Method of variations of potential of quasi-periodic Schrödinger equation

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Abstract

We study the one-dimensional discrete quasi-periodic Schrödinger equation

$$-\varphi(n+1)-\varphi(n-1)+\lambda V(x+n\omega)\varphi(n)=E\varphi(n), \qquad n\in\mathbb{Z}$$

We show that for "typical" C^3 potential V, if the coupling constant λ is large, then for most frequencies ω, the Lyapunov exponent is positive for all energies E, and the corresponding eigenfunctions φ decay exponentially.

1 Introduction

Given any function $V:\mathbb{T}\to\mathbb{R}$, we have a family of quasi-periodic discrete Schröinger equations

$$-\varphi(n+1)-\varphi(n-1)+\lambda V(x+n\omega)\varphi(n)=E\varphi(n), \qquad n\in\mathbb{Z} \tag{1.1}$$

where $(x,\omega)\in\mathbb{T}\times\mathbb{T}$ are parameters.

Equation (1.1) can be rewritten as a first order difference equation:

$$\begin{pmatrix}\varphi(n+1)\\ \varphi(n)\end{pmatrix}=\begin{pmatrix}\lambda V(x+n\omega)-E & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\varphi(n)\\ \varphi(n-1)\end{pmatrix}.$$

The *monodromy* matrix of this equation is

$$M_{[a,b]}(x,\omega,E)=\begin{pmatrix}\lambda V(x+b\omega)-E & -1\\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix}\lambda V(x+a\omega)-E & -1\\ 1 & 0\end{pmatrix}.$$

To study the properties of the spectrum and eigenfunctions of equation (1.1) for generic (x,ω), we make use of the *Lyapunov exponent*. It is defined by

$$L(\omega,E)=\lim_{n\to\infty}\frac{1}{n}\int_{\mathbb{T}}\left\|M_{[1,n]}(x,\omega,E)\right\|dx$$

By Kingman's subadditive ergodic theorem, if the shift $x \mapsto x + \omega$ is ergodic, then the limit exists and

$$L(\omega, E) = \lim_{n\to\infty} \frac{1}{n} \big\| M_{[1,n]}(x, \omega, E) \big\|$$

for almost all $x \in \mathbb{T}$.

For analytic potential V, Bourgain, Goldstein[BG] and Goldstein, Schlag[GS1] showed that the Lyapunov exponent is positive for large $|\lambda|$. Furthermore, if $L(\omega_0, E_0) > 0$, then for most ω close to ω_0, there exists $\delta > 0$ such that

> the spectrum of (1.1) in $(E_0 - \delta, E_0 + \delta)$ is pure point and the corresponding eigenfunctions decay exponentially.

This property is called *Anderson localization*. (See the recent monograph by Bourgain[Bo] for more details.) Generalization of the method of [BG] and [GS] for potential in a Gevrey-class was established by Klein[Kl].

In this paper, we use Goldstein and Schlag's methods introduced in a recent work[GS2] to study the spectrum and eigenfunctions of equation (1.1). We develop these methods for smooth potentials in the perturbative regime.

It is well known that the monodromy matrix is of the form

$$M_{[a,b]}(x, \omega, E) = \begin{pmatrix} f_{[a,b]}(x, \omega, E) & -f_{[a+1,b]}(x, \omega, E) \\ f_{[a,b-1]}(x, \omega, E) & -f_{[a+1,b-1]}(x, \omega, E) \end{pmatrix}$$

where

$$f_{[p,q]}(x, \omega, E) = \det\big[H_{[p,q]}(x, \omega) - E\big]$$

$$H_{[p,q]}(x, \omega) = \begin{pmatrix} \lambda V(x + p\omega) & -1 & & & \\ -1 & \lambda V\big(x + (p+1)w\big) & -1 & & \\ & -1 & & \ddots & \\ & \ddots & & & -1 \\ & & & -1 & \lambda V(x + q\omega) \end{pmatrix}$$

The spectrum of the Schrödinger equation

$$-\varphi(n+1) - \varphi(n-1) + \lambda V(x + n\omega)\varphi(n) = E\varphi(n),$$

with zero boundary conditions $\varphi(a-1) = 0$ and $\varphi(b+1) = 0$, consists of the eigenvalues of $H_{[a,b]}(x, \omega)$.

If $V(x + m\omega) \approx V(x + n\omega)$ for some $m, n \in [a, b]$, $m \neq n$, one says that the equation exhibits "resonance". Difficulties in studying the spectrum of $H_{[a,b]}(x, \omega)$ arise when resonances occur; two eigenvalues of $H_{[a,b]}(x, \omega)$ can be very close and the corresponding eigenfunctions need not be "localized".

In this paper, we develop methods to deal with resonances. We define some functions $E^{(s)}(x,\omega)$ which take their values in $\operatorname{sp} H_{[-N_s,N_s]}(x,\omega)$, for appropriate $N_{s+1} \asymp e^{N_s^\alpha}$, $s = 1, 2, \ldots$ In Part I, for C^1 potential, we eliminate a small part of $\mathbb{T}\times\mathbb{T}$ from the domain of $E^{(s)}(x,\omega)$ at each step. This allows us to exclude the case when two eigenvalues of $H_{[a,b]}(x,\omega)$, $b-a \asymp N_{s+1}$, become too close. We are then able to define exponentailly decaying eigenfunctions. The result is the following theorem.

Theorem 1.1. *Let $\mathcal{J} = V(\mathbb{T})$. There exists $\lambda_0 = \lambda_0(V)$ such that, for any $|\lambda| > \lambda_0$, the following holds:*
There is $\Omega = \Omega(V,\lambda) \in [0,1]$, $\operatorname{mes}\big([0,1]\setminus\Omega\big) \lesssim \lambda^{-1/2}$, such that for any $\omega \in \Omega$ there exists $\mathcal{E}_\omega \subset \lambda\mathcal{J}$, $\operatorname{mes}\big(\mathcal{J}\setminus\lambda^{-1}\mathcal{E}_\omega\big) \lesssim \lambda^{-1/2}$, with $L(\omega,E) \gtrsim \log\lambda$ for all $E \in \mathcal{E}_\omega$. Also, for any $E \in \mathcal{E}_\omega$, there is $x \in \mathbb{T}$ and $\{\varphi(n)\}$, $|\varphi(n)| \lesssim e^{-c|n|}$ such that

$$-\varphi(n+1) - \varphi(n-1) + V(x+n\omega)\varphi(n) = E\varphi(n).$$

In a recent paper, Bjerklöv[Bj] has obtained similar results.

The major drawback of the method in Part I is that part of the spectrum is also eliminated. To include all spectral values, we use variations of potential in Part II. Given a C^3 potential V, any C^3 function $\tilde{V}$ satisfying the conditions

$$\max_{x\in\mathbb{T}} |V(x) - \tilde{V}(x)| < \delta$$
$$\max_{x\in\mathbb{T}} |V'(x) - \tilde{V}'(x)| < \delta$$
$$\max_{x\in\mathbb{T}} |V''(x) - \tilde{V}''(x)| < \delta$$

can be written, near $x = 0$, in the form

$$\tilde{V}(x) = V(x) + \eta + \xi x + \frac{1}{2}\theta x^2 + x^3 R(x)$$

where $|\eta|, |\xi|, |\theta| < \delta$, $R \in C^3$, $|\partial_\alpha R| \lesssim 1$ for any index $|\alpha| \le 2$. More generally, since $\mathbb{T}$ is compact, we can find large integer T so that

$$\tilde{V}(x) = V(x) + \sum_{m=1}^{T}\Big[\eta_m + \xi_m\big(x - \frac{m}{T}\big) + \frac{1}{2}\theta_m\big(x-\frac{m}{T}\big)^2 + \big(x-\frac{m}{T}\big)^3 R_m\big(x-\frac{m}{T}\big)\Big] \tag{1.2}$$

for all $x \in \mathbb{T}$, where $\eta = (\eta_1,\ldots,\eta_T)$, $\xi = (\xi_1,\ldots,\xi_T)$, $\theta = (\theta_1,\ldots,\theta_T) \in \prod_1^T[-\delta,\delta]$, and $R_m \in C^3$, $|\partial_\alpha R_m| \lesssim 1$ for any index $|\alpha| \le 2$. This motivate the following definition.
Definition. Let T be a large integer, $0 < \delta \ll \frac{1}{T^5}$. Suppose $R_m(\eta_m,\xi_m,\theta_m;x)$ are C^3 functions,

$m = 1, 2, \ldots, T$, $(\eta, \xi, \theta) \in \prod_1^{3T}[-\delta, \delta]$, $x \in \mathbb{T}$, satisfying the following conditions:

$$|\partial_\alpha R_m(\eta_m, \xi_m, \theta_m; x)| \lesssim \frac{1}{T} \qquad \text{for any index } |\alpha| \le 3 \tag{1.3}$$

$$R_m(0, 0, 0; x) \equiv 0 \tag{1.4}$$

$$R_m(\eta_m, \xi_m, \theta_m; x) = -x^{-3}\big(\eta_m + \xi_m x + \frac{1}{2}\theta_m x^2\big) \qquad \text{for } |x| \ge \frac{1}{2T} \tag{1.5}$$

Define a (T, δ)–variation of potential by

$$W(\eta, \xi, \theta, \{R_m\}; x) = \sum_{m=1}^{T} v_m\Big(\eta_m, \xi_m, \theta_m; x - \frac{m}{T}\Big)$$

where

$$v_m(\eta_m, \xi_m, \theta_m; x) = \eta_m + \xi_m x + \frac{1}{2}\theta_m x^2 + x^3 R_m(\eta_m, \xi_m, \theta_m; x)$$

By (1.4) and (1.5),

$$v_m(0, 0, 0; x) \equiv 0$$

and

$$v_m(\eta_m, \xi_m, \theta_m; x) = 0 \qquad \text{for } |x| \ge \frac{1}{2T}\ .$$

Denote the collection of (T, δ)–variations of potential by $\mathcal{S}(T, \delta)$. The set of parameters (η, ξ, θ) has measure $(2\delta)^{3T}$. We want to define a notion of "typical" potential by using the normalized measure on this set of parameters. Hence, a set $S \subset \mathcal{S}(T, \delta)$ is called $(1 - \varepsilon)$-typical if

$$|S| := \min_{\{R_m\}} \frac{1}{(2\delta)^{3T}} \operatorname{mes}\big\{(\eta, \xi, \theta) \in [-\delta, \delta]^{3T} : W(\eta, \xi, \theta, \{R_m\}; .) \in S\big\} \ge 1 - \varepsilon$$

The main result in this paper, summarized in the following theorem, is that for typical C^3 potential $\tilde{V}$, we have positive Lyapunov exponent for all energies E.

Theorem 1.2. *Given any $V \in C^3(\mathbb{T})$, there is $\lambda_0 = \lambda_0(V)$ such that for $|\lambda| > \lambda_0$, one has a collection of pertubed potentials $\{S_\ell = S_\ell(V, \lambda)\}_{\ell=1}^{\infty}$, $S_\ell \subset \mathcal{S}(T^{(\ell)}, \delta_\ell)$, $\log T^{(\ell+1)} \asymp \big(T^{(\ell)}\big)^\alpha$, $0 < \alpha \ll 1$, $\sum_{\ell=1}^{\infty}\big(1 - |S_\ell|\big) \le \lambda^{-\beta}$, so that for any potential*

$$\widetilde{V}(x) = V(x) + \sum_{\ell=1}^{\infty} W^{(\ell)}\big(\eta^{(\ell)}, \xi^{(\ell)}, \theta^{(\ell)}, \{R_m^{(\ell)}\}; x\big)$$

where $W^{(\ell)} \in S_\ell$, there exists $\Omega = \Omega(\lambda, \tilde{V})$, $\operatorname{mes}\Omega \le \lambda^{-\beta}$, so that the Lyapunov exponent $L(\omega, E) \ge \frac{1}{4}\log\lambda$ for any $\omega \in \Omega$, $E \in \mathbb{R}$. Furthermore, the corresponding eigenfunctions are exponentially localized.

There are two central technical problems which one has to deal with in order to establish Theorem 1.2. The first one consists of the splitting of eigenvalues of the problem (1.1) on a finite interval $[-N, N]$. The technology for this splitting was developed in the recent work by Goldstein and Schlag[GS2] in the case of analytic potential. It is based on avalanche principle[GS1], elimination of resonances and localized eigenfunctions on a finite interval. We modify this method for smooth potential and prove the following result:

Proposition 1.3. *Using the notation of Theorem 1.2, there exists integers T'_s, $\log T'_s \asymp \log T^{(s)}$, such that for any nested sequence of intervals $\mathcal{F}_{s,k_s} = [\frac{k_s}{T_s}, \frac{k_s+1}{T_s})$, and $x \in \mathbb{T}, \omega \in \Omega$, there is a sequence integers $\{N_s = N_s(x,\omega)\}$, with $\log N_s \asymp \log T'_s$, so that*

$$|E_1 - E_2| > \exp(-N_s^\tau) \tag{1.6}$$

for distinct eigenvalues $E_1, E_2 \in \big(\operatorname{sp} H_{[-N_s,N_s]}(x,\omega)\big) \cap \mathcal{F}_{s,k_s}$.

The second problem is as follows. The eigenvalues of the problem (1.1) on a finite interval [1,N] have a smooth parametrization as $E_1(x) < E_2(x) < \ldots < E_N(x),\ x \in \mathbb{T}$. This general result is due to the self adjointness of the the problem (1.1) and nondegeneracy of the the eigenvalues of (1.1) restricted on a finite interval. The problem is how to evaluate the quantity

$$|\partial_x E_j| + |\partial_{xx} E_j| \tag{1.7}$$

from below.

This problem was also studied in [GS2]; for analytic potential, the problem was solved using discriminants of polynomials and Sard-type arguments. This method has no modification for smooth potentials. To solve this problem, we need to introduce variations of the potential. The most basic idea of our method is as follows.

"Typical" smooth functions $F(x)$ are Morse functions, i.e. the quantity

$$|\partial_x F| + |\partial_{xx} F| \tag{1.8}$$

has a "good" lower bound, gauged according to the "size" of F. On the other hand, there is a basic relation between $\partial_x E_j$ and the potential $V(x)$:

$$\partial_x E_j = \sum_{k=1}^{N} V'(x + k\omega)|\varphi_j(x)(k)|^2 \tag{1.9}$$

where $\varphi_j(x)(.)$ is a normalized eigenfunction of (1.1) on the interval $[1, N]$ corresponding to $E_j(x)$. Relation (1.9) enables one to express the "genericity" of the potential V in terms of the lower bound for (1.7), provided $\varphi_j(x)(.)$ is exponentially localized. Ultimately, we elimate the "bad" cases varying the frequencies ω. We use Sard-type arguments to show that the total mass of those ω for which there is no "response" in (1.9) under the variations of V, is extremely small.

In section 6, we first use a Diophantine condition to obtain an upper bound of the number of entries that can lead to resonance. Using this bound, we can find N_1 such that the entries

where resonance occur is far away from the edge of $[-N_1, N_1]$. This will give us exponentially decaying eigenfunctions.

We prove the base case of Proposition 1.3 in Section 7. In section 8, we introduce variations of potential. Using the results from Appendix E, we show that for typical potential, (1.7) has a "good" lower bound. This forms the major part of the inductive hypothesis used in Section 9 and Section 10, where we define inductively C^3 functions $E^{(s)}(x,\omega)$ which take values in $\operatorname{sp} H_{[-N_s,N_s]}(x,\omega)$.

In Section 11, we take $\tilde{V}$ as a limit of the varied potential. We show that lower estimate for (1.7), which we have established for $E^{(s)}(x,\omega)$, still holds for the eigenvalues of $H_{[-N_s,N_s]}(x,\omega)$ with the limiting potential $\tilde{V}$. Then, we will be able to prove Theorem 1.2.

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Part I

2 Elimination of flat slope

In Part I, we consider a C^1 potential $V(x)$. For any $n \in \mathbb{N}$, if
$|V'(x)|$ is small, then $|V(x)-V(x+n\omega)|$ will be small for most frequencies ω. To avoid resonance, we eliminate the part of the torus where the slope of V is flat, i.e. $|V'(x)|$ is small. We then eliminate a set of frequencies that lead to resonance.

Let $V \in C^1(\mathbb{T})$, $\max_{x\in\mathbb{T}} V(x) - \min_{x\in\mathbb{T}} V(x) \le \max_{x\in T} |V'(x)| \le C_0$. Suppose V has m_0 monotonicity intervals. Set $A = \{x \in \mathbb{T} : |V'(x)| < \varepsilon\}$. Next two lemmas give an upper bound of the measure of the spectral values that we have to exclude due to the elimination of flat slope.

Lemma 2.1. $\operatorname{mes} V(A) \le \varepsilon$

Proof. $\varepsilon \ge \int_A |V'(x)|dx = \int_{\mathbb{R}} \#\{x \in A : V(x) = E\}dE \ge \operatorname{mes} V(A)$. □

Take $\mathcal{J} = [\min_{x\in\mathbb{T}} V(x), \max_{x\in\mathbb{T}} V(x)]$. Let $\mathcal{J} \setminus V(A) = \bigcup_i [\alpha_i, \beta_i]$, $h_i = \beta_i - \alpha_i$. Define $h : [0,\infty) \to [0,\infty)$ by $h(y) = \sum_{i:h_i\le y} h_i$.

h is increasing, continuous from the right, and $h(0) = 0$. Hence, there is $\delta > 0$ such that $h(\delta) < \varepsilon$. Fix $0 < \delta < \varepsilon^{10} \ll \frac{1}{C_0}$ such that $h(\delta) < \varepsilon$. Set $R = \{i : h_i > \delta\}$. Let

$$\mathcal{E}^{(0)} = \bigcup_{i\in R} [\alpha_i + 2\delta^2,\ \beta_i - 2\delta^2]$$

Lemma 2.2. $\#R \le C_0\delta^{-1}$, $\operatorname{mes}(\mathcal{J} \setminus \mathcal{E}^{(0)}) \le 2\varepsilon + 4C_0\delta$.

Proof.

$$(\#R)\delta \le \sum_{i\in R} h_i \le C_0$$

$$\operatorname{mes}(\mathcal{J}\setminus\mathcal{E}^{(0)}) = \operatorname{mes} V(A) + h(\delta) + (\#R)(4\delta^2) \le 2\varepsilon + 4C_0\delta$$

□

Next, we eliminate some frequencies to avoid resonances. Having eliminate flat slope, we can obtain an upper bound of the measure of the set of frequencies we eliminate.

For any $i \in R$, let $\alpha_i = y_{i0} < y_{i1} < \cdots < y_{i,n_i} = \beta_i$, $\delta^2 \le y_{ik} - y_{i,k-1} < 2\delta^2$ for $2 < k < n_i - 1$ and $y_{i1} - y_{i0} = y_{i2} - y_{i1} = \delta^2 = y_{i,n_i-1} - y_{i,n_i-2} = y_{i,n_i} - y_{i,n_i-1}$. For $1 \le k \le n_i$, consider $T_{ik} = V^{-1}\big([y_{i,k-1}, y_{ik}]\big)$. T_{ik} consists of at most m_0 intervals, each with length less than $(2\delta^2)\varepsilon^{-1} < 2\delta$.

Say $T_{ik} = \bigcup_{j=1}^{J_k} [a_{ikj}, b_{ikj}]$, $J_k \le m_0$. Set $\tilde{B}_{ikj} = \bigcup_{|\ell-k|\le 2} \bigcup_{1\le m\le J_\ell} \big\{[a_{i\ell m}, b_{i\ell m}] - [a_{ikj}, b_{ikj}]\big\}$,

$$\tilde{B}_{ikj}(n) = \Big\{\omega \in [0,1] : \{n\omega\} \in \widetilde{B}_{ikj}\Big\},$$

$$B_{ikj} = \bigcup_{0<|n|\le N_1^2} \widetilde{B}_{ikj}(n) \qquad \text{where } N_1 = \lfloor \delta^{-1/4} \rfloor$$

$$\mathcal{D}_i^{(1)} = \bigcup_{1<k<n_i} \bigcup_{j=1}^{J_k} (a_{ikj}, b_{ikj}) \times \big([0,1]\setminus B_{ikj}\big)$$

Lemma 2.3. *Let $E \in [\alpha_i + 2\delta^2, \beta_i - 2\delta^2]$, $i \in R$. Suppose $(x,\omega) \in \mathcal{D}_i^{(1)}$ such that $|V(x) - E| \le \delta^2$. Then $|V(x + j\omega) - E| > \delta^2$ for $0 < |j| \le N_1^2$.*

Proof. $E \in [y_{i,k-1}, y_{ik}]$ for some k, $2 < k < n_i - 1$. $|V(x) - E| \le \delta^2 \Longrightarrow x \in T_{i\ell}$, $|\ell - k| \le 1$. Say $x \in [a_{i\ell m}, b_{i\ell m}]$. If $|V(x + j\omega) - E| \le \delta^2$ then $x + j\omega \in T_{i\tilde{\ell}}$, $|\tilde{\ell} - k| \le 1$. Thus, we have $|\ell - \tilde{\ell}| \le 2 \Longrightarrow j\omega \in \widetilde{B}_{i\ell m} \Longrightarrow \omega \in \widetilde{B}_{i\ell m}(j)$. This contradicts the construction of $\mathcal{D}_i^{(1)}$ if $0 < |j| \le N_1^2$. Hence $|V(x + j\omega) - E| > \delta^2$ for $0 < |j| \le N_1^2$. □

Define

$$\mathcal{D}^{(1)} = \bigcup_{i\in R} \mathcal{D}_i^{(1)}$$

Corollary 2.4. *If $E \in \mathcal{E}^{(0)}$, $(x,\omega) \in \mathcal{D}^{(1)}$, $|V(x) - E| \le \delta^2$. Then $|V(x + j\omega) - E| > \delta^2$ for $0 < |j| \le N_1^2$.*

For later sections, we also need an upper bound of the number of components in $\mathcal{D}^{(1)}$.

Definition. A set $U_1 \subset \mathbb{R}$ (or $U_2 \subset \mathbb{R}^2$) is said to be K–simple if $U_1 = \bigcup_{k=1}^{K} \mathcal{J}_k$ where $\mathcal{J}_k$ are intervals ($U_2 = \bigcup_{k=1}^{K} D_k$ where D_k are rectangles). If U_1 is K–simple, write $\operatorname{compl} U_1 \le K$.

Lemma 2.5. *For any $i \in R$,* $\operatorname{compl} B_{ikj} \le 5m_0\delta^{-1}$, $\operatorname{mes} B_{ikj} < 40m_0\delta^{1/2}$.

Proof. $\{[a_{i\ell m}, b_{i\ell m}] - [a_{ikj}, b_{ikj}]\}$ is an interval. Hence $\operatorname{compl} \widetilde{B}_{ikj} \le \sum_{|\ell-k|\le 2} J_\ell \le 5m_0$

$$\operatorname{compl} \widetilde{B}_{ikj}(n) \le n(5m_0)$$

$$\operatorname{compl} B_{ikj} \le 5m_0 \sum_{0<|n|\le N_1^2} n \le 5m_0(N_1^2)^2 \le 5m_0\delta^{-1}$$

$$\operatorname{mes}\{[a_{i\ell m}, b_{i\ell m}] - [a_{ikj}, b_{ikj}]\} \le (b_{i\ell m} - a_{i\ell m}) + (b_{ikj} - a_{ikj}) < 4\delta$$

$$\operatorname{mes} \widetilde{B}_{ikj}(n) = \operatorname{mes} \widetilde{B}_{ikj} < (5m_0)(4\delta) = 20m_0\delta$$

$$\operatorname{mes} B_{ikj} < (2N_1^2)(20m_0\delta) \le 40m_0\delta^{1/2}$$

□

Corollary 2.6. $\operatorname{compl} \mathcal{D}^{(1)} \le 5m_0^2 C_0 \delta^{-3}$

Proof.

$$\operatorname{compl} \mathcal{D}_i^{(1)} \le n_i m_0 (5m_0\delta^{-1})$$

$$\operatorname{compl} \mathcal{D}^{(1)} \le 5m_0^2\delta^{-1} \sum_{i\in R} n_i \le 5m_0^2\delta^{-1}(C_0\delta^{-2})$$

□

Corollary 2.7. *Let* $L = \sum_{i\in R} \sum_{1<k<n_i} \sum_j (b_{ikj} - a_{ikj})$. *Then*

$$\operatorname{mes}\left(\bigcup_{i\in R} \bigcup_{1<k<n_i} \bigcup_j (a_{ikj}, b_{ikj}) \times [0,1] \setminus \mathcal{D}^{(1)} \right) \le 40m_0\delta^{1/2}L$$

To simplify notation, henceforth we will write $\mathcal{D}^{(1)} = \bigcup_k \bigcup_\ell (a_k^{(1)}, b_k^{(1)}) \times (\underline{\omega}_{k\ell}^{(1)}, \bar{\omega}_{k\ell}^{(1)})$.

3 Eigenvalues and Eigenfunctions at first scale

We now construct a function $E^{(1)}(x, w)$, which take values in $\operatorname{sp} H_{[-N_1,N_1]}(x,\omega)$, and the corresponding eigenfunction $\varphi^{(1)}(x,\omega)$. We will also derive some properties of $E^{(1)}$ and $\varphi^{(1)}$.

Take $\lambda = \delta^{-4}$,

$$H_{[-N,N]}(x,\omega) = \begin{pmatrix} \lambda V(x - N\omega) & 1 & & & \\ -1 & \lambda V\big(x-(N-1)\omega\big) & 1 & & \\ & -1 & & & \\ & & & \ddots & \\ & & & & -1 \\ & & & -1 & \lambda V(x+N\omega) \end{pmatrix}$$

Lemma 3.1. *Let $(x,\omega) \in \mathcal{D}^{(1)}$. Then $H_{[-N_1,N_1]}(x,\omega)$ has a unique eigenvalue in $\big[\lambda V(x) - 2, \lambda V(x) + 2\big]$.*

Proof. $(x,\omega) \in \mathcal{D}^{(1)} \Longrightarrow V(x) \in \mathcal{E}^{(0)}$, $|V(x+j\omega) - V(x)| > \delta^2 = \lambda^{-1/2}$ for $0 < |j| \le N_1$. By Lemma B.4, $H_{[-N_1,N_1]}(x,\omega)$ has a unique eigenvalue in $\big[\lambda V(x) - 2, \lambda V(x) + 2\big]$. □

Denote the eigenvalue of $H_{[-N_1,N_1]}(x,\omega)$ in Lemma 3.1 by $E^{(1)}(x,\omega)$.

Lemma 3.2. *$E^{(1)}(x,\omega)$ is C^1 in $\mathcal{D}^{(1)}$. There are C^1 functions $\big\{\varphi^{(1)}(x,\omega)(n)\big\}_{|n|\le N_1}$ defined in $\mathcal{D}^{(1)}$ such that $\sum_{n=-N_1}^{N_1} \big|\varphi^{(1)}(x,\omega)(n)\big|^2 = 1$,*

$$H_{[-N_1,N_1]}(x,\omega)\varphi^{(1)}(x,\omega) = E^{(1)}(x,\omega)\varphi^{(1)}(x,\omega) \ .$$

Also,

$$\partial_x E^{(1)}(x,\omega) = \sum_{n=-N_1}^{N_1} V'(x+n\omega)|\varphi^{(1)}(x,\omega)(n)|^2$$

$$\partial_\omega E^{(1)}(x,\omega) = \sum_{n=-N_1}^{N_1} nV'(x+n\omega)|\varphi^{(1)}(x,\omega)(n)|^2$$

Proof. These follow from general results in Appendix A. □

Lemma 3.3. $\big|\varphi^{(1)}(x,\omega)(n)\big| < \lambda^{-\frac{1}{3}|n|}\big|\varphi^{(1)}(x,\omega)(0)\big|$.

Proof. We prove the case for $n > 0$; the case for $n < 0$ is similar.

By Poisson's formula,

$$\varphi^{(1)}(x,\omega)(n) = \big[H_{[1,N_1]} - E^{(1)}(x,\omega)\big]^{-1}(n,1)\varphi^{(1)}(x,\omega)(0)$$

Since

$$\begin{aligned} |\lambda V(x+m\omega) - E^{(1)}(x,\omega)| &\ge |\lambda V(x+m\omega) - \lambda V(x)| - |\lambda V(x) - E^{(1)}(x,\omega)| \\ &> \lambda^{1/2} - 2 \qquad \text{for } 1 \le m \le N_1 \ , \end{aligned}$$

and $N_1 \le \delta^{-1/4} = \lambda^{1/16}$, by Corollary C.8, one has

$$|\varphi^{(1)}(x,\omega)(n)| < \lambda^{-\frac{1}{3}|n|}|\varphi^{(1)}(x,\omega)(0)| \ .$$

□

Corollary 3.4.

$$\begin{aligned} \frac{1}{2}\lambda^{39/40} < |\partial_x E^{(1)}(x,\omega)| &\le C_0\lambda \\ |\partial_\omega E^{(1)}(x,\omega)| &\le C_0 N_1 \lambda \end{aligned}$$

Proof.

$$\sum_{n\neq 0} |\varphi^{(1)}(x,\omega)(n)|^2 \leq 2\sum_{n=1}^{\infty} \lambda^{-\frac{1}{3}n} \leq 4\lambda^{-\frac{1}{3}}$$
$$(x,\omega) \in \mathcal{D}^{(1)} \Longrightarrow |V'(x)| \geq \varepsilon > \delta^{\frac{1}{10}} = \lambda^{-\frac{1}{40}}$$

$$\begin{aligned}|\partial_x E^{(1)}(x,\omega)| &\geq \lambda |V'(x)|\, |\varphi^{(1)}(x,\omega)(0)|^2 \\ &\quad - \lambda \sum_{n\neq 0} |V'(x+n\omega)|\, |\varphi^{(1)}(x,\omega)(n)|^2 \\ &\geq \lambda\varepsilon(1-4\lambda^{-\frac{1}{3}}) - \lambda C_0(4\lambda^{-\frac{1}{3}}) > \frac{1}{2}\lambda^{\frac{39}{40}}\end{aligned}$$

□

We now construct a set of frequencies such that the image of $E^{(1)}(.,\omega)$ contains most of the spectral values. We will need this in Section 4 when we eliminate the situation when $\mathrm{sp}\big(H_{[-N_1,N_1]}(x,\omega)\big)$ and $\mathrm{sp}\big(H_{[-N_1,N_1]}(x+j\omega,\omega)\big)$ are close to each other.

Lemma 3.5. *There exists $\Omega^{(1)} \subset [0,1]$, $\mathrm{mes}\big([0,1]\setminus\Omega^{(1)}\big) \leq \delta^{1/4}$ such that for any $w \in \Omega^{(1)}$, one has*

$$\mathrm{mes}\big(V^{-1}(\mathcal{E}^{(0)}) \setminus \mathcal{D}_\omega^{(1)}\big) \leq 40 m_0 \delta^{1/4} L$$
$$\mathrm{compl}\big(V^{-1}(\mathcal{E}^{(0)}) \setminus \mathcal{D}_\omega^{(1)}\big) \leq 40 m_0 C_0 \delta^{-7/4} L\ .$$

Proof. By Corollary 2.7,

$$\int_0^1 \mathrm{mes}\big(V^{-1}(\mathcal{E}^{(0)}) \setminus \mathcal{D}_\omega^{(1)}\big) d\omega = \mathrm{mes}\big[\bigcup_k (a_k^{(1)}, b_k^{(1)}) \times [0,1] \setminus \mathcal{D}^{(1)}\big] \leq 40 m_0 \delta^{1/2} L\ .$$

Let $B^{(1)} = \Big\{\omega \in [0,1] : \mathrm{mes}\big(V^{-1}(\mathcal{E}^{(0)}) \setminus \mathcal{D}_w^{(1)}\big) \geq 40 m_0 \delta^{1/4} L\Big\}$. Then $\mathrm{mes}\, B^{(1)} \leq \delta^{1/4}$.

Take $\Omega^{(1)} = [0,1] \setminus B^{(1)}$. If $\omega \in \Omega^{(1)}$, then $\mathrm{mes}\big(V^{-1}(\mathcal{E}^{(0)}) \setminus \mathcal{D}_\omega^{(1)}\big) < 40 m_0 \delta^{1/4} L$. Each interval $(a_k^{(1)}, b_k^{(1)})$ is contained in either $\mathcal{D}_\omega^{(1)}$ or $V^{-1}\big(\mathcal{E}^{(0)}\big) \setminus \mathcal{D}_\omega^{(1)}$. Since $b_k^{(1)} - a_k^{(1)} \geq \delta^2 C_0^{-1}$

$$\#\Big\{(a_k^{(1)}, b_k^{(1)}) \subset \big(V^{-1}(\mathcal{E}^{(0)}) \setminus \mathcal{D}_\omega^{(1)}\big)\Big\} \leq (40 m_0 \delta^{1/4} L)(\delta^2 C_0^{-1})^{-1}\ .$$

□

For $\omega \in \Omega^{(1)}$, let $V\big(V^{-1}(\mathcal{E}^{(0)}) \setminus \mathcal{D}_\omega^{(1)}\big) = \bigcup_{r=1}^{R_\omega} (\underline{E}_{\omega r}, \overline{E}_{\omega r})$, $R_\omega \leq 40 m_0 C_0 \delta^{-7/4} L$. Define

$$\mathcal{E}_\omega^{(1)} = \mathcal{E}^{(0)} \setminus \bigcup_r (\underline{E}_{\omega r} - \delta^2, \overline{E}_{\omega r} + \delta^2)$$

Lemma 3.6. *Let $\omega \in \Omega^{(1)}$.*

$$\operatorname{mes}\big(\mathcal{E}^{(0)} \setminus \mathcal{E}^{(1)}_\omega\big) \le 120 m_0 C_0 \delta^{1/4}$$

Suppose $|V(x) - E| \le \delta^2$, $E \in \mathcal{E}^{(1)}_\omega$. Then $x \in \mathcal{D}^{(1)}_\omega$.

Proof.

$$\begin{aligned}\operatorname{mes}\big(\mathcal{E}^{(0)} \setminus \mathcal{E}^{(1)}_\omega\big) &\le (40 m_0 \delta^{1/4}) C_0 + (40 m_0 C_0 \delta^{-7/4})(2\delta^2) \\ &= 120 m_0 C_0 \delta^{1/4}\end{aligned}$$

The second assertion follows from the construction of $\mathcal{E}^{(1)}_\omega$. □

Next two lemmas provide us some flexibility concerning the size of the monodromy matrices we can use. We will use them in Section 4 to otain the hypothesis of the Avalanche Principle.

Lemma 3.7. *Let $\omega \in \Omega^{(1)}$, $\lambda^{-1}E \in \mathcal{E}^{(1)}_\omega$. Suppose $N_1 \le b - a \le 5N_1$. Then*

$$\log \big| f_{[a,b]}(x, \omega, E)| \ge (b-a) \log(\lambda^{1/2} - 2) + \log \operatorname{dist}\big(E, \operatorname{sp} H_{[a,b]}(x,\omega)\big)$$

Proof. Let $\operatorname{sp} H_{[a,b]}(x,\omega) = \big\{\mu_n\big\}_{n=a}^{b}$, $|\lambda V(x + n\omega) - \mu_n| \le 2$. Say

$$|E - \mu_{n_0}| = \operatorname{dist}\big(E, \operatorname{sp} H_{[a,b]}(x,\omega)\big) \ .$$

If $|E - \mu_{n_0}| > \lambda^{1/2} - 2$ then $|E - \mu_n| > \lambda^{1/2} - 2$ for all n. Hence

$$\begin{aligned}\log \big| f_{[a,b]}(x,\omega,E)\big| &= \sum_{n \ne n_0} \log |E - \mu_n| + |E - \mu_{n_0}| \\ &\ge (b-a)\log(\lambda^{1/2} - 2) + \log \operatorname{dist}\big(E, \operatorname{sp} H_{[a,b]}(x,\omega)\big) \ .\end{aligned}$$

If $|E - \mu_{n_0}| \le \lambda^{1/2} - 2$ then $|E - \lambda V(x + n_0\omega)| \le \lambda^{1/2} \Longrightarrow x + n_0\omega \in \mathcal{D}^{(1)}_\omega$. Therefore, $|E - \lambda V(x + n\omega)| > \lambda^{1/2}$ for $n \ne n_0 \Longrightarrow |E - \mu_n| > \lambda^{1/2} - 2$ for $n \ne n_0$.

Hence

$$\log |f_{[a,b]}(x,\omega,E)| \ge (b-a)\log(\lambda^{1/2} - 2) + \log \operatorname{dist}\big(E, \operatorname{sp} H_{[a,b]}(x,\omega)\big)$$

□

Lemma 3.8. *Let $E \in \operatorname{sp} H_{[a,b]}(x,\omega)$, $N_1 \le b - a \le 5N_1$, $w \in \Omega^{(1)}$, $\lambda^{-1}E \in \mathcal{E}^{(1)}_w$. If $|\lambda^{-1}E - V(x+j\omega)| < \lambda^{-1/2}$ for some j, $a + N_1^\vartheta < j < b - N_1^\vartheta$ then $|E - E^{(1)}(x + j\omega, \omega)| < 10\lambda^{-\frac{1}{3}N_1^\vartheta}$.*

Proof. $|\lambda^{-1}E - V(x + j\omega)| < \lambda^{-1/2} = \delta^2 \Rightarrow x + j\omega \in \mathcal{D}^{(1)} \Rightarrow E^{(1)}(x + j\omega, \omega)$ is defined. Also, similar to the construction of $E^{(1)}$, E is the only eigenvalue of $H_{[a,b]}(x,\omega)$ in $[V(x + j\omega) - 2, V(x + j\omega) + 2]$

Let

$$p = \max\{a, j - N_1\} \ , \quad q = \min\{b, j + N_1\},$$

$$Y = \left[\sum_{\ell=p}^{q} \left|\varphi^{(1)}(x + j\omega, \omega)(\ell - j)\right|^2\right]^{1/2}$$

$$\tilde{\varphi}(\ell) = \begin{cases} Y^{-1}\varphi^{(1)}(x + j\omega, \omega)(\ell - j) & \text{if } \ell \in [p, q] \\ 0 & \ell \in [a, b] \setminus [p, q] \end{cases}$$

Then $\|\tilde{\varphi}\| = 1$.

$$p + N_1^{\vartheta} < j < q - N_1^{\vartheta} \Longrightarrow 1 - Y^2 \le 2 \sum_{n=N_1^{\vartheta}}^{\infty} |\lambda^{-\frac{1}{3}n}|^2 \le \frac{1}{2}$$

$$\begin{aligned} \left\| \left[H_{[a,b]}(x, \omega) - E^{(1)}(x + j\omega, \omega)\right]\tilde{\varphi}\right\| &\le Y^{-1}[|\varphi(x + j\omega, \omega)(p - 1)| + |\varphi(x + j\omega, \omega)(q + 1)|] \\ &< 10\lambda^{-\frac{1}{3}N_1^{\vartheta}} \end{aligned}$$

Using Lemma A.5, since E is the only eigenvalue of $H_{[a,b]}(x, \omega)$ in $[V(x+j\omega)-2, V(x+j\omega)+2]$, we have $|E - E^{(1)}(x + j\omega, \omega)| < 10\lambda^{-\frac{1}{3}N_1^{\vartheta}}$ □

4 Elimination of Resonances

In this section, we use the result from Appendix E to eliminate the situation when $\operatorname{sp}\big(H_{[-N_1,N_1]}(x, \omega)\big)$ and $\operatorname{sp}\big(H_{[-N_1,N_1]}(x + j\omega, \omega)\big)$ are close to each other. Then we construct a function $E^{(2)}(x, \omega)$ which takes value in $\operatorname{sp} H_{[-\tilde{N}^{(2)},N^{(2)}]}(x, \omega)$, $\tilde{N}^{(2)}, N^{(2)} \asymp N_1^{\tau}$, and the corresponding eigenfunctions $\varphi^{(2)}(x, \omega)$.

Recall that a set $U \subset \mathbb{R}^2$ is called K–simple, denoted $\operatorname{compl} U \le K$, if $U = \bigcup_{k=1}^{K} D_k$ where D_k are rectangles.

Lemma 4.1. *There is m–simple set $\tilde{\mathcal{D}}^{(2)} \subset \mathcal{D}^{(1)}$, $m \le e^{2N_1^{\vartheta}}$, which satisfies the following conditions:*

(1) $\operatorname{mes}\big(\mathcal{D}^{(1)} \setminus \tilde{\mathcal{D}}^{(2)}\big) \le e^{-\frac{1}{2}N_1^{\vartheta}}L$

(2) If $(x, \omega) \in \tilde{\mathcal{D}}^{(2)}$ and $(x + j\omega, \omega) \in \mathcal{D}^{(1)}$ for some j, $0 < |j| \le N_2^2$, where $N_2 = \lfloor e^{N_1^{\tau}} \rfloor$, $\tau < \vartheta$, then $\left|E^{(1)}(x, \omega) - E^{(1)}(x + j\omega, \omega)\right| > 4e^{-N_1^{\vartheta}}$.

Proof. For $0 < |j| \le N_1^2$, by construction of $\mathcal{D}^{(1)}$, $x \in \mathcal{D}_{\omega}^{(1)} \Longrightarrow |V(x) - V(x + j\omega)| > \lambda^{-\frac{1}{2}}$. If $x + j\omega \in \mathcal{D}_{\omega}^{(1)}$, then

$$\begin{aligned} &\left|E^{(1)}(x, \omega) - E^{(1)}(x + j\omega, \omega)\right| \\ &\quad \ge |\lambda V(x) - \lambda V(x + j\omega)| - |\lambda V(x) - E^{(1)}(x, \omega)| - |\lambda V(x + j\omega) - E^{(N_1)}(x + j\omega, \omega)| \\ &\quad \ge \lambda^{1/2} - 4 > 4e^{-N_1^{\vartheta}} \ . \end{aligned}$$

Consider $D_k = \Big(\big(a_k^{(1)}, b_k^{(1)}\big) \times [0,1]\Big) \cap \mathcal{D}^{(1)}$. Let $a_k^{(1)} = x_0 < x_1 < \cdots < x_n = b_{\ell k}$, $x_i - x_{i-1} < e^{-N_1^\vartheta}$, $n \le 2e^{N_1^\vartheta}(b_k^{(1)} - a_k^{(1)})$.

For any $|j| > N_1^2$, let $T_k(j) = \Big\{(x,\omega) \in D_k : x + j\omega \in \mathcal{D}_\omega^{(1)}\Big\}$, $T_k(j)$ consists of at most $|j|C_0\delta^{-2}$ components. On each of these components, the function

$$F_j(x,\omega) = E^{(1)}(x,\omega) - E^{(1)}(x + j\omega, \omega)$$

is C^1.

$$\begin{aligned}
\partial_x F_j(x,\omega) &= \partial_x E^{(1)}(x,\omega) - \partial_x E^{(1)}(x + j\omega, \omega)\\
|\partial_x F_j(x,\omega)| &\le 2C_0\lambda
\end{aligned}$$

$$\begin{aligned}
\partial_\omega F_j(x,\omega) &= \partial_\omega E^{(1)}(x,\omega) - \partial_\omega E^{(1)}(x + j\omega, \omega) - j\partial_x E^{(1)}(x + j\omega, \omega)\\
|\partial_\omega F_j(x,\omega)| &\ge |j|\big(\frac{1}{2}\lambda^{39/40}\big) - 2C_0N_1\lambda > \frac{1}{4}|j|\lambda^{39/40} > 0
\end{aligned}$$

Apply Proposition E.1 to each of the components of $T_k(j)$. (Note that even though the components are not rectangles, the bounding consists of two verticle lines and two lines with slope $\frac{-1}{j}$; $\frac{1}{|j|} \ll (2C_0\lambda)\big(\frac{1}{4}|j|\lambda^{39/40}\big)^{-1}$.) We get rectangles $[x_{i-1}, x_i] \times [\underline{y}_{i\ell}, \bar{y}_{i\ell}]$ such that

$$\begin{aligned}
&\big\{(x,\omega) \in T_k(j) : |F_j(x,\omega)| \le 2e^{-N_1^\vartheta}\big\}\\
&\qquad\qquad \subset \bigcup_i \bigcup_\ell [x_{i-1}, x_i] \times [\underline{y}_{i\ell}, \bar{y}_{i\ell}] =: D_k(j)
\end{aligned}$$

$$\begin{aligned}
\operatorname{mes} D_k(j) &\le |j|C_0\delta^{-2}(b_k^{(1)} - a_k^{(1)}) \left[\frac{2(4e^{-N_1^\vartheta})}{\frac{1}{4}|j|\lambda^{39/40}} + 2e^{-N_1^\vartheta}\frac{2C_0\lambda}{\frac{1}{4}|j|\lambda^{39/40}}\right]\\
&< (b_k^{(1)} - a_k^{(1)})e^{-\frac{3}{4}N_1^\vartheta}
\end{aligned}$$

$$\operatorname{compl} D_k(j) \le 2e^{N_1^\vartheta}(b_k^{(1)} - a_k^{(1)})|j|C_0\delta^{-2}$$

Set

$$\tilde{\mathcal{D}}^{(2)} = \mathcal{D}^{(1)} \setminus \left(\bigcup_k \bigcup_{N_1^2 < |j| \le N_2^2} D_k(j)\right)$$

$$\operatorname{mes}\big(\mathcal{D}^{(1)} \setminus \tilde{\mathcal{D}}^{(2)}\big) \le \sum_k \sum_{N_1^2 < |j| \le N_2^2} (b_k^{(1)} - a_k^{(1)})e^{-\frac{3}{4}N_1^\vartheta} < e^{-\frac{1}{2}N_1^\vartheta}L\ .$$

$$m = \operatorname{compl}\tilde{\mathcal{D}}^{(2)} \le \sum_k \sum_{N_1^2 < |j| \le N_2^2} 2e^{N_1^\vartheta}(b_k^{(1)} - a_k^{(1)})|j|C_0\delta^{-2} \le e^{2N_1^\vartheta}$$

□

By taking away a set of measure zero in $\tilde{\mathcal{D}}^{(2)}$, we obtain

$$\mathcal{D}^{(2)} = \bigcup_k \bigcup_\ell (a_k^{(2)}, b_k^{(2)}) \times (\underline{\omega}_{k\ell}^{(2)}, \bar{\omega}_{k\ell}^{(2)}) \subset \tilde{\mathcal{D}}^{(2)}$$

where $b_k^{(2)} - a_k^{(2)} < e^{-N_1^\vartheta}$, $\bar{\omega}_{k\ell} - \underline{\omega}_{k\ell} < e^{-N_1^\vartheta}$, $\operatorname{compl} \mathcal{D}^{(2)} \le e^{2N_1^\vartheta}$

We now show that $E^{(1)}(x,\omega)$ and $\operatorname{sp} H_{[a,b]}(x,\omega)$, $b-a \asymp N_1$, are separted, provide that the entries near the edge of the interval $[a,b]$ are separated from $V(x)$, i.e. $|V(x)-V(x+j\omega)| > \lambda^{1/2}$ for $j \in [a, a+N_1^\vartheta] \cup [b-N_1^\vartheta, b]$. The hypothesis of the Avalanche Principle will follow the following lemma.

Lemma 4.2. *Let $(x,\omega) \in \mathcal{D}^{(2)}$, $\omega \in \Omega^{(1)}$, $\lambda^{-1}E \in \mathcal{E}_\omega^{(1)}$, $\left|E - E^{(1)}(x,\omega)\right| < 2e^{-N_1^\vartheta}$. Suppose $|V(x+j\omega) - \lambda^{-1}E| > \lambda^{-1/2}$ for any j, $a \le j \le a + N_1^\vartheta$ or $b - N_1^\vartheta \le j \le b + N_1^\vartheta$ where $1 \le a \le a + N_1 \le b < a + 5N$. Then $\operatorname{dist}\big(\operatorname{sp} H_{[a,b]}(x,\omega), E\big) \ge e^{-N_1^\vartheta}$.*

Proof. Let $\operatorname{sp} H_{[a,b]} = \{\mu_j\}_{j=a}^b$, $|\mu_j - \lambda V(x+j\omega)| \le 2$. If $|E - \lambda V(x+j\omega)| > \lambda^{1/2}$ for all $j \in [a,b]$ then $|E-\mu_j| > \lambda^{1/2} - 2 \Longrightarrow \operatorname{dist}\big(\operatorname{sp} H_{[a,b]}(x,\omega), E\big) > \lambda^{1/2} - 2$.

Suppose $|E - \lambda V(x+j_0\omega)| \le \lambda^{1/2}$, $a + N_1^\vartheta < j_0 < b - N_1^\vartheta$. Since $E \in \mathcal{E}_\omega^{(1)}$, $x + j_0\omega \in \mathcal{D}_\omega^{(1)}$. So $|E - \lambda V(x+j\omega)| > \lambda^{1/2}$ for all $j \ne j_0$, $a \le j \le b$, $\operatorname{dist}\big(\operatorname{sp} H_{[a,b]}(x,\omega), E\big) = |E - \mu_{j_0}|$. By Lemma 3.8, $\left|\mu_{j_0} - E^{(N_1)}(x+j_0\omega)\right| < 10\lambda^{-\frac{1}{3}N_1^\vartheta} < e^{-N_1^\vartheta}$.

$$\begin{aligned} x \in \mathcal{D}_\omega^{(2)} &\Longrightarrow \left|E^{(1)}(x,\omega) - E^{(1)}(x+j_0\omega)\right| > 4e^{-N_1^\vartheta} \\ &\Longrightarrow \left|E - \mu_{j_0}\right| \ge \left|E^{(1)}(x,\omega) - E^{(1)}(x+j_0\omega)\right| \\ &\qquad - \left|E^{(1)}(x,\omega) - E\right| - \left|E^{(1)}(x+j_0\omega) - \mu_{j_0}\right| \\ &\quad \ge e^{-N_1^\vartheta} \end{aligned}$$

□

Corollary 4.3. *Let $(x,\omega) \in \mathcal{D}^{(2)}$, $\omega \in \Omega^{(1)}$, $E \in \mathcal{E}_\omega^{(1)}$, $|E - E^{(N-1)}(x,\omega)| < e^{-N_1^\vartheta}$. Suppose $|V(x+j\omega) - \lambda^{-1}E| > \lambda^{-1/2}$ for any j,*

$$a \le j \le a + N_1^\vartheta \quad \text{or} \quad b - N_1^\vartheta \le j \le b + N_1^\vartheta \quad \text{or} \quad c - N_1^\vartheta \le j \le c$$

where

$$1 \le a \le a + N_1 \le b \le b + 1 + N_1 \le c \le a + 5N_1 \le N_2^2$$

Then

$$\begin{aligned} \log \|M_{[a,b]}(x,\omega,E)\| &\ge (b-a)\log(\lambda^{1/2} - 2) - N_1^\vartheta \\ \log \|M_{[b+1,c]}(x,\omega,E)\| &\ge (c-b-1)\log(\lambda^{1/2} - 2) - N_1^\vartheta \end{aligned}$$

and

$$\log \|M_{[a,b]}(x,\omega,E)\| + \log \|M_{[b+1,c]}(x,\omega,E)\| - \log \|M_{[a,c]}(x,\omega,E)\| \le 20\log(\lambda C_0) + 20N_1^\vartheta \ .$$

Proof. This follows from Lemma 3.7, Lemma C.10 and Lemma 4.2. □

Let $x \in \mathcal{D}_\omega^{(2)}$, $\omega \in \Omega^{(1)}$, $\lambda^{-1}E \in \mathcal{E}_\omega^{(1)}$, $|E - E^{(1)}(x,\omega)| < 2e^{-N_1^\vartheta}$. Suppose $0 < a < b \le N_2$ such that $|V(x+j\omega) - \lambda^{-1}E| \ge \lambda^{-1/2}$ for $a \le j \le a + N_1^\vartheta$ and $b - N_1^\vartheta < j \le b$, $b - a \ge N_1$.

Let $a < j_1 < j_2 < \cdots < j_m < b$ be such that $|V(x+j_i\omega) - \lambda^{-1}E| \le \lambda^{-1/2}$. By construction of $\mathcal{D}^{(1)}$, $j_i - j_{i-1} > N_1^2$. So choose $a = k_0 < k_1 < \cdots < k_n = b$ such that $\min\limits_{i,\ell} |j_i - k_\ell| > N_1^\vartheta$, $N_1 < k_\ell - k_{\ell-1} \le 2N_1$.

Let

$$A_1 = M_{[a,k_1]}(x,\omega,E)\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$$

$$A_\ell = M_{[k_{\ell-1},k_\ell]}(x,\omega,E)\ , \quad \ell = 2,3,\dots,n-1$$

$$A_n = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} M_{[k_{n-1},b]}(x,\omega,E)$$

Lemma 4.4. *With notation as above,*

$$\log|f_{[a,b]}(x,\omega,E)| = \sum_{\ell=2}^{n} \log\|A_\ell A_{\ell-1}\| - \sum_{\ell=2}^{n-1} \log\|A_\ell\| + O(\lambda^{-cN_1})\ .$$

Proof. If $[k_{\ell-1},k_\ell]$ does not contain any j_i, then $|\lambda V(x+j\omega) - E| \ge \lambda^{1/2}$ for all $j \in [k_{\ell-1},k_\ell]$. Hence $\log\|A_\ell\| \ge N_1\log(\lambda^{1/2}-2)$.

If $j_i \in [k_{\ell-1},k_\ell]$, since $k_{\ell-1} + N_1^\vartheta < j < k_\ell - N_1^\vartheta$, $\operatorname{dist}\big(\operatorname{sp} H_{[k_{\ell-1},k_\ell]}(x,\omega,E) \ge e^{-N_1^\vartheta}$ (Lemma 4.2). By Lemma 3.7, $\log\|A_\ell\| \ge N_1\log(\lambda^{1/2}-2) - N_1^\vartheta$.

Also, similar to Lemma C.10,

$$\log\|A_\ell\| + \log\|A_{\ell-1}\| - \log\|A_\ell A_{\ell-1}\| \le 20\big(\log(\lambda C_0) + N_1^\vartheta\big)\ .$$

The assertion follows from the Avalanche Principle. □

Next, we show that there is $N \approx N_2$ such that the Green's function $\big(H_{[1,N]}(x,\omega) - E\big)^{-1}(k,1)$ decay exponentially. In the next two lemmas, we find N such that the entries near the edge of $[1,N]$ are separated from $V(x)$.

Lemma 4.5. *Suppose $x \in \mathbb{T}$, $\omega \in \Omega^{(1)}$, $E \in \mathcal{E}_\omega^{(1)}$, $N \gg N_1$. Then either*

(a) $|\lambda V(x+j\omega) - E| > \frac{1}{2}\lambda^{1/2}$ *for* $N - N_1 < j \le N$; *or*

(b) $|\lambda V(x+j\omega) - E| > \frac{1}{2}\lambda^{1/2}$ *for* $N < j \le N + N_1$.

Proof. If $|\lambda V(x+j_0\omega) - E| < \frac{1}{2}\lambda^{1/2}$ for some $N - N_1 < j_0 \le N$ then $(x+j_0\omega,\omega) \in \mathcal{D}^{(1)}$ since $E \in \mathcal{E}_\omega^{(1)}$. Hence $|\lambda V(x+j\omega) - \lambda V(x+j_0\omega)| > \lambda^{1/2}$ for $|j - j_0| \le N_1^2$. In particular, this is true for all j, $N < j \le N + N_1$. Thus $|\lambda V(x+j\omega) - E| > \frac{1}{2}\lambda^{1/2}$ for $N < j \le N + N_1$. □

For $N_2 - N_1 < j \le N_2 + N_1$. Let $(x_1,\omega_1),(x_2,\omega_2) \in \big(a_k^{(2)}, b_k^{(2)}\big) \times \big(\underline{\omega}_{k\ell}^{(2)}, \bar{\omega}_{k\ell}^{(2)}\big) \subset \mathcal{D}^{(2)}$,

$$\begin{aligned} |(x_1 + jw_1) - (x_2 + j\omega_2)| &\le |x_1 - x_2| + |j|\,|\omega_1 - \omega_2| \\ &\le \big(b_k^{(2)} - a_k^{(2)}\big) + (N_2 + N_1)\big(\underline{\omega}_{k\ell}^{(2)} - \bar{\omega}_{k\ell}^{(2)}\big) \\ &\lesssim e^{-N_1^\vartheta} \ . \end{aligned}$$

Suppose $|\lambda V(x_1 + j\omega_1) - E_1| > \frac{1}{2}\lambda^{1/2}$ for $N_2 - N_1 < j \le N_2$. For $|E_2 - E_1| < \frac{1}{4}\lambda^{1/2}$, we have

$$|\lambda V(x_2 + j\omega_2) - E_2| > \frac{\lambda^{1/2}}{3}$$

for $N_2 - N_1 < j \le N_2$.

Lemma 4.6. *For some $\omega \in \big(\underline{\omega}_{k\ell}^{(2)}, \bar{\omega}_{k\ell}^{(2)}\big)$, take $E_w \in \mathcal{E}_\omega^{(1)}$, $|E_{\omega_1} - E_{\omega_2}| < \frac{1}{4}\lambda^{1/2}$. Then for $\big(a_k^{(2)}, b_k^{(2)}\big) \times \big(\underline{\omega}_{k\ell}^{(2)}, \bar{\omega}_{k\ell}^{(2)}\big) \subset \mathcal{D}^{(2)}$, either*

(a) $|\lambda V(x + j\omega) - E_\omega| > \frac{\lambda^{1/2}}{3}$ for $N_2 - N_1 < j \le N_2$ for all $(x,\omega) \in \big(a_k^{(2)}, b_k^{(2)}\big) \times \big(\underline{\omega}_{k\ell}^{(2)}, \bar{\omega}_{k\ell}^{(2)}\big)$; or

(b) $|\lambda V(x + j\omega) - E_\omega| > \frac{\lambda^{1/2}}{3}$ for $N_2 < j \le N_2 + N_1$ for all $(x,\omega) \in \big(a_k^{(2)}, b_k^{(2)}\big) \times \big(\underline{\omega}_{k\ell}^{(2)}, \bar{\omega}_{k\ell}^{(2)}\big)$.

With entries near the edge of $[1, N]$ separted from $V(x)$, we can use the results in Appendix C to verify the hypothesis of the Avalanche Principle is satisfyied, and obtain the following lemma.

Lemma 4.7. *Let $(x,\omega) \in \mathcal{D}_\omega^{(2)}$, $\omega \in \Omega^{(1)}$, $\lambda^{-1}E \in \mathcal{E}_\omega^{(1)}$, $|E - E^{(1)}(x,\omega)| < 2e^{-N_1^\vartheta}$. Then there is $N, |N - N_2| \le N_1$, such that $E \notin \operatorname{sp} H_{[1,N]}(x,\omega)$ and*

$$\log \big|\big[H_{[1,N]}(x,\omega) - E\big]^{-1}(k,1)\big| \le -\frac{1}{5}k \log \lambda$$

for $k > N_1 - N_1^\vartheta$.

Proof. We can choose p_k, q_k such that

$$\begin{aligned} &N_1 \le q_k - p_k \le 2N_1 \ , && p_k < q_{k-1} < p_{k+1} < q_k \\ &q_{k-1} - p_k > N_1^\vartheta \ , && [1,N] = \bigcup_{k=1}^{K} [p_k, q_k] \ , \\ &|V(x + j\omega) - \lambda^{-1}E| > \lambda^{-1/2} \quad \text{for} && |j - p_k| \le N_1^\vartheta \text{ or } |j - q_k| \le N_1^\vartheta \end{aligned}$$

(except for $q_K = N_2$).

The first assertion follows from Lemma C.4 and C.9 if we can show that $\operatorname{dist}\big(H_{[p_k,q_k]}(x,\omega), E\big) \ge e^{-N_1^\vartheta}$. For $k < K$, this follows from Lemma 4.2. Hence, we only need to consider the case $k = K$.

If $|V(x+j\omega)-\lambda^{-1}E|>\lambda^{-1/2}$ for $j\in[N_2-N_1^\vartheta,N_2]$, then by Lemma 4.2, we have $\operatorname{dist}\big(H_{[p_k,q_k]}(x,\omega)-E\big)\geq e^{-N_1^\vartheta}$. So assume $|V(x+j_0\omega)-\lambda^{-1}E|\leq\lambda^{-1/2}$ for some $j_0\in[N_2-N_1^\vartheta,N_2]$. Then $|V(x+j\omega)-\lambda^{-1}E|>\lambda^{-1/2}$ for $j\in[p_K,q_K]\setminus\{j_0\}$.

Let $\operatorname{sp}H_{[p_K,q_K]}(x,\omega)=\big\{\mu_j\big\}_{j=p_K}^{q_K}$, $|\mu_j-\lambda V(x+j\omega)|\leq 2$.

$$\begin{aligned}\operatorname{dist}\big(\operatorname{sp}H_{[p_K,q_K]}(x,\omega),E\big)&=|E-\mu_{j_0}|\\&\geq|\lambda V(x)-\lambda V(x+j_0\omega)|-|\lambda V(x+j_0\omega)-\mu_{j_0}|-|\lambda V(x)-E|\\&\geq 5-2-(2+e^{-N_1^\vartheta})\geq e^{-N_1^\vartheta}\ .\end{aligned}$$

Applying Lemma 4.4, and note that we can choose k_i such that $\min\limits_i|k-k_i|=|k-k_{i_0}|\leq N_1^\vartheta$, we get

$$\begin{aligned}\log\big|\big[H_{[1,N_2]}(x,\omega)-E\big]^{-1}(k_{i_0},1)\big|&=\log|f_{[k_{i_0}+1,N_2]}|-\log|f_{[1,N_2]}|\\&=-\sum_{\ell=2}^{i_0+1}\big(\log\|A_\ell A_{\ell-1}\|-\|A_\ell\|\big)+O\big(\lambda^{-cN_1}\big)\\&=-\sum_{\ell=1}^{i_0}\log\|A_\ell\|+\sum_{\ell=2}^{i_0+1}\big(\log\|A_\ell\|+\log\|A_{\ell-1}\|-\\&\qquad\log\|A_\ell A_{\ell-1}\|\big)+O(\lambda^{-cN_1})\\&\leq -k_{i_0}\log(\lambda^{1/2}-2)+i_0N_1^\vartheta+i_0\big(20(\log(\lambda C_0)+N_1^\vartheta)\big)\\&\leq-\frac14 k_{i_0}\log\lambda\end{aligned}$$

$$\begin{aligned}\log\big|\big[H_{[1,\tilde N_{k\ell}]}(x,\omega)-E\big]^{-1}(k,1)\big|&\leq-\frac14 k_{i_0}\log\lambda+N_1^\vartheta\log(\lambda C_0+2)\\&\leq-\frac15 k\log\lambda\end{aligned}$$

□

For each component $(a_k^{(2)},b_k^{(2)})\times(\underline\omega_{k\ell}^{(2)},\bar\omega_{k\ell}^{(2)})$, fix $\tilde N_{k\ell},\bar N_{k\ell}\in[N_2,N_2+N_1]$ such that the Green's functions $\big(H_{[1,\tilde N_{k\ell}]}-E\big)^{-1}(j,1)$ and $\big(H_{[-\bar N_{k\ell},-1]}-E\big)^{-1}(j,-1)$ decay exponentially for all $(x,\omega)\in(a_k^{(2)},b_k^{(2)})\times(\underline\omega_{k\ell}^{(2)},\bar\omega_{k\ell}^{(2)})$. We can now construct C^1 function $E^{(2)}(x,\omega)$ which take values in $\operatorname{sp}H_{[-\bar N_{k\ell},\tilde N_{k\ell}]}(x,\omega)$.

Lemma 4.8. *If $(x,\omega)\in(a_k^{(2)},b_k^{(2)})\times(\underline\omega_{k\ell}^{(2)},\bar\omega_{k\ell}^{(2)})$, then*

$$\operatorname{sp}H_{[-\bar N_{k\ell},\tilde N_{k\ell}]}(x,\omega)\cap\big(E^{(1)}(x,\omega)-e^{-N_1}\ ,E^{(1)}(x,\omega)+e^{-N_1}\big)\neq\emptyset\ .$$

Proof. Let

$$\psi(j)=\begin{cases}\varphi^{(1)}(x,\omega)(j) & |j|\leq N_1\\ 0 & |j|>N_1\end{cases}$$

$$\begin{aligned}\big\|\big[H_{[-\tilde N_{k\ell},\tilde N_{k\ell}]}(x,\omega)-E^{(1)}(x,\omega)\big]\psi\big\| &\le \big|\varphi^{(1)}(x,\omega)(1)\big|+\big|\varphi^{(1)}(x,\omega)(-N_1)\big|\\ &\le 2\big(\lambda^{-\frac13 N_1}\big)<e^{-N_1}\ .\end{aligned}$$

The assertion now follows from Lemma A.4. □

Lemma 4.9. *Suppose* $E_1,E_2\in \operatorname{sp} H_{[-\tilde N_{k\ell},\tilde N_{k\ell}]}(x,\omega)\cap\big(E^{(1)}(x,\omega)-e^{-N_1^\vartheta},E^{(1)}(x,\omega)+e^{-N_1^\vartheta}\big)$, $H_{[-\tilde N_{k\ell},\tilde N_{k\ell}]}(x,\omega)\psi_i=E_i\psi_i$, $\|\psi_i\|=1$. *If* $|\psi_i(k)|\le e^{-c|k|}$ *for* $|k|\ge N_1$ *then* $E_1=E_2$.

Proof. Assume $E_1\neq E_2$. Let

$$\varphi_i(k)=\begin{cases}\psi_i(k)\Big[\sum_{|j|\le N_1}|\psi_i(j)|^2\Big]^{-1/2} & \text{for } |k|\le N_1\\ 0 & |k|>N_1\end{cases}$$

$\|\varphi_i\|=1$, $\big\|\big[H_{[-N_1,N_1]}(x,w)-E_i\big]\varphi_i\big\|\le 2(2e^{-N_1})$. Since

$$\operatorname{sp} H_{[-N_1,N_1]}(x,\omega)\cap\big(E^{(N_1)}-\frac12\lambda^{1/2},E^{(N_1)}+\frac12\lambda^{1/2}\big)=\big\{E^{(N_1)}(x,\omega)\big\}\ ,$$

by Lemma A.5

$$\min_{|c|=1}\big\|\varphi^{(N_1)}(x,\omega)-c\varphi_i\big\|<\sqrt2(4e^{-N_1})\big(\frac12\lambda^{1/2}\big)^{-1}\ .$$

Hence, there is $|c|=1$ such that

$$\begin{aligned}400\lambda^{-1}e^{-2N_1}>\|\varphi_1-c\varphi_2\|^2&=2+2\operatorname{Re}\bar c\langle\varphi_1,\varphi_2\rangle\\ &\Longrightarrow |\langle\varphi_1,\varphi_2\rangle>|1-200\lambda^{-1}e^{-2N_1}\end{aligned}$$

Since $H_{[-\tilde N_{k\ell},\tilde N_{k\ell}]}(x,\omega)$ is hermitian,

$$\begin{aligned}0&=|\langle\psi_1,\psi_2\rangle|\\ &\ge\Big|\sum_{|k|\le N_1}\psi_1(k)\overline{\psi_2(k)}\Big|-\sum_{|k|>N_1}\big|\psi_1(k)\overline{\psi_2(k)}\big|\\ &\gtrsim|\langle\varphi_1,\varphi_2\rangle|-\sum_{|k|>N_1}e^{-2k}>0\end{aligned}$$

□

Lemma 4.7, Lemma 4.8 and Lemma 4.9 show that $H_{[-\tilde N_{k\ell},\tilde N_{k\ell}]}(x,\omega)$ has unique eigenvalue $E^{(2)}(x,\omega)\in\big(E^{(1)}(x,\omega)-\frac12e^{-N_1^\vartheta},E^{(1)}(x,\omega)+\frac12e^{-N_1^\vartheta}\big)$ for $\omega\in\Omega^{(1)}$, $x\in\mathcal D_\omega^{(2)}$, $\big|E^{(2)}(x,\omega)-E^{(1)}(x,\omega)\big|<e^{-N_1}$, the eigenfunction $\varphi^{(2)}(x,\omega)$ corresponding to $E^{(2)}(x,\omega)$ decay exponentially, $\|\varphi^{(2)}(x,\omega)\|=1$. By multiplying c to $\varphi^{(2)}$, $|c|=1$, we can choose

$$\|\tilde\varphi^{(1)}(x,\omega)-\varphi^{(2)}(x,\omega)\|\le e^{-N_1}$$

where

$$\tilde{\varphi}^{(1)}(x,\omega)(k) = \begin{cases} \varphi^{(1)}(x,\omega;k) & |k| \le N_1 \\ 0 & |k| > N_1 \end{cases}$$

For future reference, we record some properties of $E^{(2)}$ and $\varphi^{(2)}$ in the following lemma.

Lemma 4.10. *$E^{(2)}, \varphi^{(2)}$ are C^1 in $\mathcal{D}^{(2)}$ and*

$$-\frac{1}{2}\lambda^{39/40} < |\partial_x E^{(2)}(x,\omega)| \le C_0\lambda$$
$$|\partial_\omega E^{(2)}(x,\omega)| \le C_0 N_2 \lambda$$

Proof. Results in Appendix A show that $E^{(2)}$ and $\varphi^{(2)}$ are C^1. The bounds of the partial derivatives of $E^{(2)}(x,\omega)$ follow, since $\varphi^(x,\omega)(n)$ decay exponentially. (See Corollary 3.4) □

5 Inductive construction of eigenvalue $E^{(s)}(x,\omega)$ and eigenfunction $\varphi^{(s)}(x,\omega)$

We construct $E^{(s)}(x,\omega)$ inductively. Finally, we derive Theorem 1.1.

Assume $E^{(1)}, \dots, E^{(s)}, \varphi^{(1)}, \dots, \varphi^{(s)}$ have been defined with the following properties:

1. The domain of $E^{(k)}$ and $\varphi^{(k)}(.,.)(j)$ is $\mathcal{D}^{(k)}$, $\mathcal{D}^{(1)} \supset \mathcal{D}^{(2)} \supset \cdots \subset \mathcal{D}^{(s)}$; $\operatorname{compl} \mathcal{D}^{(k)} < e^{2N_{k-1}^{\vartheta}}$, $\operatorname{mes}\big(\mathcal{D}^{(k-1)} \setminus \mathcal{D}^{(k)}\big) < e^{-N_{k-1}^{\vartheta}}$ for $1 \le k \le s$ where $N_k \asymp e^{N_{k-1}^{\tau}}$ $(\tau < \vartheta)$.

2. $H_{[-\tilde{N}_{ij}^{(k)}, N_{ij}^{(k)}]}(x,\omega)\varphi^{(k)}(x,\omega) = E^{(k)}(x,\omega)\varphi^{(k)}(x,\omega)$.

3. $\sum_j \big|\varphi^{(k)}(x,\omega)(j)\big|^2 = 1$; $\big|\varphi^{(k)}(x,\omega)(j)\big| \le \lambda^{-\frac{1}{5}|j|}$

4. $\big|E^{(k)}(x,\omega) - E^{(k-1)}(x,\omega)\big| < e^{-N_{k-1}}$, $\big\|\varphi^{(k)}(x,\omega) - \tilde{\varphi}^{(k-1)}(x,\omega)\big\| < e^{-N_{k-1}}$ where

$$\tilde{\varphi}^{(k-1)}(x,\omega)(j) = \begin{cases} \varphi^{(k-1)}(x,\omega)(j) & \text{if } |j| \le N_{ij}^{(k)} \\ 0 & |j| > N_{ij}^{(k)} \end{cases}$$

5. $\big|E^{(k-1)}(x,\omega) - E^{(k-1)}(x+j\omega,\omega)\big| \ge 2e^{-N_{k-1}^{\vartheta}}$ if $(x,\omega) \in \mathcal{D}^{(k)}$ and $(x+j\omega,\omega) \in \mathcal{D}^{(k-1)}$, $|j| \le N_k^2$

6. $\operatorname{sp} H_{[-\tilde{N}_{ij}^{(k)}, N_{ij}^{(k)}]}(x,\omega) \cap \big(E^{(k)}(x,\omega) - e^{-N_{k-1}^{\vartheta}}, E^{(k)}(x,\omega) + E^{-N_{k-1}^{\vartheta}}\big) = \big\{E^{(N_k)}(x,\omega)\big\}$

7. There is $\Omega^{(1)} \supset \Omega^{(2)} \supset \dots \Omega^{(s-1)}$, $\operatorname{mes}\big(\Omega^{(k-1)} \setminus \Omega^{(k)}\big) < e^{-\frac{1}{2}N_{k-1}^{\vartheta}}$, such that $\operatorname{mes}\big(\mathcal{D}_\omega^{(k-1)} \setminus \mathcal{D}_\omega^{(k)}\big) < e^{-\frac{1}{2}N_{k-1}^{\vartheta}}$ for all $\omega \in \Omega^{(k)}$

8. For $\omega \in \Omega^{(k)}$, there is $\mathcal{E}_\omega^{(k)} \subset \mathcal{E}_\omega^{(k-1)} \subset \cdots \subset \varepsilon_\omega^{(1)}$, $\operatorname{mes}\big(\mathcal{E}_\omega^{(k-1)} \setminus \mathcal{E}_\omega^{(k)}\big) < e^{-\frac{1}{2}N_{k-1}^\vartheta}$, such that if $x \in \mathcal{D}_\omega^{(k-1)}$, $E \in \mathcal{E}_\omega^{(k)}$, $\big|E - E^{(N_{k-1})}(x,\omega)\big| < e^{-N_{k-1}^\vartheta}$ then $x \in \mathcal{D}_\omega^{(k)}$

9. $E \in \mathcal{E}_\omega^{(k)}$, $\omega \in \Omega^{(k)}$, $x + j\omega \in \mathcal{D}_\omega^{(k)}$ for $|j| \le 4N_k \Longrightarrow |E - E^{(k)}(x,\omega)| > \frac{1}{2}e^{-N_k^\vartheta}$. Then

$$\log |f_{[-\tilde{N}_{ij}^{(k)}, N_{ij}^{(k)}]}(x,\omega,E)| \ge \frac{1}{4}(2N_k)\log\lambda - 10N_{k-1}\log\lambda$$

and

$$\begin{aligned}\log \|M_{[-N_k,N_k]}(x,\omega,E)\| + \log \|M_{[N_{k+1},3N_{k+1}]}(x,\omega,E)\| \\ - \log \|M_{[-N_k,3N_{k+1}]}(x,\omega,E)\| \le 10N_{s-1}\log\lambda\end{aligned}$$

10. $e^{-2N_{k+1}^\vartheta} < \big|E^{(k)}(x,\omega) - E\big| < e^{-2N_k^\vartheta} \Longrightarrow \log |f_{[-\tilde{N}_{ij}^{(k)}, N_{ij}^{(k)}]}(x,\omega,E)| > \frac{N+\tilde{N}}{4}\log\lambda$

First, we need to eliminate the case when $\operatorname{sp} H_{[-N_s,N_s]}(x,\omega)$ and $\operatorname{sp} H_{-N_s,N_s]}(x+j\omega,\omega)$ are close.

Lemma 5.1. *There is $\mathcal{D}^{(s+1)} \subset \mathcal{D}^{(s)}$, $\mathcal{D}^{(s+1)} = \bigcup_{j=1}^{M} D_j^{(s+1)}$ where $\mathcal{D}_j^{(s+1)} = \Big(a_j^{(s+1)}, b_j^{(s+1)}\Big) \times \Big(\underline{\omega}_{j\ell}^{(s+1)}, \bar{\omega}_{j\ell}^{(s+1)}\Big)$, $b_j^{(s+1)} - a_j^{(s+1)} < e^{-N_s^\eta}$, $\bar{\omega}_{j\ell}^{(s+1)} - \underline{\omega}_{j\ell}^{(s+1)} < e^{-N_s^\eta}$, $M \le e^{2N_s^\eta}$, which satisfies the following conditions:*

1. $\operatorname{mes}\big(\mathcal{D}^{(s)} \setminus \mathcal{D}^{(s+1)}\big) \le e^{-\frac{1}{2}N_s^\vartheta}L$
2. *If $(x,\omega) \in \mathcal{D}^{(s+1)}$ and $(x+j\omega,\omega) \in \mathcal{D}^{(s)}$ for some $0 < |j| \le N_{s+1}$ where $N_{s+1} = \lfloor e^{N_s^\tau} \rfloor$, $\tau < \vartheta$, then $\big|E^{(s)}(x,\omega) - E^{(s)}(x+j\omega,\omega)\big| > 4e^{-N_s^\vartheta}$.*

Proof. Essentially same as proof of Lemma 4.1. □

We now show that there is $N \asymp N_{s+1}$ such that the Green's function $\big(H_{[1,N]}(x,\omega) - E\big)^{-1}(k,1)$ decays exponentially.

Lemma 5.2. *Let $(x,\omega) \in \mathcal{D}^{(s+1)}$, $E \in \mathcal{E}_\omega^{(s+1)}$, $\big|E - E^{(s)}(x,\omega)\big| < e^{-N_s^\vartheta}$. Then there is N, $|N - N_{s+1}| \le N_s$, such that $E \notin \operatorname{sp} H_{[1,N]}(x,\omega)$ and $\big[H_{[1,N]}(x,\omega) - E\big]^{-1}(k,1) \le -\frac{1}{5}k\log\lambda$ for $k > N_s$.*

Proof. Let $1 < j_1 < j_2 < \cdots < j_m \le N_{s+1}$ be such that $x + j_i\omega \in \mathcal{D}_\omega^{(s)}$ and $\big|E - E^{(s)}(x+j_i\omega,\omega)\big| < \frac{1}{2}e^{-N_s^\vartheta}$. Then

$$\begin{aligned}&\big|E^{(s-1)}(x+j_i\omega,\omega) - E^{(s-1)}(x+j_{i+1}\omega,\omega)\big| \\ &\le \big|E^{(s-1)}(x+j_i\omega,\omega) - E^{(s)}(x+j_i\omega,\omega)\big| \\ &\quad + \big|E^{(s)}(x+j_i,\omega,\omega) - E\big| + \big|E - E^{(s)}(x+j_{i+1}\omega,\omega)\big| \\ &\quad + \big|E^{(s)}(x+j_{i+1}\omega,\omega) - E^{(s-1)}(x+j_{i+1}\omega,\omega)\big| \\ &< e^{-N_s} + \frac{1}{2}e^{-N_s^\vartheta} + \frac{1}{2}e^{-N_s^\vartheta} + e^{-N_s} < 2e^{-N_s^\vartheta}\end{aligned}$$

By construction of $\mathcal{D}^{(s)}$, $|j_i - j_{i+1}| > N_s^2$. Also, $j_1 > N_s^2$.

We can choose $N, |N - N_{s+1}| \le N_s$, such that $[1, N] = \bigcup_{k=1}^{K} [a'_k, b'_k]$, $N_s \le b'_k - a'_k \le 3N_s$, $a'_k < b'_{k-1} < a'_{k+1} < b'_k$, $b'_{k-1} - a'_k > N_s^{\vartheta}$, $|j_i - b'_k| > N_s^{\vartheta}$, $|j_i - a'_k| > N_s^{\vartheta}$ for all i, k. The first assertion follows form condition (3) and Lemma C.9. By condition (10) and (11), we can apply the Avalanche Principle to obtain second assertion.

Take $(x_0, \omega_0) \in \mathcal{D}^{(s+1)}$. By Lemma 5.2, there are N, $|N - N_{s+1}| \le N_s$, and $\widetilde{N}$, $|\widetilde{N} - N_{s-1}| \le N_s$, such that $H_{[-\widetilde{N},N]}(x, \omega)$ has a unique eigenvalue $E^{(s+1)}(x_0, \omega_0)$ in $\Big(E^{(s)}(x, \omega) - 2e^{-N_s^{\vartheta}}, E^{(s)} + 2e^{-N_s^{\vartheta}}\Big)$, $\big|E^{(s+1)}(x_0, \omega_0) - E^{(s)}(x_0, \omega_0)\big| < e^{-N_s}$ (see Lemma 4.7, 4.8).

For $(x, \omega) \in \mathcal{D}^{(s+1)}$, $-\widetilde{N} \le j \le N$,

$$|\lambda V(x + j\omega) - \lambda V(x_0 + j\omega_0)| \lesssim C_0 \lambda \big(e^{-N_s^{\eta}} + N_{s+1} e^{-N_s^{\eta}}\big) < e^{-\frac{1}{2} N_s^{\vartheta}} .$$

Also

$$\big|E^{(s)}(x, \omega) - E^{(s)}(x_0, \omega_0)\big| \lesssim C_0 \lambda e^{-N_s^{\eta}} + C_0 N_s \lambda e^{-N_s^{\eta}} < e^{-\frac{1}{2} N_s^{\vartheta}} .$$

Hence

$$\begin{aligned} H_{[-\widetilde{N},N]}(x, \omega) - e^{-\frac{1}{2} N_s^{\vartheta}} I &\le H_{[-\widetilde{N},N]}(x_0, \omega_0) \\ &\le H_{[-\widetilde{N},N]}(x, \omega) + e^{-\frac{1}{2} N_s^{\vartheta}} I . \end{aligned}$$

Let $\operatorname{sp} H_{[-\widetilde{N},N]}(x, \omega) = \big\{\mu_\ell(x, \omega)\big\}_{\ell=-\widetilde{N}}^{N}$, $\mu_{-\widetilde{N}} < \mu_{-\widetilde{N}+1} < \cdots < \mu_N$. Then

$$\mu_\ell(x, \omega) - e^{-\frac{1}{2} N_s^{\eta}} < \mu_\ell(x_0, \omega_0) < \mu_\ell(x, \omega) + e^{-\frac{1}{2} N_s^{\eta}} .$$

Say $E^{(s+1)}(x_0, \omega_0) = \mu_{\ell_0}(x_0, \omega_0)$. Then

$$\begin{aligned} \big|\mu_{\ell_0+1}(x, \omega) - E^{(s)}(x, \omega)\big| &\ge \big|\mu_{\ell_0+1}(x_0, \omega_0) - E^{(s)}(x_0, \omega_0)\big| \\ &\quad - \big|\mu_{\ell_0+1}(x, \omega) - \mu_{\ell_0+1}(x_0, \omega_0)\big| \\ &\quad - \big|E^{(s)}(x, \omega) - E^{(s)}(x_0, \omega_0)\big| \\ &> \frac{3}{2} e^{-N_s^{\vartheta}} - e^{-\frac{1}{2} N_s^{\vartheta}} - e^{-\frac{1}{2} N_s^{\vartheta}} > e^{-N_s^{\vartheta}} . \end{aligned}$$

Similarly, $\big|\mu_{\ell_0-1}(x, \omega) - E^{(s)}(x, \omega)\big| > e^{-N_s^{\vartheta}}$.

Therefore, $H_{[-\widetilde{N},N]}(x, \omega)$ has a unique eigenvalue $E^{(s+1)}(x, \omega)$ in $\big(E^{(s)}(x, \omega) - e^{-N_s^{\vartheta}}, E^{(s)}(x, \omega) + e^{-N_s^{\vartheta}}\big)$ for all $(x, \omega) \in \mathcal{D}_j^{(s+1)} \cdot \big|E^{(s+1)}(x, \omega) - E^{(s)}(x, \omega)\big| < e^{-N_s^{\vartheta}}$.

We can define $\varphi^{(s+1)}(x,\omega)(n)$ such that

$$\begin{aligned} H_{[-\widetilde{N},N]}(x,\omega)\varphi^{(s+1)}(x,\omega) &= E^{(s+1)}(x,\omega)\varphi^{(s+1)}(x,\omega)\\ \|\varphi^{(s)}(x,\omega)-\varphi^{(s+1)}(x,\omega)\| &\lesssim e^{-N_s}e^{N_s^{\vartheta}} < e^{\frac{1}{2}N_s}\\ |\varphi^{(s+1)}(x,\omega)(n)| &< \lambda^{-\frac{1}{5}\lambda}\\ \frac{1}{2}\lambda^{39/40} < |\partial_x E^{(s+1)}(x,\omega)| &\le C_0\lambda\\ |\partial_\omega E^{(s+1)}(x,\omega)| &\le C_0 N_{s+1}\lambda\\ \sum_n |\varphi^{(s+1)}(x,\omega)(n)| &\le 1 \end{aligned}$$

□

As in Lemma 3.5, there exist $\Omega^{(s+1)} \subset \Omega^{(s)}$ such that for any $\omega \in \Omega^{(s+1)}$, one has

$$\mathrm{mes}\Big(E^{(s)}(.,\omega)^{-1}(\mathcal{E}^{(s)} \setminus \mathcal{D}_\omega^{(s+1)})\Big) \lesssim e^{-N_s^{\vartheta/2}} .$$

For $\omega \in \Omega^{(s+1)}$, suppose $\mathcal{D}_\omega^{(s+1)} = \bigcup_r \big(a_r^{(s)}(\omega), b_r^{(s)}(\omega)\big)$. Let

$$\tilde{\mathcal{E}}_{\omega,r}^{(s+1)} = RanE^{(s)}(.,\omega)\Big|_{\mathcal{D}_\omega^{(s+1)}\cap\left(a_r^{(s)}(\omega),b_r^{(s)}(\omega)\right)} = \bigcup_k \big(\underline{E}_{r,k}, \bar{E}_{r,k}\big)$$

$$\tilde{\mathcal{E}}_{\omega}^{(s+1)} = \bigcup_r\bigcup_k \big(\underline{E}_{r,k} + e^{-N_s^{\vartheta}}, \bar{E}_{r,k} - e^{-N_s^{\vartheta}}\big)$$

$$\mathcal{E}_{\omega}^{(s+1)} = \bigcup_r\bigcup_k \big(\underline{E}_{r,k} + 2e^{-N_s^{\vartheta}}, \bar{E}_{r,k} - 2e^{-N_s^{\vartheta}}\big)$$

Then $\mathrm{mes}\big(\mathcal{E}^{(s)} \setminus \mathcal{E}^{(s+1)}\big) < e^{-N_s^{\vartheta/4}}$.

In the next two lemmas, we prove the hypothesis of the Avalanche Principle in the $s+1$ scale.

Lemma 5.3. *Let $\omega \in \Omega^{(s+1)}$, $E \in \mathcal{E}_\omega^{(s+1)}$. Suppose $\big|E - E^{(s+1)}(x+j\omega,\omega)\big| \ge e^{-2N_s^{\vartheta}}$ whenever $(x+j\omega,\omega) \in \mathcal{D}^{(s+1)}$ for $|j| \le 4N_{s+1}$. Then*

$$\log|f_{[-\widetilde{N},N]}(x,\omega,E)| \ge \frac{N_{ij}^{(s+1)} + \tilde{N}_{ij}^{(s+1)}}{4}\log\lambda$$

and

$$\begin{aligned} &\log\big\|M_{[-N_{s+1},N_{s+1}]}(x,\omega,E)\big\| + \log\big\|M_{[N_{s+1}+1,3N_{s+1}+1]}(x,\omega,E)\big\|\\ &\quad - \log\big\|M_{[-N_{s+1},3N_{s+1}+1]}(x,\omega,E)\big\| \lesssim N_s\log\lambda . \end{aligned}$$

Proof. For $\tilde{x} = x + j_0\omega$, $|j_0| \le 3N_{s+1}$, if $\big(\tilde{x}+j\omega,\omega\big) \in \mathcal{D}^{(s)}$, $|j| \le 4N_s$, then either

1. $(\tilde{x}+j\omega,\omega)\notin\mathcal{D}^{(s+1)}$, or

2. $(\tilde{x}+j\omega,\omega)\in\mathcal{D}^{(s+1)}$, $|E-E^{(s+1)}(x+j\omega,\omega)|>e^{-2N_s^\vartheta}$

In (1), $E\in\mathcal{E}_\omega^{(s+1)}\Longrightarrow|E^{(s)}(\tilde{x}+j\omega,\omega)-E|>\frac{1}{2}e^{-N_s^\vartheta}$. In (2),
$|E-E^{(s)}(x,\omega)|\geq|E-E^{(s+1)}(x,\omega)|-|E^{(s+1)}(x,\omega)-E^{(s)}(x,\omega)|>\frac{1}{2}e^{-2N_s^\vartheta}$.

By induction hypothesis (10), (11), we can apply Avalanche Principle to obtain the assertion. □

Lemma 5.4. *If $E\in\mathcal{E}_\omega^{(s+1)}$, $(x,\omega)\in(a_i^{(s+1)},b_i^{(s+1)})\times(\underline{\omega}_{ij},\bar{\omega}_{ij})$, and $e^{-2N_{s+1}^\vartheta}<|E-E^{(s+1)}(x,\omega)|<e^{-2N_s^\vartheta}$. Then there is $N_{ij}^{(s+1)},|N_{ij}^{(s+1)}-N_{s+1}|\leq N_s$, and $\widetilde{N}_{ij}^{(s+1)},|\widetilde{N}_{ij}^{(s+1)}-N_{s+1}|\leq N_s$, such that*

$$\log|f_{[-\tilde{N}_{ij}^{(s+1)},N_{ij}^{(s+1)}]}(x,\omega,E)|>\frac{N_{ij}^{(s+1)}+\tilde{N}_{ij}^{(s+1)}}{4}\log\lambda-2N_{s+1}^\vartheta$$

Proof. By construction

$$\begin{aligned}
&\operatorname{sp}H_{[-\tilde{N},N]}(x,\omega)\cap\big(E^{(s)}(x,\omega)-e^{-N_s^\vartheta},E^{(s)}(x,\omega)+e^{-N_s^\vartheta}\big)=\big\{E^{(s+1)}(x,\omega)\big\}\\
&\big|E^{(s)}(x,\omega)-E^{(s+1)}(x,\omega)\big|<e^{-N_s}\\
&\Longrightarrow\big|E-E^{(s)}(x+j\omega,\omega)\big|\geq e^{-N_s^\vartheta}\quad\text{for }(x+j\omega,\omega)\in\mathcal{D}^{(s)},\ 0<|j|\leq N_{s+1}^2\\
&\Longrightarrow\text{ there exist }|\widetilde{E}-E|\leq e^{-2N_s^\vartheta}\text{ such that }\big|\widetilde{E}-E^{(s)}(x+j\omega,\omega)\big|\geq e^{-2N_s^\vartheta},\ |j|\leq N_{s+1}^s
\end{aligned}$$

By Lemma 5.3, $\log|f_{[-\tilde{N},N]}(x,\omega,\widetilde{E})|>\frac{N_{ij}^{(s+1)}+\tilde{N}_{ij}^{(s+1)}}{4}\log\lambda$. Applying Lemma B.5 gives

$$\log|f_{[-\tilde{N},N]}(x,\omega,E)|>\frac{N_{ij}^{(s+1)}+\tilde{N}_{ij}^{(s+1)}}{4}\log\lambda-2N_{s+1}\ .$$

□

By taking the limit as s goes to infinity, we can prove Theorem 1.1:

Theorem 5.5. *There is $\Omega=\Omega(V,\lambda)\in[0,1]$, $\operatorname{mes}\big([0,1]\setminus\Omega\big)\lesssim\lambda^{-\frac{1}{2}}$, such that for any $\omega\in\Omega$ there exists $\mathcal{E}_\omega\subset\lambda\mathcal{J}$, $\operatorname{mes}\big(\mathcal{J}\setminus\lambda^{-1}\mathcal{E}_\omega\big)\lesssim\lambda^{-\frac{1}{2}}$, with $\lambda(\omega,E)\gtrsim\log\lambda$ for all $E\in\mathcal{E}_\omega$. Also, there is $x\in\mathbb{T}$ and $\{\varphi(n)\}$, $|\varphi(n)|\lesssim e^{-c|n|}$ such that*

$$-\varphi(n+1)+\varphi(n-1)+V(x+n\omega)\varphi(n)=E\varphi(n).$$

Proof. Take $\Omega=\bigcap\Omega^{(k)}$, $\mathcal{E}_\omega=\bigcap\mathcal{E}_\omega^{(k)}$. For any $\omega\in\Omega$ and $E\in\mathcal{E}_\omega$, one has

$$\frac{1}{N_{ij}^{(k)}+\tilde{N}_{ij}^{(k)}}\log\big\|M_{[-\tilde{N}_{ij}^{(k)},N_{ij}^{(k)}]}(x_k,\omega,E)\big\|\gtrsim\log\lambda,$$

$k=1,\dots$ where $E=E^{(k)}(x_k,\omega)$, $x_k\in\mathcal{D}_\omega^{(k)}$.

A subsequent of $\{x_k\}$ converges. Without loss of generality, we may assume $x=\lim x_k$. Take $\varphi(n)=\lim\varphi^{(k)}(x_k,\omega)(n)$. □

Part II

6 Resonance at first scale

In Part II, we introduce variations of potential (Section 8), and show that for typical variation, the resulting potential leads to positive Lyapunov exponent for all spectral value. In this section, we get an upper estimate of the number of terms where resonances can occur.

Let $V \in C^3(\mathbb{T})$ be such that $|V'(x)| + |V''(x)| \geq 2c > 0$ for all $x \in \mathbb{T}$. Note that the assumption implies V has finitely many critical points.

Let $x_1, \ldots, x_n$ be the critical points of V. There exists $\delta > 0$ such that

$$|V'(y)| \geq c\,|y - x_i| \qquad \text{whenever } |y - x_i| < \delta$$

for any $i = 1, 2, \ldots, n$.

Suppose

$$C_0 = \min\big\{|V'(y)| : y \in \mathbb{T} \setminus \bigcup_j (x_j - \delta, x_j + \delta)\big\} > 0\ ,$$

$$C_1 = \max_{y \in \mathbb{T}} (|V'(y)| + |V''(y)|)\ .$$

and V has m_0 monotonicity intervals.

In the next two lemmas, we find an upper bound of the number of entries that can lead to resonance.

Lemma 6.1. *Take any $E \in \mathbb{R}$, $0 < \varepsilon < \min\{\frac{1}{6}c\delta^2, C_0^2\}$. Then*

$$V^{-1}(E - \varepsilon, E + \varepsilon) = \bigcup_{i=1}^{m} (a_i, b_i), \quad m \leq m_0, \quad b_i - a_i \lesssim \varepsilon^{1/2}$$

Proof. Each monotonicity interval of V can intersect with at most one component (a_i, b_i) of $V^{-1}(E - \varepsilon, E + \varepsilon)$ so $m \leq m_0$.

$$\int_{x_j}^{x_j + \delta} |V'(y)| dy \geq \int_0^{\delta} ctdt = \frac{c}{2}\delta^2$$

so (a_i, b_i) does not contain any $(x_j, x_j + \delta)$. Similarly, (a_i, b_i) does not contain any $(x_j - \delta, x_j)$. Therefore, (a_i, b_i) intersects with at most one $(x_{j_1}, x_{j_1} + \delta)$ and one $(x_{j_2} - \delta, x_{j_2})$. Suppose $(a_i, b_i) \cap (x_{j_1}, x_{j_1} + \delta) = (y_1, y_2)$

$$\begin{aligned} 2\varepsilon &\geq \int_{y_1}^{y_2} |V'(y)| dy \\ &\geq c \int_{y_1}^{y_2} (y - x_{j_1}) dy \\ &= \frac{c}{2}\left[(y_2 - x_{j_1})^2 - (y_1 - x_{j_1})^2\right] \\ &\geq \frac{c}{2}(y_2 - y_1)^2 \end{aligned}$$

Similarly, if $(a_i, b_i) \cap (x_{j_2} - \delta, x_{j_2}) = (y_3, y_4)$ then $2\varepsilon \geq \frac{c}{2}(y_4 - y_3)^2$.

$$\begin{aligned} b_i - a_i &= (y_2 - y_1) + (y_4 - y_3) + \operatorname{mes}\Big[(a_i, b_i) \setminus \bigcup_j (x_j - \delta, x_j + \delta)\Big] \\ &\leq 2\Big(\frac{4\varepsilon}{c}\Big)^{1/2} + \frac{2\varepsilon}{C_0} \end{aligned}$$

□

Fix $N \gg 1$, $\lambda = N^{8 \cdot 2^{m_0}}$, such that

$$\lambda^{-1/2} < \min\{\frac{1}{6} c\delta^2, C_0^2\} \ .$$

Lemma 6.2. *For any $x \in \mathbb{T}$, $E \in \mathbb{R}$, if ω satisfies the Diophantine condition*

$$\|\ell\omega\| \gtrsim |\ell|^{-\beta} \tag{6.1}$$

$1 < \beta < 2$, then

$$\#\Big\{|\ell| \leq N^{2^{m_0}} : x + \ell\omega \in V^{-1}\big(E - \lambda^{-\frac{1}{2}}, E + \lambda^{-\frac{1}{2}}\big)\Big\} \leq m_0$$

provided $\delta < \delta_0(m_0, \beta)$.

Proof. Suppose, for contradiction, that

$$x + \ell_1\omega, x + \ell_2\omega \in (a_i, b_i) \subset V^{-1}(E - \lambda^{-1/2}, E + \lambda^{-1/2}) \ .$$

Then, by Lemma 6.1,

$$N^{-\beta 2^{m_0}} < |\ell_1 - \ell_2|^{-\beta} \lesssim \|(\ell_1 - \ell_2)\omega\| < b_i - a_i \lesssim \Big(\lambda^{-1/2}\Big)^{1/2} = N^{-2 \cdot 2^{m_0}}$$

This is false for sufficiently large N (or equivalently, for sufficiently small δ). □

With the upper bound provided by Lemma 6.2, we can now find $N_{(1)}$ such that all the resonance term is far away from the edge of $[-(N^{(1)})^2, (N^{(1)})^2]$. In this case, the Green's function $H_{[1,(N^{(1)})^2]}(x, \omega)(k, N^{(1)} + 1)$ will decaly exponentilly and we get exponentially decaying eigenfunctions.

Lemma 6.3. *For any $x \in \mathbb{T}$, and any ω satisfying condition* (6.1)*, there is $N^{(1)}(x, \omega) \leq N^{2^{m_0 - 1}}$ such that*

$$|V(x + j\omega) - V(x)| \geq \lambda^{-1/2}$$

for all

$$j \in \Big[-\big(N^{(1)}(x, \omega)\big)^2, -N^{(1)}(x, \omega) - 1\Big] \cup \Big[N^{(1)}(x, \omega) + 1, \big(N^{(1)}(x, \omega)\big)^2\Big] \ .$$

Proof. If $|V(x+j\omega)-V(x)|\geq\lambda^{-1/2}$ for all $j\in[-N^2,-N-1]\cup[N+1,N^2]$ then we can take $N^{(1)}(x,\omega)=N$.

So assume there is $j_1\in[-N^2,-N-1]\cup[N+1,N^2]$ such that $|V(x+j_1\omega)-V(x)|<\lambda^{-1/2}$. If $|V(x+j\omega)-V(x)|\geq\lambda^{-1/2}$ for all $j\in[-N^4,-N^2-1]\cup[N^2+1,N^4]$ then we can choose $N^{(1)}(x,\omega)=N^2$.

So assume there is $j_2\in[-N^4,-N^2-1]\cup[N^2+1,N^4]$ such that $|V(x+j_2\omega)-V(x)|<\lambda^{-1/2}$. Continue this process if there is $j_i\in[-N^{2^i},-N^{2^{i-1}}-1]\cup[N^{2^{i-1}}+1,N^{2^i}]$ such that $|V(x+j_i\omega)-V(x)|<\lambda^{-1/2}$. Since

$$\#\left\{0<|j|\leq N^{2^{m_0}}:|V(x+j\omega)-V(x)|<\lambda^{-1/2}\right\}\leq m_0-1$$

we can find $N^{(1)}(x,\omega)\leq N^{2^{m_0-1}}$ such that

$$|V(x+j\omega)-V(x)|\geq\lambda^{-1/2}$$

for all

$$j\in\left[-\left(N^{(1)}(x,\omega)\right)^2,-N^{(1)}(x,\omega)-1\right]\cup\left[N^{(1)}(x,\omega)+1,\left(N^{(1)}(x,\omega)\right)^2\right]\ .$$

□

Take $K,L\in\mathbb{N}$ such that $20C_1\lambda^{1/2}<K<30C_1\lambda^{1/2}$, $20C_1\lambda^{5/8}<L<30C_1\lambda^{5/8}$. Let $\tilde{x}_i=\frac{i}{K}$, $i=1,2,\ldots,K$ and $\tilde{\omega}_j=\frac{j}{L}$, $j=1,2,\ldots,L$. Define

$$U_i=(\tilde{x}_i-\frac{1}{32C_1}\lambda^{-1/2},\tilde{x}_i+\frac{1}{32C_1}\lambda^{-1/2}),\qquad V_j=(\tilde{\omega}_j-\frac{1}{32C_1}\lambda^{-5/8},\tilde{\omega}_j+\frac{1}{32C_1}\lambda^{-5/8})$$

Then $\mathbb{T}\times\mathbb{T}=\bigcup_{i,j}U_i\times V_j$. Moreover, there is $\hat{\omega}_j\in V_j$ satisying condition (6.1) for each j.

Lemma 6.4. *For each i,j, there is $N_{ij}^{(1)}\leq N^{2^{m_0-1}}$ such that $|V(x+k\omega)-V(x)|\geq\frac{7}{8}\lambda^{-1/2}$ for all $(x,\omega)\in U_i\times V_j$, $k\in\left[-\left(N_{ij}^{(1)}\right)^2,-N_{ij}^{(1)}-1\right]\cup\left[N_{ij}^{(1)}+1,\left(N_{ij}^{(1)}\right)^2\right]$.*

Proof. Let $N_{ij}^{(1)}=N^{(1)}(\tilde{x}_i,\hat{\omega}_j)$. For any $(x,\omega)\in U_i\times V_j$, $k\in\left[-\left(N_{ij}^{(1)}\right)^2,-N_{ij}^{(1)}-1\right]\cup\left[N_{ij}^{(1)}+1,\left(N_{ij}^{(1)}\right)^2\right]$, we have

$$\begin{aligned}|V(x+k\omega)-V(x)| &\geq |V(\tilde{x}_i+k\hat{\omega}_j)-V(\tilde{x}_i)|-|V(\tilde{x}_i+k\hat{\omega}_j)-V(\tilde{x}_i+k\omega)|\\ &\quad -|V(\tilde{x}_i+k\omega)-V(x+k\omega)|-|V(x)-V(\tilde{x}_i)|\\ &\geq \lambda^{-1/2}-C_1|k(\hat{\omega}_j-\omega)|-2C_1|\tilde{x}_i-x|\\ &\geq \frac{7}{8}\lambda^{-1/2}\end{aligned}$$

□

Notation: $\Lambda_{ij}^{(1)} = \Big[-\big(N_{ij}^{(1)}\big)^2, \big(N_{ij}^{(1)}\big)^2\Big]$.

Let

$$\begin{aligned} J_{ij}^{(1)} &:= \#\left\{\ell \in \Lambda_{ij}^{(1)} : |V(\tilde{x}_i + \ell\hat{\omega}_j) - V(\tilde{x}_i)| < \frac{1}{2}\lambda^{-1/2}\right\} \\ &\leq \#\left\{|\ell| \leq N^{2m_0} : |V(\tilde{x}_i + \ell\hat{\omega}_j) - V(\tilde{x}_i)| < \frac{1}{2}\lambda^{-1/2}\right\} \leq m_0 \ . \end{aligned}$$

We have distinct eigenvalues

$$E_{ijk}^{(1)}(\tilde{x}_i, \hat{\omega}_j) \in \operatorname{sp} H_{\Lambda_{ij}^{(1)}}(\tilde{x}_i, \hat{\omega}_j) \ , \quad 1 \leq k \leq J_{ij}^{(1)} \ ,$$

such that there is $\ell_k \in \big[-N_{ij}^{(1)}, N_{ij}^{(1)}\big]$, $\big|E_{ijk}^{(1)}(\tilde{x}_i, \hat{\omega}_j) - \lambda V(\tilde{x}_i + \ell_k\hat{\omega}_j)\big| \leq 2$,
$|V(\tilde{x}_i + \ell_k\hat{\omega}_j) - V(\tilde{x}_i)| < \frac{1}{2}\lambda^{-1/2}$. By perturbation theory and Lemma A.1, we can define C^3 functions $E_{ijk}^{(1)}(x,\omega)$, $\varphi_{ijk}^{(1)}(x,\omega)(\ell)$, $\ell \in \Lambda_{ij}^{(1)}$, for $(x,\omega) \in U_i \times V_j$ such that

$$\begin{aligned} H_{\Lambda_{ij}^{(1)}}(x,\omega)\varphi_{ijk}^{(1)}(x,\omega) &= E_{ijk}^{(1)}(x,\omega)\varphi_{ijk}^{(1)}(x,\omega) \\ \partial_x E_{ijk}^{(1)}(x,\omega) &= \lambda \sum_{\ell \in \Lambda_{ij}^{(1)}} V'(x+\ell\omega)\big|\varphi_{ijk}^{(1)}(x,\omega)(\ell)\big|^2 \\ \partial_\omega E_{ijk}^{(1)}(x,\omega) &= \lambda \sum_{\ell \in \Lambda_{ij}^{(1)}} \ell\ V'(x+\ell\omega)\big|\varphi_{ijk}^{(1)}(x,\omega)(\ell)\big|^2 \\ \sum_{\ell \in \Lambda_{ij}^{(1)}} \big|\varphi_{ijk}^{(1)}(x,\omega)(\ell)\big|^2 &= 1 \end{aligned}$$

Furthermore,

$$\begin{aligned} |E_{ijk}^{(1)}(x,\omega) - \lambda V(x)| &\leq |E_{ijk}^{(1)}(x,\omega) - E_{ijk}^{(1)}(\tilde{x}_i,\omega)| + |E_{ijk}^{(1)}(\tilde{x}_i,\omega) - E_{ijk}^{(1)}(\tilde{x}_i,\hat{\omega})| \\ &\quad + |E_{ijk}^{(1)}(\tilde{x}_i,\hat{\omega}) - \lambda V(x+\ell_k\hat{\omega})| + |\lambda V(x+\ell\hat{\omega}) - \lambda V(x)| \\ &\leq (\lambda C_1)|x - \tilde{x}_i| + (\lambda N^{2m_0} C_1)|\omega - \hat{\omega}| + 2 + \frac{1}{2}\lambda^{1/2} \\ &\leq \frac{3}{4}\lambda^{3/4} \end{aligned}$$

Lemma 6.5. *For any $(x,\omega) \in U_i \times V_j$, $N_{ij}^{(1)} < \ell \leq (N_{ij}^{(1)})^2$, one has*

$$\left(H_{\left[N_{ij}^{(1)}+1, \left(N_{ij}^{(1)}\right)^2\right]}(x,\omega) - E_{ijk}^{(1)}(x,\omega)\right)^{-1}\big(N_{ij}^{(1)}+1, \ell\big) \lesssim \lambda^{-\frac{1}{2}(\ell - N_{ij}^{(1)})} \ .$$

Proof.

$$\begin{aligned}|\lambda V(x+m\omega)-E^{(1)}_{ijk}(x,\omega)| &\geq |\lambda V(x+m\omega)-\lambda V(x)|-|\lambda V(x)-E^{(1)}_{ijk}(x,\omega)|\\ &\geq \frac{7}{8}\lambda^{1/2}-\big(\frac{3}{4}\lambda^{1/2}\big)\geq \frac{1}{8}\lambda^{1/2}\end{aligned}$$

for any $m\in\big[N^{(1)}_{ij}+1,(N^{(1)}_{ij})^2\big]$. Also $(N^{(1)}_{ij})^2\leq N^{2m_0}\leq\lambda^{1/2}$. The assertion follows from Corollary C.8. □

Corollary 6.6. $\big|\varphi^{(1)}_{ijk}(x,\omega)(\ell)\big|\leq\lambda^{-\frac{1}{2}(|\ell|-N^{(1)}_{ij})}$, $N^{(1)}_{ij}<|\ell|\leq(N^{(1)}_{ij})^2$.

Lemma 6.7. *For any* $(x,\omega)\in U_i\times V_j$,

$$\operatorname{sp}H_{\Lambda^{(1)}_{ij}}(x,\omega)\cap\left(\lambda V(x)-\frac{1}{4}\lambda^{1/2},\lambda V(x)+\frac{1}{4}\lambda^{1/2}\right)\subset\bigcup_k\left\{E^{(1)}_{ijk}(x,\omega)\right\}.$$

Proof. By construction, $\operatorname{sp}H_{\Lambda^{(1)}_{ij}}(\tilde{x}_i,\hat{\omega}_j)\cap\big(\lambda V(\tilde{x}_i)-\frac{1}{2}\lambda^{1/2}+2,\lambda V(\tilde{x}_i)+\frac{1}{2}\lambda^{1/2}-2\big)\subset\bigcup_k\left\{E^{(1)}_{ijk}(\tilde{x}_i,\hat{\omega}_j)\right\}$. Suppose

$$\mu(x_0,\omega_0)\in\operatorname{sp}H_{\Lambda^{(1)}_{ij}}(x_0,\omega_0)\cap\left(\lambda V(x_0,\omega_0)-\frac{1}{4}\lambda^{1/2},\lambda V(x_0,\omega_0)+\frac{1}{4}\lambda^{1/2}\right)$$

for some $(x_0,\omega_0)\in U_i\times V_j$. Then we can define C^1 function

$$\mu(x,\omega)\in\operatorname{sp}H_{\Lambda^{(1)}_{ij}}(x,\omega)\ ,\quad |\partial_x\mu(x,\omega)|\leq\lambda C_1,\ |\partial_\omega\mu(x,\omega)|\leq\lambda N^{2m_0}C_1$$

on $U_i\times V_j$. But

$$\begin{aligned}|\mu(\tilde{x}_i,\hat{\omega}_j)-\lambda V(\tilde{x}_i)| &\leq |\mu(\tilde{x}_i,\hat{\omega}_j)-\mu(x_0,\hat{\omega}_j)|+|\mu(x_0,\hat{\omega}_j)-\mu(x_0,\omega_0)|\\ &\quad+|\mu(x_0,\omega_0)-\lambda V(x_0)|+|\lambda V(x_0)-\lambda V(\tilde{x}_i)|\\ &\leq\lambda C_1|\tilde{x}_i-x_0|+\lambda N^{2m_0}C_1|\hat{\omega}_j-\omega_0|+\frac{1}{4}\lambda^{1/2}+\lambda C_1|x_0-\tilde{x}_i|\\ &\leq\frac{1}{2}\lambda^{1/2}-2\end{aligned}$$

Hence $\mu(\tilde{x}_i,\hat{\omega}_j)=E^{(1)}_{ijk}(\tilde{x}_i,\hat{\omega}_j)$ for some k. Therefore $\mu(x,\omega)=E^{(1)}_{ijk}(x,\omega)$. □

7 Separation of eigenvalues at first scale

We develop a method to find a lower estimate for the separtion between eigenvalues of $H_{\Lambda_{ij}}(x,\omega)$. This method is first introduced in [GS2] for analytic potential.

Lemma 7.1. *Let $(x,\omega) \in U_i \times V_j$, $E_1, E_2 \in \big(\lambda V(x) - \frac{3}{4}\lambda^{1/2}, \lambda V(x) + \frac{3}{4}\lambda^{1/2}\big)$. Then for any $n \in \big[-(N_i^{(1)})^2, -N_i^{(1)} - 1\big]$, one has*

$$\big|\log|f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega,E_1)| - \log|f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega,E_2)|\big| \le |E_1 - E_2| \ .$$

Proof. Let $\big\{\mu_m\big\}_{m=-(N_{ij}^{(1)})^2}^{n} = \operatorname{sp} H_{[-(N_{ij}^{(1)})^2,n]}(x,\omega)$, $|\mu_m - \lambda V(x+m\omega)| \le 2$. Then for any $E \in \big(\lambda V(x) - \frac{3}{4}\lambda^{1/2}, \lambda V(x) + \frac{3}{4}\lambda^{1/2}\big)$

$$\begin{aligned}
|\mu_m - E| &\ge |\lambda V(x+m\omega) - \lambda V(x)| - |\lambda V(x+m\omega) - \mu_m| - |\lambda V(x) - E| \\
&\ge \frac{7}{8}\lambda^{1/2} - 2 - \frac{3}{4}\lambda^{1/2} \\
&\ge \frac{1}{16}\lambda^{1/2} \ .
\end{aligned}$$

For each m, there exists $E \in (E_1, E_2) \subset \big(\lambda V(x) - \frac{1}{4}\lambda^{1/2}, \lambda V(x) + \frac{1}{4}\lambda^{1/2}\big)$ such that

$$\big|\log|\mu_m - E_1| - \log|\mu_m - E_2|\big| = \frac{|E_1 - E_2|}{|\mu_m - E|} \le 2\lambda^{-1/2}|E_1 - E_2|$$

$$\begin{aligned}
&\big|\log|f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega,E_1)| - \log|f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega,E_2)|\big| \\
&\le \sum_m \big|\log|\mu_m - E_1| - \log|\mu_m - E_2|\big| \\
&\le (N_{ij}^{(1)})^2\big[16\lambda^{-1/2}|E_1 - E_2|\big] \\
&\le \lambda^{1/8}\big[16\lambda^{-1/2}|E_1 - E_2|\big] \\
&\le |E_1 - E_2|
\end{aligned}$$

□

Corollary 7.2. *Let $(x,\omega), E_1, E_2, n$ be as in Lemma 7.1. Then*

$$\big|f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega,E_1) - f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega,E_2)\big| \le |E_1 - E_2| e^{|E_1 - E_2|}\big|f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega,E_1)\big| \ .$$

Proof. With $\big\{\mu_m\big\}$ as in proof of Lemma 7.1, note that $\operatorname{sgn}(\mu_j - E_1) = \operatorname{sgn}(\mu_j - E_2)$. So

$$\operatorname{sgn} f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega,E_1) = \operatorname{sgn} f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega,E_2) \ .$$

Hence, there is Γ between $|f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega,E_1)|$ and $|f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega,E_2)|$ such that

$$\begin{aligned}
&\big|f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega,E_1) - f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega,E_2)\big| \\
&\quad = \big||f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega,E_1)| - |f_{[-(N_{ij}^{(1)})^2-n]}(x,\omega,E_2)|\big| \\
&\quad = \big|\log|f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega,E_1)| - \log|f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega,E_2)|\big| \cdot \Gamma \\
&\quad \le |E_1 - E_2| \cdot \Gamma \ .
\end{aligned}$$

But

$$\Gamma \leq \max\left\{|f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega,E_1)|,\ |f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega,E_2)|\right\} \leq e^{|E_1-E_2|}|f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega,E_1)| \ .$$

□

Lemma 7.3. *For any* $(x,\omega) \in U_i \times V_j$, $1 \leq k \leq J_{ij}^{(1)}$,

$$\sum_{N_{ij}^{(1)}<|n|\leq(N_{ij}^{(1)})^2} \left|f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega,E_{ijk}^{(1)}(x,\omega))\right|^2 < 4\lambda^{-1} \sum_{n\in\Lambda_{ij}^{(1)}} \left|f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega,E_{ijk}^{(1)}(x,\omega))\right|^2 \ .$$

Proof. $\left\{f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega,E_{ijk}^{(1)}(x,\omega))\right\}_{n\in\Lambda_{ij}^{(1)}}$ satisfies the Schrödinger equation with zero boundary condition. Hence, there is $\mu \in \mathbb{R}$ such that

$$f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega,E_{ijk}^{(1)}(x,\omega)) = \mu\,\varphi_{ijk}^{(1)}(x,\omega)(n) \ .$$

By Corollory 6.6,

$$\begin{aligned}
\sum_{N_{ij}^{(1)}<|n|\leq(N_{ij}^{(1)})^2} \left|f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega,E_{ijk}^{(1)}(x,\omega))\right|^2 &= \mu^2 \sum_{N_{ij}^{(1)}<|n|\leq(N_{ij}^{(1)})^2} \left|\varphi_{ijk}^{(1)}(x,\omega)(n)\right|^2 \\
&\leq 2\mu^2 \sum_{n>N_{ij}^{(1)}} \lambda^{-(|n|-N_{ij}^{(1)})} \\
&\leq 2\mu^2 \frac{\lambda^{-1}}{1-\lambda^{-1}} \\
&< 4\lambda^{-1}\mu^2 \\
&= 4\lambda^{-1} \sum_{n\in\Lambda_{ij}^{(1)}} \left|f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega,E_{ijk}^{(1)}(x,\omega))\right|^2 \ .
\end{aligned}$$

□

Lemma 7.4. *For any* $(x,\omega) \in U_i \times V_j$, $\left|E_{ijk}^{(1)}(x,\omega) - E_{ij\ell}^{(1)}(x,\omega)\right| > e^{-|\Lambda_{ij}^{(1)}|^{3/4}}$ *if* $k \neq \ell$.

Proof. For any $n \in \big[-N_{ij}^{(1)}, N_{ij}^{(1)}\big]$,

$$
\begin{aligned}
&\big|f_{[-(N_{ij}^{(1)})^2,n]}\big(x,\omega,E_{ijk}^{(1)}(x)\big) - f_{[-(N_{ij}^{(1)})^2,n]}\big(x,\omega,E_{ij\ell}^{(1)}(x,\omega)\big)\big| \\
&\le \left\| \begin{pmatrix} f_{[-(N_{ij}^{(1)})^2,n]}\big(x,\omega,E_{ijk}^{(1)}(x,\omega)\big) \\ f_{[-(N_{ij}^{(1)})^2,n-1]}\big(x,\omega,E_{ijk}^{(1)}(x,\omega)\big) \end{pmatrix} - \begin{pmatrix} f_{[-(N_{ij}^{(1)})^2,n]}\big(x,\omega,E_{ij\ell}^{(1)}(x,\omega)\big) \\ f_{[-(N_{ij}^{(1)})^2,n-1]}\big(x,\omega,E_{ij\ell}^{(1)}(x,\omega)\big) \end{pmatrix} \right\| \\
&= \left\| M_{[-N_{ij}^{(1)}-1,n-1]}\big(x,\omega,E_{ijk}^{(1)}(x,\omega)\big) \begin{pmatrix} f_{[-(N_{ij}^{(1)})^2,-N_{ij}^{(1)}-1]}\big(x,\omega,E_{ijk}^{(1)}(x,\omega)\big) \\ f_{[-(N_{ij}^{(1)})^2,-N_{ij}^{(1)}-2]}\big(x,\omega,E_{ijk}^{(1)}(x,\omega)\big) \end{pmatrix} \right. \\
&\quad \left. - M_{[-N_{ij}^{(1)}-1,n-1]}\big(x,\omega,E_{ij\ell}^{(1)}(x,\omega)\big) \begin{pmatrix} f_{[-(N_{ij}^{(1)})^2,-N_{ij}^{(1)}-1]}\big(x,\omega,E_{ij\ell}^{(1)}(x,\omega)\big) \\ f_{[-(N_{ij}^{(1)})^2,-N_{ij}^{(1)}-2]}\big(x,\omega,E_{ij\ell}^{(1)}(x,\omega)\big) \end{pmatrix} \right\| \\
&\le \left\| M_{[-N_{ij}^{(1)}-1,n-1]}\big(x,\omega,E_{ijk}^{(1)}(x,\omega)\big) - M_{[-N_{ij}^{(1)}-1,n-1]}\big(x,\omega,E_{ij\ell}^{(1)}(x,\omega)\big) \right\| \cdot \\
&\quad \left\| \begin{pmatrix} f_{[-(N_{ij}^{(1)})^2,-N_{ij}^{(1)}-1]}\big(x,\omega,E_{ijk}^{(1)}(x,\omega)\big) \\ f_{[-(N_{ij}^{(1)})^2,-N_{ij}^{(1)}-2]}\big(x,\omega,E_{ijk}^{(1)}(x,\omega)\big) \end{pmatrix} \right\| + \left\| M_{[-N_{ij}^{(1)}-1,n-1]}\big(x,\omega,E_{ij\ell}^{(1)}(x,\omega)\big) \right\| \cdot \\
&\quad \left\| \begin{pmatrix} f_{[-(N_{ij}^{(1)})^2,-N_{ij}^{(1)}-1]}\big(x,\omega,E_{ijk}^{(1)}(x,\omega)\big) \\ f_{[-(N_{ij}^{(1)})^2,-N_{ij}^{(1)}-2]}\big(x,\omega,E_{ijk}^{(1)}(x,\omega)\big) \end{pmatrix} - \begin{pmatrix} f_{[-(N_{ij}^{(1)})^2,-N_{ij}^{(1)}-1]}\big(x,\omega,E_{ij\ell}^{(1)}(x,\omega)\big) \\ f_{[-(N_{ij}^{(1)})^2,-N_{ij}^{(1)}-2]}\big(x,\omega,E_{ij\ell}^{(1)}(x,\omega)\big) \end{pmatrix} \right\| \\
&\le \Big|E_{ijk}^{(1)}(x,\omega) - E_{ij\ell}^{(1)}(x,\omega)\Big| (\lambda C_2)^{n+N_{ij}^{(1)}} |\mu| \\
&\quad + (\lambda C_2)^{n+N_{ij}^{(1)}+1} \Big(\big|E_{ijk}^{(1)}(x,\omega) - E_{ij\ell}^{(1)}(x,\omega)\big| e^{|E_{ijk}^{(1)}(x,\omega) - E_{ij\ell}^{(1)}(x,\omega)|} |\mu| \Big)
\end{aligned}
$$

where $C_2 = \log\big(2 \max_{y\in\mathbb{T}} |V(y)| + 1\big)$, $|\mu|^2 = \sum_{m\in\Lambda_{ij}^{(1)}} |f_{[-(N_{ij}^{(1)})^2,n]}(x,\omega,E_{ijk}^{(1)}(x,\omega))|^2$. ($\mu$ is the same as in the proof of Lemma 7.3)

If $\big|E_{ijk}^{(1)}(x,\omega) - E_{ij\ell}^{(1)}(x,\omega)\big| \le e^{-|\Lambda_{ij}^{(1)}|^{3/4}} = e^{-(N_{ij}^{(1)})^{3/2}}$ then

$$
\big|f_{[-(N_{ij}^{(1)})^2,n]}\big(x,\omega,E_{ijk}^{(1)}(x,\omega)\big) - f_{[-(N_{ij}^{(1)})^2,n]}\big(x,\omega,E_{ij\ell}^{(1)}(x,\omega)\big)\big| \le e^{-\frac{1}{2}|\Lambda_{ij}^{(1)}|^{-3/4}} |\mu|
$$

for all $n \in [-N_{ij}^{(1)}, N_{ij}^{(1)}]$.

Therefore,

$$\begin{aligned}
&\sum_{n\in\Lambda_{ij}^{(1)}} \Big| f_{[-(N_{ij}^{(1)})^2,n]}\big(x,\omega,E_{ijk}^{(1)}(x,\omega)\big) - f_{[-(N_{ij}^{(1)})^2,n]}\big(x,\omega,E_{ij\ell}^{(1)}(x,\omega)\big)\Big|^2 \\
&= \sum_{N_{ij}^{(1)}<|n|\le(N_{ij}^{(1)})^2} \Big| f_{[-(N_{ij}^{(1)})^2,n]}\big(x,\omega,E_{ijk}^{(1)}(x,\omega)\big) - f_{[-(N_{ij}^{(1)})^2,n]}\big(x,\omega,E_{ij\ell}^{(1)}(x,\omega)\big)\Big|^2 \\
&\quad + \sum_{|n|\le N_{ij}^{(1)}} \Big| f_{[-(N_{ij}^{(1)})^2,n]}\big(x,\omega,E_{ijk}^{(1)}(x,\omega)\big) - f_{[-(N_{ij}^{(1)})^2,n]}\big(x,\omega,E_{ij\ell}^{(1)}(x,\omega)\big)\Big|^2 \\
&\le \sum_{N_{ij}^{(1)}<|n|\le(N_{ij}^{(1)})^2} \Big[\big| f_{[-(N_{ij}^{(1)})^2,n]}\big(x,\omega,E_{ijk}^{(1)}(x,\omega)\big)\big|^2 + \big| f_{[-(N_{ij}^{(1)})^2,n]}\big(x,\omega,E_{ij\ell}^{(1)}(x,\omega)\big)\big|^2 \Big] \\
&\quad + \big(2N_{ij}^{(1)}+1\big)e^{-|\Lambda_{ij}^{(1)}|^{3/4}}|\mu|^2 \\
&\le 4\lambda^{-1} \sum_{n\in\Lambda_{ij}^{(1)}} \Big[\big| f_{[-(N_{ij}^{(1)})^2,n]}\big(x,\omega,E_{ijk}^{(1)}(x,\omega)\big)\big|^2 + \big| f_{[-(N_{ij}^{(1)})^2,n]}\big(x,\omega,E_{ij\ell}^{(1)}(x,\omega)\big)\big|^2 \Big] \\
&\quad + \big(2N_{ij}^{(1)}+1\big)e^{-|\Lambda_{ij}^{(1)}|^{3/4}}|\mu|^2 \\
&\le \lambda^{-1/2} \sum_{n\in\Lambda_{ij}^{(1)}} \Big[\big| f_{[-(N_{ij}^{(1)})^2,n]}\big(x,\omega,E_{ijk}^{(1)}(x,\omega)\big)\big|^2 + \big| f_{[-(N_{ij}^{(1)})^2,n]}\big(x,\omega,E_{ij\ell}^{(1)}(x,\omega)\big)\big|^2 \Big]
\end{aligned}$$

But $E_{ijk}^{(1)}(x,\omega) \neq E_{ij\ell}^{(1)}(x,\omega) \Longrightarrow \Big\{ f_{[-(N_{ij}^{(1)})^2,n]}\big(x,\omega,E_{ijk}^{(1)}(x,\omega)\big)\Big\}_{n\in\Lambda_{ij}^{(1)}}$ and $\Big\{ f_{[-(N_{ij}^{(1)})^2,n]}\big(x,w,E_{ij\ell}^{(1)}(x,\omega)\big)\Big\}_{n\in\Lambda_{ij}^{(1)}}$ are orthogonal. Hence

$$\begin{aligned}
&\sum_{n\in\Lambda_{ij}^{(1)}} \big| f_{[-(N_{ij}^{(1)})^2,n]}\big(x,\omega,E_{ijk}^{(1)}(x,\omega)\big) - f_{[-(N_{ij}^{(1)})^2,n]}\big(x,\omega,E_{ij\ell}^{(1)}(x,\omega)\big)\big|^2 \\
&= \sum_{n\in\Lambda_{ij}^{(1)}} \Big[\big| f_{[-(N_{ij}^{(1)})^2,n]}\big(x,\omega,E_{ijk}^{(1)}(x,\omega)\big)\big|^2 + \big| f_{[-(N_{ij}^{(1)})^2,n]}\big(x,\omega,E_{ij\ell}^{(1)}(x,\omega)\big)\big|^2 \Big] .
\end{aligned}$$

This contradiction shows that

$$\big|E_{ijk}^{(1)}(x,\omega) - E_{ij\ell}^{(1)}(x,\omega)\big| > e^{-|\Lambda_{ij}^{(1)}|^{3/4}} .$$

□

8 Variations of potential

We introduce variation of potential and the notion of "typical" variations. We then derive some properties of $E_{ijk}^{(1)}$ for typical variations.

Definition. Let T be a large integer, $0 < \delta \ll \frac{1}{T^5}$. Suppose $R_m(\eta_m, \xi_m, \theta_m; x)$ are C^3 functions, $m = 1, 2, \ldots, T$, $(\eta, \xi, \theta) \in \prod_1^{3T}[-\delta, \delta]$, $x \in \mathbb{T}$, satisfying the following conditions:

$$|\partial_\alpha R_m(\eta_m, \xi_m, \theta_m; x)| \lesssim \frac{1}{T} \qquad \text{for any index } |\alpha| \le 3 \tag{8.1}$$

$$R_m(0, 0, 0; x) \equiv 0 \tag{8.2}$$

$$R_m(\eta_m, \xi_m, \theta_m; x) = -x^{-3}\big(\eta_m + \xi_m x + \frac{1}{2}\theta_m x^2\big) \qquad \text{for } |x| \ge \frac{1}{2T} \tag{8.3}$$

Define a (T, δ)–variation of potential by

$$W(\eta, \xi, \theta, \{R_m\}; x) = \sum_{m=1}^{T} v_m\Big(\eta_m, \xi_m, \theta_m; x - \frac{m}{T}\Big)$$

where

$$v_m(\eta_m, \xi_m, \theta_m; x) = \eta_m + \xi_m x + \frac{1}{2}\theta_m x^2 + x^3 R_m(\eta_m, \xi_m, \theta_m; x)$$

By (8.2) and (8.3),

$$v_m(0, 0, 0, \{R_m\}; x) \equiv 0$$

and

$$v_m(\eta_m, \xi_m, \theta_m, \{R_m\}; x) = 0 \qquad \text{for } |x| \ge \frac{1}{2T}\ .$$

Denote the collection of (T, δ)–variations of potential by $\mathcal{S}(T, \delta)$. A set $S \subset \mathcal{S}(T, \delta)$ is called $(1 - \varepsilon)$-typical if

$$|S| := \min_{\{R_m\}} \frac{1}{(2\delta)^{3T}} \operatorname{mes}\big\{(\eta, \xi, \theta) \in [-\delta, \delta]^{3T} : v(\eta, \xi, \theta, \{R_m\}; .) \in S\big\} \ge 1 - \varepsilon$$

Remark 8.1. We assume $\delta \ll \lambda^{-1/2}$. Since $\max_{y \in \mathbb{T}} \big|\widetilde{V}(y) - V(y)\big| \lesssim \delta$, one has

$$\big|\widetilde{V}(x + j\omega) - \widetilde{V}(x)\big| \ge \frac{7}{8}\lambda^{-1/2} \qquad \text{for } j \in \Lambda_{ij}^{(1)} \setminus \big[-N_{ij}^{(1)}, N_{ij}^{(1)}\big]$$

for any $(x, \omega) \in U_i \times V_j$. Hence, we can define $\widetilde{E}_{ijk}^{(1)}(x, \omega)$ and $\tilde{\varphi}_{ijk}^{(1)}(x, \omega)$ on $U_i \times V_j$, such that

$$\widetilde{H}_{\Lambda_{ij}^{(1)}}(x, \omega)\tilde{\varphi}_{ijk}^{(1)}(x, \omega) = \widetilde{E}_{ijk}^{(1)}(x, \omega)\tilde{\varphi}_{ijk}^{(1)}(x, \omega)\ , \quad \|\tilde{\varphi}_{ijk}^{(1)}(x, \omega)\| = 1$$

$\widetilde{E}_{ijk}^{(1)}(x, \omega)$ and $\tilde{\varphi}_{ijk}^{(1)}(x, \omega)$ have the same properties of $E_{ijk}^{(1)}(x, \omega)$ and $\varphi_{ijk}^{(1)}(x, \omega)$, shown in Section 6 and Section 7.

Remark 8.2. These functions depend on $\{\eta_m\}, \{\xi_m\}, \{\theta_m\}$. It is for the simplicity of notations that the dependence is not explicitly written.

For the remaining of this section, we fix $(\tilde{x}, \tilde{\omega}) \in U_i \times V_j$, $\tilde{\omega}$ satisfies condition (6.1).

Remark 8.3. We can define an one-to-one correspondence between $\Lambda_{ij}^{(1)}$ and a subset of $[1, T]$ as follow: For any $\ell \in \Lambda_{ij}^{(1)}$, there is unique $\ell' \in [1, T] \cap \mathbb{N}$ such that $-\frac{1}{2T} < \{\tilde{x} + \ell\tilde{\omega}\} - \frac{\ell'}{T} \leq \frac{1}{2T}$. If $m, \ell \in \Lambda_{ij}^{(1)}$, $m \neq \ell$, then the Diophantine condition implies $m' \neq \ell'$.

Remark 8.4. If $p \neq q'$ for all $q \in \Lambda_{ij}^{(1)}$, then $\partial_{\xi_p} \widetilde{H}_{\Lambda_{ij}^{(1)}}(\tilde{x}, \tilde{\omega}) = 0$. Let $\varrho = \frac{1}{2} \operatorname{dist}\big(\widetilde{E}_{ijk}^{(1)}(\tilde{x}, \tilde{\omega}), \operatorname{sp} \widetilde{H}_{\Lambda_{ij}^{(1)}}(\tilde{x}, \tilde{\omega}) \setminus \big\{\widetilde{E}_{ijk}^{(1)}(\tilde{x}, \tilde{\omega})\big\}\big) > \frac{1}{2} e^{-|\Lambda_{ij}^{(1)}|^{3/4}}$.

Lemma 8.5.

$$\big\|\partial_{\xi_p} \tilde{\varphi}_{ijk}^{(1)}(\tilde{x}, \tilde{\omega})\big\| \leq \begin{cases} \frac{\lambda}{\varrho T} & \text{if } \exists\ q \in \Lambda_{ij}^{(1)} \text{ such that } q' = p \\ 0 & \text{otherwise} \end{cases}$$

$$\big\|\partial_{\theta_p} \tilde{\varphi}_{ijk}^{(1)}(\tilde{x}, \tilde{\omega})\big\| \leq \begin{cases} \frac{\lambda}{\varrho T^2} & \text{if } \exists\ q \in \Lambda_{ij}^{(1)} \text{ such that } q' = p \\ 0 & \text{otherwise} \end{cases}$$

$$\big\|\partial_{\theta_p} \partial_x \tilde{\varphi}_{ijk}^{(1)}(\tilde{x}, \tilde{\omega})\big\| \leq \begin{cases} \frac{\lambda}{\varrho T} & \text{if } \exists\ q \in \Lambda_{ij}^{(1)} \text{ such that } q' = p \\ 0 & \text{otherwise} \end{cases}$$

Proof. By Lemma A.2,

$$\begin{aligned} \big\|\partial_{\xi_p} \tilde{\varphi}_{ijk}^{(1)}(\tilde{x}, \tilde{\omega})\big\| &\leq \Big\| \frac{-1}{2\pi i} \oint_{|z - \widetilde{E}_{ijk}^{(1)}(\tilde{x}, \tilde{\omega})| = \varrho} \big[z - \widetilde{H}_{\Lambda_{ij}^{(1)}}(\tilde{x}, \tilde{\omega})\big]^{-1} \partial_{\xi_p} \widetilde{H}_{\Lambda_{ij}^{(1)}}(\tilde{x}, \tilde{\omega}) \big[z - \widetilde{H}_{\Lambda_{ij}^{(1)}}(\tilde{x}, \tilde{\omega})\big]^{-1} dz \Big\| \\ &\leq \begin{cases} \frac{1}{2\pi}(2\pi\varrho)\frac{\lambda/T}{\varrho^2} = \frac{\lambda}{\varrho T} & \text{if } \exists\ q \in \Lambda_{ij}^{(1)} \text{ such that } q' = p \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

The proofs for the other two inequalities are similar. □

Fix $\eta = (\eta_1, \ldots, \eta_T) \subset (-\delta, \delta)^{|\Lambda_{ij}^{(1)}|}$.

Proposition 8.6. *For* $\lambda^2 \frac{|\Lambda_{ij}^{(1)}|^2 \delta}{\varrho\ T} \ll \varepsilon \ll 1$, *one has*

$$\operatorname{mes}\Big\{(\xi, \theta) \in (-\delta, \delta)^{2|\Lambda_{ij}^{(1)}|} : \big|\partial_x \widetilde{E}_{ijk}^{(1)}(\tilde{x}, \tilde{\omega})\big| \leq \varepsilon, \big|\partial_{xx} \widetilde{E}_{ijk}^{(1)}(\tilde{x}, \tilde{\omega})\big| \leq \varepsilon\Big\} \leq \Big(|\Lambda_{ij}^{(1)}| \frac{\varepsilon}{\lambda\delta}\Big)^2$$

Proof. Let $\tilde\varphi_{00}(x,\omega)=\tilde\varphi^{(1)}_{ijk}(x,\omega)\Big|_{(\xi,\theta)=(0,0)}$, $\widetilde E_{00}(x,\omega)=\widetilde E^{(1)}_{ijk}(x,\omega)\Big|_{(\xi,\theta)=(0,0)}$

$$\begin{aligned}\lambda^{-1}\partial_x\widetilde E^{(1)}_{ijk}(\tilde x,\tilde\omega)&=\sum_{\ell\in\Lambda^{(1)}_{ij}}\partial_x\widetilde V(\tilde x+\ell\tilde\omega)\big|\tilde\varphi^{(1)}_{ijk}(\tilde x,\tilde\omega)(\ell)\big|^2\\&=\sum_{\ell\in\Lambda^{(1)}_{ij}}\Big[V'(\tilde x+\ell\tilde\omega)+\xi_{\ell'}+O\big(\frac{\delta}{T}\big)\Big]\Big[\big|\tilde\varphi_{00}(\tilde x,\tilde\omega)(\ell)\big|^2+O\big(\frac{|\Lambda^{(1)}_{ij}|\delta\lambda}{\varrho\,T}\big)\Big]\\&=\lambda^{-1}\partial_x\widetilde E_{00}(\tilde x,\tilde\omega)+\sum_{\ell\in\Lambda^{(1)}_{ij}}\xi_{\ell'}\big|\tilde\varphi_{00}(\tilde x,\tilde\omega)(\ell)\big|^2+O\big(\frac{|\Lambda^{(1)}_{ij}|^2\delta\lambda}{\varrho\,T}\big)\end{aligned}$$

By Lemma F.1,

$$\begin{aligned}&(2\delta)^{-T}\operatorname{mes}\left\{\xi:\Big|\lambda^{-1}\partial_x\widetilde E_{00}(\tilde x,\tilde\omega)+\sum_{\ell\in\Lambda^{(1)}_{ij}}\xi_{\ell'}\big|\tilde\varphi_{00}(\tilde x,\tilde\omega)(\ell)\big|^2\Big|\le\lambda^{-1}\varepsilon\right\}\\&=(2\delta)^{-|\Lambda^{(1)}_{ij}|}\operatorname{mes}\left\{\{\xi_{m'}\}_{m\in\Lambda^{(1)}_{ij}}:\big|\lambda^{-1}\partial_x\widetilde E_{00}(\tilde x,\tilde\omega)+\sum_{\ell\in\Lambda^{(1)}_{ij}}\xi_{\ell'}\big|\tilde\varphi_{00}(\tilde x,\tilde\omega)(\ell)\big|^2\le\lambda^{-1}\varepsilon\right\}\\&\le|\Lambda^{(1)}_{ij}|\frac{\lambda^{-1}\varepsilon}{\delta}\ .\end{aligned}$$

This expression does not depend on θ. So

$$\Big\{(\xi,\theta):\big|\partial_x\widetilde E^{(1)}_{ijk}(\tilde x,\tilde\omega)\big|\le\varepsilon\Big\}\subset\Xi\times(-\delta,\delta)^{|\Lambda^{(1)}_{ij}|}$$

where $(2\delta)^{-T}\operatorname{mes}\Xi\le|\Lambda^{(1)}_{ij}|\frac{\lambda^{-1}\varepsilon}{\delta}$.

Fix $\xi_0 \in \Xi$. Let $\tilde{\varphi}_0(x,\omega) = \tilde{\varphi}^{(1)}_{ijk}(x,\omega)\Big|_{\theta=0}$, $\widetilde{E}_0(x,\omega) = \widetilde{E}^{(1)}_{ijk}(x,\omega)\Big|_{\theta=0}$

$$\begin{aligned}
\lambda^{-1}\partial_{xx}\widetilde{E}^{(1)}_{ijk}(\tilde{x},\tilde{\omega}) &= \sum_{\ell\in\Lambda^{(1)}_{ij}} \partial_{xx}\widetilde{V}(\tilde{x}+\ell\tilde{\omega})\big|\tilde{\varphi}^{(1)}_{ijk}(\tilde{x},\tilde{\omega})(\ell)\big|^2 \\
&\quad + 2\sum_{\ell\in\Lambda^{(1)}_{ij}} \partial_x\widetilde{V}(\tilde{x}+\ell\tilde{\omega})\tilde{\varphi}^{(1)}_{ijk}(\tilde{x},\tilde{\omega})(\ell)\partial_x\tilde{\varphi}_{ijk}(\tilde{x},\tilde{\omega})(\ell) \\
&= \sum_{\ell\in\Lambda^{(1)}_{ij}} \Big[V''(\tilde{x}+\ell\tilde{\omega}) + \theta_{\ell'} + O\big(\frac{1}{T}\big)\Big]\Big[\big|\tilde{\varphi}_0(\tilde{x},\tilde{\omega})(\ell)\big|^2 + O\big(\frac{|\Lambda^{(1)}_{ij}|\delta\lambda}{\varrho T^2}\big)\Big] \\
&\quad + 2\sum_{\ell\in\Lambda^{(1)}_{ij}} \Big[V'(\tilde{x}+\ell\tilde{\omega}) + \xi_{\ell'} + O\big(\frac{\delta}{T}\big)\Big]\Big[\tilde{\varphi}_0(\tilde{x},\tilde{\omega})(\ell) + O\big(\frac{|\Lambda^{(1)}_{ij}|\delta\lambda}{\varrho T^2}\big)\Big] \\
&\qquad \cdot \Big[\partial_x\tilde{\varphi}_0(\tilde{x},\tilde{\omega})(\ell) + O\big(\frac{|\Lambda^{(1)}_{ij}|\delta\lambda}{\varrho T}\big)\Big] \\
&= \lambda^{-1}\partial_{xx}\widetilde{E}_0(\tilde{x},\tilde{\omega}) + \sum_{\ell\in\Lambda^{(1)}_{ij}} \theta_{\ell'}\big|\tilde{\varphi}_0(\tilde{x},\tilde{\omega})(\ell)\big|^2 + O\big(\frac{|\Lambda^{(1)}_{ij}|^2\delta\lambda}{\varrho T}\big)
\end{aligned}$$

Similar calculation shows that

$$(2\delta)^{-T}\,\mathrm{mes}\Big\{\theta : \big|\partial_{xx}\widetilde{E}^{(1)}_{ijk}(\tilde{x},\tilde{\omega})\big| \le \varepsilon \quad \text{for fix } \xi_0 \in \Xi\Big\} \le |\Lambda^{(1)}_{ij}|\frac{\lambda^{-1}\varepsilon}{\delta}$$

□

Let $V_j = \bigcup_\ell V^{(2)}_{j\ell}$, where $V^{(2)}_{j\ell} = (\underline{\omega}^{(2)}_{j\ell}, \bar{\omega}^{(2)}_{j\ell})$, $\bar{\omega}^{(2)}_{j\ell} - \underline{\omega}^{(2)}_{j\ell} \approx exp(-N^A)$.

Corollary 8.7. *For each ℓ,*

$$(2\delta)^{-3T}\,\mathrm{mes}\Big\{(\eta,\xi,\theta) : \min_{(x,\omega)\in U_i\times V^{(2)}_{j\ell}}\Big[\big|\partial_x\widetilde{E}^{(1)}_{ijk}(x,\omega)\big| + \big|\partial_{xx}\widetilde{E}^{(1)}_{ijk}(x,\omega)\big|\Big] \le \varepsilon \text{ for some } i,k\Big\} \lesssim m_0\frac{|\Lambda_{ij}|^2}{\varrho}\frac{\varepsilon}{\delta^2}$$

provided $\varepsilon > exp(-N^A/2)$.

Proof. We have

$$\begin{aligned}
\big|\partial_{xx}\widetilde{E}^{(1)}_{ijk}(x,\omega)\big| &\lesssim \frac{\lambda}{\varrho}, \qquad \big|\partial_{xxx}\widetilde{E}^{(1)}_{ijk}(x,\omega)\big| \lesssim \frac{\lambda}{\varrho^2} \\
\big|\partial_{x\omega}\widetilde{E}^{(1)}_{ijk}(x,\omega)\big| &\lesssim |\Lambda^{(1)}_{ij}|\frac{\lambda}{\varrho}, \qquad \big|\partial_{xx\omega}\widetilde{E}^{(1)}_{ijk}(x,\omega)\big| \lesssim |\Lambda^{(1)}_{ij}|\frac{\lambda}{\varrho^2}
\end{aligned}$$

Let $K \asymp \big(\frac{\varrho\varepsilon}{\lambda|\Lambda^{(1)}_{ij}|}\big)^{-1} \in \mathbb{N}$, $x_m = \frac{m}{K}$. Also, we can pick $\tilde{\omega} \in V^{(2)}_{j\ell}$ satisfying condition (6.1).

If $\left|\partial_x \widetilde{E}^{(1)}_{ijk}(x_m,\tilde{\omega})\right| > 2\varepsilon$ then $\left|\partial_x \widetilde{E}^{(1)}_{ijk}(x,\omega)\right| > \varepsilon$ for all $(x,\omega) \in \left[\left(x_m - \frac{1}{2T},\ x_m + \frac{1}{2T}\right) \cap U_i\right] \times V^{(2)}_{j\ell}$.
If $\left|\partial_{xx} \widetilde{E}^{(1)}_{ijk}(x_m,\tilde{\omega})\right| > 2\varepsilon$ then $\left|\partial_{xx} \widetilde{E}^{(1)}_{ijk}(x,\omega)\right| > \varepsilon$ for all $(x,\omega) \in \left[\left(x_m - \frac{1}{2T},\ x_m + \frac{1}{2T}\right) \cap U_i\right] \times V^{(2)}_{j\ell}$.

The assertion follows from Proposition 8.6. □

9 Eigenvalues and eigenfunctions at second scale

Using the properites of $E^{(1)}_{ijk}$ derived in Section 8, for typical variations, we define eigenvalues and eigenfunctions at second scale. We also get an upper estimate of the number of eigenvalues in an interval.

From Sections 6–8, we have the following

Proposition 9.1. *There are* $\{U_i\}, \{V^{(2)}_{j\ell}\}$, $\mathbb{T}\times\mathbb{T} = \bigcup\limits_{i,j,\ell} U_i \times V^{(2)}_{j\ell}$, $\Lambda^{(1)}_{ij} = \left[-\left(N^{(1)}_{ij}\right)^2, \left(N^{(1)}_{ij}\right)^2\right]$, $\frac{\log N^{(1)}_{ij}}{\log\lambda} \asymp 1$. *Furthermore, for any* j,ℓ, *and for typical*

$$\widetilde{V}^{(1)}(x) = V(x) + W^{(1)}(\eta,\xi,\theta,\{R_m\};x)\Big) \ ,$$

one has C^3 *functions* $\widetilde{E}^{(1)}_{ijk}(x,\omega)$, $\tilde{\varphi}^{(1)}_{ijk}(x,\omega)(n)$ *defined on* $U_i \times V_j$, $k = 1,2,\ldots,J^{(1)}_{ij}$, $J^{(1)}_{ij} \le m_0$, *such that*

(1) $\widetilde{H}_{\Lambda^{(1)}_{ij}}(x,\omega)\tilde{\varphi}^{(1)}_{ijk}(x,\omega) = \widetilde{E}^{(1)}_{ijk}(x,\omega)\tilde{\varphi}^{(1)}_{ijk}(x,\omega)$

(2) $\left|\tilde{\varphi}^{(1)}_{ijk}(x,\omega)(n)\right| \le \lambda^{-\frac{1}{2}(|n|-N^{(1)}_{ij})}$ *if* $N^{(1)}_{ij} < |n| \le \left(N^{(1)}_{ij}\right)^2$

(3) $\left|\widetilde{E}^{(1)}_{ijk}(x,\omega) - \widetilde{E}^{(1)}_{ij\ell}(x,\omega)\right| > e^{-|\Lambda^{(1)}_{ij}|^{3/4}}$ *if* $k \ne \ell$

(4) $\left|\partial_x \widetilde{E}^{(1)}_{ijk}(x,\omega)\right| + \left|\partial_{xx} \widetilde{E}^{(1)}_{ijk}(x,\omega)\right| \ge e^{-\lambda^A/2}$ *for all* $(x,w) \in U_i \times V^{(2)}_{j\ell}$

Fix j. For simplicity of notations, we will simply write $V^{(2)}_\ell$ for $V^{(2)}_{j\ell}$. Also, we write $E_{ijk}(x,\omega)$ for $\widetilde{E}_{ijk}(x,\omega)$ and $\varphi_{ijk}(x,\omega)$ for $\tilde{\varphi}_{ijk}(x,\omega)$. Let $N^{(2)} \asymp e^{N^\alpha}$.

Next, we use results from Appendix E to eliminate a set of frequencies to avoid the situation where the distance between $\operatorname{sp} H_{\Lambda^{(1)}ij}(x,\omega)$ and $\operatorname{sp} H_{\Lambda^{(1)}_{pj}}(x+n\omega,\omega)$ is small. After that, we will be able to apply the Avalanche Principle.

Lemma 9.2. *There is* $\Omega^{(2)} \subset \mathbb{T}$, $\operatorname{mes}\mathbb{T}\setminus\Omega^{(2)} \le e^{\frac{-1}{2}N^\beta}$ *such that for each* $U_i \times V^{(2)}_\ell$, *there is* $N^{(2)}_{i\ell}$, $\frac{\log N^{(2)}_{i\ell}}{\log N^{(2)}} \asymp 1$, *and for any* $x \in U_i$, $x+n\omega \in U_p$, $\omega \in V^{(2)}_\ell \cap \Omega^{(2)}$, *one has*
$\operatorname{dist}\left(E, \operatorname{sp} H_{\Lambda^{(1)}_{pj}}(x+n\omega,\omega)\right) > \frac{1}{2}e^{-(N^{(1)})^\beta} =: \frac{\varepsilon}{2}$ *whenever* $\left|E - E^{(1)}_{ijk}(x,\omega)\right| < \frac{\varepsilon}{2}$.

Proof. If $\left|E^{(1)}_{ijk}(x,\omega)-E^{(1)}_{pjr}(x+n\omega,\omega)\right| > \varepsilon$ for all $N^{(2)}_{ij} < |n| \le \left(N^{(2)}_{ij}\right)^2 p, r$, $N^{(2)} < |n| \le \left(N^{(2)}\right)^2$, $(x,w) \in U_i \times V^{(2)}_\ell$ then choose $N^{(2)}_{i\ell} = N^{(2)}$. Otherwise, let

$$F_1(x,\omega) = E^{(1)}_{ijk}(x,\omega) - E^{(1)}_{pjr}(x+n_1\omega,\omega)$$

$$F_2(x,\omega) = E^{(1)}_{pjr}(x+n_1\omega,\omega) - E^{(1)}_{ajc}(x+n_2\omega,\omega)$$

where $N^{(2)} \le |n_1| \le \left(N^{(2)}\right)^2$, $\left(N^{(2)}\right)^4 \le |n_2| \le \left(N^{(2)}\right)^8$.

Since $\left|\partial_x E^{(1)}(x,\omega)\right| + \left|\partial_{xx} E^{(1)}(x,\omega)\right| \ge e^{-N^A/2}$ by Theorem E.6,

$$\operatorname{mes}\left\{\omega \in V^{(2)}_\ell : \begin{array}{l} \exists x \in U_i,\ x+n_1\omega \in U_k \text{ such that} \\ \left|F_1(x,w)\right| \le \varepsilon/2\ ,\ \left|F_2(x+n_1\omega,\omega)\right| \le \varepsilon/2 \end{array}\right\} \le \varepsilon^\vartheta |U_i|\,|V^{(2)}_\ell|$$

$$\begin{aligned} \operatorname{mes}(\mathbb{T}\setminus\Omega^{(2)}) &\le \sum_{i,j,\ell} \varepsilon^\vartheta |U_i|\,|V^{(2)}_\ell| \\ &\le \lambda\varepsilon^\vartheta \le \varepsilon^{\vartheta/2}\ . \end{aligned}$$

□

Denote $\Lambda^{(2)}_{ij} = \left[-N^{(2)}_{ij}, N^{(2)}_{ij}\right]$.

Lemma 9.3. $\log\left|f_{[a,b]}(x+n\omega,\omega,E)\right| > \frac{1}{4}(b+a)\log\lambda$ *if* $x \in U_i$, $\omega \in V_j \cap \Omega^{(2)}$, $N^{(2)}_{ij} < |n| \le \left(N^{(2)}_{ij}\right)^2$, $\operatorname{dist}\left(E, \operatorname{sp} H_{\Lambda^{(1)}_{ij}}(x,\omega)\right) < \frac{1}{2}e^{-N^\beta}$, $a, b \asymp N^{(1)}$.

Proof. Let $\tilde{x} = x + n\omega$. By Lemma C.9, $\operatorname{dist}\left(E, \operatorname{sp} H_{[a,b]}(x,w)\right) < \frac{1}{2}e^{-(N^{(1)})}$

$$\#\left\{\ell \in [a,b] : |V(\tilde{x}+\ell\omega) - V(x)| < \lambda^{-\frac{1}{2}}\right\} \le m_0$$

$$\log\left|f_{[a,b]}(x+n\omega,\omega,E)\right| \ge \left((b+a)-m_0\right)\log\lambda^{\frac{1}{2}} + m_0\log\left(\frac{1}{2}e^{-N}\right)$$

□

Lemma 9.4. *Let E as in Lemma 9.3, $-\left(N^{(2)}_{ij}\right)^2 \le c \le d < N^{(2)}_{ij}$, $d-c > 10N$ then we can choose $c = a_0 < a_1 < \cdots < a_K = d$, $2N^{(2)} < a_k - a_{k-1} < 5N^{(2)}$,*

$$A_k = M_{[a_{k-1},a_k]}(x,\omega,E) \quad k = 2,\dots,\kappa-1$$

$$A_1 = M_{[a_0,a_1]}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$a_K = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} M_{[a_{K-1},a_K]}$$

such that

$$\log\left|f_{[c,d]}(x,\omega,E)\right| = \sum_{k=1}^{K} \log\left\|A_k A_{k-1}\right\| - \sum_{k=2}^{K-1} \log\|A_k\| + O\left(-N^{(2)}\lambda^{-N}\right)\ .$$

Proof. The hypothesis of Avalanche principle is verified by Lemmas 9.3 and C.10. □

Corollary 9.5. $\Big|\big[H_{[N_{ij}^{(2)}+1,(N_{ij}^{(2)})^2]}(x,\omega)-E\big]^{-1}(k,\ell)\Big|\le\lambda^{-\frac{1}{4}|k-\ell|}$ *for* $|k-\ell|>10N^{(1)}$, $\operatorname{dist}\big(E,\operatorname{sp}H_{\Lambda_{ij}^{(1)}}(x,\omega)\big)<\frac{1}{2}e^{-(N_{ij}^{(1)})}$.

Similar to Section 7, we need now to show that the spectrum of $H_{\Lambda_{ij}^{(2)}}(x,\omega)$ near $E_{ijk}^{(1)}(x,\omega)$ is separated.

Lemma 9.6. *If* $E_1,E_2\in\left(E_{ijk}^{(1)}(x,\omega)-\frac{1}{2}e^{-\left(N^{(1)}\right)^\beta},E_{ijk}^{(1)}(x,\omega)+\frac{1}{2}e^{-\left(N^{(1)}\right)^\beta}\right)$ *then for* $-\big(N_{ij}^{(2)}\big)^2\le a<b<-N_{ij}^{(2)}$, $|\Lambda_{pq}^{(1)}|<b-a\le 5|\Lambda_{pq}^{(1)}|$, *one has*

$$\Big|\log\big|f_{[a,b]}(x,\omega,E_1)\big|-\log\big|f_{[a,b]}(x,\omega,E_2)\big|\Big|\lesssim m_0e^{\left(N^{(1)}\right)^\beta}|E_1-E_2|\ .$$

Proof. Let $J=\big\{j\in[a,b],|V(x+j\omega)-V(x)|<\lambda^{-1/2}\big\}$. Then $\#J\le m_0$.
Let $\{\mu_j\}_{j=a}^b=\operatorname{sp}H_{[a,b]}(x,\omega)$, $|\mu_j-\lambda V(x+j\omega)|\le 2$

$$\begin{aligned}&\Big|\log\big|f_{[a,b]}(x,\omega,E_1)\big|-\log\big|f_{[a,b]}(x,\omega,E_2)\big|\Big|\\&\le\sum_{j\in J}\big|\log|\mu_j-E_1|-\log|\mu_j-E_2|\big|+\sum_{j\notin J}\big|\log|\mu_j-E_1|-\log|\mu_j-E_2|\big|\\&\le m_0\Big(\frac{1}{2}e^{-\left(N^{(1)}\right)^\beta}\Big)^{-1}|E_1-E_2|+(b-a)\big(\frac{1}{2}\lambda^{1/2}\big)^{-1}|E_1-E_2|\ .\end{aligned}$$

□

Corollary 9.7. E_1,E_2 *as in Lemma 9.6, then*

$$\begin{aligned}&\Big|\log\big|f_{[-(N_{ij}^{(2)})^2,n]}(x,\omega,E_1)\big|-\log\big|f_{[-(N_{ij}^{(2)})^2,n]}(x,\omega,E_2)\big|\Big|\\&\lesssim\frac{\big(N_{ij}^{(2)}\big)^2}{N^{(1)}}m_0e^{(N^{(1)})^\beta}|E_1-E_2|\end{aligned}$$

Corollary 9.8. *If* $E_1,E_2\in\Big(E_{ijk}^{(1)}(x,\omega)-\frac{1}{2}e^{-N^\beta},E_{ijk}^{(1)}(x,\omega)+\frac{1}{2}e^{-N^\beta}\Big)\cap\operatorname{sp}H_{\Lambda_{ij}^{(2)}}(x,\omega)$, $E_1\ne E_2$, *then* $|E_1-E_2|>e^{-(N_{ij}^{(2)})^\vartheta}$.

Proof. The proof is essentially the same as in Lemmas 7.2, 7.3, 7.4. □

Applying the same method as in Section 8, we can show the following proposition, which is the main result in this section.

Proposition 9.9. *For typical* $\widetilde{V}^{(2)}(x)=\widetilde{V}^{(1)}(x)+W^{(2)}\left(\eta^{(2)},\xi^{(2)},\theta^{(2)},\{R_m^{(2)}\};x\right)$ *there are* C^3 *functions* $E_{ijk}^{(2)}(x,\omega)$, $\varphi_{ijk}^{(2)}(x,\omega)$, $x\in U_i$, $\omega\in V_j\cap\Omega^{(2)}$ *such that*

(1) $H_{\Lambda_{ij}^{(2)}}(x,\omega)\varphi_{ijk}^{(2)}(x,\omega) = E_{ijk}^{(2)}(x,\omega)\varphi_{ijk}^{(2)}(x,\omega)$

(2) $\operatorname{sp} H_{\Lambda_{ij}^{(2)}}(x,\omega) \cap \bigcup_{\ell} \Big(E_{ijk}^{(2)}(x,\omega) - \frac{1}{2}e^{-(N^{(1)})^{\beta}}, E_{ijk}^{(2)} + \frac{1}{2}e^{-(N^{(1)})^{\beta}}\Big) = \bigcup_{k} \Big\{E_{ijk}^{(2)}(x,\omega)\Big\}$

(3) $|E_{ijk}^{(2)}(x,\omega) - E_{ij\ell}^{(2)}(x,\omega)| > e^{-|\Lambda_{ij}^{(2)}|^{1/4}}$ *if* $\ell \neq k$

(4) $|\varphi_{ijk}^{(2)}(x,\omega)(\ell)| \leq \lambda^{-\frac{1}{4}|\ell| - N_{ij}^{(2)})}$ *for* $N_{ij}^{(2)} < |\ell| \leq \big(N_{ij}^{(2)}\big)^2$

(5) $\big|\partial_x E_{ijk}^{(2)}(x,\omega)\big| + \big|\partial_{xx} E_{ijk}^{(2)}(x,\omega)\big| \geq e^{-(N^{(2)})^{\vartheta}}$

For the general inductive step, we need to count the number of eignevalues of $H_{\Lambda_{ij}^{(2)}}(x,\omega)$ near $E_{ijk}^{(1)}(x,\omega)$.

Lemma 9.10. *If* $\omega \in \Omega^{(2)}$, *then*

$$\#\left\{E \in \operatorname{sp} H_{\Lambda_{ij}^{(2)}}(x,\omega) : \operatorname{dist}\big(E, H_{\Lambda_{ij}^{(1)}}(x,\omega)\big) < \frac{1}{2}e^{-(N^{(1)})^{\beta}}\right\} \lesssim \big|\Lambda_{ij}^{(2)}\big|^{1/2}$$

Proof. By Corollary 9.5, if $H_{\Lambda_{ij}^{(2)}}\varphi = E\varphi$, $\|\varphi\| = 1$, $\operatorname{dist}\big(E, H_{\Lambda_{ij}^{(1)}}(x,\omega)\big) < \frac{1}{2}e^{-(N^{(1)})^{\beta}}$ then $|\varphi(n)| \lesssim e^{-|n|-N_{ij}^{(2)}}$ if $|n| > N_{ij}^{(2)}$.

Let $\mathcal{L} = \bigoplus_{\operatorname{dist}\left(E, H_{\Lambda_{ij}^{(1)}}(x,\omega)\right) < \frac{1}{2}e^{-(N^{(1)})^{\beta}}} \ker\big(H_{\Lambda_{ij}^{(2)}} - EI\big)$. Then $|\varphi(n)| \lesssim e^{-|n|-N_{ij}^{(2)}}$ if $|n| > N_{ij}^{(2)}$ for all $\varphi \in \mathcal{L}$, $\|\varphi\| = 1$. Hence there is $\psi \in \mathbb{R}^{[-2N_{ij}^{(2)}, 2N_{ij}^{(2)}]}$ such that $\varphi \cdot \psi \neq 0$.

But by Lemma A.6, $\dim \mathcal{L} \lesssim N_{ij}^{(2)} \lesssim \big|\Lambda_{ij}^{(2)}\big|^{1/2}$. □

Lemma 9.11. *If* $w \in \Omega^{(2)}$, $e^{-(N^{(2)})^{\gamma}} < \operatorname{dist}\big(E, \operatorname{sp} H_{\Lambda_{ij}^{(2)}}(x,\omega)\big)$ *then* $\log\big|f_{\Lambda_{ij}^{(1)}}(x,\omega)\big| > \frac{|\Lambda_{ij}^{(2)}|}{4}\log\lambda$.

Proof. If $\big|E - E_{pjk}^{(1)}(x + n\omega, \omega)\big| \geq \frac{1}{2}e^{-(N^{(1)})^{\beta}}$ for n, p, k then the assertion follows immediately from the Avalanche Principle.

So suppose $\big|E - E_{p_0jk_0}^{(1)}(x + n_0\omega, \omega)\big| < \frac{1}{2}e^{-(N^{(1)})^{\beta}}$, $\omega \in \Omega^{(2)}$. Then $\big|E - E_{pjk}^{(1)}\big(x + (n_0 + n)\omega, \omega\big)\big| \geq \frac{1}{2}e^{-(N^{(1)})^{\beta}}$, $N_{pj}^{(2)} < |n| \leq \big(N_{pj}^{(2)}\big)^2$.

Take $\widetilde{E} = |E - \widetilde{E}| \lesssim e^{-(N^{(1)})^{\beta}}$ such that $\big|\widetilde{E} - E_{pjk}^{(1)}(x + n\omega, \omega)\big| \geq \frac{1}{4}e^{-(N^{(1)})^{\beta}}$. By Lemma 9.10 and Lemma B.5,

$$\log\big|f_{\Lambda_{ij}^{(2)}}(E)\big| > \log\big|f_{\Lambda_{ij}^{(2)}}(\widetilde{E})\big| - \big|\Lambda_{ij}^{(2)}\big|^{1/2}\big(N^{(2)}\big)^{\gamma} .$$

□

10 Inductive construction of eigenvalues $E^{(s)}_{ijk}(x,\omega)$ and eigenfunctions $\varphi^{(s)}_{ijk}(x,\omega)$

Suppose we have defined C^3–functions $E^{(s-1)}_{ijk}(x,\omega)$, $\varphi^{(s-1)}_{ijk}(x,\omega)$, for $x \in U_i$, $\omega \in V_j \cap \Omega^{(s-1)}$, such that

1. $H_{\Lambda^{(s-1)}_{ij}}(x,\omega)\varphi^{(s-1)}_{ijk}(x,\omega) = E^{(s-1)}_{ijk}(x,\omega)\varphi^{(s-1)}_{ijk}(x,\omega)$

2. $\Big\{E \in \operatorname{sp} H_{\Lambda^{(s-1)}_{ij}}(x,\omega) : \operatorname{dist}\big(E, \operatorname{sp} H_{\Lambda^{(s-2)}_{ij}}(x,\omega)\big) < e^{-(N^{(s-2)})^\beta}\Big\} = \bigcup\limits_{k=1}^{J^{(s-1)}_{ij}} \Big\{E^{(s-1)}_{ijk}(x,\omega)\Big\}$, $J^{(s-1)}_{ij} \lesssim \big|\Lambda^{(s-1)}_{ij}\big|^{1/2}$

3. $\big|E^{(s-1)}_{ijk}(x,\omega) - E^{(s-1)}_{ij\ell}(x,\omega)\big| > e^{-(N^{(s-1)})^\vartheta}$ if $\ell \neq k$

4. $\big|\varphi^{(s-1)}_{ijk}(x,\omega)(\ell)\big| \le \lambda^{-\frac{1}{4}\left(|\ell| - N^{(s-1)}_{ij}\right)}$ for $N^{(s-1)}_{ij} < |\ell| \le \big(N^{(s-1)}_{ij}\big)^2$

5. $\big|\partial_x E^{(s-1)}_{ijk}(x,\omega)\big| + \big|\partial_{xx} E^{(s-1)}_{ijk}\big| \ge e^{-(N^{(s-1)})^\vartheta}$

6. If $\operatorname{dist}\big(E, \operatorname{sp} H_{\Lambda^{(s-1)}_{ij}}(x,\omega)\big) > e^{-(N^{(s-1)})^\gamma}$ then $\log\big|f_{\Lambda^{(s-1)}_{ij}}(x,\omega)\big| > \frac{|\Lambda^{(s-1)}_{ij}|}{4}\log\lambda$

Let $N^{(s)} \asymp e^{-(N^{(s-1)})^\alpha}$.

Lemma 10.1. *There is $\Omega^{(s)} \subset \Omega^{(s-1)} \subset \mathbb{T}$, $\operatorname{mes}\Omega^{(s-1)} \setminus \Omega^{(s)} \le e^{-\frac{1}{2}(N^{(s-1)})^\beta}$ such that for each $U_i \times V_j$, there is $\log N^{(s)}_{ij} \asymp \log N^{(s)}$ and for any $x \in U_i$, $x + nw \in U_i$, $w \in V_j \cap \Omega^{(s)}$, $N^{(s)}_{ij} < |n| \le \big(N^{(s)}_{ij}\big)^2$, one has*

$$\operatorname{dist}\big(E, \operatorname{sp} H_{\Lambda^{(s-1)}_{pj}}(x + n\omega, \omega)\big) > \frac{1}{2}e^{-(N^{(s-1)})^\beta}$$

whenever $\big|E - E^{(s-1)}_{ijk}(x,\omega)\big| < \frac{1}{2}e^{-(N^{(s-1)})^\beta}$.

Proof. Same as in Lemma 9.2. □

Denote $\Lambda^{(s)}_{ij} = \Big[-\big(N^{(s)}_{ij}\big)^2, \big(N^{(s)}_{ij}\big)^2\Big]$.

Lemma 10.2. $\log\big|f_{[-a,b]}(x + nw, w, E)\big| > \frac{1}{4}(b+a)\log\lambda$ *if $x \in U_i$, $w \in V_j \cap \Omega^{(s)}$, $N^{(s)}_{ij} < |n| \le \big(N^{(s)}_{ij}\big)^2$, $\operatorname{dist}\big(E, \operatorname{sp} H_{\Lambda^{(s-1)}_{ij}}(x,\omega)\big) < \frac{1}{2}e^{-(N^{(s-1)})^\beta}$, $a, b \asymp N^{(s-1)}$.*

Proof. By Lemma C.9, $\operatorname{dist}\big(E, \operatorname{sp} H_{[a,b]}(x,\omega)\big) > \frac{1}{2}e^{-(N^{(s)})^\beta}$. The assertion follows from inductive hypothesis (6). □

Lemma 10.3. $\Big|\big[H_{[N_{ij}^{(s)}+1,(N_{ij}^{(s)})^2]}(x,\omega)-E\big]^{-1}(k,\ell)\Big| \le \lambda^{-\frac{1}{4}|k-\ell|}$ *for* $|k-\ell|>10N^{(s-1)}$, $\operatorname{dist}\big(E,\operatorname{sp} H_{\Lambda_{ij}^{(s-1)}}(x,w)\big)<\frac{1}{2}e^{-(N^{(s-1)})}$.

Proof. This follows from Avalanche Principle (see Lemma 9.4 and Corollary 9.5). □

Lemma 10.4. *If* $E_1,E_2\in\big(E_{ijk}^{(s-1)}(x,\omega)-\frac{1}{2}e^{-(N^{(s-1)})^\gamma},E_{ijk}^{(s-1)}(x,\omega)+\frac{1}{2}e^{-(N^{(s-1)})^\gamma}\big)$ *then for* $-\big(N_{ij}^{(s)}\big)^2\le a<b<-N_{ij}^{(s)}$, $|\Lambda_{pq}^{(s-1)}|<b-a\le 5|\Lambda_{pq}^{(s-1)}|$ *then*

$$\Big|\log\big|f_{[a,b]}(x,\omega,E_1)\big|-\log\big|f_{[a,b]}(x,\omega,E_2)\big|\Big|\lesssim e^{-(N^{(s-1)})^\beta}|E_1-E_2|$$

Proof. The hypothesis of the Avalanche Principle is satisfied by induction hypothesis (6), Lemma C.9 and Lemma C.10. □

Corollary 10.5. *If* $E_1,E_2\in\big(E_{ijk}^{(s-1)}(x,\omega)-\frac{1}{2}e^{-(N^{(s)})^\gamma},E_{ijk}^{(s-1)}(x,\omega)+\frac{1}{2}e^{-(N^{(s)})^\vartheta}\big)\cap\operatorname{sp}H_{\Lambda_{ij}^{(s)}}(x,\omega)$, $E_1\neq E_2$ *then* $|E_1-E_2|>e^{-(N_{ij}^{(s)})^\vartheta}$.

Proof. The assertion follows from Lemma 10.4. (See Corollary 9.7 and Corollary 9.8) □

Lemma 10.6. *If* $w\in\Omega^{(s)}$, *then*

$$\#\left\{E\in\operatorname{sp}H_{\Lambda_{ij}^{(s)}}(x,\omega):\operatorname{dist}\big(E,H_{\Lambda_{ij}^{(s-1)}}(x,\omega)\big)<\frac{1}{2}e^{-(N^{(s)})^\beta}\right\}\lesssim\big|\Lambda_{ij}^{(s)}\big|^{1/2}.$$

Proof. Same as in Lemma 9.10. □

Lemma 10.7. *If* $\omega\in\Omega^{(s)}$, $e^{-(N^{(s)})^\gamma}<\operatorname{dist}\big(E,\operatorname{sp}H_{\Lambda_{ij}^{(s)}}(x,\omega)\big)$ *then* $\log\big|f_{\Lambda_{ij}^{(s)}}(x,\omega)\big|>\frac{|\Lambda_{ij}^{(s)}|}{4}\log\lambda$.

Proof. Same as in Lemma 9.11. □

11 Proof of Main Theorem

We have defined C^3 functions $V^{(1)},V^{(2)},\dots$, $\big\|V^{(\ell+1)}-V^{(\ell)}\big\|\lesssim e^{-(N^{(\ell)})^\nu}$, and $\mathbb{T}\supset\Omega^{(2)}\supset\Omega^{(3)}\supset\dots$, $\operatorname{mes}\big(\Omega^{(\ell+1)}\setminus\Omega^{(\ell)}\big)\lesssim e^{-(N^{(\ell)})}$. Let $\widehat{V}=\lim V^{(\ell)}\in C^3(\mathbb{T})$, $\Omega=\bigcap\Omega^{(\ell)}$,

$$\widehat{H}_{\Lambda_{ij}^{(\ell)}}(x,\omega)=\begin{pmatrix}\widehat{V}\big(x-|\Lambda_{ij}^{(\ell)}|\omega\big) & -1 & \\ -1 & \widehat{V}\big(x-\big(|\Lambda_{ij}^{(\ell)}|-1\big)\omega\big) & -1 \\ & -1 & \ddots \\ & \ddots & -1 \\ & -1 & \widehat{V}\big(x+|\Lambda_{ij}^{(\ell)}|\omega\big)\end{pmatrix}$$

Then $\big\|\widehat{H}_{\Lambda_{ij}^{(\ell)}}(x,\omega)-H_{\Lambda_{ij}^{(\ell)}}(x,\omega)\big\|\lesssim e^{-(N^{(\ell)})^\nu}$.

To establish Theorem 1.2, we need to show that the eigenvalues and eigenfunctions of $\widehat{H}_{\Lambda_{ij}^{(\ell)}}(x,\omega)$ have the same properties shared by the eigenvalues and eigenfunctions of the potential $V^{(\ell)}$. We do this by comparing the corresponding eigenvalues and eigenfunctions.

By Lemma B.4, we can define

$$\widehat{E}_{ijk}^{(\ell)}(x,\omega) \in \operatorname{sp} \widehat{H}_{\Lambda_{ij}^{(\ell)}}(x,\omega) \ , \quad \omega \in \Omega \ , \ x \in U_{ij}$$

such that $\left|\widehat{E}_{ijk}^{(\ell)}(x,\omega) - E_{ijk}^{(\ell)}(x,\omega)\right| \lesssim e^{-(N^{(\ell)})^{\nu}}$.

Lemma 11.1. $\operatorname{dist}\left(\widehat{E}_{ijk}^{(\ell)}(x,\omega), \operatorname{sp} \widehat{H}_{\Lambda_{ij}^{(\ell)}}(x,\omega) \setminus \left\{\widehat{E}_{ijk}^{(\ell)}(x,\omega)\right\}\right) > e^{-(N_{ij}^{(\ell)})^{\nu}}$.

Proof. This follows from Corollary 10.5, Lemma B.4 and the definition of $\widehat{E}_{ijk}^{(\ell)}(x,\omega)$ above. □

Lemma 11.2. *There is* $\left\|\hat{\varphi}_{ijk}^{(\ell)}(x,\omega)\right\| = 1$ *such that*

$$\widehat{H}_{\Lambda_{ij}^{(\ell)}}(x,\omega)\hat{\varphi}_{ijk}^{(\ell)}(x,\omega) = \widehat{E}_{ijk}^{(\ell)}(x,\omega)\hat{\varphi}_{ijk}^{(\ell)}(x,\omega),$$
$$\left\|\hat{\varphi}_{ijk}^{(\ell)}(x,\omega) - \varphi_{ijk}^{(\ell)}(x,\omega)\right\| \lesssim e^{-(N^{(\ell)})^{\nu}}$$

Proof.

$$\begin{aligned}
&\left\| \left[\widehat{H}_{\Lambda_{ij}^{(\ell)}}(x,\omega) - \widehat{E}_{ijk}^{(\ell)}(x,\omega)\right]\varphi_{ijk}^{(\ell)}(x,\omega)\right\| \\
&\leq \left\|\widehat{H}_{\Lambda_{ij}^{(\ell)}}(x,\omega) - H_{\Lambda_{ij}^{(\ell)}}(x,\omega)\right\| + \left\| \left[H_{\Lambda_{ij}^{(\ell)}}(x,\omega) - E_{ijk}^{(\ell)}(x,\omega)\right]\varphi_{ijk}^{(\ell)}(x,\omega)\right\| \\
&+ \left|E_{ijk}^{(\ell)}(x,\omega) - \widehat{E}_{ijk}^{(\ell)}(x,\omega)\right| \lesssim e^{-(N^{(\ell)})^{\nu}}
\end{aligned}$$

The assertions follows from Lemma A.5 and Lemma 11.1. □

Corollary 11.3. $w \in \Omega$. *If* $\left|E - \widehat{E}_{ijk}^{(\ell)}(x,\omega)\right| < \frac{1}{2}e^{-(N^{(\ell)})^{\beta}}$ *then*

$$\operatorname{dist}\left(E, \operatorname{sp} \widehat{H}_{\Lambda_{pj}^{(\ell)}}(x+n\omega,\omega)\right) > \frac{1}{2}e^{-(N^{(\ell)})^{\beta}}$$

for all $N^{(\ell+1)} < |n| \leq \left(N^{(\ell+1)}\right)^2$. *Also,*

$$\#\left(E \in \operatorname{sp} \widehat{H}_{\Lambda_{ij}^{(\ell+1)}}(x,\omega) : \operatorname{dist}\left(E, \operatorname{sp} \widehat{H}_{\Lambda_{ij}^{(\ell)}}(x,\omega)\right) < \frac{1}{2}e^{-(N^{(\ell)})^{\beta}}\right) \lesssim \left|\Lambda_{ij}^{(\ell+1)}\right|^{1/2}$$

Proof. Same as in Lemma 10.1 and Lemma 10.6. □

Let $f_{\Lambda_{ij}^{(\ell)}}(x,\omega,E) = \det\left(\widehat{E}_{\Lambda_{ij}^{(\ell)}}(x,\omega) - E\right)$.

Lemma 11.4. *If* $\omega \in \Omega \cap V_j^{(\ell)}$, $x \in U_i$

$$\operatorname{dist}\left(E, \operatorname{sp} H_{\Lambda_{ij}^{(\ell)}}(x,\omega)\right) > \frac{1}{2}e^{-(N^{(\ell)})^{\gamma}}$$

Then $\log\left|\hat{f}_{\Lambda_{ij}^{(\ell)}}(x,\omega,E)\right| > \frac{|\Lambda_{ij}^{(\ell)}|}{4}\log\lambda$.

Proof. For $\ell = 1$, let $\{\mu_k\} = \operatorname{sp} H_{\Lambda_{ij}^{(2)}}(x,\omega)$, $\big|\lambda\widehat{V}(x+k\omega) - \mu_k\big| \le 2$.

Let $K = \big\{k : \big|E - \mu_k\big| \le \lambda^{1/2}\big\}$. If $k \in K$, then

$$\begin{aligned}\big|V(x+k\omega) - \lambda^{-1}E\big| &\le \big|V(x+k\omega) - \widehat{V}(x+k\omega)\big| + \big|\widehat{V}(x+k\omega) - \lambda^{-1}E\big| \\ &\lesssim \lambda^{-1/2}\ .\end{aligned}$$

By Lemma 6.2, $\#K \le m_0$. Hence

$$\begin{aligned}\log\big|\hat{f}_{\Lambda_{ij}^{(1)}}(x,\omega,E)\big| &> \big[\Lambda_{ij}^{(1)}(x,\omega) - m_0\big]\log\lambda^{1/2} - m_0\big(N^{(1)}\big)^{\gamma} \\ &> \frac{|\Lambda_{ij}^{(1)}|}{4}\log\lambda\ .\end{aligned}$$

For $\ell > 1$, there is $|\widetilde{E} - E| \lesssim e^{-(N^{(\ell-1)})^{\beta}}$ such that $\operatorname{dist}\Big(\widetilde{E}, \operatorname{sp} H_{\Lambda_{pj}^{(\ell-1)}}(x+n\omega)\Big) > e^{-N^{(\ell-1)})^{\beta}}$. By the Avalanche Principle

$$\log\big|\hat{f}_{\Lambda_{ij}^{(\ell)}}(x,\omega,\widetilde{E})\big| > \frac{\Lambda_{ij}^{(\ell)}|}{4}\log\lambda\ .$$

In view of Corollary 11.3, we get

$$\log\big|\hat{f}_{\Lambda_{ij}^{(\ell)}}(x,\omega,E)\big| > \frac{|\Lambda_{ij}^{(\ell)}|}{4}\log\lambda$$

(see Lemma 10.7). □

Lemma 11.5. $\big|\partial_x\widehat{E}_{ijk}^{(\ell)}(x,\omega) - \partial_x E_{ijk}^{(\ell)}(x,\omega)\big| \lesssim e^{-(N^{(\ell)})^{\nu}}$.

Proof.

$$\begin{aligned}&\big|\partial_x\widehat{E}_{ijk}^{(\ell)}(x,\omega) - \partial_x E_{ijk}^{(\ell)}(x,\omega)\big| \\ &\quad\le \lambda\sum_{n\in\Lambda_{ij}^{(\ell)}}\big|\widehat{V}'(x+n\omega)\big|\,\big|\hat{\varphi}_{ijk}^{(\ell)}(x,\omega)(n)\big|^2 - V'(x+n\omega)\big|\varphi_{ijk}^{(\ell)}(x,\omega)(n)\big|^2 \\ &\quad\le \lambda\sum_n \widehat{V}'(x+n\omega) - V^{(2)'}(x+n\omega)\big|\,\big|\hat{\varphi}_{ijk}^{(\ell)}(x,\omega)(n)\big|^2 \\ &\qquad + \lambda\sum_n\big|V^{(2)}(x+n\omega)\big|\,\big|\hat{\varphi}_{ijk}^{(\ell)}(x,\omega)(n)\big|^2 - \big|\varphi_{ijk}^{(\ell)}(x,\omega)(n)\big|^2 \\ &\quad\lesssim e^{-(N^{(\ell)})^2}\end{aligned}$$

□

Lemma 11.6. $\big\|\partial_x\hat{\varphi}_{ijk}^{(\ell)}(x,\omega) - \partial_x\varphi_{ijk}^{(\ell)}(x,\omega)\big\| \lesssim e^{-\frac{1}{2}(N^{(\ell)})^{\nu}}$.

Proof.

$$\partial_x\varphi_{ijk}^{(\ell)}(x,\omega) = \begin{bmatrix} \oint_{|z-E_{ijk}^{(\ell)}|=\varrho}\big(H_{\Lambda_{ij}^{(\ell)}}(x,\omega) - z\big)^{-1}\big[\partial_x H_{\Lambda_{ij}^{(\ell)}}(x,\omega)\big]\big(H_{\Lambda_{ij}^{(\ell)}} - z\big)^{-1}dz \\ \varphi_{ijk}^{(\ell)}(x,\omega)\end{bmatrix}$$

where $\varrho = \frac{1}{2}e^{-(N^{(\ell)})^\nu}$

$$\begin{aligned}
&\big\|\partial_x\hat\varphi^{(\ell)}_{ijk}(x,\omega) - \partial_x\varphi^{(\ell)}_{ijk}(x,\omega)\big\| \\
&\quad\le \oint \big\|\big(\widehat H - z\big)^{-1}\big(\partial_x\widehat H\big)\big(\widehat H - z\big)^{-1} - \big(H-z\big)^{-1}\big(\partial_x H\big)\big(H-z\big)^{-1}\big\|dz \\
&\qquad + \Big[\oint \big\|\big(H-z\big)^{-1}\big(\partial_x H\big)\big(H-z\big)^{-1}\big\|dz\Big]\big\|\hat\varphi^{(\ell)}_{ijk}(x,\omega) - \varphi^{(\ell)}_{ijk}(x,\omega)\big\| \\
&\big\|\big(\widehat H - z\big)^{-1}\big(\partial_x\widehat H\big)\big(\widehat H - z\big)^{-1} - \big(H-z\big)^{-1}\big(\partial_x H\ bigr\big)\big(H-z\big)^{-1}\big\| \\
&\quad\le \big\|\big(\widehat H - z\big)^{-1} - \big(H-z\big)^{-1}\big\|\,\big\|\big(\partial_x\widehat H\big)\big(\widehat H - z\big)^{-1}\big\| \\
&\qquad + \big\|\big(H-z\big)^{-1}\big\|\,\big\|\partial_x H - \partial_x\widehat H\big\|\,\big\|\big(\widehat H - z\big)^{-1}\big\| \\
&\qquad + \big\|\big(H-z\big)^{-1}\big\|\,\big\|\partial_x H\big\|\,\big\|\big(\widehat H - z\big)^{-1} - \big(H-z\big)^{-1}\big\| \\
&\quad\lesssim e^{-\frac{1}{2}(N^{(\ell)})^\nu}
\end{aligned}$$

since $\big(\widehat H - z\big)^{-1} - \big(H-z\big)^{-1} = \big(\widehat H - z\big)^{-1}\Big[\big(H-z\big) - \big(\widehat H - z\big)\Big]\big(H-z\big)^{-1}$ and $\big\|\big(\widehat H - z\big)^{-1}\big\| \lesssim e^{(N^{(\ell)})^\nu}$, $\big\|\big(H-z\big)^{-1}\big\| \lesssim e^{(N^{(\ell)})^\nu}$. □

Lemma 11.7. $\big|\partial_{xx}\widehat E^{(\ell)}_{ijk}(x,\omega) - \partial_{xx}E^{(\ell)}_{ijk}(x,\omega)\big| \lesssim e^{-\frac{1}{2}(N^{(\ell)})^\nu}$.

Proof.

$$\begin{aligned}
&\big|\partial_{xx}\widehat E^{(\ell)}_{ijk}(x,\omega) - \partial_{xx}E^{(\ell)}_{ijk}(x,\omega)\big| \\
&\quad\le \lambda\sum_{n\in\Lambda^{(\ell)}_{ij}}\Big|\widehat V''(x+n\omega)\big|\hat\varphi^{(\ell)}_{ijk}(x,\omega)(n)\big|^2 - V^{(\ell)''}(x+n\omega)\big|\varphi^{(\ell)}_{ijk}(x,\omega)(n)\big|^2\Big| \\
&\qquad + 2\lambda\sum_{n\in\Lambda^{(\ell)}_{ij}}\Big|\widehat V'(x+n\omega)\hat\varphi^{(\ell)}_{ijk}(x,\omega)\partial_x\hat\varphi^{(\ell)}_{ijk} - V^{(\ell)'}(x+n\omega)\varphi^{(\ell)}_{ijk}(x,\omega)\partial_x\varphi^{(\ell)}_{ijk}\Big| \\
&\quad\lesssim e^{-\frac{1}{2}(N^{(\ell)})^\nu}
\end{aligned}$$

□

Corollary 11.8. $\big|\partial_x\widehat H^{(\ell)}_{ijk}(x,\omega)\big| + \partial_{xx}\widehat H^{(\ell)}_{ijk}(x,\omega)\big| \ge e^{-\frac{1}{2}(N^{(\ell)})^\nu}$.

Proof. Since $\big|\partial_x E^{(\ell)}_{ijk}(x,\omega)\big| + \big|\partial_{xx}E^{(\ell)}_{ijk}(x,\omega)\big| \ge e^{-(N^{(\ell)})^\nu}$, the assertion follows from Lemma 11.5 and Lemma 11.7. □

Theorem 11.9. *Given any $V\in C^3(\mathbb{T})$, there is $\lambda_0 = \lambda_0(V)$ such that for $|\lambda| > \lambda_0$, one has a collection of pertubed potentials $\{S_\ell = S_\ell(V,\lambda)\}_{\ell=1}^\infty$, $S_\ell \subset \mathcal{S}(T^{(\ell)},\delta_\ell)$, $\log T^{(\ell+1)} \asymp \big(T^{(\ell)}\big)^\alpha$, $0<\alpha\ll 1$, $\sum\limits_{\ell=1}^\infty\big(1-|S_\ell|\big) \le \lambda^{-\beta}$, so that for any potential*

$$\widetilde V(x) = V(x) + \sum_{\ell=1}^\infty W^{(\ell)}\big(\eta^{(\ell)},\xi^{(\ell)},\theta^{(\ell)},\{R^{(\ell)}_m\};x\big)$$

where $W^{(\ell)} \in S_\ell$, there exists $\Omega = \Omega(\lambda, \tilde{V})$, $\operatorname{mes}\Omega \le \lambda^{-\beta}$, so that the Lyapunov exponent $L(\omega, E) \ge \frac{1}{4}\log\lambda$ for any $\omega \in \Omega$, $E \in \mathbb{R}$. Furthermore, the corresponding eigenfunctions are exponentially localized.

Proof. Fix E, ω. For each ℓ, from Corollary 11.8, we have

$$\operatorname{mes}\Big\{x \in \mathbb{T} : \operatorname{dist}\big(E, \operatorname{sp} H_{\Lambda_{ij}^{(\ell)}}(x,\omega)\big) \le e^{-(N^{(\ell)})^r}\Big\} \le e^{-c(N^{(\ell)})^\nu} \ .$$

Let $\mathcal{J} = \bigcup_{\ell=1}^{\infty}\Big\{x \in \mathbb{T} : \operatorname{dist}\big(E, \operatorname{sp} H_{\Lambda_{ij}^{(\ell)}}(x,\omega)\big) \le e^{-(N^{(\ell)})^r}\Big\}$. If $x \in \mathbb{T}\setminus\mathcal{J}$ then

$$\begin{aligned}\frac{1}{|\Lambda_{ij}|}\log\big\|M_{\Lambda_{ij}}(x,\omega,E)\big\| &\ge \frac{1}{|\Lambda_{ij}|}\log\big|f_{\Lambda_{ij}}(x,\omega,E)\big| \\ &\ge \frac{1}{4}\log\lambda \ .\end{aligned}$$

From above, $\operatorname{mes}\mathbb{T}\setminus\mathcal{J} > 0$. But $\frac{1}{|\Lambda_{ij}|}\log\big\|M_{ij}(x,\omega,E)\big\| \to L(\omega,E)$ for almost all x since $\omega \in \mathbb{R}\setminus\mathbb{Q}$. Therefore $L(\omega,E) \ge \frac{1}{4}\log\lambda$. □

A Matrix Functions

Let $A : (a,b) \to M_{m\times m}(\mathbb{C})$ be C^1, $A(x)$ self-adjoint for every $x \in (a,b)$. Suppose $E : (a,b) \to \mathbb{R}$ is such that $E(x)$ is a simple eigenvalue of $A(x)$, and

$$\operatorname{sp}A(x) \cap \Big[E_0 - \frac{3\delta}{2},\ E_0 + \frac{3\delta}{2}\Big] = \{E(x)\} \subset \Big(E_0 - \frac{\delta}{2},\ E_0 + \frac{\delta}{2}\Big) \ , \quad \delta > 0 \ .$$

Let $P(x)$ be the orthogonal projection on the eigenspace of $A(x)$ with eigenvalue $E(x)$. By Riesz formula for orthogonal projection of self-adjoint matrix

$$P(x) = \frac{1}{2\pi i}\oint_{|z-E_0|=\delta}\big(zI - A(x)\big)^{-1}dz$$

$$P'(x) = \frac{1}{2\pi i}\oint_{|z-E_0|=\delta}\big(zI - A(x)\big)^{-1}A'(x)\big(zI - A(x)\big)^{-1}dz$$

Hence $P : (a,b) \to M_{m\times m}(\mathbb{C})$ is C^1

$$\|P'(x)\| \le \frac{1}{2\pi}\big(\frac{\delta}{2}\big)^{-1}\|A'(x)\|\big(\frac{\delta}{2}\big)^{-1}(2\pi\delta) = \frac{1}{4\delta}\|A'(x)\| \ .$$

Lemma A.1. *$A : (a,b) \to M_{m\times m}(\mathbb{C})$, $E : (a,b) \to \mathbb{R}$ as above. Then E is C^1 and $E'(x) = \big(A'(x)\varphi(x), \varphi(x)\big)$ where $\varphi(x)$ is a normalized eigenvector of $A(x)$ with eigenvalue $E(x)$.*

Proof. For any $x_0 \in (a,b)$, choose $\varphi_0 \in \operatorname{Ran} P(x_0)$, $\|\varphi_0\| = 1$. Then $\big(P(x_0)\varphi_0, \varphi_0\big) = 1$. There exists neighborhood U of x_0 such that $\big(P(x)\varphi_0, \varphi_0\big) > 0$ for every $x \in U$.

$$E(x) = \frac{\big(A(x)P(x)\varphi_0, \varphi_0\big)}{\big(P(x)\varphi_0, \varphi_0\big)} \quad \text{for } x \in U$$

E is C^1 since A and P are C^1.

$$\begin{aligned} E'(x_0) &= \frac{\big(A'(x_0)P(x_0)\varphi_0, \varphi_0\big) + \big(A(x_0)P'(x_0)\varphi_0, \varphi_0\big)}{\big(P(x_0)\varphi_0, \varphi_0\big)} \\ &\qquad - \big(A(x_0)P(x_0)\varphi_0, \varphi_0\big)\ \frac{\big(P'(x_0)\varphi_0, \varphi_0\big)}{\big(P(x_0)\varphi_0, \varphi_0\big)^2} \\ &= \big(A'(x_0)\varphi_0, \varphi_0\big) + \big(P'(x_0)\varphi_0, A(x_0)\varphi_0\big) - \big(A(x_0)\varphi_0, \varphi_0\big)\big(P'(x_0)\varphi_0, \varphi_0\big) \\ &= \big(A'(x_0)\varphi_0, \varphi_0\big) + E(x_0)\big(P'(x_0)\varphi_0, \varphi_0\big) - E(x_0)\big(P'(x_0)\varphi_0, \varphi_0\big) \\ &= \big(A'(x_0)\varphi_0, \varphi_0\big)\ . \end{aligned}$$

□

Lemma A.2. *Let x_0, φ_0, U be as in proof of Lemma A.1. Define $\varphi(x) = \big(P(x)\varphi_0, \varphi_0\big)^{-1/2}\big(P(x)\varphi_0\big)$ for $x \in U$. Then $\varphi(x)$ is a normalized eigenvector of $A(x)$ with eigenvalue $E(x)$. $\varphi : U \to \mathbb{R}^n$ is C^1 and $\varphi'(x_0) = P'(x_0)\varphi_0$.*

Proof. Since $P(x)$ is an orthogonal projection,

$$\big(P(x)\varphi_0, \varphi_0\big) = \big(P(x)^2\varphi_0, \varphi_0\big) = \big(P(x)\varphi_0, P(x)\varphi_0\big) = \|P(x_0)\varphi_0\|^2\ .$$

Therefore, $\|\varphi(x)\| = 1$.

Since P is C^1, φ is C^1,

$$\begin{aligned} \varphi'(x_0) &= -\frac{1}{2}\big(P(x_0)\varphi_0, \varphi_0\big)^{-\frac{3}{2}}\big(P'(x_0)\varphi_0, \varphi_0\big)\big(P(x_0)\varphi_0\big) \\ &\qquad + \big(P(x_0)\varphi_0, \varphi_0\big)^{-\frac{1}{2}}\big(P'(x_0)\varphi_0\big) \\ &= -\frac{1}{2}\big(P'(x_0)\varphi_0, \varphi_0\big)\varphi_0 + P'(x_0)\varphi\ . \end{aligned}$$

It remains to show $\big(P'(x_0)\varphi_0, \varphi_0\big) = 0$.

$$\begin{aligned} P(x) &= P(x)^2 \\ P'(x_0) &= P'(x_0)P(x_0) + P(x_0)P'(x_0) \\ \big(P'(x_0)\varphi_0, \varphi_0\big) &= \big(P'(x_0)P(x_0)\varphi_0, \varphi_0\big) + \big(P(x_0)P'(x_0)\varphi_0, \varphi_0\big) \\ \big(P'(x_0)\varphi_0, \varphi_0\big) &= 2\big(P'(x_0)\varphi_0, \varphi_0\big)\ . \end{aligned}$$

Hence $\big(P'(x_0)\varphi_0, \varphi_0\big) = 0$. □

Lemma A.3. *$A : (a,b) \to M_{m\times m}(\mathbb{C})$, $E : (a,b) \to \mathbb{R}$ as above. If A is C^2, then E is C^2 and*

$$|E''(x)| \le \|A''(x)\| + \frac{1}{2\delta}\|A'(x)\|^2 \ .$$

Proof. For any $x_0 \in (a,b)$, define $\varphi(x)$ in a neighborhood U of x_0 as in Lemma A.2, $\|\varphi(x)\| = 1$,

$$\begin{aligned} A(x)\varphi(x) &= E(x)\varphi(x) \qquad\qquad \text{for every} x \in U \\ E'(x) &= \big(A'(x)\varphi(x), \varphi(x)\big) \quad \text{for every} x \in U \ . \end{aligned}$$

Since $\varphi(x)$ is C^1, if A is C^2 then E' is C^1. So E is C^2.

$$\begin{aligned} E''(x_0) &= \big(A''(x_0)\varphi(x_0), \varphi(x_0)\big) + \big(A'(x_0)P'(x_0)\varphi(x_0), \varphi(x_0)\big) \\ &\quad + \big(A'(x_0)\varphi(x_0), P'(x_0)\varphi(x_0)\big) \\ &= \big(A''(x_0)\varphi(x_0), \varphi(x_0)\big) + 2\big(A'(x_0)P'(x_0)\varphi(x_0), \varphi(x_0)\big) \\ |E''(x_0)| &\le \|A''(x_0)\| + 2\|A'(x_0)\|\,\|P'(x_0)\| \\ &\le \|A''(x_0)\| + \frac{1}{2\delta}\|A'(x_0)\|^2 \end{aligned}$$

Let $V : \mathbb{R} \to \mathbb{R}$ be continuous, piecewise smooth, 1–periodic. Define

$$H_{[m,n]}(x,w) = \begin{pmatrix} V(x+mw) & -1 & & \\ -1 & V\big(x+(m+1)w\big) & -1 & \\ & -1 & & \ddots \\ & \ddots & & -1 \\ & & -1 & V(x+nw) \end{pmatrix}$$

Then $H_{[m,n]}(x,w)$ is continuous; $H_{[m,n]}(.,w)$ and $H_{[m,n]}(x,.)$ are piecewise smooth. So the eigenvalues $E^{(m)}_{[m,n]}(x,w) \le \cdots \le E^{(n)}_{[m,n]}(x,w)$ of $H_{[m,n]}(x,w)$ are continuous. Also, it is well-known that the eigenvalues of $H_{[m,n]}(x,w)$ are simple.

Let $\Big\{\varphi^{(\ell)}_{[m,n]}(x,\omega)(j)\Big\}_{m\le j\le n}$ be a normalized eigenvector of $H_{[m,n]}(x,\omega)$ with eigenvalue $E^{(\ell)}_{[m,n]}(x,\omega)$. Suppose $\partial_x H_{[m,n]}(x_0,\omega_0)$ exists. For fixed k, let

$$\begin{aligned} E_0 &= E^{(k)}_{[m,n]}(x_0,\omega_0), \\ \delta &= \frac{1}{2}\operatorname{dist}\Big(E^{(k)}_{[m,n]}(x_0,\omega_0), \big\{E^{(\ell)}_{[m,n]}(x_0,\omega_0)\big\}_{\ell\ne k}\Big) > 0 \ . \end{aligned}$$

Since $E^{(\ell)}_{[m,n]}$ are continuous, there exists neighborhood U of x_0 such that for every $x \in U$

$$\operatorname{sp} H_{[m,n]}(x,\omega_0) \cap \Big[E_0 - \frac{3\delta}{2}, E_0 + \frac{3\delta}{2}\Big] = \big\{E^{(k)}_{[m,n]}(x,\omega_0)\big\} \subset \Big(E_0 - \frac{\delta}{2}, E_0 + \frac{\delta}{2}\Big) \ .$$

By Lemma A.1,

$$\partial_x E^{(k)}_{[m,n]}(x_0,\omega_0) = \sum_{j=m}^{n} V'(x_0+j\omega_0)\big|\varphi^{(k)}_{[m,n]}(x_0,\omega_0)(j)\big|^2 \ .$$

Similarly,

$$\partial_\omega E^{(k)}_{[m,n]}(x_0,\omega_0) = \sum_{j=m}^{n} jV'(x_0+j\omega_0)\big|\varphi^{(k)}_{[m,n]}(x_0,\omega_0)(j)\big|^2 \ .$$

□

Lemma A.4. *If A is $n\times n$, self-adjoint, and $\|A\varphi\|<\varepsilon$ for some φ, $\|\varphi\|=1$. Then there exists $\lambda\in \operatorname{sp} A\cap(-\varepsilon,\varepsilon)$.*

Proof. If A is not invertible, then take $\lambda=0$. If A^{-1} exists, then $1=\|\varphi\|=\|A^{-1}A\varphi\|<\|A^{-1}\|\varepsilon$. Let $\lambda_1,\dots,\lambda_n$ be the eigenvalues of A,

$$\frac{1}{\varepsilon}<\|A^{-1}\|=\frac{1}{\min|\lambda_i|}\Longrightarrow \min|\lambda_i|<\varepsilon \ .$$

□

Lemma A.5. *Let A be $n\times n$, self-adjoint. Suppose $\|A\varphi\|<\varepsilon$, $A\psi=0$, $\|\varphi\|=\|\psi\|=1$. If $|\mu|>\delta$ for every $\mu\in\operatorname{sp}A\setminus\{0\}$ then $\min\limits_{|c|=1}\|\varphi-c\psi\|\lesssim\varepsilon/\delta$ provided 0 is simple eigenvalue.*

Proof. Let ψ_μ be normalized eigenvectors of A with eigenvalues $\mu\neq0$,

$$\varepsilon^2>\|A\varphi\|^2=\lambda^2|(\varphi,\psi)|^2+\sum_{\mu\neq0}\mu^2|(\varphi,\psi_\mu)|^2>\delta^2\sum_{\mu\neq0}|(\varphi,\psi_\mu)|^2$$

$$\begin{aligned}
\|\varphi-c\psi\|^2 &= \|\varphi\|^2-\bar c(\varphi,\psi)-c(\psi,\varphi)+|c|^2\|\psi\|^2\\
&= 2\left[1-\operatorname{Re}\bar c(\varphi,\psi)\right] \qquad [|c|=1]\\
\min_{|c|=1}\|\varphi-c\psi\|^2 &= 2\left[1-|(\varphi,\psi)|\right]\\
&\le 2\left[1-|(\varphi,\psi)|^2\right] \qquad [|(\psi,\varphi)|\le1]\\
&= 2\|\varphi-(\varphi,\psi)\psi\|^2\\
&= 2\sum_{\mu\neq0}|(\varphi,\psi_\mu)|^2\\
&< 2\varepsilon^2/\delta^2
\end{aligned}$$

□

Lemma A.6. *Let $\mathcal{L},\mathcal{M}$ be subspaces of $\mathbb{R}^n$. If $\dim\mathcal{L}>\dim\mathcal{M}$, then there is $\psi\in\mathcal{L}$, $\|\psi\|=1$, such that $(\psi,\varphi)=0$ for all $\varphi\in\mathcal{M}$.*

Proof. Suppose, for contradiction, that $\mathcal{L}\cap\mathcal{M}^\perp=\{0\}$. Since $\mathbb{R}=\mathcal{M}\oplus\mathcal{M}^\perp$, we have $\mathcal{L}\subset\mathcal{M}$. But this is impossible because $\dim\mathcal{L}>\dim\mathcal{M}$. Hence, there is $\psi\in\mathcal{L}\cap\mathcal{M}^\perp$, $\|\psi\|=1$. □

B Mini-max principle

Mini-max Principle: Let A be $n \times n$ hermitian matrix, with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Then

$$\begin{aligned}\lambda_j &= \min_{\dim M = j} \max_{\substack{x \in M \\ \|x\|=1}} (Ax, x) \\ &= \max_{\dim M = n-j+1} \min_{\substack{x \in M \\ \|x\|=1}} (Ax, x) \ .\end{aligned}$$

Lemma B.1. *Let A, B be $n \times n$ hermitian matrices, with eigenvalues $\lambda_1 \leq \ldots \lambda_n$ and $\mu_1 \leq \cdots \leq \mu_n$ respectively. Suppose there is $\alpha > 0$, $y \in \mathbb{R}^n$ such that $Ax = Bx + \alpha(x, y)y$ for all $x \in \mathbb{R}^n$. Then the eigenvalues of A and B interlace,*
i.e. $\mu_1 \leq \lambda_1 \leq \mu_2 \leq \lambda_2 \leq \cdots \leq \mu_n \leq \lambda_n$.

Proof.

$$\mu_j = \min_{\substack{x \in M_j \\ \|x\|=1}} (Bx, x) \qquad \text{for some } M_j, \dim M_j = n - j + 1$$

$$\begin{aligned}\lambda_j &= \max_{\dim M = n-j+1} \min_{\substack{x \in M \\ \|x\|=1}} (Ax, x) \\ &\geq \min_{\substack{x \in M_j \\ \|x\|=1}} (Ax, x) = \min_{\substack{x \in \widetilde{M}_j \\ \|x\|=1}} \left[(Bx, x) + \alpha |(x, y)|^2\right] \geq \mu_j\end{aligned}$$

$$\lambda_j = \min_{\substack{x \in M_j \\ \|x\|=1}} (Ax, x) \qquad \text{for some } \widetilde{M}_j, \dim \widetilde{M}_j = n - j + 1$$

Let $U_j \subset \widetilde{M}_j \cap (\mathbb{R}y)^\perp$, $\dim U_j = n - j$ for $j < n$.

$$\begin{aligned}\mu_{j+1} &= \max_{\dim M = n-j} \min_{\substack{x \in M \\ \|x\|=1}} (Bx, x) \\ &\geq \min_{\substack{x \in U_j \\ \|x\|=1}} (Bx, x) = \min_{\substack{x \in U_j \\ \|x\|=1}} (Ax, x) \geq \lambda_j \ .\end{aligned}$$

□

Corollary B.2. *Let A, B be hermitian matrices, with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ and $\mu_1 \leq \cdots \leq \mu_n$ respectively. Suppose* $\operatorname{rank}(A - B) = k$. *Then*

$$\begin{aligned}\mu_j &\leq \lambda_{j+k} \\ \lambda_j &\leq \mu_{j+k} \ .\end{aligned}$$

Lemma B.3. *Let A, B be $n \times n$ hermitian matrices,* $\operatorname{rank}(A-B) = k > 0$, $B \neq 0$. *Then*

$$\log|\det B| - \log|\det A| \leq 2k \log\|B\| - 2k \log \operatorname{dist}(\operatorname{sp} A, 0) \ .$$

Proof. $\det A = 0 \iff \operatorname{dist}(\operatorname{sp} A, 0) = 0$. In this case, the inequality is trivial. So assume $\det A \neq 0$.

Let $\lambda_1 \leq \cdots \leq \lambda_n$ and $\mu_1 \leq \cdots \leq \mu_n$ be the eigenvalues of A and B respectively. Suppose $\mu \leq 0 < \mu_{i+1}$.

By Lemma B.2, $\lambda_{j-k} \leq \mu_j < 0$ for $k < j \leq i$; $0 < \mu_\ell \leq \lambda_{\ell+k}$ for $i < \ell \leq n-k$.

$$\begin{aligned}
\log|\det B| - \log|\det A| = &\sum_{k<j\leq i} \big(\log|\mu_j| - \log|\lambda_{j-k}|\big) + \\
&\sum_{i<j\leq n-k} \big(\log|\mu_j| - \log|\lambda_{j+k}|\big) + \\
&\sum_{j\leq k} \log|\mu_j| + \sum_{j>n-k} \log|\mu_j| - \sum_{i-k<j\leq i+k} \log|\lambda_j| \\
&\leq 2k \log\|B\| - 2k \log \operatorname{dist}(\operatorname{sp} A, 0) \ .
\end{aligned}$$

□

Lemma B.4. *Let A, B be $n \times n$ hermitian matrices, with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ and $\mu_1 \leq \cdots \leq \mu_n$ respectively. Then $|\lambda_j - \mu_j| \leq \|A - B\|$, $j = 1, \ldots, n$.*

Proof.

$$\begin{aligned}
\mu_j &= \max_{\substack{x\in M_j \\ \|x\|=1}} (Bx, x) \quad \text{for some } \overline{M}_j, \dim \overline{M}_j = j \\
\lambda_j &= \min_{\dim M = j} \max_{\substack{x\in M \\ \|x\|=1}} (Ax, x) \leq \max_{\substack{x\in \overline{M}_j \\ \|x\|=1}} (Ax, x) \\
&\leq \max_{\substack{x\in \overline{M}_j \\ \|x\|=1}} (Bx, x) + \max_{\substack{x\in \overline{M}_j \\ \|x\|=1}} \big((A-B)x, x\big) \\
&\leq \mu_j + \|A - B\| \ .
\end{aligned}$$

Similarly, $\mu_j \leq \lambda_j + \|A - B\|$. □

Lemma B.5. *Let A be an $n \times n$ hermitian matrix. Given an interval (E', E''), let $\underline{E} = E' - n(E'' - E')$, $\overline{E} = E'' + n(E'' - E')$. If $m = \#\big(\operatorname{sp} A \cap (\underline{E}, \overline{E})\big)$, then for any $E_1, E_2 \in (E', E'')$ one has*

$$\log|\det(A - E_2)| - \log|\det(A - E_1)| \leq 1 + m \log\|A - E_2\| - m \log \operatorname{dist}(\operatorname{sp} A, E_1)$$

Proof. We prove the case for $E_1 < E_2$. (The case for $E_2 < E_1$ is similar.) Let $\lambda_1 \le \cdots \le \lambda_n$ be the eigenvalues of A, $\lambda_r \le \underline{E} < \lambda_{r+1}$, $\lambda_s < \overline{E} \le \lambda_{s+1}$.

$$\begin{aligned}
&\text{For } j > s, \qquad |E_2 - \lambda_j| < |E_1 - \lambda_j| \\
&\text{For } j \le r, \qquad \frac{|E_2 - \lambda_j|}{|E_1 - \lambda_j|} \le \frac{|E_2 - E_1| + |E_1 - \lambda_j|}{|E_1 - \lambda_j|} \\
&\qquad\qquad\qquad\qquad \le \frac{E'' - E'}{n(E'' - E')} + 1 = 1 + \frac{1}{n} \\
&\text{For } r < j \le s, \frac{|E_2 - \lambda_j|}{|E_1 - \lambda_j|} \le \frac{\|A - E_2\|}{\operatorname{dist}(\operatorname{sp} A, E_1)}
\end{aligned}$$

$$\begin{aligned}
\log|\det(A - E_2)| &= \log \prod_{j=1}^{n} |E_2 - \lambda_j| \\
&= \log \prod_{j=1}^{n} |E_1 - \lambda_j| + \sum_{j \le r} \log \frac{|E_2 - \lambda_j|}{|E_1 - \lambda_j|} \\
&\quad + \sum_{r < j \le s} \log \frac{|E_2 - \lambda_j|}{|E_1 - \lambda_j|} + \sum_{j > s} \log \frac{|E_2 - \lambda_j|}{|E_1 - \lambda_j|} \\
&\le \log|\det(A - E_1)| + n \log\bigl(1 + \frac{1}{n}\bigr) \\
&\quad + m\Bigl(\log\|A - E_2\|\bigr| - \log \operatorname{dist}(\operatorname{sp} A, E_1)\Bigr)
\end{aligned}$$

The assertion now follows since $n \log\bigl(1 + \frac{1}{n}\bigr) \le 1$. □

C 1 dimensional difference Schrödinger equation

Consider the 1 dimensional difference Schrödinger equation:

$$-\varphi(k-1) - \varphi(k+1) + \lambda v(k)\varphi(k) = E\varphi(k) \ , \qquad k \in \mathbb{Z}$$

where $v(k) \in \mathbb{R}$, $|v(k)| \le C$.

The eigenvalues of this equation with Dirichlet boundary conditions, $\varphi(a-1) = \varphi(b+1) = 0$, are the same as the eigenvalues of

$$H_{[a,b]} = \begin{pmatrix} \lambda v(a) & -1 & & \\ -1 & \lambda v(a+1) & -1 & \\ & -1 & \ddots & \\ & \ddots & & -1 \\ & & -1 & \lambda v(b) \end{pmatrix}$$

Given initial condition $\varphi(a-1), \varphi(a)$, the solution of this equation, for $b \geq a$, can be written as

$$\begin{pmatrix} \varphi(b+1) \\ \varphi(b) \end{pmatrix} = M_{[a,b]}(E) \begin{pmatrix} \varphi(a) \\ \varphi(a-1) \end{pmatrix}$$

where $M_{[a,b]}(E)$ is the monodromy matrix

$$M_{[a,b]}(E) = \prod_{k=b}^{a} A_k, \qquad A_k = \begin{pmatrix} \lambda v(k) - E & -1 \\ 1 & 0 \end{pmatrix}$$

The entries of the monodromy M_n are

$$M_{[a,b]}(E) = \begin{pmatrix} f_{[a,b]}(E) & -f_{[a+1,b]}(E) \\ f_{[a,b-1]}(E) & -f_{[a+1,b-1]}(E) \end{pmatrix}$$

where $f_{[a,b]}(E) = \det\big(H_{[a,b]} - E\big)$.

If $E \notin \operatorname{sp} H_{[a,b]}$, then by Cramer's rule,

$$\begin{aligned} \big(H_{[a,b]} - E\big)^{-1}(k,\ell) &= \big(H_{[a,b]} - E\big)^{-1}(\ell,k) \qquad [a \leq k \leq \ell \leq b] \\ &= \frac{f_{[a,k-1]}(E) f_{[\ell+1,b]}(E)}{f_{[a,b]}(E)} \ . \end{aligned}$$

By convention, $f_{[a,a-1]}(E) = f_{[b+1,b]}(E) = 1$.

Poisson's Formula: If $-\varphi(n-1) - \varphi(n+1) + v(n)\varphi(n) = E\varphi(n)$, $E \notin \operatorname{sp} H_{[a,b]}$. Then for $a \leq m \leq b$

$$\varphi(m) = \big(H_{[a,b]} - E\big)^{-1}(m,a)\varphi(a-1) + \big(H_{[a,b]} - E\big)^{-1}(m,b)\varphi(b+1) \ .$$

Notation:

$$\begin{aligned} T_{[a,b]} &= H_{[a,b]} - \operatorname{diag}\big(\lambda v(a), \dots, \lambda v(b)\big) \\ &= \begin{pmatrix} 0 & -1 & & & \\ -1 & 0 & -1 & & \\ & -1 & 0 & \ddots & \\ & & & \ddots & -1 \\ & & & -1 & 0 \end{pmatrix} \end{aligned}$$

Note that $\big\|T_{[a,b]}\big\| \leq 2$.

Fix a, b. Let $n = b - a + 1$.

Lemma C.1. *If* $\mu = \frac{1}{2} \min\limits_{a \leq j \leq b} |\lambda v(j) - E| > 2$, *then* $\operatorname{dist}(\operatorname{sp} H_{[a,b]}, E) > \mu$ *and* $\log |f_{[a,b]}(E)| > n \log \mu$.

Proof. Let $\mu_1 \le \cdots \le \mu_n$ be the eigenvalues of $\big(H_{[a,b]} - E\big)$,

$$\tilde\mu_j = \lambda v(i_j) - E \ , \qquad \tilde\mu_1 \le \cdots \le \tilde\mu_n \ .$$

By Lemma B.4, $|\mu_j - \tilde\mu_j| \le \big\|T_{[a,b]}\big\| \le 2$. Hence $|\mu_j| \ge 2\mu - 2 > \mu$

$$\log|f_{[a,b]}(E)| = \Sigma \log|\mu_j| > n \log \mu \ .$$

□

Lemma C.2. *If* $\mu = \frac{1}{2}\min\limits_{j\neq j_0}|\lambda v(j) - E| > 2$, *then* $\#\big(\mathrm{sp}(H_{[a,b]} - E) \cap [-\mu, \mu]\big) \le 1$ *and*

$$\log|f_{[a,b]}(E)| > (n-1)\log\mu + \log \mathrm{dist}(\mathrm{sp}\, H_{[a,b]}, E) \ .$$

Proof. Let $\mu_1 \le \cdots \le \mu_n$ be the eigenvalues of $\big(H_{[a,b]} - E\big)$,

$$\tilde\mu_j = \lambda v(i_j) - E \ , \quad \tilde\mu_1 \le \cdots \le \tilde\mu_n \ , \quad \tilde\mu_{\bar j} = \lambda v(j_0) - E \ .$$

Then $|\mu_j| > \mu$ for $j \neq \bar j$

$$\begin{aligned}\log|f_{[a,b]}(E)| &= \sum_{j\neq\bar j} \log|\mu_j| + \log|\mu_{\bar j}| \\ &> (n-1)\log\mu + \log \mathrm{dist}\big(\mathrm{sp}\, H_{[a,b]}, E\big)\end{aligned}$$

□

Lemma C.3. *If* $\mu = \frac{1}{2}\min|\lambda v(j) - E| > 2$, $|E| \le \max|\lambda v(j)| = \lambda C$ *then* $\log\big|(H_{[a,b]} - E)^{-1}(k,\ell)\big| < -\frac{1}{2}|\ell - k|\log\mu$ *provided* $|\ell - k| > \lambda^\vartheta$, $\lambda > \lambda_0(\vartheta, C)$.

Proof. Without loss of generality, assume $\ell > k$. Let

$$H_{[a,b;k,\ell]} = \begin{pmatrix} H_{[a,k-1]} & & \\ & H_{[k,\ell]} & \\ & & H_{[\ell+1,b]} \end{pmatrix}$$

$$f_{[a,b;k,\ell]}(E) = \det\big(H_{[a,b;k,\ell]} - E\big)$$

$\mathrm{rank}\big(H_{[a,b]} - H_{[a,b;k,\ell]}\big) \le 4$. Since $\log \mathrm{dist}(\mathrm{sp}\, H_{[a,b]}, E) > \log\mu > 0$, by Lemma B.3

$$\log|f_{[a,b;k,\ell]}(E)| - \log|f_{[a,b]}(E)| \le 8\log\|H_{[a,b;k,\ell]} - E\|$$

$$\begin{aligned}\log\big|(H_{[a,b]} - E)^{-1}(k,\ell)\big| &= \log\big|f_{[a,k-1]}(E)\big| + \log\big|f_{[\ell+1,b]}(E)\big| - \log\big|f_{[a,b]}(E)\big| \\ &= \log\big|f_{[a,b;k,\ell]}(E)\big| - \log\big|f_{[a,b]}(E)\big| - \log\big|f_{[k,\ell]}(E)\big| \\ &\le 8\log\big\|H_{[a,b;k,\ell]} - E\big\| - \log\big|f_{[k,\ell]}(E)\big| \\ &\le 8\log(2\lambda C + 2) - (\ell - k + 1)\log\mu \\ &< -\frac{1}{2}(\ell - k)\log\mu \qquad \text{provided } (\ell - k) > \lambda^\vartheta\end{aligned}$$

□

Lemma C.4. *If* $\mu = \frac{1}{2}\min_{j\neq j_0}|\lambda v(j) - E| > 2$, $|E| \leq \max|\lambda v(j)| = \lambda C$, $E \notin \mathrm{sp}H_{[a,b]}$. *Then*

$$\log\big|(H_{[a,b]} - E)^{-1}(k,\ell)\big| < -\frac{1}{2}|k-\ell|\log\mu + 8\big|\log\mathrm{dist}(\mathrm{sp}\,H_{[a,b]}, E)\big|$$

provided $|k-\ell| > \lambda^{\vartheta}$, $\lambda > \lambda_0(\vartheta, C)$.

Proof. Without loss of generality, assume $k < \ell$. If $j_0 < k < \ell$ or $k < \ell < j_0$, then

$$\begin{aligned}
\log\big|(H_{[a,b]} - E)^{-1}(k,\ell)\big| &= \log\big|f_{[a,k-1]}(E)\big| + \log\big|f_{[\ell+1,b]}(E)\big| - \log\big|f_{[a,b]}(E)\big| \\
&= \log\big|f_{[a,b;k,\ell]}(E)\big| - \log\big|f_{[a,b]}(E)\big| - \log\big|f_{[k,\ell]}(E)\big| \\
&\leq 8\log\big\|H_{[a,b;k,\ell]} - E\big\| + 8\big|\log\mathrm{dist}(\mathrm{sp}\,H_{[a,b]}, E)\big| - (\ell - k + 1)\log\mu \\
&< -\frac{1}{2}|k-\ell|\log\mu + 8|\log\mathrm{dist}(\mathrm{sp}\,H_{[a,b]}, E)|
\end{aligned}$$

Suppose $k \leq j_0 \leq \ell$. Let

$$\begin{aligned}
\widetilde{H}_{[a,b]}(E) &= \mathrm{diag}\big(\lambda v(a), \ldots, \lambda v(j_0 - 1), 2\mu + E, \lambda v(j_0+1), \ldots, \lambda v(b)\big) + T_{[a,b]} \\
\tilde{f}_{[a,b]}(E) &= \det\big(\widetilde{H}_{[a,b]}(E) - E\big)\ .
\end{aligned}$$

By Lemma C.3,

$$\log\big|(\widetilde{H}_{[a,b]}(E) - E)^{-1}(k,\ell)\big| \leq 8\log(2\lambda C + 2) - (\ell - k + 1)\log\mu$$

$\mathrm{rank}\big(\widetilde{H}_{[a,b]}(E) - H_{[a,b]}\big) = 1$. By Lemma B.3

$$\begin{aligned}
\log\big|\tilde{f}_{[a,b]}(E)\big| - \log\big|f_{[a,b]}(E)\big| &\leq 2\log\big\|\widetilde{H}_{[a,b]}(E) - E\big\| - 2\log\mathrm{dist}(\mathrm{sp}\,H_{[a,b]}, E) \\
&\leq 2\log(2\lambda C + 2) + 2\big|\log\mathrm{dist}(\mathrm{sp}\,H_{[a,b]}, E)\big|
\end{aligned}$$

$$\begin{aligned}
\log\big|(H_{[a,b]} - E)^{-1}(k,\ell)\big| &= \log\big|f_{[a,k-1]}(E)\big| + \log\big|f_{[\ell+1,b]}(E)\big| - \log\big|f_{[a,b]}(E)\big| \\
&= \log\big|(\widetilde{H}_{[a,b]}(E) - E)^{-1}(k,\ell)\big| + \log\big|\tilde{f}_{[a,b]}(E)\big| - \log\big|f_{[a,b]}(E)\big| \\
&\leq 10\log(2\lambda C + 2) - (\ell - k + 1)\log\mu + 2\big|\log\mathrm{dist}(\mathrm{sp}\,H_{[a,b]}, E)\big| \\
&< -\frac{1}{2}|k-\ell|\log\mu + 8\big|\log\mathrm{dist}(\mathrm{sp}\,H_{[a,b]}, E)\big|
\end{aligned}$$

□

Lemma C.5. *Let* $K \in M_{n\times n}(\mathbb{C})$. *Then* $\big|\log|\det(I+K)|\big| \lesssim n\|K\|$ *provided* $\|K\| < \frac{1}{2}$.

Proof. $\mathrm{sp}(I+K) \subset \{z \in \mathbb{C} : |z-1| \leq \|K\|\}$, $\det(I+K) = \Pi\lambda$ where the product runs over the eigenvalues of $I+K$ with corresponding algebraic multiplicity. So

$$\big(\frac{1}{2}\big)^n < \big(1 - \|K\|\big)^n \leq |\det(I+K)| \leq \big(1 + \|K\|\big)^n$$

if $\|K\| < \frac{1}{2}$. Therefore

$$\big|\log|\det(I+K)|\big| \lesssim n\|K\|\ .$$

□

Lemma C.6.

$$\left|\log\left|\det\begin{pmatrix} a_1 & 1 & & & \\ 1 & a_2 & 1 & & \\ & 1 & \ddots & & \\ & & & & 1 \\ & & & 1 & a_n \end{pmatrix}\right| - \sum_{j=1}^{n}\log|a_j|\right| \lesssim \frac{n}{\min|a_j|}$$

provided $\min|a_j| > 2$.

Proof.

$$\det\begin{pmatrix} a_1 & 1 & & & \\ 1 & a_2 & 1 & & \\ & 1 & \ddots & & \\ & & & & 1 \\ & & & 1 & a_n \end{pmatrix} = \left(\prod_{j=1}^{n} a_j\right)\begin{pmatrix} 1 & \frac{1}{a_1} & & \\ \frac{1}{a_2} & 1 & \frac{1}{a_2} & \\ & \frac{1}{a_3} & \ddots & \\ & & \ddots & \end{pmatrix}$$

$$\left|\log\left|\det\begin{pmatrix} a_1 & 1 & & & \\ 1 & a_2 & 1 & & \\ & 1 & \ddots & & \\ & & & & 1 \\ & & & 1 & a_n \end{pmatrix}\right| - \sum_{j=1}^{n}\log|a_j|\right|$$

$$= \left|\log\left|\det\left[I + \begin{pmatrix} 0 & \frac{1}{a_1} & \\ \frac{1}{a_2} & 0 & \frac{1}{a_2} \\ & & \ddots \end{pmatrix}\right]\right|\right| \lesssim \frac{n}{\min|a_j|}$$

provided $\min|a_j| > 2$. □

Lemma C.7. *If* $\min\limits_{a\le k\le b}|\lambda v(k) - E| \ge \lambda^A$, $a \le i < j < b$. *Then*

$$\left|\frac{f_{[a,i]}(E)f_{[j,b]}(E)}{f_{[a,b]}(E)}\right| \le \lambda^{-A|j-i-1|}$$

provided $n = b - a + 1 \lesssim \lambda^A$.

Proof.

$$\log\left|f_{[a,i]}(E)\right| \le \sum_{k=a}^{i}\log|\lambda v(k) - E| + n\lambda^{-A}$$

$$\log\left|f_{[j,b]}(E)\right| \le \sum_{k=j}^{b}\log|\lambda v(k) - E| + n\lambda^{-A}$$

$$\log\left|f_{[a,b]}(E)\right| \ge \sum_{k=a}^{b}\log|\lambda v(k) - E| - n\lambda^{-A}$$

$$\log\Big|\frac{f_{[a,i]}(E)f_{[j,b]}(E)}{f_{[a,b]}(E)}\Big| \le -\sum_{i<k<j}\log|\lambda v(k)-E| + 3n\lambda^{-A}$$
$$\le -|j-i-1|\log\lambda^A + 3n\lambda^{-A}$$

□

Corollary C.8. $\big|(H_{[a,b]}-E)^{-1}(i,j)\big| \le \lambda^{-A(|i-j|+1)}$ *if* $\min\limits_{a\le k\le b}|\lambda v(k)-E| \ge \lambda^A$, $n \lesssim \lambda^A$.

Lemma C.9. *Suppose* $[a,b]=\bigcup\limits_{k=1}^{K}[a'_k,b'_k]$, $a'_1=a$, $b'_K=b$, $a'_k<b'_{k-1}<a'_{k+1}<b'_k$ *for* $k=2,3,\dots,K-1$, $b'_{k-1}-a'_k>n^\tau$ *for* $k=2,3,\dots,K$. *Assume that*

$$\big|(H_{[a',b']}-E)^{-1}(i,j)\big| \le \exp(-\gamma|i-j|)$$

when $|i-j|\ge\frac{1}{2}n^\tau$ *for all* k. *Then* $E\notin \operatorname{sp} H_{[a,b]}$ *provided* $n>n_0(\tau,\gamma)$.

Proof. Suppose, for contradiction, that $H_{[a,b]}\varphi=E\varphi$, $\|\varphi\|=1$. Let

$$x_k=\lfloor\frac{1}{2}(a'_k+b'_{k-1})\rfloor\ ,\quad k=2,\dots,K\ .$$

Then $a'_k<x_k<x_{k+1}<b_k$. For any $j\in[x_k,x_{k+1}]$, by Poisson's formula

$$\varphi(j)=\big(H_{[a'_k,b'_k]}-E\big)^{-1}(j,a'_k)\varphi(a'_k-1)+\big(H_{[a'_k,b'_k]}-E\big)^{-1}(j,b'_k)\varphi(b'_k+1)$$
$$|\varphi(j)|\le 2\exp\big[-\frac{\gamma}{2}n^\tau\big]$$

since $|j-a'_k|\ge\frac{1}{2}n^\tau$, $|j-b'_k|\ge\frac{1}{2}n^\tau$.

Similarly, for $j\in[a,x_2]\cup[x_k,b]$,

$$|\varphi(j)|\le\exp\big[-\frac{\gamma}{2}n^\tau\big]$$

Since

$$[a,b]=\Big(\bigcup_{k=2}^{K-1}[x_k,x_{k+1}]\Big)\cup[a,x_2]\cup[x_k,b]$$

$$1=\sum_{j=a}^{b}|\varphi(j)|^2\le n\big[4\exp(-\gamma n^\tau)\big]$$

But this is false for sufficiently large n. □

Lemma C.10. *Suppose* $|E| \le \lambda \max_{a\le n\le c}|v(n)| =: \lambda C_0$, $\operatorname{dist}\big(\operatorname{sp} H_{[a,c]}, E\big) = \kappa$. *If* $a < b < c$ *then*

$$\log\big\|M_{[a,b]}(E)\big\| + \log\big\|M_{[b+1,c]}(E)\big\| - \log\big\|M_{[a,c]}(E)\big\| \le 20\big[\log(\lambda C_0) - \log\kappa\big]$$

provided $\lambda C_0 \gg 1$.

Proof.

$$\begin{aligned}
\|M_{[a,c]}(E)\| &\ge |f_{[a,c]}(E)| \\
\|M_{[a,b]}(E)\| &\le |f_{[a,b]}(E)| + |f_{[a,b-1]}(E)| + |f_{[a+1,b]}(E)| + |f_{[a+1,b-1]}(E)| \\
\|M_{[b+1,c]}(E)\| &\le |f_{[b+1,c]}(E)| + |f_{[b+1,c-1]}(E)| + |f_{[b+2,c]}(E)| + |f_{[b+1,c-1]}(E)|
\end{aligned}$$

$$\frac{\|M_{[a,b]}(E)\|\,\|M_{[b+1,c]}(E)\|}{\|M_{[a,c]}(E)\|} \le \frac{\big(|f_{[a,b]}(E) + \cdots\big)\big(|f_{[b+1,c]}(E)| + \cdots\big)}{|f_{[a,c]}(E)|} \tag{C.1}$$

Expand the numerator, we get 16 terms. One of the terms is

$$\begin{aligned}
&|f_{[a+1,b-1]}(E)|\,|f_{[b+2,c-1]}(E)| \\
&= \left|\det\begin{pmatrix} I_1 & & & & \\ & H_{[a+1,b-1]} & & & \\ & & I_2 & & \\ & & & H_{[b+2,c-1]} & \\ & & & & I_1 \end{pmatrix}\right| \\
&=: |\det \tilde{H}|
\end{aligned}$$

$\operatorname{rank}\big(H_{[a,c]} - \tilde{H}\big) = 8$. By Lemma B.3

$$\begin{aligned}
&\log|f_{[a+1,b-1]}(E) f_{[b+1,c-1]}(E)| - \log|f_{[a,c]}(E)| \\
&\le 16\log\|\tilde{H}\| - 16\log\operatorname{dist}(\operatorname{sp} H_{[a,c]}, E) \\
&\Longrightarrow \frac{|f_{[a+1,b-1]}(E)|\,|f_{[b+1,c-1]}(E)|}{|f_{[a,c]}(E)|} \le \left(\frac{2\lambda C_0 + 2}{\kappa}\right)^{16}
\end{aligned}$$

Same estimate holds for other terms in (C.1).

$$\frac{\|M_{[a,b]}(E)\|\,\|M_{[b+1,c]}(E)\|}{\|M_{[a,c]}(E)\|} \le 16\left(\frac{2\lambda C_0 + 2}{\kappa}\right)^{16}$$

Hence

$$\log\|M_{[a,b]}(E)\| + \log\|M_{[b+1,c]}(E)\| - \log\|M_{[a,c]}(E)\| \le \log 16 + 16\log\left(\frac{2\lambda C_0 + 2}{\kappa}\right)$$

□

D Avalanche Principle

Proposition D.1. ***(Goldstein, Schlag)****[GS1] Let $A_1, \dots, A_n$ be a sequence of 2×2 matrices whose determinants satisfy*

$$\max_{1 \le j \le n} |\det A_j| \le 1 \ .$$

Suppose that

$$\min_{1 \le j \le n} \|A_j\| \ge \mu > n$$

and

$$\max_{1 \le j < n} \big[\log \|A_{j+1}\| + \log \|A_j\| - \log \|A_{j+1} A_j\|\big] < \frac{1}{2} \log \mu \ .$$

Then

$$\big| \log \|A_n \dots A_1\| + \sum_{j=2}^{n-1} \log \|A_j\| - \sum_{j=1}^{n-1} \log \|A_{j+1} A_j\| \big| \lesssim \frac{n}{\mu} \ .$$

Proposition D.2. ***(Goldstein, Schlag)****[GS2] For $a_0 < a_1 < \dots < a_n$, if the monodromy matrices $M_{[a_{j-1}, a_j]} =: A_j$ satisfy the conditions of Proposition 1, then*

$$\Big| \log \|M_{[a_0, a_n]}\| + \sum_{j=2}^{n-1} \log \|M_{[a_{j-1}, a_j]}\| - \sum_{j=2}^{n} \log \|M_{[a_{j-2}, a_j]}\| \Big| \lesssim \frac{n}{\mu} \ .$$

Moreover, if $A_j = M_{[a_{j-1}, a_j]}$ for $1 < j < n$

$$A_1 = M_{[a_0, a_1]} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \textit{and} \quad A_n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} M_{[a_{n-1}, a_n]}$$

satisfy the conditions of the proposition, then

$$\Bigg| \log f_{[a_0, a_n]}(E) + \sum_{j=2}^{n-1} \log \|M_{[a_{j-1}, a_j]}\| - \sum_{j=3}^{n-1} \log \|M_{[a_{j-2}, a_j]}\| \\ - \log \left\| M_{[a_0, a_1]} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\| - \log \left\| \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} M_{[a_{n-1}, a_n]} \right\| \Bigg| \lesssim \frac{n}{\mu}$$

E Implicit Function

Definition. $D_0 \subset \mathbb{R}^2$ is n–simple if $D_0 = \bigcup_{j=1}^{J_0} D_j$, where D_j are rectangles, $J_0 \le n$.

Proposition E.1. *Let $f(x, y)$ be C^1 on $(a, b) \times (c, d) =: D$. Assume $\partial_y f(x, y) \ge \mu > 0$, $\big|\partial_x f(x, y)\big| \le \kappa$ forall $(x, y) \in D$. Given $\rho > 0$, there exists n–simple set $D_0 \subset D$, $n = 2m$, $|D \setminus D_0| \le (b - a)\big(\frac{2\rho}{\mu} + \frac{1}{m} \frac{\kappa}{\mu}\big)$ such that $|f(x, y)| \ge \rho$ on D_0.*

Proof. For $x \in (a,b)$, define $s^+(x) = \{y \in (c,d) : f(x,y) \geq \rho\}$, $s^-(x) = \{y \in (c,d) : f(x,y) \leq -\rho\}$ and

$$y^+(x) = \begin{cases} d & \text{if } s^+ = \emptyset \\ \inf s^+ & s^+ \neq \emptyset \end{cases}$$

$$y^-(x) = \begin{cases} c & \text{if } s^- = \emptyset \\ \sup s^- & s^- \neq \emptyset \end{cases}$$

Let

$$J_k = \left(a + \frac{k-1}{m}(b-a),\ a + \frac{k}{m}(b-a)\right), \qquad x_k = a + \frac{2k-1}{2m}(b-a), \qquad k = 1, 2, \ldots, m,$$

$$y_{k,1} = \min\left\{y^+(x_k) + \frac{1}{2m}\frac{\kappa}{\mu}, d\right\}$$

$$y_{k,2} = \max\left\{y^-(x_k) - \frac{1}{2m}\frac{\kappa}{\mu}, c\right\}$$

Take

$$D_{k,1} = J_k \times (y_{k,1}, d)$$

$$D_{k,2} = J_k \times (c, y_{k,2})$$

$$D_0 = \bigcup_k (D_{k,1} \cup D_{k,2})$$

Then $|f(x,y)| \geq \rho$ on D_0 and

$$0 < y_{k,1} - y_{k,2} \leq \frac{2\rho}{\mu} + \frac{1}{m}\frac{\kappa}{\mu} \Longrightarrow |D \setminus D_0| \leq (b-a)\big(\frac{2\rho}{\mu} + \frac{1}{m}\frac{\kappa}{\mu}\big)$$

□

Lemma E.2. *Suppose $f : [a,b] \to \mathbb{R}$ is C^2, monotone. If $|f''(x)| \geq C$ for all $x \in (a,b)$, then*

$$\operatorname{mes}\{x \in [a,b] : |f(x)| \leq \varepsilon\} \lesssim \big(\frac{\varepsilon}{C}\big)^{1/2}.$$

Proof. Without loss of generality, assume f is increasing. (Otherwise, consider $-f$.)

Let $(\tilde{a}, \tilde{b}) \subset [\bar{a}, \bar{b}] = \big\{x \in [a,b] : |f(x)| \leq \varepsilon\big\}$. If $f''(x) \geq C$ then there is $\tilde{c} \in (\tilde{a}, \tilde{b})$ such that

$$f(\tilde{b}) - f(\tilde{a}) = f'(\tilde{a})(\tilde{b} - \tilde{a}) + \frac{1}{2}f''(\tilde{c})(\tilde{b} - \tilde{a})^2$$

$$2\varepsilon \geq \frac{1}{2}C(\tilde{b} - \tilde{a})^2$$

$$\tilde{b} - \tilde{a} \lesssim \big(\frac{\varepsilon}{C}\big)^{1/2}.$$

If $f''(x) \le -C$ then

$$f(\tilde{a}) - f(\tilde{b}) = f'(\tilde{b})(\tilde{a}-\tilde{b}) + \frac{1}{2}f''(\tilde{c})(\tilde{b}-\tilde{a})^2$$

$$f(\tilde{b}) - f(\tilde{a}) = f'(\tilde{b})(\tilde{b}-\tilde{a}) + \frac{1}{2}\big[-f''(\tilde{c})\big](\tilde{b}-\tilde{a})^2$$

$$2\varepsilon \ge \frac{1}{2}C(\tilde{b}-\tilde{a})^2$$

$$\tilde{b}-\tilde{a} \lesssim \big(\frac{\varepsilon}{C}\big)^{1/2}$$

$\operatorname{mes}\big\{x \in [a,b] : |f(x)| \le \varepsilon\big\} = \sup(\tilde{b}-\tilde{a}) \lesssim \big(\frac{\varepsilon}{C}\big)^{1/2}$. □

Corollary E.3. *Suppose $f : [a,b] \to \mathbb{R}$ is C^3, such that $|f'(x)| + |f''(x)| \ge C$, $|f''(x)| + |f''(x)| \le \kappa$. Then for $\varepsilon \lesssim C \lesssim \kappa(b-a)$, one has*

$$\operatorname{mes}\{x \in [a,b] : |f(x)| \le \varepsilon\} \lesssim \frac{\kappa}{C}\big(\frac{\varepsilon}{C}\big)^{1/2}(b-a).$$

Proof. Let $x_i = a + \frac{b-a}{N}i$, $i = 0,1,2,\dots,N$ where $\frac{C}{8\kappa} \le \frac{b-a}{N} \le \frac{C}{4\kappa}$. On each x_i, either $|f'(x_i)| \ge \frac{C}{2}$ or $|f''(x_i)| \ge \frac{C}{2}$.

If $|f''(x_i)| \ge \frac{C}{2}$ then $|f''(x)| \ge \frac{C}{4}$ for all $x \in [x_{i-1}, x_i]$ since $\operatorname{sgn} f''$ does not change in $[x_{i-1}, x_i]$, f has at most two monotonicity intervals in $[x_{i-1}, x_i]$. Hence
$\operatorname{mes}\{x \in [x_{i-1}, x_i] : |f(x)| \le \varepsilon\} \lesssim \big(\frac{\varepsilon}{C}\big)^{1/2}$.

If $|f'(x_i)| \ge \frac{C}{2}$ then $|f'(x)| \ge \frac{C}{4}$ for all $x \in [x_{i-1}, x_i]$. So
$\operatorname{mes}\{x \in [x_{i-1}, x_i] : |f(x)| \le \varepsilon\} \le \frac{2\varepsilon}{C/4} \lesssim \big(\frac{\varepsilon}{C}\big)^{1/2}$. Therefore

$$\operatorname{mes}\{x \in [a,b] : |f(x)| \le \varepsilon\} \lesssim N\big(\frac{\varepsilon}{C}\big)^{1/2} \lesssim \frac{\kappa}{C}\big(\frac{\varepsilon}{C}\big)^{1/2}(b-a)\ .$$

□

Let $F_\ell(x,y)$ be C^3 on $(a_\ell, b_\ell) \times (c,d)$, where $(a_\ell, b_\ell),\ (c,d) \subset \mathbb{T}$, $\ell \in \{1,2\}$. Suppose

$$0 < \mu \le |\partial_y F_1(x,y)|$$

$$|\partial_y F_\ell|, |\partial_x F_\ell|, |\partial_{xx} F_\ell|, |\partial_{yy} F_\ell|, |\partial_{xy} F_\ell| \le \kappa$$

Since $\partial_y F_1 \ne 0$, we can define $y(.,t)$ such that $F_1\big(x, y(x,t)\big) = t$ for $x \in \pi_1\big(F_1^{-1}(t)\big)$. Furthermore,

$$|\partial_x y(x,t)| \le \frac{\kappa}{\mu}, \qquad |\partial_{xx} y(x,t)| \le \frac{4\kappa^3}{\mu^2}$$

Let $x_i = a_1 + \frac{i}{N}$, $i = 0,1,2,\dots,\lfloor N(b_1-a_1)\rfloor =: N_1$. Define

$$\xi_i = \min\{|\partial_x y(x,t)| : (x,t) \in [x_{i-1}, x_i] \times [-\varepsilon, \varepsilon]\}\ .$$

Suppose $\xi_i \le \varrho$. Then there is $t \in [-\varepsilon, \varepsilon]$ such that

$$\operatorname{mes}\big\{y\big(\big[x_{i-1}, x_i\big], t\big)\big\} \le \frac{\varrho}{N} + \frac{4\kappa^3}{\mu^3 N^2}\ .$$

Since $|y(x,s) - y(x,t)| \le \frac{|s-t|}{\mu}$,

$$\bigcup_{t\in[-\varepsilon,\varepsilon]} \big\{y\big(\big[x_{i-1}, x_i\big], t\big)\big\} \subset \big[\tilde{c}_i, \tilde{d}_i\big]$$

$\tilde{d}_i - \tilde{c}_i \le \frac{\varrho}{N} + \frac{4\kappa^3}{\mu^2 N^2} + \frac{2\varepsilon}{\mu}$. Write $(c,d) \setminus \bigcup_{i:\xi_i\le\varrho} [\tilde{c}_i, \tilde{d}_i] = \bigcup_{j=1}^{J} (c_j, d_j)$, $J \le N_1 + 1$. On each (c_j, d_j), define the inverse functions of y, i.e. $F_1\big(\phi_{jk}(y,t), y\big) = t$ for $y \in (c_j, d_j) \cap \pi_2\big(F_1^{-1}(t)\big)$, $k = 1, 2, \ldots, K_j$, $K_j \le N_1 + \frac{b_1 - a_1}{d-c}\,\frac{\kappa}{\mu} \le 2N_1$ provided $N \ge \frac{1}{d-c}\,\frac{\kappa}{\mu}$.

Lemma E.4. *Let $y_0 \in (c_j + \frac{2\varepsilon}{\mu}, d_j - \frac{2\varepsilon}{\mu})$. Then $|\phi_{jk}(y_0, s) - \phi_{jk}(y_0, t)| \le \frac{2\varepsilon}{\mu\varrho}$ for any $s, t \in [-\varepsilon, \varepsilon]$ such that $\phi_{jk}(y_0, s)$ and $\phi_{jk}(y_0, t)$ are defined.*

Proof. Let $a = \phi_{jk}(y_0, s)$, $b = \phi_{jk}(y_0, t)$. Then $F_1(a, y_0) = s$. Since $|\partial_y F_1(x,y)| > \mu$ and $|s - t| \le 2\varepsilon$, there is $y_1 \in [y_0 - \frac{2\varepsilon}{\mu}, y_0 + \frac{2\varepsilon}{\mu}] \subset (c_j, d_j)$ such that $F_1(a, y_1) = t$. Hence, $y(x,t)$ is defined for all x between a and b. Therefore, there exists $\tilde{x}$ between a and b such that

$$\frac{2\varepsilon/\mu}{|\phi_{jk}(y_0,s) - \phi_{jk}(y_0,t)|} \ge \frac{|y(a,s) - y(a,t)|}{|\phi_{jk}(y_0,s) - \phi_{jk}(y_0,t)|} = |\partial_x y(\tilde{x}, t)| > \varrho$$

□

Lemma E.5. *Suppose $b_1 - a_1 > \varepsilon^{1/100}$ and $d - c > \varepsilon^{1/100}$. If*

$$|\partial_x F_2(x,y)| + |\partial_{xx} F_2(x,y)| \ge C > \varepsilon^{1/100}$$

for all $(x,y) \in (a_2, b_2) \times (c,d)$, then for $\kappa \le \varepsilon^{-1/100}$, $\mu \ge \varepsilon^{1/100}$, one has

$$\operatorname{mes}\left\{y \in (c,d) : \begin{array}{l} \exists\ x \in (a_1, b_1),\ n > \varepsilon^{-1/10} \\ \text{such that } |F_1(x,y)| \le \varepsilon,\ |F_2(x + ny, y)| \le \varepsilon \end{array}\right\} \lesssim \varepsilon^{\vartheta}(d-c)(b_1 - a_1)\ .$$

Proof. Choose $\varrho = \varepsilon^{1/25}$, $N \asymp \varepsilon^{-1/10}$. Fix jk, t, n. Let $f(y) = F_2\big(\phi_{jk}(y,t) + ny, y\big)$. Then

$$\begin{aligned} f'(y) &= (\partial_x F_2)(\partial_y \phi_{jk} + n) + \partial_y F_2 \\ f''(y) &= (\partial_{xx} F_2)(\partial_y \phi_{jk} + n)^2 + (\partial_x F_2)(\partial_{yy}\phi_{jk}) + 2(\partial_{xy} F_2)(\partial_y \phi_{jk} + n) + \partial_{yy} F_2 \end{aligned}$$

for all y where f is defined.

$$|f'(y)|+|f''(y)| \;\geq\; \begin{cases} \frac{C}{2}(n-\varrho^{-1})-\kappa & \text{if } |\partial_x F_2|\geq \frac{C}{2} \\ \frac{C}{2}(n-\varrho^{-1})^2-\kappa\frac{4\kappa^3}{\mu^2\varrho^3}-2\kappa(\kappa+n)-\kappa & \text{if } |\partial_{xx} F_2|\geq \frac{C}{2} \end{cases}$$
$$\geq \;\frac{nC}{4}$$

By Corollary E.3,

$$\operatorname{mes}\left\{y\in(c_j+\frac{2\varepsilon}{\mu},d_j-\frac{2\varepsilon}{\mu}) : |f(y)|\leq \varepsilon^{1/2}\right\} \lesssim \frac{\kappa}{nC}\big(\frac{\varepsilon^{1/2}}{nC}\big)^{1/2}(d_j-c_j)$$

Also, by Lemma E.4,

$$|F_2(\phi_{jk}(y,s)+ny,y)-F_2(\phi_{jk}(y,t)+ny,y)|\leq \kappa|\phi_{jk}(y,s)-\phi_{jk}(y,t)|\leq 2\frac{\kappa\varepsilon}{\mu\varrho}$$

for all s such that $\phi_{jk}(y,s)$ is defined. If $|f(y)|>\varepsilon^{1/2}$ then

$$|F_2\big(\phi_{jk}(y,s)+ny,y\big)|>|f(y)|-2\frac{\kappa\varepsilon}{\mu\varrho}>\varepsilon$$

Hence,

$$\begin{aligned} &\operatorname{mes}\left\{y\in(c,d): \begin{array}{l}\exists\ x\in(a_1,b_1),\ n>\varepsilon^{-1/10} \\ \text{such that } |F_1(x,y)|\leq\varepsilon,\ |F_2(x+ny,y)|\leq\varepsilon\end{array}\right\} \\ &\lesssim (N_1+1)(2N_1)\frac{\kappa}{C}\Big(\frac{\varepsilon^{1/2}}{C}\Big)^{1/2}(d-c)\sum_{n>\varepsilon^{-1/10}}\frac{1}{n^{3/2}}+N\Big(\frac{\varrho}{N}+\frac{4\kappa^3}{\mu^2N^2}+\frac{6\varepsilon}{\mu}\Big) \\ &\lesssim \varepsilon^{\vartheta}(d-c)(b_1-a_1) \end{aligned}$$

□

Fix $n_1\ll n_2$. Let

$$F_1(x,\omega)=E_1(x,\omega)-E_2(x+n_1\omega,\omega) \qquad \text{for } (x,\omega)\in(a_1,b_1)\times(c,d)$$
$$F_2(\tilde{x},\omega)=E_2(\tilde{x},\omega)-E_3(\tilde{x}+n_2\omega,\omega) \qquad \text{for } (\tilde{x},\omega)\in(a_2,b_2)\times(c,d)$$

where $|\partial_x E_\ell(x,\omega)|+|\partial_{xx}E_\ell(x,\omega)|>\delta$. Also, assume

$$|\partial_x E_\ell|,\ |\partial_\omega E_\ell|,\ |\partial_{xx}E_\ell|,\ |\partial_{x\omega}E_\ell|,\ |\partial_{\omega\omega}E_\ell|,\ |\partial_{xxx}E_\ell|,\ |\partial_{xx\omega}E_\ell|\leq\kappa$$

Theorem E.6. *For $\varepsilon\ll n_1\delta$,*

$$\operatorname{mes}\left\{\omega\in(c,d): \begin{array}{l}\exists\ x\in(a_1,b_1),\ x+n_1\omega\in(a_2,b_2) \\ \textit{such that } |F_1(x,\omega)|\leq\varepsilon,\ |F_2(x+n\omega,\omega)|\leq\varepsilon\end{array}\right\}\lesssim\varepsilon^{\vartheta}(d-c)(b_1-a_1)\ .$$

Proof. Let $\tilde{x}_i = a_1 + \frac{n_1\delta}{4\kappa} i,\ \tilde{\omega}_j = c + \frac{n_1\delta}{4\kappa} j,\ D_{ij} = [\tilde{x}_{i-1}, \tilde{x}_i] \times [\tilde{\omega}_{j-1}, \tilde{\omega}_j]$. Either

(1) $|\partial_\omega F_1| > \frac{n_1\delta}{4}$ for all $(x,\omega) \in D_{ij}$; or

(2) $|\partial_{\omega\omega} F_1| > \frac{n_1\delta}{4}$ for all $(x,\omega) \in D_{ij}$.

Choose μ so that $\left(\frac{n_1\delta}{4}\right)^{100} > \mu > \varepsilon^{1/100}$. In case (1),

$$\operatorname{mes}\left\{\omega \in (\tilde{\omega}_{j-1}, \tilde{\omega}_j) : \begin{array}{l} \exists x \in (\tilde{x}_{i-1}, \tilde{x}_i) \\ |F_1(x,\omega)| \le \varepsilon,\ |F_2(x + n\omega, \omega)| \le \varepsilon \end{array}\right\} \lesssim \varepsilon^{\vartheta}(\tilde{\omega}_j - \tilde{\omega}_{j-1})(x_i - x_{i-1})$$

In case (2), consider $G_1 = \partial_\omega F_1$, $G_2 = F_2$. Applying Lemma E.5 to G_1, G_2 gives

$$\begin{aligned} \operatorname{mes}\Omega_{ij} &:= \operatorname{mes}\left\{\omega \in (\tilde{\omega}_{j-1}, \tilde{\omega}_j) : \begin{array}{l} \exists x \in (\tilde{x}_{i-1}, \tilde{x}_i) \\ |G_1(x,\omega)| \le \mu\ ,\ |G_2(x + n_1\omega, \omega)| \le \mu \end{array}\right\} \\ &\lesssim \mu^{\vartheta}(\tilde{\omega}_j - \tilde{\omega}_{j-1})(x_i - x_{i-1}) \end{aligned}$$

Also,

$$\operatorname{mes}\left\{\begin{array}{l} \omega \in (\tilde{\omega}_{j-1}, \tilde{\omega}_j) \setminus \Omega_{ij} : \exists x \in (\tilde{x}_{i-1}, \tilde{x}_i) \\ |F_1(x,\omega)| \le \varepsilon\ ,\ |F_2(x + n_1\omega, \omega)| \le \varepsilon \end{array}\right\} \lesssim \varepsilon^{\vartheta}(\tilde{\omega}_j - \tilde{\omega}_{j-1})(x_i - x_{i-1})\ .$$

Hence,

$$\begin{aligned} &\operatorname{mes}\{\omega \in (c,d) : \exists x \in (a_1, b_1), |F_1(x,\omega)| \le \varepsilon,\ |F_2(x + n_1\omega, \omega)| \le \varepsilon\} \\ &\lesssim \varepsilon^{\vartheta}(d-c)(b_1 - a_1) \end{aligned}$$

□

F Volume of Hyperplane

Given $0 \neq a = (a_1, \ldots, a_N) \in \mathbb{R}^N$, $C \in \mathbb{R}$. Let $P(a,C) = \left\{x \in \mathbb{R}^N, a \cdot x = C\right\}$. Suppose $a_i \neq 0$. For any $\delta > 0$ define

$$P_{i,\delta}(a,C) = \{x \in P(a,C) : |x_j| \le \delta \quad \text{for } j \neq i\}\ .$$

Then $\mathrm{Vol_{N-1}}\big(\mathrm{P_{i,\delta}(a,C)}\big) = \frac{\|a\|}{|a_i|}(2\delta)^{N-1}$.

Lemma F.1. *Suppose $a_j \ge 0$ for all $j = 1, \ldots, N$, $\Sigma a_j = 1$. For any $C \in \mathbb{R}$, $\delta, \varepsilon > 0$, let*

$$P_\delta(a, C, \varepsilon) = \{\xi \in (\xi_1, \ldots, \xi_N) : |\xi_j| \le \delta \quad \textit{for all } j,\ |a \cdot \xi - C| \le \varepsilon\}$$

Then $(2\delta)^{-N} \operatorname{mes} P_\delta(a, C, \varepsilon) \le N\frac{\varepsilon}{\delta}$.

Proof. Write $\xi = t\frac{a}{\|a\|} + \eta$ where $\eta \cdot a = 0$. Then $\xi \cdot a = t\|a\|$.

From the hypothesis, there is $|a_i| \ge \frac{1}{N}$

$$\begin{aligned} \xi \in P_\delta(a,C,\varepsilon) &\Longrightarrow t \in \Big[\frac{C}{\|a\|} - \frac{\varepsilon}{\|a\|}, \frac{C}{\|a\|} + \frac{\varepsilon}{\|a\|}\Big] =: J \\ \operatorname{mes} P_\delta(a,C,\varepsilon) &\le \int_J P_{i,\delta}(a,y)dy = \frac{2\varepsilon}{|a_i|}(2\delta)^{N-1} \end{aligned}$$

□

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