Absolutely continuous spectrum for the Anderson model on a tree:

a geometric proof of Klein's theorem

Richard Froese

Department of Mathematics
University of British Columbia
Vancouver, British Columbia, Canada

David Hasler

Department of Mathematics
University of Virginia
Charlottesville, Virginia, USA

Wolfgang Spitzer

Department of Physics
International University Bremen
Bremen, Germany

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Abstract

We give a new proof of a version of Klein's theorem on the existence of absolutely continuous spectrum for the Anderson model on the Bethe Lattice at weak disorder.

Model and Statement of Main Results

It is widely believed that the Anderson model [An] should exhibit absolutely continuous spectrum at weak disorder in dimensions three and higher. But it is only for the Bethe lattice $\mathbb{B}$, or Cayley tree, that this has been established. The first proof was given by Klein [K1, K2], and his remained the only result of this kind until the recent work of Aizenman, Sims and Warzel [ASW]. These authors proved a stability result for absolutely continuous spectrum for the Anderson model on $\mathbb{B}$ that implies the existence of an absolutely continuous component in the spectrum for perturbations of the Anderson model on $\mathbb{B}$, also in the presence of a periodic background potential.

For the related problem of proving absolutely continuous spectrum for slowly decaying potentials there has been recent progress ([D], [KLS], [SS]). It is interesting to note that for the Bethe lattice, localization for large disorder has not yet been established at the band edge, but only for strictly larger energies (see [Ai], [AM]). For more information about this model, and further references we recommend the discussion in [ASW].

In this paper we give a new proof of a variant of Klein's theorem, Theorem 1 below. Our proof is quite different from either of the two previous approaches. It is based on [FHS] where we proved existence of absolutely continuous spectrum for a class of deterministic potentials whose radial behaviour was restricted only by an ℓ^∞ bound. That proof was based on the contracting properties of the map ϕ, defined below, that arises in the recurrence relation for the Green's function, thought of as a map between hyperbolic spaces. Our Lemma 4 is a version of this, adapted to the probabilistic setting. Klein is able to handle some random potentials that we cannot, since we require that the single site distribution has a finite fourth moment. On the other hand, our Theorem 6 quantifies how the finiteness of higher moments of the single site distribution leads to more decay in the probability distribution of the Green's function.

The Bethe Lattice, $\mathbb{B}$, or Cayley tree of degree k is the infinite connected graph with no closed loops where each vertex has k nearest neighbours. In this paper, we set $k = 3$. We believe a similar proof should work for all k. But a proof along our lines for larger k would involve, at the least, greater notational complexity.

The Anderson model on $\mathbb{B}$ is given by the random Hamiltonian

$$H = \Delta + q$$

on the Hilbert space

$$\mathcal{H} = \ell^2(\mathbb{B}) = \left\{ \varphi : \mathbb{B} \to \mathbb{C} \,:\, \sum_{x\in\mathbb{B}} |\varphi(x)|^2 < \infty \right\} ,$$

where q denotes a random potential, such that for each $x \in \mathbb{B}$, $q(x)$ is an independently distributed real random variable with probability distribution ν, and Δ is the Laplacian defined by

$$(\Delta\varphi)(x) = \sum_{y:\mathrm{d}(x,y)=1} \varphi(y) \,.$$

Here $\mathrm{d}(x, y)$ denotes the distance in the graph, that is, the number of edges in the shortest path joining x and y. The spectrum of the free Laplacian is $\sigma(\Delta) = [-2\sqrt{2}, 2\sqrt{2}]$. The main theorem which we will prove is the following.

Theorem 1 *For any E, with $0 < E < 2\sqrt{2}$, there exist $\delta_1 > 0$ and $\delta_2 > 0$, such that for all ν with*

$$\int_{|q|\geq\delta_1} (1 + |q|^4)\, d\nu(q) \leq \delta_2 \,,$$

the spectrum of H is purely absolutely continuous in $[-E,E]$ with probability one, i.e., we have almost surely

$$\Sigma_{\rm ac}\cap[-E,E]=[-E,E]\ ,\ \ \Sigma_{\rm pp}\cap[-E,E]=\emptyset\ ,\ \ \Sigma_{\rm sc}\cap[-E,E]=\emptyset\ .$$

As was shown in [K], Theorem 1 follows from the following fact. Let $R(E,\epsilon)$ be the strip in the complex plane defined by

$$R(E,\epsilon)=\{z\in\mathbb{H}\,:\,{\rm Re}z\in[-E,E],\,0<{\rm Im}z\le\epsilon\}\,.$$

Theorem 2 *Let $x\in\mathbb{B}$. Under the hypotheses of Theorem 1,*

$$\sup_{\lambda\in R(E,\epsilon)}\mathbb{E}\left(\left|\langle x|(H-\lambda)^{-1}|x\rangle\right|^2\right)<\infty$$

for some $\epsilon>0$.

Here is a brief outline of the paper. Our main technical result is the contraction estimate in Lemma 4. This and the companion result Lemma 5 are used in Theorem 6 to prove that the probability distribution on the hyperbolic plane for $Z^{(x|y)}(\lambda)$, defined below, decays at infinity. This decay then implies the decay of the Green's function required in Theorem 2.

Let $\mathbb{H}=\{z\in\mathbb{C}:{\rm Im}(z)>0\}$ denote the complex upper half plane. For convenience we fix an arbitrary site in $\mathbb{B}$ to be the origin and denote it by 0. Given two nearest neighbour sites $x,y\in\mathbb{B}$, we will denote by $\mathbb{B}^{(x|y)}$ the graph obtained by removing from $\mathbb{B}$ the branch emanating from x that passes through y. We will write $H^{(x|y)}$ for H when restricted to $\mathbb{B}^{(x|y)}$ and set

$$G^{(x|y)}(\lambda)=\langle x|(H^{(x|y)}-\lambda)^{-1}|x\rangle\ .$$

We will use the following recursion relations. For a proof see [K] or [FHS].

Proposition 3 *For any $\lambda\in\mathbb{H}$,*

$$G(0,0,\lambda)=\langle 0|(H-\lambda)^{-1}|0\rangle=-\left(\sum_{x:{\rm dist}(x,0)=1}G^{(x|0)}(\lambda)+\lambda-q(0)\right)^{-1}\ ,\tag{1}$$

and for any two nearest neighbour sites $x,y\in\mathbb{B}$

$$G^{(x|y)}(\lambda)=-\left(\sum_{x':\,{\rm d}(x,x')=1,x'\neq y}G^{(x'|x)}(\lambda)+\lambda-q(x)\right)^{-1}\ .\tag{2}$$

It will turn out to be convenient to study the sum of two Green's functions, i.e., for two nearest neighbour sites $x,y\in\mathbb{B}$ we set

$$Z^{(x|y)}(\lambda)=\sum_{x':\,{\rm d}(x,x')=1,x'\neq y}G^{(x'|x)}(\lambda)\ .\tag{3}$$

Using the recursion relation for $G^{(x|y)}(\lambda)$ we obtain the following recursion relation

$$Z^{(x|y)}(\lambda) = -\sum_{x' \colon \mathrm{d}(x,x')=1, x'\neq y} \Big(Z^{(x'|x)}(\lambda) + \lambda - q(x') \Big)^{-1} .$$

This leads to the investigation of the transformation $\phi : \mathbb{H} \times \mathbb{H} \times \mathbb{R} \times \mathbb{R} \times \mathbb{H} \to \mathbb{H}$ defined by

$$\phi(z_1, z_2, q_1, q_2, \lambda) = \frac{-1}{z_1 + \lambda - q_1} + \frac{-1}{z_2 + \lambda - q_2} . \tag{4}$$

If $\mathrm{Im}\lambda > 0$, the transformation $z \mapsto \phi(z, z, 0, 0, \lambda)$ has a unique fixed point, z_λ, in the upper half plane, i.e, with $\mathrm{Im} z_\lambda > 0$. Explicitly,

$$z_\lambda = -\lambda/2 + \sqrt{(\lambda/2)^2 - 2} ,$$

where we will always make the choice $\mathrm{Im}\sqrt{\cdot} \geq 0$ (and $\sqrt{a} > 0$ for $a > 0$). This fixed point as a function of $\lambda \in \mathbb{H}$ extends continuously onto the real axis. This extension yields for $\mathrm{Im}(\lambda) = 0$ and $|\lambda| < 2\sqrt{2}$ the fixed point

$$z_\lambda = -\lambda/2 + i\sqrt{2 - (\lambda/2)^2} ,$$

lying on an arc of the circle $|z| = \sqrt{2}$. When $\mathrm{Im}(\lambda) = 0$ and $|\lambda| \leq E < 2\sqrt{2}$, the arc is strictly contained in the upper half plane. Thus when λ lies in the strip $R(E, \epsilon)$ with $0 < E < 2\sqrt{2}$ and ϵ sufficiently small, $\mathrm{Im}(z_\lambda)$ is bounded below and $|z_\lambda|$ is bounded above by a positive constant.

We will use the weight function $\mathrm{cd}(z)$ defined by

$$\mathrm{cd}(z) = 2\,\mathrm{Im}(z_\lambda)(\cosh(\mathrm{dist}_{\mathbb{H}}(z, z_\lambda)) - 1) = \frac{|z - z_\lambda|^2}{\mathrm{Im}(z)} . \tag{5}$$

Up to constants, $\mathrm{cd}(z)$ is the hyperbolic cosine of the hyperbolic distance from z to z_λ, provided $\lambda \in R(E, \epsilon)$ with $0 < E < 2\sqrt{2}$ and ϵ sufficiently small. This notation suppresses the λ dependence.

To prove Theorem 2, we will study the following function

$$\mu_{3,p}(z_1, z_2, z_3, q_1, q_2, q_3, q_4, \lambda) = \sum_\sigma \frac{\mathrm{cd}^p(\phi(z_{\sigma_1}, \phi(z_{\sigma_2}, z_{\sigma_3}, q_{\sigma_2}, q_{\sigma_3}, \lambda), q_{\sigma_1}, q_4, \lambda))}{\mathrm{cd}^p(z_1) + \mathrm{cd}^p(z_2) + \mathrm{cd}^p(z_3)} , \tag{6}$$

where σ runs over the cyclic permutations of $(1, 2, 3)$, i.e., $(\sigma_1, \sigma_2, \sigma_3) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$. Note that $\mu_{3,p}$ is well defined as long as $(z_1, z_2, z_3) \neq (z_\lambda, z_\lambda, z_\lambda)$. The proof of Theorem 1 is based on the following bounds, which will be proved in the next section. For small $|q_i|$ we have

Lemma 4 *For any E, $0 < E < 2\sqrt{2}$ and any $p > 1$, there exist positive constants ϵ , δ, ϵ_0 and a compact set $K \subset \mathbb{H}^3$ such that*

$$\mu_{3,p}\big|_{K^c \times [-\delta,\delta]^4 \times R(E,\epsilon_0)} \leq 1 - \epsilon . \tag{7}$$

Here K^c denotes the complement $\mathbb{H}^3 \backslash K$.

This theorem also holds for $p = 1$, but the proof is more involved. We will also need the following bounds, that hold for all $|q_i|$.

Lemma 5 *For any E, $0 < E < 2\sqrt{2}$ and any $p \geq 1$, there exist positive constants ϵ_0, C and a compact set $K \subset \mathbb{H}^3$ such that*

$$\mu_{3,p}\big|_{K^c \times \mathbb{R}^4 \times R(E,\epsilon_0)} \leq C\left(1 + \sum_{i=1}^{4} |q_i|^{2p}\right). \tag{8}$$

Similarly, if we define

$$\mu'_{3,p}(z_1, z_2, z_3, q, \lambda) = \frac{\mathrm{cd}(-(z_1 + z_2 + z_3 + \lambda - q)^{-1})^p}{\mathrm{cd}(z_1)^p + \mathrm{cd}(z_2)^p + \mathrm{cd}(z_3)^p},$$
$$\mu'_{1,p}(z, q, \lambda) = \frac{\mathrm{cd}(-(z + \lambda - q)^{-1})^p}{\mathrm{cd}(z)^p},$$

then

$$\mu'_{3,p}\big|_{K^c \times \mathbb{R}^4 \times R(E,\epsilon_0)} \leq C\left(1 + |q|^{2p}\right),$$
$$\mu'_{1,p}\big|_{K^c \times \mathbb{R}^4 \times R(E,\epsilon_0)} \leq C\left(1 + |q|^{2p}\right).$$

Let ρ be the probability distribution for $Z^{(0|x)}(\lambda)$ on the hyperbolic plane given by

$$\rho(A) = \mathrm{Prob}(Z^{(0|x)}(\lambda) \in A).$$

Although it is suppressed in the notation, ρ depends on λ, and for $\mathrm{Im}(\lambda) > 0$ the support of ρ is bounded. This follows, for example from the fact that it is contained in the range of ϕ. Given Lemma 4 and Lemma 5, we can prove that the decay of ρ at infinity is preserved as $\mathrm{Im}(\lambda)$ becomes small, provided ν has enough finite moments and is concentrated near 0.

Theorem 6 *Let x be a nearest neighbour of 0. For any E, $0 < E < 2\sqrt{2}$ and $p > 1$, there exist $\delta_1 > 0$, $\delta_2 > 0$ and $\epsilon > 0$, such that for all ν satisfying*

$$\int_{|q| \geq \delta_1} (1 + |q|^{2p}) d\nu(q) \leq \delta_2,$$

we have

$$\sup_{\lambda \in R(E,\epsilon)} \mathbb{E}\left(\mathrm{cd}^p(Z^{(0|x)}(\lambda))\right) < \infty.$$

Proof: Let δ_1 be the δ given by Lemma 4, and choose ϵ_0 and K that work in both Lemma 4 and Lemma 5. For $(z_1, z_2, z_3) \in K^c$, we estimate

$$\begin{aligned}
&\int_{\mathbb{R}^4} \mu_{3,p}(z_1, z_2, z_3, q_1, \ldots, q_4, \lambda)\, d\nu(q_1) \ldots d\nu(q_4) \\
&\quad \leq (1-\epsilon) \int_{[-\delta_1,\delta_1]^4} d\nu(q_1) \ldots d\nu(q_4) + C \int_{\mathbb{R}^4 \setminus [-\delta_1,\delta_1]^4} (1 + \sum_{i=1}^{4} |q_i|^{2p})\, d\nu(q_1) \ldots d\nu(q_4) \\
&\quad \leq (1-\epsilon) + C\left(1 + 4M_{2p} - \left(\int_{[-\delta_1,\delta_1]} d\nu(q)\right)^4 - 4\left(\int_{[-\delta_1,\delta_1]} d\nu(q)\right)^3 \int_{[-\delta_1,\delta_1]} |q|^{2p} d\nu(q)\right) \\
&\quad \leq (1-\epsilon) + C\big(1 + 4M_{2p} - (1-\delta_2)^4 - 4(1-\delta_2)^3(M_{2p} - \delta_2)\big) \\
&\quad \leq 1 - \epsilon/2
\end{aligned}$$

provided δ_2 is sufficiently small. Here M_{2p} denotes the moment $\int |q|^{2p}\, d\nu(q)$.

The recursion relation for $Z^{(0|x)}(\lambda)$ implies that for any continuous function $w(z)$

$$\int_{\mathbb{H}} w(z) d\rho(z) = \int_{\mathbb{H}\times\mathbb{H}\times\mathbb{R}\times\mathbb{R}} w(\phi(z_1, z_2, q_1, q_2, \lambda))\, d\rho(z_1) d\rho(z_2)\, d\nu(q_1) d\nu(q_2).$$

Using this relation (twice) and the estimate above, we obtain for $\lambda \in R(E, \epsilon_0)$,

$$\begin{aligned}
&\mathbb{E}\left(\mathrm{cd}^p(Z^{(0|x)})(\lambda)\right) \\
&\quad = \int \mathrm{cd}^p(z)\, d\rho(z) \\
&\quad = \int \mathrm{cd}^p(\phi(z_1, z_2, q_1, q_2, \lambda)\, d\rho(z_1) d\rho(z_2)\, d\nu(q_1) d\nu(q_2) \\
&\quad = \int \mathrm{cd}^p(\phi(z_1, \phi(z_2, z_3, q_2, q_3, \lambda), q_1, q_4, \lambda))\, d\rho(z_1) \ldots d\rho(z_3)\, d\nu(q_1) \ldots d\nu(q_4) \\
&\quad = \int \frac{1}{3} \sum_{\sigma} \mathrm{cd}^p\left(\phi(z_{\sigma_1}, \phi(z_{\sigma_2}, z_{\sigma_3}, q_{\sigma_2}, q_{\sigma_3}, \lambda) q_{\sigma_1}, q_{\sigma_4}, \lambda))\right) d\rho(z_1) \ldots d\rho(z_3)\, d\nu(q_1) \ldots d\nu(q_4) \\
&\quad = \frac{1}{3} \int_{K^c} \left(\int_{\mathbb{R}^4} \mu_{3,p}(z_1, z_2, z_3, q_1, \ldots, q_4, \lambda)\, d\nu(q_1) \ldots d\nu(q_4) \right) \\
&\qquad \times \left(\mathrm{cd}^p(z_1) + \mathrm{cd}^p(z_2) + \mathrm{cd}^p(z_3)\right) d\rho(z_1) \ldots d\rho(z_3) + C \\
&\quad \le (1 - \epsilon/2) \int \mathrm{cd}^p(z)\, d\rho(z) + C\,,
\end{aligned}$$

where C is some finite constant, only depending on the choice of K. This implies that for all $\lambda \in R(E, \epsilon_0)$,

$$\mathbb{E}\left(\mathrm{cd}^p(Z^{(0|x)})\right) \le \frac{2C}{\epsilon}\,.$$

□

Now we show how this theorem for $p = 2$ implies Theorem 2. We must transfer our decay estimate for the distribution ρ for $Z^{(0|x)}(\lambda)$ to the distributions ρ_g for $G^{(0|x)}(\lambda)$ and finally to ρ_G for $G(0, 0, \lambda)$, where these probability distributions are defined by by

$$\begin{aligned}
\rho_G(A) &= \mathrm{Prob}\{G(0, 0, \lambda) \in A\}\,, \\
\rho_g(A) &= \mathrm{Prob}\{G^{(0|x)}(\lambda) \in A\}\,.
\end{aligned}$$

Proof of Theorem 2: We will use the following inequality:

$$|z| \le 4 \frac{|z - w|^2}{\mathrm{Im} z} + 2|w|\,. \tag{9}$$

The inequality clearly holds for $|z| \le 2|w|$. In the complementary case, we have $|z| > 2|w|$ and thus $|z - w| \ge ||z| - |w|| \ge |w|$, implying

$$|z| \mathrm{Im} z \le |z|^2 \le 2|z - w|^2 + 2|w|^2 \le 4|z - w|^2$$

and further $|z| \leq 4|z-w|^2/\mathrm{Im} z$. This proves (9).

Using (9) with $w = z_\lambda$ yields that for $\lambda \in R(E, \epsilon)$

$$|z| \leq 4\mathrm{cd}(z) + C\,,$$

where C depends only on E and ϵ. To transfer the estimate on ρ_g to one on ρ_G we use the relation (1) and the estimate on $\mu'_{3,2}$ given by Lemma 5. Let R denote $R(E,\epsilon)$. Then

$$\begin{aligned}
&\sup_{\lambda\in R}\mathbb{E}\left(\left|\langle 0|(H-\lambda)^{-1}|0\rangle\right|^2\right)\\
&\qquad = \sup_{\lambda\in R}\int |z|^2\, d\rho_G(z)\\
&\qquad \leq 32\sup_{\lambda\in R}\int \mathrm{cd}^2(z)\, d\rho_G(z) + C\\
&\qquad = 32\sup_{\lambda\in R}\int \mathrm{cd}^2(-1/(z_1+z_2+z_3+\lambda-q))\, d\rho_g(z_1)d\rho_g(z_2)d\rho_g(z_3)\, d\nu(q) + C\\
&\qquad \leq 32\sup_{\lambda\in R}\int_{K^c\times\mathbb{R}} \mu'_{3,2}(z_1,z_2,z_3,q,\lambda)\\
&\qquad\qquad \times (\mathrm{cd}^2(z_1)+\mathrm{cd}^2(z_2)+\mathrm{cd}^2(z_3))\, d\rho_g(z_1)d\rho_g(z_2)\, d\rho_g(z_3)\, d\nu(q) + C\\
&\qquad \leq C\int_{\mathbb{H}\times\mathbb{R}} (1+|q|^4)\mathrm{cd}^2(z)\, d\rho_g(z)\, d\nu(q) + C\\
&\qquad \leq C\int \mathrm{cd}^2(z)\, d\rho_g(z) + C\,.
\end{aligned}$$

A completely analogous argument, using the relations (2) and (3) and the estimate of $\mu'_{1,p}$ in Lemma 5 yields

$$\int \mathrm{cd}^2(z)\, d\rho_g(z) \leq C\int \mathrm{cd}^2(z)\, d\rho(z) + C$$

and completes the proof. □

Analysis of μ_2

To analyze the function $\mu_{3,p}$ we will write it in terms of μ_2, defined by

$$\mu_2(z_1,z_2,q_1,q_2,\lambda) = \frac{2\mathrm{cd}(\phi(z_1,z_2,q_1,q_2,\lambda))}{\mathrm{cd}(z_1)+\mathrm{cd}(z_2)}$$

initially as a function from $\mathbb{H}^2\backslash\{(z_\lambda,z_\lambda)\}\times\mathbb{R}^2\times R\to\mathbb{R}$. In this section $R = R(E,\epsilon)$ for some $0 < E < 2\sqrt{2}$ and $\epsilon > 0$. (Note that here and throughout this paper we are using $\mathbb{H}^n$ to denote a product of hyperbolic planes, and not n-dimensional hyperbolic space.)

Proposition 7 *For all $z_1, z_2 \in \mathbb{H}^2\backslash\{(z_\lambda,z_\lambda)\}$ and $\lambda\in R$,*

$$\mu_2(z_1,z_2,0,0,\lambda) < 1\,.$$

Proof: For $z, w \in \mathbb{H}$ set

$$\mathrm{c}(w,z) = 2(\cosh(\mathrm{dist}_{\mathbb{H}}(w,z)) - 1) = \frac{|w-z|^2}{\mathrm{Im}(w)\,\mathrm{Im}(z)}\,.$$

Note that $z \mapsto \mathrm{c}(w,z)$ is strictly convex. This can be seen for example by noting that its Hessian has strictly positive eigenvalues. Also, $\mathrm{c}(w,z)$ is invariant under hyperbolic isometries. Thus

$$\mathrm{c}(2w, z_1+z_2) = \mathrm{c}(w, \frac{z_1+z_2}{2}) \le \frac{1}{2}\mathrm{c}(w,z_1) + \frac{1}{2}\mathrm{c}(w,z_2)\,.$$

Substituting $-(z_1-\lambda)^{-1}$ for z_1 and $-(z_2-\lambda)^{-1}$ for z_2 yields

$$\begin{aligned}\mathrm{c}(2w, \phi(z_1,z_2,0,0,\lambda)) &\le \frac{1}{2}\mathrm{c}(w, -(z_1-\lambda)^{-1}) + \frac{1}{2}\mathrm{c}(w,-(z_2-\lambda)^{-1})\\ &= \frac{1}{2}\mathrm{c}(2w, -2(z_1-\lambda)^{-1}) + \frac{1}{2}\mathrm{c}(2w,-2(z_2-\lambda)^{-1})\,.\end{aligned}$$

Now choose $2w = z_\lambda$. Since z_λ is the fixed point of $z \mapsto -2(z-\lambda)^{-1}$ we obtain

$$\mathrm{cd}(\phi(z_1,z_2,0,0,\lambda)) \le \frac{1}{2}\mathrm{cd}(z_1) + \frac{1}{2}\mathrm{cd}(z_2)\,.$$

If equality holds then strict convexity in the first estimate above implies $z_1 = z_2$. Then, since $\mathrm{Im}(\lambda) > 0$, $z \mapsto \phi(z,z,0,0,\lambda)$ is a strict contraction with fixed point z_λ (see [FHS]). This implies that the common value of z_1 and z_2 must be z_λ. $\square$

We need to understand the behaviour of $\mu_2(z_1,z_2,q_1,q_2,\lambda)$ as z_1 and z_2 approach infinity, and λ approaches the real axis. We know from Proposition 7 that the value of μ_2 is at most one, and wish to determine at what points it is equals one. Thus it is natural to introduce the compactification $\overline{\mathbb{H}}^2 \times \mathbb{R}^2 \times \overline{R}$. Here $\overline{R}$ denotes the closure, and $\overline{\mathbb{H}}$ is the compactification of $\mathbb{H}$ obtained by adjoining the boundary at infinity. (The word compactification is not quite accurate here because of the factors of $\mathbb{R}$, but we will use the term nevertheless.)

The boundary at infinity is defined as follows. Cover the upper half plane model of the hyperbolic plane $\mathbb{H}$ with two coordinate patches, one where $|z|$ is bounded below and one where $|z|$ is bounded above. On the patch where $|z| > C$ we use the co-ordinate function $w = -1/z$. Each chart looks like a semi-circle in the complex plane of the form $\{z \in \mathbb{C} : \mathrm{Im}(z) > 0, |z| < C\}$. The boundary at infinity consists of the sets $\{\mathrm{Im}(z) = 0\}$ and $\{\mathrm{Im}(w) = 0\}$ in the respective charts. The compactification $\overline{\mathbb{H}}$ is the upper half plane with the boundary at infinity adjoined. We will use $i\infty$ to denote the point where $w = 0$.

We now think of μ_2 as being defined in the interior of the compactification $\overline{\mathbb{H}}^2 \times \mathbb{R}^2 \times \overline{R}$ and ask how it behaves near the boundary. It turns out that in the co-ordinates introduced above, μ_2 is a rational function. At most points on the boundary the denominator does not vanish in the limit, and μ_2 has a continuous extension. There are, however, points where both numerator and denominator vanish, and at these singular points the limiting value of μ_2 depends on the direction of approach. By blowing up the singular points, it would be possible to define a compactification of $\mathbb{H}^2 \times \mathbb{R}^2 \times R$ to which μ_2 extends continuously. However, this is more than we need for our proof. We will do a partial resolution of the singularities of μ_2, consisting of two blow-ups of the simplest kind, and then extend μ_2 to an upper semi-continuous function on the resulting compactification.

The reciprocal of the function $\mathrm{cd}(z)$,

$$\chi(z) = 1/\mathrm{cd}(z) = \frac{\mathrm{Im}(z)}{|z - z_\lambda|^2}$$

is a boundary defining function for $\mathbb{H}$. This means that in each of the two charts above, χ is positive near infinity and vanishes exactly to first order on the boundary at infinity.

We will now describe our compactification of $\mathbb{H}^2 \times \mathbb{R}^2 \times R$. Start with $\overline{\mathbb{H}}^2 \times \mathbb{R}^2 \times \overline{R}$. The first blowup consists of writing $\chi(z_1), \chi(z_2)$ in polar co-ordinates. Thus we introduce new variables r_1, ω_1 and ω_2 and impose the equations

$$\begin{aligned} \chi(z_1) &= r_1 \omega_1 \,, \\ \chi(z_2) &= r_1 \omega_2 \,, \end{aligned} \tag{10}$$

and

$$\omega_1^2 + \omega_2^2 = 1 \,. \tag{11}$$

The blown up space is the variety in $\overline{\mathbb{H}}^2 \times \mathbb{R}^2 \times \overline{R} \times \mathbb{R}^3$ containing all points $(z_1, z_2, q_1, q_2, \lambda, r_1, \omega_1, \omega_2)$ that satisfy (10) and (11).

In the region where $|z_1|$ and $|z_2|$ are bounded, we could use $\chi(z_1)$, $\chi(z_2)$, $\mathrm{Re}(z_1)$, $\mathrm{Re}(z_2)$, q_1, q_2, λ as local co-ordinates for the original space $\overline{\mathbb{H}}^2 \times \mathbb{R}^2 \times \overline{R}$. The image of such a co-ordinate chart near the boundary would be $[0, \epsilon)^2 \times I^2 \times \mathbb{R}^2 \times \overline{R}$ for some interval I. Local co-ordinates for the blown up space would be r_1, θ, $\mathrm{Re}(z_1)$, $\mathrm{Re}(z_2)$, q_1, q_2, λ where $\omega_1 = \cos(\theta)$ and $\omega_2 = \sin(\theta)$. The image of such a chart in the blown up space would be $[0, \epsilon) \times [0, \pi/2] \times I^2 \times \mathbb{R}^2 \times \overline{R}$. Similarly, we could write local co-ordinates in the other regions. The singular locus for the first blowup is the corner $\partial_\infty(\overline{\mathbb{H}}) \times \partial_\infty(\overline{\mathbb{H}}) \times \mathbb{R}^2 \times \overline{R}$ in $\overline{\mathbb{H}}^2 \times \mathbb{R}^2 \times \overline{R}$, defined by $\chi(z_1) = \chi(z_2) = 0$. Corresponding to each point in the singular locus is a quarter circle of points in the blown up space, parametrized by ω_1, ω_2. Away from the singular locus the original space and the blown up space are essentially the same, since we can solve for r_1, ω_1, ω_2 in terms of the original variables.

For the second blowup we introduce an additional real variable r_2 and two additional complex variables η_1 and η_2. We impose

$$\begin{aligned} z_1 + \mathrm{Re}(\lambda) - q_1 &= r_2\, \eta_1 \,, \\ z_2 + \mathrm{Re}(\lambda) - q_2 &= r_2\, \eta_2 \,, \end{aligned} \tag{12}$$

with

$$|\eta_1|^2 + |\eta_2|^2 = 1 \,, \tag{13}$$

and

$$r_2 \geq 0\,.$$

The variables of the first and second blowups are not independent when $r_1, r_2 \neq 0$. In fact, since $\chi(z_1) = r_2 \operatorname{Im}(\eta_1)/|r_2\eta_1 - \operatorname{Re}(\lambda) + q_1 - z_\lambda|^2 = r_1\omega_1$ we find that $r_1 r_2 \operatorname{Im}(\eta_1)\omega_2|r_2\eta_2 - \operatorname{Re}(\lambda) + q_2 - z_\lambda|^2$ and $r_1 r_2 \operatorname{Im}(\eta_2)\omega_1|r_2\eta_1 - \operatorname{Re}(\lambda) + q_1 - z_\lambda|^2$ are equal so that, when $r_1, r_2 \neq 0$,

$$\operatorname{Im}(\eta_1)\omega_2|r_2\eta_2 - \operatorname{Re}(\lambda) + q_2 - z_\lambda|^2 = \operatorname{Im}(\eta_2)\omega_1|r_2\eta_1 - \operatorname{Re}(\lambda) + q_1 - z_\lambda|^2\,. \tag{14}$$

We will require that this equation be satisfied everywhere. Otherwise, there would be points in the blown up space (where $r_2 = 0$ and (14) is not satisfied) that are not in the closure of the interior of the original space.

As before, the twice blown up space is essentially the same as the once blown up space away from the singular locus $z_1 = -\operatorname{Re}(\lambda) + q_1$, $z_2 = -\operatorname{Re}(\lambda) + q_2$. Local co-ordinates for the twice blown up space near the singular locus are given by r_2, ω_1, ω_2, $\operatorname{Re}(\eta_1)$, $\operatorname{Re}(\eta_2)$, q_1, q_2, λ.

Define K to be the space obtained from $\overline{\mathbb{H}}^{\,2} \times \mathbb{R}^2 \times \overline{R}$ by the two blowups described above. The topology is the one given by the local description as a closed subset of Euclidean space. The boundary at infinity is defined to be

$$\partial_\infty K = \{\chi(z_1) = 0\} \cup \{\chi(z_2) = 0\} = \{r_1 = 0\} \cup \{\omega_1 = 0\} \cup \{\omega_2 = 0\}\,.$$

The set $K \backslash \partial_\infty K$ can be identified with $\mathbb{H}^2 \times \mathbb{R}^2 \times \overline{R}$.

Extend μ_2 to an upper semi-continuous function on K by defining, for points $k \in \partial_\infty K$,

$$\mu_2(k) = \limsup_{\substack{k_n \to k \\ k_n \in K\backslash\partial_\infty K}} \mu_2(k_n)\,.$$

Here $k_n \to k$ means convergence in K. More explicitly, k_n is a point $(z_{1,n}, z_{2,n}, q_{1,n}, q_{2,n}, \lambda_n) \in \mathbb{H}^2 \times \mathbb{R}^2 \times R$, and not only do these co-ordinates approach limiting values $(z_1, z_2, q_1, q_2, \lambda)$ in $\overline{\mathbb{H}}^2 \times \mathbb{R}^2 \times \overline{R}$, but also the co-ordinates r_1, ω_1 and ω_2 defined by (10) and (11) and the co-ordinates r_2, η_1 and η_2 defined by (12) and (13) approach limiting values as well. Of course, the co-ordinates r_2, η_1 and η_2 are only defined in the region where $|z_1|$ and $|z_2|$ are bounded. But, for these co-ordinates, we really care only about the point where $z_i = -\operatorname{Re}(\lambda) + q_i$, $i = 1, 2$, since away from the singular locus, the blowup co-ordinates are determined by the base co-ordinates z_i, q_i and λ.

Lemma 8 *Let Σ be the subset of K where $\mu_2 = 1$. Let K_0 denote the subset of $\partial_\infty K$ where $\lambda \in (-2\sqrt{2}, 2\sqrt{2})$, $q_1 = q_2 = 0$. Then*

$$\Sigma \cap K_0 = (\Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Sigma_4) \cap K_0\,, \tag{15}$$

where

$$\Sigma_1 = \{z_1 \neq -\lambda, z_2 \neq -\lambda, z_1 = z_2 \in \partial_\infty \overline{\mathbb{H}}, \omega_1 = \omega_2\}\,,$$

$$\Sigma_2 = \{z_1 = -\lambda, z_2 \neq -\lambda, \omega_1 = 0\}\,,$$

$$\Sigma_3 = \{z_1 \neq -\lambda, z_2 = -\lambda, \omega_2 = 0\}\,,$$

$$\Sigma_4 = \{z_1 = -\lambda, z_2 = -\lambda, \eta_1 = e^{i\psi}\omega_1, \eta_2 = e^{i\psi}\omega_2 \text{ for some } \psi \in [0,\pi]\}\,.$$

Remark: In fact we will only use this theorem when both z_1 and z_2 are in $\partial_\infty \overline{\mathbb{H}}$.

Proof: Assume for the moment that $(z_1, z_2, q_1, q_2, \lambda) \in \mathbb{H}^2 \times \mathbb{R}^2 \times \overline{R}$. Since

$$\chi(\phi(z_1, z_2, q_1, q_2, \lambda)) = \frac{\mathrm{Im}(z_1+\lambda)|z_2+\lambda-q_2|^2 + \mathrm{Im}(z_2+\lambda)|z_1+\lambda-q_1|^2}{|z_1+\lambda-q_1+z_2+\lambda-q_2+z_\lambda(z_1+\lambda-q_1)(z_2+\lambda-q_2)|^2},$$

the function μ_2 is given by

$$\begin{aligned}
&\mu_2(z_1, z_2, q_1, q_2, \lambda)\\
&\quad= \frac{2\chi(z_1)\chi(z_2)}{\chi(\phi(z_1, z_2, \lambda, q_1, q_2))(\chi(z_1)+\chi(z_2))}\\
&\quad= \frac{2\chi(z_1)\chi(z_2)|z_1+\lambda-q_1+z_2+\lambda-q_2+z_\lambda(z_1+\lambda-q_1)(z_2+\lambda-q_2)|^2}{(\mathrm{Im}(z_1+\lambda)|z_2+\lambda-q_2|^2+\mathrm{Im}(z_2+\lambda)|z_1+\lambda-q_1|^2)(\chi(z_1)+\chi(z_2))}\\
&\quad= \frac{2\chi(z_1)\chi(z_2)|z_1+\lambda-q_1+z_2+\lambda-q_2+z_\lambda(z_1+\lambda-q_1)(z_2+\lambda-q_2)|^2}{(p_1\chi(z_1)|z_1-z_\lambda|^2|z_2+\lambda-q_2|^2+p_2\chi(z_2)|z_2-z_\lambda|^2|z_1+\lambda-q_1|^2)(\chi(z_1)+\chi(z_2))}\,,
\end{aligned}$$

where

$$p_i = 1 + \mathrm{Im}(\lambda)/\mathrm{Im}(z_i)\,.$$

Define μ_2^* by setting $p_1 = p_2 = 1$ in this formula, that is,

$$\begin{aligned}
&\mu_2^*(z_1, z_2, q_1, q_2, \lambda)\\
&\quad= \frac{2\chi(z_1)\chi(z_2)|z_1+\lambda-q_1+z_2+\lambda-q_2+z_\lambda(z_1+\lambda-q_1)(z_2+\lambda-q_2)|^2}{(\chi(z_1)|z_1-z_\lambda|^2|z_2+\lambda-q_2|^2+\chi(z_2)|z_2-z_\lambda|^2|z_1+\lambda-q_1|^2)(\chi(z_1)+\chi(z_2))}\\
&\quad= \frac{2\omega_1\omega_2|z_1+\lambda-q_1+z_2+\lambda-q_2+z_\lambda(z_1+\lambda-q_1)(z_2+\lambda-q_2)|^2}{(\omega_1|z_1-z_\lambda|^2|z_2+\lambda-q_2|^2+\omega_2|z_2-z_\lambda|^2|z_1+\lambda-q_1|^2)(\omega_1+\omega_2)}\,.
\end{aligned} \tag{16}$$

Clearly $\mu_2 \leq \mu_2^*$.

Now let $k \in \Sigma \cap K_0$. To show the inclusion $\subseteq$ in (15) we must show that k is in Σ_i for some $i \in \{1, 2, 3, 4\}$. Let the co-ordinates of k be given by the base co-ordinates z_1, z_2, $q_1 = 0$, $q_2 = 0$, $\lambda \in (-2\sqrt{2}, 2\sqrt{2})$, the first blow-up co-ordinates r_1, ω_1 and ω_2 and, if $z_1, z_2 \neq i\infty$, the second blow-up co-ordinates r_2, η_1 and η_2. Since $k \in \partial_\infty K$, $\mu_2(k)$ is defined as a $\limsup$.

The points of continuity of μ_2^* are the points where the denominator of (16) does not vanish. Thus k satisfies one of the following four mutually disjoint conditions:

(i) k is a point of continuity for μ_2^*,

(ii) $z_1 = -\lambda$ and $z_2 = -\lambda$,

(iii) $z_1 = -\lambda$, $z_2 \neq -\lambda$ and $\omega_1 = 0$,

(iv) $z_1 \neq -\lambda$, $z_2 = -\lambda$ and $\omega_2 = 0$.

If conditions (iii) or (iv) hold then k lies in Σ_2 or Σ_3 and we are done.

Suppose that (i) holds. Then

$$1 = \mu_2(k) = \limsup_{\substack{k_n \to k \\ k_n \in K \setminus \partial_\infty K}} \mu_2(k_n) \leq \limsup_{\substack{k_n \to k \\ k_n \in K \setminus \partial_\infty K}} \mu_2^*(k_n) \leq 1 \,.$$

The last inequality holds because at a point of continuity, the $\limsup$ is actually a limit which can be evaluated in any order. If we take the limit in λ and q_i first, we may use the fact that for $\lambda \in (-2\sqrt{2}, 2\sqrt{2})$, $\mu_2 = \mu_2^*$, and from Proposition 7 we see that the remaining limit in z_1 and z_2 can be at most 1.

Thus we have $\mu_2^*(k) = 1$ and we need to show that if (i) holds, and (ii), (iii) and (iv) do not, then k lies in one of the sets Σ_i. Let us first consider the case where $z_1 = z_2 = i\infty$. In this case we must introduce new variables $w_i = -1/z_i$, substitute into (16) and send w_1 and w_2 to zero. Using $|z_\lambda|^2 = 2$ for $\lambda \in (-2\sqrt{2}, 2\sqrt{2})$ we find that at this point $\mu_2^* = 4\omega_1\omega_2/(\omega_1+\omega_2)^2$. So $\mu_2^* = 1$ implies that $\omega_1 = \omega_2$ and thus that $k \in \Sigma_1$.

Continuing with case (i) let us consider next the possibility that $z_1 \in \partial_\infty \overline{\mathbb{H}}$ and $z_2 \notin \partial_\infty \overline{\mathbb{H}}$. Then $\omega_1 = 0$ and $\omega_2 = 1$ and the numerator in (16) is zero. Away from points described by (ii), (iii) and (iv) the denominator is not zero so $\mu_2^*(k) = 0$. This is impossible. When $z_1 = i\infty$ we should first replace z_1 with $-1/w_1$ and send w_1 to zero. This leads to the same conclusion.

Thus we are left to consider points satisfying (i) where z_1 and z_2 are both in $\partial_\infty \overline{\mathbb{H}}$ and not the point at infinity. In other words z_1 and z_2 are real but not equal to $-\lambda$. In this case the condition $\mu_2^* = 1$ can be rewritten as

$$\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}^T M \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = 0 \,, \tag{17}$$

with

$$M = \begin{bmatrix} m_{1,1} & m_{1,2} \\ m_{2,1} & m_{2,2} \end{bmatrix} ,$$

where $m_{1,2} = m_{2,1}$ and

$$m_{1,1} = |z_1 - z_\lambda|^2 |z_2 + \lambda|^2 \,,$$

$$m_{1,2} = (|z_1 - z_\lambda|^2|z_2+\lambda|^2 + |z_2 - z_\lambda|^2|z_1+\lambda|^2)/2 - |z_1 + \lambda + z_2 + \lambda + z_\lambda(z_1+\lambda)(z_2+\lambda)|^2 \,,$$

$$m_{2,2} = |z_2 - z_\lambda|^2 |z_1 + \lambda|^2 \,.$$

Setting $s_1 = z_1 + \lambda$ and $s_2 = z_2 + \lambda$ and using $s_1, s_2 \in \mathbb{R}$ we find

$$M = \begin{bmatrix} (s_1^2 - \lambda s_1 + 2)s_2^2 & -s_1 s_2(s_1 s_2 - \lambda(s_1+s_2)/2 + 2 \\ -s_1 s_2(s_1 s_2 - \lambda(s_1+s_2)/2 + 2 & (s_2^2 - \lambda s_2 + 2)s_1^2 \end{bmatrix} .$$

Since $\mathrm{tr}(M) \geq 0$ the condition (17) requires

$$\det(M) = s_1^2 s_2^2 (s_1 - s_2)^2 (2 - \lambda^2/4) = 0 \,.$$

But s_1 and s_2 are not zero. Therefore we conclude that $s_1 = s_2$ and thus $z_1 = z_2$. In addition,

$$M = s^2(s^2 - \lambda s + 2) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

where s is the common value of s_1 and s_2. For $\lambda \in (-2\sqrt{2}, 2\sqrt{2})$, $s^2 - \lambda s + 2 > 0$. Thus (17) implies that $\omega_1 = \omega_2$ and so $k \in \Sigma_1$.

Finally, we must deal with case (ii). Let $(z_{1,n}, z_{2,n}, q_{1,n}, q_{2,n}, \lambda_n) \in \mathbb{H}^2 \times \mathbb{R}^2 \times \overline{R}$ be a sequence that realizes the $\limsup$ in the definition of $\mu_2(k)$. Define $\tilde{r}_{2,n}$, $\tilde{\eta}_{1,n}$ and $\tilde{\eta}_{2,n}$ via

$$z_{1,n} + \lambda_n - q_{1,n} = \tilde{r}_{2,n}\tilde{\eta}_{1,n},$$

$$z_{2,n} + \lambda_n - q_{2,n} = \tilde{r}_{2,n}\tilde{\eta}_{2,n},$$

and

$$|\tilde{\eta}_{1,n}|^2 + |\tilde{\eta}_{2,n}|^2 = 1.$$

By going to a subsequence if needed, we may assume that $\tilde{r}_{2,n}$, $\tilde{\eta}_{1,n}$ and $\tilde{\eta}_{2,n}$ converge to 0, $\tilde{\eta}_1$ and $\tilde{\eta}_2$ respectively. We may also assume that $p_{i,n} = 1 + \mathrm{Im}(\lambda_n)/\mathrm{Im}(z_{i,n})$ converge to p_i for $i = 1, 2$.

Then we find that

$$1 = \mu_2(k) = \frac{\omega_1\omega_2|\tilde{\eta}_1 + \tilde{\eta}_2|^2}{(p_1\omega_1|\tilde{\eta}_2|^2 + p_2\omega_2|\tilde{\eta}_1|^2)(\omega_1 + \omega_2)} \leq \frac{\omega_1\omega_2|\tilde{\eta}_1 + \tilde{\eta}_2|^2}{(\omega_1|\tilde{\eta}_2|^2 + \omega_2|\tilde{\eta}_1|^2)(\omega_1 + \omega_2)} \leq 1, \tag{18}$$

unless the denominator is zero, that is, $\omega_1 = 0$, $\tilde{\eta}_1 = 0$ or $\omega_2 = 0$, $\tilde{\eta}_2 = 0$, which we assume for the moment is not the case. The last inequality holds because it is equivalent to

$$\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}^T \begin{bmatrix} |\tilde{\eta}_2|^2 & -\mathrm{Re}(\overline{\tilde{\eta}_1}\tilde{\eta}_2) \\ -\mathrm{Re}(\overline{\tilde{\eta}_1}\tilde{\eta}_2) & |\tilde{\eta}_1|^2 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \geq 0,$$

and the matrix in this formula is positive semi-definite.

Still under the assumption that neither $\omega_1 = 0$, $\tilde{\eta}_1 = 0$ nor $\omega_2 = 0$, $\tilde{\eta}_2 = 0$ hold, we see that at least one of p_1 or p_2 must equal 1. Otherwise we would have a strict inequality in (18) which is impossible. If $p_i = 1$ then $\mathrm{Im}(\lambda_n)/\mathrm{Im}(z_{i,n}) \to 0$. This implies that $\tilde{r}_{2,n}/r_{2,n} \to 1$, because $r_{2,n} \geq \mathrm{Im}(z_{i,n})$ implies $\mathrm{Im}(\lambda_n)/r_{2,n} \to 0$ and

$$\tilde{r}_{2,n}^2 = \sum_{i=1}^{2} \mathrm{Re}(z_{i,n} - q_{i,n} + \lambda_n)^2 + \mathrm{Im}(z_{i,n} + \lambda_n)^2.$$

Now, from $\tilde{r}_{2,n}\tilde{\eta}_{i,n} = r_{2,n}\eta_{i,n} + \mathrm{Im}(\lambda_n)$ we conclude that $\tilde{\eta}_{i,n}$ and $\eta_{i,n}$ have the same limit η_i. So, in fact, we have that (17) holds with

$$M = \begin{bmatrix} |\eta_2|^2 & -\mathrm{Re}(\overline{\eta}_1\eta_2) \\ -\mathrm{Re}(\overline{\eta}_1\eta_2) & |\eta_1|^2 \end{bmatrix}.$$

Since $\mathrm{tr}(M) > 0$ this requires

$$\det(M) = |\eta_1|^2|\eta_2|^2\big(1 - \cos(2(\arg(\eta_1) - \arg(\eta_2)))\big) = 0.$$

This means either $\eta_1 = 0$, $\eta_2 = 0$, $\arg(\eta_1) = \arg(\eta_2)$ or $\arg(\eta_1) = \arg(\eta_2) + \pi$. If $\eta_1 = 0$ then $\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \in \mathrm{Ker}(M)$ requires $\omega_1 = 0$ and $k \in \Sigma_4$. Similarly, if $\eta_2 = 0$ then $k \in \Sigma_4$. If $\arg(\eta_1) = \arg(\eta_2) = \psi$ then

$$M = \begin{bmatrix} |\eta_2|^2 & -|\eta_1||\eta_2| \\ -|\eta_1||\eta_2| & |\eta_1|^2 \end{bmatrix}$$

and (17) and the fact that $\omega_1, \omega_2 \geq 0$ implies that $\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} |\eta_1| \\ |\eta_2| \end{bmatrix}$. Thus $\eta_1 = e^{i\psi}\omega_1$ and $\eta_2 = e^{i\psi}\omega_2$, and again we have $k \in \Sigma_4$. The remaining possibility is that $\arg(\eta_1) = \arg(\eta_2) + \pi$. Since both η_1 and η_2 lie in the upper half plane, this implies that they are both real with opposite signs. Equation (17) then requires $\omega_1 = \eta_1$ and $\omega_2 = \eta_2$. But this is impossible, as ω_1 and ω_2 are both non-negative.

To complete the proof we must return to the possibility that $\omega_1 = 0$, $\tilde{\eta}_1 = 0$ or $\omega_2 = 0$, $\tilde{\eta}_2 = 0$. Clearly, at most one of these can hold. Suppose $\omega_1 = 0$, $\tilde{\eta}_1 = 0$. (The other possibility is handled similarly.) Introduce one more set of variables s_n, $\alpha_{1,n}$ and $\alpha_{2,n}$ satisfying

$$\omega_{1,n} = s_n^2 \alpha_{1,n} \,,$$

$$\tilde{\eta}_{1,n} = s_n^2 \alpha_{2,n}$$

and

$$\alpha_{1,n}^2 + |\alpha_{2,n}|^2 = 1 \,.$$

Then $s_n \to 0$ and going to a subsequence we may assume that $\alpha_{1,n}$ and $\alpha_{2,n}$ converge to α_1 and α_2. Then $\mu_2(k) = \alpha_1/(p_1\alpha_1 + p_2|\alpha_2|^2) = 1$ so that $\alpha_2 = 0$ and $p_1 = 1$. But $p_1 = 1$ implies $\tilde{\eta}_1 = \eta_1$ by the argument above. Thus $\eta_1 = \tilde{\eta}_1 = 0$ and $\omega_1 = 0$ which implies that $k \in \Sigma_4$. $\square$

Proofs of Lemma 4 and Lemma 5

Proof of Lemma 4: Extend $\mu_{3,p}$ to an upper semi-continuous function on $\overline{\mathbb{H}}^3 \times \mathbb{R}^4 \times \overline{R}$ by setting, at points Z_0, Q_0, λ_0 where it is not already defined,

$$\mu_{3,p}(Z_0, Q_0, \lambda_0) = \limsup_{Z \to Z_0, Q \to Q_0, \lambda \to \lambda_0} \mu_{3,p}(Z, Q, \lambda) \,.$$

Here the limsup is taken over points in $\mathbb{H}^3 \times \mathbb{R}^4 \times R$, and we are using the notation $Z = (z_1, z_2, z_3)$ for points in $\mathbb{H}^3$ and $Q = (q_1, q_2, q_3, q_4)$ for points in $\mathbb{R}^4$. The points Z, Q and λ are approaching their limits in the topology of $\overline{\mathbb{H}}^3 \times \mathbb{R}^4 \times \overline{R}$.

To prove the theorem it is then enough to show that

$$\mu_{3,p}(Z, Q, \lambda) < 1 \tag{19}$$

for (Z, Q, λ) in the compact set $\partial_\infty(\overline{\mathbb{H}}^3) \times \{0\}^4 \times [-E, E]$, since this implies that for some $\epsilon > 0$, the upper semi-continuous function $\mu_{3,p}(Z, Q, \lambda)$ is bounded by $1 - 2\epsilon$ on the set, and by $1 - \epsilon$ in some neighbourhood.

We will rewrite $\mu_{3,p}$ in terms of the simpler function μ_2. Define

$$\nu_i(Z) = \frac{\mathrm{cd}(z_i)}{\mathrm{cd}(z_1) + \mathrm{cd}(z_2) + \mathrm{cd}(z_3)}\,,$$

and the maps ξ_σ and τ_σ from $\mathbb{H}^3 \times \mathbb{R}^4 \times R$ to $\mathbb{H} \times \mathbb{H} \times \mathbb{R}^2 \times L$ labelled by a permutation σ of $(1, 2, 3)$ and given by

$$\begin{aligned}
\xi_\sigma(Z, Q, \lambda) &= (z_{\sigma_2}, z_{\sigma_3}, q_{\sigma_2}, q_{\sigma_3}, \lambda)\,,\\
\tau_\sigma(Z, Q, \lambda) &= (z_{\sigma_1}, \phi(z_{\sigma_2}, z_{\sigma_3}, q_{\sigma_2}, q_{\sigma_3}, \lambda), q_{\sigma_1}, q_4, \lambda)\,.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\mu_{3,p}(Z, Q, \lambda)\\
&\quad= \sum_\sigma \frac{\mathrm{cd}^p(\phi(z_{\sigma_1}, \phi(z_{\sigma_2}, z_{\sigma_3}, q_{\sigma_2}, q_{\sigma_3}, \lambda), q_{\sigma_1}, q_{\sigma_4}, \lambda))}{\mathrm{cd}^p(z_1) + \mathrm{cd}^p(z_2) + \mathrm{cd}^p(z_3)}\\
&\quad= \sum_\sigma \left(\frac{\mathrm{cd}(\phi(z_{\sigma_1}, \phi(z_{\sigma_2}, z_{\sigma_3}, q_{\sigma_2}, q_{\sigma_3}, \lambda), q_{\sigma_1}, q_{\sigma_4}, \lambda))}{\mathrm{cd}(z_1) + \mathrm{cd}(z_2) + \mathrm{cd}(z_3)}\right)^p \frac{1}{\nu_1^p + \nu_2^p + \nu_3^p} &(20)\\
&\quad= \sum_\sigma \left(\mu_2(\tau_\sigma(Z, Q, \lambda)) \left(\frac{1}{2}\nu_{\sigma_1} + \frac{1}{4}\mu_2(\xi_\sigma(Z, Q, \lambda))(\nu_{\sigma_2} + \nu_{\sigma_3})\right)\right)^p \frac{1}{\nu_1^p + \nu_2^p + \nu_3^p}\,.
\end{aligned}$$

Let R_1, Ω_1, Ω_2 and Ω_3 be three dimensional polar co-ordinates defined as functions of $Z = (z_1, z_2, z_3) \in \mathbb{H}^3$ by

$$\begin{aligned}
\chi(z_1) &= R_1\Omega_1\,,\\
\chi(z_2) &= R_1\Omega_2\,,\\
\chi(z_3) &= R_1\Omega_3
\end{aligned}$$

and

$$\Omega_1^2 + \Omega_2^2 + \Omega_3^2 = 1\,.$$

Notice that for any permutation σ of $(1, 2, 3)$,

$$\nu_{\sigma_1} = \frac{\Omega_{\sigma_2}\Omega_{\sigma_3}}{\Omega_1\Omega_2 + \Omega_1\Omega_3 + \Omega_2\Omega_3}\,. \tag{21}$$

Next, let $r_1(z_1, z_2)$, $\omega_1(z_1, z_2)$ and $\omega_2(z_1, z_2)$ be the co-ordinates defined by (10) and (11). Then, for any permutation σ of $(1, 2, 3)$ and any $Z = (z_1, z_2, z_3) \in \mathbb{H}^3$,

$$\begin{aligned}
\Omega_{\sigma_2}^2 &= \left(\Omega_{\sigma_2}^2 + \Omega_{\sigma_3}^2\right)\omega_1^2(z_{\sigma_2}, z_{\sigma_3})\,,\\
\Omega_{\sigma_3}^2 &= \left(\Omega_{\sigma_2}^2 + \Omega_{\sigma_3}^2\right)\omega_2^2(z_{\sigma_2}, z_{\sigma_3})\,,
\end{aligned} \tag{22}$$

where each Ω_{σ_i} is evaluated at Z. To see this note that, since $\chi(z_{\sigma_2}) = r_1(z_{\sigma_2}, z_{\sigma_3})\omega_1(z_{\sigma_2}, z_{\sigma_3}) = R_1\Omega_{\sigma_2}$ and $\chi(z_{\sigma_3}) = r_1(z_{\sigma_2}, z_{\sigma_3})\omega_2(z_{\sigma_2}, z_{\sigma_3}) = R_1\Omega_{\sigma_3}$ we have

$$r_1^2(z_{\sigma_2}, z_{\sigma_3}) = r_1^2(z_{\sigma_2}, z_{\sigma_3})(\omega_1^2(z_{\sigma_2}, z_{\sigma_3}) + \omega_2^2(z_{\sigma_2}, z_{\sigma_3})) = R_1^2(\Omega_{\sigma_2}^2 + \Omega_{\sigma_3}^2)\,.$$

Thus

$$R_1^2\Omega_{\sigma_2}^2 = r_1^2(z_{\sigma_2}, z_{\sigma_3})\omega_1^2(z_{\sigma_2}, z_{\sigma_3}) = R_1^2(\Omega_{\sigma_2}^2 + \Omega_{\sigma_3}^2)\omega_1^2(z_{\sigma_2}, z_{\sigma_3})$$

and since $R_1 \neq 0$ for $Z \in \mathbb{H}^3$ the first equality of (22) follows. The second equality is proved in the same way. A similar argument also shows

$$\begin{aligned} \Omega_{\sigma_1}^2 &= \left(\Omega_{\sigma_1}^2 + F^2(\Omega_{\sigma_2}^2 + \Omega_{\sigma_3}^2)\right)\omega_1^2(z_{\sigma_1}, \phi(z_{\sigma_2}, z_{\sigma_3}, q_{\sigma_2}, q_{\sigma_3}, \lambda))\,, \\ F^2(\Omega_{\sigma_2}^2 + \Omega_{\sigma_3}^2) &= \left(\Omega_{\sigma_1}^2 + F^2(\Omega_{\sigma_2}^2 + \Omega_{\sigma_3}^2)\right)\omega_2^2(z_{\sigma_1}, \phi(z_{\sigma_2}, z_{\sigma_3}, q_{\sigma_2}, q_{\sigma_3}, \lambda))\,. \end{aligned} \tag{23}$$

Here each Ω_{σ_i} is evaluated at Z and

$$\begin{aligned} F &= F(z_{\sigma_2}, z_{\sigma_3}, q_{\sigma_2}, q_{\sigma_3}, \lambda) \\ &= \frac{\chi(\phi(z_{\sigma_2}, z_{\sigma_3}, q_{\sigma_2}, q_{\sigma_3}, \lambda))}{r_1(z_{\sigma_2}, z_{\sigma_3})} \\ &= \frac{2\omega_1(z_{\sigma_2}, z_{\sigma_3})\omega_2(z_{\sigma_2}, z_{\sigma_3})}{\mu_2(z_{\sigma_2}, z_{\sigma_3}, q_{\sigma_2}, q_{\sigma_3}, \lambda)(\omega_1(z_{\sigma_2}, z_{\sigma_3}) + \omega_2(z_{\sigma_2}, z_{\sigma_3}))}\,. \end{aligned} \tag{24}$$

We will prove (19) by contradiction. For this suppose that $\mu_{3,p}(Z, Q, \lambda) = 1$ for some $(Z, Q, \lambda) \in \partial_\infty(\overline{\mathbb{H}}^3) \times \{0\}^4 \times [-E, E]$. Then there must exist a sequence (Z_n, Q_n, λ_n) with $Z_n \to Z$ in $\overline{\mathbb{H}}^3$ $Q_n \to (0, 0, 0, 0)$ and $\lambda_n \to \lambda \in [-E, E]$ such that

$$\lim \mu_{3,p}(Z_n, Q_n, \lambda_n) = 1\,.$$

From now on $Z = (z_1, z_2, z_3)$ and λ will denote the limiting values of the sequence Z_n and λ_n. Similarly, we will denote by ν_i and Ω_i the limits of $\nu_i(Z_n)$ and $\Omega_i(Z_n)$. We claim that

$$\nu_1 = \nu_2 = \nu_3 = \frac{1}{3}\,. \tag{25}$$

This follows from (20), the bound $\mu_2 \leq 1$ proved in Proposition 7 and convexity of $x \mapsto x^p$ which imply

$$\begin{aligned} 1 &\leq \sum_\sigma \left(\frac{1}{2}\nu_{\sigma_1} + \frac{1}{4}(\nu_{\sigma_2} + \nu_{\sigma_3})\right)^p \frac{1}{\nu_1^p + \nu_2^p + \nu_3^p} \\ &\leq \sum_\sigma \left(\frac{1}{2}\nu_{\sigma_1}^p + \frac{1}{4}(\nu_{\sigma_2}^p + \nu_{\sigma_3}^p)\right) \frac{1}{\nu_1^p + \nu_2^p + \nu_3^p} \\ &= 1\,, \end{aligned}$$

so the inequalities must actually be equalities. Since $p > 1$, strict convexity implies that equality only holds if $\nu_1 = \nu_2 = \nu_3$. Since their sum is 1, their common value must be $1/3$.

By going to a subsequence, we may assume that $\Omega_i(Z_n)$ converge. Then (25) and (21) imply that their limiting values along the sequence must be

$$\Omega_1 = \Omega_2 = \Omega_3 = 1/\sqrt{3}\,. \tag{26}$$

One consequence is that

$$z_i \in \partial_\infty \overline{\mathbb{H}} \tag{27}$$

for $i = 1, 2, 3$.

Now consider the values of $\xi_\sigma(Z_n, Q_n, \lambda_n)$ and $\tau_\sigma(Z_n, Q_n, \lambda_n)$. Since these vary in a compact region in K we may, again by going to a subsequence, assume that they converge in K to values which we will denote ξ_σ and τ_σ. Returning to (20) and using (25), the upper semi-continuity of μ_2 and the bound $\mu_2 \leq 1$, we find that

$$\begin{aligned} 1 &= \lim_{n\to\infty} \frac{1}{3} \sum_\sigma \left(\frac{\mu_2(\tau_\sigma(Z_n, Q_n, \lambda_n))(1 + \mu_2(\xi_\sigma(Z_n, Q_n, \lambda_n)))}{2} \right)^p \\ &\leq \frac{1}{3} \sum_\sigma \left(\frac{\mu_2(\tau_\sigma)(1 + \mu_2(\xi_\sigma))}{2} \right)^p \\ &\leq 1\,. \end{aligned}$$

This implies that for every σ occurring in the sum we have

$$\mu_2(\xi_\sigma) = \mu_2(\tau_\sigma) = 1\,.$$

This and (27) imply that for each σ, ξ_σ and τ_σ lie in the set Σ of Lemma 8.

Now consider the co-ordinates ω_1 and ω_2 for the the point ξ_σ. These are the limiting values of $\omega_i(z_{\sigma_2}, z_{\sigma_3})$ along our sequence. Equations (22) and (26) then imply that these limiting values are $\omega_1 = \omega_2 = 1/\sqrt{2}$. Examining the description of Σ in Lemma 8, we conclude that the $\overline{\mathbb{H}}$ co-ordinates of ξ_σ, namely the limiting values of z_{σ_2} and z_{σ_3}, must be equal. Since this is true for every σ we conclude that

$$z_1 = z_2 = z_3 \in \partial_\infty \overline{\mathbb{H}}\,.$$

Let z denote their common value.

We first consider the possibility that $z \neq -\lambda$. The two $\overline{\mathbb{H}}$ co-ordinates of the point $\tau_{(1,2,3)}$ are z and the limiting value of $\phi(z_2, z_3, q_2, q_3, \lambda)$. This limiting value is simply $\phi(z, z, 0, 0, \lambda) = -2/(z+\lambda)$ and is easily seen to be not equal to z. The only way that τ_σ with $\overline{\mathbb{H}}$ co-ordinates z and $\phi(z, z, 0, 0, \lambda)$ can lie in Σ with $z \neq -\lambda$ is that $\phi(z, z, 0, 0, \lambda) = -\lambda$ and that the ω_2 co-ordinate is 0. The ω_2 co-ordinate is the limiting value of $\omega_2(z_1, \phi(z_2, z_3, q_2, q_3, \lambda))$ which we may use in taking the limit of equation (23). The limiting value of F in that equation can be computed from (24), since we know that the values of ω_i in that formula are $1/\sqrt{2}$ and the value of μ_2 in that formula is 1. This gives $F = 1/\sqrt{2}$ and so the second equation of (23) yields $1/3 = 0$ in the limit, which is impossible.

This leaves the possibility that $z = -\lambda$. Again, the $\overline{\mathbb{H}}$ co-ordinates for the point $\tau_{(1,2,3)}$ are z and the limiting value of $\phi(z_2, z_3, q_2, q_3, \lambda)$. By going to a subsequence, we have assumed that this limiting value exists. However, in this case it is not clear what the value is, since $(-\lambda, -\lambda, 0, 0, \lambda)$ is the point where ϕ is not continuous. In fact, we will see that the limiting value, possibly after going one more time to a subsequence, is $i\infty$. To see this we write

$$\begin{aligned} \phi(z_{2,n}, z_{3,n}, q_{2,n}, q_{3,n}, \lambda_n) &= \frac{-(z_{2,n} + \lambda_n - q_{2,n}) - (z_{3,n} + \lambda_n - q_{3,n})}{(z_{2,n} + \lambda_n - q_{2,n})(z_{3,n} + \lambda_n - q_{3,n})} \\ &= \frac{-r_{2,n}(\eta_{1,n} + \eta_{2,n}) - 2i\,\mathrm{Im}(\lambda_n)}{(r_{2,n}\eta_{1,n} + i\,\mathrm{Im}(\lambda_n))(r_{2,n}\eta_{2,n} + i\,\mathrm{Im}(\lambda_n))}\,, \end{aligned}$$

where $r_{2,n}$, $\eta_{1,n}$ and $\eta_{1,n}$ are the co-ordinates defined by (12) and (13). Since for our sequence, $z_{2,n}, z_{3,n} \to -\lambda$, $q_{2,n}, q_{3,n} \to 0$ we have $r_{2,n} \to 0$. We also have $\operatorname{Im}\lambda_n \to 0$ so if we write $(r_{2,n}, \operatorname{Im}(\lambda_n))$ in polar co-ordinates, that is, $r_{2,n} = s_n\alpha_{1,n}$ and $\operatorname{Im}(\lambda_n) = s_n\alpha_{2,n}$ with $\alpha_{1,n}^2+\alpha_{2,n}^2 = 1$, then $s_n \to 0$. By going to a subsequence we may assume $\alpha_{1,n}$ and $\alpha_{2,n}$ converge to non-negative values α_1 and α_2. Then

$$\phi(z_{2,n}, z_{3,n}, q_{2,n}, q_{3,n}, \lambda_n) = \frac{-\alpha_{1,n}(\eta_{1,n} + \eta_{2,n}) - 2i\alpha_{2,n}}{s_n(\eta_{1,n} + i\alpha_{1,n})(\eta_{2,n} + i\alpha_{2,n})}\,.$$

The denominator of this expression converges to 0. The numerator converges to $-\alpha_1(\eta_1+\eta_2)-2i\alpha_2$ where η_1 and η_2 are co-ordinates in K for $\xi_{(1,2,3)}$. Since $\xi_{(1,2,3)}$ lies in Σ with $\overline{\mathbb{H}}$ co-ordinates $(-\lambda, -\lambda)$ we must have $\eta_1 + \eta_2 = e^{-i\psi}\sqrt{2}$ with $\psi \in [0, \pi]$. Here we used that the ω co-ordinates for $\xi_{(1,2,3)}$ are both $1/\sqrt{2}$. But now we see that it is impossible that $\alpha_1(\eta_1 + \eta_2) + 2i\alpha_2 = 0$, since that imaginary part being zero forces $\alpha_2 = 0$ and $\psi \in \{0, \pi\}$ in which case $\alpha_1 = 1$ so that $\alpha_1(\eta_1 + \eta_2) + 2i\alpha_2 = \pm\sqrt{2} \neq 0$. This implies that the limiting value of ϕ is $i\infty$.

Now we know that the point $\tau_{(1,2,3)}$ has $\overline{\mathbb{H}}$ co-ordinates $-\lambda$ and $i\infty$. Thus $\tau_{(1,2,3)} \in \Sigma$ requires that the ω_1 co-ordinate of $\tau_{(1,2,3)}$ be zero. Arguing as above, we find that in the limit, the first equation of (23) reads $1/3 = 0$. This contradiction concludes the proof of the theorem. $\square$

Proof of Lemma 5: Each term in the sum appearing in $\mu_{3,p}$ can be estimated

$$\begin{aligned}\frac{\operatorname{cd}^p(\phi(\cdots\cdots))}{\operatorname{cd}^p(z_1) + \operatorname{cd}^p(z_2) + \operatorname{cd}^p(z_3)} &= \frac{(\operatorname{cd}(z_1) + \operatorname{cd}(z_2) + \operatorname{cd}(z_3))^p}{\operatorname{cd}^p(z_1) + \operatorname{cd}^p(z_2) + \operatorname{cd}^p(z_3)}\left(\frac{\operatorname{cd}(\phi(\cdots\cdots))}{\operatorname{cd}(z_1) + \operatorname{cd}(z_2) + \operatorname{cd}(z_3)}\right)^p \\ &\leq 3^{p-1}\left(\frac{\operatorname{cd}(\phi(\cdots\cdots))}{\operatorname{cd}(z_1) + \operatorname{cd}(z_2) + \operatorname{cd}(z_3)}\right)^p,\end{aligned}$$

where $\phi(\cdots\cdots)$ denotes $\phi(z_{\sigma_1}, \phi(z_{\sigma_2}, z_{\sigma_3}, q_{\sigma_2}, q_{\sigma_3}, \lambda), q_{\sigma_1}, q_4, \lambda)$. Therefore it is enough to prove

$$\frac{\operatorname{cd}(\phi(\cdots\cdots))}{\operatorname{cd}(z_1) + \operatorname{cd}(z_2) + \operatorname{cd}(z_3)} \leq C(1 + \sum_{i=1}^{4} |q_i|^2)\,. \tag{28}$$

Let $\phi(\cdots)$ denote $\phi(z_{\sigma_2}, z_{\sigma_3}, q_{\sigma_2}, q_{\sigma_3}, \lambda)$. Then

$$\begin{aligned}\operatorname{Im}(\phi(\cdots)) &= \frac{\operatorname{Im}(z_{\sigma_2} + \lambda)}{|z_{\sigma_2} + \lambda - q_{\sigma_2}|^2} + \frac{\operatorname{Im}(z_{\sigma_3} + \lambda)}{|z_{\sigma_3} + \lambda - q_{\sigma_3}|^2} \\ &\geq \frac{\operatorname{Im}(z_{\sigma_2})}{|z_{\sigma_2} + \lambda - q_{\sigma_2}|^2}\,.\end{aligned}$$

Thus we have

$$\frac{\operatorname{cd}(\phi(\cdots\cdots))}{\operatorname{cd}(z_1) + \operatorname{cd}(z_2) + \operatorname{cd}(z_3)}$$

$$
\begin{aligned}
&= \frac{\left|-\dfrac{1}{z_{\sigma_1}+\lambda-q_{\sigma_1}}-\dfrac{1}{\phi(\cdots)+\lambda-q_{\sigma_4}}-z_\lambda\right|^2}{\operatorname{Im}\left(-\dfrac{1}{z_{\sigma_1}+\lambda-q_{\sigma_1}}-\dfrac{1}{\phi(\cdots)+\lambda-q_{\sigma_4}}\right)}\frac{1}{\sum_{i=1}^3|z_i-z_\lambda|^2/\operatorname{Im}(z_i)}\\
&= \frac{|(z_{\sigma_1}+\lambda-q_{\sigma_1})+(\phi(\cdots)+\lambda-q_{\sigma_4})+z_\lambda(z_{\sigma_1}+\lambda-q_{\sigma_1})(\phi(\cdots)+\lambda-q_{\sigma_4})|^2}{\operatorname{Im}(z_{\sigma_1}+\lambda)|\phi(\cdots)+\lambda-q_{\sigma_4}|^2+\operatorname{Im}(\phi(\cdots)+\lambda)|z_{\sigma_1}+\lambda-q_{\sigma_1}|^2}\\
&\quad\times\frac{1}{\sum_{i=1}^3|z_i-z_\lambda|^2/\operatorname{Im}(z_i)}\\
&\le \left(\frac{3}{\operatorname{Im}(\phi(\cdots))}+\frac{3+3|z_{\sigma_1}+\lambda-q_{\sigma_1}|^2}{\operatorname{Im}(z_{\sigma_1})}\right)\frac{1}{\sum_{i=1}^3|z_i-z_\lambda|^2/\operatorname{Im}(z_i)}\\
&\le \left(\frac{3|z_{\sigma_2}+\lambda-q_{\sigma_2}|^2}{\operatorname{Im}(z_{\sigma_2})}+\frac{3+3|z_{\sigma_1}+\lambda-q_{\sigma_1}|^2}{\operatorname{Im}(z_{\sigma_1})}\right)\frac{1}{\sum_{i=1}^3|z_i-z_\lambda|^2/\operatorname{Im}(z_i)}\,.
\end{aligned}
$$

Choose the compact set K so that $\sum_{i=1}^3|z_i-z_\lambda|^2/\operatorname{Im}(z_i)\ge C>0$ for some constant C if $(z_1,z_2,z_3)\in K^c$. Then we can estimate each term depending on whether z_{σ_i} is close to z_λ. If it is sufficiently close, then $\operatorname{Im}(z_{\sigma_i})$ is bounded below and $|z_{\sigma_i}|$ is bounded above by a constant. Thus

$$\operatorname{Im}(z_{\sigma_i})\sum_{i=1}^3|z_i-z_\lambda|^2/\operatorname{Im}(z_i)\ge\operatorname{Im}(z_{\sigma_i})C\ge C'>0$$

and $|z_{\sigma_i}+\lambda-q_{\sigma_i}|^2\le C(1+|q_{\sigma_i}|^2)$, so we are done. Otherwise

$$\operatorname{Im}(z_{\sigma_i})\sum_{i=1}^3|z_i-z_\lambda|^2/\operatorname{Im}(z_i)\ge|z_{\sigma_i}-z_\lambda|^2\ge C(1+|z_{\sigma_i}|^2)$$

so that $|z_{\sigma_i}+\lambda-q_{\sigma_i}|^2/\left(\operatorname{Im}(z_{\sigma_i})\sum_{i=1}^3|z_i-z_\lambda|^2/\operatorname{Im}(z_i)\right)\le C(1+|q_{\sigma_i}|^2)$ in this case too.

The estimates for $\mu'_{3,p}$ and $\mu'_{1,p}$ are very similar. We omit the details. $\square$

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References

[Ai] M. Aizenman, *Localization at weak disorder: some elementary bounds*, Rev. Math. Phys., **6**, 1163–1182, 1994.

[AM] M. Aizenman and S. Molchanov, *Localization at large disorder and at extreme energies: an elementary derivation*, Commun. Math. Phys., **157**, 245–278, 1993.

[ASW] M. Aizenman, R. Sims and S. Warzel, *Stability of the Absolutely Continuous Spectrum of Random Schrödinger Operators on Tree Graphs,* to appear in Prob. Theor. Rel. Fields, `math-phys/0502006`.

[An] P.W. Anderson, *Absence of diffusion in certain random lattices*, Phys. Rev., **109**, 1492–1505, 1958.

[D] S.A. Denisov, *On the preservation of absolutely continuous spectrum for Schrodinger operators*, `math-phys/0501046`.

[FHS] R. Froese, D. Hasler and W. Spitzer, *Transfer matrices, hyperbolic geometry and absolutely continuous spectrum for some discrete Schrödinger operators on graphs*, to appear in J. Func. Anal., `mp-arc:04-244`.

[K1] A. Klein, *Extended States in the Anderson Model on the Bethe Lattice*, Advances in Math., **133**, 163–184, 1998, `mp-arc:94-236`.

[K2] A. Klein, *Spreading of wave packets in the Anderson model on the Bethe lattice*, Commun. Math. Phys., **177**, 755–773, 1996, `mp-arc:95-481`.

[KLS] A. Laptev, S. Naboko and O. Safronov, *Absolutely continuous spectrum of Schrödinger operators with slowly decaying and oscillating potentials*, Commun. Math. Phys., **253**, 611–631, 2005, `mp-arc:03-383`.

[SS] O. Safronov and G. Stolz, *Absolutely continuous spectrum of Schrödinger operators with potentials slowly decaying inside a cone*, `mp-arc:05-316`.