Inverse spectral problem for radial Schrödinger operator on $[0,1]$.

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Abstract

For a class of singular Sturm-Liouville equations on the unit interval with explicit singularity $a(a+1)/x^2, a \in \mathbb{N}$, we consider an inverse spectral problem. Our goal is the global parametrization of potentials by spectral data noted by λ^a, and some norming constants noted by κ^a. For $a = 0$ and 1, $\lambda^a \times \kappa^a$ was already known to be a global coordinate system on $L^2_{\mathbb{R}}(0,1)$. With the help of transformation operators, we extend this result to any non-negative integer a and give a description of isospectral sets.

1 Introduction

Inverse spectral problem for Schrödinger operator with radial potential $\mathrm{H} := -\Delta + q(\|X\|)$ acting on the unit ball of $\mathbb{R}^3$ leads by separation of variables (see [10], p. $160-161$) to consider a collection of singular differential operators $\mathrm{H}_a(q)$, $a \in \mathbb{N}$ acting on $L^2_{\mathbb{R}}(0,1)$, defined by

$$\mathrm{H}_a(q)y(x) := \left(-\frac{d^2}{dx^2} + \frac{a(a+1)}{x^2} + q(x)\right) y(x) = \lambda y(x), \quad x \in [0,1], \lambda \in \mathbb{C}, \tag{1.1}$$

with Dirichlet boundary conditions

$$y(0) = y(1) = 0. \tag{1.2}$$

Our purpose is the stability of the inverse spectral problem for H_a. For this, we construct for each $a \in \mathbb{N}$, a standard map $\lambda^a \times \kappa^a$ for potentials $q \in L^2_{\mathbb{R}}(0,1)$ with spectral data λ^a and some terminal velocities κ^a.

Borg [1] and Levinson [7] proved that $\lambda^0 \times \kappa^0$ is one-to-one on $L^2_{\mathbb{R}}(0,1)$. Pöschel and Trubowitz [9] improved this result obtaining $\lambda^0 \times \kappa^0$ as a global real-analytic coordinate system on $L^2_{\mathbb{R}}(0,1)$. Guillot and Ralston [6] extended their results to $\lambda^1 \times \kappa^1$. Next Carlson [3] (see also Zhornitskaya and Serov [12]) proved that for all real $a \geq -1/2$, $\lambda^a \times \kappa^a$ is one-to-one on $L^2_{\mathbb{R}}(0,1)$.

We complete these works, to any integer a, with the realization of $\lambda^a \times \kappa^a$ as a local real-analytic coordinate system on $L^2_{\mathbb{R}}(0,1)$. Moreover, we obtain a description for the set of potentials with same spectral data : these so-called isospectral sets are proved to be regular manifolds, expression for their tangent and normal spaces are given. The key point of the proof lies in the use of transformation operators: they help us to handle Bessel functions naturally underlying in this problem.

2 The direct spectral problem.

Part of the following properties are deduced or came from [6], [3]-[2] and [12]; nevertheless main structure is given by [9].

Define $\omega := \lambda^{1/2}$ (the square root determination is pointless because of the parity of functions using it) and for all $n \in \mathbb{N}$

$$(2n+1)!! := 1 \cdot 3 \cdots (2n+1).$$

A fundamental system of solutions for (1.1) when $q = 0$ is given by

$$u(x,\lambda) = \frac{(2a+1)!!}{\omega^{a+1}}\, j_a(\omega x), \quad v(x,\lambda) = -\frac{\omega^a}{(2a+1)!!}\, \eta_a(\omega x),$$

where j_a and η_a are spherical Bessel functions (see section 5). This family is called fundamental since its wronskian $\mathcal{W}(u,v)$ is equal to 1. From their behavior near $x = 0$, $u(x,\lambda)$ is called the regular solution, it is analytic on $[0,1] \times \mathbb{C}$; $v(x,\lambda)$ is called the singular solution, it is analytic on $(0,1] \times \mathbb{C}$.

Following Guillot and Ralston [6] but also Zhornitskaya and Serov [12], we construct solutions for (1.1) by a Picard's iteration method from u and v.

Let φ and $\widetilde{\psi}$ be defined by

$$\varphi(x,\lambda,q) = \sum_{k\geq 0} \varphi_k(x,\lambda,q), \quad \widetilde{\psi}(x,\lambda,q) = \sum_{k\geq 0} \widetilde{\psi}_k(x,\lambda,q)$$

with

$$\begin{cases} \varphi_0(x,\lambda,q) = u(x,\lambda), \\ \varphi_{k+1}(x,\lambda,q) = \displaystyle\int_0^x \mathcal{G}(x,t,\lambda)q(t)\varphi_k(t,\lambda,q)dt, \quad k \in \mathbb{N}; \end{cases} \tag{2.1}$$

$$\begin{cases} \widetilde{\psi}_0(x,\lambda,V) = v(x,\lambda), \\ \widetilde{\psi}_{k+1}(x,\lambda,q) = -\displaystyle\int_x^1 \mathcal{G}(x,t,\lambda)q(t)\widetilde{\psi}_k(t,\lambda,q)dt, \quad k \in \mathbb{N}. \end{cases} \tag{2.2}$$

$\mathcal{G}$ is called Green function and is given by

$$\mathcal{G}(x,t,\lambda) = v(x,\lambda)u(t,\lambda) - u(x,\lambda)v(t,\lambda), \quad (x,t) \in (0,1) \times (0,1).$$

Now follows the expected result:

Lemma 2.1. *Series defined in (2.1), respectively in (2.2) uniformly converge on bounded sets of $[0,1] \times \mathbb{C} \times L^2_{\mathbb{C}}(0,1)$, respectively of $(0,1] \times \mathbb{C} \times L^2_{\mathbb{C}}(0,1)$ towards solutions of (1.1). Moreover, they satisfy the integral equations*

$$\varphi(x,\lambda,q) = u(x,\lambda) + \int_0^x \mathcal{G}(x,t,\lambda)q(t)\varphi(t,\lambda,q)dt, \tag{2.3}$$

$$\widetilde{\psi}(x,\lambda,q) = v(x,\lambda) - \int_x^1 \mathcal{G}(x,t,\lambda)q(t)\widetilde{\psi}(t,\lambda,q)dt, \tag{2.4}$$

and the estimates

$$\begin{aligned} |\varphi(x,\lambda,q)| &\le Ce^{|\operatorname{Im}\omega|x}\left(\frac{x}{1+|\omega|x}\right)^{a+1}, \\ \left|\widetilde{\psi}(x,\lambda,q)\right| &\le Ce^{|\operatorname{Im}\omega|(1-x)}\left(\frac{1+|\omega|x}{x}\right)^{a}, \end{aligned}$$

with C uniform on bounded set of $L^2_{\mathbb{C}}(0,1)$.

Proof. The proof is given for φ, it is the same for $\widetilde{\psi}$. Estimates (5.4) for Bessel functions gives

$$|u(x,\lambda)| \le Ce^{|\operatorname{Im}\omega|x}\left(\frac{x}{1+|\omega|x}\right)^{a+1}. \tag{2.5}$$

Iterative relation (2.1) leads to

$$\varphi_1(x,\lambda,q) = \int_0^x \mathcal{G}(x,t,\lambda)q(t)u(t,\lambda)dt,$$

which, combining (2.5) and the Green function estimate (5.6), is bounded by

$$|\varphi_1(x,\lambda,q)| \le C^2 e^{|\operatorname{Im}\omega|x}\left(\frac{x}{1+|\omega|x}\right)^{a+1}\int_0^x \frac{t|q(t)|}{1+|\omega|t}dt,$$

By successive iterations and recurrence, for all positive integer n, we get

$$|\varphi_n(x,\lambda,q)| \le \frac{C^{n+1}}{n!}e^{|\operatorname{Im}\omega|x}\left(\frac{x}{1+|\omega|x}\right)^{a+1}\left(\int_0^x \frac{t|q(t)|}{1+|\omega|t}dt\right)^n.$$

This proves uniform convergence on bounded sets of $[0,1]\times\mathbb{C}\times L^2_{\mathbb{C}}(0,1)$ for φ and the estimate. Integral relation follows from (2.1). □

Corollary 2.1. *φ and $\widetilde{\psi}$, solutions for (1.1) follow the estimates*

$$|\varphi(x,\lambda,q)-u(x,\lambda)| \le Ce^{|\operatorname{Im}\omega|x}\left(\frac{x}{1+|\omega|x}\right)^{a+1}\int_0^x \frac{t|q(t)|}{1+|\omega|t}dt, \tag{2.6}$$

$$\left|\widetilde{\psi}(x,\lambda,q)-v(x,\lambda)\right| \le Ce^{|\operatorname{Im}\omega|(1-x)}\left(\frac{1+|\omega|x}{x}\right)^{a}\int_x^1 \frac{t|q(t)|}{1+|\omega|t}dt. \tag{2.7}$$

Let $'$ denote the derivative with respect to x, we also obtain

$$\begin{aligned} \varphi'(x,\lambda,q) &= u'(x,\lambda) + \int_0^x \frac{\partial\mathcal{G}}{\partial x}(x,t,\lambda)q(t)\varphi(t,\lambda,q)dt, \\ \widetilde{\psi}'(x,\lambda,q) &= v'(x,\lambda) - \int_x^1 \frac{\partial\mathcal{G}}{\partial x}(x,t,\lambda)q(t)\widetilde{\psi}(t,\lambda,q)dt, \end{aligned}$$

and

$$|\varphi'(x,\lambda,q)-u'(x,\lambda)| \le Ce^{|\operatorname{Im}\omega|x}\left(\frac{x}{1+|\omega|x}\right)^{a}\int_0^x \frac{t|q(t)|}{1+|\omega|t}dt, \tag{2.8}$$

$$\left|\widetilde{\psi}'(x,\lambda,q)-v'(x,\lambda)\right| \le Ce^{|\operatorname{Im}\omega|(1-x)}\left(\frac{1+|\omega|x}{x}\right)^{a+1}\int_x^1 \frac{t|q(t)|}{1+|\omega|t}dt, \tag{2.9}$$

with C uniform on bounded sets of $L^2_{\mathbb{C}}(0,1)$.

Proof. For the first estimates, we just have to add those from the proof of the preceding lemma beginning at $k = 1$. Integral equations follow by derivation and the last estimates follow using estimates (5.7) and (5.9) for $\mathcal{G}$ given in annexe 5. □

Remark 1. Notice that for $\omega \neq 0$,

$$0 \leq \frac{t}{1+|\omega|t} = \frac{1}{|\omega|}\frac{|\omega|t}{1+|\omega|t} \leq \frac{1}{|\omega|}.$$

Thus, bound from the corollary are asymptotic estimates when $\omega \to \infty$.

Remark 2. Theses estimates lead to

$$\mathcal{W}(\lambda, q) := \mathcal{W}(\varphi(\cdot,\lambda,q), \widetilde{\psi}(\cdot,\lambda,q)) = 1 + \mathcal{O}\left(\frac{1}{\omega}\right), \quad \omega \to \infty. \tag{2.10}$$

According to [8] and [2], following [6], there exists a function $\psi(x,\lambda,q)$ such that $x^a\psi(x,\lambda,q)$ is analytic on $\mathbb{C}\times L^2_{\mathbb{C}}(0,1)$ for all $x \in [0,1]$ and such that $\{\varphi,\psi\}$ is a basis for the solutions of (1.1), with the normalization

$$\mathcal{W}(\varphi,\psi) = 1.$$

Behavior near $x = 0$ for φ is thus inherited from u and with the wronskian relation this leads to the following boundary conditions at $x = 0$

$$\lim_{x\to 0^+} \frac{\varphi(x,\lambda,q)}{x^{a+1}} = 1, \quad \lim_{x\to 0^+} x^a\psi(x,\lambda,q) = 1.$$

Thus, ψ is called as the singular solution of (3.1).

With the estimate (2.10), $\mathcal{W}(\lambda,q)$ has no zeros near the real axis and we can take for this singular solution the one defined by

$$\psi(x,\lambda,q) = \frac{\widetilde{\psi}(x,\lambda,q)}{\mathcal{W}(\lambda,q)}. \tag{2.11}$$

For real values of a, it is interesting to read [3] and [12] (precisely when $a \geq \frac{-1}{2}$), particularly non integer values allow to study the spectral problem for the radial Schrödinger operator acting on the unit ball in a even dimension space (in our case, the operator acts on $\mathbb{R}^3$).

Uniform convergence for series in the previous lemma give regularity for φ and $\tilde{\psi}$, expressed as bellow

Proposition 2.1 (Analyticity of solutions)**.**

(a) For all $x \in [0,1]$, $\varphi(x,\lambda,q)$ is analytic on $\mathbb{C}\times L^2_{\mathbb{C}}(0,1)$. Moreover, it is real valued on $\mathbb{R}\times L^2_{\mathbb{R}}(0,1)$.

(b) The map

$$\varphi : (\lambda, q) \mapsto \varphi(\cdot,\lambda,q)$$

is analytic from $\mathbb{C}\times L^2_{\mathbb{C}}(0,1)$ to $H^2([0,1],\mathbb{C})$.

(c) For all $x \in (0,1]$, $\tilde{\psi}(x,\lambda,q)$ is analytic on $\mathbb{C}\times L^2_{\mathbb{C}}(0,1)$ and real valued on $\mathbb{R}\times L^2_{\mathbb{R}}(0,1)$.

Consequently, $\mathcal{W}(\lambda, q)$ is an analytic function on $\mathbb{C} \times L^2_{\mathbb{C}}(0,1)$ and thus ψ defined by (2.11) has the needed regularity.

Regularity leads to existence of derivatives, precisely we obtain expression for $L^2_{\mathbb{C}}(0,1)$-gradients with respect to λ and q. For this, following [9], we calculate differential for φ with respect to q:

$$[d_q\varphi(x,\lambda,q)](v) = \int_0^x \tilde{\mathcal{G}}(x,t,\lambda,q)v(t)\varphi(t,\lambda,q)dt,$$

where

$$\tilde{\mathcal{G}}(x,t,\lambda,q) = \psi(x,\lambda,q)\varphi(t,\lambda,q) - \varphi(x,\lambda,q)\psi(t,\lambda,q).$$

Rewriting this relation as a scalar product, we obtain

Proposition 2.2. *[Gradients for the regular solution]*
For all $v \in L^2_{\mathbb{C}}(0,1)$, we have

$$\nabla_q\,\varphi(x,\lambda,q)(t) = \varphi(t,\lambda,q)\left[\psi(x,\lambda,q)\varphi(t,\lambda,q) - \varphi(x,\lambda,q)\psi(t,\lambda,q)\right]\mathbb{1}_{[0,x]}(t),$$
$$\frac{\partial\varphi}{\partial\lambda}(x,\lambda,q) = -\left[d_q\varphi(x,\lambda,q)\right](1),$$
$$\nabla_q\,\varphi'(x,\lambda,q)(t) = \varphi(t,\lambda,q)\left[\psi'(x,\lambda,q)\varphi(t,\lambda,q) - \varphi'(x,\lambda,q)\psi(t,\lambda,q)\right]\mathbb{1}_{[0,x]}(t).$$

2.1 Spectral data

From this point, potential q has real values, that is

$$q \in L^2_{\mathbb{R}}(0,1).$$

2.1.1 Eigenvalues

Spectrum localization and normalization for eigenfunctions is made following [9] and [6]. Boundary conditions given by (1.2) define spectra of (1.1)-(1.2) as the zeros set of the entire function $\lambda \mapsto \varphi(1,\lambda,q)$. Results from Zhornitskaya and Serov [12] give simplicity and localization similarly to [9] and [6]. Regularity and expression for the gradient for each eigenvalue follows from [2].

Notations. Let $(\lambda_{a,n}(q))_{n\geq 1}$ be the set of eigenvalues representing the spectrum $\mathrm{H}_a(q)$. For every integer $n \geq 1$, let $g_n(t,q)$ be the eigenvector with respect to $\lambda_{a,n}(q)$ defined by the normalization

$$g_n(t,q) = \frac{\varphi(t,\lambda_{a,n}(q),q)}{\|\varphi(\cdot,\lambda_{a,n}(q),q)\|_{L^2_{\mathbb{R}}(0,1)}}.$$

Now recall previous results with the following :

Theorem 2.1. *Let $q \in L^2_{\mathbb{R}}(0,1)$, the spectrum for the problem (1.1) with Dirichlet boundary conditions (1.2) is a strictly increasing sequence of eigenvalues $\lambda^a(q) = (\lambda_{a,n}(q))_{n\geq 1}$ which are all real-analytic on $L^2_{\mathbb{R}}(0,1)$ and verify*

$$\lambda_{a,n}(q) = \left(n + \frac{a}{2}\right)^2 \pi^2 \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), \tag{2.12}$$

$$\nabla_q\lambda_{a,n}(t) = g_n(t,q)^2.$$

Corollary 2.2.
Let $q \in L^2_{\mathbb{R}}(0,1)$, the following asymptotics

$$g_n(t,q) = \sqrt{2} j_a(\omega_{a,n}(q)\, t) + \mathcal{O}\left(\frac{1}{n}\right), \tag{2.13}$$

$$\nabla_q \lambda_{a,n}(t) = 2 j_a(\omega_{a,n}(q)\, t)^2 + \mathcal{O}\left(\frac{1}{n}\right), \tag{2.14}$$

$$\lambda_{a,n}(q) = \left(n + \frac{a}{2}\right)^2 \pi^2 + \mathcal{O}(1), \tag{2.15}$$

are uniform on bounded sets in $L^2_{\mathbb{R}}(0,1)$.

Proof. Rewriting relation (2.6) gives for $\omega \in \mathbb{R}$,

$$\varphi(x,\lambda,q) = \frac{(2a+1)!!}{\omega^{a+1}} j_a(\omega x) + \mathcal{O}\left(\frac{1}{\omega}\left(\frac{x}{1+|\omega|x}\right)^{a+1}\right). \tag{2.16}$$

Thus,

$$\int_0^1 \varphi(t,\lambda,q)^2 dt = \frac{[(2a+1)!!]^2}{\omega^{2a+2}} \int_0^1 \left[j_a(\omega t) + \mathcal{O}\left(\frac{1}{\omega}\left(\frac{|\omega|x}{1+|\omega|x}\right)^{a+1}\right)\right]^2 dt.$$

Estimate (5.4) for Bessel function j_a leads to

$$\left[j_a(\omega t) + \mathcal{O}\left(\frac{1}{\omega}\left(\frac{|\omega|x}{1+|\omega|x}\right)^{a+1}\right)\right]^2 = j_a(\omega t)^2 + \mathcal{O}\left(\frac{1}{\omega}\left(\frac{|\omega|x}{1+|\omega|x}\right)^{2a+2}\right),$$

then

$$\int_0^1 \varphi(t,\lambda,q)^2 dt = \frac{[(2a+1)!!]^2}{\omega^{2a+2}} \left[\int_0^1 j_a(\omega t)^2 dt + \mathcal{O}\left(\frac{1}{\omega}\right)\right].$$

Relation (5.14) gives

$$\int_0^1 \varphi(t,\lambda,q)^2 dt = \frac{1}{2}\frac{[(2a+1)!!]^2}{\omega^{2a+2}} \left[1 + \mathcal{O}\left(\frac{1}{\omega}\right)\right]. \tag{2.17}$$

Using both (2.16) and (2.17), knowing that $\omega_{a,n}(q) = \left(n + \frac{a}{2}\right)\pi + \mathcal{O}(1)$, relations (2.13) and (2.14) follow.
For (2.15), we write

$$\begin{aligned}\lambda_{a,n}(q) - \lambda_{a,n}(0) &= \int_0^1 \frac{d}{dt}\left(\lambda_{a,n}(tq)\right) dt = \int_0^1 \langle \nabla_{tq}\lambda_{a,n}, q\rangle\, dt,\\ &= \iint_{[0,1]^2} 2 j_a(\omega_{a,n}(tq)x)^2 q(x)dx\, dt + \mathcal{O}\left(\frac{1}{n}\right).\end{aligned}$$

Thus,

$$\begin{aligned}\lambda_{a,n}(q) = \lambda_{a,n}(0) &+ \int_0^1 q(t)dt\\ &+ \iint_{[0,1]^2} \left[2 j_a(\omega_{a,n}(tq)x)^2 - 1\right] q(x)dx\, dt + \mathcal{O}\left(\frac{1}{n}\right).\end{aligned} \tag{2.18}$$

Eq. (5.4) shows the boundedness for the above second integral. According to [11] Eq. (2.10), we have

$$\lambda_{a,n}(0) = \left(n + \frac{a}{2}\right)^2 \pi^2 - a(a+1) + \ell^2(n),$$

which proves (2.15). □

The asymptotic for eigenvalues is far from optimal, but at this point we cannot do better unless more work to deal with the second integral in (2.18).

2.1.2 Terminal velocities

Solving inverse spectral problem implies the knowledge of other spectral data. Indeed even for the regular case, the eigenvalues don't give the associated potential q. Nevertheless, partial results (as Borg theorem) exist. Thus, let's define the needed additional data as in [9] and [6].

Definition 2.1. Let $\kappa_{a,n}$, called terminal velocity, be defined for all integer $n \geq 1$ by

$$\kappa_{a,n}(q) = \ln \left| \frac{\varphi'(1, \lambda_{a,n}(q), q)}{u'(1, \lambda_{a,n}(0))} \right|.$$

Moreover, let $a_n(t,q)$ be the function given by

$$a_n(t,q) = \varphi(t, \lambda_{a,n}(q), q)\psi(t, \lambda_{a,n}(q), q).$$

Give some properties for these items:

Theorem 2.2. *For all $n \in \mathbb{N}$, $q \mapsto \kappa_{a,n}(q)$ is real-analytic on $L^2_{\mathbb{R}}(0,1)$. Its gradient is expressed by*

$$\nabla_q \kappa_{a,n}(t) = -a_n(t,q) + \nabla_q \lambda_{a,n}(t) \int_0^1 a_n(s,q)ds, \tag{2.19}$$

and follows the estimate

$$\nabla_q \kappa_{a,n}(t) = \frac{1}{\omega_{a,n}} j_a(\omega_{a,n}t)\eta_a(\omega_{a,n}t) + \mathcal{O}\left(\frac{1}{n^2}\right), \tag{2.20}$$

uniformly on bounded sets in $[0,1] \times L^2_{\mathbb{R}}(0,1)$.

Preuve. Regularity of $\kappa_{a,n}$ comes from regularity of $\lambda_{a,n}$ and φ. Expression for its gradient is a straightforward calculation using Proposition 2.2.

Determine the asymptotic value of the gradient. Estimations (2.6)-(2.7) and (2.10) lead to

$$a_n(x,q) = \left[u(x, \lambda_{a,n}) + \mathcal{O}\left(\frac{1}{\omega_{a,n}} \left(\frac{x}{1+|\omega_{a,n}x|}\right)^{a+1}\right)\right] \times \left[v(x, \lambda_{a,n}) + \mathcal{O}\left(\frac{1}{\omega_{a,n}} \left(\frac{1+|\omega_{a,n}x|}{x}\right)^{a}\right)\right],$$

then, definitions of u and v give

$$a_n(x,q) = \frac{-1}{\omega_{a,n}}\left[j_a(\omega_{a,n}x) + \mathcal{O}\left(\frac{1}{\omega_{a,n}}\left[\frac{|\omega_{a,n}|x}{1+|\omega_{a,n}x|}\right]^{a+1}\right)\right]$$
$$\times\left[\eta_a(\omega_{a,n}x) + \mathcal{O}\left(\frac{1}{\omega_{a,n}}\left[\frac{1+|\omega_{a,n}x|}{|\omega_{a,n}|x}\right]^{a}\right)\right].$$

From (5.4) and (5.5), we deduce

$$a_n(t,q) = \frac{-1}{\omega_{a,n}} j_a(\omega_{a,n}x)\eta_a(\omega_{a,n}x) + \mathcal{O}\left(\frac{1}{{\omega_{a,n}}^2}\right).$$

Now, we have

$$\int_0^1 a_n(t,q)dt = \frac{-1}{\omega_{a,n}}\int_0^1 j_a(\omega_{a,n}t)\eta_a(\omega_{a,n}t)dt + \mathcal{O}\left(\frac{1}{{\omega_{a,n}}^2}\right).$$

Relation (5.15) gives the result. □

As noticed previously, on the contrary to the regular case, good estimations for terminal velocities are harder to obtain. Indeed, we have the relation

$$\kappa_{a,n}(q) = \int_0^1 \frac{d}{dt}\left(\kappa_{a,n}(tq)\right)dt = \int_0^1 \langle \nabla_{tq}\kappa_{a,n}, q\rangle_{L^2_{\mathbb{R}}(0,1)}\, dt.$$

Using Eq. (2.20), we get

$$\kappa_{a,n}(q) = \iint_{[0,1]^2} \frac{j_a\big(\omega_{a,n}(tq)x\big)\eta_a\big(\omega_{a,n}(tq)x\big)}{\omega_{a,n}(tq)}\, q(x)dx\, dt + \mathcal{O}\left(\frac{1}{n^2}\right). \tag{2.21}$$

Thus, follows

$$\kappa_{a,n}(q) = \mathcal{O}\left(\frac{1}{n}\right),$$

losing the $\mathcal{O}(\frac{1}{n^2})$ accuracy. As Guillot and Ralston [6] for $a = 1$, we'll use transformation operator to avoid this problem.

We finish our study of the direct problem some properties of the gradients.

2.2 Orthogonality

Proposition 2.3. *For all $(n,m) \in \mathbb{N}^2$, $n,m \geq 1$, we have*

1. $\left\langle {g_n}^2, \frac{d}{dx}{g_m}^2\right\rangle = 0,$
2. $\left\langle a_n, \frac{d}{dx}{g_m}^2\right\rangle = \frac{1}{2}\delta_{n,m},$
3. $\left\langle a_n, \frac{d}{dx}a_m\right\rangle = 0.$

Proof. Formally, the proof follows as in [9], thus we just prove the first one. Let $(n,m) \in \mathbb{N}^2$, with $n,m \geq 1$, integration by parts gives

$$\left\langle {g_n}^2, \frac{d}{dx}{g_m}^2 \right\rangle = \frac{1}{2}\int_0^1 \left[{g_n}^2\left({g_m}^2\right)' - {g_m}^2\left({g_n}^2\right)'\right] dt = \int_0^1 g_n g_m \mathcal{W}(g_n, g_m) dt.$$

When $n = m$, the first relation is true. Now suppose $n \neq m$, we use the identity

$$\frac{d}{dx}\mathcal{W}(g_n, g_m) = (\lambda_{a,n} - \lambda_{a,m}) g_n g_m$$

to obtain

$$\left\langle {g_n}^2, \frac{d}{dx}{g_m}^2 \right\rangle = \frac{1}{2(\lambda_{a,n} - \lambda_{a,m})}\left[\mathcal{W}(g_n, g_m)^2\right]_0^1 = 0.$$

□

Now, we deduce algebraic properties for gradients. For this, recall the definition of the linear independence in an Hilbert space.

Definition 2.2. An infinite vector family $(u_k)_{k\geq 1}$ in an Hilbert space is called free or its vectors are linearly independent if each vector of this family doesn't belong to the closed span of the others. Precisely

$$\forall k \geq 1\,, \quad u_k \notin \overline{\mathrm{Vect}\,(u_j | j \geq 1, j \neq k)}.$$

Corollary 2.3. *[Gradients and orthogonality]*

(a) 1, $\left\{{g_n}^2 - 1\right\}_{n\geq 1}$ *are linearly independent, and so are* $\left\{\frac{d}{dx}{g_n}^2\right\}_{n\geq 1}$*. Moreover, these two families are orthogonal.*

(b) For all $(n,m) \in \mathbb{N}^2$, $n,m \geq 1$ *we have*

(i) $\left\langle \nabla_q \kappa_{a,n}, \frac{d}{dx}\left(\nabla_q \kappa_{a,m}\right)\right\rangle = 0,$

(ii) $\left\langle \nabla_q \kappa_{a,n}, \frac{d}{dx}\left(\nabla_q \lambda_{a,m}\right)\right\rangle = \frac{1}{2}\delta_{n,m},$

(iii) $\left\langle \nabla_q \lambda_{a,n}, \frac{d}{dx}\left(\nabla_q \lambda_{a,m}\right)\right\rangle = 0.$

2.3 The spectral map

We define the spectral map $\lambda^a \times \kappa^a : L^2_{\mathbb{R}}(0,1) \longrightarrow \mathbb{R} \times \ell^\infty_{\mathbb{R}} \times \ell^\infty_{\mathbb{R}}$ with

$$\left[\lambda^a \times \kappa^a\right](q) = \left(\int_0^1 q(t)dt, \left(\widetilde{\lambda}_{a,n}(q)\right)_{n\geq 1}, \left(n\kappa_{a,n}(q)\right)_{n\geq 1}\right), \tag{2.22}$$

where $\widetilde{\lambda}_{a,n}(q)$ be given, through (2.18), by

$$\lambda_{a,n}(q) = \left(n + \frac{a}{2}\right)^2 \pi^2 + \int_0^1 q(t)dt - a(a+1) + \widetilde{\lambda}_{a,n}(q), \quad n \geq 1.$$

Regularity for $\lambda^a \times \kappa^a$ follows as in [9] (see also [6] when $a = 1$). We give this property with the following

Theorem 2.3. *The map $\lambda^a \times \kappa^a$ is real-analytic from $L^2_{\mathbb{R}}(0,1)$ in $\mathbb{R} \times \ell^\infty_{\mathbb{R}} \times \ell^\infty_{\mathbb{R}}$. Its differential is given by*

$$d_q(\lambda^a \times \kappa^a)(v) = \Big(\langle 1, v \rangle \, , \big(\langle \nabla_q \widetilde{\lambda}_{a,n}, v \rangle\big)_{n\geq 1}, \big(\langle n \nabla_q \kappa_{a,n}, v \rangle\big)_{n\geq 1} \Big).$$

Keep in mind that this result is not optimal for the arrival space. To improve this, we use the following transformation operators.

3 Transformation operators

This idea is the key point for such an inverse spectral problem. It was first introduced by Guillot and Ralston for $a = 1$. Their goal was to transform scalar products with the first Bessel functions into scalar products with trigonometric functions. Successfully used in [2] and [4], Rundell and Sacks in [11] present theses operators stepwise for any integer a: First they construct elementary operator S_a which "maps" Bessel function related to $\mathrm{H}_a(q)$ into Bessel function related to $\mathrm{H}_{a-1}(q)$. Then, they just chain these operators to reach Bessel function for $a = 0$, in other words, trigonometric functions. The following results extend their results.

Notations. Let Φ_a and Ψ_a be defined by

$$\Phi_a(x) = j_a(x)^2 \quad \text{et} \quad \Psi_a(x) = j_a(x)\eta_a(x), \quad x \in [0,1].$$

Our first add, as Guillot and Ralston for $a = 1$, is a transformation property for the family $(\Psi_a)_a$ related to the terminal velocities. A second and new improvement is some transformation properties for the families $({\Phi_a}')_a$ and $({\Psi_a}')_a$ related to the dual family appeared in corollary 2.3. We use it further to give precision for isospectral sets.

Lemma 3.1. *For all $a \in \mathbb{N}$, $a \geq 1$ define S_a acting from $L^2_{\mathbb{C}}(0,1)$ in $L^2_{\mathbb{C}}(0,1)$ by*

$$S_a[f](x) = f(x) - 4a\, x^{2a-1} \int_x^1 \frac{f(t)}{t^{2a}} dt \tag{3.1}$$

We have the following properties:

(i) The adjoint operator for S_a is

$$S_a^*[g](x) = g(x) - \frac{4a}{x^{2a}} \int_0^x t^{2a-1} g(t) dt. \tag{3.2}$$

(ii) The family $\{S_a\}$ pairwise commute:

$$\forall (a,b) \in \mathbb{N}^2, \quad S_a S_b = S_b S_a.$$

(iii) S_a is a bounded operator in $L^2_{\mathbb{C}}(0,1)$.

(iv) S_a is a Banach isomorphism between $L^2_{\mathbb{C}}(0,1)$ and $\big(x \mapsto x^{2a}\big)^\perp$. Its inverse is given by the bounded operator on $L^2_{\mathbb{C}}(0,1)$ defined by

$$A_a[g](x) = g(x) - \frac{4a}{x^{2a+1}} \int_0^x t^{2a} g(t) dt.$$

(v) Φ_a *and* Ψ_a *check the properties*

$$\Phi_a = -S_a^*[\Phi_{a-1}], \quad \Psi_a = -S_a^*[\Psi_{a-1}], \tag{3.3}$$

$$\Phi_a' = -A_a[\Phi_{a-1}'], \quad \Psi_a' = -A_a[\Psi_{a-1}']. \tag{3.4}$$

Lemma 3.2. *For all* $a \in \mathbb{N}^*$ *let* T_a *be defined by*

$$T_a = (-1)^{a+1} S_a S_{a-1} \cdots S_1.$$

Then

(i) T_a *is a bounded one-to-one operator on* $L^2_{\mathbb{C}}(0,1)$ *such that :*
For all $q \in L^2_{\mathbb{C}}(0,1)$ *and all* $\lambda \in \mathbb{C}$,

$$\int_0^1 \left[2\Phi_a(\lambda t) - 1\right] q(t)dt = \int_0^1 \cos(2\lambda t) T_a[q](t)dt, \tag{3.5}$$

$$\int_0^1 \Psi_a(\lambda t) q(t)dt = -\frac{1}{2}\int_0^1 \sin(2\lambda t) T_a[q](t)dt. \tag{3.6}$$

(ii) *The adjoint of* T_a *check*

$$2\Phi_a(\lambda x) - 1 = T_a^* \left[\cos(2\lambda x)\right], \quad \Psi_a(\lambda x) = T_a^* \left[-\frac{1}{2}\sin(2\lambda x)\right] \tag{3.7}$$

and its kernel is

$$\ker(T_a^*) = \operatorname{Span}\left\{x^2, x^4, \ldots, x^{2a}\right\}.$$

(iii) T_a *define a Banach isomorphism between* $L^2_{\mathbb{C}}(0,1)$ *and* $(\ker(T_a^*))^{\perp}$.
Its inverse is given by the bounded operator in $L^2_{\mathbb{C}}(0,1)$ *defined by*

$$B_a[f] := (-1)^{a+1} A_a A_{a-1} \cdots A_1,$$

moreover, we have

$$\Phi_a'(\lambda x) = B_a\left[-\sin(2\lambda x)\right], \quad \Psi_a'(\lambda x) = B_a\left[-\cos(2\lambda x)\right] \tag{3.8}$$

Proof of theses results lies on properties of Bessel functions for the calculation part and lies on Hardy inequalities for the bounded properties and range of presented operators. For a detailed proof, we send back to [11], the new facts follow similarly.

The first use of these operators is to improve spectral data estimations.

Proposition 3.1. *Uniformly on bounded sets in* $L^2_{\mathbb{R}}(0,1)$, *we have*

$$\lambda_{a,n}(q) = \left(n + \frac{a}{2}\right)^2 \pi^2 + \int_0^1 q(t)dt - a(a+1) + \ell^2(n),$$

$$n\kappa_{a,n}(q) = \ell^2(n).$$

Proof. According to (2.18) and (2.21), we just have to find the behavior for integral terms I_n and J_n defined by

$$I_n(q) = \iint_{[0,1]^2} \left[2j_a(\omega_{a,n}(tq)x)^2 - 1\right] q(x)dx\,dt,$$

$$J_n(q) = \iint_{[0,1]^2} \frac{j_a\big(\omega_{a,n}(tq)x\big)\eta_a\big(\omega_{a,n}(tq)x\big)}{\omega_{a,n}(tq)}\, q(x)dx\,dt.$$

We use (3.5) and (3.6) to get

$$I_n(q) = \iint_{[0,1]^2} \cos\big(2\omega_{a,n}(tq)x\big) T_a[q](x)dx\,dt,$$

$$J_n(q) = -\iint_{[0,1]^2} \frac{\sin\big(2\omega_{a,n}(tq)x\big)}{2\omega_{a,n}(tq)}\, T_a[q](x)dx\,dt.$$

From (2.12), we have $\omega_{a,n} = \left(n + \frac{a}{2}\right)\pi + \mathcal{O}\left(\frac{1}{n}\right)$. Hence

$$I_n(q) = \int_0^1 \cos\big((2n+a)\pi x\big) T_a[q](x)dx\, + \mathcal{O}\left(\frac{1}{n}\right),$$

$$J_n(q) = \frac{-1}{(2n+a)\pi} \int_0^1 \sin\big((2n+a)\pi x\big)\, T_a[q](x)dx + \mathcal{O}\left(\frac{1}{n^2}\right).$$

These relations lead to the result. □

Give two more estimates for the following functions

Notations. For all integer $n \geq 1$, we define

$$V_{a,n}(x,q) := 2\left[\frac{d}{dx}\nabla_q \lambda_{a,n}\right], \quad W_{a,n}(x,q) := -2\left[\frac{d}{dx}\nabla_q \kappa_{a,n}\right]. \tag{3.9}$$

Proposition 3.2. *Uniformly the bounded sets in* $[0,1] \times L^2_{\mathbb{R}}(0,1)$*, we have*

$$V_{a,n}(x,q) = 4\omega_{a,n}\left({\Phi_a}'(\omega_{a,n}x) + \mathcal{O}\left(\frac{1}{n}\right)\right) \tag{3.10}$$

$$W_{a,n}(x,q) = -2{\Psi_a}'(\omega_{a,n}x) + \mathcal{O}\left(\frac{1}{n}\right). \tag{3.11}$$

Proof. We only proof the first one, the second follows similarly. We have, according to the eigenvalue gradient expression,

$$V_{a,n}(x,q) = 2\frac{d}{dx}\left(g_n(x,q)^2\right) = 4g_n(x,q)g_n'(x,q).$$

Estimate (2.8) combined with (2.17) and (2.10) leads to

$${g_n}'(x,q) = \sqrt{2}\omega_{a,n}{j_a}'(\omega_{a,n}x) + \mathcal{O}(1)\,.$$

We conclude that to (2.13) which allows us to write

$$V_{a,n}(x,q) = 4\omega_{a,n}\left(2j_a(\omega_{a,n}x){j_a}'(\omega_{a,n}x) + \mathcal{O}\left(\frac{1}{n}\right)\right).$$

□

4 The inverse spectral problem

Now, the map $\lambda^a \times \kappa^a$ defined by (2.22) is a real analytic map from $L^2_{\mathbb{R}}(0,1)$ into $\mathbb{R} \times \ell^2_{\mathbb{R}} \times \ell^2_{\mathbb{R}}$. Give the main result of this paper:

Theorem 4.1. *For all integer $a \geq 1$, $d_q(\lambda^a \times \kappa^a)$ is a Banach isomorphism from $L^2_{\mathbb{R}}(0,1)$ onto $\mathbb{R} \times \ell^2_{\mathbb{R}} \times \ell^2_{\mathbb{R}}$.*

From the results of Zhornitskaya, Zerov [12] and Carlson [3], for all $a \geq -\frac{1}{2}$, the map $\lambda^a \times \kappa^a$ is one-to-one on $L^2_{\mathbb{R}}(0,1)$. We immediately deduce the following property

Corollary 4.1. *For all integer $a \geq 1$, the map $\lambda^a \times \kappa^a$ is a global real analytic coordinate system on $L^2_{\mathbb{R}}(0,1)$.*

Proof of the theorem. The main arguments of this proof are extensions of those in [6]. From orthogonality relations in corollary 2.3 the family

$$\{1\} \cup \left\{\nabla_q \widetilde{\lambda}_{a,n}\right\}_{n\geq 1} \cup \left\{\nabla_q \kappa_{a,n}\right\}_{n\geq 1}$$

is free in $L^2_{\mathbb{R}}(0,1)$. Let r_n and s_n be defined by

$$r_n(x) = \nabla_q \widetilde{\lambda}_{a,n}(x) - \left(2\Phi_a(\omega_{a,n}x) - 1\right), \tag{4.1}$$

$$s_n(x) = \nabla_q \kappa_{a,n}(x) - \frac{1}{\omega_{a,n}} \Psi_a(\omega_{a,n}x). \tag{4.2}$$

Using lemma 3.2, we have, for all $v \in L^2_{\mathbb{R}}(0,1)$,

$$\left\langle \nabla_q \widetilde{\lambda}_{a,n}, v \right\rangle = \int_0^1 \left(\cos\left(2\omega_{a,n}t\right) + R_n(t)\right) T_a[v](t)dt, \tag{4.3}$$

$$\left\langle \nabla_q \kappa_{a,n}, v \right\rangle = \frac{-1}{2\omega_{a,n}} \int_0^1 \left(\sin\left(2\omega_{a,n}t\right) + S_n(t)\right) T_a[v](t)dt, \tag{4.4}$$

with

$$R_n = B_a^*[r_n] \text{ et } \frac{-S_n}{2\omega_{a,n}} = B_a^*[s_n]. \tag{4.5}$$

Moreover, recall that $T_a[1] = -1$, particularly

$$\langle 1, v \rangle = \int_0^1 v(t)dt = \int_0^1 -T_a[v](t)dt.$$

We denote F the operator defined by

$$F(w) = \Bigg(\langle -1, w \rangle, \left\{\langle \cos\left(2\omega_{a,n}t\right) + R_n(t), w \rangle\right\}_{n\geq 1}, \left\{ \left\langle \frac{-n}{2\omega_{a,n}} \left(\sin\left(2\omega_{a,n}t\right) + S_n(t)\right), w \right\rangle \right\}_{n\geq 1} \Bigg),$$

such that $d_q(\lambda^a \times \mu^a)(v) = F \circ T_a[v]$. From lemma 3.2, T_a is invertible from $L^2_{\mathbb{R}}(0,1)$ onto $\left(\text{Span}\left\{x^2, x^4, \ldots, x^{2a}\right\}\right)^\perp$. Thus, we have to show that F is invertible from $\left(\text{Span}\left\{x^2, x^4, \ldots, x^{2a}\right\}\right)^\perp$ onto $\mathbb{R} \times \ell^2_{\mathbb{R}} \times \ell^2_{\mathbb{R}}$. Following [6], we have to show the invertibility of $\mathbf{F}$ the operator from $L^2_{\mathbb{R}}(0,1)$ into $\mathbb{R} \times \ell^2_{\mathbb{R}} \times \ell^2_{\mathbb{R}}$, sending functions in $L^2_{\mathbb{R}}(0,1)$ on their Fourier coefficients (or, more simply, their scalar products against each element of the considered family) with respect to the family

$$\mathcal{F} = \{1\} \cup \left\{t^{2j}\right\}_{j \in [\![1,a]\!]} \cup \{\cos(2\omega_{a,n}t) + R_n(t)\}_{n \geq 1} \cup \left\{\frac{-n}{2\omega_{a,n}}\left(\sin(2\omega_{a,n}t) + S_n(t)\right)\right\}_{n \geq 1}. \quad (4.6)$$

This last statement will come from [9]: Appendix D, theorem 3:

Lemma 4.1. *Let $\mathcal{F} = \{f_n\}_{n \in \mathbb{N}}$ be a sequence in an Hilbert space H, with the two following properties*

(a) there exists an orthogonal basis $\mathcal{E} = \{e_n\}_{n \in \mathbb{N}}$ in H for which

$$\sum \|f_n - e_n\|_2^2 < \infty.$$

(b) the f_n are linearly independent.

Then $\{f_n\}_{n \in \mathbb{N}}$ is a basis of H and the map $\mathbf{F} : x \mapsto \{(f_n, x)\}_{n \in \mathbb{N}}$ is a linear isomorphism between H and $\ell^2_{\mathbb{R}}$.

Considering the orthonormal basis

$$\mathcal{E} = \left\{\sqrt{2}\cos \pi x, \sqrt{2}\sin \pi x, \dots, \sqrt{2}\cos(2n+1)\pi x, \sqrt{2}\sin(2n+1)\pi x, \dots\right\},$$

when $a = 2\mathfrak{a} + 1$ with $\mathfrak{a} \in \mathbb{N}$ (as in [6] for $a = 1$), and

$$\mathcal{E} = \left\{1, \sqrt{2}\cos 2\pi x, \sqrt{2}\sin 2\pi x, \dots, \sqrt{2}\cos 2n\pi x, \sqrt{2}\sin 2n\pi x, \dots\right\},$$

when $a = 2\mathfrak{a}$ with $\mathfrak{a} \in \mathbb{N}$ (as in [9] for $a = 0$), estimates (2.14) and (2.20) used with (4.1) and (4.2) show that

$$\|r_n\|_{L^2_{\mathbb{R}}(0,1)} = \mathcal{O}\left(\frac{1}{n}\right), \quad \|s_n\|_{L^2_{\mathbb{R}}(0,1)} = \mathcal{O}\left(\frac{1}{n^2}\right),$$

then the boundedness of B_a^* and relations given in (4.5) give

$$\|R_n\|_{L^2_{\mathbb{R}}(0,1)} = \mathcal{O}\left(\frac{1}{n}\right), \quad \|S_n\|_{L^2_{\mathbb{R}}(0,1)} = \mathcal{O}\left(\frac{1}{n}\right),$$

which implies condition (a) after normalization. Lemma 4.2 will give condition (b), thus the proof will be complete. □

Now, give the lemma implying the theorem. Ingredients of the proof are similar to [6], we improve the method in a systematic way (for each a).

Lemma 4.2. *$\mathcal{F}$ defined by (4.6) is a free family in $L^2_{\mathbb{R}}(0,1)$.*

Proof. Rewrite relations (4.3) and (4.4) as

$$T_a^* \left(\cos\left(2\omega_{a,n}t\right) + R_n(t)\right) = \nabla_q \widetilde{\lambda}_{a,n}$$

and

$$T_a^* \left(\frac{-1}{2\omega_{a,n}} \left(\sin\left(2\omega_{a,n}t\right) + S_n(t)\right)\right) = \nabla_q \kappa_{a,n}.$$

Since T_a^* is bounded and the famille $\{1\} \cup \left\{\nabla_q \widetilde{\lambda}_{a,n}\right\}_{n\geq 1} \cup \{\nabla_q \kappa_{a,n}\}_{n\geq 1}$ is free, we deduce the linear independency for the following family

$$\{1\} \cup \{\cos\left(2\omega_{a,n}t\right) + R_n\}_{n\geq 1} \cup \left\{\frac{-1}{2\omega_{a,n}} \left(\sin\left(2\omega_{a,n}t\right) + S_n\right)\right\}_{n\geq 1}.$$

Let $k \in [\![1, a]\!]$, denote W_k the function $W_k(t) = t^{2k}$. Let us show that W_k is not in the closure of $\operatorname{Vect}\left(\mathcal{F} \setminus \{W_k\}\right)$. (We might show iteratively that $W_k \notin \overline{\operatorname{Vect}\left(\mathcal{F} \setminus \{W_j, j \in [\![k, a]\!]\}\right)}$, but it is not necessary by taking $\alpha_m^{(j)} = 0$ for any $m \in [\![k, a]\!]$ in the next expression.)
Suppose the contrary: there exists for $j \in \mathbb{N}$ a sequence of vector

$$\begin{aligned} W_k^{(j)}(t) = \alpha_0^{(j)} + \sum_{\substack{m\in[\![1,a]\!] \\ m\neq k}} \alpha_m^{(j)}\, W_m(t) + \sum_{n\in[\![1,N_j]\!]} a_n^{(j)} \left(\cos\left(2\omega_{a,n}t\right) + R_n(t)\right) \\ + \sum_{n\in[\![1,N_j]\!]} b_n^{(j)} \frac{-1}{2\omega_{a,n}} \left(\sin\left(2\omega_{a,n}t\right) + S_n(t)\right), \end{aligned}$$

with $N_j < \infty$, $\alpha_m^{(j)}, a_n^{(j)}, b_n^{(j)} \in \mathbb{R}$ such that

$$W_k^{(j)} \underset{j\to\infty}{\longrightarrow} W_k \text{ in } L^2_{\mathbb{R}}(0,1).$$

Since $T_a^*(W_m) = 0$ for $m = 1, \ldots, a$, the sequence

$$w^{(j)} := T_a^*(W_k^{(j)}) = -\alpha_0^{(j)} + \sum_{n\in[\![1,N_j]\!]} a_n^{(j)} \nabla_q \widetilde{\lambda}_{a,n} + b_n^{(j)} \nabla_q \kappa_{a,n}$$

tends to 0 in $L^2_{\mathbb{R}}(0,1)$ when $j \to \infty$. Thus, corollary 2.3 gives

$$\alpha_0^{(j)} = \int_0^1 w^{(j)}(t)dt \underset{j\to\infty}{\longrightarrow} 0, \tag{4.7}$$

$$a_n^{(j)} = -2\int_0^1 w^{(j)}(t) \frac{d}{dx}\left(\nabla_q \kappa_{a,n}\right) dt \underset{j\to\infty}{\longrightarrow} 0, \tag{4.8}$$

$$b_n^{(j)} = -2\int_0^1 w^{(j)}(t) \frac{d}{dx}\left(\nabla_q \lambda_{a,n}\right) dt \underset{j\to\infty}{\longrightarrow} 0. \tag{4.9}$$

Now consider $\omega \in \mathcal{C}_0^\infty([0,1], \mathbb{R})$, supported in $[\delta, 1]$ with $\delta > 0$, such that

$$\langle \omega, W_m \rangle = \delta_{k,m}, \quad m \in [\![1, a]\!]$$

and

$$\langle B_a[\omega], 1 \rangle = 0 \quad \text{i.e.} \quad \langle \omega, 1 \rangle = c(k, a) \langle \omega, W_k \rangle.\text{[i]}$$

[i] $c(k,a)$ represents the W_k component of $B_a^*[1]$. It can by computed by induction on a: $B_a^*[1] = -1 + \sum_{m=1}^a c(m,a) W_m$. Particularly $c(1,a) = a(a+1)$ (idem [6] when $a = 1$).

Smoothness and support of ω imply that

$$\int_0^1 \omega(t)\cos(2\omega_{a,n}t)dt, \int_0^1 \omega(t)\sin(2\omega_{a,n}t)dt = \mathcal{O}\left(\frac{1}{n^N}\right), \quad \forall N \in \mathbb{N}$$

and that $B_a[\omega]$ is $\mathcal{C}^\infty([0,1],\mathbb{R})$ supported in $[\delta,1]$.
Now, plug estimation (2.6) in the integral expression (2.3), then use (2.17), controls (5.4) and (5.6) to obtain the uniform estimate on $[0,1]$

$$r_n(x) = \frac{1}{\|\varphi_n\|_2^2}\left(2u(x,\lambda_{a,n})\int_0^x \mathcal{G}(x,t,\lambda_{a,n})q(t)u(t,\lambda_{a,n})dt\right) + \left(\frac{1}{\|\varphi_n\|_2^2} - \frac{2(\lambda_{a,n})^{a+1}}{((2a+1)!!)^2}\right)u(x,\lambda_{a,n})^2 + \mathcal{O}\left(\frac{1}{n^2}\right). \quad (4.10)$$

Thanks to lemma 4.3, this implies that the following family is summable

$$\left\{\left\langle\omega, \left[\cos(2\omega_{a,n}t) + R_n(t)\right]\right\rangle\right\}_{n\geq 1}.$$

Now turn to s_n. From relations (4.2) and (2.19), we get

$$\begin{aligned} s_n(x) &= -a_n(x,q) + \nabla_q\lambda_{a,n}(x)\int_0^1 a_n(t,q)dt - \frac{1}{\omega_{a,n}}\Psi_a(\omega_{a,n}x), \\ &= -a_n(x,q) + (2\Phi_a(\omega_{a,n}x) + r_n(x))\int_0^1 a_n(t,q)dt - \frac{1}{\omega_{a,n}}\Psi_a(\omega_{a,n}x), \\ &= -a_n(x,q) + 2\Phi_a(\omega_{a,n}x)\int_0^1 a_n(t,q)dt \\ &\qquad + r_n(x)\int_0^1 a_n(t,q)dt - \frac{1}{\omega_{a,n}}\Psi_a(\omega_{a,n}x). \end{aligned}$$

Respectfully insert (2.6) and (2.7) in integral expressions (2.3) and (2.4); then use (5.4), (5.5), (5.6), (5.8) and wronskian estimate (2.10) to obtain the uniform estimate on $[0,1]$

$$\begin{aligned} s_n(x) = &\left.\begin{aligned} &- v(x,\lambda_{a,n})\int_0^x \mathcal{G}(x,t,\lambda_{a,n})q(t)u(t,\lambda_{a,n})dt \\ &+ u(x,\lambda_{a,n})\int_x^1 \mathcal{G}(x,t,\lambda_{a,n})q(t)v(t,\lambda_{a,n})dt \\ &+ 2\Phi_a(\omega_{a,n}x)\int_0^1 a_n(t,q)dt \end{aligned}\right\}\tilde{s}_n(x) \\ &+ r_n(x)\int_0^1 a_n(t,q)dt \\ &+ \frac{1}{\omega_{a,n}}\Psi_a(\omega_{a,n}x)\left(W^{-1}-1\right) + \mathcal{O}\left(\frac{1}{n^3}\right). \end{aligned} \quad (4.11)$$

With the help of lemma 4.4, we deduce the summability of the family

$$\left\{n\int_0^1 \omega(t)\frac{-1}{2\omega_{a,n}}\left(\sin(2\omega_{a,n}t) + S_n(t)\right)dt\right\}_{n\geq 1}.$$

We finish the proof writing

$$\left\langle \omega, W_k^{(j)} \right\rangle = \alpha_0^{(j)} \langle \omega, 1 \rangle + \sum_{n \in [\![1,N_j]\!]} a_n^{(j)} \langle \omega, (\cos(2\omega_{a,n}t) + R_n(t)) \rangle$$
$$+ \sum_{n \in [\![1,N_j]\!]} b_n^{(j)} \left\langle \omega, \frac{-1}{2\omega_{a,n}} (\sin(2\omega_{a,n}t) + S_n(t)) \right\rangle .$$

Indeed, the following estimates, deduced from (3.10)-(3.11) and (4.7)-(4.9),

$$\left|a_n^{(j)}\right| \le C, \quad \left|b_n^{(j)}\right| \le Cn, \quad \forall (n,j) \in \mathbb{N}^2,$$

imply that

$$\left\langle \omega, W_k^{(j)} \right\rangle \underset{j \to \infty}{\longrightarrow} 0,$$

which is contradictory with the choice of ω. □

Lemma 4.3.

$$\{\langle \omega, R_n \rangle\}_{n \ge 1} \in \ell^1_{\mathbb{R}}.$$

Proof. Let $r_{n,1}$ and $r_{n,2}$ be the first and second terms in(4.10). Recall that

$$\langle \omega, R_n \rangle = \langle \omega, B_a^*[r_n] \rangle = \langle B_a[\omega], r_n \rangle,$$

thus, we just have to show that $\{\langle B_a[\omega], r_{n,j} \rangle\} \in \ell^1_{\mathbb{R}}$, $j = 1, 2$. First consider $r_{n,2}$:

$$\begin{aligned} \langle B_a[\omega], r_{n,2} \rangle &= \left\langle B_a[\omega], \left(\frac{1}{\|\varphi_n\|_2^2} - \frac{2(\lambda_{a,n})^{a+1}}{((2a+1)!!)^2} \right) u(x, \lambda_{a,n})^2 \right\rangle, \\ &= \left\langle B_a[\omega], \left(\frac{((2a+1)!!)^2}{2(\lambda_{a,n})^{a+1} \|\varphi_n\|_2^2} - 1 \right) 2j_a(\omega_{a,n}x)^2 \right\rangle. \end{aligned}$$

With (2.17), we have

$$\langle B_a[\omega], r_{n,2} \rangle = \mathcal{O}\left(\frac{1}{\omega_{a,n}} \right) \left\langle B_a[\omega], 2j_a(\omega_{a,n}x)^2 \right\rangle,$$

then, hypothesis upon $B_a[\omega]$ gives

$$\langle B_a[\omega], r_{n,2} \rangle = \mathcal{O}\left(\frac{1}{\omega_{a,n}} \right) \left\langle B_a[\omega], 2j_a(\omega_{a,n}x)^2 - 1 \right\rangle.$$

Lemma 3.2 leads to the result.
Now consider $r_{n,1}$ and write as before

$$\begin{aligned} r_{n,1}(x) &= \frac{1}{\|\varphi_n\|_2^2} \left(2u(x, \lambda_{a,n}) \int_0^x \mathcal{G}(x, t, \lambda_{a,n}) q(t) u(t, \lambda_{a,n}) dt \right) \\ &= \frac{2((2a+1)!!)^2}{(\omega_{a,n})^{2a+2} \|\varphi_n\|_2^2} j_a(\omega_{a,n}x) \int_0^x \mathcal{G}(x, t, \lambda_{a,n}) q(t) j_a(\omega_{a,n}t) dt. \end{aligned}$$

With (5.4) and (5.6) we get

$$\int_0^x \mathcal{G}(x,t,\lambda_{a,n})q(t)j_a(\omega_{a,n}t)dt = \mathcal{O}\left(\frac{1}{\omega_{a,n}}\right),$$

and, thanks to (2.17), we have

$$r_{n,1}(x) = 4j_a(\omega_{a,n}x)\int_0^x \mathcal{G}(x,t,\lambda_{a,n})q(t)j_a(\omega_{a,n}t)dt + \mathcal{O}\left(\frac{1}{n^2}\right).$$

From the expression of $\mathcal{G}(x,t,\lambda)$, we can write

$$\begin{aligned}\langle B_a[\omega], r_{n,1}\rangle = \\ &\frac{4}{\omega_{a,n}}\int_0^1 B_a[\omega](x)j_a(\omega_{a,n}x)^2\int_0^x q(t)j_a(\omega_{a,n}t)\eta_a(\omega_{a,n}t)dtdx \\ &-\frac{4}{\omega_{a,n}}\int_0^1 B_a[\omega](x)j_a(\omega_{a,n}x)\eta_a(\omega_{a,n}x)\int_0^x q(t)j_a(\omega_{a,n}t)^2dtdx \\ &+\mathcal{O}\left(\frac{1}{n^2}\right).\end{aligned}$$

But, $B_a[\omega]$ is supported in $[\delta,1]$, $\delta>0$. Relations (5.2) and (5.3) thus give

$$\begin{aligned}\langle B_a[\omega], r_{n,1}\rangle = \\ &\frac{2}{\omega_{a,n}}\int_0^1 B_a[\omega](x)(1-\cos(2\omega_{a,n}x-a\pi))\int_0^x q(t)j_a(\omega_{a,n}t)\eta_a(\omega_{a,n}t)dtdx \\ &-\frac{2}{\omega_{a,n}}\int_0^1 B_a[\omega](x)\sin(2\omega_{a,n}x-a\pi)\int_0^x q(t)j_a(\omega_{a,n}t)^2dtdx \\ &+\mathcal{O}\left(\frac{1}{n^2}\right).\end{aligned}$$

Integrating by parts in all terms having factors of $\cos(\omega_{a,n}x)$ or $\sin(\omega_{a,n}x)$, we get

$$\begin{aligned}\langle B_a[\omega],\, r_{n,1}\rangle =& \frac{2}{\omega_{a,n}}\int_0^1 B_a[\omega](x)\int_0^x q(t)j_a(\omega_{a,n}t)\eta_a(\omega_{a,n}t)dtdx \\ &+\mathcal{O}\left(\frac{1}{n^2}\right).\end{aligned}$$

Finally, interchanging the order of integration and using properties of the transformation operator, we obtain the concluding relation

$$\langle B_a[\omega],\, r_{n,1}\rangle = \frac{-1}{\omega_{a,n}}\int_0^1 \sin(2\omega_{a,n}t)T_a\left[t\mapsto q(t)\int_t^1 B_a[\omega](x)dx\right]dt + \mathcal{O}\left(\frac{1}{n^2}\right).$$

□

Lemma 4.4.

$$\{\langle\omega, S_n\rangle\}_{n\geq 1}\in \ell^1_{\mathbb{R}}.$$

Proof. Recall

$$\langle \omega, S_n \rangle = 2\omega_{a,n} \langle \omega, B_a^*[s_n] \rangle = 2\omega_{a,n} \langle B_a[\omega], s_n \rangle.$$

The proof is similar to lemma 4.3 for $\tilde{s}_n$ (see (4.11)): we change in the scalar product $\langle B_a[\omega], \tilde{s}_n \rangle$, $2\Phi_a(\lambda_{a,n}x)$ by $2\Phi_a(\lambda_{a,n}x) - 1$ and use the transformation operator; for the two following terms in $\tilde{s}_n$, as previously, we have to use the support of $B_a[\omega]$, asymptotics deduced from (5.2) and (5.3), integrate by parts terms with $\cos(2\omega_{a,n}x)$ or $\sin(2\omega_{a,n}x)$ and finally invert integration order to use transformation operators.

The term following $\tilde{s}_n$ in (4.11) is controlled thanks to r_n; denote $\hat{s}_n$ the remaining term. But, we have

$$\begin{aligned} \langle B_a[\omega], \hat{s}_n \rangle &= \mathcal{O}\left(\frac{1}{n^2}\right) \int_0^1 B_a[\omega](t) j_a(\lambda_{a,n}t)\, \eta_a(\lambda_{a,n}t)\, dt \\ &= \mathcal{O}\left(\frac{1}{n^2}\right) \int_\delta^1 B_a[\omega](t) \sin(2\omega_{a,n}t) dt + \mathcal{O}\left(\frac{1}{n^3}\right). \end{aligned}$$

Thus $\left\{ n \langle B_a[\omega], \hat{s}_n \rangle \right\}_{n \geq 1}$ is in $\ell^1_{\mathbb{R}}$ and the proof is completed. □

Now comes the corollary (see (3.9) for the definition of V_n and W_n).

Corollary 4.2. *$\lambda^a \times \kappa^a$ is a local real analytic isomorphism on $L^2_{\mathbb{R}}(0,1)$. Moreover, the inverse of $d_q(\lambda^a \times \kappa^a)$ is the bounded linear map from $\mathbb{R} \times \ell^2_{\mathbb{R}} \times \ell^2_{\mathbb{R}}$ onto $L^2_{\mathbb{R}}(0,1)$ given by*

$$(d_q(\lambda^a \times \kappa^a))^{-1}(\eta_0, \eta, \xi) = \eta_0 + \sum_{n \geq 1} \eta_n W_{a,n} + \sum_{n \geq 1} \frac{\xi_n}{n} V_{a,n}.$$

Proof. The first part of the corollary comes from the theorem 4.1 and from the inverse function theorem. Now consider $(\eta_0, \eta, \xi) \in \mathbb{R} \times \ell^2_{\mathbb{R}} \times \ell^2_{\mathbb{R}}$ and define

$$u = \eta_0 + \sum_{n \geq 1} \eta_n W_{a,n} + \sum_{n \geq 1} \frac{\xi_n}{n} V_{a,n}.$$

Since B_a is bounded, estimations (3.10) and (3.11), with relations (3.8) give

$$\frac{1}{n} V_{a,n}(x,q) = B_a \left[\frac{4\omega_{a,n}}{n} \sin(2\omega_{a,n}x) + \mathcal{O}\left(\frac{1}{n}\right) \right]$$

and

$$W_{a,n}(x,q) = B_a \left[-2\cos(2\omega_{a,n}x) + \mathcal{O}\left(\frac{1}{n}\right) \right].$$

Definition of ξ and η, estimation of eigenvalues and boundedness of B_a imply the convergence in $L^2_{\mathbb{R}}(0,1)$ for the series defining u.

From corollary 2.3, we have

$$\langle 1, u \rangle = \eta_0,$$

and for all integer $n \geq 1$

$$\left\langle \nabla_q \widetilde{\lambda}_{a,n}, u \right\rangle = \eta_n, \quad \langle n \nabla_q \kappa_{a,n}, u \rangle = \xi_n.$$

Thus, we have $d_q(\lambda^a \times \kappa^a)(u) = (\eta_0, \eta, \xi)$, which proofs the corollary. □

We finish with the description of isospectral sets. For $q_0 \in L^2_{\mathbb{R}}(0,1)$, we define the set of potentials with same Dirichlet spectrum as q_0, called isospectral set, by $\mathrm{Iso}(q_0, a) = \left\{q \in L^2_{\mathbb{R}}(0,1) : \lambda^a(q) = \lambda^a(q_0)\right\}$. The new fact of the following result is to explicit tangent and normal spaces.

Theorem 4.2. *Let $q_0 \in L^2_{\mathbb{R}}(0,1)$, then*

(a) $\mathrm{Iso}(q_0, a)$ is a real-analytic manifold of $L^2_{\mathbb{R}}(0,1)$ of infinite dimension and codimension, lying in the hyperplane of all functions with mean $\int_0^1 q_0(t)dt$.

(b) At every point q in $\mathrm{Iso}(q_0, a)$, the tangent space is

$$T_q \,\mathrm{Iso}(q_0, a) = \left\{ \sum_{n\geq 1} \frac{\xi_n}{n} V_{a,n} : \xi \in \ell^2_{\mathbb{R}} \right\}$$

and the normal space is

$$N_q \,\mathrm{Iso}(q_0, a) = \left\{ \eta_0 + \sum_{n\geq 1} \eta_n ({g_n}^2 - 1) : (\eta_0, \eta) \in \mathbb{R} \times \ell^2_{\mathbb{R}} \right\}.$$

Proof. It is straightforward from [9].

(a) The first part of the assertion comes from [2] Theorem 1.3, the second is direct from (3.1).

(b) Since

$$T_q \,\mathrm{Iso}(q_0, a) = \left(d_q\left(\lambda^a \times \kappa^a\right)\right)^{-1} \left(\left\{0_{\mathbb{R}\times\ell^2_{\mathbb{R}}}\right\} \times \ell^2_{\mathbb{R}}\right),$$

corollary 4.2 gives expression of the tangent space. According to corollary 2.3, $\left\{1, {g_n}^2 - 1 : n \geq 1\right\}$ is free, orthogonal to the free family $(V_{a,n})_{n\in\mathbb{Z}}$. Thus, we have

$$\left\{ \eta_0 + \sum_{n\geq 1} \eta_n (g_n^2 - 1) : (\eta_0, \eta) \in \mathbb{R} \times \ell^2_{\mathbb{R}} \right\} \subset N_q \,\mathrm{Iso}(q_0).$$

Moreover, any vector orthogonal to $\left\{1, {g_n}^2 - 1 : n \geq 1\right\}$ is, with regards to the Fréchet derivative of λ^a, in the null space of $d_q\lambda^a$. Thus, we get the other inclusion and then the proof.

□

To finish, we recall that the characterization of the spectra of each operator $\mathrm{H}_a(q)$ was obtained by Carlson ([2] Theorem 1.1).

5 Bessel functions

Spherical Bessel functions j_a and η_a are defined through

$$j_a(z) = \sqrt{\frac{\pi z}{2}} J_{a+1/2}(z), \quad \eta_a(z) = (-1)^a \sqrt{\frac{\pi z}{2}} J_{-a-1/2}(z), \tag{5.1}$$

where J_ν is the first kind Bessel Bessel function of order ν (see [5] for precisions).

From [5] formulas (1) and (2) section 7.11 p.78, they behave like

$$j_a(z) = \sin\left(z - \frac{a\pi}{2}\right) + \mathcal{O}\left(\frac{e^{|\operatorname{Im} z|}}{|z|}\right), \quad |z| \to \infty, \tag{5.2}$$

$$\eta_a(z) = \cos\left(z - \frac{a\pi}{2}\right) + \mathcal{O}\left(\frac{e^{|\operatorname{Im} z|}}{|z|}\right), \quad |z| \to \infty. \tag{5.3}$$

The following estimates can be found in [3]

- Uniform estimates on $\mathbb{C}$:

$$|j_a(z)| \leq Ce^{|\operatorname{Im} z|}\left(\frac{|z|}{1+|z|}\right)^{a+1}, \tag{5.4}$$

$$|\eta_a(z)| \leq Ce^{|\operatorname{Im} z|}\left(\frac{1+|z|}{|z|}\right)^{a}. \tag{5.5}$$

- Estimations for the Green function $G(x,t,\lambda)$ when $0 \leq t \leq x$:

$$|\mathcal{G}(x,t,\lambda)| \leq C\left(\frac{x}{1+|\omega|x}\right)^{a+1}\left(\frac{1+|\omega|t}{t}\right)^{a}\exp\left(|\operatorname{Im}\omega|(x-t)\right), \tag{5.6}$$

$$\left|\frac{\partial \mathcal{G}}{\partial x}(x,t,\lambda)\right| \leq C\left(\frac{x}{1+|\omega|x}\right)^{a}\left(\frac{1+|\omega|t}{t}\right)^{a}\exp\left(|\operatorname{Im}\omega|(x-t)\right). \tag{5.7}$$

- Estimations for the Green function $G(x,t,\lambda)$ when $0 \leq x \leq t \leq 1$:

$$|\mathcal{G}(x,t,\lambda)| \leq C\left(\frac{1+|\omega|x}{x}\right)^{a}\left(\frac{t}{1+|\omega|t}\right)^{a+1}\exp\left(|\operatorname{Im}\omega|(t-x)\right), \tag{5.8}$$

$$\left|\frac{\partial \mathcal{G}}{\partial x}(x,t,\lambda)\right| \leq C\left(\frac{1+|\omega|x}{x}\right)^{a+1}\left(\frac{t}{1+|\omega|t}\right)^{a+1}\exp\left(|\operatorname{Im}\omega|(t-x)\right). \tag{5.9}$$

Rewriting relations (54) – (56) in [5] section 7.2.8 pp.11 – 12, we get

$$x{j_a}'(x) = xj_{a-1}(x) - aj_a(x), \tag{5.10}$$

$$x{j_{a-1}}'(x) = aj_{a-1}(x) - xj_a(x), \tag{5.11}$$

$$x\eta_a'(x) = x\eta_{a-1}(x) - a\eta_a(x), \tag{5.12}$$

$$x\eta_{a-1}'(x) = a\eta_{a-1}(x) - x\eta_a(x). \tag{5.13}$$

We also deduce the following uniform estimates with $\omega \in \mathbb{R}$, $|\omega| \to +\infty$

$$\int_0^1 j_a(\omega t)^2 dt = \frac{1}{2}\left[1 + \mathcal{O}\left(\frac{1}{\omega}\right)\right], \tag{5.14}$$

$$\int_0^1 j_a(\omega t)\eta_a(\omega t)dt = \mathcal{O}\left(\frac{1}{\omega}\right). \tag{5.15}$$

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