LOCALIZATION NEAR FLUCTUATION BOUNDARIES VIA FRACTIONAL MOMENTS AND APPLICATIONS

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Abstract. We present a new, short, self-contained proof of localization properties of multi-dimensional continuum random Schrödinger operators in the fluctuation boundary regime. Our method is based on the recent extension of the fractional moment method to continuum models in [2], but does not require the random potential to satisfy a covering condition. Applications to random surface potentials and potentials with random displacements are included.

1. Introduction

1.1. **Motivation.** We are concerned here with proving localization properties of multi-dimensional continuum random Schrödinger operators in the fluctuation boundary regime.

Such results were first found via the method of multiscale analysis, which had been developed in the 80s to handle lattice models and was later extended to the continuum (for a rather complete history and list of references on multiscale analysis see [30] and for some of the more recent developments [12]).

Later, the fractional moment method was developed [3] as an alternative approach to the same problem, also initially for lattice models. It leads to a stronger form of dynamical localization than multiscale analysis (see [1, 4]) and has provided much shorter and more transparent proofs in the lattice case, for example [13].

It was recently shown in [2] that all the main features of the fractional moment approach also apply to continuum random Schrödinger operators. This extension required substantial new input from operator theory and harmonic analysis. The paper [2] provides a framework of necessary and sufficient criteria for localization in terms of fractional moment bounds, which can be verified for a rather broad range of regimes.

One of our goals here is to complement the general framework from [2] by focusing exclusively on presenting a short and self-contained proof of localization properties via fractional moments for one specific regime, where the technical effort remains minimal.

For this we pick a fairly general setting we label the *fluctuation boundary regime*. This is described by a random Schrödinger operator of Anderson-type in $L^2(\mathbb{R}^d)$, where our approach allows for quite arbitrary background potentials and geometries of the random impurities, provided the ground state energy is induced by rare events (fluctuations) and therefore sensitive

Date: September 16, 2005.

to changes in the random parameters. The goal is to prove localization in the vicinity of the bottom of the spectrum. Of course, various versions of the fluctuation boundary regime have been studied in many works and we borrowed the term from [27].

Another motivation for our work is that we want to extend the fractional moment method to situations where the random potential does not satisfy a covering condition, i.e. where the individual impurity potentials have small supports which do not cover all of $\mathbb{R}^d$. This condition, which was required for the technical approach to the continuum found in [2], is not natural in the fluctuation boundary regime and should not be needed there as has already been verified via multiscale analysis. Particularly interesting examples are random surface potentials which act in a small portion of space only. Nevertheless, they lead to a fluctuation boundary by creating new "surface spectrum" below the "bulk spectrum".

In our main result, Theorem 1 below, the fluctuation boundary regime will be described in form of an abstract condition. For random surface potentials, which will be discussed as an application, this condition follows in an appropriate setting from a result proven in [24] in order to derive Lifshits tails. Another application concerns models with additional random displacements as were originally studied in [10].

Let us confess that we require absolutely continuous distribution of random couplings. While it might be possible to relax this to Hölder continuous distribution (as has been done in the lattice case, e.g. [4]), the fractional moment method is so far less flexible in that respect than the multiscale technique. In particular, see the variant of multiscale analysis adapted to Bernoulli-Anderson models recently developed in [6] and an application of similar ideas to Poisson models announced in [11].

1.2. **Results.** Let us now describe our results in more detail after introducing some notation: On $\mathbb{R}^d$ we often consider the supremum norm $|x| := \max_{i=1,\dots,d}|x_i|$ and write

$$\Lambda_r(x) := \left\{ y \in \mathbb{R}^d : |x-y| < \frac{r}{2} \right\}$$

for the d-dimensional cube with sidelength r centered at x. For an open set $G \subset \mathbb{R}^d$ we denote the restriction of the Schrödinger operator H to $L^2(G)$ with Dirichlet boundary conditions by H^G. In our results we assume $d \leq 3$ and rely upon the following assumptions, which guarantee self-adjointness and lower semi-boundedness of all the Schrödinger operators appearing in this paper:

(A1) The background potential $V_0 \in L^2_{\mathrm{loc,unif}}(\mathbb{R}^d)$ is real-valued, $H_0 := -\Delta + V_0$.
(A2) The set $\mathcal{I} \subset \mathbb{R}^d$, where the random impurities are located, is uniformly discrete, i.e., $\inf\{|\alpha - \beta| : \alpha \neq \beta \in \mathcal{I}\} =: r_{\mathcal{I}} > 0$.
(A3) The random couplings η_α, $\alpha \in \mathcal{I}$, are independent random variables supported in $[0, \eta_{\max}]$ for some $\eta_{\max} > 0$ and with absolutely continuous distribution of bounded density ρ_α with a uniform bound $\sup_\alpha \|\rho_\alpha\|_\infty =: M_\rho < \infty$.

The single site potentials U_α, $\alpha \in \mathcal{I}$ satisfy

$$c_U \chi_{\Lambda_{r_U}(\alpha)} \le U_\alpha \le C_U \chi_{\Lambda_{R_U}(\alpha)}$$

for all α with $c_U, C_U, r_U, R_U > 0$ independent of α.

$$V_\omega(x) = \sum_{\alpha \in \mathcal{I}} \eta_\alpha(\omega) U_\alpha(x)$$

and

$$H := H(\omega) := H_0 + V_\omega \text{ in } L^2(\mathbb{R}^d).$$

The most important condition expresses the fact that the ground state energy comes from those realizations of the potential that vanish on large sets:

(A4) Denote $E_0 := \inf \sigma(H_0) \le \inf \sigma(H(\omega))$ and let

$$H_F := H_0 + \eta_{\max} \sum_{\alpha \in \mathcal{I}} U_\alpha,$$

the subscript F standing for full coupling.
Assume that E_0 is a *fluctuation boundary* in the sense that
  (i) $E_F := \inf \sigma(H_F) > E_0$, and
  (ii) There is $m \in (0,2)$ and L^* such that for $m_d := 42 \cdot d$, all $L \ge L^*$ and $x \in \mathbb{Z}^d$

$$\mathbb{P}\big(\sigma(H^{\Lambda_L(x)}(\omega) \cap [E_0, E_0 + L^{-m}] \ne \emptyset\big) \le L^{-m_d}.$$

By χ_x we denote the characteristic function of the unit cube centered at x. In the following it is understood that $\chi_x (H^G - E - \mathrm{i}\varepsilon)^{-1} \chi_y = 0$ if $\Lambda_1(x) \cap G$ or $\Lambda_1(y) \cap G$ have measure zero.

Our main result is

Theorem 1. *Let $d \le 3$ and assume* (A1)-(A4). *Then there exist $\delta > 0$, $0 < s < 1$, $\mu > 0$ and $C < \infty$ such that for $I := [E_0, E_0 + \delta]$, all open sets $G \subset \mathbb{R}^d$ and $x, y \in \mathbb{R}^d$,*

$$\sup_{E \in I, \varepsilon > 0} \mathbb{E}(\|\chi_x (H^G - E - \mathrm{i}\varepsilon)^{-1} \chi_y\|^s) \le C\, \mathrm{e}^{-\mu|x-y|}. \tag{1}$$

Exponential decay of fractional moments of the resolvent as described by (1) implies spectral and dynamical localization in the following sense:

Theorem 2. *Let $d \le 3$, assume* (A1)-(A4) *and let I be given as in Theorem 1. Then:*

(a) *For all open sets $G \subset \mathbb{R}^d$ the spectrum of H^G in I is almost surely pure point with exponentially decaying eigenfunctions.*

(b) *There are $\mu > 0$ and $C < \infty$ such that for all $x, y \in \mathbb{R}^d$ and open $G \subset \mathbb{R}^d$,*

$$\mathbb{E}\big(\sup \|\chi_x g(H^G) P_I(H^G) \chi_y\|\big) \le C \mathrm{e}^{-\mu|x-y|}. \tag{2}$$

where the supremum is taken over all Borel measurable functions g which satisfy $|g| \le 1$ pointwise and $P_I(H^G)$ is the spectral projection for H^G onto I.

Dynamical localization should be considered as the special case $g(\lambda) = \mathrm{e}^{\mathrm{i}t\lambda}$ in (b), with the supremum taken over $t \in \mathbb{R}$.

The proof of Theorem 1 is given in Section 2. This will be done by a self-contained presentation of a new version of the continuum fractional moment method. While we use many of the same ideas as [2], due to the lack of a covering condition we can not rely any more on the concept of "averaging over local environments", heavily exploited in [2]. It is interesting to note that, in some sense, we instead use a global averaging procedure. Technically, this actually leads to some simplifications compared to the method in [2], as repeated commutator arguments can be replaced by simpler iterated resolvent identities. We also mention that exponential decay in (1) will follow from smallness of the fractional moments at a suitable initial length scale (the localization length) via an abstract contraction property.

As technical tools we need Combes-Thomas bounds (in operator norm as well as in Hilbert-Schmidt norm) and a weak-L^1-type bound for the boundary values of resolvents of maximally dissipative operators, which is based on results from [26] and was also central to the argument in [2]. We collect these tools in an appendix.

That Theorem 2 follows from Theorem 1 was essentially shown in [2], Section 2. In Section 3 below we will briefly discuss the changes which arise due to our somewhat different set-up. In particular, the argument in [2] for proving (2) uses the covering condition

$$0 < C_1 \leq \sum U_\alpha \leq C_2 < \infty \tag{3}$$

in one occasion. But this is easily circumvented.

In Sections 4 and 5 we apply our main result to concrete models by verifying assumption (A4) for these models. In Section 4 we consider Anderson-type random potentials supported in the vicinity of a lower-dimensional surface. The "usual" fully stationary Anderson model is considered in Section 5. The fact that we don't have to assume a covering condition leads to high flexibility in the geometry of the random scatterers. We could use this to go for far reaching generalizations of Anderson models. Instead, we restrict ourselves to the treatment of additional random displacements as was done in [10].

1.3. **Remarks.** We could have extended Theorem 1 in at least two different ways, but refrained from doing so to keep the proofs as transparent as possible:

(i) The restriction to $d \leq 3$ is not necessary. We use it because in this case the abstract fractional moment bound in Corollary 17 is more directly applicable to our proof of Theorem 1 than in higher dimension (which technically can be traced back to the fact that $\chi_x(-\Delta+1)^{-1}$ is a Hilbert-Schmidt operator only for $d \leq 3$). In higher dimension more iterations of resolvent identities would be needed to yield the Hilbert-Schmidt multipliers required by Corollary 17, leading to more involved summations in the arguments of Section 2.

(ii) Instead of bounded U_α we can work with relatively Δ-bounded U_α, i.e. allow for suitable L^p-type singularities in the single site potentials. In the course of our proofs they could be "absorbed" into resolvents using standard arguments from relative perturbation theory.

In principle, our arguments can also be used to prove localization at fluctuation type band edges more general than the bottom of the spectrum without using a covering condition as in [2]. But this would require to be much more specific with settings and assumptions and, in particular, with the geometry of the impurity set. Inconvenience would also arise from having to work with boundary conditions other than Dirichlet.

We mention that the applications in Section 4 improve the results on continuum random surface potentials of [7, 24], obtained through the use of multiscale analysis:

(i) The exponentially decaying correlations of the time evolution, shown as a special case of Theorem 2(b), are stronger than the dynamical bounds which follow from multiscale analysis.
(ii) Due to the use of the recent result of [24] on Lifshitz tails for surface potentials, we do not need a condition on the smallness of the distribution of the η_α near the minimum of their support as in [7], a progress that had been achieved in [24].
(iii) We can allow for more flexibility concerning the geometry of the scatterers.

Of course, due to using fractional moments we cannot include single site measures as singular as the ones considered in [7, 24] but instead have to assume absolute continuity of the η_α.

2. Localization near fluctuation boundaries

This section is entirely devoted to the proof of Theorem 1. For a convenient normalization write

$$\begin{aligned}&\xi_\alpha(\omega) := \eta_{\max} - \eta_\alpha(\omega)\\ &\text{for } \omega = (\omega_\alpha)_{\alpha\in\mathcal{I}} = (\eta_\alpha(\omega))_{\alpha\in\mathcal{I}} \in \Omega := [0,\eta_{\max}]^{\mathcal{I}},\end{aligned}$$

and denote the product measure $\otimes_{\alpha\in\mathcal{I}} \mathrm{d}\eta_\alpha\rho_\alpha(\eta_\alpha)$ on Ω by $\mathbb{P}$. We write

$$W(x) := W_\omega(x) := \sum_{\alpha\in\mathcal{I}} \xi_\alpha(\omega) U_\alpha(x).$$

Note that $W_\omega \geq 0$ and that

$$H = H(\omega) = H_F - W_\omega.$$

Fixing an open set $G \subset \mathbb{R}^d$ we write

$$\begin{aligned}R^G &= R^G_z = (H^G - z)^{-1},\\ R^G_F &= R^G_{F,z} = (H^G_F - z)^{-1}\end{aligned}$$

whenever $z = E+\mathrm{i}\varepsilon$. Since $H^G_F \geq H_F$ due to our choice of Dirichlet boundary conditions, and $E_F = \inf\sigma(H_F)$ we know that $(-\infty, E_F) \subset \rho(H^G_F)$.

The resolvent equation yields

$$R^G = R^G_F + R^G_F W R^G_F + R^G_F W R^G W R^G_F, \tag{4}$$

an identity that will be used over and again. The other workhorse result is the following averaging estimate, that follows from Corollary 17 in the appendix below, taking into account the uniform boundedness of the densities ρ_α.

Lemma 3. *For all $s \in [0,1)$ there is $c(s)$ such that*

$$\int \mathrm{d}\eta_\alpha \rho_\alpha(\eta_\alpha) \int \mathrm{d}\eta_\beta \rho_\beta(\eta_\beta) \| M_1 U_\alpha^{1/2} (H^G - E - \mathrm{i}\varepsilon)^{-1} U_\beta^{1/2} M_2 \|_{\mathrm{HS}}^s \leq c(s) \|M_1\|_{\mathrm{HS}}^s \|M_2\|_{\mathrm{HS}}^s.$$

As a warm-up, we prove boundedness of fractional moments:

Lemma 4. *Let $E_1 < E_F$, $I = [E_0, E_1]$ and $s \in [0,1)$. Then*

$$\sup\{\mathbb{E}\, \|\chi_x R^G_{E+i\varepsilon} \chi_y\|^s \mid E \in I, \varepsilon > 0, x, y \in \mathbb{R}^d, G \subset \mathbb{R}^d \text{ open}\} < \infty\,. \quad (5)$$

Proof. We use (4) above and write, suppressing the superscript G and the subscript $z = E + \mathrm{i}\varepsilon$ mostly:

$$\chi_x R \chi_y = \chi_x R_F \chi_y + \chi_x R_F W R_F \chi_y + \chi_x R_F W R W R_F \chi_y.$$

The first two terms on the r.h.s. of this equation obey an exponential bound due to the Combes-Thomas estimate, see subsection A.1 below:

$$\|\chi_x R_F \chi_y\| \leq c\, \mathrm{e}^{-\mu_0 |x-y|}$$

and

$$\begin{aligned} \|\chi_x R_F W R_F \chi_y\| &\leq \eta_{\max} \sum_{\alpha \in \mathcal{I}} \|\chi_x R_F U_\alpha^{1/2}\| \cdot \|U_\alpha^{1/2} R_F \chi_y\| \\ &\leq C \sum_{\alpha \in \mathcal{I}} \mathrm{e}^{-\mu_0|x-\alpha|} \mathrm{e}^{-\mu_0|\alpha - y|} \leq C \mathrm{e}^{-\mu_1 |x-y|} \end{aligned}$$

with μ_0 and $\mu_1 = \mu_0/2$ depending on E_1 only. In the last estimate we have used that $\mathcal{I}$ is uniformly discrete.

For the third term, expand $W = \sum_\alpha \xi_\alpha U_\alpha$ and use the boundedness of the ξ_α and the fact that

$$\left(\sum a_n\right)^s \leq \sum a_n^s$$

to estimate

$$\|\chi_x R_F W R W R_F \chi_y\|^s \leq c \sum_{\alpha,\, \beta \in \mathcal{I}} \|\chi_x R_F U_\alpha R U_\beta R_F \chi_y\|^s.$$

We now fix $\alpha, \beta \in \mathcal{I}$ and use the workhorse Lemma 3 to conclude

$$\begin{aligned} \int \mathrm{d}\eta_\alpha \rho_\alpha(\eta_\alpha) \int \mathrm{d}\eta_\beta \rho_\beta(\eta_\beta) &\|\chi_x R_F U_\alpha R U_\beta R_F \chi_y\|^s \\ &\leq c(s) \|\chi_x R_F U_\alpha^{1/2}\|_{\mathrm{HS}}^s \|U_\beta^{1/2} R_F \chi_y\|_{\mathrm{HS}}^s \\ &\leq c(s) \cdot \mathrm{e}^{-s\mu_0|x-\alpha|} \cdot \mathrm{e}^{-s\mu_0|y-\beta|} \end{aligned}$$

by the HS-norm Combes-Thomas bound from Proposition 15 and since $\mathrm{dist}(x, \operatorname{supp} U_\alpha) \geq |x - \alpha| - R_U$ where R_U majorizes the size of the support of U_α according to assumption (A3).

Note that here and in the following we use the convention that c, $c(s)$, etc. denote constants that only depend on non-crucial quantities and may change from line to line. In particular, the constants are independent of $\varepsilon > 0$ and the random background.

Now, we can sum up the last terms and get the assertion. □

Remarks. (i) In this proof it is still quite easy to see how to extend to arbitrary dimension through iterations of the resolvent identity. It will be harder to keep track of this later.

(ii) Note that due to the α, β-summations, averaging over the η_α is required for all α, i.e., is global. In [2], due to the covering condition, an argument is provided that only requires averaging over local environments of x and y and proves Lemma 4 for arbitrary finite intervals $I = [E_0, E_1]$, i.e. without requiring $E_1 < E_F$.

(iii) The above proof shows that (5) also holds in HS-norm, but this will not be used below.

We will now start an iterative procedure that will show exponential decay of $\mathbb{E}(\|\chi_x R \chi_y\|^s)$ in $|x-y|$ for energies sufficiently close to E_0. Clearly, it suffices to consider $x, y \in \mathbb{Z}^d$. In view of the preceding lemma the following quantity is finite:

$$\tau_{x,y} := \sup\{\mathbb{E}\,\|\chi_x R^G_{E+i\varepsilon} \chi_y\|^s \mid E \in I,\, \varepsilon > 0 \text{ and } G \subset \mathbb{R}^d \text{ open}\}\,.$$

Moreover, we should actually keep in mind the dependence on the interval $I = [E_0, E_1]$. In fact, E_1 will later be chosen small enough.

In order to use that E_0 appears rarely as an eigenvalue for boxes of side length L we exploit the resolvent identity and what is sometimes called the Simon-Lieb inequality in a way visualized in Figure 1!

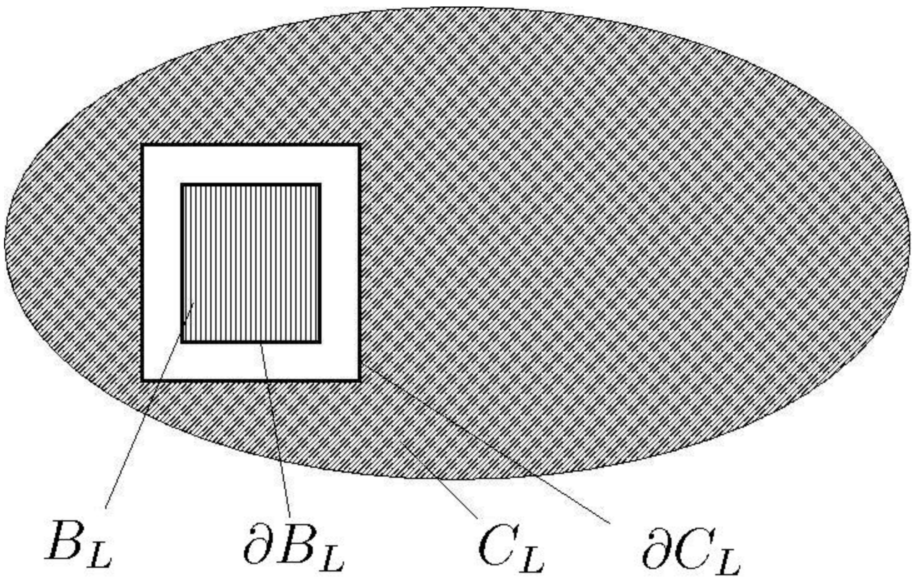

FIGURE 1. The geometry of the induction step

Consider

$$B_L := \Lambda_L(x) \cap G,$$
$$\partial B_L = (\Lambda_L(x) \setminus \Lambda_{L-2}(x)) \cap G, \text{ and } \chi_L^- := \chi_{\partial B_L}.$$

Furthermore, with R_U as in assumption (A3), define

$$C_L := G \setminus \overline{\Lambda_{2R_U+L}(x)},$$
$$\partial C_L = \left(\Lambda_{2R_U+L+2}(x) \setminus \overline{\Lambda_{2R_U+L}(x)}\,\right) \cap G, \text{ and } \chi_L^+ := \chi_{\partial C_L}.$$

The geometry is chosen in such a way that R^{B_L} and R^{C_L} are stochastically independent. For R^{B_L} we can use the fluctuation boundary assumption to get small fractional moments and the right size of L will be adjusted. But all that later...

Thus, by the Simon-Lieb inequality (e.g. [30], Sect. 2.5)

$$\|\chi_x R^G \chi_y\| \le C \|\chi_x R^{B_L} \chi_L^-\| \cdot \|\chi_L^- R^G \chi_L^+\| \cdot \|\chi_L^+ R^{C_L} \chi_y\| \tag{SLI}$$

where C only depends on $\sup_{\{\eta_\alpha | \alpha \in \mathcal{I}\}} \|V\|_\infty$ and the interval I.

The basic idea for proving exponential decay of $\tau_{x,y}$ is to establish a recurrence inequality for energies sufficiently close to E_0. This recurrence inequality is described in Proposition 6 below and allows to apply a discrete Gronwall-type argument found in Lemma 7 below. To this end we exploit smallness of fractional moments of the first factor on the r.h.s. of (SLI) for energies close to E_0 and sufficiently large, but fixed, L: This will follow from (A4)(ii) as is presented in the following Lemma 5. Fractional moments of the second factor are bounded due to Lemma 4 (up to a polynomial factor in L). Finally, we use the third factor to start an iteration (with x replaced by sites x' covering the layer $\Lambda_{L+R_U+2} \setminus \Lambda_{L+R_U}$). By construction, the first and third factor on the r.h.s. of (SLI) are probabilistically independent. Unfortunately, the second factor introduces a correlation which prevents us from simply factoring the expectation. We will rely on a version of the re-sampling procedure developed in [2] to solve this problem. Moreover, we will not use Lemma 4, but apply Lemma 3 directly to bound certain conditional expectations. This will result in Proposition 6 below.

Lemma 5. *For m as in* (A4) *and $s \in (0, \frac{1}{3})$ there is $L^* = L^*(m,s)$ such that for all $L \ge L^*$, open $B \subset \Lambda_L(x)$, $E \in I := [E_0, E_0 + L^{-\frac{1}{2}m}]$, $\varepsilon > 0$ and $u, v \in \mathbb{Z}^d$ with $|u - v| \ge \frac{L}{4}$ we have*

$$\mathbb{E}(\|\chi_u (H^B - E - \mathrm{i}\varepsilon)^{-1} \chi_v\|^s) \le L^{-\frac{1}{2}m_d},$$

where $m_d = 42 \cdot d$.

Proof. Divide Ω into the good and bad sets

$$\Omega_{\text{good}} := \{\omega \mid \operatorname{dist}(\sigma(H^B), E_0) > L^{-m}\}, \quad \Omega_{\text{bad}} = \Omega \setminus \Omega_{\text{good}}.$$

Since $H^B \ge H^{\Lambda_L(x)}$ by our choice of Dirichlet boundary conditions, (A4) implies that

$$\mathbb{P}(\Omega_{\text{bad}}) \le L^{-m_d}.$$

We split the expectation into contributions from the good and bad sets: By the improved Combes-Thomas bound Subsection A.1 we get, for $\omega \in \Omega_{\text{good}}$,

$E \in I$:

$$\|\chi_u R^B_{E+\mathrm{i}\varepsilon}\chi_v\|^s \le CL^{\frac{1}{2}ms}\mathrm{e}^{-cs|u-v|L^{-\frac{1}{2}m}} .$$

This gives a uniform bound of the same type for the expectation over Ω_{good}. For the bad set Hölder with $t \in (s,1)$ gives

$$\begin{aligned}\mathbb{E}(\|\chi_u R^B_{E+\mathrm{i}\varepsilon}\chi_v\|^s \chi_{\Omega_{\mathrm{bad}}}) &\le \left(\mathbb{E}(\|\chi_u R^B_{E+\mathrm{i}\varepsilon}\chi_v\|^t)\right)^{\frac{s}{t}} \mathbb{P}(\Omega_{\mathrm{bad}})^{1-\frac{s}{t}} \\ &\le c(t)^{\frac{s}{t}} L^{(1-\frac{s}{t})m_d}.\end{aligned}$$

Now we choose $t = \frac{1}{2}(s+1)$ so that $(1-\frac{s}{t}) > \frac{1}{2}$ if $s < \frac{1}{3}$. Putting things together, we get

$$\mathbb{E}(\|\chi_u R^B_{E+\mathrm{i}\varepsilon}\chi_v\|^s) \le C(s)\Big(L^{\frac{1}{2}ms}\mathrm{e}^{-cs|u-v|\cdot L^{-\frac{1}{2}m}} + L^{(1-\frac{s}{t})m_d}\Big).$$

If L is large enough we can use $\frac{1}{2}m < 1$ and $|u-v| \ge \frac{L}{4}$ to see that the r.h.s. is bounded as asserted. □

The exponential decay of the $\tau_{x,y}$ will follow from the following result, whose proof will take most of the present section.

Proposition 6. *There exist L^*, $\kappa > 0$, $c > 0$ and $C > 0$, all depending on $s, m, R_U, r_U, M_\rho, E_0, E_F, \eta_{\max}$, such that for $L \ge L^*$ and $I = [E_0, E_0 + L^{-\frac{1}{2}m}]$ the above defined $\tau_{x,y}$ satisfy:*

$$\tau_{x,y} \le L^{-2d-\kappa} \sum_{x',\,y' \in \mathbb{Z}^d} \mathrm{e}^{-c(|x-x'|+|y-y'|)/L}\,\tau_{x',y'} + C\mathrm{e}^{-c|x-y|/L}. \tag{6}$$

Proof of Proposition 6. We now restrict to the energy interval $I = [E_0, E_0 + L^{-\frac{1}{2}m}]$ assuming L is large enough to guarantee that $I \subset [E_0, E_F)$. Using (SLI) above and denoting

$$\begin{aligned}T_{x,L} &= \chi_x R^{B_L}\chi_L^-,\\ S_{x,L} &= \chi_L^- R^G \chi_L^+,\\ Q_{x,L} &= \chi_L^+ R^{C_L}\chi_y\end{aligned}$$

we get that

$$\mathbb{E}(\|\chi_x R^G \chi_y\|^s) \le C\,\mathbb{E}(\|T_{x,L}\|^s \|S_{x,L}\|^s \|Q_{x,L}\|^s).$$

Note that $\|T_{x,L}\|^s$ and $\|Q_{x,L}\|^s$ are stochastically independent. Unfortunately, they are correlated via $\|S_{x,L}\|^s$.

Fix $s \in (0, \frac{1}{3})$ to estimate $\mathbb{E}(\|T_{x,L}\|^s)$. Using the preceding Lemma, we get that

$$\begin{aligned}\mathbb{E}(\|T_{x,L}\|^s) &\le \sum_{z \in \operatorname{supp}\chi_L^-} \mathbb{E}(\|\chi_x R^{B_L}\chi_z\|^s)\\ &\le CL^{d-1}\cdot L^{-\frac{1}{2}m_d},\end{aligned}$$

for L large enough. We get that

$$\mathbb{E}(\|T_{x,L}\|^s) \le L^{d-\frac{1}{2}m_d}.$$

We can now expand $S_{x,L}$ to split off a uniformly bounded (in ω) term:

$$S_{x,L} = \underbrace{\chi_L^- R_F^G \chi_L^+ + \chi_L^- R_F^G W R_F^G \chi_L^+}_{S_{1,L}} + \underbrace{\chi_L^- R_F^G W R^G W R_F^G \chi_L^+}_{S_{2,L}}.$$

Since $I \subset [E_0, E_F)$ we have that $\|S_{1,L}\|^s$ is uniformly bounded. Thus, we get

$$\begin{aligned}\mathbb{E}(\|\chi_x R_{E+\mathrm{i}\varepsilon}^G \chi_y\|^s) &\leq C\big(\mathbb{E}(\|T_{x,L}\|^s \cdot \|Q_{x,L}\|^s) + \Sigma_2\big)\\ &= C\big(\mathbb{E}(\|T_{x,L}\|^s) \cdot \mathbb{E}(\|Q_{x,L}\|^s) + \Sigma_2\big)\end{aligned}$$

as $\|T_{x,L}\|^s$ and $\|Q_{x,L}\|^s$ are independent. Here

$$\Sigma_2 := \mathbb{E}(\|T_{x,L}\|^s \|S_{2,L}\|^s Q_{x,L}\|^s).$$

Expanding χ_L^+ we get, for some $c > 0$, that

$$\ldots \leq C L^{d-\frac{1}{2}m_d} \sum_{x' \in \partial C_L} \mathrm{e}^{-c|x-x'|/L}\, \mathbb{E}(\|\chi_{x'} R^{C_L} \chi_y\|^s) + C\,\Sigma_2,$$

whence

$$\tau_{x,y} \leq L^{d-\frac{1}{2}m_d} \sum_{x' \in \partial C_L} \mathrm{e}^{-c|x-x'|/L}\, \tau_{x',y} + C \sup_{\substack{E \in I,\, \varepsilon > 0\\ G \subset \mathbb{R}^d}} \Sigma_2. \tag{7}$$

To estimate Σ_2 we begin by expanding

$$\begin{aligned} T_{x,L} &= \chi_x R^{B_L} \chi_L^-\\ &= \chi_x R_F^{B_L} \chi_L^- + \chi_x R_F^{B_L} W R_F^{B_L} \chi_L^- + \chi_x R_F^{B_L} W R^{B_L} W R_F^{B_L} \chi_L^-. \end{aligned} \tag{8}$$

Since I has positive distance from $\sigma(H_F)$, we have the Combes-Thomas bound $C\mathrm{e}^{-\mu_0 L/2}$ for the norm of the first two terms on the r.h.s. of (8), see Appendix A.1. Here $C < \infty$ and $\mu_0 > 0$ are uniform in the randomness, $E \in I$, $\varepsilon > 0$ and $x \in \mathbb{Z}^d$. Expanding the third term and using boundedness of the ξ's yields

$$\|T_{x,L}\|^s \leq C\left(\mathrm{e}^{-\mu_0 \cdot s \cdot \frac{L}{2}} + \sum_{\beta,\gamma \in \mathcal{I} \cap \Lambda_{L+R_U}(x)} \|T_{\beta,\gamma}\|^s\right),$$

setting $T_{\beta,\gamma} = \chi_x R_F^{B_L} U_\beta R^{B_L} U_\gamma R_F^{B_L} \chi_L^-$, and only summing over those β, γ for which the corresponding U-terms touch B_L.

A similar argument applied to $Q_{x,L}$ leads to

$$\|Q_{x,L}\|^s \leq C\left(\mathrm{e}^{-\mu_0 \cdot s \cdot (|x-y| - \frac{L}{2})} + \sum_{\beta',\gamma' \in \mathcal{I} \cap \Lambda_{L+R_U}^c(x)} \|Q_{\beta',\gamma'}\|^s\right),$$

where we have chosen $Q_{\beta',\gamma'} = \chi_L^+ R_F^{C_L} U_{\gamma'} R^{C_L} U_{\beta'} R_F^{C_L} \chi_y$.

Finally, expand

$$S_{2,L} = \chi_L^- R_F^G W R^G W R_F^G \chi_L^+ = \sum_{\alpha,\alpha' \in \mathcal{I}} S_{\alpha,\alpha'},$$

where $S_{\alpha,\alpha'} = \chi_L^- R_F^G \xi_\alpha U_\alpha R^G \xi_{\alpha'} U_{\alpha'} R_F^G \chi_L^+$.

Combining all this we get that

$$\Sigma_2 \le C\Bigg(e^{-\mu_0\cdot s\cdot\frac{L}{2}} \sum_{\alpha,\alpha'} \mathbb{E}\,\|S_{\alpha,\alpha'}\|^s e^{-\mu_0\cdot s\cdot(|x-y|-\frac{L}{2})} + e^{-\mu_0\cdot s\cdot\frac{L}{2}} \sum_{\alpha,\alpha',\beta',\gamma'} \mathbb{E}(\|S_{\alpha,\alpha'}\|^s\cdot\|Q_{\beta',\gamma'}\|^s) + \sum_{\alpha,\alpha',\beta,\gamma} \mathbb{E}(\|T_{\beta,\gamma}\|^s\cdot\|S_{\alpha,\alpha'}\|^s) e^{-\mu_0\cdot s\cdot(|x-y|-\frac{L}{2})} + \sum_{\alpha,\alpha',\beta,\beta',\gamma,\gamma'} \mathbb{E}(\|T_{\beta,\gamma}\|^s\cdot\|S_{\alpha,\alpha'}\|^s\cdot\|Q_{\beta',\gamma'}\|^s)\Bigg).$$

The most complicated of these terms is the last one; it will be obvious how to estimate the first three once we have established a bound for the last one according to the assertion of Proposition 6. Thus we have to estimate

$$\Sigma_3 := \sum_{\alpha,\alpha',\beta,\beta',\gamma,\gamma'} A_{\alpha,\alpha',\beta,\beta',\gamma,\gamma'},$$

where

$$A_{\alpha,\alpha',\beta,\beta',\gamma,\gamma'} = \mathbb{E}\left(\|T_{\beta,\gamma}\|^s\cdot\|S_{\alpha,\alpha'}\|^s\cdot\|Q_{\beta',\gamma'}\|^s\right).$$

If it weren't for the $S_{\alpha,\alpha'}$-terms, the $T_{\beta,\gamma}$ and $Q_{\beta',\gamma'}$ would be independent, leading to an estimate like in (7) above. We will reinforce a certain kind of independence through *re-sampling*. For fixed

$$\mathcal{J} := \{\alpha,\alpha',\gamma,\gamma'\}$$

we introduce new independent random variables $\widehat{\xi}_j$, $j\in\mathcal{J}$, independent of the ξ_ζ, $\zeta\in\mathcal{I}$, and with the same distribution as the ξ_ζ. We denote the corresponding space by $\widehat{\Omega}$, the corresponding probability by $\widehat{\mathbb{P}}$ and the expectation with respect to $\widehat{\mathbb{P}}$ by $\widehat{\mathbb{E}}$. Consider

$$\widehat{H}(\omega,\widehat{\omega}) = H(\omega) + \underbrace{\sum_{j\in\mathcal{J}}(\xi_j(\omega)-\widehat{\xi}_j(\widehat{\omega}))U_j}_{\widehat{W}}$$

and note that $\widehat{H}$ doesn't depend on the ξ_j, $j\in\mathcal{J}$. The resolvent identity for $\widehat{R}^G_z = (\widehat{H}^G - z)^{-1}$ gives

$$R^G_z = \widehat{R}^G_z + \widehat{R}^G_z\widehat{W}R^G_z.$$

We insert this for $T_{\beta,\gamma}$ and $Q_{\beta',\gamma'}$ and get

$$T_{\beta,\gamma} = \underbrace{\chi_x R_F^{B_L} U_\beta \widehat{R}^{B_L} U_\gamma R_F^{B_L}\chi_L^-}_{\widehat{T}_{\beta,\gamma}} + \underbrace{\chi_x R_F^{B_L} U_\beta \widehat{R}^{B_L}\widehat{W}R^{B_L} U_\gamma R_F^{B_L}\chi_L^-}_{\widetilde{T}_{\beta,\gamma}}$$

and, similarly,

$$Q_{\beta',\gamma'} = \widehat{Q}_{\beta',\gamma'} + \widetilde{Q}_{\beta',\gamma'}.$$

Now we can estimate

$$A_{\alpha,\alpha',\beta,\beta',\gamma,\gamma'} \le \widehat{\mathbb{E}}\,\mathbb{E}\left[(\|\widehat{T}_{\beta,\gamma}\|^s + \|\widetilde{T}_{\beta,\gamma}\|^s)\|S_{\alpha,\alpha'}\|^s(\|\widehat{Q}_{\beta',\gamma'}\|^s + \|\widetilde{Q}_{\beta',\gamma'}\|^s)\right]. \tag{9}$$

This gives a sum of four terms we have to control. Let's start with the easiest one

$$A^1_{\alpha,\alpha',\beta,\beta',\gamma,\gamma'} := \widehat{\mathbb{E}}\,\mathbb{E}\big[\|\widehat{T}_{\beta,\gamma}\|^s\|S_{\alpha,\alpha'}\|^s\|\widehat{Q}_{\beta',\gamma'}\|^s\big].$$

Denote

$$\mathbb{E}(X|\alpha,\alpha') = \int \mathrm{d}\xi_\alpha\rho_\alpha(\xi_\alpha)\int \mathrm{d}\xi_{\alpha'}\rho_{\alpha'}(\xi_{\alpha'})\,X(\xi)$$

for a random variable on $\Omega\times\widehat{\Omega}$, so that $\mathbb{E}(X|\alpha,\alpha')$ is nothing but the conditional expectation with respect to the σ-field generated by the family $(\xi_\beta \mid \beta\in\mathcal{I}\setminus\{\alpha,\alpha'\})$. According to the usual rules for conditional expectations:

$$\begin{aligned} A^1_{\alpha,\alpha',\beta,\beta',\gamma,\gamma'} &= \widehat{\mathbb{E}}\,\mathbb{E}\left[\mathbb{E}(\|\widehat{T}_{\beta,\gamma}\|^s\|S_{\alpha,\alpha'}\|^s\|\widehat{Q}_{\beta',\gamma'}\|^s|\alpha,\alpha')\right]\\ &= \widehat{\mathbb{E}}\,\mathbb{E}\left[\|\widehat{T}_{\beta,\gamma}\|^s\|\widehat{Q}_{\beta',\gamma'}\|^s\,\mathbb{E}(\|S_{\alpha,\alpha'}\|^s|\alpha,\alpha')\right] \qquad (10)\end{aligned}$$

since the $\widehat{T}$ and $\widehat{Q}$ are independent of $\xi_\alpha,\xi_{\alpha'}$. Using the workhorse Lemma 3 and the Combes-Thomas estimate Proposition 15 we get

$$\begin{aligned}\mathbb{E}(\|S_{\alpha,\alpha'}\|^s|\alpha,\alpha') &\le c(s)\|\chi_L^- R_F^G U_\alpha^{\frac12}\|^s_{\mathrm{HS}}\|U_{\alpha'}^{\frac12}R_F^G\chi_L^+\|^s_{\mathrm{HS}}\\ &\le c(s)L^{2s(d-1)}\mathrm{e}^{-\mu_1 s(|\frac{L}{2}-|\alpha-x||+|\frac{L}{2}-|\alpha'-x||)},\end{aligned}$$

where the extra $L^{2s(d-1)}$ term comes from covering ∂B_L and ∂C_L. We have

$$\widehat{\mathbb{E}}\,\mathbb{E}\left[\|\widehat{T}_{\beta,\gamma}\|^s\|\widehat{Q}_{\beta',\gamma'}\|^s\right] = \mathbb{E}\left[\|T_{\beta,\gamma}\|^s\|Q_{\beta',\gamma'}\|^s\right] = \mathbb{E}\left[\|T_{\beta,\gamma}\|^s\right]\mathbb{E}\left[\|Q_{\beta',\gamma'}\|^s\right]$$

since the $\widehat{\xi}$'s have the same distribution as the ξ's and the T's and Q's are independent. Inserting into (10) gives

$$A^1_{\alpha,\alpha',\beta,\beta',\gamma,\gamma'} \le c(s)L^{2s(d-1)}\mathrm{e}^{-\mu_1 s(|\frac{L}{2}-|\alpha-x||+|\frac{L}{2}-|\alpha'-x||)}\,\mathbb{E}\left[\|T_{\beta,\gamma}\|^s\right]\mathbb{E}\left[\|Q_{\beta',\gamma'}\|^s\right].$$

We will now treat the latter two terms separately:

Step 1. Denote by $Z(\gamma') = \{y'\in\mathbb{Z}^d \mid \chi_{y'}\cdot U_{\gamma'}\neq 0\}$ those lattice points whose 1-cubes support $U_{\gamma'}$. By Combes-Thomas once again:

$$\begin{aligned}\|Q_{\beta',\gamma'}\|^s &= \|\chi_L^+ R_F^{C_L}U_{\gamma'}R^{C_L}U_{\beta'}R_F^{C_L}\chi_y\|^s\\ &\le C\sum_{x'\in Z(\gamma')}\sum_{y'\in Z(\beta')}\|\chi_{x'}R^{C_L}\chi_{y'}\|^s\mathrm{e}^{-\mu_1 s(|x-x'|-L)}\mathrm{e}^{-\mu_1 s|y-y'|}.\end{aligned}$$

By assumption on the size of the support of $U_{\gamma'}$ we see that $\#Z(\gamma')$ is uniformly bounded. This and uniform discreteness of $\mathcal{I}$ gives

$$\sum_{\beta',\gamma'}\mathbb{E}\,\|Q_{\beta',\gamma'}\|^s \le C\sum_{x',\,y'\in\mathbb{Z}^d\cap C_L}\mathrm{e}^{-\mu_1 s(|x-x'|-\frac{L}{2})}\mathrm{e}^{-\mu_1 s|y-y'|}\,\tau_{x',y'}.$$

Step 2. For the $T_{\beta,\gamma}$-term we have

$$\begin{aligned}\|T_{\beta,\gamma}\|^s &= \|\chi_x R_F^{B_L} U_\beta R^{B_L} U_\gamma R_F^{B_L} \chi_L^-\|^s \\ &\leq C \sum_{u\in Z(\beta)\cap B_L} \sum_{v\in Z(\gamma)\cap B_L} \|\chi_x R_F^{B_L}\chi_u\|^s \|\chi_u R^{B_L}\chi_v\|^s \|\chi_v R_F^{B_L}\chi_L^-\|^s.\end{aligned}$$

If $|u-v| \geq \frac{1}{4}L$, Lemma 5 gives

$$\mathbb{E}(\|\chi_u R^{B_L}\chi_v\|^s) \leq C\cdot L^{-\frac{1}{2}m_d}.$$

If, on the other hand, $|u-v| \leq \frac{1}{4}L$ then $\operatorname{dist}(v,\partial B_L) \geq \frac{1}{8}L$ or $|x-u| \geq \frac{1}{8}L$, so that the uniform bound of Lemma 4 for $\mathbb{E}(\|\chi_u R^{B_L}\chi_v\|^s)$ together with the Combes-Thomas bound for $\|\chi_v R_F^{B_L}\chi_L^-\|^s$, resp. $\|\chi_x R_F^{B_L}\chi_u\|^s$ gives, for L large enough,

$$\begin{aligned}\mathbb{E}\big(\|\chi_x R_F^{B_L}\chi_u\|^s \|\chi_u R^{B_L}\chi_v\|^s \|\chi_v R_F^{B_L}\chi_L^-\|^s\big) &\leq C\mathrm{e}^{-\frac{1}{8}\mu_0 sL} \\ &\leq L^{-\frac{1}{2}m_d}.\end{aligned}$$

Combined we get that, again for L sufficiently large,

$$\sum_{\beta,\gamma}\mathbb{E}(\|T_{\beta,\gamma}\|^s) \leq CL^{2d-\frac{1}{2}m_d},$$

where an extra factor L^{2d} arises through the number of terms considered.

Joining Step 1, Step 2 and the bound

$$\sum_{\alpha,\alpha'} \mathrm{e}^{-\mu_1 s(|\frac{L}{2}-|\alpha-x||+|\frac{L}{2}-|\alpha'-x||)} \leq C(s)L^{2d}$$

we arrive at

$$\begin{aligned}&\sum_{\alpha,\alpha',\beta,\beta',\gamma,\gamma'} A^1_{\alpha,\alpha',\beta,\beta',\gamma,\gamma'} \\ &\qquad \leq C(s)L^{6d-\frac{1}{2}m_d} \sum_{x',\,y'\in\mathbb{Z}^d\cap C_L} \mathrm{e}^{-\mu_1 s(|x-x'|-\frac{L}{2})}\mathrm{e}^{-\mu_1 s|y-y'|}\,\tau_{x',y'},\end{aligned}$$

which is a contribution to Σ_3 (and therefore Σ_2) bounded by one of the type asserted in Proposition 6.

A look back at (9) shows that we still have to estimate three terms similar to $A^1_{\alpha,\alpha',\beta,\beta',\gamma,\gamma'}$ of which the last one,

$$A^4_{\alpha,\alpha',\beta,\beta',\gamma,\gamma'} := \widehat{\mathbb{E}}\,\mathbb{E}\left[\|\widetilde{T}_{\beta,\gamma}\|^s \|S_{\alpha,\alpha'}\|^s \|\widetilde{Q}_{\beta',\gamma'}\|^s\right]$$

is the most complicated one. Using Steps 1 and 2 above as well as the steps below it will be clear how to treat the two remaining terms.

Step 3. We start taking the conditional expectation:

$$\begin{aligned}A^4_{\alpha,\alpha',\beta,\beta',\gamma,\gamma'} &= \widehat{\mathbb{E}}\,\mathbb{E}\left[\mathbb{E}(\|\widetilde{T}_{\beta,\gamma}\|^s \|S_{\alpha,\alpha'}\|^s \|\widetilde{Q}_{\beta',\gamma'}\|^s | \alpha,\alpha',\gamma,\gamma')\right] \\ &\leq \widehat{\mathbb{E}}\,\mathbb{E}\Big[\mathbb{E}(\|\widetilde{T}_{\beta,\gamma}\|^{3s}|\alpha,\alpha',\gamma,\gamma')^{\frac{1}{3}} \\ &\qquad\qquad \cdot\mathbb{E}(\|S_{\alpha,\alpha'}\|^{3s}|\alpha,\alpha',\gamma,\gamma')^{\frac{1}{3}}\cdot\mathbb{E}(\|\widetilde{Q}_{\beta',\gamma'}\|^{3s}|\alpha,\alpha',\gamma,\gamma')^{\frac{1}{3}}\Big]\end{aligned}$$

by Hölder's inequality. Like above, the middle term can, up to $CL^{2s(d-1)}$, be estimated by

$$e_{\alpha,\alpha'} := \mathrm{e}^{-\mu_1 s||\alpha-x|-\frac{L}{2}|}\mathrm{e}^{-\mu_1 s||\alpha'-x|-\frac{L}{2}|}.$$

Recall that

$$\begin{aligned}\|\widetilde{Q}_{\beta',\gamma'}\|^{3s} &= \Big\|\chi_L^+ R_F^{C_L} U_{\gamma'} R^{C_L} \sum_{j\in\mathcal{J}\setminus\{\gamma\}} (\xi_j - \widehat{\xi}_j) U_j \widehat{R}^{C_L} U_{\beta'} R_F^{C_L} \chi_y\Big\|^{3s} \\ &\le C \cdot \sum_{j\in\mathcal{J}\setminus\{\gamma\}} \|\chi_L^+ R_F^{C_L} U_{\gamma'} R^{C_L} U_j \widehat{R}^{C_L} U_{\beta'} R_F^{C_L} \chi_y\|^{3s},\end{aligned}$$

where γ can be excluded from the summation as U_γ doesn't touch C_L. Integration over ξ_j and $\xi_{\gamma'}$ gives a uniform bound by the workhorse Lemma 3:

$$\begin{aligned}&\mathbb{E}(\|\widetilde{Q}_{\beta',\gamma'}\|^{3s}|\alpha,\alpha',\gamma,\gamma') \\ &\qquad\le \sum_{j\in\mathcal{J}\setminus\{\gamma\}} \mathbb{E}(\|\chi_L^+ R_F^{C_L} U_{\gamma'} R^{C_L} U_j \widehat{R}^{C_L} U_{\beta'} R_F^{C_L} \chi_y\|^{3s}|\alpha,\alpha',\gamma,\gamma') \\ &\qquad\le C(s)\cdot \sum_{j\in\mathcal{J}\setminus\{\gamma\}} \|\chi_L^+ R_F^{C_L} U_{\gamma'}^{\frac{1}{2}}\|_{\mathrm{HS}}^{3s} \cdot \|U_j^{\frac{1}{2}} \widehat{R}^{C_L} U_{\beta'} R_F^{C_L} \chi_y\|_{\mathrm{HS}}^{3s}\end{aligned}$$

so that, as the sum has only three terms,

$$\mathbb{E}(\|\widetilde{Q}_{\beta',\gamma'}\|^{3s}|\alpha,\alpha',\gamma,\gamma')^{\frac{1}{3}} \le C(s)\cdot \underbrace{\sum_{j\in\mathcal{J}\setminus\{\gamma\}} \|\chi_L^+ R_F^{C_L} U_{\gamma'}^{\frac{1}{2}}\|_{\mathrm{HS}}^{s} \cdot \|U_j^{\frac{1}{2}} \widehat{R}^{C_L} U_{\beta'} R_F^{C_L} \chi_y\|_{\mathrm{HS}}^{s}}_{\Sigma_Q}.$$

Similarly,

$$\mathbb{E}(\|\widetilde{T}_{\beta,\gamma}\|^{3s}|\alpha,\alpha',\gamma,\gamma')^{\frac{1}{3}} \le C(s)\cdot \underbrace{\sum_{j\in\mathcal{J}\setminus\{\gamma'\}} \|\chi_x R_F^{B_L} U_\beta \widehat{R}^{B_L} U_j^{\frac{1}{2}}\|_{\mathrm{HS}}^{s} \cdot \|U_\gamma^{\frac{1}{2}} R_F^{B_L} \chi_L^-\|_{\mathrm{HS}}^{s}}_{\Sigma_T}.$$

Now Σ_T and Σ_Q are independent so that

$$A^4_{\alpha,\alpha',\beta,\beta'\gamma,\gamma'} \le C(s) L^{2s(d-1)} \cdot \widehat{\mathbb{E}}\,\mathbb{E}[\Sigma_T] \cdot \widehat{\mathbb{E}}\,\mathbb{E}[\Sigma_Q] \cdot e_{\alpha,\alpha'}. \tag{11}$$

Since the ξ_j and the $\widehat{\xi}_j$ have the same distribution, we can omit the hats in $\widehat{R}^{C_L}$ and $\widehat{R}^{B_L}$ and replace $\widehat{\mathbb{E}}\,\mathbb{E}$ by $\mathbb{E}$ in the bounds for $\widehat{\mathbb{E}}\,\mathbb{E}[\Sigma_T]$ and $\widehat{\mathbb{E}}\,\mathbb{E}[\Sigma_Q]$ to be derived below.

Step 4. We start with the Q-term. Combes-Thomas, Proposition 15 gives

$$\|\chi_L^+ R_F^{C_L} U_{\gamma'}^{\frac{1}{2}}\|_{\mathrm{HS}}^{s} \le CL^{s(d-1)}\mathrm{e}^{-\mu_1 s||\gamma'-x|-\frac{L}{2}|}.$$

This will be used to deal with the term for $j = \gamma'$ which appears in the sum over $\mathcal{J} \setminus \{\gamma\}$; since $\|AB\|_{\mathrm{HS}} \le \|A\|\,\|B\|_{\mathrm{HS}}$ we get that:

$$\begin{aligned}
&\|\chi_L^+ R_F^{C_L} U_{\gamma'}^{\frac12}\|_{\mathrm{HS}}^s \cdot \|U_{\gamma'}^{\frac12} R^{C_L} U_{\beta'} R_F^{C_L} \chi_y\|_{\mathrm{HS}}^s \\
&\qquad \le CL^{s(d-1)} \mathrm{e}^{-\mu_1 s||\gamma'-x|-\frac{L}{2}|} \|U_{\gamma'}^{\frac12} R^{C_L} U_{\beta'}^{\frac12}\|^s \cdot \|U_{\beta'}^{\frac12} R_F^{C_L} \chi_y\|_{\mathrm{HS}}^s \\
&\qquad \le CL^{s(d-1)} \sum_{\substack{x'\in Z(\gamma')\\ y'\in Z(\beta')}} \mathrm{e}^{-\frac{c}{L}|x-x'|-\mu_1 s|y-y'|} \|\chi_{x'} R^{C_L} \chi_{y'}\|^s. \qquad (12)
\end{aligned}$$

For the terms $j = \alpha$ and $j = \alpha'$ in the sum we borrow from $e_{\alpha,\alpha'}$ above and use that

$$e_{\alpha,\alpha'}^{\frac13} \le C \cdot \mathrm{e}^{-c|x-x'|/L}$$

if $j \in \{\alpha, \alpha'\}$ and $x' \in Z(j)$:

$$\begin{aligned}
&e_{\alpha,\alpha'}^{\frac13} \|\chi_L^+ R_F^{C_L} U_{\gamma'}^{\frac12}\|^s \cdot \|U_j^{\frac12} R^{C_L} U_{\beta'} R_F^{C_L} \chi_y\|_{\mathrm{HS}}^s \\
&\quad \le CL^{s(d-1)} \mathrm{e}^{-\mu_1 s||\gamma'-x|-\frac{L}{2}} \sum_{\substack{x'\in Z(\gamma')\\ y'\in Z(\beta')}} \mathrm{e}^{-\frac{c}{L}|x-x'|-\mu_1 s|y-y'|} \|\chi_{x'} R^{C_L} \chi_{y'}\|^s. \qquad (13)
\end{aligned}$$

Summing each of the three contributions from (12) and (13) to $e_{\alpha,\alpha}^{\frac13}\Sigma_Q$ over β', γ' (and extending the x'-sum in (13) to all of $\mathbb{Z}^d$) gives

$$\sum_{\beta',\gamma'\in\mathcal{I}} e_{\alpha,\alpha'}^{\frac13}\, \widehat{\mathbb{E}}\, \mathbb{E}[\Sigma_Q] \le CL^{2(d-1)} \sum_{x',y'\in\mathbb{Z}^d} \mathrm{e}^{-\frac{c}{L}|x-x'|-\mu_1 s|y-y'|}\, \tau_{x',y'}. \qquad (14)$$

We now show that summation over $\alpha, \alpha', \beta, \beta'$ gives a small prefactor. By exponential decay:

$$\sum_{\alpha,\alpha'} e_{\alpha,\alpha'}^{\frac13} \le CL^{2d}. \qquad (15)$$

Step 5. We analyze

$$\|\,\mathbb{E}(\chi_x R_F^{B_L} U_\beta R^{B_L} U_j^{\frac12}\|_{\mathrm{HS}}^s) \le \|\chi_x R_F^{B_L} U_\beta^{\frac12}\|_{\mathrm{HS}}^s \cdot \mathbb{E}(\|U_\beta^{\frac12} R^{B_L} U_j^{\frac12}\|^s).$$

If $|\beta - j| < \frac14 L$ then either $|x-\beta| \ge \frac18 L$ or $\operatorname{dist}(j, \partial C_L) \ge \frac18 L$. Since $j \in \mathcal{J} \setminus \{\gamma'\}$

$$\text{either } \mathbb{E}(\|\chi_x R_F^{B_L} U_\beta R^{B_L} U_j^{\frac12}\|_{\mathrm{HS}}^s), \quad e_{\alpha,\alpha'}^{\frac13} \quad \text{or} \quad \|U_\gamma^{\frac12} R_F^{B_L} \chi_L^-\|_{\mathrm{HS}}^s$$

is bounded by $L^{-\frac12 m_d}$; see Step 2 above. If, on the other hand $|\beta - j| \ge \frac14 L$ we can use Lemma 5 above to estimate

$$\mathbb{E}(\|U_\beta^{\frac12} R^{B_L} U_j^{\frac12}\|^s) \le C \cdot L^{-\frac12 m_d}.$$

Summing up these terms we get that

$$\sum_{\beta,\gamma\in\mathcal{I}} e_{\alpha,\alpha'}^{\frac13}\, \widehat{\mathbb{E}}\, \mathbb{E}[\Sigma_T] \le CL^{3d-1-\frac12 m_d} \qquad (16)$$

since β, γ run through at most cL^d different points of $\mathcal{I}$ in B_L. Putting the estimates from (14),(16),(15) together we arrive at:

$$\sum_{\alpha,\alpha',\beta,\beta',\gamma,\gamma'} A^4_{\alpha,\alpha',\beta,\beta',\gamma,\gamma'} \leq C \cdot L^{9d-5-\frac{1}{2}m_d} \sum_{x',y' \in \mathbb{Z}^d} \mathrm{e}^{-\frac{c}{L}|x-x'|-\mu_1 s|y-y'|}\, \tau_{x',y'}$$

which is the desired bound. To deal with the other terms appearing in $A_{\alpha,\alpha',\beta,\beta',\gamma,\gamma'}$ we just combine the corresponding steps to control the T and Q-sums respectively.

This concludes the proof of Proposition 6. □

For energies sufficiently close to E_0 we will now complete the proof of exponential decay of $\tau_{x,y}$, and thus of Theorem 1, by applying a discrete Gronwall-type argument to the recursion inequality established in Proposition 6.

For $\mu > 0$ consider the weighted ℓ^∞-space

$$X = \ell^\infty(\mathbb{Z}^{2d}; \mathrm{e}^{\mu|x-y|/2}),$$

i.e., for $\psi = (\psi_{x,y})$,

$$\|\psi\|_X = \sup_{x,y \in \mathbb{Z}^d} \mathrm{e}^{\mu|x-y|/2} |\psi_{x,y}|.$$

Lemma 7. *The operator A defined by*

$$(A\psi)_{x,y} = \sum_{x',y'} \mathrm{e}^{-\mu(|x-x'|+|y-y'|)} \psi_{x',y'}$$

is bounded as an operator in X as well as an operator in $\ell^\infty(\mathbb{Z}^{2d})$ with

$$\|A\|_X \leq C(d)\mu^{-2d} \quad \text{and} \quad \|A\|_{\ell^\infty} \leq C(d)\mu^{-2d}. \tag{17}$$

Proof of Lemma 7. The norm of A in X is the same as the norm of the operator $\hat{A}$ in $\ell^\infty(\mathbb{Z}^{2d})$ with kernel

$$\hat{A}_{xyx'y'} = \mathrm{e}^{\mu|x-y|/2} \mathrm{e}^{-\mu(|x-x'|+|y-y'|)} \mathrm{e}^{-\mu|x'-y'|/2}.$$

Thus

$$\begin{aligned}
\|A\|_X = \|\hat{A}\|_{\ell^\infty} &= \sup_{x,y} \sum_{x',y'} \hat{A}_{xyx'y'} \\
&\leq C \sup_{x,y} \iint \mathrm{d}x' \mathrm{d}y' \mathrm{e}^{\mu|x-y|/2} \mathrm{e}^{-\mu(|x-x'|+|y-y'|)} \mathrm{e}^{-\mu|x'-y'|/2} \\
&= C \sup_{\Delta} \iint \mathrm{d}s\, \mathrm{d}p\, \mathrm{e}^{\mu|\Delta|/2} \mathrm{e}^{-\mu(|s|+|p-\Delta|)-\mu|p-s|/2},
\end{aligned}$$

with the substitutions $s = x - x'$, $p = y' - x$, $\Delta = y - x$.

Bound the latter exponent through

$$\begin{aligned}
&\mu(|s| + |p-\Delta|) + \frac{\mu}{2}|p-s| \\
&\qquad = \left(\mu - \frac{\mu}{2}\right)(|s| + |p-\Delta|) + \frac{\mu}{2}(|\Delta - p| + |p-s| + |s|) \\
&\qquad \geq \frac{\mu}{2}(|s| + |p-\Delta|) + \frac{\mu}{2}|\Delta|.
\end{aligned}$$

After cancellation the integral factorizes and gives (17) for $\|A\|_X$ after scaling. The bound for $\|A\|_{\ell^\infty}$ is found more directly. □

This may be applied to the situation of Proposition 6 as it shows that for L sufficiently large the operator A with kernel

$$A_{xyx'y'} = L^{-2d-\kappa}\,\mathrm{e}^{-c(|x-x'|+|y-y'|)/L}$$

has norm less than one, both as an operator in $X = \ell^\infty(\mathbb{Z}^{2d}; \mathrm{e}^{c|x-y|/2L})$ and an operator in $\ell^\infty(\mathbb{Z}^{2d})$. Fix this L and choose $\delta = L^{-m}$, $I = [E_0, E_0 + \delta]$ in Theorem 1 and the definition of $\tau_{x,y}$.

The recursion inequality (6) now takes the form

$$\tau_{x,y} \le (A\tau)_{x,y} + b_{x,y} \tag{18}$$

with $b_{x,y} := C\mathrm{e}^{-c|x-y|/L}$. The conclusion of the proof of Theorem 1 is now the content of

Lemma 8.

$$\tau = (\tau_{x,y}) \in X.$$

Proof of Lemma 8. With $\mu = \frac{c}{L}$ define the diagonal operator

$$\mathcal{D} = \operatorname{diag}(\mathrm{e}^{\mu|x-y|/2}),$$

which is an isometry from X to $\ell^\infty(\mathbb{Z}^{2d})$. Let $\hat{\tau} = \mathcal{D}\tau$ and $\hat{b} = \mathcal{D}b \in \ell^\infty$. Let $\hat{A} = \mathcal{D}A\mathcal{D}^{-1}$. Then (18) implies that componentwise

$$\hat{\tau} \le \hat{A}\hat{\tau} + \hat{b}. \tag{19}$$

Since $\tau = \mathcal{D}^{-1}\hat{\tau}$ is bounded and A a bounded operator in $\ell^\infty(\mathbb{Z}^{2d})$, we have that $\hat{\tau} \in Y := \ell^\infty(\mathbb{Z}^{2d}; \mathrm{e}^{-\mu|x-y|/2})$ and $\hat{A}$ is a bounded operator in Y with non-negative kernel. Thus we obtain from (19) that

$$\hat{A}^n\hat{\tau} \le \hat{A}^{n+1}\hat{\tau} + \hat{A}^n\hat{b}$$

holds with finite components. Summation yields

$$\hat{\tau} \le \hat{A}^{N+1}\hat{\tau} + \sum_{n=0}^{N} \hat{A}^n\hat{b}$$

and thus

$$\tau \le A^{N+1}\tau + \sum_{n=0}^{N} A^n b$$

for all N.

$A\colon \ell^\infty \to \ell^\infty$ is a contraction and $\tau \in \ell^\infty$. Thus $A^{N+1}t \to 0$ in ℓ^∞ and componentwise. Also, $A\colon X \to X$ is a contraction and $b \in X$. Thus $\sum_{n=0}^{N} A^n b \to (I-A)^{-1}b \in X$ and componentwise as $N \to \infty$. We conclude

$$\tau \le (I-A)^{-1}b \in X.$$

Lemma 8 is proved. □

3. On the proof of Theorem 2

That the localization properties stated in Theorem 2 follow from the fractional moment bound for the resolvent established in Theorem 1 was demonstrated in [2]. Here we want to comment on two minor changes in the argument which are due to our somewhat different set-up.

First we note that spectral and dynamical localization as established in parts (a) and (b) of Theorem 2 hold for restrictions of H to arbitrary open domains G, and, in particular, that the exponential decay established in equation (2) holds with respect to the standard distance $|x-y|$ rather than the domain adapted distance $\mathrm{dist}_G(x,y)$ used in [2]. Given that the corresponding bound (1) in Theorem 1 is true for arbitrary G and in standard distance, this follows with exactly the same proof as in Section 2 of [2] (with one exception discussed below). That the authors of [2] chose to work with the domain adapted distance was in order to include more general regimes in which extended surface states might exist. This is not the case in the regime considered here.

Second, let us provide a few details on how to eliminate the use of the covering condition (3) from the proof of (2) provided in Section 2 of [2]. As done there one first considers bounded open $\Lambda \subset \mathbb{R}^d$ and defines

$$Y_\Lambda(I;x,y) := \sup_{f\in C_c(I),\,|f|\le 1} \|\chi_x f(H^\Lambda)\chi_y\|.$$

If E_n and ψ_n are the eigenvalues and corresponding eigenfunctions of H^Λ and f is as above, then $f(H^\Lambda) = \sum_{n\colon E_n\in I} f(E_n)\langle\psi_n,\,\cdot\,\rangle\psi_n$ readily implies

$$Y_\Lambda(I;x,y) \le \sum_{n\colon E_n\in I} \|\chi_x\psi_n\|\cdot\|\chi_y\psi_n\|.$$

At this point we modify the argument of [2] and write

$$\begin{aligned}\chi_y\psi_n &= \chi_y(H_F^\Lambda - E_n)^{-1}(H_F^\Lambda - E_n)\psi_n\\ &= \chi_y(H_F^\Lambda - E_n)^{-1}W\psi_n\\ &= \sum_{\alpha\in\mathcal{I}} \xi_\alpha\chi_y(H_F^\Lambda - E_n)^{-1}U_\alpha\psi_n.\end{aligned}$$

As all $E_n\in I$ have a uniform distance from $\inf\sigma(H_F^\Lambda)$ we get from Combes-Thomas Proposition 14 that

$$\begin{aligned}\|\chi_y\psi_n\| &\le C\sum_\alpha \|\chi_y(H_F^\Lambda - E_n)^{-1}U_\alpha^{1/2}\|\cdot\|U_\alpha^{1/2}\psi_n\|\\ &\le C\sum_\alpha \mathrm{e}^{-\mu_0|y-\alpha|}\|U_\alpha^{1/2}\psi_n\|.\end{aligned}$$

Inserting above yields

$$Y_\Lambda(I;x,y) \le C\sum_\alpha \mathrm{e}^{-\mu_0|y-\alpha|}Q_1(I;x,\alpha)$$

with $Q_1(I;x,\alpha) = \sum_{n\colon E_n\in I}\|\chi_x\psi_n\|\cdot\|U_\alpha^{1/2}\psi_n\|$ defined as in [2], where the bound $\mathbb{E}(Q_1(I;x,\alpha)) \le C\mathrm{e}^{-\mu_1|x-\alpha|}$ is established without any further references to the covering condition. Thus we conclude

$$\mathbb{E}\big(Y_\Lambda(I;x,y)\big) \le C\mathrm{e}^{-\mu_2|x-y|}. \tag{20}$$

The rest of the proof of Theorem 2, in particular the extension of (20) to infinite volume and a supremum over arbitrary Borel functions, follows the argument in [2] without change.

4. Localization for continuum random surface models

Random surface models have attracted quite some interest with most of the work dealing with the discrete case [9, 14, 15, 17, 16, 18, 19] and some with the continuum case [20, 24, 7, 8], as we do here. Our aim in this section is to show that under suitable conditions such surface models obey condition (A4) above. To achieve it, we combine recent results from [24] with a technique from [29].

As usual, the background is assumed to be partially periodic:

(B1) Fix $1 \leq d_1 \leq d$ and write $\mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, $x = (x_1, x_2)$; assume that $V_0 \in L^2_{\text{loc,unif}}(\mathbb{R}^d)$ is real-valued and periodic with respect to the first variable, i.e.,

$$V_0(x_1 + m, x_2) = V_0(x_1, x_2) \text{ for } m \in \mathbb{Z}^{d_1}.$$

Denote $H_0 := -\Delta + V_0$.

In order to state our second requirement, let us recall some facts from Bloch theory. For more details, see [24]. For V_0, H_0 as in (B1) we get a direct integral decomposition

$$H_0 = (2\pi)^{-d_1} \int_{\mathbb{T}^{d_1}}^{\oplus} h_\theta \, \mathrm{d}\theta,$$

where $\mathbb{T}^{d_1} = \mathbb{R}^{d_1}/(2\pi\mathbb{Z})^{d_1}$ is the d_1-dimensional torus and

$$h_\theta = -\Delta + V_0 \text{ in } L^2(S_1)$$

with θ-periodic boundary conditions on the unit strip $S_1 = \Lambda_1(0) \times \mathbb{R}^{d_2}$. We now fix the assumption

(B2)

$$\inf \sigma(h_0) < \inf \sigma_{\text{ess}}(h_0).$$

It is well known that under (B2) we have that

$$E_0 := \inf \sigma(H_0) = \inf \sigma(h_0)$$

and there is a positive eigensolution ψ_0 of the distributional equation

$$H_0 \psi_0 = E_0 \psi_0,$$

see [24, 23] and the references therein. Finally, our random perturbation is assumed to satisfy

(B3) The set $\mathcal{I} \subset \mathbb{R}^d$, where the random impurities are located, is uniformly discrete, i.e., $\inf\{|\alpha - \beta| : \alpha \neq \beta \in \mathcal{I}\} =: r_{\mathcal{I}} > 0$. Moreover $\mathcal{I}$ is dense near the surface $\mathbb{R}^{d_1} \times \{0\}$ in the sense that there exist $R_\perp, c_\perp > 0$ such that for L large enough and $x_1 \in \mathbb{R}^{d_1}$:

$$\#\big[\mathcal{I} \cap \big(\Lambda_L(x_1) \times \Lambda_{R_\perp}(0)\big)\big] \geq c_\perp L^{d_1}.$$

We will see that (B1)-(B3) ensure (A4) from Section 1. Of course, there might be other ways to verify (A4) for surface-like potentials so that Theorems 1 and 2 could, in principle, be used for other examples.

Theorem 9. *Assume* (B1)-(B3) *and* (A3). *Then there exist* $\delta > 0$, $0 < s < 1$, $\mu > 0$ *and* $C < \infty$ *such that for* $I := [E_0, E_0 + \delta]$, *all open sets* $G \subset \mathbb{R}^d$ *and* $x, y \in \mathbb{R}^d$

$$\sup_{E \in I,\, \varepsilon > 0} \mathbb{E}(\|\chi_x (H^G - E - \mathrm{i}\varepsilon)^{-1} \chi_y\|^s) \leq C\, \mathrm{e}^{-\mu|x-y|}. \tag{21}$$

In particular, the following consequences hold:

(a) *The spectrum of* H^G *in* I *is almost surely pure point with exponentially decaying eigenfunctions.*

(b) *There are* $\mu > 0$ *and* $C < \infty$ *such that for all* $x, y \in \mathbb{Z}^d$,

$$\mathbb{E}\big(\sup_{t \in \mathbb{R}} \|\chi_x \mathrm{e}^{-\mathrm{i}tH^G} P_I(H^G) \chi_y\|\big) \leq C \mathrm{e}^{-\mu|x-y|}. \tag{22}$$

The rest of this section is devoted to deducing (A4) under the assumptions of the Theorem. Note that this will be accomplished once we have shown the following, where

$$S_L = S_L(x_1) := \Lambda_L(x_1) \times \mathbb{R}^{d_2}$$

denotes the strip of side length L centered at $x_1 \in \mathbb{R}^{d_1}$ perpendicular to the "surface" $\mathbb{R}^{d_1} \times \{0\}$.

Proposition 10. *For all* $\gamma, \xi > 0$ *there exists* $L(\gamma, \xi)$ *such that for all odd integers* $L \geq L(\gamma, \xi)$ *and* $x_1 \in \mathbb{Z}^{d_1}$*:*

$$\mathbb{P}\big\{\sigma(H^{S_L(x_1)}) \cap [E_0, E_0 + L^{-\gamma}] \neq \emptyset\big\} \leq L^{-\xi}. \tag{23}$$

In fact, (A4)(ii) then follows, since $H^{\Lambda_L(x)} \geq H^{S_L(x_1)}$ and therefore $E_0 \leq \inf \sigma(H^{S_L(x_1)}) \leq \inf \sigma(H^{\Lambda_L(x)})$.

We will actually prove the analogue of Proposition 10 with Dirichlet boundary conditions replaced by suitable Robin boundary conditions that are defined using the periodic ground state ψ_0 introduced above. Assume, for later convenience, that

$$\int_{S_1} |\psi_0(x)|^2 \mathrm{d}x = 1.$$

We consider on S_L, $L \in 2\mathbb{N} - 1$, *Mezincescu boundary conditions*, given as follows. Let

$$\chi(x) := -\frac{1}{\psi_0(x)} \nabla_n \psi_0(x),$$

where ∇_n denotes the outer normal derivatives. The Mezincescu boundary condition can be thought of as the following requirement for functions ϕ in the domain of $H_\chi^{S_L}$:

$$\nabla_n \phi(x) = -\chi(x)\phi(x) \text{ for } x \in \partial S_L.$$

For the formal definition of $H_\chi^{S_L}$ via quadratic forms and more background, see Mezincescu's original paper [25] as well as [23, 24]. In particular, we immediately get the following important relations in the sense of the corresponding quadratic forms:

$$H_\chi^{S_L} \leq H^{S_L} \tag{24}$$

as well as

$$H_\chi^{S_L} \geq \bigoplus_{k=1}^{n} H_\chi^{S_{l_k}(y_k)}, \tag{25}$$

whenever the strip S_L is divided into disjoint strips $S_{l_k}(y_k)$ whose closures exhaust the closure of S_L.

Proof of Proposition 10. Due to the form inequality (24) above it remains to prove the estimate for $H_\chi^{S_L}$.

Denoting the bottom eigenvalue of an operator H by $E_1(H)$ (*caution*: here our notation differs from the one in [23, 24], where the second eigenvalue is denoted by $E_1(H)$) we see that

$$\sigma(H_\chi^{S_L}) \cap [E_0, E_0 + L^{-\gamma}] \neq \emptyset \Longleftrightarrow E_1(H_\chi^{S_L}) \leq E_0 + L^{-\gamma}.$$

Step 1. There exist $b, K, \beta > 0$ such that

$$\mathbb{P}\big\{E_1(H_\chi^{S_L}) \leq E_0 + bL^{-2}\big\} \leq K \cdot \exp(-K \cdot L^{d_1}). \tag{26}$$

We use here the method from [29]. Denote $H(t,\omega) := (H_0 + tV_\omega)_\chi^{S_L}$, and its first eigenvalue by $E_1(t,\omega)$. Since $E_1(t,\omega)$ increases in t the event in (26) implies that $E_1(t,\omega)$ be small for all $t \leq 1$ which in turn implies that $E_1'(0,\omega)$ must be small.

We infer from [24], Theorem 3.25 that the gap between the first two eigenvalues satisfies

$$E_2(0,\omega) - E_1(0,\omega) \geq \text{const.}\, L^{-2}.$$

As in [29], Lemma 2.3 this gives that

$$|E_1(t,\omega) - (E_0 + t \cdot E_1'(0,\omega))| \leq KL^2 \cdot t^2 \text{ for } 0 \leq t \ \leq \tau \cdot L^{-2}. \tag{27}$$

Now assume that

$$E_1(H_\chi^{S_L}) \leq E_0 + bL^{-2}$$

for $b > 0$. From (27) we get that

$$E_1'(0,\omega) \leq c(b)$$

with $c(b) \to 0$ for $b \to 0$.

On the other hand

$$E_1'(0,\omega) = (V_\omega \psi_{0,L} | \psi_{0,L})$$

where $\psi_{0,L}$ is the normalized ground state of $H_{0,\chi}^{S_L}$. Now, the boundary condition of $H_{0,\chi}^{S_L}$ is defined so as to make sure that ψ_0 is an eigenfunction; see the discussion in [24]. Therefore $\psi_{0,L} = L^{-d_1/2}\psi_0$ and we get

$$\begin{aligned} E_1'(0,\omega) &= (V_\omega \psi_{0,L} | \psi_{0,L}) \\ &= L^{-d_1} \sum_{\alpha \in \mathcal{I}} \eta_\alpha(\omega) \cdot \int_{S_L} U_\alpha(x) |\psi_0(x)|^2 \mathrm{d}x \\ &\geq L^{-d_1} \sum_{\alpha \in \mathcal{I} \cap S_{L-r_U}} \eta_\alpha(\omega) \cdot c_U \cdot \int_{\Lambda_{r_U}(\alpha)} |\psi_0(x)|^2 \mathrm{d}x. \end{aligned}$$

Since, by (B3), there are at least $c_\perp(L-r_U)^{d_1}$ elements of $\mathcal{I}\cap S_{L-r_U}$ in $\mathbb{R}^{d_1}\times\Lambda_{R_\perp}(0)$ and

$$\inf_{(x_1,x_2)\in\mathbb{R}^{d_1}\times\Lambda_{R_\perp}(0)}\int_{\Lambda_{r_U}(x_1,x_2)}|\psi_0(x)|^2\mathrm{d}x>0$$

we arrive at

$$E_1'(0,\omega)\geq c_1\cdot\frac{1}{|\mathcal{I}_\perp|}\sum_{\alpha\in\mathcal{I}_\perp}\eta_\alpha(\omega) \tag{28}$$

with $c_1>0$ and independent variables η_α running through an index set $\mathcal{I}_\perp$ of cardinality at least $c_2L^{d_1}$. If we now choose $b>0$ so small that $\frac{c(b)}{c_1}<M$ where M is smaller than the mean of all the η_α's we get that:

$$\begin{aligned}\mathbb{P}\big\{E_1(H_\chi^{S_L})\leq E_0+bL^{-2}\big\}&\leq\mathbb{P}\big\{c_1\cdot\frac{1}{|\mathcal{I}_\perp|}\sum_{\alpha\in\mathcal{I}_\perp}\eta_\alpha(\omega)\leq c(b)\big\}\\&\leq K\cdot\exp(-\beta_0|\mathcal{I}_\perp|)\\&\leq K\cdot\exp(-\beta L^{d_1}),\end{aligned}$$

by a standard large deviation estimate; see [21] or [31], Theorem 1.4. This finishes the proof of Step 1.

Step 2. To deduce the desired bound from Step 1 we divide the strip S_L into disjoint strips $S_{l_k}(y_k)$ whose closures exhaust the closure of S_L and such that

$$L^{-\gamma}\leq b\cdot l_k^{-2}\leq 42\cdot L^{-\gamma},\quad l_k\in 2\mathbb{N}+1$$

which is possible for L large enough.

Their number n is at most const. $L^{(1-\frac{\gamma}{2})d_1}$. By (25) we know that

$$E_1\big(H_\chi^{S_L}\big)\geq\min_{1\leq k\leq n}E_1\big(H_\chi^{S_{l_k}(y_k)}\big)$$

so that

$$\begin{aligned}\mathbb{P}\big\{E_1(H_\chi^{S_L})\leq E_0+L^{-\gamma}\big\}&\leq\mathbb{P}\big\{\min_{1\leq k\leq n}E_1(H_\chi^{S_{l_k}(y_k)})\leq E_0+L^{-\gamma}\big\}\\&\leq\sum_{k=1}^n\mathbb{P}\big\{E_1(H_\chi^{S_{l_k}(y_k)})\leq E_0+L^{-\gamma}\big\}\\&\leq\sum_{k=1}^n\mathbb{P}\big\{E_1(H_\chi^{S_{l_k}(y_k)})\leq E_0+b\cdot l_k^{-2}\big\}\\&\leq n\cdot K\cdot\exp(-\beta {l_k}^{d_1})\\&\leq L^{-\xi}\end{aligned}$$

provided L is large enough. □

Remarks. (1) In cases where the operator H is ergodic, a stronger bound than (23) is provided in [24, Proposition 5.2]. Their bound is in terms of the integrated density of states for which [24] establishes Lifshits asymptotics. As we are only interested in localization properties here, the bound (23) suffices and allows to handle the non-ergodic random potentials defined in (B3) and (A3).

(2) We have established localization near the bottom of the spectrum for the random surface models considered in this section. If $d_1 = 1$ one expects for physical reasons that the entire spectrum of H below $\inf \sigma(H_F)$ (see (A4)) is localized. A corresponding result for lattice operators has been proven in [18] (in situations where H_F is the discrete Laplacian and $d_2 = 1$). To show this for continuum models remains an open problem.

5. Anderson models with displacement

By considering the special case $d_1 = d$, the results of the previous Section also cover "usual" Anderson models, sometimes also called alloy models. Note that in this case (B2) becomes trivial. Let us nevertheless state the assumptions and result again for this case, mainly because we want to point out below that the obtained bounds hold uniformly in the geometric parameters describing the random potential. This will then be applied to models with random displacements. Here are the assumptions we rely upon:

(D1) $V_0 \in L^2_{\mathrm{loc,unif}}(\mathbb{R}^d)$ is real-valued and periodic.

(D2) The set $\mathcal{I} \subset \mathbb{R}^d$, where the random impurities are located, is uniformly discrete, i.e., $\inf\{|\alpha - \beta| : \alpha \neq \beta \in \mathcal{I}\} =: r_{\mathcal{I}} > 0$ and uniformly dense, i.e., there exists $R_{\mathcal{I}} > 0$ such that $\Lambda_{R_{\mathcal{I}}}(x) \cap \mathcal{I} \neq \emptyset$ for every $x \in \mathbb{R}^d$.

Theorem 11. *Assume* (D1), (D2) *and* (A3). *Then there exist* $\delta > 0$, $0 < s < 1$, $\mu > 0$ *and* $C < \infty$ *such that for* $I := [E_0, E_0 + \delta]$, *all open sets* $G \subset \mathbb{R}^d$ *and* $x, y \in \mathbb{R}^d$

$$\sup_{E \in I, \varepsilon > 0} \mathbb{E}(\|\chi_x (H^G - E - \mathrm{i}\varepsilon)^{-1} \chi_y\|^s) \leq C\, \mathrm{e}^{-\mu|x-y|}. \tag{29}$$

In particular, the following consequences hold:

(a) *The spectrum of* H^G *in* I *is almost surely pure point with exponentially decaying eigenfunctions.*

(b) *There are* $\mu_1 > 0$ *and* $C_1 < \infty$ *such that for all* $x, y \in \mathbb{Z}^d$,

$$\mathbb{E}\big(\sup_{t \in \mathbb{R}} \|\chi_x \mathrm{e}^{-\mathrm{i}tH^G} P_I(H^G) \chi_y\|\big) \leq C_1 \mathrm{e}^{-\mu_1|x-y|}. \tag{30}$$

Here all the constants $\delta, s, C, \mu, C_1, \mu_1$ *can be chosen to only depend on the potential through the parameters* $V_0, \eta_{\max}, M_\rho, c_U, C_U, r_U, R_U, r_{\mathcal{I}}, R_{\mathcal{I}}$.

To this end we first observe that (D1), (D2) and (A3) imply (A4) with constants E_F, m and L^* only depending on the listed parameters:

Proposition 12. *Assume* (D1), (D2) *and* (A3). *Then there exist*

$$E_1 = E_1(V_0, \eta_{\max}, M_\rho, c_U, C_U, r_U, R_U, r_{\mathcal{I}}, R_{\mathcal{I}}) > E_0,$$
$$m = m(V_0, \eta_{\max}, M_\rho, c_U, C_U, r_U, R_U, r_{\mathcal{I}}, R_{\mathcal{I}}) \in (0, 2)$$

and $L^* = L^*(\ldots)$ *such that*

(1) $E_F \geq E_1$.

(2) *For* $m_d := 42 \cdot d$, *all* $L \geq L^*$ *and* $x \in \mathbb{Z}^d$:

$$\mathbb{P}\big(\sigma(H^{\Lambda_L(x)}(\omega)) \cap [E_0, E_0 + L^{-m}] \neq \emptyset\big) \leq L^{-m_d}.$$

Proof. First we show that (D2) implies that there exist $c_{\mathcal{I}}, C_{\mathcal{I}}$ and $L_{\mathcal{I}}$ depending only on $r_{\mathcal{I}}, R_{\mathcal{I}}$ such that for all $L \geq L_{\mathcal{I}}$:

$$c_{\mathcal{I}} \cdot L^d \leq \# (\mathcal{I} \cap \Lambda_L(x)) \leq C_{\mathcal{I}} \cdot L^d. \tag{31}$$

The upper bound follows from uniform discreteness:

$$\# (\mathcal{I} \cap \Lambda_L(x)) \cdot |B_{r_{\mathcal{I}/2}}| \leq |\Lambda_{L+r_{\mathcal{I}/2}}| \leq (2L)^d,$$

provided $L \geq r_{\mathcal{I}}/2$. For the lower bound use uniform denseness: Divide $\Lambda_L(x)$ into disjoint boxes of side length $R_{\mathcal{I}}$. If $L \geq 2R_{\mathcal{I}}$ there are at least $(L/2R_{\mathcal{I}})^d$ of them each of which contains at least one point from $\mathcal{I}$.

Now we can use the analysis of the preceding Section. Since the relevant quantities depend only on the indicated parameters, the assertions follow. □

With this uniform version of (A4) and the proofs provided in Sections 2 and 3 we also get corresponding uniform versions of Theorems 1 and 2, i.e. Theorem 11.

As a specific application of the previous observation, we can start from an Anderson model as above and additionally vary the set $\mathcal{I}$ in a random way, as long as $r_{\mathcal{I}}$ and $R_{\mathcal{I}}$ obey uniform upper and lower bounds. Instead of formulating the most general result in this direction we look at models that were introduced in [10] and further studied in [32].

(D3) Let η_j, $j \in \mathbb{Z}^d$ be independent random couplings, defined on a probability space Ω with distribution ρ_j and U_j as in (A3).

(D4) Let x_j, $j \in \mathbb{Z}^d$ be independent random vectors of length at most $\frac{1}{3}$ in $\mathbb{R}^d$; denote the corresponding probability space by $\widetilde{\Omega}$.

Define

$$H(\omega, \widetilde{\omega}) := -\Delta + V_0 + \sum_{j \in \mathbb{Z}^d} \eta_j(\omega) U_j(\,\cdot - j - x_j(\widetilde{\omega})).$$

Corollary 13. *Assume* (D1), (D3), (D4). *Then, for $H(\omega, \widetilde{\omega})$ as above there exist $\delta > 0$, $0 < s < 1$, $\mu > 0$ and $C < \infty$ such that for $I := [E_0, E_0 + \delta]$, all open sets $G \subset \mathbb{R}^d$ and $x, y \in \mathbb{R}^d$*

$$\sup_{E \in I, \varepsilon > 0} \widetilde{\mathbb{E}}\, \mathbb{E}(\|\chi_x (H^G - E - \mathrm{i}\varepsilon)^{-1} \chi_y\|^s) \leq C\, \mathrm{e}^{-\mu|x-y|}. \tag{32}$$

In particular, the following consequences hold:

(a) *The spectrum of H^G in I is almost surely pure point with exponentially decaying eigenfunctions.*

(b) *There are $\mu > 0$ and $C < \infty$ such that for all $x, y \in \mathbb{Z}^d$,*

$$\widetilde{\mathbb{E}}\, \mathbb{E}\big(\sup_{t \in \mathbb{R}} \|\chi_x \mathrm{e}^{-\mathrm{i}tH^G} P_I(H^G) \chi_y\|\big) \leq C \mathrm{e}^{-\mu|x-y|}. \tag{33}$$

Proof. The corresponding inequality holds uniformly in $\widetilde{\omega}$ by what we proved above. □

Note that in this last Corollary we have not assumed that the random perturbations cover the whole space. In that respect our result provides substantial progress as compared to [10, 32].

Appendix A. Some technical tools

Here we collect some technical background which was used in Section 2 above. All of this is known. We either provide references or, for convenience, in some cases sketch the proof.

A.1. **Combes-Thomas bounds.** Proofs of the following improved Combes-Thomas bound can be found in [5] (where it was first observed) and [30]. We state it here under assumptions which are sufficient for our applications. In particular, we assume $d \le 3$, while the result holds in arbitrary dimension for a suitably modified class of potentials. As above, for an open $G \subset \mathbb{R}^d$ we denote by H^G the restriction of $-\Delta + V$ to $L^2(G)$ with Dirichlet boundary conditions.

Proposition 14. *Let $d \le 3$, $V \in L^2_{\mathrm{loc,unif}}(\mathbb{R}^d)$ with $\sup_x \|V\chi_{\Lambda_1(x)}\|_2 \le M$. Let $M \ge 1$ and $R > 0$. Then there exist $c_1 = c_1(M,R)$ and $c_2 = c_2(M,R)$ such that the following conditions*

(i) *$G \subset \mathbb{R}^d$ open, $A, B \subset G$, $\mathrm{dist}(A,B) =: \delta > 0$,*
(ii) *$(r,s) \subset \rho(H^G) \cap (-R,R)$, $E \in (r,s)$ and $\eta := \mathrm{dist}(E,(r,s)^c) > 0$,*

imply the estimate

$$\sup_{\varepsilon\in\mathbb{R}} \|\chi_A(H^G - E - \mathrm{i}\varepsilon)^{-1}\chi_B\| \le \frac{c_1}{\eta}\mathrm{e}^{-c_2\sqrt{s-r}\,\eta^{1/2}\delta}. \tag{34}$$

Note that the results in [5] and [30] are stated for $\varepsilon = 0$, but the proofs are easily adjusted to show that the bounds are uniform in the additional imaginary part.

A.2. **Combes-Thomas bounds in Hilbert-Schmidt norm.** A consequence of (34) is that $\|\chi_x(H^{(G)} - E - \mathrm{i}\varepsilon)^{-1}\chi_y\|$ decays exponentially in $|x-y|$. Due to the restriction to $d \le 3$ this is also true in Hilbert-Schmidt norm:

Proposition 15. *Let $d \le 3$, $V \in L^2_{\mathrm{loc,unif}}(\mathbb{R}^d)$, $H = -\Delta + V$ in $L^2(\mathbb{R}^d)$ and $I \subset (-\infty, \inf\sigma(H))$ a compact interval. Then there exist $C < \infty$ and $\mu > 0$ such that*

$$\sup_{\substack{E\in I,\,\varepsilon>0\\ G\subset\mathbb{R}^d\,open}} \|\chi_x(H^G - E - \mathrm{i}\varepsilon)^{-1}\chi_y\|_{\mathrm{HS}} \le C\mathrm{e}^{-\mu|x-y|} \tag{35}$$

for all $x, y \in \mathbb{R}^d$.

Proof. Let us sketch the proof by combining several well known facts. To this end, let S_p denote the p-th Schatten class, i.e. the set of all bounded operators A such that $\|A\|_p := (\mathrm{tr}\,|A|^p)^{1/p} < \infty$. As $d \le 3$, by Theorem B.9.3 of [28] we have

$$\|\chi_x(H-E)^{-1/2}\|_p \le C_1 < \infty \tag{36}$$

for each $p > 3$ and $E < \inf\sigma(H)$. The proof provided in [28] shows that C_1 can be chosen uniform in $x \in \mathbb{R}^d$ and $E \in I$. In the sense of quadratic forms it holds that $H^G \ge H$ for each open $G \subset \mathbb{R}^d$, i.e. $\|(H-E)^{1/2}(H^G -$

$E)^{-1/2}\| \le 1$ for all $E < \inf \sigma(H)$, see e.g. Section VI.2 of [22]. Thus

$$\begin{aligned} \|\chi_x(H^G - E)^{-1/2}\|_p &\le \|\chi_x(H - E)^{-1/2}\|_p \, \|(H - E)^{1/2}(H^G - E)^{-1/2}\| \\ &\le C_1 < \infty. \end{aligned} \tag{37}$$

The Hölder property of Schatten classes implies

$$\|\chi_x(H^G - E)^{-1}\chi_y\|_{p/2} \le C_1^2 \tag{38}$$

uniformly in $x, y \in \mathbb{R}^d$, $E \in I$ and $G \subset \mathbb{R}^d$ open. From the resolvent identity

$$\chi_x(H^G - E - \mathrm{i}\varepsilon)^{-1}\chi_y = \chi_x(H^G - E)^{-1}\chi_y + \mathrm{i}\varepsilon\chi_x(H^G - E - \mathrm{i}\varepsilon)^{-1}(H^G - E)^{-1}\chi_y$$

we easily see that

$$\|\chi_x(H^G - E - \mathrm{i}\varepsilon)^{-1}\chi_y\|_{p/2} \le C_2 < \infty \tag{39}$$

holds uniformly also in the additional parameter $\varepsilon \in \mathbb{R}$. By Proposition 14 we also have $C_3 < \infty$ and $\mu_1 > 0$ such that

$$\|\chi_x(H^G - E - \mathrm{i}\varepsilon)^{-1}\chi_y\| \le C_3 \mathrm{e}^{-\mu_1|x-y|} \tag{40}$$

uniform in G, $E \in I$ and $\varepsilon \in \mathbb{R}$. As we may choose $p/2 \in (3/2, 2)$, (35) follows from (39) and (40) by interpolation, more precisely from the fact that $\|\cdot\|_{\mathrm{HS}} = \|\cdot\|_2$ and $\|A\|_2^2 = \operatorname{tr}|A|^2 = \operatorname{tr}(|A|^{p/2}|A|^{2-p/2}) \le \|A\|^{2-p/2}\|A\|_{p/2}^{p/2}$. □

A.3. **A fractional-moment bound.** The next result and its proof are found in [2], where it played a central role in the extension of the fractional-moment method to Anderson-type random Schrödinger operators in the continuum.

Recall that an operator A is called dissipative if $\mathrm{Im}\langle A\varphi, \varphi\rangle \ge 0$ for all $\varphi \in D(A)$. It is called maximally dissipative if it has no proper dissipative extension. Below we also use the notation $|\cdot|$ for Lebesgue measure in $\mathbb{R}^2$.

Proposition 16. *There exists a universal constant $C < \infty$ such that for every separable Hilbert space $\mathcal{H}$, every maximally dissipative operator A in $\mathcal{H}$ with strictly positive imaginary part (i.e. $\mathrm{Im}\langle A\varphi, \varphi\rangle \ge \delta\|\varphi\|^2$ for some $\delta > 0$ and all $\varphi \in D(A)$), for arbitrary Hilbert-Schmidt operators M_1, M_2 in $\mathcal{H}$, for arbitrary bounded non-negative operators U_1, U_2 in $\mathcal{H}$, and for all $t > 0$ the following holds:*

$$\begin{aligned} \big|\big\{(v_1, v_2) \in [0,1]^2 : \|M_1 U_1^{1/2}(A - v_1U_1 - v_2U_2)^{-1}U_2^{1/2}M_2\|_{\mathrm{HS}} > t\big\}\big| \\ \le C\|M_1\|_{\mathrm{HS}}\|M_2\|_{\mathrm{HS}} \cdot \frac{1}{t}. \end{aligned} \tag{41}$$

The weak-L_1-type bound (41) yields a fractional moment bound:

Corollary 17. *Let $s \in (0,1)$. Then for the constant C and operators A, M_1, M_2, U_1, U_2 as in Proposition 16,*

$$\begin{aligned} \int_0^1 \mathrm{d}v_1 \int_0^1 \mathrm{d}v_2 \|M_1 U_1^{1/2}(A - v_1U_1 - v_2U_2)^{-1}U_2^{1/2}M_2\|_{\mathrm{HS}}^s \\ \le \frac{C^s}{1-s}\|M_1\|_{\mathrm{HS}}^s\|M_2\|_{\mathrm{HS}}^s. \end{aligned} \tag{42}$$

This follows with layer-cake integration, which gives for the l.h.s. of (42)

$$\int_0^1 \mathrm{d}v_1 \int_0^1 \mathrm{d}v_2 \|\ldots\|^s \leq \int_0^\infty \left|\{(v_1, v_2) \in [0,1]^2 : \|\ldots\| > t^{1/s}\}\right| \mathrm{d}t.$$

The integrand is bounded by the minimum of 1 and a bound following from (41). Splitting the integral accordingly leads to (42).

Remarks. (1) The use of the interval $[0, 1]$ as support of v_1, v_2 in Proposition 16 and Corollary 17 is not essential. Using shifting and scaling it can be replaced by an arbitrary compact interval K, with constants becoming K-dependent.

(2) In our applications maximally dissipative operators arise in the form $A = -(S - E - \mathrm{i}\varepsilon)$ for self-adjoint operators S, with $\varepsilon > 0$ providing a strictly positive imaginary part.

(3) Note that, as seen from the argument in [2], a bound like (42) also holds in the "diagonal" case, i.e. for $\int_0^1 \mathrm{d}v \, \|MU^{1/2}(A - vU)^{-1}U^{1/2}M\|_{\mathrm{HS}}^s$.

Acknowledgement. S. N. , P. S. and G. S. gratefully acknowledge financial support by the University Paris 7 Denis Diderot, where part of this work was done. S. N. and G. S. also received support through US-NSF grant no. DMS-0245210, P. S. received support from the DFG.

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