JACOBI MATRICES WITH RAPIDLY GROWING WEIGHTS HAVING ONLY DISCRETE SPECTRUM

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ABSTRACT. We establish sufficient conditions for self-adjointness on a class of unbounded Jacobi operators defined by matrices with main diagonal sequence of very slow growth and rapidly growing off-diagonal entries. With some additional assumptions, we also prove that these operators have only discrete spectrum.

1. INTRODUCTION

Let $\{b_n\}_{n\in\mathbb{N}}$ be a real sequence whose elements are given by

$$b_n = n^{\alpha}\left(1+\frac{c_n}{n}+h_n\right)\,,\quad \alpha>1\,,\quad 1+\frac{c_n}{n}+h_n\neq 0\quad \forall n\in\mathbb{N}\,, \tag{1}$$

where $\{c_n\}_{n\in\mathbb{N}}$ is bounded with $\liminf|c_n|>0$, and $\{h_n\}_{n\in\mathbb{N}}$ is such that $h_n=O(n^{-\gamma})$ when $n\to\infty$, for some $\gamma>1$.

Let $\{q_n\}_{n\in\mathbb{N}}$ be a real sequence that grows not faster than logarithmically, viz.,

$$q_n = O(\log n)\quad \text{as } n\to\infty\,. \tag{2}$$

In the Hilbert space $l_2(\mathbb{N})$, we define the Jacobi operator J as the minimal closed operator such that

$$(Je_n,e_n)=q_n\,,\quad (Je_n,e_{n+1})=(Je_{n+1},e_n)=b_n\,,\quad \forall n\in\mathbb{N}\,, \tag{3}$$

where $\{e_n\}_{n\in\mathbb{N}}$ is the canonical basis in $l_2(\mathbb{N})$. Clearly, there are closed operators satisfying (3), thus J is correctly defined. This operator is

2000 *Mathematics Subject Classification.* Primary 47B36; Secondary 47A10.

Key words and phrases. Unbounded Jacobi Operators; Jacobi Matrices; Discrete Spectrum.

Research partially supported by Universidad Nacional Autónoma de México under Project PAPIIT-DGAPA IN 101902 and by CONACYT under Project P42553F.

unbounded and we have defined it in such a way that the semi-infinite Jacobi matrix

$$\begin{pmatrix} q_1 & b_1 & 0 & 0 & \cdots \\ b_1 & q_2 & b_2 & 0 & \cdots \\ 0 & b_2 & q_3 & b_3 & \\ 0 & 0 & b_3 & q_4 & \ddots \\ \vdots & \vdots & & \ddots & \ddots \end{pmatrix} \tag{4}$$

is the matrix representation of J with respect to the basis $\{e_n\}_{n\in\mathbb{N}}$ (we refer to [2] for a discussion on matrix representation of unbounded symmetric operators).

For $\alpha > 1$ fixed, different sequences $\{c_n\}_{n\in\mathbb{N}}$, $\{h_n\}_{n\in\mathbb{N}}$ and $\{q_n\}_{n\in\mathbb{N}}$ define different concrete realizations of J. Throughout this paper, we use J to denote anyone of these realizations prescribed by (1)–(3).

Since the off-diagonal entries (weights) of (4) grow fast ($\alpha > 1$), J is not always self-adjoint. It is known, nevertheless, that certain conditions on the weights ensure self-adjointness when $\{q_n\}_{n\in\mathbb{N}}$ is a bounded sequence. Indeed, the class of operators studied here is a generalization of an example suggested by Kostyuchenko and Mirzoev in [12] as an illustration of a Jacobi operator whose matrix representation does not satisfy the Carleman criterion [1] but nevertheless, under certain conditions, can be self-adjoint. The case of operators J for which $\{c_n\}_{n\in\mathbb{N}}$ is even-periodic and $\{q_n\}_{n\in\mathbb{N}} \equiv \{h_n\}_{n\in\mathbb{N}} \equiv 0$ has been studied in [6, 14]. There are formulae in [6] for the asymptotic behavior of the generalized eigenvectors and it is proven there, by means of the Subordinacy Theory [5, 10], that the spectrum of J is pure point in the self-adjoint case.

We carry out the spectral analysis of J on basis of the asymptotic behavior of its generalized eigenvectors. These asymptotics are obtained by means of a Levinson-type theorem for discrete linear systems. We use transfer matrices to study the generalized eigenequation of J and arrive to a system that turns out to be asymptotically diagonal in the sense that it satisfies the assumptions of the Benzaid-Lutz theorem [3]. The fact that we can study the generalized eigenequation through a discrete system, for which the Benzaid-Lutz theorem yields the asymptotic behavior of the solutions, is a remarkable property of this class of operators.

We impose certain mild conditions over the average behavior of the sequence $\{c_n\}_{n\in\mathbb{N}}$ in order to ensure self-adjointness. These conditions are far less restrictive that those used in previous works and contain

them as special cases. Also, these conditions turn out to be sufficient for the absence of continuous spectrum. In passing, we note that in [6, 14] a much stronger assumption of periodicity of the sequence $\{c_n\}_{n\in\mathbb{N}}$ was required to gain spectral information from the asymptotic behavior of the generalized eigenvectors of operator J. As it will become apparent later, the sequences $\{q_n\}_{n\in\mathbb{N}}$ and $\{h_n\}_{n\in\mathbb{N}}$, with the asymptotic behavior specified above, play no role concerning both self-adjointness and spectral properties of J. Unlike the methods used in [6, 14], our main results do not make use of Subordinacy Theory. Instead we mainly use general facts for Jacobi operators and Spectral Theory.

To the knowledge of the authors, the spectral analysis of Jacobi operators corresponding to matrices with rapidly growing weights has been treated in few occasions [6, 13, 14]. In contrast, literature on Jacobi matrices with slowly growing weights is more abundant (for instance, [4, 7, 8, 9, 17]). In particular, Jacobi operators with zero main diagonal and $b_n \sim n^\alpha$, with $1/2 < \alpha \leq 1$, has been studied in [7, 8, 9]. In these papers it is proven that, depending on the concrete realization, this kind of Jacobi operators may have either purely absolutely continuous spectrum or absolutely continuous spectrum with a gap where the spectrum is pure point.

This paper is organized as follows. In Section 2, we lay down the notation and some preparatory facts. In Section 3, we derive the asymptotic behavior of solutions of the discrete system associated to the generalized eigenequation for J. In Section 4, we establish sufficient conditions for J to be either self-adjoint (Theorem 4.1) or non self-adjoint (Theorem 4.2), and give examples. Finally, in Section 5, we characterize the spectrum (Corollary 5.3 and Theorem 5.5).

2. Preliminaries

In this section we introduce the notation and main concepts that will be used throughout this paper.

The elements of a scalar sequence will be denoted by Latin lowercase letters. For sequences of vectors and matrices we will use Greek lowercase letters and Latin capital letters, respectively. Besides the Hilbert space $l_2(\mathbb{N}) = l_2(\mathbb{N}, \mathbb{C})$, we shall consider the space $l_2(\mathbb{N}, \mathbb{C}^d)$ consisting of sequences of d-dimensional vectors $\{\xi_n\}_{n\in\mathbb{N}}$ such that $\sum_{n=1}^{\infty} \|\xi_n\|^2_{\mathbb{C}^d}$ is convergent, where $\|\cdot\|_{\mathbb{C}^d}$ denotes some norm in $\mathbb{C}^d$. Should no confusion arise, we will denote $\|\xi\|_{l_2(\mathbb{C}^d)}$ (for the norm in $l_2(\mathbb{N}, \mathbb{C}^d)$) simply as $\|\xi\|$. Keeping the notation used in the Introduction, the canonical basis in $l_2(\mathbb{N})$ is denoted by $\{e_n\}_{n\in\mathbb{N}}$, while $\{\delta^{(k)}\}_{k=1}^d$ will stand for the canonical basis in $\mathbb{C}^d$. Given a scalar sequence $\{f_n\}_{n\in\mathbb{N}}$, the standard asymptotic

notation $O(f_n)$ ($o(f_n)$) as $n \to \infty$, depending on the context, will be used to denote the n-th element of a sequence of either scalars, vectors, or matrices, whose norm behaves as $O(f_n)$ ($o(f_n)$) as $n \to \infty$. All products of matrices will be taken in "chronological" order, i. e.,

$$\prod_{s=p}^{m} A_s = A_m A_{m-1} \dots A_p \,.$$

Our approach to the spectral analysis of J is based on the study of the asymptotics of solutions of the generalized eigenvalue equation

$$b_{n-1}x_{n-1}(\zeta) + q_n x_n(\zeta) + b_n x_{n+1}(\zeta) = \zeta x_n(\zeta)\,, \quad n > 1\,,\ \zeta \in \mathbb{C}\,. \tag{5}$$

Clearly, if a solution $x(\zeta) = \{x_n(\zeta)\}_{n\in\mathbb{N}}$ satisfies the "boundary condition"

$$q_1 x_1(\zeta) + b_1 x_2(\zeta) = \zeta x_1(\zeta) \tag{6}$$

and turns out to be in $\mathrm{Dom}(J)$, then $x(\zeta)$ is an eigenvector of J corresponding to the eigenvalue ζ. To simplify somewhat the notation, in what follows we will not indicate the dependency of x on ζ.

Equation (5) can be written as follows

$$\xi_{n+1} = B_n(\zeta)\xi_n\,, \qquad n > 1\,, \quad \zeta \in \mathbb{R}\,, \tag{7}$$

where $\xi_n = \begin{pmatrix} x_{n-1} \\ x_n \end{pmatrix}$ and $B_n(\zeta) = \begin{pmatrix} 0 & 1 \\ -\frac{b_{n-1}}{b_n} & \frac{\zeta - q_n}{b_n} \end{pmatrix}$ is the so-called transfer matrix. From (7) it follows that

$$\xi_n = \prod_{k=n_0}^{n-1} B_k(\zeta)\xi_{n_0}\,, \qquad n_0 \geq 2\,. \tag{8}$$

Since clearly we have $\|x\|^2 < \|\xi\|^2 \leq 2\|x\|^2$, the asymptotic behavior (in norm) of solutions of (7) governs the asymptotic behavior of those of (5) (and vice-versa). There are various methods for obtaining the asymptotic behavior of solutions of discrete linear systems. As already mentioned, the main tools are the so-called discrete Levinson-type theorems. For the purpose of the present work we use the Levinson-type theorem proved by Benzaid and Lutz in [3] since, as it will be shown in the next section, system (7) can be rewritten as a discrete linear system which satisfies the conditions of that theorem. For the sake of completeness, we state the Benzaid-Lutz theorem below.

Definition 2.1. A sequence $Q = \{Q_n\}_{n=n_0}^{\infty}$ ($n_0 \in \mathbb{N}$) of $d \times d$ diagonal complex matrices,

$$Q_n = \mathrm{diag}\{q_n^{(k)}\}_{k=1}^{d}\,,$$

is said to satisfy the Levinson condition for k (denoted $Q \in \mathcal{L}(k)$) if there exists a natural number $N \geq n_0$ and a constant $M > 1$ such that, k being fixed, each j $(1 \leq j \leq d)$ falls into one and only one of the two classes I_1 or I_2, where

(a) $j \in I_1$ if

$$\frac{\left|\prod_{i=N}^{n} q_i^{(k)}\right|}{\left|\prod_{i=N}^{n} q_i^{(j)}\right|} \to \infty \quad \text{as } n \to \infty\,, \quad \text{and}$$

$$\frac{\left|\prod_{i=n_1}^{n_2} q_i^{(k)}\right|}{\left|\prod_{i=n_1}^{n_2} q_i^{(j)}\right|} > \frac{1}{M}\,, \quad \forall\, n_2\,, n_1 \text{ such that } n_2 > n_1 \geq N\,.$$

(b) $j \in I_2$ if

$$\frac{\left|\prod_{i=n_1}^{n_2} q_i^{(k)}\right|}{\left|\prod_{i=n_1}^{n_2} q_i^{(j)}\right|} < M\,, \quad \forall\, n_2\,, n_1 \text{ such that } n_2 > n_1 \geq N\,.$$

Theorem 2.2 (Benzaid-Lutz)**.** *Let $Q = \{Q_n\}_{n=n_0}^{\infty}$ be a sequence of $d \times d$ diagonal complex matrices,*

$$Q_n = \operatorname{diag}\{q_n^{(k)}\}_{k=1}^{d}\,.$$

Consider also another sequence $R = \{R_n\}_{n=n_0}^{\infty}$ $(n_0 \in \mathbb{N})$ of $d \times d$ complex matrices. Suppose that the following conditions hold:

i. *The sequence $Q \in \mathcal{L}(k)$, for all $k = 1, \ldots, d$,*

ii. $\displaystyle\sum_{n=n_0}^{\infty} \frac{\|R_n\|}{|q_n^{(k)}|} < \infty$,

iii. $\det(Q_n + R_n) \neq 0$ *for all* $n \geq n_0$.

Then one can find a natural number $n_1 \geq n_0$ such that there exists a basis $\varphi^{(k)}$ $(k = 1 \ldots, d)$ in the space of solutions of the system

$$\varphi_{n+1} = (Q_n + R_n)\varphi_n\,, \qquad n \geq n_1\,,$$

such that

$$\left\| \frac{\varphi_n^{(k)}}{\prod_{i=n_1}^{n-1} q_i^{(k)}} - \delta^{(k)} \right\| \to 0, \quad \text{as } n \to \infty\,, \quad \text{for } k = 1, \ldots, d\,.$$

3. Asymptotic behavior of solutions

Let us define the matrices

$$A_m(\zeta) = B_{2m+1}(\zeta)B_{2m}(\zeta)\,, \quad \forall m \in \mathbb{N}\,. \tag{9}$$

Clearly, system (7) is equivalent to

$$\eta_{m+1} = A_m(\zeta)\eta_m\,, \quad \forall m \in \mathbb{N}\,, \tag{10}$$

where $\eta_m = \xi_{2m}$. A straightforward computation shows that

$$B_{n+1}(\zeta)B_n(\zeta) = -I + \begin{pmatrix} \frac{c_n - c_{n-1} + \alpha}{n} & 0 \\ 0 & \frac{c_{n+1} - c_n + \alpha}{n} \end{pmatrix} + O(n^{-1-\epsilon}) \tag{11}$$

as $n \to \infty$, where I denotes the identity matrix and $\epsilon > 0$. We notice that the term $O(n^{-1-\epsilon})$ contains all the dependencies on the parameter ζ and sequences $\{q_n\}_{n\in\mathbb{N}}$ and $\{h_n\}_{n\in\mathbb{N}}$. Up to the same order in n, we may write

$$B_{n+1}(\zeta)B_n(\zeta) = -\begin{pmatrix} \exp\frac{c_{n-1} - c_n - \alpha}{n} & 0 \\ 0 & \exp\frac{c_n - c_{n+1} - \alpha}{n} \end{pmatrix}\left[I + O(n^{-1-\epsilon})\right],$$

where again the dependency on ζ is contained exclusively in the term $O(n^{-1-\epsilon})$. Thus, $A_m(\zeta)$ admits the decomposition

$$A_m(\zeta) = G_m + R_m(\zeta)\,,$$

where, for notational convenience, we define

$$G_m := -\begin{pmatrix} \exp g_m^{(1)} & 0 \\ 0 & \exp g_m^{(2)} \end{pmatrix}$$

with

$$g_m^{(1)} := \frac{c_{2m-1} - c_{2m} - \alpha}{2m} \quad \text{and} \quad g_m^{(2)} := \frac{c_{2m} - c_{2m+1} - \alpha}{2m}\,,$$

and the residual term is given by

$$R_m(\zeta) := G_m\, O(m^{-1-\epsilon})\,.$$

The r. h. s. of the last identity must be understood as the product of G_m times a matrix that depends on ζ and whose norm behaves as specified by the last factor as $m \to \infty$.

Lemma 3.1. *System (10) satisfies the assumptions of Theorem 2.2. Therefore, one can find $n_1 \in \mathbb{N}$ for which there exists a basis $\eta^{(k)}$ ($k = 1, 2$) in the space of solutions of (10) such that, for $k = 1$ and $k = 2$,*

$$\left\| \eta_m^{(k)} \Big/ \prod_{s=n_1}^{m-1} (-1) \exp g_s^{(k)} - \delta^{(k)} \right\| \to 0, \tag{12}$$

as $m \to \infty$.

Proof. We shall verify the fulfillment of conditions (i), (ii) and (iii) of Theorem 2.2, in that order.

Suppose that $\{G_m\}_{m\in\mathbb{N}} \notin \mathcal{L}(1)$, i. e., it does not satisfy the Levinson condition for $k = 1$. Then, it must occur that

$$\liminf_{m\to\infty} \sum_{s=1}^{m} \left(\frac{c_{2s-1} - c_{2s}}{2s} - \frac{c_{2s} - c_{2s+1}}{2s} \right) \neq +\infty$$

and

$$\limsup_{m\to\infty} \sum_{s=1}^{m} \left(\frac{c_{2s-1} - c_{2s}}{2s} - \frac{c_{2s} - c_{2s+1}}{2s} \right) = +\infty\,.$$

Let $\{\widetilde{m}_i\}_{i\in\mathbb{N}}$ be the sequence that contains all the sequences $\{m_i\}_{i\in\mathbb{N}}$ for which

$$\lim_{i\to\infty} \sum_{s=1}^{m_i} \left(\frac{c_{2s-1} - c_{2s}}{2s} - \frac{c_{2s} - c_{2s+1}}{2s} \right) = +\infty\,.$$

Clearly, $\{\widetilde{m}_i\}_{i\in\mathbb{N}}$ is such that:

(1) For any $i_0 \in \mathbb{N}$, there exists an $i > i_0$ such that $\widetilde{m}_i - 1 > \widetilde{m}_{i-1}$.
(2) For all sequences $\{m_i\}_{i\in\mathbb{N}}$ with the property that $m_i \notin \{\widetilde{m}_n\}_{n\in\mathbb{N}}$ for every i greater than some i_0, there exists M_0 (independent of the sequence) such that

$$\sup_{i\in\mathbb{N}} \left\{ \sum_{s=1}^{m_i} \frac{c_{2s-1} + c_{2s+1} - 2c_{2s}}{2s} \right\} < M_0\,.$$

Pick any $M > 0$ and choose i (of property (1)) such that

$$\sum_{s=1}^{m_i} \frac{c_{2s-1} + c_{2s+1} - 2c_{2s}}{2s} > M\,.$$

However, since $m_i - 1 \notin \{\widetilde{m}_i\}_{i\in\mathbb{N}}$, we also have

$$\sum_{s=1}^{m_i-1} \frac{c_{2s-1} + c_{2s+1} - 2c_{2s}}{2s} < M_0\,.$$

Thus,

$$M - M_0 < \frac{c_{2m_i-1} + c_{2m_i+1} - 2c_{2m_i}}{2m_i} < \frac{v_0}{2m_i}$$

valid for some constant v_0 (because $\{c_n\}_{n\in\mathbb{N}}$ is bounded). Clearly, the last inequality makes no sense since $M - M_0$ can be done as big as we want. We therefore conclude that condition (i) of Theorem 2.2 is satisfied for $k = 1$. By a similar reasoning, the same assertion holds for $k = 2$.

Now, we have

$$\frac{v_1}{2m} \leq g_m^{(k)} \leq \frac{v_2}{2m}$$

for $k = 1, 2$ and some constants v_1 and v_2. It follows that

$$\frac{\|R_m(\zeta)\|}{\exp g_m^{(k)}} \leq r_1 m^{-1-\epsilon} \frac{\|G_m\|}{\exp g_m^{(k)}} \leq r_2 m^{-1-\epsilon} \exp \frac{v_2 - v_1}{2m}$$

for some positive constants r_1 and r_2. This implies that the l. h. s. of the inequality above is summable, hence condition (ii) of Theorem 2.2 holds.

It remains to verify condition (iii) of Theorem 2.2. Since the determinant of each transfer matrix is different from zero, the same is true for the determinant of each A_m (which equals $G_m + R_m$). □

Up to a normalization constant, statement (12) is equivalent to saying that

$$\eta_{m+1}^{(k)} = (-1)^m \exp\left(\sum_{s=1}^{m} g_s^{(k)}\right) \left[\delta^{(k)} + \tau_m^{(k)}\right] , \tag{13}$$

where $\tau_m^{(k)}$ is $o(1)$ as $m \to \infty$. We will use this expression in the sequel.

Solutions of (10) obviously depend on ζ and the specific form of the sequences $\{q_n\}_{n\in\mathbb{N}}$ and $\{h_n\}_{n\in\mathbb{N}}$. However, equation (12) (or (13)) shows that the main asymptotic term does not.

Remark 1. We have the following consequence of Lemma 3.1: Let $\{m_j\}_{j\in\mathbb{N}}$ be a monotone growing sequence of natural numbers. Then we have, from (13),

$$\xi_{2m_j+2}^{(k)} = (-1)^{m_j} \exp\left(\sum_{s=1}^{m_j} g_s^{(k)}\right) \left[\delta^{(k)} + \tau_{m_j}^{(k)}\right] .$$

Moreover,

$$\xi_{2m_j+2p+2}^{(k)} = (-1)^{m_j+p} \exp\left(\sum_{s=1}^{m_j+p} g_s^{(k)}\right) \left[\delta^{(k)} + \tau_{m_j+p}^{(k)}\right]$$

for $p \in \mathbb{N}$. Thus, by splitting the exponential factor and working out residual terms, we obtain

$$\xi_{2m_j+2p+2}^{(k)} = (-1)^{p} \exp\left(\sum_{s=m_j+1}^{m_j+p} g_s^{(k)}\right) \left[\xi_{2m_j+2}^{(k)} + \phi_{j,p}^{(k)}\right] , \tag{14}$$

where the norm of $\phi_{j,p}^{(k)}$ equals the product of the norm of $\xi_{2m_j+2}^{(k)}$ times a function $o(1)$ as $j \to \infty$, uniformly in p.

We close this section noticing the fact that, in order to obtain the asymptotic behavior of solutions of (10), it was sufficient to consider those conditions on $\{c_n\}_{n\in\mathbb{N}}$, $\{h_n\}_{n\in\mathbb{N}}$, and $\{q_n\}_{n\in\mathbb{N}}$ given in the Introduction. However, without further assumptions on $\{c_n\}_{n\in\mathbb{N}}$, these asymptotics yields no clue about the decaying behavior of these solutions. This information is necessary to determine whether the operator is self-adjoint or not and, in the latter case, to classify the spectrum (by using, for instance, Subordinacy Theory). Theses issues will be attended, in the following sections, by requiring certain regularity on average of $\{c_n\}_{n\in\mathbb{N}}$ over sequences $\{m_j\}_{j\in\mathbb{N}}$. This will give rise to nice asymptotic behavior for the norm of $\xi^{(k)}_{2m_j+2}$ from which, the asymptotic behavior of the norm of $\xi^{(k)}_{2m+2}$ will be interpolated by resorting to (14).

4. Self-adjointness

It is well known that, for Jacobi matrices, we have the discrete analogue of Weyl's limit point–limit circle theory for self-adjoint extensions of differential operators. In particular, it is true that if two linearly independent solutions of (5) are in $l_2(\mathbb{N})$, then J is not self-adjoint (limit circle case). If, however, one solution of (5) is not in $l_2(\mathbb{N})$ then $J = J^*$ (limit point case) [1, 15].

Next, we establish sufficient conditions for J to be self-adjoint.

Theorem 4.1. *Define* $w^{(1)} = \min\{0, \inf_s\{c_{2s-1} - c_{2s} - \alpha\}\}$ *and* $w^{(2)} = \min\{0, \inf_s\{c_{2s} - c_{2s+1} - \alpha\}\}$. *Fix the value of* k. *Consider a monotone growing sequence* $\{m_j\}_{j\in\mathbb{N}}$ *of natural numbers that obeys*

$$\sum_{j=1}^{\infty} \frac{m_{j+1} - m_j}{m_j} \left(\frac{m_{j+1}}{m_j}\right)^{w^{(k)}} = \infty\,. \tag{15}$$

Suppose now that, for this sequence, the following holds:

$$\sum_{s=1}^{m_j} g_s^{(k)} \geq -\frac{1}{2}\log(m_j) + O(1), \quad \textit{when } j \to \infty\,. \tag{16}$$

Then, there exists a solution of (5) that is not in $l_2(\mathbb{N})$ *and, therefore,* $J = J^*$.

Proof. We prove the assertion under the assumption that $k = 1$. The case $k = 2$ follows analogously. We first note that (16) and (13) imply

$$\left\|\xi^{(1)}_{2m_j+2}\right\|^2 \geq r_1 m_j^{-1}\,, \quad \forall j \geq j_0, \tag{17}$$

where r_1 is some positive constant and j_0 is sufficiently big. Moreover, from (14) we obtain

$$\begin{aligned} \left\|\xi^{(1)}_{2m_j+2p+2}\right\|^2 &\geq r_2 \exp\left(2\sum_{s=m_j+1}^{m_j+p} g_s^{(1)}\right)\left\|\xi^{(1)}_{2m_j+2}\right\|^2 \\ &\geq r_3\left(\frac{m_j+p}{m_j+1}\right)^{w^{(1)}}\left\|\xi^{(1)}_{2m_j+2}\right\|^2, \end{aligned} \tag{18}$$

for some positive constants r_2, r_3 and $0 < p < m_{j+1} - m_j$. Inequality (18) follows from the fact that

$$\sum_{s=m_j+1}^{m_j+p} g_s^{(1)} \geq \frac{w^{(1)}}{2}\sum_{s=m_j+1}^{m_j+p}\frac{1}{s} = \frac{w^{(1)}}{2}\log\left(\frac{m_j+p}{m_j+1}\right) + O(1)\,,$$

as $j \to \infty$.

Now,

$$\begin{aligned} \left\|\xi^{(1)}\right\|^2 &\geq \sum_{m=0}^{\infty}\left\|\xi^{(1)}_{2m+2}\right\|^2 \\ &\geq \sum_{j=j_0}^{\infty}\left(\left\|\xi^{(1)}_{2m_j+2}\right\|^2 + \sum_{p=1}^{m_{j+1}-m_j-1}\left\|\xi^{(1)}_{2m_j+2p+2}\right\|^2\right) \\ &\geq r_1 r_3 \sum_{j=j_0}^{\infty}\frac{m_{j+1}-m_j-1}{m_{j+1}}\left(\frac{m_{j+1}-1}{m_j+1}\right)^{w^{(1)}}, \end{aligned}$$

where the last inequality follows from (17) and (18). Clearly, $\xi^{(1)}$ is not in $l_2(\mathbb{N},\mathbb{C}^2)$ if (15) is fulfilled. Since $\|\xi\| \asymp \|x\|$, it follows that $x^{(1)}$ does not belong to $l_2(\mathbb{N})$ (so J is in the limit point case). □

Remark 2. For $w^{(k)} = 0$, condition (15) seems to be met for any monotone growing sequence $\{m_j\}_{j\in\mathbb{N}}$. In any case, we could not find counterexamples. However, this condition is more stringent for $w^{(k)} < 0$, although it is satisfied for sequences of fairly fast growth, for instance, $m_j \sim \exp j$ (for any $w^{(k)} < 0$) and also $m_j \sim j^j$ (for $-2 \leq w^{(k)} < 0$). Clearly, sequences of polynomial growth, like $m_j \sim j^l$ with $l \geq 1$, also satisfy (15) for any $w^{(k)} \leq 0$.

Remark 3. Theorem 4.1 applies quite easily to the particular case where the sequence $\{c_n\}_{n\in\mathbb{N}}$ is even-periodic (say, of period $2L$). In this case $\left|\sum_{n=1}^{2L}(-1)^n c_n\right| \geq (\alpha-1)L$ implies (16) and therefore self-adjointness. As already mentioned, this model, with $\{q_n\}_{n\in\mathbb{N}} \equiv \{h_n\}_{n\in\mathbb{N}} \equiv 0$, was discussed in [6, 14].

Example 1. As a more involved example of application of Theorem 4.1, we now proceed with the construction of a self-adjoint operator J with no periodicity in $\{c_n\}_{n\in\mathbb{N}}$. Let the sequence $\{c_n\}_{n\in\mathbb{N}}$ be given by c_1, c_2, and c_3 as follows:

$$\underbrace{\underbrace{c_1\, c_2\, c_1\, c_2 \cdots c_1\, c_2\, c_1\, c_2}_{2p_1} c_2\, c_3\, c_2\, c_3 \cdots c_2\, c_3\, c_2\, c_3\, c_1\, c_2\, c_1\, c_2 \cdots}_{2p_2} \tag{19}$$

(we pray allowance for the abuse of notation), where $\{p_j\}_{j\in\mathbb{Z}^+}$ is a sequence with polynomial growth of the form $p_j = Cj^a$, with $a \in \mathbb{N}\backslash\{1\}$ and $C > 0$. Define $m_j = p_{2j}$; this sequence clearly satisfies (15). Next, we show that, for a suitable choice of c_1 and c_3, inequality (16) holds so $J = J^*$.

Set $k = 1$, then

$$\sum_{s=1}^{m_j} g_s^{(1)} = \frac{1}{2}\sum_{n=1}^{j}\sum_{s=m_{n-1}+1}^{m_n} \frac{c_{2s-1} - c_{2s}}{s} - \frac{\alpha}{2}\log(m_j) + O(1)\,. \tag{20}$$

The sequences $\{m_j\}_{j\in\mathbb{Z}^+}$ and $\{c_n\}_{n\in\mathbb{N}}$ are such that

$$\sum_{s=m_{n-1}+1}^{m_n} \frac{c_{2s-1} - c_{2s}}{s} = \sum_{s=p_{2n-2}+1}^{p_{2n-1}} \frac{c_1 - c_2}{s} + \sum_{s=p_{2n-1}+1}^{p_{2n}} \frac{c_2 - c_3}{s}\,.$$

Furthermore,

$$\begin{aligned}
&\sum_{s=p_{2n-2}+1}^{p_{2n-1}} \frac{c_1 - c_2}{s} + \sum_{s=p_{2n-1}+1}^{p_{2n}} \frac{c_2 - c_3}{s} \\
&\quad= \frac{1}{2}\Bigg(\sum_{s=p_{2n-2}+1}^{p_{2n-1}} \frac{c_1 - c_2}{s} + \sum_{s=p_{2n-2}+1}^{p_{2n-1}} \frac{c_2 - c_3}{s + p_{2n-1} - p_{2n-2}} \\
&\qquad + \sum_{s=2p_{2n-1}-p_{2n-2}+1}^{p_{2n}} \frac{c_2 - c_3}{s} + \sum_{s=p_{2n-1}+1}^{p_{2n}} \frac{c_1 - c_2}{s - p_{2n-1} + p_{2n-2}} \\
&\qquad - \sum_{s=p_{2n-1}+1}^{p_{2n-1}-p_{2n-1}+p_{2n-2}} \frac{c_1 - c_2}{s} + \sum_{s=p_{2n-1}+1}^{p_{2n}} \frac{c_2 - c_3}{s}\Bigg).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sum_{s=m_{n-1}+1}^{m_n} \frac{c_{2s-1}-c_{2s}}{s} \\
&= \frac{c_1-c_3}{2} \sum_{s=m_{n-1}+1}^{m_n} \frac{1}{s} \\
&\quad + \frac{c_2-c_3}{2}\left(\sum_{s=p_{2n-2}+1}^{p_{2n-1}} \frac{1}{s^2}\left(\frac{p_{2n-2}-p_{2n-1}}{1+\frac{p_{2n-1}-p_{2n-2}}{s}} \right) + \sum_{s=2p_{2n-1}-p_{2n-2}+1}^{p_{2n}} \frac{1}{s} \right) \\
&\quad + \frac{c_1-c_2}{2}\left(\sum_{s=p_{2n-1}+1}^{p_{2n}} \frac{1}{s^2}\left(\frac{p_{2n-1}-p_{2n-2}}{1+\frac{p_{2n-2}-p_{2n-1}}{s}} \right) + \sum_{s=p_{2n-1}+1}^{p_{2n}-p_{2n-1}+p_{2n+2}} \frac{1}{s} \right).
\end{aligned}$$

It is now easy to verify that

$$\sum_{1}^{m_j} g_s^{(1)} = \frac{c_1-c_3-2\alpha}{4}\log(m_j)+O(1)\,, \tag{21}$$

since, as a consequence of the polynomial growth of p_n, we have

$$(p_{2n-1}-p_{2n-2}) \sum_{s=m_{n-1}+1}^{m_n} \frac{1}{s^2} = O(n^{-2})$$

and

$$\sum_{s=2p_{2n-1}-p_{2n-2}+1}^{p_{2n}} \frac{1}{s} + \sum_{s=p_{2n-1}+1}^{p_{2n}-p_{2n-1}+p_{2n+2}} \frac{1}{s} = O(n^{-2})$$

as $n \to \infty$. From (21) it follows that if

$$c_1 - c_3 \geq 2(\alpha-1)\,,$$

then Theorem 4.1 holds. Since a similar result must follow when c_1 and c_3 are switched over (or from doing an analogous computation for $k=2$), we conclude that the inequality

$$|c_1-c_3| \geq 2(\alpha-1)$$

is sufficient for self-adjointness. We note the fact that the value of c_2 plays no role in this example. The computation done above shows also that this construction may be generalized to the cases where $\{p_j\}_{j\in\mathbb{Z}^+}$ grows faster that polynomially, with probably different conditions imposed upon c_1, c_2 and c_3 to ensure self-adjointness.

We now establish sufficient conditions for J to be non-selfadjoint.

Theorem 4.2. *Define v as the largest among 0, $\sup_s\{c_{2s} - c_{2s+1} - \alpha\}$ and $\sup_s\{c_{2s-1} - c_{2s} - \alpha\}$. Consider a monotone growing sequence $\{m_j\}_{j\in\mathbb{N}}$ that satisfies*

$$\sum_{j=1}^{\infty} \frac{m_{j+1} - m_j}{m_j^{\beta}} \left(\frac{m_{j+1}}{m_j}\right)^{v} < \infty\,, \tag{22}$$

for some $\beta > 1$. Now suppose that the inequality

$$\sum_{s=1}^{m_j} g_s^{(k)} \leq -\frac{\beta}{2}\log(m_j) + O(1)\,, \quad \textit{when } j \to \infty, \tag{23}$$

holds for both $k = 1$ and $k = 2$. Then all solutions of (5) are in $l_2(\mathbb{N})$ and, therefore, $J \neq J^*$.

Proof. Firstly, (14) and (23) imply

$$\left\|\xi^{(k)}_{2+2m_j}\right\|^2 < r_0 m_j^{-\beta}\,, \quad \forall j \geq j_1,$$

where r_0 is some positive constant. Without loss we may assume $j_1 = 1$ to avoid unnecessary complications. Secondly, (14) implies

$$\begin{aligned}\left\|\xi^{(k)}_{2m_j+2p+2}\right\|^2 &\leq r_1 \exp\left(2\sum_{s=m_j+1}^{m_j+p} g_s^{(k)}\right)\left\|\xi^{(k)}_{2m_j+2}\right\|^2 \\ &\leq r_2\left(\frac{m_j+p}{m_j+1}\right)^{v}\left\|\xi^{(k)}_{2m_j+2}\right\|^2,\end{aligned}$$

for $0 < p < m_{j+1} - m_j$ and some positive constants r_1 and r_2, where the last inequality follows from an argument analogous to the one yielding (18). Finally, we note that the transfer matrices $B_n(\zeta)$ are uniformly norm-bounded with respect to the index n. Let $r_3 > 0$ denote the square of such a bound.

Now, bearing these facts in mind, we have

$$\begin{aligned}
\left\|\xi^{(k)}\right\|^2 &\leq r_3 \sum_{m=m_1}^{\infty} \left\|\xi^{(k)}_{2m+2}\right\|^2 \\
&= r_3 \sum_{j=1}^{\infty} \sum_{p=0}^{m_{j+1}-m_j-1} \left\|\xi^{(k)}_{2m_j+2p+2}\right\|^2 \\
&\leq r_3 \sum_{j=1}^{\infty} \left\|\xi^{(k)}_{2m_j+2}\right\|^2 \left[1 + r_2 \sum_{j=1}^{m_{j+1}-m_j-1} \left(\frac{m_j+p}{m_j+1}\right)^v\right] \\
&\leq r_1 r_3 \sum_{j=1}^{\infty} \frac{1}{m_j^\beta} + r_1 r_2 r_3 \sum_{j=1}^{\infty} \frac{1}{m_j^\beta} \sum_{p=1}^{m_{j+1}-m_j-1} \left(\frac{m_j+p}{m_j+1}\right)^v .
\end{aligned}$$

The first term in the last inequality is clearly convergent, since $m_j \geq j$ and $\beta > 1$. The second term is also convergent since (22) is fulfilled.

Thus, we have proved that $\xi^{(1)}$ and $\xi^{(2)}$ belong to $l_2(\mathbb{N}, \mathbb{C}^2)$. It follows (by the argumentation given in the proof of Theorem 4.1) that $x^{(1)}$ and $x^{(2)}$ belong to $l_2(\mathbb{N})$. Moreover, they are linearly independent as a consequence of the linear independency of $\xi^{(1)}$ and $\xi^{(2)}$. The lack of self-adjointness now follows immediately. □

Remark 4. Theorem 4.2 does not allow to obtain the converse of Theorem 4.1.

Remark 5. A fairly large set of sequences $\{m_j\}_{j\in\mathbb{N}}$ satisfies condition (22), ranging from those of linear growth, i. e. $m_j \sim j$, up to those of very fast growth like $m_j \sim j^j$.

As we have mentioned previously, Jacobi operators of the form proposed in the present work has been already studied for the case of even-periodic sequences $\{c_n\}_{n\in\mathbb{N}}$. No results are known (to the authors anyway) concerning the odd-periodic case, although it has been suggested that, in this case, these operators would not be self-adjoint [14]. Theorem 4.2 allows us to prove that assertion easily.

Corollary 4.3. *Let $\{c_n\}_{n\in\mathbb{N}}$ be an odd-periodic sequence. Then the operator J, given by the matrix representation (4) with entries defined by (1) and (2), is not self-adjoint.*

Proof. Let T be the period of the sequence. Define $m_j = jT$ for $j \in \mathbb{N}$. Clearly, $\{m_j\}_{j\in\mathbb{N}}$ satisfies (22) for any $\beta > 1$ and $v \geq 0$. Thus, it only remains to verify (23) for $k = 1, 2$.

We have,

$$\begin{aligned}\sum_{s=1}^{m_j} g_s^{(1)} &= \frac{1}{2}\sum_{s=1}^{m_j}\frac{c_{2s-1}-c_{2s}-\alpha}{s}\\ &= \frac{1}{2}\sum_{n=1}^{j}\sum_{s=m_{n-1}+1}^{m_n}\frac{c_{2s-1}-c_{2s}}{s}-\frac{\alpha}{2}\sum_{s=1}^{m_j}\frac{1}{s}\\ &= \frac{1}{2}\sum_{n=1}^{j}\sum_{s=1}^{T}\frac{c_{2s-1}-c_{2s}}{s+(j-1)T}-\frac{\alpha}{2}\log(m_j)+O(1)\,, \qquad (24)\end{aligned}$$

where the first term in the last equality follows from the periodicity of $\{c_n\}_{n\in\mathbb{N}}$. Now, for $j\geq 2$,

$$\begin{aligned}\sum_{s=1}^{T}\frac{c_{2s-1}-c_{2s}}{s+(j-1)T} &= \sum_{s=1}^{T}\frac{c_{2s-1}-c_{2s}}{(j-1)T}\\ &\quad+\sum_{s=1}^{T}(c_{2s-1}-c_{2s})\left[\frac{1}{s+(j-1)T}-\frac{1}{(j-1)T}\right].\end{aligned}$$

The first term above equals zero because of the odd-periodicity of the sequence, while the second term is $O(j^{-2})$. Therefore, the first term in (24) is also $O(1)$ thus yielding the expected inequality (with $\beta=\alpha$) for $\sum_{s=1}^{h_m} g_s^{(1)}$. A similar computation shows that also $\sum_{s=1}^{h_m} g_s^{(2)}$ fulfills the same inequality. □

The proof of Corollary 4.3 tells us that an odd-periodic sequence $\{c_n\}_{n\in\mathbb{N}}$ produces a non-self-adjoint operator because one can find a suitable sequence $\{m_j\}_{j\in\mathbb{N}}$ for which, the contribution of $c_{2s-1}-c_{2s}$ to the l. h. s. of (23) is nearly canceled (and the same occurs for $c_{2s}-c_{2s+1}$). This idea may be used to construct non-selfadjoint Jacobi operators out of certain non-periodic sequences. As a matter of fact, the following example is defined essentially in the same way as the one discussed previously, having exactly one modification.

Example 2. Consider a sequence $\{c_n\}_{n\in\mathbb{N}}$ defined by two given numbers c_1 and c_2 arranged as follows:

$$\underbrace{\underbrace{c_1\,c_2\,c_1\,c_2\cdots c_1\,c_2\,c_1\,c_2}_{2p_1}\,c_2\,c_1\,c_2\,c_1\cdots c_2\,c_1\,c_2\,c_1\,c_1\,c_2\,c_1\,c_2\cdots}_{2p_2}, \qquad (25)$$

where $\{p_j\}_{j\in\mathbb{Z}^+}$ is a sequence with polynomial growth of the form $p_j=Cj^a$, with $a\in\mathbb{N}\setminus\{1\}$ and $C>0$. That is, we use the sequence defined by (21) with $c_3=c_1$. Alternatively, we may think of this sequence

as one obtained from a two-periodic sequence by inserting dislocations at sites $p_j + 1$. We claim that the Jacobi operator defined by (25) is not self-adjoint. To prove it, we first define $m_j = p_{2j}$. This sequence satisfies (22) for any $\beta > 1$, as a straightforward computation shows. Next, we must verify (23) for $k = 1, 2$. For $k = 1$, (20) holds true. The computation done in Example 1 shows that

$$\begin{aligned}
&\sum_{s=m_{n-1}+1}^{m_n} \frac{c_{2s-1} - c_{2s}}{s} \\
&= \frac{c_2 - c_1}{2} \left(\sum_{s=p_{2n-2}+1}^{p_{2n-1}} \frac{1}{s^2} \left(\frac{p_{2n-2} - p_{2n-1}}{1 + \frac{p_{2n-1} - p_{2n-2}}{s}} \right) + \sum_{s=2p_{2n-1}-p_{2n-2}+1}^{p_{2n}} \frac{1}{s} \right) \\
&\quad + \frac{c_1 - c_2}{2} \left(\sum_{s=p_{2n-1}+1}^{p_{2n}} \frac{1}{s^2} \left(\frac{p_{2n-1} - p_{2n-2}}{1 + \frac{p_{2n-2} - p_{2n-1}}{s}} \right) + \sum_{s=p_{2n-1}+1}^{p_{2n} - p_{2n-1} + p_{2n+2}} \frac{1}{s} \right).
\end{aligned}$$

By what has been shown before, the r. h. s. of this equality is $O(n^{-2})$ as $n \to \infty$. This, together with (20), yields

$$\sum_{s=1}^{m_j} g_s^{(1)} = -\frac{\alpha}{2} \log(m_j) + O(1)\,.$$

In a similar fashion, we can show that the same equality is true for $k = 2$. Hence, we have verified that this example fulfills the hypotheses of Theorem 4.2.

We note that the case $p_m \sim m$ can not be included here since the summability of

$$\sum_{j=1}^{\infty} \sum_{s=m_{j-1}+1}^{m_j} \frac{c_{2s-1} - c_{2s}}{s}$$

is no longer ensured. This is to be expected because, for $p_m \sim m$, the sequence $\{c_n\}_{n\in\mathbb{N}}$ is again even-periodic so the associated Jacobi operator may be self-adjoint (see Remark 3).

Other examples of non-self-adjoint Jacobi operators (of the class discussed in this work) could be devised along similar ideas. We think that the example given above is of some interest because it is derived from an operator defined by a even-periodic sequence and therefore self-adjoint (provided that $|c_1 - c_2| \geq \alpha - 1$, although this condition plays no role in our example) [6, 14].

5. Discreteness of spectrum

In this section we show that, if the assumptions of Theorem 4.1 are fulfilled and $\{m_j\}_{j\in\mathbb{N}}$ obeys certain further conditions, then J has only discrete spectrum.

To simplify proofs, we will assume that the conditions of Theorem 4.1 are satisfied for $k = 1$ (so $\xi^{(1)}$ is not in $l_2(\mathbb{N}, \mathbb{C}^2)$).

We first prove an easy consequence of inequality (16).

Lemma 5.1. *If (16) is true for $k = 1$, then*

$$\sum_{s=1}^{m_j} g_s^{(2)} < -\frac{\alpha}{2}\log(m_j) + O(1)\,, \quad \text{when } m \to \infty\,. \tag{26}$$

An analogous assertion holds if the roles of $k = 1$ and $k = 2$ are interchanged.

Proof. We have

$$\sum_{s=1}^{m_j} g_s^{(2)} = -\sum_{s=1}^{m_j} g_s^{(1)} + \sum_{s=1}^{m_j} \frac{c_{2s-1} - c_{2s+1}}{2s} - \alpha \sum_{s=1}^{m_j} \frac{1}{s}\,. \tag{27}$$

Furthermore,

$$\sum_{s=1}^{m_j} \frac{c_{2s-1} - c_{2s+1}}{2s} = \sum_{s=2}^{m_j} \frac{c_{2s-1}}{2}\left(\frac{1}{s} - \frac{1}{s-1}\right) + \frac{c_1}{2} - \frac{c_{2m_j+1}}{2m_j}\,.$$

Thus, the second term in (27) is $O(1)$ because the sequence $\{c_n\}_{n\in\mathbb{N}}$ is bounded and $(s^{-1} - (s-1)^{-1})$ is $O(s^{-2})$. Since moreover the last term in (27) equals $\alpha\log(m_j)$ up to a term $O(1)$, we obtain

$$\begin{aligned}
\sum_{s=1}^{m_j} g_s^{(2)} &= -\sum_{s=1}^{m_j} g_s^{(1)} - \alpha\log(m_j) + O(1) \\
&\le \frac{1}{2}\log(m_j) - \alpha\log(m_j) + O(1) \\
&< -\frac{\alpha}{2}\log(m_j) + O(1)\,,
\end{aligned}$$

where the last inequality holds since $\alpha > 1$. □

Theorem 5.2. *Suppose that the hypotheses of Theorem 4.1 are satisfied for $k = 1$ (then $x^{(1)}$ is not in $l_2(\mathbb{N})$). In addition, assume that $\{m_j\}_{j\in\mathbb{N}}$ satisfies*

$$\sum_{j=1}^{\infty} \frac{m_{j+1} - m_j}{m_j^{\alpha}} \left(\frac{m_{j+1}}{m_j}\right)^{\check{v}} < \infty\,, \tag{28}$$

with $\check{v} = \max\{0, \sup_s\{c_{2s} - c_{2s+1} - \alpha\}\}$. *Then* $x^{(2)}$ *is in* $l_2(\mathbb{N})$. *The assertion holds also if we interchange* $k = 1$ *and* $k = 2$ *and take* $\check{v} = \max\{0, \sup_s\{c_{2s-1} - c_{2s} - \alpha\}\}$.

Proof. By Lemma 5.1 we have (26) and then, taking into account (28) and reasoning as in Theorem 4.2, we obtain that $\xi^{(2)}$ is in $l_2(\mathbb{N}, \mathbb{C}^2)$. □

Remark 6. In view of Remarks 2 and 5, the set of sequences $\{m_j\}_{j\in\mathbb{N}}$ that fulfill both (15) and (28) is far from being empty.

Corollary 5.3. *J has only pure point spectrum whenever the assumptions of Theorem 5.2 are met.*

Proof. Under the assumptions of the previous theorem we have that $x^{(1)} \notin l_2(\mathbb{N})$ and $x^{(2)} \in l_2(\mathbb{N})$. This behavior of the basis $\{x^{(1)}, x^{(2)}\}$ in the space of solutions of (5) is the same for any $\zeta \in \mathbb{C}$. Thus, we always have a subordinate solution for the generalized eigenequation of the self-adjoint operator J. By applying Subordinacy Theory [10, Theorem 3], we conclude that the spectrum is pure point. □

Remark 7. Notice that Theorem 4.1 does not necessarily imply Theorem 5.2. However, in account of what is said in Remark 6, the implication is true for a large set of Jacobi operators J.

We notice that (15) along with (26) imply a certain correlation between the slow decay of $x^{(1)}$ and the fast decay of $x^{(2)}$ (or vice-versa). We shall use this to show that the resolvent $(J - \zeta I)^{-1}$ is Hilbert-Schmidt. Unfortunately, (16) is not good enough for that purpose. The next technical result accounts for what is needed.

Lemma 5.4. *Suppose that for a certain sequence* $\{m_j\}_{j\in\mathbb{N}}$ *one of the following holds:*

i. For any natural i *greater than some* $i_0 \in \mathbb{N}$,

$$\sum_{s=m_i+1}^{m_{i+1}} g_s^{(1)} \geq -\frac{1}{2}\log\left(\frac{m_{i+1}}{m_i}\right) + f_i\,,$$

where $\sum_{i=j}^{l} f_i$ *has an upper bound for any* j *and* l $(i_0 \leq j < l)$.

ii. For any natural numbers j *and* l, *where* $i_0 \leq j < l$ *and* $i_0 \in \mathbb{N}$,

$$\sum_{s=m_j+1}^{m_l} \frac{c_{2s} - c_{2s+1}}{s}$$

has an upper bound.

Then,

$$\exp\left(\sum_{s=m_j+1}^{m_l} g_s^{(2)}\right) \leq r\left(\frac{m_l}{m_j}\right)^{-\frac{\alpha}{2}},$$

for some constant r. A similar result holds if the roles of $k=1$ and $k=2$ are interchanged and $c_{2s}-c_{2s+1}$ is substituted by $c_{2s-1}-c_{2s}$ in (ii)

Proof. Let us assume (i) first. From (27), it follows that

$$\exp\left(\sum_{s=m_j+1}^{m_l} g_s^{(2)}\right) = \exp\left(-\sum_{s=m_j+1}^{m_l} g_s^{(1)} + \sum_{s=m_j+1}^{m_l} \frac{c_{2s-1}-c_{2s+1}}{2s} - \alpha \sum_{s=m_j+1}^{m_l} \frac{1}{s}\right)$$

As in the proof of Lemma 5.1, one has

$$\sum_{s=m_j+1}^{m_l} \frac{c_{2s-1}-c_{2s+1}}{2s} = \sum_{s=m_j+2}^{m_l} \frac{c_{2s-1}}{2}\left(\frac{1}{s}-\frac{1}{s-1}\right) + \frac{c_{2m_j+1}}{2m_j+2} - \frac{c_{2m_l+1}}{2m_l},$$

where the first term above is clearly $O(m_j^{-1})$. Therefore, there is a constant r_1 such that, for any pair $j<l$,

$$\exp\left(\sum_{s=m_j+1}^{m_l} \frac{c_{2s-1}-c_{2s+1}}{2s}\right) < r_1\,.$$

We thus conclude that there exists a constant r_2 such that

$$\exp\left(\sum_{s=m_j+1}^{m_l} g_s^{(2)}\right) \leq r_2\left(\frac{m_l}{m_j}\right)^{-\frac{\alpha}{2}} \exp\left(\sum_{i=j}^{l} f_i\right)$$

which proves our assertion.

In the case where (ii) holds, we have the result straightforwardly. □

Remark 8. Condition (i) of last lemma clearly implies (16).

The results proven above allow us to give conditions ensuring the discreteness of J's spectrum when it is self-adjoint.

Theorem 5.5. *Consider a sequence $\{m_j\}_{j\in\mathbb{N}}$ that complies with (15) and*

$$\sum_{j=1}^{\infty} \frac{m_{j+1}-m_j}{m_j^{\alpha}}\left(\frac{m_{j+1}}{m_j}\right)^{\hat{v}} < \infty\,, \tag{29}$$

where $\hat{v} = \max\{0, \sup_s\{c_{2s} - c_{2s+1} - \alpha\}, \sup_s\{c_{2s-1} - c_{2s} - \alpha\}\}$. *Suppose moreover the fulfillment of either condition (i) of Lemma 5.4, or condition (ii) of the same lemma and (16). Then the resolvent*

$$(J - \zeta I)^{-1}, \qquad \zeta \in \mathbb{C} \setminus \sigma(J),$$

is Hilbert-Schmidt and, therefore, the spectrum of J *is discrete.*

Proof. We first note that (29) implies (28) since $\hat{v} \geq \check{v}$. Thus, the hypotheses of the theorem (along with Remark 8) imply that both Theorems 4.1 and 5.2 hold true. Thence, there exists a basis $\{x^{(1)}, x^{(2)}\}$ in the space of solutions of (5), with one element in $l_2(\mathbb{N})$ and the other outside it. Let $x^{(2)}$ be the element in $l_2(\mathbb{N})$.

In what follows, $r_1, r_2, \ldots, r_{12}$ will stand for certain suitable constants that will appear in estimations along the way.

Let us denote by $P_n(\zeta)$ the n-th orthogonal polynomial of the first kind associated to J. Define the sequence $u = \{u_n\}_{n\in\mathbb{N}}$ in $l_2(\mathbb{N})$ by

$$u = (J - \zeta I)^{-1} e_1 .$$

According to [11, 16], the Green function is given by

$$G_{m\,n}(\zeta) = \begin{cases} P_m(\zeta)u_n(\zeta) & \text{when } m \leq n , \\ P_n(\zeta)u_m(\zeta) & \text{otherwise .} \end{cases}$$

The sequence $\{P_n(\zeta)\}_{n\in\mathbb{N}}$ is not square-summable because $J = J^*$ [1, 15]. Since it belongs to the space of solutions of (5), it may be expressed as a linear combination of the basis $\{x^{(1)}, x^{(2)}\}$. However, the solution $u = \{u_n\}_{n\in\mathbb{N}}$ of (5) must be proportional to $x^{(2)}$. Noticing that $\|\xi_m\|^2 = |x_{m-1}|^2 + |x_m|^2 > |x_m|^2$, we have (dependencies on ζ are dropped from now on)

$$|G_{mn}| < \begin{cases} r_0 \left\|\xi_m^{(1)}\right\| \left\|\xi_n^{(2)}\right\| + r_1 \left\|\xi_m^{(2)}\right\| \left\|\xi_n^{(2)}\right\| & \text{when } m \leq n , \\ r_0 \left\|\xi_m^{(2)}\right\| \left\|\xi_n^{(1)}\right\| + r_1 \left\|\xi_m^{(2)}\right\| \left\|\xi_n^{(2)}\right\| & \text{otherwise .} \end{cases}$$

Since $\xi_m = B_{m-1}\xi_{m-1}$ and B_n is uniformly norm-bounded in n, we have $\|\xi_{2m+1}\| \leq r_2 \|\xi_{2m}\|$. It thus follows that

$$\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty} |G_{mn}|^2 < \sum_{k_0\leq k\leq j}^{\infty} \left(r_3 \left\|\xi_{2k+2}^{(1)}\right\| \left\|\xi_{2j+2}^{(2)}\right\| + r_4 \left\|\xi_{2k+2}^{(2)}\right\| \left\|\xi_{2j+2}^{(2)}\right\| \right)^2$$

$$+ \sum_{j_0\leq j<k}^{\infty} \left(r_3 \left\|\xi_{2k+2}^{(2)}\right\| \left\|\xi_{2j+2}^{(1)}\right\| + r_4 \left\|\xi_{2k+2}^{(2)}\right\| \left\|\xi_{2j+2}^{(2)}\right\| \right)^2, \tag{30}$$

for some large m_0, n_0, k_0 and j_0.

Let us consider any term in the first sum above. Since $k = m_q + p$ and $j = m_t + l$ for some m_q, m_t, $0 \le p < m_{q+1} - m_q$ and $0 \le l < m_{t+1} - m_t$, these terms may be written as

$$r_3 \left\| \xi^{(1)}_{2m_q+2p+2} \right\| \left\| \xi^{(2)}_{2m_t+2l+2} \right\| + r_4 \left\| \xi^{(2)}_{2m_q+2p+2} \right\| \left\| \xi^{(2)}_{2m_t+2l+2} \right\|. \tag{31}$$

From (14), it follows that

$$\begin{aligned} \left\| \xi^{(k)}_{2m_q+2p+2} \right\| &\le r_5 \exp\left(\sum_{s=m_q+1}^{m_q+p} g_s^{(k)} \right) \left\| \xi^{(k)}_{2m_q+2} \right\| \\ &\le r_6 \left(\frac{m_q+p}{m_q} \right)^{\frac{\hat{v}}{2}} \left\| \xi^{(k)}_{2m_q+2} \right\| \end{aligned}$$

holds for $k = 1, 2$. Thus, (31) is bounded from above by

$$r_7 \left(\frac{m_q+p}{m_q} \right)^{\frac{\hat{v}}{2}} \left(\frac{m_t+l}{m_t} \right)^{\frac{\hat{v}}{2}} \left(\left\| \xi^{(1)}_{2m_q+2} \right\| \left\| \xi^{(2)}_{2m_t+2} \right\| + \left\| \xi^{(2)}_{2m_q+2} \right\| \left\| \xi^{(2)}_{2m_t+2} \right\| \right).$$

Let us now look at the sum between parentheses in the last expression. On one hand, since Lemma 5.1 holds, we have

$$\left\| \xi^{(2)}_{2m_q+2} \right\| \left\| \xi^{(2)}_{2m_t+2} \right\| \le r_8 \, m_q^{-\frac{\alpha}{2}} m_t^{-\frac{\alpha}{2}}.$$

On the other hand, since $k \le j$ implies $m_q \le m_t$,

$$\begin{aligned} \left\| \xi^{(1)}_{2m_q+2} \right\| \left\| \xi^{(2)}_{2m_t+2} \right\| &\le r_9 \exp\left(\sum_{s=1}^{m_q} g_s^{(1)} + \sum_{s=1}^{m_q} g_s^{(2)} \right) \exp\left(\sum_{s=m_q+1}^{m_t} g_s^{(2)} \right) \\ &\le r_9 \exp\left[-\alpha \log(m_q) + O(1) \right] \left(\frac{m_t}{m_q} \right)^{-\frac{\alpha}{2}} \\ &\le r_{10} \, m_q^{-\frac{\alpha}{2}} m_t^{-\frac{\alpha}{2}}, \end{aligned}$$

where the second inequality above follows from the proof of Lemma 5.1 along with Lemma 5.4. Therefore, we have

$$r_3 \left\| \xi^{(1)}_{2m_q+2p+2} \right\| \left\| \xi^{(2)}_{2m_t+2l+2} \right\| + r_4 \left\| \xi^{(2)}_{2m_q+2p+2} \right\| \left\| \xi^{(2)}_{2m_t+2l+2} \right\| \le r_{11} \, m_q^{-\frac{\alpha}{2}} m_t^{-\frac{\alpha}{2}}.$$

Similar estimates show that the same bound holds for the terms of the second sum in (30). By rearranging sums, we finally obtain

$$\begin{aligned}
&\sum_{m=m_0}^{\infty}\sum_{n=n_0}^{\infty}|G_{mn}|^2 \\
&< r_{11}\sum_{m=q_0}^{\infty}\sum_{p=0}^{m_{q+1}-m_q-1} m_q^{-\alpha}\left(\frac{m_q+p}{m_q}\right)^{\hat{v}}\sum_{t=t_0}^{\infty}\sum_{l=0}^{m_{t+1}-m_t-1} m_t^{-\alpha}\left(\frac{m_t+l}{m_t}\right)^{\hat{v}} \\
&< r_{12}\sum_{q=q_0}^{\infty}\frac{m_{q+1}-m_q}{m_q^{\alpha}}\left(\frac{m_{q+1}}{m_q}\right)^{\hat{v}}\sum_{t=t_0}^{\infty}\frac{m_{t+1}-m_t}{m_t^{\alpha}}\left(\frac{m_{t+1}}{m_t}\right)^{\hat{v}}.
\end{aligned}$$

Since (29) holds, we conclude that $\mathrm{Tr}\,(G^*G)<\infty$. □

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