(N)\otimes 1 + 1\otimes(H_I - H(N)) \tag{4.11}$$

where $H_I - H(N)$ acts on $\Gamma_{\mathrm{s}}(\mathfrak{h}_N^\perp)$ and is defined as H_I but with $h\lceil_{\mathfrak{h}_N^\perp}$ and $v\lceil_{\mathfrak{h}_N^\perp}$ instead of h and v.

The positivity of the classical symbol of $H(N)$ then writes

$$\forall(z_0,\cdots,z_N)\in\mathbb{C}^{N+1}, \quad \sum_{n=0}^{N}\mathrm{Re}(h_n|z_n|^2 + v_n z_n^2) \geq 0.$$

In particular this implies that the h_n are positive. It remains to prove that h is trace class.

Let $\epsilon > 0$, there exists $\Psi_N \in \mathcal{D}(H(N)) \subset \Gamma_{\mathrm{s}}(\mathfrak{h}_N)$ such that $\langle\Psi_N, H(N)\Psi_N\rangle \leq \inf H(N) + \epsilon = -\frac{1}{2}\sum_{n=0}^{N} h_n + \epsilon$. Let now $\Phi_N := \Psi_N\otimes\Omega_N^\perp$ where $\Omega_N^\perp$ denotes the vacuum of $\Gamma_{\mathrm{s}}(\mathfrak{h}_N^\perp)$. Using (4.11), it is then easy to see that $\Phi_N\in\mathcal{D}(H_I)$ and

$$\langle\Phi_N, H_I\Phi_N\rangle \leq -\frac{1}{2}\sum_{n=0}^{N} h_n + \epsilon.$$

Since the above inequality holds for all N and $\epsilon > 0$, and since H_I is bounded from below, this proves that h is trace class and that

$$\inf H_I \leq -\frac{\mathrm{Tr}(h)}{2}. \tag{4.12}$$

Finally, (4.7) and (4.12) prove that $H_{II} = H_I + \frac{\mathrm{Tr}(h)}{2}$. □

Note that if h is positive but is not trace class, we have an example of a unitarily implementable group $R(t)$ which has a positive classical symbol but which is not type II.

4.2.3 The case $0 < |h|^2 - |v|^2 \leq \frac{1}{2}$

In this section we now assume that for all n, $0 < |h_n|^2 - |v_n|^2 \leq \frac{1}{2}$.

Proposition 4.4. *(i) $R(t)$ defines a strongly continuous group if and only if v is bounded.*

(ii) $R(t)$ is unitarily implementable if and only if v is Hilbert-Schmidt.

(iii) All unitarily implementable symplectic groups are of type I.

(iv) A unitarily implementable symplectic group is of type II if and only if $h \geq 0$ and $|v|^2h^{-1}$ is trace class.

Proof. (i) If v is bounded the result follows once again from Proposition 3.5.

Suppose now that $R(t)$ is a strongly continuous group. A densely defined closed operator A is the generator of strongly continuous group if and only if [Da] there exists $M \geq 1$ and $\omega \geq 0$ such that

- $]-\infty,-\omega[\cup]\omega,+\infty[\subset\rho(A)$, where $\rho(A)$ denotes the resolvent set of A,

- For all $\lambda \in]-\infty, -\omega[\cup]\omega, +\infty[$ and all $m \in \mathbb{N}$, $\|(A-\lambda)^{-m}\| \le \frac{M}{(|\lambda|-\omega)^m}$.

It is easy to see that the operator $A = \mathrm{i}\begin{pmatrix} h & -v \\ \bar{v} & -\bar{h} \end{pmatrix}$ is closed and densely defined. Let us denote $A_n := \mathrm{i}\begin{pmatrix} h_n & -v_n \\ \bar{v_n} & -\bar{h_n} \end{pmatrix} \in M_2(\mathbb{C})$. Since $|h_n| > |v_n|$, $A_n - \lambda$ is invertible for any $\lambda \in \mathbb{R}$, and $\|(A_n-\lambda)^{-1}\| = \frac{\sqrt{\lambda^2+|h_n|^2+|v_n|^2}}{\lambda^2+|h_n|^2-|v_n|^2}$. A necessary condition so that A generates a strongly continuous group is thus

$$\sup_{n\in\mathbb{N}} \frac{\sqrt{\lambda^2+|h_n|^2+|v_n|^2}}{\lambda^2+|h_n|^2-|v_n|^2} \le \frac{M}{|\lambda|-\omega}, \tag{4.13}$$

for some $M \ge 1$, $\omega \ge 0$ and for any $|\lambda| > \omega$. The boundedness of v follows directly from (4.13) and the fact that $|h_n|^2 - |v_n|^2 \le \frac{1}{2}$.

(*ii*) The sufficient condition follows once again from the general theory (Theorem 3.6). Suppose now that $R(t)$ is unitarily implementable. In particular $Q(t)$ has to be Hilbert-Schmidt for all t, i.e. $\forall t \in \mathbb{R}$,

$$\sum \left| |v_n|^2 \frac{\sin^2(t\sqrt{h_n^2-|v_n|^2})}{h_n^2-|v_n|^2} \right| < +\infty. \tag{4.14}$$

Take $t = \pi$. Since $0 < h_n^2 - |v_n|^2 \le \frac{1}{2}$, one has, for all n, $\sin^2(\pi\sqrt{h_n^2-|v_n|^2}) \ge 2(h_n^2-|v_n|^2)$. Inserting this in (4.14) proves that v is Hilbert-Schmidt.

(*iii*) Once again the result follows from Theorem 3.12.

(*iv*) The proof is the same as for the case $|h_n| = |v_n|$ and using the fact that $\sum(h_n - \sqrt{h_n^2-|v_n|^2}) < +\infty \Leftrightarrow \sum \frac{|v_n|^2}{|h_n|} < +\infty$. □

4.2.4 The case $|h|^2 - |v|^2 > \frac{1}{2}$

Finally, in this section we assume that for all n, $|h_n|^2 - |v_n|^2 > \frac{1}{2}$.

Proposition 4.5. *(i) $R(t)$ defines a strongly continuous group if and only if $\frac{|v|}{\sqrt{|h|^2-|v|^2}}$ is bounded.*

(ii) $R(t)$ is unitarily implementable if and only if $\frac{|v|}{\sqrt{|h|^2-|v|^2}}$ is Hilbert-Schmidt.

(iii) A unitarily implementable symplectic group is of type I if and only if $|v|^2h^{-1}$ is trace class.

(iv) A unitarily implementable symplectic group is of type II if and only if $h \ge 0$.

Proof. (*i*) Suppose $\frac{|v|}{\sqrt{|h|^2-|v|^2}}$ is bounded. Thus v is h-bounded with relative bound strictly less than one. Writing, as in Section 3.3, $A = A_0 + V$ we get that V is A_0 bounded with relative bound strictly less than one. Since A_0 generates a strongly continuous group (A_0 is antiselfadjoint) this proves that A generates a strongly continuous group [Da].

Suppose now that A generates a strongly continuous group. Using the same argument as in the previous section (see (4.13)), there are constants $M \ge 1$ and $\omega \ge 0$ such that for all $\lambda > \omega$, and all n,

$$\frac{1}{\sqrt{2}(\sqrt{\lambda^2+h_n^2}-|v_n|)} \le \frac{\sqrt{\lambda^2+|h_n|^2+|v_n|^2}}{\lambda^2+|h_n|^2-|v_n|^2} \le \frac{M}{\lambda-\omega},$$

which one can rewrite as

$$\lambda^2(2M^2-1) - 2\lambda(\sqrt{2}M|v_n|-\omega) + 2M^2h_n^2 - (\sqrt{2}M|v_n|-\omega)^2 \ge 0, \qquad \forall\lambda \ge \omega.$$

The result follows easily from the above inequality and the assumption $|h_n|^2 - |v_n|^2 > \frac{1}{2}$.

(*ii*) Suppose $\frac{|v|}{\sqrt{|h|^2-|v|^2}}$ is Hilbert-Schmidt. Therefore so is vh^{-1}, and hence, using the fact that v and h commute, Assumption 3.C is satisfied, so that $R(t)$ is unitarily implementable by Theorem 3.6.

Suppose now that $R(t)$ is unitarily implementable. Hence the map $t \mapsto \mathrm{Tr}(\log(1 - K(t)^*K(t)))$ is continuous (see the proof of Theorem 3.4) and thus locally integrable. Using (4.4) we get

$$\int_0^T \mathrm{Tr}(\log(1-K(t)^*K(t)))\mathrm{d}t = -\int_0^T \sum_n \log\left(1 + \frac{|v_n|^2}{h_n^2 - |v_n|^2}\sin^2(t\sqrt{h_n^2 - |v_n|^2})\right)\mathrm{d}t,$$

Then, using (*i*), we know that the sequence $\frac{|v_n|^2}{h_n^2-|v_n|^2}$ is bounded. Hence there exists $C > 0$ such that for all $n \in \mathbb{N}$ and $t \in \mathbb{R}$,

$$\log\left(1 + \frac{|v_n|^2}{h_n^2 - |v_n|^2}\sin^2(t\sqrt{h_n^2 - |v_n|^2})\right) \geq C\frac{|v_n|^2}{h_n^2 - |v_n|^2}\sin^2(t\sqrt{h_n^2 - |v_n|^2}),$$

and hence

$$\sum \int_0^T \frac{|v_n|^2}{h_n^2 - |v_n|^2}\sin^2(t\sqrt{h_n^2 - |v_n|^2})\mathrm{d}t = \sum \frac{|v_n|^2}{h_n^2 - |v_n|^2}\left(\frac{T}{2} - \frac{\sin(2T\sqrt{h_n^2 - |v_n|^2})}{4\sqrt{h_n^2 - |v_n|^2}}\right) < +\infty, \quad \text{for all } T.$$

Using $\sqrt{h_n^2 - |v_n|^2} > \frac{1}{2}$ and choosing T large enough, we get $\sum \frac{|v_n|^2}{h_n^2-|v_n|^2} < +\infty$.

(*iii*) If $|v|^2h^{-1}$ is trace class, then Assumption 3.D is satisfied so $R(t)$ is of type I. Suppose now that $R(t)$ is of type I. Then by definition, $P(t)\mathrm{e}^{-\mathrm{i}th} - 1$ is trace class for all t. Using (3.6) and the fact that all the operators involved here commute one gets $P(t)\mathrm{e}^{-\mathrm{i}th} = \mathrm{e}^{\mathrm{i}\int_0^t \bar{Q}(s)\bar{v}P(s)^{-1}\mathrm{d}s}$. Therefore we have, for all t,

$$\sum_n \left|\int_0^t \bar{Q}_n(s)\bar{v}_n P_n(s)^{-1}\mathrm{d}s\right| < +\infty. \tag{4.15}$$

Using (4.4) we get

$$\begin{aligned} \bar{Q}_n(s)\bar{v}_n P_n(s)^{-1} \quad = \quad & -h_n|v_n|^2 \frac{\sin^2(s\sqrt{h_n^2 - |v_n|^2})}{h_n^2 - |v_n|^2\cos^2(s\sqrt{h_n^2 - |v_n|^2})} \\ & -\mathrm{i}|v_n|^2\sqrt{h_n^2 - |v_n|^2}\frac{\sin(s\sqrt{h_n^2 - |v_n|^2})\cos(s\sqrt{h_n^2 - |v_n|^2})}{h_n^2 - |v_n|^2\cos^2(s\sqrt{h_n^2 - |v_n|^2})}, \end{aligned} \tag{4.16}$$

so that, in particular,

$$\sum \left| h_n|v_n|^2 \int_0^t \frac{\sin^2(s\sqrt{h_n^2 - |v_n|^2})}{h_n^2 - |v_n|^2\cos^2(s\sqrt{h_n^2 - |v_n|^2})}\mathrm{d}s\right| < +\infty. \tag{4.17}$$

The above integral can be explicitly computed and one gets

$$\begin{aligned} & h_n|v_n|^2 \int_0^t \frac{\sin^2(s\sqrt{h_n^2 - |v_n|^2})}{h_n^2 - |v_n|^2\cos^2(s\sqrt{h_n^2 - |v_n|^2})}\mathrm{d}s \\ = \quad & th_n\left(1 - \sqrt{1 - \frac{|v_n^2|}{h_n^2}}\right) + \frac{h_n}{|h_n|}\left[r_n(t)\sqrt{h_n^2 - |v_n|^2} - \mathrm{arccotan}\left(\sqrt{1 - \frac{|v_n^2|}{h_n^2}}\mathrm{cotan}(r_n(t)\sqrt{h_n^2 - |v_n|^2})\right)\right], \end{aligned} \tag{4.18}$$

where $r = t - \frac{\pi}{\sqrt{h_n^2-|v_n|^2}}\mathrm{E}(\frac{t\sqrt{h_n^2-|v_n|^2}}{\pi}) \in [0, \frac{\pi}{\sqrt{h_n^2-|v_n|^2}}[$ and where E denotes the entire part.

To prove (iii) it suffices to prove that for all $t \in \mathbb{R}$,

$$\sum \left| r_n(t)\sqrt{h_n^2 - |v_n|^2} - \operatorname{arccotan}\left(\sqrt{1 - \frac{|v_n^2|}{h_n^2}} \operatorname{cotan}(r_n(t)\sqrt{h_n^2 - |v_n|^2}) \right) \right| < +\infty. \tag{4.19}$$

Indeed, if (4.19) holds, then one has $\sum \left| h_n \left(1 - \sqrt{1 - \frac{|v_n^2|}{h_n^2}}\right)\right| < +\infty$, from which the result follows directly using $|h_n|^2 > |v_n|^2 + \frac{1}{2} \geq \frac{1}{2}$.

Now, it is not difficult to show that for all $r \in [0, \frac{\pi}{\sqrt{h_n^2 - |v_n|^2}}[$ one has

$$\left| r\sqrt{h_n^2 - |v_n|^2} - \operatorname{arccotan}\left(\sqrt{1 - \frac{|v_n^2|}{h_n^2}} \operatorname{cotan}(r\sqrt{h_n^2 - |v_n|^2}) \right) \right| \leq \frac{\pi}{2} - 2\arctan\left(\left(1 - \frac{|v_n|^2}{h_n^2}\right)^{1/4} \right).$$

Since $R(t)$ is unitarily implementable, $\sum \frac{|v_n|^2}{h_n^2 - |v_n|^2} < +\infty$ and hence $\sum \frac{|v_n|^2}{h_n^2} < +\infty$. Equation (4.19) follows from this and the above majoration.

(iv) We use the notation introduced in the proof of Proposition 4.3. Using (4.16)-(4.18)-(4.19), one can see that for all t

$$U_{\mathrm{ren}}(t) := \mathrm{e}^{\frac{\mathrm{i}}{2}\sum_n \left(\operatorname{Re}\int_0^t Q_n(\tau) v_n \bar{P}_n(\tau)^{-1} \mathrm{d}\tau + t h_n \left(1 - \sqrt{1 - \frac{|v_n|^2}{h_n^2}}\right)\right)} U_{\mathrm{nat}}(t) \tag{4.20}$$

is a well defined Bogoliubov implementer of $R(t)$.

In the same way as in the proof of Theorem 3.8, to prove that it forms a one paremeter unitary group, it suffices to prove that

$$\langle \Omega | U_{\mathrm{ren}}(t) U_{\mathrm{ren}}(s) \Omega \rangle = \langle \Omega | U_{\mathrm{ren}}(t+s)\Omega \rangle. \tag{4.21}$$

Since the operators h and v are both diagonal with respect to the basis $(e_n)_n$, so are the operators $P(t), Q(t), K(t), L(t)$. Hence one can write both the left and right hand side of (4.21) as a product over n. It therefore suffices to show that, for all n,

$$\begin{aligned} & \exp\left(\frac{1}{2}\int_0^{t_1} Q_n(s) v_n \bar{P}_n(s)^{-1} \mathrm{d}s + \mathrm{i}\lambda_n t_1 \right) \exp\left(\frac{1}{2}\int_0^{t_2} Q_n(s) v_n \bar{P}_n(s)^{-1} \mathrm{d}s + \mathrm{i}\lambda_n t_2 \right) \\ & \qquad \times \det(1 - L_n^*(t_1) K_n(t_2))^{-1/2} \\ = \; & \exp\left(\frac{1}{2}\int_0^{t_1+t_2} Q_n(s) v_n \bar{P}_n(s)^{-1} \mathrm{d}s + \mathrm{i}\lambda_n (t_1 + t_2) \right), \end{aligned}$$

where $\lambda_n = h_n \left(1 - \sqrt{1 - \frac{h_n^2}{|v_n|^2}}\right)$. This follows from Theorem 3.12 applied to the generator $\begin{pmatrix} \mathrm{i}h_n & -\mathrm{i}v_n \\ \mathrm{i}\bar{v}_n & -\mathrm{i}h_n \end{pmatrix}$ considered on the space $\mathbb{R}^2$. Hence $R(t)$ is of type II if and only if the generator H_{ren} of $U_{\mathrm{ren}}(t)$ is bounded from below.

Suppose h is positive. Let

$$H_{II,n} := H_n - \inf H_n = H_n + \frac{1}{2} h_n \left(1 - \sqrt{1 - \frac{|v_n|^2}{h_n^2}}\right) \quad \text{and} \quad H_{II}(N) := \sum_{n=0}^{N} H_{II,n}.$$

For any N, $H_{II}(N)$ is selfadjoint and $\inf H_{II}(N) = 0$. Moreover, using the same argument as in the proof of Proposition 4.3, we prove that $H_{II}(N)$ converges to H_{ren} in the strong resolvent sense so that

$$\inf H_{\mathrm{ren}} \geq \lim_{N\to\infty} \inf H_{II}(N) = 0. \qquad \square$$

4.2.5 Putting all together

It is easy to see that in each of the four situations the necessary and sufficient conditions which appear in Theorem 4.1 are equivalent to the corresponding conditions used in the various propositions. The only points which are not immediate are the definition of the operator $\Lambda_{\rm ren}$ in (ii) and of the series which appears in (iv).

The obvious definition of $\Lambda_{\rm ren}$ would be to replace the sum over $\{n \mid |h_n| > 1\}$ by the same sum but over $\{n \mid |h_n|^2 > |v_n|^2 + \frac{1}{2}\}$, and similarly for the series in (iv). To prove that these definitions are equivalent, we have to prove that $\sum_{n\in\mathcal{N}} \frac{|v_n|^2}{|h_n|} < +\infty$ where

$$\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2 := \{n \mid 1 < |h_n|^2 \leq |v_n|^2 + \frac{1}{2}\} \cup \{n \mid |v_n|^2 + \frac{1}{2} < |h_n|^2 \leq 1\}.$$

Since $R(t)$ is unitarily implementable,we have $\sum_{\mathcal{N}} \frac{|v_n|^2}{1+|h_n|^2} < +\infty$. Using this, it is then clear that $\sum_{n\in\mathcal{N}_2} \frac{|v_n|^2}{|h_n|} < +\infty$. On the other hand, if $n \in \mathcal{N}_1$, one easily gets that $\frac{|v_n|^2}{1+|h_n|^2} \geq \frac{1}{4}$. The implementability of $R(t)$ thus gives that $\mathcal{N}_1$ is actually a finite set. □

References

[A] Araki H., On quasifree states of the canonical commutation relations (II), Publ. RIMS, Kyoto Univ. **7**, 121-152 (1971).

[AY] Araki H., Yamagami S., On quasi-equivalence of quasifree states of the canonical commutation relations, Publ. RIMS, Kyoto Univ. **18**, 283-338 (1982).

[Ar] Arai A., On a model of a harmonic oscillator coupled to a quantized, massless, scalar field I, J. Math. Phys. **22**, 2539-2548 (1981).

[BFS] Bach V., Fröhlich J., Sigal I, Quantum electrodynamics of confined non-relativistic particles, Adv.Math. **137**, 299-395 (1998).

[Be] Berezin E. A., *The method of second quantization*, Academic presse, 1966.

[BR] Bratteli O., Robinson D. W., *Operators Algebras and Quantum Statistical Mechanics*, vol. I and II, Springer-Verlag, Berlin, 1981.

[Ca] Carleman, *Les fonctions quasi-analytiques*, Gauthier-Villars, Paris, 1926.

[Da] Davies E. B., *One-parameter Semigroups*, Academic Press, London, 1980.

[De] Dereziński J., Van Hove Hamiltonians-Exactly solvable models of the infrared and ultraviolet problem, Ann. Inst. H. Poincaré. **4**, 713-738 (2003).

[DG] Dereziński J., Gérard C., Asymptotic completeness in quantum field theory. Massive Pauli-Fierz Hamiltonians, Rev. Math. Phys. **11**, 383-450 (1999).

[DJ] Dereziński J., Jakšić V., Spectral theory of Pauli-Fierz operators, J. Func. Anal. **180**, 243-327 (2001).

[GJ] Glimm J., Jaffe A., *Quantum Physics. A Functional Integral Point of View*, 2nd ed., Springer-Verlag, Berlin, 1987.

[IH] Ito K. R., Hiroshima F., Local exponents and infinitesimal generators of canonical transformations on Boson Fock spaces, preprint mp-arc 03-338 (2003).

[La] Langmann E., Cocycles for boson and fermion Bogoliubov transformations, J. Math. Phys. **35**, 96-112 (1994).

[Ne] Neretin Y. A. *Categories of Symmetries and Infinite-Dimensional Groups*, Clarendon Press, Oxford, 1996.

[RS1] Reed M., Simon B., *Methods of modern Mathematical Physics, I*, Academic Press, London, 1976.

[RS2] Reed M., Simon B., *Methods of modern Mathematical Physics, II*, Academic Press, London, 1976.

[Ru1] Ruijsenaars S. N. M., On Bogoliubov transformations for systems of relativistic charged particles, J. Math. Phys. **18**, 517-526 (1976).

[Ru2] Ruijsenaars S. N. M., On Bogoliubov transforms II. The general case., Ann. Phys. **116**, 105-134 (1978).

[Sh] Shale D., Linear symmetries of free boson fields, Trans. Amer. Math. Soc. **103**, 149-167 (1962).