

Blow-Up for Nonlinear Wave Equations describing Boson Stars

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Abstract

We consider the nonlinear wave equation

$$i\partial_t u = \sqrt{-\Delta + m^2}u - (|x|^{-1} * |u|^2)u \quad \text{on } \mathbb{R}^3$$

modelling the dynamics of (pseudo-relativistic) boson stars. For spherically symmetric initial data, $u_0(x) \in C_c^\infty(\mathbb{R}^3)$, with negative energy, we prove blow-up of $u(t, x)$ in $H^{1/2}$ -norm within a finite time. Physically, this phenomenon describes the onset of “gravitational collapse” of a boson star. We also study blow-up in external, spherically symmetric potentials and we consider more general Hartree-type nonlinearities. As an application, we exhibit instability for ground state solitary waves of the equation with $m = 0$.

1 Introduction

In this paper, we prove blow-up of solutions of the nonlinear wave equation

$$i\partial_t u = \sqrt{-\Delta + m^2}u - \left(\frac{1}{|x|} * |u|^2\right)u \quad \text{on } \mathbb{R}^3 \quad (1.1)$$

arising as an effective description of pseudo-relativistic *boson stars*, as recently shown in [ES05], see also [Sch05]. Here $u(t, x)$ is a complex-valued wave field (a one-particle wave function). The operator $\sqrt{-\Delta + m^2}$, which is defined via its symbol $\sqrt{k^2 + m^2}$ in Fourier space, describes the kinetic and rest energy of a relativistic particle of mass $m \geq 0$, and $|x|^{-1}$ is the Newtonian gravitational potential in appropriate physical units. Moreover, the symbol $*$ stands for convolution on \mathbb{R}^3 .

Apart from its applications in theoretical astrophysics, equation (1.1) is of considerable interest from the PDE’s point of view: It is a “semi-relativistic” nonlinear Schrödinger (or Hartree) equation with focusing, L^2 -critical nonlinearity. In particular, there exist ground state solitary wave solutions, referred to as *ground state solitons*; an extensive study can be found in [FJL05].

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The purpose of the present paper is to address the problem of proving blow-up of solutions, and we will establish the following result: Any spherically symmetric initial datum, $u_0(x) \in C_c^\infty(\mathbb{R}^3)$, with negative energy,

$$\mathcal{E}(u_0) < 0, \quad (1.2)$$

gives rise to a solution, $u(t, x)$, of (1.1) that blows up within a finite time, i. e., we have that

$$\lim_{t \nearrow T} \|u(t, \cdot)\|_{H^{1/2}} = \infty, \quad \text{for some } 0 < T < \infty. \quad (1.3)$$

Here the energy functional, $\mathcal{E}(u)$, is given by

$$\mathcal{E}(u) = \frac{1}{2} \int_{\mathbb{R}^3} \bar{u} \sqrt{-\Delta + m^2} u \, dx - \frac{1}{4} \int_{\mathbb{R}^3} \left(\frac{1}{|x|} * |u|^2 \right) |u|^2 \, dx, \quad (1.4)$$

and $\|\cdot\|_{H^{1/2}}$ in (1.3) denotes the norm of the Sobolev space $H^{1/2}(\mathbb{R}^3)$. In more generality, this blow-up result is described in Theorem 1 below, which also takes external potentials and other types of Hartree nonlinearities into account. In physical terms, finite-time blow-up of $u(t, x)$ is indicative of the onset of “*gravitational collapse*” of a boson star modelled by (1.1).

We begin with a brief recapitulation of some important results for (1.1) that have been derived so far. As shown in [Len05], the Cauchy problem for (1.1) is locally well-posed for initial data, $u_0(x)$, that belong to the Sobolev space $H^s(\mathbb{R}^3)$, with $s \geq 1/2$, where $H^{1/2}(\mathbb{R}^3)$ is the energy space for (1.1). Moreover, we have that $u(t, x)$ extends to all times, $t \geq 0$, provided that the initial datum satisfies

$$\int_{\mathbb{R}^3} |u_0|^2 \, dx < N_c, \quad (1.5)$$

where $N_c > 4/\pi$ is a universal constant; see [Len05] for more details. In particular, we point out that condition (1.5) implies that $\mathcal{E}(u_0) > 0$ holds. Thus, blow-up phenomena for (1.1) can only occur for large initial data, $u_0(x)$, that do *not* satisfy condition (1.5). Concerning its physical interpretation, the universal constant N_c (which defines the scale of large and small initial data) can be viewed as the “*Chandrasekhar limit mass*” for boson stars modelled by (1.1). Furthermore, the boson star equation (1.1) has been rigorously derived in [ES05] from many-body quantum mechanics; we refer to [LY87] for an earlier result on the time-independent problem.

As an application of our blow-up result, we can study the *stability of ground state solitons*,

$$u_{\text{sol}}(t, x) = e^{i\omega t} Q(x), \quad (1.6)$$

where $Q(x) \in H^{1/2}(\mathbb{R}^3)$ satisfies

$$\sqrt{-\Delta + m^2} Q - \left(\frac{1}{|x|} * |Q|^2 \right) Q = -\omega Q, \quad (1.7)$$

for some $\omega > 0$. Indeed, existence and spherical symmetry of solutions $Q(x)$, which we define as minimizers of $\mathcal{E}(u)$ subject to $\int_{\mathbb{R}^3} |u|^2 \, dx = N$, follows from the discussion in [LY87]; see also [Len05] for $m = 0$. Furthermore, it turns out that the mass parameter, $m \geq 0$, plays a decisive role summarized as follows; see also [FJL05] for more details.

For $\left\{ \begin{array}{l} m > 0 \\ m = 0 \end{array} \right\}$, $Q(x)$ exists if and only if $\left\{ \begin{array}{l} \int_{\mathbb{R}^3} |Q|^2 \, dx < N_c \\ \int_{\mathbb{R}^3} |Q|^2 \, dx = N_c \end{array} \right\}$, and then $\left\{ \begin{array}{l} \mathcal{E}(Q) > 0 \\ \mathcal{E}(Q) = 0 \end{array} \right\}$.

Here N_c is the universal constant appearing in (1.5). For $m = 0$, we can prove existence of blow-up solutions with initial data $u_0(x)$ arbitrarily close to $Q(x)$; see Theorem 2 below, for a precise statement. In contrast to this instability result, ground state solitons turn out to be orbitally *stable* whenever $m > 0$ holds; see [FJL05] for a proof of this fact and a detailed study of the effective dynamics of these ground state solitons in an external potential.

Commenting the proof of our main result, we remark that its key ingredient is a virial-type argument for the nonnegative quantity, $M(t)$, defined as

$$M(t) = \int_{\mathbb{R}^3} \sum_{k=1}^3 \bar{u}(t, x) x_k \sqrt{-\Delta + m^2} x_k u(t, x) dx. \quad (1.8)$$

By restricting to spherically symmetric solutions, $u(t, x)$, of equation (1.1), various estimates (which become decisively clear when using commutators) and conservation laws, we will derive the inequality

$$0 \leq M(t) \leq 2\mathcal{E}(u_0)t^2 + C_1t + C_2, \quad (1.9)$$

for some finite constants C_1 and C_2 . Thus, if $\mathcal{E}(u_0)$ is negative, we conclude that $u(t, x)$ cannot exist for all times $t \geq 0$, which implies statement (1.3), by results taken from [Len05].

It would obviously be of considerable interest to overcome our restriction to spherically symmetric blow-up solutions of (1.1) and to arrive at a state of affairs comparable to that known for L^2 -critical nonlinear Schrödinger equations; see, e.g., [Caz03] for an overview. Especially, it would be important to get insight into (upper) bounds on blow-up rates and blow-up profiles.

Notation. Throughout this text, $L^p(\mathbb{R}^3)$, with norm $\|\cdot\|_p$ and $1 \leq p \leq \infty$, denotes the Lebesgue L^p -space of complex-valued functions on \mathbb{R}^3 . The complex scalar product on $L^2(\mathbb{R}^3)$ is given by

$$\langle u, v \rangle := \int_{\mathbb{R}^3} \bar{u}v dx.$$

We employ inhomogeneous Sobolev spaces, $H^s(\mathbb{R}^3)$, of fractional order $s \in \mathbb{R}$, which are defined as

$$H^s(\mathbb{R}^3) := \left\{ u \in L^2(\mathbb{R}^3) : \|u\|_{H^s}^2 := \int_{\mathbb{R}^3} (1 + |k|^2)^s |\hat{u}(k)|^2 dk < \infty \right\},$$

where $\hat{u}(k)$ is the Fourier transform of $u(x)$. Furthermore, we make use of inhomogeneous Sobolev spaces, $W^{k, \infty}(\mathbb{R}^3)$, of integer order $k \in \mathbb{N}$, which are given by

$$W^{k, \infty}(\mathbb{R}^3) := \left\{ u \in L^\infty(\mathbb{R}^3) : \|u\|_{W^{k, \infty}} := \sum_{\alpha \in \mathbb{N}^3, |\alpha| \leq k} \|\partial^\alpha u\|_\infty < \infty \right\}.$$

The space of complex-valued, smooth functions on \mathbb{R}^3 with compact support is denoted by $C_c^\infty(\mathbb{R}^3)$. Moreover, the operator $\sqrt{-\Delta + m^2}$ is defined via its symbol $\sqrt{k^2 + m^2}$ in Fourier space, and the letter C appearing in inequalities denotes constants.

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2 Main Results

Generalizing equation (1.1), we introduce the following initial value problem

$$\begin{cases} i\partial_t u = (\sqrt{-\Delta + m^2} + V)u - \left(\frac{e^{-\mu|x|}}{|x|} * |u|^2\right)u, \\ u(0, x) = u_0(x), \quad u : [0, T) \times \mathbb{R}^3 \rightarrow \mathbb{C}, \end{cases} \quad (2.1)$$

where $m \geq 0$ and $\mu \geq 0$ are parameters, and $V : \mathbb{R}^3 \rightarrow \mathbb{R}$ denotes an external potential. We recall from [Len05] that, under fairly general assumptions on $V(x)$, we have local well-posedness of (2.1) in energy space

$$X := \{u \in H^{1/2}(\mathbb{R}^3) : V|u|^2 \in L^1(\mathbb{R}^3)\}. \quad (2.2)$$

This means that, for any $u_0 \in X$, there exists a unique solution, $u \in C^0([0, T); X)$, of (2.1) with maximal time of existence $T > 0$. In addition, we have conservation of charge, $\mathcal{N}(u)$, and energy, $\mathcal{E}(u)$, which are given by

$$\mathcal{N}(u) = \int_{\mathbb{R}^3} |u|^2 dx, \quad (2.3)$$

$$\mathcal{E}(u) = \frac{1}{2} \int_{\mathbb{R}^3} \bar{u} \sqrt{-\Delta + m^2} u dx + \frac{1}{2} \int_{\mathbb{R}^3} V|u|^2 dx - \frac{1}{4} \int_{\mathbb{R}^3} \left(\frac{e^{-\mu|x|}}{|x|} * |u|^2\right) |u|^2 dx. \quad (2.4)$$

2.1 Blow-Up

We now consider external potentials, $V(x)$, that satisfy the following condition.

Assumption 1. *Suppose that $V \in W^{1,\infty}(\mathbb{R}^3)$ is real-valued with $|V(x)| \leq C(1 + |x|)^{-1}$ and $|\nabla V(x)| \leq C(1 + |x|)^{-2}$, for some finite constant C .*

In view of (2.2), Assumption 1 obviously implies that the energy space, X , equals $H^{1/2}(\mathbb{R}^3)$.

Our first main result shows that spherically symmetric initial data, $u_0(x) \in C_c^\infty(\mathbb{R}^3)$, with sufficiently negative energy lead to blow-up of $u(t, x)$ within a finite time.

Theorem 1. *Let $m \geq 0$, $\mu \geq 0$, and suppose that $V(x)$ is spherically symmetric and satisfies Assumption 1. Furthermore, we define the bounded function $U := \min\{V + x \cdot \nabla V, 0\}$.*

Then any spherically symmetric initial datum, $u_0(x) \in C_c^\infty(\mathbb{R}^3)$, satisfying

$$\mathcal{E}(u_0) < -\frac{1}{2} \|U\|_\infty \|u_0\|_2^2 \quad (2.5)$$

gives rise to a solution, $u(t, x)$, of (2.1) that blows up within a finite time, i. e., we have that

$$\lim_{t \nearrow T} \|u(t, \cdot)\|_{H^{1/2}} = \infty, \quad \text{for some } 0 < T < \infty. \quad (2.6)$$

Remarks. 1) The condition that $u_0(x)$ belongs to $C_c^\infty(\mathbb{R}^3)$ can be relaxed to weaker regularity and decay assumptions. But for simplicity of our presentation, we do not pursue this point in more detail.

2) By scaling properties of $\mathcal{E}(u)$, it is easy to see that condition (2.5) holds for $u_0(x) = \lambda v(x)$ whenever $v \in H^{1/2}(\mathbb{R}^3)$, with $v \not\equiv 0$, and $|\lambda|$ is sufficiently large.

3) Theorem 1 can be viewed as a quantum-mechanical extension of the blow-up result derived by [GS85] for the *relativistic Vlasov–Poisson system* which models classical, relativistic particles with Newtonian gravitational interactions.

2.2 An Instability Result

As mentioned in Sect. 1, we can address the problem of stability of ground state solitary wave solutions,

$$u_{\text{sol}}(t, x) = e^{i\omega t} Q(x), \quad \text{for some } Q \in H^{1/2}(\mathbb{R}^3), \quad (2.7)$$

which we refer to as *ground state solitons*. We now set $m = 0$, $\mu = 0$, and $V \equiv 0$ in (2.1). Then $Q(x)$ has to solve the nonlinear equation

$$\sqrt{-\Delta} Q - \left(\frac{1}{|x|} * |Q|^2\right) Q = -\omega Q \quad (2.8)$$

for some $\omega > 0$; see Sect. 1 and references given there, for existence and spherical symmetry of $Q(x)$. We will prove the following instability result.

Theorem 2. *Set $m = \mu = 0$ and $V \equiv 0$ in (2.1). Then the ground state solitons (2.7) are unstable in the following sense. For any $\varepsilon > 0$, there exists $u_0 \in C_c^\infty(\mathbb{R}^3)$ such that $\|u_0 - Q\|_{H^{1/2}} < \varepsilon$ holds and the corresponding solution, $u(t, x)$, of (2.1) blows up within a finite time.*

Remark. In contrast to this result, ground state solitons are (orbitally) *stable* whenever the mass parameter is positive, i. e., $m > 0$ in (2.1). We refer to [FJL05] for a proof of this result and a detailed analysis of the effective dynamics of such solitons in an external potential.

3 Proof of Main Results

3.1 Proof of Theorem 1

The proof of Theorem 1 is organized in four steps.

Step 1: Preliminaries

We begin with an observation that immediately follows from the field equation (2.1): Let B be some time-independent operator on $L^2(\mathbb{R}^3)$ and let $u(t)$ be a solution of (2.1). Then the time derivative of the expected value,

$$\langle u(t), Bu(t) \rangle, \quad (3.1)$$

of B is (formally, at least) given by *Heisenberg's formula*:

$$\frac{d}{dt} \langle u(t), Bu(t) \rangle = i \langle u(t), [H, B]u(t) \rangle. \quad (3.2)$$

Here $[H, B] \equiv HB - BH$ denotes the commutator of B with the time-dependent Hamiltonian $H = H(t)$ given by

$$H(t) := \sqrt{-\Delta + m^2} + V + V_s(t), \quad \text{where } V_s(t) := -\left(\frac{e^{-\mu|x|}}{|x|} * |u(t)|^2\right). \quad (3.3)$$

If (3.2) is applied for purpose of rigorous arguments, we have to verify that $\langle u(t), HBu(t) \rangle$ etc. are well-defined in the case at hand.

Next, we note that the proof presented in [Len05] for local well-posedness of (2.1) with $V \equiv 0$ and initial data in $H^s(\mathbb{R}^3)$, with $s \geq 1/2$, carries over, with only cosmetic changes, for real-valued $V \in W^{1,\infty}(\mathbb{R}^3)$ and initial data belonging to $H^s(\mathbb{R}^3)$, with $2 \geq s \geq 1/2$.

Therefore we have that, for every $u_0 \in H^2(\mathbb{R}^3)$, there exists a unique solution

$$u \in C^0([0, T]; H^2(\mathbb{R}^3)) \cap C^1([0, T]; H^1(\mathbb{R}^3)), \quad (3.4)$$

where $T > 0$ denotes the maximal time of existence. Moreover, the following blow-up alternative in energy norm holds: Either $T = \infty$ or $T < \infty$ and $\lim_{t \nearrow T} \|u(t)\|_{H^{1/2}} = \infty$. In addition, we have that $|x|^2 u_0(x) \in L^2(\mathbb{R}^3)$ implies that

$$|x|^2 u(t) \in L^2(\mathbb{R}^3), \quad \text{for } t \in [0, T], \quad (3.5)$$

as shown by Lemma 3 in Sect. A.1. In what follows, we will make use of the regularity and decay properties of $u(t)$ stated in (3.4) and (3.5).

Finally, we mention spherical symmetry of $u(t, x)$ follows from (2.1) whenever $u_0(x)$ and $V(x)$ exhibit this property. For conservation of charge, $\mathcal{N}(u) = \|u\|_2^2$, (notice that V is real-valued) and energy, $\mathcal{E}(u)$, we refer to [Len05].

Step 2: Dilatation Estimate

The first crucial step in proving Theorem 1 is to estimate the time evolution for the expected value of the generator of dilatations,

$$A := \frac{1}{2}(x \cdot p + p \cdot x), \quad (3.6)$$

where, from now on, we employ the following notation

$$p := -i\nabla. \quad (3.7)$$

Lemma 1. *Let the assumptions on m , μ , and V stated in Theorem 1 (except for spherical symmetry of V) be satisfied. Furthermore, suppose that $u_0(x) \in C_c^\infty(\mathbb{R}^3)$ holds. Then the map $t \mapsto \langle u(t), Au(t) \rangle$ satisfies the inequality*

$$\frac{1}{2} \frac{d}{dt} \langle u(t), Au(t) \rangle \leq \mathcal{E}(u_0) + \frac{1}{2} \|U\|_\infty \|u_0\|_2^2, \quad (3.8)$$

for all $0 \leq t < T$, where $U := \min\{V + x \cdot \nabla V, 0\}$.

Proof of Lemma 1. Since $u(t) \in H^2(\mathbb{R}^3)$ and $|x|^2 u(t) \in L^2(\mathbb{R}^3)$ for all $t \in [0, T]$, as discussed in Step 1 above, it is legitimate to apply (3.2); notice that, e. g., $|\langle u(t), (x \cdot p)Hu(t) \rangle| \leq \| |x|u(t) \|_2 \|u(t)\|_{H^2}$ is finite.

For brevity, we will write u , instead of $u(t, x)$, etc. Using the definition of H in (3.3), an elementary calculation with commutators leads to

$$[H, A] = \frac{-ip^2}{\sqrt{p^2 + m^2}} + ix \cdot \nabla V + ix \cdot \nabla V_s, \quad (3.9)$$

which, by Heisenberg's formula (3.2), implies that

$$\frac{d}{dt}\langle u, Au \rangle = \langle u, \frac{p^2}{\sqrt{p^2 + m^2}}u \rangle - \langle u, (x \cdot \nabla V)u \rangle - \langle u, (x \cdot \nabla V_s)u \rangle. \quad (3.10)$$

Notice that, by our assumptions on V , we have that $|\langle u, (x \cdot \nabla V)u \rangle| \leq C\langle u, |x|u \rangle$ is finite.

Next, we observe the following identity (using Fubini's theorem and interchanging differentiation and integration, which can be justified easily)

$$\begin{aligned} \langle u, (x \cdot \nabla V_s)u \rangle &= \int_{\mathbb{R}^3 \times \mathbb{R}^3} x \cdot \left(\frac{e^{-\mu|x-y|}}{|x-y|^2} \frac{x-y}{|x-y|} + \mu \frac{e^{-\mu|x-y|}}{|x-y|} \frac{x-y}{|x-y|} \right) |u(t, x)|^2 |u(t, y)|^2 dx dy \\ &= \frac{1}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \left(\frac{e^{-\mu|x-y|}}{|x-y|} + \mu e^{-\mu|x-y|} \right) |u(t, x)|^2 |u(t, y)|^2 dx dy \\ &= -\frac{1}{2} \langle u, V_s u \rangle + \frac{\mu}{2} \langle u, (e^{-\mu|x|} * |u|^2)u \rangle, \end{aligned} \quad (3.11)$$

where the second equation follows from a simple symmetry argument. Furthermore, thanks to the obvious fact that $p^2/(p^2 + m^2)^{1/2} = (p^2 + m^2)^{1/2} - m^2/(p^2 + m^2)^{1/2}$, we find that (3.10) can be expressed as follows

$$\frac{d}{dt}\langle u, Au \rangle = 2\mathcal{E}(u_0) - \langle u, (V + x \cdot \nabla V)u \rangle - \langle u, \frac{m^2}{\sqrt{p^2 + m^2}}u \rangle - \frac{\mu}{2} \langle u, (e^{-\mu|x|} * |u|^2)u \rangle,$$

where we use that $\mathcal{E}(u_0)$ is conserved and given by (2.4). Since the last two terms in (3.12) are nonpositive, we can deduce inequality (3.8) by applying Hölder's inequality and noticing that

$$-\langle u, (V + x \cdot \nabla V)u \rangle \leq \|U\|_\infty \|u_0\|_2^2, \quad (3.12)$$

where $U := \min\{V + x \cdot \nabla V, 0\}$. This completes our proof of Lemma 1. \square

Step 3: Variance-Type Estimate

For our next step in the proof of Theorem 1, we introduce

$$M := x\sqrt{-\Delta + m^2}x := \sum_{k=1}^3 x_k \sqrt{-\Delta + m^2}x_k, \quad (3.13)$$

for $m \geq 0$. Note that M is nonnegative, i. e., we have that $\langle u, Mu \rangle \geq 0$. The time evolution of its expected value is estimated by the following lemma, whose proof rests on the restriction to initial data and external potentials that are spherically symmetric.

Lemma 2. *Let the assumptions on m , μ , and V stated in Theorem 1 be satisfied, and suppose, further, that $u_0(x) \in C_c^\infty(\mathbb{R}^3)$ is spherically symmetric. Then the map $t \mapsto \langle u(t), Mu(t) \rangle$ satisfies the inequality*

$$\frac{1}{2} \frac{d}{dt} \langle u(t), Mu(t) \rangle \leq \langle u(t), Au(t) \rangle + C, \quad (3.14)$$

for all $0 \leq t < T$, where C is some constant depending only on $\|u_0\|_2^2$ and V .

Proof of Lemma 2. As in the proof of Lemma 1, we also write u instead of $u(t, x)$, etc.

Since $u(t) \in H^2(\mathbb{R}^3)$ and $|x|^2 u(t) \in L^2(\mathbb{R}^3)$ for all $t \in [0, T]$, by Step 1 discussed above, Heisenberg's formula (3.2) can be applied in a rigorous way. To see this, we note, e.g., that $[x, \sqrt{p^2 + m^2}] = ip/\sqrt{p^2 + m^2}$ holds, which shows that $|\langle u, M\sqrt{p^2 + m^2}u \rangle| \leq \| |x|^2 u \|_2 \|u\|_{H^2} + \| |x|u \|_2 \|u\|_{H^1}$ is finite.

Applying now (3.2), we find that

$$\frac{d}{dt} \langle u, Mu \rangle = i \langle u, [\sqrt{p^2 + m^2}, M]u \rangle + i \langle u, [V, M]u \rangle + i \langle u, [V_s, M]u \rangle, \quad (3.15)$$

where V_s is defined in (3.3). Next, we note the identity

$$\begin{aligned} [V, M] &= [V, x\sqrt{p^2 + m^2}x] = Vx\sqrt{p^2 + m^2}x - x\sqrt{p^2 + m^2}xV \\ &= [Vx^2, \sqrt{p^2 + m^2}] + \frac{ip}{\sqrt{p^2 + m^2}} \cdot xV + Vx \cdot \frac{ip}{\sqrt{p^2 + m^2}}, \end{aligned} \quad (3.16)$$

which is, of course, also true if V is replaced by V_s . Using (3.16) and applying Lemma 4 in Sect. A.2, we conclude that the first commutator in (3.16) is a bounded operator on $L^2(\mathbb{R}^3)$ with norm

$$\| [Vx^2, \sqrt{p^2 + m^2}] \|_{L^2 \rightarrow L^2} \leq C \|\nabla(Vx^2)\|_\infty \leq C_V, \quad (3.17)$$

for some finite constant C_V , thanks to Assumption 1 for $V(x)$. Furthermore, by noticing that $ip/(p^2 + m^2)^{1/2}$ is a bounded operator on $L^2(\mathbb{R}^3)$ and that $\|Vx\|_\infty \leq C$ holds (which follows from Assumption 1), we deduce that the remaining commutators in (3.16) are also bounded operators on $L^2(\mathbb{R}^3)$. Thus, we can estimate

$$|\langle u, [V, M]u \rangle| \leq C_V \|u\|_2^2 = C_V \|u_0\|_2^2, \quad (3.18)$$

by conservation of the L^2 -norm, with some finite constant C_V .

To find an estimate for $[V_s, M]$ similar to (3.18), we now show that

$$\|\nabla(x^2 V_s)\|_\infty + \| |x| V_s \|_\infty \leq C_0 \|u_0\|_2^2, \quad (3.19)$$

with some universal constant C_0 . In fact, we prove the pointwise estimates

$$|V_s(t, x)| \leq \frac{\|u_0\|_2^2}{|x|}, \quad |\nabla V_s(t, x)| \leq \frac{\|u_0\|_2^2}{|x|^2}, \quad (3.20)$$

for all $x \in \mathbb{R}^3$ and $0 \leq t < T$, which obviously imply (3.19).

To prove (3.20), we recall from (3.3) that $V_s = V_s(t, x)$ is given by

$$V_s(t, x) = - \int_{\mathbb{R}^3} \frac{e^{-\mu|x-y|}}{|x-y|} |u(t, y)|^2 dy, \quad (3.21)$$

Since $u = u(t, x)$ is spherically symmetric (and hence so is $V_s(t, x)$), we can write $u(t, r)$ and $V_s(t, r)$ for $r = |x|$ with some abuse of notation. For the moment, let us assume that $\mu = 0$ in (3.21). Then, due to the spherical symmetry of $u = u(t, r)$, we can invoke *Newton's theorem* (see, e.g., [LL01, Theorem 9.7]), which yields the bound

$$|V_s(t, r)| \leq \frac{1}{r} \int_{\mathbb{R}^3} |u(t, |y|)|^2 dy = \frac{\|u_0\|_2^2}{r}, \quad (3.22)$$

by conservation of the L^2 -norm. Moreover, we have the following explicit formula

$$V_s(t, r) = -\frac{1}{r} \int_{|y| \leq r} |u(t, |y|)|^2 dy - \int_{|y| > r} \frac{|u(t, |y|)|^2}{|y|} dy. \quad (3.23)$$

Taking the derivative with respect to $r = |x|$ (which is allowed by the regularity properties of $|u(t, r)|^2$), we obtain that

$$\frac{d}{dr} V_s(t, r) = -\frac{1}{r^2} \int_{|y| \leq r} |u(t, |y|)|^2 dy. \quad (3.24)$$

Hence we conclude that

$$|\nabla V_s(t, x)| = \left| \frac{d}{dr} V_s(t, r) \right| \leq \frac{1}{r^2} \int_{|y| \leq r} |u(t, |y|)|^2 dy \leq \frac{\|u_0\|_2^2}{r^2}. \quad (3.25)$$

This proves estimate (3.20) for $\mu = 0$.

It remains to show (3.20) for $\mu > 0$. Here we apply the following simple trick: We see that V_s satisfies the Yukawa equation, $(\Delta - \mu^2)V_s = 4\pi|u|^2$, which can be rewritten as follows

$$\Delta V_s = 4\pi|u|^2 + \mu^2 V_s, \quad \text{with } V_s \rightarrow 0 \text{ as } |x| \rightarrow \infty. \quad (3.26)$$

By linearity, we have that $V_s = V_1 + V_2$, where $\Delta V_1 = 4\pi|u|^2$ and $\Delta V_2 = \mu^2 V_s$, with V_1 and V_2 vanishing at infinity. Since $V_s \leq 0$ by (3.21), we have that $V_2 \geq 0$. Estimating V_1 has already been accomplished above. To bound V_2 and ∇V_s , we proceed in the same way replacing only $|u|^2$ by $\frac{\mu^2}{4\pi} V_s$. This yields

$$|V_2(t, r)| \leq \frac{1}{r} \frac{\mu^2}{4\pi} \int_{\mathbb{R}^3} |V_s(t, |y|)| dy \leq \frac{1}{r} \frac{\mu^2}{4\pi} \left\| \frac{e^{-\mu r}}{r} \right\|_1 \|u\|_2^2 = \frac{\|u_0\|_2^2}{r}, \quad (3.27)$$

where we use Young's inequality and the fact $\int_{\mathbb{R}^3} e^{-\mu r} r^{-1} dx = 4\pi/\mu^2$ whenever $\mu > 0$. Similarly, we find that

$$\left| \frac{d}{dr} V_2(t, r) \right| \leq \frac{\|u_0\|_2^2}{r^2}. \quad (3.28)$$

Finally, we note that $V_1 \leq V \leq V_2$, as well as $\frac{d}{dr} V_1 \leq \frac{d}{dr} V \leq \frac{d}{dr} V_2$ hold, and we eventually arrive at (3.19).

Returning to (3.15), we may now proceed as follows

$$\frac{d}{dt} \langle u, Mu \rangle \leq i \langle u, [\sqrt{p^2 + m^2}, M]u \rangle + C, \quad (3.29)$$

for some constant C depending only on V and on $\|u_0\|_2^2$. To complete our proof of Lemma 2, we note that, by a simple calculation,

$$[\sqrt{p^2 + m^2}, M] = [\sqrt{p^2 + m^2}, x\sqrt{p^2 + m^2}x] = -i(x \cdot p + p \cdot x) = -2iA, \quad (3.30)$$

where A is the generator of dilatations introduced in (3.6). This completes our proof of Lemma 2. \square

Step 4: Completing the Proof of Theorem 1

The assertion of Theorem 1 follows by integrating and combining the differential inequalities (3.8) and (3.14) stated in Lemma 1 and 2, respectively. We find that

$$0 \leq \langle u(t), Mu(t) \rangle \leq 2(\mathcal{E}(u_0) + \frac{1}{2}\|U\|_\infty\|u_0\|_2^2)t^2 + C_1t + C_2, \quad \text{for } t \in [0, T], \quad (3.31)$$

where C_1 and C_2 are some finite constants. If $\mathcal{E}(u_0) < -\frac{1}{2}\|U\|_\infty\|u_0\|_2^2$, we conclude that the maximal time of existence obeys $T < \infty$. By the blow-up alternative proved in [Len05], this implies that $\|u(t, \cdot)\|_{H^{1/2}} \rightarrow \infty$, as $t \nearrow T$. This completes the proof of Theorem 1.

3.2 Proof of Theorem 2

We set $m = \mu = 0$ and $V \equiv 0$ in (2.1). As recalled in the introduction (see references given there), we have that any ground state $Q(x)$ is spherically symmetric with respect to some point x_0 , which we can assume to be the origin. Recalling that $\mathcal{E}(Q) = 0$ holds, we see by elementary arguments that

$$\begin{aligned} \mathcal{E}((1 + \delta)Q) &= \mathcal{E}(Q) + \langle \mathcal{E}'(Q), \delta Q \rangle_{H^{-1/2}, H^{1/2}} + \mathcal{O}(\delta^2) \\ &= -\delta\omega\mathcal{N}(Q) + \mathcal{O}(\delta^2) < 0, \end{aligned} \quad (3.32)$$

for $\delta > 0$ sufficiently small. By density of $C_c^\infty(\mathbb{R}^3) \subset H^{1/2}(\mathbb{R}^3)$ and by continuity of \mathcal{E} , there exists $u_0 \in C_c^\infty(\mathbb{R}^3)$ such that $\mathcal{E}(u_0) < 0$, and u_0 is arbitrarily close (in $H^{1/2}$ -norm) to $(1 + \delta)Q$. Thus, given any $\varepsilon > 0$, we can choose $\delta > 0$ sufficiently small and $u_0 \in C_c^\infty(\mathbb{R}^3)$ in such a way that $\|u_0 - Q\|_{H^{1/2}} < \varepsilon$ and $\mathcal{E}(u_0) < 0$ holds. Since u_0 is spherically symmetric and $u \in C_c^\infty(\mathbb{R}^3)$, we can invoke Theorem 1 to find that $u(t)$ blows up within finite time. This completes the proof of Theorem 2.

A Appendix

A.1 Propagation of Moments in x -Space

Lemma 3. *Let $m \geq 0$, $\mu \geq 0$ in (2.1), and suppose that V and $u_0(x)$ satisfy the assumptions of Theorem 1 (except for spherical symmetry). Then we have that $|x|^2u(t) \in L^2(\mathbb{R}^3)$ holds on $[0, T)$.*

Proof of Lemma 3. First, we show that the second moment of $u(t)$ in x -space stays finite along the flow, i. e.,

$$|x|u(t) \in L^2(\mathbb{R}^3), \quad \text{on } [0, T). \quad (\text{A.1})$$

To prove this claim, we introduce the regularized quantity

$$f_\varepsilon(t) := \langle u(t), |x|^2e^{-2\varepsilon|x|}u(t) \rangle, \quad (\text{A.2})$$

where $\varepsilon > 0$. For $f_\varepsilon(t)$ and $t \in [0, T)$, a well-defined calculation yields

$$\begin{aligned}
f_\varepsilon(t) &= f_\varepsilon(0) + \int_0^t f'_\varepsilon(s) \, ds \\
&= f_\varepsilon(0) + i \int_0^t \langle u(s), [\sqrt{p^2 + m^2}, |x|^2 e^{-2\varepsilon|x|}] u(s) \rangle \, ds \\
&= f_\varepsilon(0) + i \int_0^t \langle u(s), x e^{-\varepsilon|x|} \cdot [\sqrt{p^2 + m^2}, x e^{-\varepsilon|x|}] \\
&\quad + [\sqrt{p^2 + m^2}, x e^{-\varepsilon|x|}] \cdot x e^{-\varepsilon|x|} u(s) \rangle \, ds.
\end{aligned} \tag{A.3}$$

Invoking the Cauchy-Schwarz inequality and noting that $\sqrt{f_\varepsilon(t)} = \| |x| e^{-\varepsilon|x|} u(t) \|_2$, we obtain

$$f_\varepsilon(t) \leq f_\varepsilon(0) + C \int_0^t \| [\sqrt{p^2 + m^2}, |x| e^{-\varepsilon|x|}] u(s) \|_2 \sqrt{f_\varepsilon(s)} \, ds. \tag{A.4}$$

By Lemma 4 in Appendix A.2 and the fact that $|x| e^{-\varepsilon|x|}$ has a uniformly bounded gradient, we have that

$$\| [\sqrt{p^2 + m^2}, |x| e^{-\varepsilon|x|}] \|_{L^2 \rightarrow L^2} \leq C \| \nabla (|x| e^{-\varepsilon|x|}) \|_\infty \leq C. \tag{A.5}$$

Thus, it is a bounded operator from $L^2(\mathbb{R}^3)$ into itself (and the bound on its operator norm is independent of ε).

Let us now return to (A.4). Appealing to conservation of the L^2 -norm of $u(t)$, we conclude that

$$f_\varepsilon(t) \leq f_\varepsilon(0) + 2C \int_0^t \sqrt{f_\varepsilon(s)} \, ds, \tag{A.6}$$

This estimate is easily seen to imply that

$$\sqrt{f_\varepsilon(t)} \leq f_\varepsilon(0) + C \int_0^t \, ds = f_\varepsilon(0) + Ct. \tag{A.7}$$

By monotone convergence, we can take the limit $\varepsilon \rightarrow 0$, which leads to

$$\sqrt{f(t)} \leq f(0) + Ct, \tag{A.8}$$

using that $f(0) = \| |x| u_0 \|_2 \leq C(\|u_0\|_2 + \| |x|^2 u_0 \|_2)$ is finite because u_0 and $|x|^2 u_0$ belong to $L^2(\mathbb{R}^3)$. This proves (A.1).

Next, we prove a similar statement for the fourth moment, i. e.,

$$|x|^2 u(t) \in L^2(\mathbb{R}^3), \quad \text{on } [0, T). \tag{A.9}$$

This follows in fact from (A.1), as we now show. In analogy to (A.2), we define

$$g_\varepsilon(t) := \langle u(t), |x|^4 e^{-2\varepsilon|x|} u(t) \rangle, \tag{A.10}$$

where $\varepsilon > 0$. A calculation similar to (A.3) yields that

$$g_\varepsilon(t) \leq g_\varepsilon(0) + 2C \int_0^t \| [\sqrt{p^2 + m^2}, |x|^2 e^{-\varepsilon|x|}] u(s) \|_2 \sqrt{g_\varepsilon(s)} \, ds. \tag{A.11}$$

Further evaluation of the commutator leads to

$$\begin{aligned} \|\llbracket \sqrt{p^2 + m^2}, |x|^2 e^{-\varepsilon|x|} \rrbracket u\|_2 &= \|(e^{-\varepsilon|x|} x \cdot [\sqrt{p^2 + m^2}, x] + [\sqrt{p^2 + m^2}, x e^{-\varepsilon|x|}] \cdot x) u\|_2 \\ &\leq C(\|x \cdot \frac{p}{\sqrt{p^2 + m^2}} u\|_2 + \|x|u\|_2), \end{aligned} \quad (\text{A.12})$$

using that $[\sqrt{p^2 + m^2}, x]$ and $[\sqrt{p^2 + m^2}, x e^{-\varepsilon|x|}]$ are bounded operators on $L^2(\mathbb{R}^3)$, see Lemma 4. Next, by a simple commutator calculation, we find that

$$x \cdot \frac{p}{\sqrt{p^2 + m^2}} = \frac{p}{\sqrt{p^2 + m^2}} \cdot x + i \left(\frac{3}{(p^2 + m^2)^{1/2}} - \frac{p^2}{(p^2 + m^2)^{3/2}} \right). \quad (\text{A.13})$$

Hence the first term on the right-hand side of (A.12) can be estimated as follows

$$\|x \cdot \frac{p}{\sqrt{p^2 + m^2}} u\|_2 \leq C(\|\frac{p}{\sqrt{p^2 + m^2}} \cdot x u\|_2 + \|\frac{1}{|p|} u\|_2) \leq C\|x|u\|_2, \quad (\text{A.14})$$

using Hardy's inequality, $\|x|^{-1} u\|_2 \leq C\|p|u\|_2$, which also holds when p and x are interchanged, by Fourier duality.

Commenting the proof given above for (A.14), we mention that we could also control appeal to weighted L^2 -estimates for the singular integral operator, $T = p/\sqrt{p^2 + m^2}$, and noting that the weight $\omega(x) = |x|^2$ belongs the class A_2 ; see, e. g., [Ste93, Section V.5].

In summary, we conclude that estimates (A.11) and (A.14) imply that

$$g_\varepsilon(t) \leq g_\varepsilon(0) + 2C \int_0^t (\|u(s)\|_2 + \|x|u(s)\|_2) \sqrt{g_\varepsilon(s)} \, ds. \quad (\text{A.15})$$

By conservation of $\|u(t)\|_2$ and estimate (A.8) for $\sqrt{f(t)} = \|x|u(t)\|_2$, we deduce, in a similar fashion as above, that

$$\sqrt{g_\varepsilon(t)} \leq g_\varepsilon(0) + C \int_0^t (1 + s) \, ds \quad (\text{A.16})$$

holds. This bound finally leads to

$$\sqrt{g(t)} \leq g(0) + C(t + \frac{t^2}{2}), \quad (\text{A.17})$$

when passing to the limit $\varepsilon \rightarrow 0$, by monotone convergence. The proof of Lemma 3 is now complete. \square

A.2 Commutator Estimate

Lemma 4. *Let $m \geq 0$, and suppose that $f(x)$ is a locally integrable. If the distributional gradient, $\nabla f(x)$, is an $L^\infty(\mathbb{R}^3)$ vector-valued function, then we have that*

$$\|\llbracket \sqrt{-\Delta + m^2}, f \rrbracket\|_{L^2 \rightarrow L^2} \leq C \|\nabla f\|_\infty,$$

for some constant C independent of m .

Remark. Although this result can be deduced by means of Calderón–Zygmund theory for singular operators and its consequences for pseudo-differential operators (see, e. g., [Ste93, Section VII.3]), we present an elementary proof which makes good use of the spectral theorem, enabling us to write the commutator in a convenient way.

Proof. For $m \geq 0$, we put

$$A := \sqrt{p^2 + m^2}, \quad \text{where } p := -i\nabla. \quad (\text{A.18})$$

Since A is a self-adjoint operator on $L^2(\mathbb{R}^3)$ (with domain $H^1(\mathbb{R}^3)$), functional calculus yields the formula

$$A^{-1} = \frac{1}{\pi} \int_0^\infty \frac{1}{\sqrt{s} A^2 + s} ds. \quad (\text{A.19})$$

Due to this fact and $A = A^2 A^{-1}$, we obtain the formula

$$[A, f] = \frac{1}{\pi} \int_0^\infty \frac{\sqrt{s}}{A^2 + s} [A^2, f] \frac{ds}{A^2 + s}. \quad (\text{A.20})$$

Clearly, we have that $[A^2, f] = [p^2, f] = p \cdot [p, f] + [p, f] \cdot p$, which leads to

$$[A, f] = \frac{1}{\pi} \int_0^\infty \frac{\sqrt{s}}{p^2 + m^2 + s} (p \cdot [p, f] + [p, f] \cdot p) \frac{ds}{p^2 + m^2 + s}. \quad (\text{A.21})$$

Moreover, since $[p, f] = -i\nabla f$ holds, we have that

$$\left\| \left[\frac{1}{p^2 + m^2 + s}, [p, f] \right] \right\|_{L^2 \rightarrow L^2} \leq \frac{2}{s} \|\nabla f\|_\infty. \quad (\text{A.22})$$

Hence we can estimate, for arbitrary Schwartz functions $\xi, \eta \in \mathcal{S}(\mathbb{R}^3)$, as follows:

$$\begin{aligned} & \left| \left\langle \xi, \int_0^\infty \frac{\sqrt{s}}{p^2 + m^2 + s} ([p, f] \cdot p) \frac{ds}{p^2 + m^2 + s} \eta \right\rangle \right| \quad (\text{A.23}) \\ & \leq \left| \left\langle [p, f] \xi, p \int_0^\infty \frac{\sqrt{s} ds}{(p^2 + m^2 + s)^2} \eta \right\rangle \right| + \left| \left\langle \xi, \int_0^\infty \left[\frac{1}{p^2 + m^2 + s}, [p, f] \right] \cdot \frac{p\sqrt{s} ds}{p^2 + m^2 + s} \eta \right\rangle \right| \\ & \leq \|[p, f] \xi\|_2 \left\| \int_0^\infty \frac{p\sqrt{s} ds}{(p^2 + m^2 + s)^2} \eta \right\|_2 + 2\|\xi\|_2 \|\nabla f\|_\infty \left\| \int_0^\infty \frac{p ds}{\sqrt{s}(p^2 + m^2 + s)} \eta \right\|_2. \end{aligned}$$

Evaluation of the s -integrals yields

$$(\text{A.23}) \leq C \|\nabla f\|_\infty \|\xi\|_2 \left\| \frac{p}{\sqrt{p^2 + m^2}} \eta \right\|_2 \leq C \|\nabla f\|_2 \|\xi\|_2 \|\eta\|_2. \quad (\text{A.24})$$

The same estimate holds if $[p, f] \cdot p$ is replaced by $p \cdot [p, f]$ in (A.23). Thus, we have found that

$$|\langle \xi, [A, f] \eta \rangle| \leq C \|\nabla f\|_\infty \|\xi\|_2 \|\eta\|_2, \quad \text{for } \xi, \eta \in \mathcal{S}(\mathbb{R}^3), \quad (\text{A.25})$$

with some constant C independent of m . Since $\mathcal{S}(\mathbb{R}^3)$ is dense in $L^2(\mathbb{R}^3)$, the assertion for the L^2 -boundedness of $[A, f]$ now follows. This completes the proof of Lemma 4. \square

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