BEC of Free Bosons on Networks

Taku Matsui

Graduate School of Mathematics
Kyushu University
1-10-6 Hakozaki, Fukuoka 812-8581
JAPAN
e-mail: matsui@math.kyushu-u.ac.jp

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Abstract: We consider free Bosons hopping on a network(infinite graph). The condition for Bose-Einstein condensation is given in terms of the random walk on a graph. In case of periodic lattices, we also consider Boson moving in an external periodic potential and obatined the criterion for Bose-Einstein condensation.

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1 Results

In this note, we consider the free Bosons hopping on networks (general graphs) and we consider the condition for Bose-Einstein condensation. Let $\Gamma = \{\mathcal{V}, \mathcal{E}\}$ be an infinite graph where $\mathcal{V}$ is the set of vertices and $\mathcal{E}$ is the set of edges. Here for simplicity, we assume that a pair of vertices i and j is not connected by more than two edges(*no multiple edge*), thus we can denote the unique edge connecting vertices i and j by (i, j). The graphs we consider is connected and not oriented. Furthermore for simplicity of presentation, we assume that the graph does not contain any self-loop, i.e. $(i, i) \notin \mathcal{E}$.

Next we introduce free Bosons hopping on vertices of Γ. Let a_j and a_k^* (j, k in $\mathcal{V}$) be the creation and annihilation operators satisfying the canonical commutation relations, $[a_j, a_k^*] = \delta_{jk}$. Smeared Boson operators, $a^*(f)$ and $a(f)$, are defined as

$$a^*(f) = \sum_{j \in \mathcal{V}} a_k^* f_k \quad , \quad a(f) = \sum_{j \in \mathcal{V}} a_k \overline{f}_k \tag{1.1}$$

where f_j is a complex function on $\mathcal{V}$. The function f_j is referred to as a test function. Obviously,

$$[a(f), a^*(g)] = (f, g)_{l^2(\mathcal{V})} 1$$

where (f, g) is the inner product of $l^2(\mathcal{V})$,

$$(f, g)_{l^2(\mathcal{V})} = \sum_{j \in \mathcal{V}} \overline{f}_j g_j .$$

To be definite, we choose the set of rapidly decreasing test functions on $\mathcal{V}$ as the test function space. Rapidly decreasing functions on $\mathcal{V}$ can be introduced as follows. By definition, the graph distance $dist(j, k)$ of two vertices j, k is the minimum of the number of edges giving rise to a path connecting j, k. Fix the origin o of the graph and a complex function f on $\mathcal{V}$ is called rapidly decreasing if

$$\sum_{j \in \mathcal{V}} dist(j, o)^n \, |f_j| < \infty$$

for any positive integer n. The set of all rapidly decreasing functions is denoted by S. Set

$$W(f) = \exp(\frac{i}{\sqrt{2}}(a^*(f) + a(f))).$$

Then,

$$W(f)W(g) = \exp(-i\frac{Im((f, g)_{l^2(\mathcal{V})})}{2})W(f + g),$$

where $Im((f,g)_{l^2)(\mathcal{V})})$ is the imaginary part of the inner product of $l^2(\mathcal{V})$. We denote $\mathfrak{A}(\Gamma)$ by the Weyl CCR C^*-algebra generated by $W(f)$ $(f \in S)$.

Now we consider the free Hamiltonians on our network. On Euclidian spaces, the time evolution of the free Boson is determined by the Laplcian acting on the test function space. Two natural candidates of the graph analogue of the Laplacian are often used. One is the discrete Laplacian and the other is the adjacency matrix. By *degree* of a vertex j we mean the number of edges connected to j and we denote the degree of j by $d(j)$. (The degree is sometimes called *coordination number.*) The discrete Laplacian $-\triangle_\Gamma$ on Γ is defined by

$$-\triangle_\Gamma f(j) = d(j)f(j) - \sum_{k:(k,j)\in\mathcal{E}} f(k), \tag{1.2}$$

where the sum is taken for all vertices k connected to j by an edge of Γ. The discrete Laplacian is a positive operator on $l^2(\mathcal{V})$. The adjacency matrix A_Γ is the off-diagonal part of the discrete Laplacian,

$$A_\Gamma f(j) = \sum_{k:(k,j)\in\mathcal{E}} f(k). \tag{1.3}$$

Thus when the degree of the graph is a constant $d(o)$, $-A_\Gamma + d(o) = -\triangle$ and there is no physical difference of these two operators. When the degree is not constant, we regard the adjacency matrix $-A_\Gamma$ as a Schrödinger operator on a graph and the degree plays a role of potential.

We introduce assumptions for graphs to describe our main result.

Assumption 1.1 *(i) We suppose that the Graph Γ is connected and that Γ is the limit of an increasing sequence of finite connected subgraphs $\Gamma_n = \{\mathcal{V}_n, \mathcal{E}_n\}$,*

$$\Gamma = \cup_{n=1}^{\infty}\Gamma_n = \lim_{n\to\infty}\Gamma_n.$$

(ii) We assume the Følner condition is valid:

$$\lim_{n\to\infty}\frac{|\partial\Gamma_n|}{|\Gamma_n|} = 0. \tag{1.4}$$

where $|\Gamma_n|$ is the number of vertices of Γ_n and $|\partial\Gamma_n|$ is the size of the boundary of Γ_n. More precisely, $|\partial\Gamma_n|$ is the number of vertices of Γ_n connected to the complement Γ_n^c of Γ_n by an edge of Γ.
(iii) The degree of the graph is bounded, $\sup d(j) < \infty$.

The conditions (i) and (ii) above are referred to as van Hove condition in statistical mechanics and they are used to handle thermodynamical quantities of infinite volume systems. The condition (iii) ensures boundedness of the discrete

Laplacians and adjacency matrices. We note that the condition (ii) excludes certain *non-amenable* graphs such as Cayley trees while the condition (iii) excludes the complete graph (all pairs of vertices connected by edges) corresponding to the mean field free model. The case of Cayley trees is studied in [13].

For a finite sub-graph Γ_n, the discrete Laplacian and the adjacency matrix are denoted by $-\triangle_{\Gamma_n}$ and A_{Γ_n}. If the graph is finite, the constant function 1 is the unique ground state for the discrete Laplacian,

$$-\triangle_\Gamma 1 = 0.$$

The unicity (up to multiplicative constant) of the ground state follows from the Perron-Frobenius theorem for positive matrices.

The quantum Hamiltonian is the second quantization $d\Gamma$ of these operators, thus $H = d\Gamma(-\triangle_\Gamma)$ or $H = d\Gamma(-A_\Gamma)$. In what follows, we consider mainly the second quantization of the discrete Laplacian and we set

$$H_\Gamma = d\Gamma(-\triangle_\Gamma) = \sum_{j\in\mathcal{V}}\{d(j)a_j^*a_j - \frac{1}{2}\sum_{k:(k,j)\in\mathcal{E}}(a_j^*a_k + a_k^*a_j)\}. \tag{1.5}$$

Then,

$$[H_\Gamma, a^*(f)] = a^*(-\triangle_\Gamma f)$$

The time evolution α_t^Γ of observables in the Weyl CCR C^*-algebra $\mathcal{A}(\Gamma)$ on Γ is determined by

$$\alpha_t^\Gamma(W(f)) = W(e^{-it\triangle_\Gamma}f).$$

For each finite sub-graph Γ_n, let $\varphi_{\Gamma_n}^{(\beta,\rho)}$ be the equilibrium state(grand canonical ensemble) at the inverse temperature β with the mean density ρ:

$$\begin{aligned}\varphi_{\Gamma_n}^{(\beta,\rho)}(W(f)) &= \exp(-\frac{1}{4}(f, \frac{1+z_ne^{\beta\triangle_\Gamma}}{1-z_ne^{\beta\triangle_\Gamma}}f)) \\ \varphi_{\Gamma_n}^{(\beta,\rho)}(a^*(f)a(g)) &= (g, \frac{z_ne^{\beta\triangle_\Gamma}}{1-z_ne^{\beta\triangle_\Gamma}}f)\end{aligned} \tag{1.6}$$

where z_n $(0 < z_n < 1)$ is determined by

$$\rho = \frac{1}{|\Gamma_n|}\sum_{k\in\mathcal{V}_n}\varphi_{\Gamma_n}^{(\beta,\rho)}(a_k^*a_k) = \frac{1}{|\Gamma_n|}\mathrm{tr}(\frac{z_ne^{\beta\triangle_{\Gamma_n}}}{1-z_ne^{\beta\triangle_{\Gamma_n}}}) \tag{1.7}$$

z_n is the exponential of the chemical potential μ_n , $z_n = \exp(-\beta\mu_n)$.

When Γ is a finite graph, we define $l_0^2(\mathcal{V}) = \{(f_j) \in l^2(\mathcal{V}) \mid \sum_{j\in\mathcal{V}} f_j = 0\}$.

By E_0 we denote the projection from $l^2(\mathcal{V})$ to $l^2_0(\mathcal{V})$. We set $\mathrm{tr}_0(Q) = \mathrm{tr}(E_0 Q E_0)$ and

$$\overline{\rho}(\beta) = \sup_{0<z<1} \limsup_n \frac{1}{|\Gamma_n|} \mathrm{tr}(\frac{z e^{\beta \triangle_{\Gamma_n}}}{1 - z e^{\beta \triangle_{\Gamma_n}}}). \tag{1.8}$$

Then,

$$\overline{\rho}(\beta) = \limsup_{n\to\infty} \frac{1}{|\Gamma_n|} \mathrm{tr}_0(\frac{e^{\beta \triangle_{\Gamma_n}}}{1 - e^{\beta \triangle_{\Gamma_n}}}) \tag{1.9}$$

because

$$\mathrm{tr}(\frac{z e^{\beta \triangle_{\Gamma_n}}}{1 - z e^{\beta \triangle_{\Gamma_n}}}) = \frac{z}{1-z} + \mathrm{tr}_0(\frac{e^{\beta \triangle_{\Gamma_n}}}{1 - e^{\beta \triangle_{\Gamma_n}}}).$$

The following can be shown in the same way of the Bose gas on the Euclidean space. (See [2] [11] and [7].)

Proposition 1.2 *(i) Suppose that $\overline{\rho}(\beta)$ is infinite. Then, for any $\rho > 0$, there exists z_∞ and a subsequence of finite graphs $\Gamma_{n(i)}$ such that $\lim_i z_{n(i)} = z_\infty < 1$ and*

$$\varphi_\Gamma^{(\beta,\rho)}(W(f)) \equiv \lim_i \varphi_{\Gamma_{n(i)}}^{(\beta,\rho)}(W(f)) = \exp(-\frac{1}{4}(f, \frac{1 + z_\infty e^{\beta\triangle_\Gamma}}{1 - z_\infty e^{\beta\triangle_\Gamma}} f)). \tag{1.10}$$

$$\varphi_\Gamma^{(\beta,\rho)}(a^*(f)a(g)) = \lim_n \varphi_{\Gamma_n}^{(\beta,\rho)}(a^*(f)a(g)) = (g, \frac{z_\infty e^{\beta\triangle_\Gamma}}{1 - z_\infty e^{\beta\triangle_\Gamma}} f). \tag{1.11}$$

(ii) Suppose that $\overline{\rho}(\beta)$ is finite.
(iia) If $\rho \le \overline{\rho}(\beta)$, there exists $\lim_n z_n = z_\infty \le 1$ and a subsequence of finite graphs $\Gamma_{n(i)}$ such that the equations (1.10) and (1.11) are valid.
(iib) Suppose $\rho > \overline{\rho}(\beta)$. Take any subsequence of finite graphs $\Gamma_{n(i)}$ satisfying

$$\overline{\rho}(\beta) = \lim_i \frac{1}{|\Gamma_{n(i)}|} \mathrm{tr}_0(\frac{e^{\beta \triangle_{\Gamma_{n(i)}}}}{1 - e^{\beta \triangle_{\Gamma_{n(i)}}}}) \tag{1.12}$$

Then,

$$\lim_i z_{n(i)} = 1 \ , \ \lim_i \frac{z_{n(i)}}{|\Gamma_{z_{n(i)}}|(1 - z_{n(i)})} = \rho - \overline{\rho}(\beta). \tag{1.13}$$

Furthermore, suppose that any $f = (f_j)$ with compact support is in the domain of $(-\triangle_\Gamma)^{-1/2}$ and that the following uniformly boundedness of matrix elements holds.

$$\sup_{i,j\in\mathcal{V}} (\delta_i, \frac{1}{(-\triangle_\Gamma)} \delta_j) < \infty. \tag{1.14}$$

Then, any rapidly decreasing f and g are in the domain of $(-\triangle_\Gamma)^{-1/2}$ (hence in the domain of $(1-e^{\beta\triangle_\Gamma})^{-1/2}$) and,

$$\begin{aligned}&\varphi_\Gamma^{(\beta,\rho)}(W(f)) \equiv \lim_n \varphi_{\Gamma_n}^{(\beta,\rho)}(W(f)) \\ =\; &\exp(-\frac{1}{2}(\rho-\overline{\rho}(\beta))|\chi_\Gamma(f)|^2)\exp(-\frac{1}{4}(f, \frac{1+e^{\beta\triangle_\Gamma}}{1-e^{\beta\triangle_\Gamma}}f)) \qquad (1.15)\end{aligned}$$

$$\begin{aligned}&\varphi_\Gamma^{(\beta,\rho)}(a^*(f)a(g)) \equiv \lim_n \varphi_{\Gamma_n}^{(\beta,\rho)}(a^*(f)a(g)) \\ =\; &(\rho-\overline{\rho}(\beta))\chi_\Gamma(f)\overline{\chi}_\Gamma(g) + (g, \frac{e^{\beta\triangle_\Gamma}}{1-e^{\beta\triangle_\Gamma}}f) \qquad (1.16)\end{aligned}$$

where

$$\chi_\Gamma(f) = \sum_{j\in\mathcal{V}} f_j. \qquad (1.17)$$

We say *Bose-Einstein condensation* occurs for the second case, $\rho > \overline{\rho}(\beta)$, in the above proposition. Due to the equation (1.15) and (1.16), we have off-diagonal long range order and the U(1) gauge symmetry breaking for the state $\varphi_\Gamma^{(\beta,\rho)}$ with a high mean density in the following sense:

$$\lim_{dist(i,j)\to\infty} \varphi_\Gamma^{(\beta,\rho)}(a_i^* a_j) = (\rho-\overline{\rho}(\beta)) \neq 0 = \varphi_\Gamma^{(\beta,\rho)}(a_i^*)\varphi_\Gamma^{(\beta,\rho)}(a_j).$$

$\varphi_\Gamma^{(\beta,\eta,\theta)}$ is decomposed into the following factor states.

$$\begin{aligned}\psi_\Gamma^{(\beta,\eta,\theta)}(W(f)) &= \exp\{i(\eta-\overline{\rho}(\beta))^{1/2}(e^{i\theta}\chi_\Gamma(f)+e^{-i\theta}\overline{\chi}_\Gamma(f))\} \\ &\quad\times\exp(-\frac{1}{4}(f, \frac{1+e^{\beta\triangle_{\Gamma_n}}}{1-e^{\beta\triangle_{\Gamma_n}}}f)) \qquad (1.18)\end{aligned}$$

In another word, $\varphi_\Gamma^{(\beta,\rho)}$ is an integral of $\varphi_\Gamma^{(\beta,\eta,\theta)}$ as functions of η and θ.

When the one-particle Hamiltonian is a discrete Laplacian, we present the condition for Bose-Einstein condensation in terms of the simple random walk on Γ. By the simple random walk, we mean that the random walk can move only to adjacent vertices and that the probability of the jump from one vertex i to another adjacent vertex is $1/d(i)$. The probability of the random walk moving to adjacent vertices are all equal.

Definition 1.3 *Let $p_N(j)$ be the probability of the random walk starting from j and returning to j at the Nth step for the first time. Let $q_N(j)$ be the probability of the random walk starting from j and returning to j at the Nth step.*
The simple random walk on Γ is recurrent if $\sum_{N=2}^{\infty} p_N(j) = 1$ for each vertex j. The simple random walk on Γ is transient if $\sum_{N=2}^{\infty} q_N(j)$ is finite for each vertex j.

It is well known that the random walk is not recurrent if and only if it is transient. (c.f. [14]) For general inhomogeneous networks, we use stronger conditions to describe our results on Bose-Einstein condensation.

Definition 1.4 *(i) The simple random walk on Γ is uniformly transient if*

$$\sup_{j\in\mathcal{V}} \sum_{N=2}^{\infty} q_N(j) < \infty. \tag{1.19}$$

(ii) The simple random walk on Γ is uniformly recurrent if for any large K there exists M such that for any j

$$K \leq \sum_{N=2}^{M} q_N(j). \tag{1.20}$$

We do not know whether uniformity conditions of (1.19) and (1.20) are used in other contexts of probability theory. The same conditions may be expressed in other terms.

Theorem 1.5 *We consider the free Bosons on a graph Γ satisfying previous conditions Assumption 1.1 and assume that Γ contains no self-loop and no multiple edge. Let the one-particle Hamiltonian be the discrete Laplacian $-\triangle_\Gamma$.*
(i) Suppose that the simple random walk on Γ is uniformly transient. Then,

$$\overline{\rho}(\beta) < \infty$$

and any rapidly decreasing f is in the domain of $(-\triangle)^{-1/2}$, hence, alll the assumptions of Proposition 1.2 (iia) and (iib) are valid and the Bose-Einstein condensation occurs.
(ii) Suppose that the simple random walk on Γ is uniformly recurrent. Then,

$$\overline{\rho}(\beta) = \infty,$$

and the conclusion of Proposition 1.2 (i) holds.

Example 1.6 (Periodic lattice) *We consider an infinite graph Γ on which the abelian group $\mathbf{Z}^\nu$ acts graph automorphisms. By τ_k we denote this $\mathbf{Z}^\nu$ action. We assume that*

(i) the isotropy group is trivial , namely, for any vertex i, $\tau_k(i) = i$ *implies* $k = 0$.
and
(ii) the $\mathbf{Z}^\nu$ *action is co-finite in the sense that the quotient graph* $\Gamma_0 = \Gamma/\mathbf{Z}^\nu$ *is a finite graph.*
A graph satisfying these conditions (i) and (ii) is referred to as a periodic lattice and Γ_0 *will be called the fundamental domain of* Γ. *In periodic lattices, by definition, the simple random walk is uniformly transient (resp. uniformly recurrent) if it is transient (resp. recurrent). It is known that the random walk is transient if and only if the dimension* ν *is greater than or equal to three,* $3 \leq \nu$. *See [10]. In the periodic case, we can replace* $\limsup$ *with* $\lim$ *in taking infinite volume limit and we do not have to take subsequences of finite graphs.*

Example 1.7 (Defects) *Now we remove edges from graphs and we call removed edges defects. We denote the set of defects by* $\mathcal{D}$. *We assume that the density of defects vanishes*

$$\lim_{n\to\infty} \frac{|\mathcal{D} \cap \mathcal{V}_n|}{|\mathcal{V}_n|} = 0.$$

Then, our proof shows that if the simple random walk is uniformly transient for the initial graph, BEC occurs for the graph with defects. If the density of defects is positive, further consideration is required and we do not know condition for occurrence or absence of BEC in this case.

Next we discuss the case when the one-particle Hamiltonian is an adjacency matrix. The transiency of the random walk does not determine the occurence of BEC. Such examples are provied in [4] and[5]. The examples discussed in [4] and[5] are star graphs and comb graphs for which the simple random walk are recurrent . (See [14].) In [4], I. Brunelli, G. Giusiano, F. P. Mancini, P. Sodano and A. Trombettoni considered the star graph and the adjacency matrix has a spectral gap. In such situation, it is easy to see the existence of the critical density and occurrence of BEC. In [5], R.Burioni e D. Cassi, M. Rasetti, P. Sodano and A. Vezzani investigated BEC on the comb graph. The vertices of a comb graph are same as $\mathbf{Z^2}$ but some edges are removed from $\mathbf{Z^2}$. In our terminology, the density of defects is strictly positive.

When the graph is a periodic lattice and the adjacency matrix is the one-particle Hamiltonian, we have the same results as the integer lattice, so BEC occurs if and only if $3 \leq \nu$. For the continuous space case on $\mathbf{R}^\nu$, this follows from a result of W.Kirsch and B.Simon in [9] (See also [3]) . W.Kirsch and

B.Simon considered the Schrödinger operator with a periodic potential and they proved that behavior of the density of states in the vicinity of the bottom of energy is same as that for free Laplacian. The argument of W.Kirsch and B.Simon works for our periodic lattice as well. We obtain the same result when we add a periodic potential to the one-particle Hamiltonian.

We set

$$\Gamma_n = \cup_{|k_i|\leq n \ (i=1,2,\ldots\nu)} \tau_k(\Gamma_0) \tag{1.21}$$

By $\Gamma_n^{(p)}$, we denote the graph obtained by the periodic boundary condition on $\Gamma_n^{(p)}$. The vertex set of $\Gamma_n^{(p)}$ is same as that of Γ_n and the edge set of $\Gamma_n^{(p)}$ is the union of Γ_n and additional edges in the way that there is a natural $(\mathbf{Z_{2n+1}})^\nu$ action on $\Gamma_n^{(p)}$ compatible with the shift of Γ. More precisely, if $i \in \mathcal{V}_n$, $j \notin \mathcal{V}_n$, $(i,j) \in \mathcal{E}$ and there exists $m \in \mathbf{Z}^\nu$ such that $|m| = 1$ and $\tau_m(i) = k$ for some k in $\mathcal{V}_n$, (i,k) is an edge of $\Gamma_n^{(p)}$. Using this periodic graph $\Gamma_n^{(p)}$, we can introduce the one-particle Hamiltonian with the periodic boundary condition.

Theorem 1.8 *Suppose the graph Γ is a periodic lattice and $v(j)$ is a (real) periodic function on the set of vertices $\mathcal{V}$. Let E_n (resp. E) be the supremum of the spectrum of $A_{\Gamma_n^{(p)}} - v$ (resp. $A_\Gamma - v$) acting on $l^2(\mathcal{V}_n)$ (resp.$l^2(\mathcal{V})$) . Consider*

$$h_n = E_n - A_{\Gamma_n^{(p)}} + v, \tag{1.22}$$

$$h = E - A_\Gamma + v. \tag{1.23}$$

Let $\varphi_n^{(\beta,\rho)}$ be the equilibrium state associated with the second quatization of h_n with a mean particle density ρ. It is determined by

$$\begin{aligned} &\varphi_n^{(\beta,\rho)}(W(f)) = \exp(-\frac{1}{4}(f, \frac{1+z_n e^{-\beta h_n}}{1-z_n e^{-\beta h_n}} f)), \\ &\frac{1}{|\Gamma_n|}\varphi_n^{(\beta,\rho)}(N_n) = \rho. \end{aligned} \tag{1.24}$$

(i) Suppose that $\nu \leq 2$. BEC does not occur. For any ρ the limits $\lim_n z_n = z < 1$ and

$$\lim_{n\to\infty} \varphi_n^{(\beta,\rho)}(W(f)) = \varphi^{(\beta,\rho)}(W(f)) = \exp(-\frac{1}{4}(f, \frac{1+ze^{-\beta h}}{1-ze^{-\beta h}} f))$$

exist. $\varphi^{(\beta,\rho)}$ is a translationally invariant factor state of $\mathfrak{A}(\Gamma)$.
(ii)Suppose that $\nu \geq 3$. Then the rapidly decreasing functions are in the domain of $h^{-1/2}$ and we can define the critical density ρ_β via the trace $\mathrm{tr}_{l^2(\mathcal{V}_0)}$ over the fundamental domain of our periodic lattice:

$$\rho_\beta = \frac{1}{|\Gamma_0|}\mathrm{tr}_{l^2(\mathcal{V}_0)}(\frac{e^{-\beta h}}{1-e^{-\beta h}}) = \frac{1}{|\Gamma_0|}\sum_{a\in\mathcal{V}_0}(\delta_a, \frac{e^{-\beta h}}{1-e^{-\beta h}}\delta_a). \tag{1.25}$$

Suppose $\rho_\beta \leq \rho$. Then, $\lim_{n\to\infty} z_n = 1$ and

$$\begin{aligned}\varphi_\Gamma^{(\beta,\rho)}(W(f)) &\equiv \lim_n \varphi_{\Gamma_n}^{(\beta,\rho)}(W(f)) \\ &= \exp(-\frac{1}{2}(\rho-\rho(\beta))|\tilde{\chi}_\Gamma(f)|^2)\exp(-\frac{1}{4}(f,\frac{1+e^{-\beta h}}{1-e^{-\beta h}}f)) \end{aligned} \tag{1.26}$$

$$\varphi_\Gamma^{(\beta,\rho)}(a^*(f)a(g)) = (\rho-\rho(\beta))\tilde{\chi}_\Gamma(f)\overline{\tilde{\chi}_\Gamma(g)} + (g,\frac{e^{-\beta h}}{1-e^{-\beta h}}f)$$

where

$$\tilde{\chi}_\Gamma(f) = \lim_{n\to\infty} |\Gamma_n|^{1/2}(\Omega_n, f)_{l^2(\mathcal{V}_n)}. \tag{1.27}$$

and Ω_n is the positive normalized ground state of h_n uniquely determined by

$$h_n\Omega_n = 0, \quad (\Omega_n,\Omega_n)_{l^2(\mathcal{V}_n)} = 1. \tag{1.28}$$

Alternatively, $\tilde{\chi}_\Gamma(f)$ is described by the inner product of the positive periodic ground state $\Omega = \{\Omega(i)\}$ for the infinite volume Hamiltonian h. Let Ω belong to l^∞ satisfying the normalization condition

$$\sum_{i\in\mathcal{V}_0} |\Omega(i)|^2 = 1 \tag{1.29}$$

Then,

$$\tilde{\chi}_\Gamma(f) = \sum_{j\in\mathcal{V}} \Omega(i) f_i = (\Omega, f)_{l^2(\mathcal{V})}, \tag{1.30}$$

which converges for any rapidly decreasing f.

Next we explain key points in our proof of Theorem 1.5 and 1.8. The first point is finiteness of the critical density $\overline{\rho}(\beta)$. For $x > 0$ and $0 < z < 1$,

$$\frac{ze^{-\beta x}}{1-ze^{-\beta x}} = \sum_{m=1}^{\infty} z^m e^{-m\beta x}.$$

Hence, to examine finiteness of $\overline{\rho}(\beta)$, we consider the trace of Gibbs weight, $\frac{1}{|\Gamma_n|}\mathrm{tr}(e^{-\beta h_n})$. Let $P_{l^2(\mathcal{V}_n)}$ be the projection $l^2(\mathcal{V})$ from to $l^2(\mathcal{V}_n)$ and d_n (resp. d) be the degree of Γ_n(resp. Γ). By Trotter-Kato product formula we will see

$$\mathrm{tr}_{l^2(\mathcal{V}_n)}(\frac{ze^{-\beta h_n}}{1-ze^{-\beta h_n}}) \leq \mathrm{tr}_{l^2(\mathcal{V})}(P_{l^2(\Gamma_n)}\frac{ze^{-\beta h_n}}{1-ze^{-\beta h_n}}P_{l^2(\Gamma_n)}) \tag{1.31}$$

Then, by use of the Følner condition (1.4), we show that we can replace d_n with d in the right-hand side in the above inequality. Then, we obtain

$$\overline{\rho}(\beta) \leq \sup_{i \in \mathcal{V}} \left(\delta_i, \frac{ze^{-\beta h}}{1 - ze^{-\beta h}} \delta_i \right)_{l^2(\mathcal{V})} \leq C \left(\delta_i, \frac{1}{h + (1-z)} \delta_i \right)_{l^2(\mathcal{V})} \tag{1.32}$$

for a positive constant C. If the diagonal of the Green function is bounded uniformly in i, we obtain finiteness of the critical density $\overline{\rho}(\beta)$. It is east to see that boundedness of the diagonal of Green function is equivalent to transiency of the simple random walk. This shows the first part of Theorem 1.5. Another inequality can be obtained by similar way and the divergence of the Green function on the diagonal is implied by recurrence of the simple random walk.

When the graph is a periodic lattice, the situation is simpler as we employed the periodic boundary condition. It is easy to see

$$\begin{aligned} & \lim_{n \to \infty} \frac{1}{|\Gamma_n|} \mathrm{tr}_{l^2(\Gamma_n)} \left(\frac{ze^{-\beta h_n}}{1 - ze^{-\beta h_n}} \right) \\ = \quad & \frac{1}{|\Gamma_0|} \left\{ \sum_{i \in \mathcal{V}_0} \left(\delta_i, \frac{ze^{-\beta h}}{1 - ze^{-\beta h}} \delta_i \right)_{l^2(\Gamma)} \right\}. \end{aligned} \tag{1.33}$$

Thus, proof of finiteness or divergence of the critical density reduces to finiteness of the diagonal of Green functions again. We show an inequality between the Dirichlet forms attached to our discrete Schrödinger operators. (c.f. Lemma 3.9) . This tells us that the behavior of the density of states at the bottom of spectrum for our discrete Schrödinger operators with a periodic potential is same as that for our discrete Laplacian.

In Theorem 1.5 we assumed a number of technical conditions on graphs. Conditions of as no self loop and of no multiple edge are not essential. If the graph has self loops we have only to consider random walks which stay on the vertex with certain probability. To handle multiple edges, we modify the definition of the adjacency matrix and its entry is the number of edges connecting vertices labeling entries. On the other hand, the Følner condition (1.4) plays an crucial rôle in our argument. Even though we are unable to prove Bose-Einstein condensation without using the Følner condition, we believe the same result is valid for graphs without our Følner condition. One such example is the Cayley tree associated with the free group. In [13] M.van den Berg, T.C.Dorlas and V.B.Priezzhev proved Bose-Einstein condensation for the free Boson hopping on a Cayley tree. Their argument is based on explicit computation of the spectrum of the discrete Laplacian.

In Section 2 we present our proof of Theorem 1.5 and Section 3 is devoted to Theorem 1.8.

2 Proof of Theorem 1.5.

In this section, we prove Theorem 1.5. Let $\mathcal{W}$ be a subset of vertices $\mathcal{V}$ Let us regard $l^2(\mathcal{W})$ as a closed subspace of $l^2(\mathcal{V})$ and we denote the orthogonal projection from $l^2(\mathcal{W})$ to $l^2(\mathcal{V})$ $P_{\mathcal{W}}$. We regard the degree $d(j)$ of Γ as a multiplication operator on $l^2(\mathcal{V})$ denoted by d, while the degree of the subgraph Γ_n is denoted by d_n. d_n is different from $d(j)$ on the boundary of Γ_n. We set

$$\overline{d} = \sup_{j\in\mathcal{V}} d_\Gamma(j) < \infty.$$

By $tr_{\mathfrak{H}}(Q)$ we denote the trace of an operator Q on a Hilbert space $\mathfrak{H}$. For simplicity, A_Γ(resp. A_{Γ_n} is denoted by A (resp. A_n).

Lemma 2.1

$$\begin{aligned}
&\mathrm{tr}_{l^2(\mathcal{V}_n)}(e^{\beta\triangle_{\Gamma_n}}) \\
&\leq \mathrm{tr}_{l^2(\mathcal{V})}(P_{\partial\Gamma_n}\beta A e^{\beta\triangle_{\Gamma_n}} P_{\partial\Gamma_n}) + \mathrm{tr}_{l^2(\mathcal{V}_n)}(\exp(P_{\Gamma_n^{int}}\beta \triangle_\Gamma P_{\Gamma_n^{int}})) \\
&\leq \mathrm{tr}_{l^2(\mathcal{V})}(P_{\partial\Gamma_n}\beta A e^{\beta\triangle_\Gamma} P_{\partial\Gamma_n}) + \mathrm{tr}_{l^2(\mathcal{V})}(e^{\beta\triangle_{\Gamma_n}} P_{\partial\Gamma_n}) \\
&+\mathrm{tr}_{l^2(\mathcal{V})}(P_{\mathcal{V}_n} e^{\beta\triangle_\Gamma} P_{\mathcal{V}_n})
\end{aligned} \tag{2.1}$$

where Γ_n^{int} *is the interior of* Γ_n, $\Gamma_n^{int} = \mathcal{V}_n - \partial\Gamma_n$.

Proof. The positive discrete Laplcian $-\triangle_{\Gamma_n}$ is composed of the diagonal term (multiplication operator) d_n and the off-diagonal part identical to the adjacency matrix A_n of. The difference of the (positive) discrete Laplcian $-\triangle_{\Gamma_n}$ and $-P_{\Gamma_n}\triangle_\Gamma P_{\Gamma_n}$ is the diagonal part. Using the Trotter-Kato product formula, we obtain

$$\mathrm{tr}_{l^2(\mathcal{V}_n)}(e^{-\beta(-\triangle_{\Gamma_n})}) = \lim_{N\to\infty} tr_{l^2(\mathcal{V}_n)}\left(\left[(1+\frac{\beta A_n}{N})e^{-\beta\frac{d_n}{N}}\right]^N\right). \tag{2.2}$$

Set

$$X(N) = \left[(1+\frac{\beta A_n}{N})e^{-\beta\frac{d_n}{N}}\right], \quad \bar{X}(N) = P_{\Gamma_n^{int}} X(N) P_{\Gamma_n^{int}}.$$

Then, the right-hand side of the equation (2.2) is

$$\begin{aligned}
&\mathrm{tr}_{l^2(\mathcal{V}_n)}(X(N)^N) = \mathrm{tr}_{l^2(\mathcal{V}_n)}(X(N)^N P_{\partial\Gamma_n}) + \mathrm{tr}_{l^2(\mathcal{V}_n)}(X(N)^N P_{\Gamma_n^{int}}) \\
=\ &\mathrm{tr}_{l^2(\mathcal{V}_n)}(X(N)^N P_{\partial\Gamma_n}) + \mathrm{tr}_{l^2(\mathcal{V}_n)}(X(N)^{N-1}\bar{X}(N)) \\
&+\mathrm{tr}_{l^2(\mathcal{V}_n)}(X(N)^{N-1} P_{\Gamma_n^{int}} X(N) P_{\partial\Gamma_n}) \\
=\ &\mathrm{tr}_{l^2(\mathcal{V}_n)}(X(N)^N P_{\partial\Gamma_n}) + \mathrm{tr}_{l^2(\mathcal{V}_n)}(X(N)^{N-1}\bar{X}(N)) \\
&+\mathrm{tr}(X(N)^{N-1} P_{\Gamma_{int}} \frac{\beta A}{N} e^{-\beta d_n} P_{\partial\Gamma_n}).
\end{aligned} \tag{2.3}$$

As all the terms in (2.3) are the trace of non-negative matrices,

$$\begin{aligned}
& \mathrm{tr}_{l^2(\mathcal{V}_n)}(X(N)^N) \\
\leq \; & \mathrm{tr}_{l^2(\mathcal{V}_n)}(X(N)^N P_{\partial\Gamma_n}) + \mathrm{tr}_{l^2(\mathcal{V}_n)}(X(N)^{N-1}\bar{X}(N)) \\
& + \frac{1}{N}\mathrm{tr}_{l^2(\mathcal{V}_n)}(X(N)^{N-1}P_{\Gamma_{int}}\beta A P_{\partial\Gamma_n} e^{-\beta d_n}) \\
\leq \; & \mathrm{tr}_{l^2(\mathcal{V}_n)}(X(N)^N P_{\partial\Gamma_n}) + \mathrm{tr}_{l^2(\mathcal{V}_n)}(X(N)^{N-2}\bar{X}(N)^2) \\
& + \frac{2}{N}\mathrm{tr}_{l^2(\mathcal{V}_n)}(X(N)^{N-1}P_{\Gamma_{int}}\beta A P_{\partial\Gamma_n} e^{-\beta d_n}) \\
\leq \; & \mathrm{tr}_{l^2(\mathcal{V}_n)}(X(N)^N P_{\partial\Gamma_n}) + \mathrm{tr}_{l^2(\mathcal{V}_n)}(\bar{X}(N)^N) \\
& + \mathrm{tr}_{l^2(\mathcal{V}_n)}(X(N)^{N-1}P_{\Gamma_n^{int}}\beta A P_{\partial\Gamma_n}).
\end{aligned} \tag{2.4}$$

Now we take N to infinity in (2.4), and we obtain

$$\begin{aligned}
& \mathrm{tr}_{l^2(\mathcal{V}_n)}(e^{\beta\triangle_{\Gamma_n}}) \\
\leq \; & \mathrm{tr}_{l^2(\mathcal{V}_n)}(e^{\beta\triangle_{\Gamma_n}}P_{\partial\Gamma_n}) + \mathrm{tr}_{l^2(\mathcal{V}_n)}(e^{\beta P_{\Gamma_n^{int}}\triangle_{\Gamma_n}P_{\Gamma_n^{int}}}) \\
& + \mathrm{tr}_{l^2(\mathcal{V}_n)}(e^{\beta\triangle_{\Gamma_n}}P_{\Gamma_n^{int}}\beta A P_{\partial\Gamma_n}) \\
\leq \; & \mathrm{tr}_{l^2(\mathcal{V}_n)}(e^{\beta\triangle_{\Gamma_n})}P_{\partial\Gamma_n}) + \mathrm{tr}_{l^2(\Gamma_n^{int})}(e^{\beta P_{\Gamma_n^{int}}\triangle_{\Gamma_n}P_{\Gamma_n^{int}}}) \\
& + \mathrm{tr}_{l^2(\mathcal{V}_n)}(e^{\beta\triangle_{\Gamma_n}}\beta A P_{\partial\Gamma_n}).
\end{aligned} \tag{2.5}$$

Next we show that for any subsets of vertices $\mathcal{W}_1$ and $\mathcal{W}_2$ satisfying $\mathcal{W}_1 \subset \mathcal{W}_2 \subset \mathcal{V}$,

$$\mathrm{tr}_{l^2(\mathcal{W}_1)}(e^{\beta P_{\mathcal{W}_1}\triangle_{\Gamma_n}P_{\mathcal{W}_1}}) \leq \mathrm{tr}_{l^2(\mathcal{W}_2)}(e^{\beta P_{\mathcal{W}_2}\triangle_{\Gamma_n}P_{\mathcal{W}_2}}) \leq \mathrm{tr}_{l^2(\mathcal{W}_2)}(e^{\beta\triangle_\Gamma}) \tag{2.6}$$

Again, this is due to the Trotter-Kato formula and non-negativity of matrix elements of the adjacency matrix.

$$\begin{aligned}
& \mathrm{tr}_{l^2(\mathcal{W}_1)}(e^{\beta P_{\mathcal{W}_1}\triangle_{\Gamma_n}P_{\mathcal{W}_1}}) = \lim_{N\to\infty}\mathrm{tr}_{l^2(\mathcal{W}_1)}((P_{\mathcal{W}_1}(1+\frac{\beta A_n}{N})P_{\mathcal{W}_1}e^{-\beta d_n/N})^N) \\
& \leq \lim_{N\to\infty}\mathrm{tr}_{l^2(\mathcal{W}_2)}((P_{\mathcal{W}_2}(1+\frac{\beta A_n}{N})P_{\mathcal{W}_2}e^{-\beta d_n/N})^N) = \mathrm{tr}_{l^2(\mathcal{W}_2)}(e^{\beta P_{\mathcal{W}_2}\triangle_{\Gamma_n}P_{\mathcal{W}_2}}) \\
& \leq \lim_{N\to\infty}\mathrm{tr}_{l^2(\mathcal{W}_2)}(((1+\frac{\beta A}{N})e^{-\beta d_n})^N) = \mathrm{tr}_{l^2(\mathcal{W}_2)}(e^{-\beta(d_n-A)})
\end{aligned}$$

As n is arbitrary, we take n to ∞ and we obtain,

$$\mathrm{tr}_{l^2(\mathcal{W}_2)}(e^{\beta P_{\mathcal{W}_2}\triangle_{\Gamma_n}P_{\mathcal{W}_2}}) \leq \mathrm{tr}_{l^2(\mathcal{W}_2)}(e^{-\beta(d-A)}) = \mathrm{tr}_{l^2(\mathcal{W}_2)}(e^{\beta\triangle_\Gamma})$$

Combined with (2.6) , the equation (2.5) implies (2.1). **End of Proof.**

Lemma 2.2 *There exists a constant C such that*

$$\mathrm{tr}_{l^2(\mathcal{V}_n)}(\frac{ze^{\beta\triangle_{\Gamma_n}}}{1-ze^{\beta\triangle_{\Gamma_n}}}) \leq C|\partial\Gamma_n|(\beta\frac{z}{(1-z)^2}+\frac{z}{1-z})+tr_{l^2(\mathcal{V})}(P_{\Gamma_n}\frac{ze^{\beta\triangle_{\Gamma}}}{1-ze^{\beta\triangle_{\Gamma}}}P_{\Gamma_n}) \tag{2.7}$$

Proof. As the heat kernels are contractive, $e^{\beta\triangle_\Gamma}$ and $e^{\beta\triangle_{\Gamma_n}}$ have the norm 1. By definition,

$$\mathrm{tr}_{l^2(\mathcal{V})}(P_{\partial\Gamma_n}) = |\partial\Gamma_n|.$$

Thus, we have

$$\left|\mathrm{tr}_{l^2(\mathcal{V})}(P_{\partial\Gamma_n}\beta Ae^{\beta\triangle_{\mathcal{V}}}P_{\partial\Gamma_n})\right| \leq \beta|\partial\mathcal{V}_n|\beta\left\|A\right\|, \tag{2.8}$$

and

$$\mathrm{tr}_{l^2(\Gamma)}(e^{\beta\triangle_{\Gamma_n}}P_{\partial\Gamma_n}) \leq |\partial\Gamma_n|. \tag{2.9}$$

On the other hand, recall that

$$\frac{ze^{\beta\triangle_{\Gamma_n}}}{1-ze^{\beta\triangle_{\Gamma_n}}} = \sum_{M=1}^{\infty} z^M e^{M\beta\triangle_{\Gamma_n}}$$

for $0 < z < 1$ and $\beta > 0$. We combine the trace of this geometric series with the inequality (2.1) (2.8) and (2.9) to get (2.7). **End of Proof.**

Lemma 2.3 *(i) Suppose that the simple random walk is transient. Then, the delta function δ_j is in the domain of $-\triangle_\Gamma^{-1/2}$*
(ii) Suppose further that the simple random walk is uniformly transient. Then,

$$\left(\delta_i, \frac{1}{-\triangle_\Gamma}\delta_j\right)_{l^2(\Gamma)} \leq Cr_{ij}^{(0)} \tag{2.10}$$

where $r_{ij}^{(0)}$ is the probability of the simple random walk starting from i arriving at j.

Proof. (i) Note that

$$q_N(j) = \left(\delta_i, (\frac{1}{d}A_\Gamma)^N\delta_i\right)_{l^2(\Gamma)} \tag{2.11}$$

Then, for positive z, we have the following Neuman expansion:

$$\left(\delta_i, \frac{1}{z - \triangle_\Gamma}\delta_i\right)_{l^2(\Gamma)} = \left(\delta_i, \frac{1}{d + z - A_\Gamma}\delta_i\right)_{l^2(\Gamma)}$$
$$= \sum_{N=0}^{\infty} \left(\delta_i, (\frac{1}{d+z}A_\Gamma)^N \frac{1}{d+z}\delta_i\right)_{l^2(\Gamma)}$$
$$\leq (1 + \sum_{N=2}^{\infty} q_N(i)\frac{1}{d(i)}) < \infty. \tag{2.12}$$

(ii) Let $r_{ij}(N)$ be the probability of the random walk starting from i and arriving at j at the Nth step and let $r_{ij}^{(0)}(N)$ be he probability of the random walk starting from i and arriving at j at the Nth step for the first time. Then,

$$r_{ij}(N) = r_{ij}^{(0)}(N) + \sum_{k=dist(i,j)}^{N-2} r_{ij}(k)p_j(N-k). \tag{2.13}$$

Set

$$\overline{r}_{ij}(z) = \sum_{N=dist(i,j)}^{\infty} r_{ij}(N)z^n, \quad \overline{r}_{ij}^{(0)}(z) = \sum_{N=dist(i,j)}^{\infty} r_{ij}^{(0)}(N)z^n,$$
$$\overline{p}_j(z) = \sum_{N=2}^{\infty} p_j(N)z^n, \quad \overline{q}_j(z) = \sum_{N=0}^{\infty} q_j(N)z^n.$$

Multiplying z^N and adding in N we have

$$\overline{r}_{ij}(z) = \overline{r}_{ij}^{(0)}(z) + \overline{r}_{ij}(z)\overline{p}_j(z) \tag{2.14}$$

If $|z| < 1$, $\overline{r}_{ij}(z)$, $\overline{r}_{ij}^{(0)}(z)$ and $\overline{p}_j(z)$ converge absolutely. By definition,

$$\overline{q}_j(z) = \overline{r}_{jj}(z), \quad r_{ij}^{(0)} = \overline{r}_{ij}^{(0)}(1),$$

and the random walk starting from j is transient if and only if $\overline{p}_j(1) < 1$. As we assumed that the random walk is uniformly transient, there exists a positive ϵ such that , for any j, $\overline{p}_j(1) < 1 - \epsilon$. As a consequence,

$$\overline{r}_{ij}(1) = \frac{r_{ij}^{(0)}}{1 - \overline{p}_j(1)} \leq \frac{r_{ij}^{(0)}}{\epsilon}. \tag{2.15}$$

Returning to the Neumann expansion of the resolvent, we have the following relation:

$$r_N(ij) = \left(\delta_i, (\frac{1}{d}A_\Gamma)^N \delta_j\right)_{l^2(\Gamma)}. \tag{2.16}$$

If i and j are different, we obtain

$$\left(\delta_i, \frac{1}{-\triangle_\Gamma}\delta_j\right)_{l^2(\Gamma)} = \sum_{N=0}^{\infty}\left(\delta_i, (\frac{1}{d}A_\Gamma)^N \frac{1}{d}\delta_j\right)_{l^2(\Gamma)} \leq \frac{\overline{r}_{ij}(1)}{d(j)} < \frac{r_{ij}^{(0)}}{\epsilon}. \tag{2.17}$$

End of Proof.

Lemma 2.4 *Set*

$$\underline{\rho}(\beta) = \limsup_n \frac{1}{|\Gamma_n|}\mathrm{tr}_{l^2(\mathcal{V})}(P_{\mathcal{V}_n}\frac{e^{\beta\triangle_\Gamma}}{1-e^{\beta\triangle_\Gamma}}P_{\mathcal{V}_n}). \tag{2.18}$$

Assume that the simple random walk is uniformly transient. Then, $\underline{\rho}(\beta)$ is finite.

Proof. As $-\triangle_\Gamma$ is a positive bounded operator, the functional calculus implies the following inequality with suitably large C.

$$\frac{e^{\beta\triangle_\Gamma}}{1-e^{\beta\triangle_\Gamma}} \leq C\frac{1}{-\triangle_\Gamma}.$$

By previous lemma, we have

$$\mathrm{tr}_{l^2(\mathcal{V})}(P_{\mathcal{V}_n}\frac{e^{\beta\triangle_\Gamma}}{1-e^{\beta\triangle_\Gamma}}P_{\mathcal{V}_n}) \leq C\sum_{i\in\mathcal{V}_n}\left(\delta_i, \frac{1}{-\triangle_\Gamma}\delta_i\right)_{l^2(\mathcal{V})}$$
$$\leq |\Gamma_n| \sup_{i\in\mathcal{V}_n}\overline{q}_i(1). \tag{2.19}$$

This implies the finiteness of $\underline{\rho}(\beta)$. **End of Proof.**

Lemma 2.5 *Assume that the simple random walk is uniformly transient. Then,*

$$\overline{\rho}(\beta) \leq \underline{\rho}(\beta) < \infty. \tag{2.20}$$

In particular,$\overline{\rho}(\beta)$ is finite.

Proof. Assuming $\underline{\rho}(\beta) < \overline{\rho}(\beta)$, we show contradiction. Take ρ satisfying $\underline{\rho}(\beta) < \rho < \overline{\rho}(\beta)$. There exists a sequence of subgraphs $\Gamma_{n(k)}$ and z_k such that the following is valid:

$$\rho = tr_{l^2(\mathcal{V}_{n(k)})}(\frac{z_k e^{\beta\triangle_{\Gamma_{n(k)}}}}{1-z_k e^{\beta\triangle_{\Gamma_{n(k)}}}}),$$
$$\lim_k z_k = z_\infty < 1. \tag{2.21}$$

However, the inequalities (2.21) and (2.7) imply that $\rho \leq \underline{\rho}(\beta)$. **End of Proof.**

To prove Theorem 1.5 (ii), we introduce a new notation. For any positive integer m we define an augmented boundary:

$$\partial_m \Gamma_n = \{ i \in \mathcal{V}_n \, | dist(i, \Gamma_n^c) \leq m \}$$

where Γ_n^c is the complement of Γ_n in Γ. Then, obviously,

$$|\partial_m \Gamma_n| \leq \overline{d}^m \, |\partial \Gamma_n| \, .$$

As a result,

$$\lim_{n \to \infty} \frac{|\partial_m \Gamma_n|}{|\Gamma_n|} = 0.$$

Lemma 2.6 *Suppose that the simple random walk is uniformly recurrent and fix large K and n_0 such that the inequality (1.20) is vaild. Take larger n_1 satisfying*

$$n_0 \leq n_1, \quad \frac{|\partial_{n_o} \Gamma_n|}{|\Gamma_n|} < 1/2$$

for $n \geq n_1$. There exists a constant C independent of K such that

$$\overline{\rho}(\beta) \geq CK. \tag{2.22}$$

Proof. First by use of Trotter Kato formula, we show

$$(\delta_i, e^{\beta \triangle_{\Gamma_n}} \delta_i)_{l^2(\mathcal{V}_n)} \geq (\delta_i, e^{\beta P_{\mathcal{V}_n} \triangle_\Gamma P_{\mathcal{V}_n}} \delta_i)_{l^2(\mathcal{V}_n)}. \tag{2.23}$$

As $d_n \leq P_{\mathcal{V}_n} d P_{\mathcal{V}_n}$ and $A_{\Gamma_n} = P_{\mathcal{V}_n} A_\Gamma P_{\mathcal{V}_n}$,

$$\begin{aligned} & (\delta_i, e^{\beta \triangle_{\Gamma_n}} \delta_i)_{l^2(\mathcal{V}_n)} = \lim_{N \to \infty} (\delta_i, (e^{-\frac{\beta d_n}{N}} e^{\frac{\beta A_{\Gamma_n}}{N}})^N \delta_i)_{l^2(\mathcal{V}_n)} \\ \geq \quad & \lim_{N \to \infty} (\delta_i, (e^{-\frac{\beta P_{\mathcal{V}_n} d P_{\mathcal{V}_n}}{N}} e^{\frac{\beta P_{\mathcal{V}_n} A_\Gamma P_{\mathcal{V}_n}}{N}})^N \delta_i)_{l^2(\mathcal{V}_n)} \\ = \quad & (\delta_i, e^{\beta P_{\mathcal{V}_n} \triangle_\Gamma P_{\mathcal{V}_n}} \delta_i)_{l^2(\mathcal{V}_n)}. \end{aligned}$$

Thus we obtained (2.23) .

Next take C_1 such that $e^{\beta x} - 1 \leq C_1 x$ for x satisfying $0 < x < \overline{d}$. Note that $e^x - z = e^x - 1 + (1 - z)$. By (2.23) we have

$$\begin{aligned} & \frac{1}{|\Gamma_n|} tr_{l^2(\mathcal{V}_n)} (\frac{z e^{\beta \triangle_{\Gamma_n}}}{1 - z e^{\beta \triangle_{\Gamma_n}}}) \geq C_1 \frac{1}{|\Gamma_n|} tr_{l^2(\mathcal{V}_n)} (\frac{z}{- \triangle_{\Gamma_n} + 1 - z}) \\ & \geq C_1 \frac{1}{|\Gamma_n|} tr_{l^2(\mathcal{V}_n)} (\frac{z}{- P_{\mathcal{V}_n} \triangle_\Gamma P_{\mathcal{V}_n} + 1 - z}) \\ & \geq C_1 z \frac{1}{|\Gamma_n|} \left\{ \sum_{i \in \Gamma_n \cap \partial_{n_1} \Gamma_n^c} (\delta_i, \frac{1}{P_{\mathcal{V}_n} (-\triangle_\Gamma) P_{\mathcal{V}_n} + 1 - z} \delta_i)_{l^2(\mathcal{V}_n)} \right\}. \end{aligned} \tag{2.24}$$

As before we use the Neumann expansion of the resolvent:

$$\begin{aligned}
& (\delta_i, \frac{1}{P_{\mathcal{V}_n}(-\triangle_\Gamma)P_{\mathcal{V}_n}+1-z}\delta_i)_{l^2(\mathcal{V}_n)} \\
= & \sum_{L=0}^{\infty}(\delta_i, \frac{1}{d+1-z}(P_{\mathcal{V}_n}A_\Gamma P_{\mathcal{V}_n}\frac{1}{d+1-z})^L\delta_i)_{l^2(\mathcal{V}_n)} \\
\geq & \sum_{L=0}^{n_1}(\delta_i, \frac{1}{d+1-z}(P_{\mathcal{V}_n}A_\Gamma P_{\mathcal{V}_n}\frac{1}{d+1-z})^L\delta_i)_{l^2(\mathcal{V}_n)} \qquad (2.25)
\end{aligned}$$

as each term in the above Neumann expansion is non-negative.
As far as i belongs to $\Gamma_n \cap \partial_{n_1}\Gamma_n^c$ and $L \leq n_1$

$$\begin{aligned}
& (\delta_i, \frac{1}{d+1-z}(P_{\Gamma_n}A_\Gamma P_{\Gamma_n}\frac{1}{d+1-z})^L\delta_i)_{l^2(\mathcal{V}_n)} \\
= & (\delta_i, \frac{1}{d+1-z}(A_\Gamma\frac{1}{d+1-z})^L\delta_i)_{l^2(\mathcal{V}_n)}. \qquad (2.26)
\end{aligned}$$

This is because the random walk starting from i cannot reach the augmented boundary $\partial_{n_1}\Gamma_n$ within L steps.
Now we combine these estimates

$$\begin{aligned}
& \frac{1}{|\Gamma_n|}tr_{l^2(\mathcal{V}_n)}(\frac{ze^{\beta\triangle_{\Gamma_n}}}{1-ze^{\beta\triangle_{\Gamma_n}}}) \\
\geq & \frac{C_1 z}{2\,|\Gamma_n \cap \partial_{n_1}\Gamma_n^c|}\sum_{i\in\Gamma_n\cap\partial_{n_1}\Gamma_n^c}\sum_{L=0}^{n_1}(\delta_i, \frac{1}{d+1-z}(A_\Gamma\frac{1}{d+1-z})^L\delta_i)_{l^2(\mathcal{V}_n)}
\end{aligned} \qquad (2.27)$$

As a consequence

$$\begin{aligned}
& \overline{\rho}(\beta) \geq \frac{C_1}{2}\inf_{i\in\mathcal{V}_n\cap\partial_{n_1}\Gamma_n^c}\sum_{L=0}^{n_1}(\delta_i, \frac{1}{d}(A_\Gamma\frac{1}{d})^L\delta_i)_{l^2(\mathcal{V}_n)} \\
= & \frac{C_1}{2}\inf_{i\in\mathcal{V}_n\cap\partial_{n_1}\Gamma_n^c} q_i(n(1)) \geq \frac{C_1}{2}K. \qquad (2.28)
\end{aligned}$$

End of Proof.
The above lemma completes our proof of Theorem 1.5.
End of Proof of Theorem 1.5.

3 Proof of Theorem1.8.

In this section, we prove Theorem 1.8. As remarked before, when the one-particle Hamiltonian is a Schrödinger operator on Euclidean spaces, the question of occurrence of the Bose-Einstein condensation is reduced to behavior of the density of states at the bottom of the spectrum. In case of periodic potentials W.Kirsch and B.Simon proved that the asymptotic behavior of the density states is same as that for the free Schrödinger operator. (See [9] .) The same problem is considered in presence of magnetic field by P.Briet, H.D.Cornean, and V.A. Zagrebnov in [3]. The basic idea of our proof for the periodic lattice is same as [9], however, there appears some difference between the periodic lattice case and the Euclidean case, which we explain below.

Let us recall that the periodic lattice $\Gamma = \{\mathcal{V}, \mathcal{E}\}$ is obtained by the fundamental domain $\Gamma_0 = \{\mathcal{V}_0, \mathcal{E}_0\}$ and its shift. The shift is denoted by τ_k as before. The choice of fundamental domain is not unique and we fix one fundamental domain $\Gamma_0 = \{\mathcal{V}_0, \mathcal{E}_0\}$ satisfying two conditions:
(i) it is a connected sub-graph of Γ.
(ii) the conditions $\tau_k(i) = j$, and $i, j \in \mathcal{V}_0$ imply $i = j$ and $k = 0$.
Furthermore, without loss of generality, we may assume
(iii) $\Gamma_0^{(p)}$ does not possess a multiple edge.
For our purpose, we may consider a larger block Γ_n as a fundamental domain and a smaller group $((2n+1)\mathbf{Z})^\nu$ as the shift on our lattice. This is the reason why we can assume (iii).

Now, we identify the vertex set $\mathcal{V}$ with $\mathcal{V}_0 \times \mathbf{Z}^\nu$ in such a way that the shift on $\mathcal{V}_0 \times \mathbf{Z}^\nu$ acts via the following formula:

$$\tau_k(a, j) = (a, j+k).$$

$l^2(\mathcal{V})$ is naturally isomorphic to $l^2(\mathcal{V}_0) \otimes l^2(\mathbf{Z}^\nu)$, We identify this Hilbert space with $l^2(\mathcal{V}_0) \otimes L^2(T^\nu)$ by use of Fourier transformation F from $l^2(\mathbf{Z}^\nu)$ to $L^2(T^\nu)$ where the torus T^d is identified with $[-\pi, \pi]^\nu$. Our convention for Fourier transform F and the inner product of $L^2(T^\nu)$ are

$$(f, g)_{L^2(T^\nu)} = \frac{1}{(2\pi)^\nu} \int_{T^\nu} \overline{f}(p) g(p) dp, \quad F\delta_k = e^{ik\cdot p} \qquad k \in \mathbf{Z}^\nu, \mathbf{p} \in [-\pi, \pi]^\nu.$$

Then any operator B commuting with shift is called *translationally invariant*. It is unitarily equivalent to a matrix valued multiplication operator $\widetilde{B}(p)$:

$$FBF^{-1} f(p) = \widetilde{B}(p) f(p) \qquad , f(p) \in l^2(\mathcal{V}_0) \otimes L^2(T^\nu).$$

Thus, we have direct integral representation of any translationally invariant operator B :

$$B = \int_{T^\nu}^{\oplus} \widetilde{B}(p)dp, \quad l^2(\mathcal{V}) = \int_{T^\nu}^{\oplus} \mathfrak{H}_p dp$$

where $\mathfrak{H}_p$ is a $|\Gamma_0|$ dimensional Hilbert space of wave functions satisfying the following twisted boundary condition:

$$\mathfrak{H}_p = \{f(a,j) \in l^\infty(\mathcal{V}) | f(a, j+k) = e^{ik\cdot p} f(a,j)\}. \tag{3.1}$$

The above observation is valid for finite periodic graphs $\Gamma_n^{(p)}$ introduce in Section 1 as well. We have only to replace the $\mathbf{Z}^\nu$ with the finite cyclic group $\mathbf{Z}_\mathbf{n}^\nu$ ($\mathbf{Z_n} = \mathbf{Z}/\mathbf{nZ}$). Following the custom of physicists, we call the variable p *quasi-momentum.*

$\widetilde{A}_\Gamma(p)$ is a matrix with entires indexed by vertices of the fundamental domain Γ_0. The matrix elements $[\widetilde{A}_\Gamma]_{ij}(p)$ of $\widetilde{A}_\Gamma(p)$ (for a fixed quasi-momentum p) are described as follows.

Set $e_i(k,j) = \delta_{i,j}$ where $k \in \mathbf{Z}^\nu$ and $i, j \in \Gamma_0$. We regard e_i a periodic function on $\mathcal{V}$. Obviously e_i is in $l^\infty(\mathcal{V})$ and $\{e_i(i \in \mathcal{V}_0)\}$ is a basis of the set of periodic functions on $\mathcal{V}$. When the quasi-momentum p is zero, $[\widetilde{A}_\Gamma]_{ij}(0)$ is determined by

$$A_\Gamma e_j = \sum_{i \in \mathcal{V}_0} [\widetilde{A}_\Gamma]_{ij}(0) e_i. \tag{3.2}$$

As the degree of graph is bounded, the adjacency matrix A_Γ is a bounded operator on $l^\infty(\mathcal{V})$ and (3.2) should be understood as an identity of $l^\infty(\mathcal{V})$.

To consider the case for non vanishing quasi-momentum p, we set

$$e_i^{(p)}(k,j) = e^{ik\cdot p}\delta_{ij}.$$

$\{e_i^{(p)}(k,j)\}$ is a basis of the space $\mathfrak{H}_p$ of functions with quasi-momentum p. Then, the matrix elements of $\widetilde{A}_\Gamma(p)_{ij}$ is determined by

$$A_\Gamma e_j^{(p)} = \sum_{i \in \mathcal{V}_0} [\widetilde{A}_\Gamma]_{ij}(p) e_i^{(p)}. \tag{3.3}$$

By construction and translational invariance, we obtain the following formulae.

Lemma 3.1 *(i) If i and j are connected by an edge inside the fundamental domain Γ_0,*

$$[\widetilde{A}_\Gamma]_{ij}(p) = [\widetilde{A}_\Gamma]_{ij}(0). \tag{3.4}$$

(ii) If i and j are connected by an edge bridging adjacent blocks of Γ,

$$[\widetilde{A}_\Gamma]_{ij}(p) = e^{i\theta_{ij}(p)}[\widetilde{A}_\Gamma]_{ij}(0). \tag{3.5}$$

Here $\theta_{ij}(p)$ is real and $\theta_{ij}(p) = -\theta_{ji}(p)$. It is a linear combination of the compnent $p(k)$ of the quasi-momentum $p = (p(1), p(2), \cdots p(\nu))$ described as follows: If the edge (i,j) of $\mathcal{E}_0^{(p)}$ corresponds to an edge (i,j) of Γ such that

$$i = ((i_1, i_2, \cdots, i_\nu), a) \ , \ j = ((j_1, \cdots, j_\nu), b) \in \mathcal{V} = \mathbf{Z}^\nu \times \mathcal{V}_0,$$
$$1 \le \sum_{n=1}^{\nu} (j_n - i_n) \le \nu, \quad i_n \le j_n \ (n = 1, 2, \cdots, \nu),$$

$$\theta_{ij}(p) = \sum_{n=1}^{\nu} p_n (j_n - i_n). \tag{3.6}$$

(iii) If (i,j) is not an edge of Γ,

$$[\widetilde{A}_\Gamma]_{ij}(p) = 0. \tag{3.7}$$

For later convenience, we set $\theta_{ij}(p) = 0$ when i and j are connected by an edge inside the fundamental domain Γ_0 and we can write

$$[\widetilde{A}_\Gamma]_{ij}(p) = e^{i\theta_{ij}(p)} [\widetilde{A}_\Gamma]_{ij}(0) \tag{3.8}$$

for any i and j. The factor $e^{i\theta_{ij}(p)}$ corresponds to an external magnetic field in physics and to a curvature in context of the discrete geometry of graphs.

Let us return to the Bose-Einstein condensation. Let l be a positive integer and we consider Γ_l of (1.21) and the finite graph $\Gamma_l^{(p)}$ obtained by the periodic boundary condition. Consider the adjacency matrices A_Γ and $A_{\Gamma_l^p}$. Let v be the periodic potential of Theorem 1.8. v is a periodic potential for the finite periodic system on $\Gamma_l^{(p)}$ as well. Set

$$h_l = E^{(l)} - A_{\Gamma_l^{(p)}} + v, \quad h = E - A_\Gamma + v, \tag{3.9}$$

where $E^{(l)}$ (resp. E) is the supremum of the spectrum of $A_{\Gamma_l^{(p)}} - v$ (resp. $A_\Gamma - v$).

Now we consider the particle density. Set

$$\rho_l(z) = \frac{1}{\left|\Gamma_l^{(p)}\right|} \mathrm{tr}_{l^2(\Gamma_l^{(p)})} \left(\frac{z e^{-\beta h_l}}{1 - z e^{-\beta h_l}} \right).$$

Due to translational invariance,

$$\rho(z) = \lim_{l \to \infty} \rho_l(z) = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \frac{1}{|\Gamma_0|} \mathrm{tr}_{l^2(\mathcal{V}_0)} \left(\frac{z e^{-\beta \widetilde{h}(p)}}{1 - z e^{-\beta \widetilde{h}(p)}} \right) dp. \tag{3.10}$$

As there exist constants $C_\beta^{(1)}$ and $C_\beta^{(2)}$such that

$$C_\beta^{(1)} \frac{1}{x+1-z} \leq \frac{ze^{-\beta x}}{1-ze^{-\beta x}} \leq C_\beta^{(2)} \frac{1}{x}$$

for positive x we have the following lemma.

Lemma 3.2 *Let $E(p)$ be the largest eigenvalue of the matrix $\widetilde{A}_\Gamma(p) - \widetilde{v}(p)$.*
(i) Suppose the following integral is finite.

$$\int_{[-\pi,\pi]^\nu} \frac{1}{E - E(p)} dp < \infty. \tag{3.11}$$

Then $\rho(1)$ is finite and for any z_n and l_n satisfying $\lim_n z_n = 1$ and $\lim_n l_n = \infty$,

$$\lim_{n\to\infty} \rho_{l_n}(z_n) = \rho(1) \tag{3.12}$$

(ii) Suppose that the following limit is infinite.

$$\lim_{z \nearrow 1} \int_{[-\pi,\pi]^\nu} \frac{1}{E - E(p) + 1 - z} dp = \infty. \tag{3.13}$$

Then, for any large positive ρ, there exists z ($0 < z < 1$) such that $\rho = \rho(\beta)$.

Note that for any periodic potential v

$$\widetilde{v}(p) = v|_{\mathcal{V}_0}$$

Thus by abuse of notation we identify $\widetilde{v}(p)$ and v.

$$\widetilde{v}(p) = v$$

(3.13) of the above Lemma implies the absence of Bose-Einstein condensation while the case (i) of the above Lemma suggests Bose-Einstein condensation for the mean particle density ρ greater than $\rho(1)$. We show that (3.13) holds if the dimension ν of our periodic lattice is one or two and that (3.11) is finite if if the dimension ν is greater than or equal to three.

Lemma 3.3

$$E = E(0) \geq E(p). \tag{3.14}$$

The proof is same as periodic Schrödinger operators on $\mathbf{R}^\nu$. For the detail of proof, see Chapter XIII,16 of [12].

Lemma 3.4 *In a neighborhood of $p = 0$,*

$$|E - E(p)| \leq C \|p\|^2 = C \sum_{k=1}^{\nu} p_k^2 \tag{3.15}$$

Proof. We set

$$\mathrm{Re}(\widetilde{A}_\Gamma(p)) = \frac{1}{2}\left(\widetilde{A}_\Gamma(p) + (\widetilde{A}_\Gamma(-p)\right).$$

When the quasi-momentum p is sufficiently small, $\mathrm{Re}(\widetilde{A}_\Gamma(p))$ and hence, $\mathrm{Re}(\widetilde{A}_\Gamma(p)) - v$ are Perron-Frobenius positive matrices and we have a positive vector as a unique eigenvector for the largest eigenvalue. Then,

$$E(p) \geq \sup_{\|f\|=1} (f, (\mathrm{Re}(\widetilde{A}_\Gamma(p)) - v)f)_{l^2(\mathcal{V}_0)} = \sup_{\|f\|=1} (|f|, \{\mathrm{Re}(\widetilde{A}_\Gamma(p)) - v\}\, |f|)_{l^2(\mathcal{V}_0)} \tag{3.16}$$

where $|f|$ is a vector $l^2(\Gamma_0)$ with the component $|f_j|$ $(j \in \Gamma_0)$.

The matrix element of $\mathrm{Re}(\widetilde{A}_\Gamma(p))$ is the real part of (3.4) and (3.5).

$$[\mathrm{Re}(\widetilde{A}_\Gamma(p))]_{ij} = \cos\theta_{ij}(p)[\mathrm{Re}(\widetilde{A}_\Gamma(0))]_{ij}.$$

When the quasi-momentum p is sufficiently small, we have a small positive constant C such that

$$\cos\theta_{ij}(p) \geq (1 - C\|p\|^2).$$

Thus

$$\begin{aligned} & \sup_{\|f\|=1} (|f|, \{\mathrm{Re}(\widetilde{A}_\Gamma(p)) - v\}\, |f|)_{l^2(\mathcal{V}_0)} \\ \geq \quad & (1 - C\|p\|^2) \sup_{\|f\|=1} (|f|, \{(\widetilde{A}_\Gamma(0)) - v\}\, |f|)_{l^2(\mathcal{V}_0)} \\ = \quad & (1 - C\|p\|^2)E(0). \end{aligned} \tag{3.17}$$

This inequality implies (3.15) . **End of Proof.**

Lemma 3.4 shows the divergence of the integral (3.13) if ν is one or two. Next we consider the case $\nu \geq 3$. To show Proposition 3.6 below, we use a graph analogue of the diamagnetic inequality.

Lemma 3.5 *For any function f on $\mathcal{V}_0$,*

$$\left(f, e^{-\beta(E - \widetilde{A}(p) + v)} f\right)_{\mathcal{V}_0} \leq \left(|f|, e^{-\beta(E - \widetilde{A}(0) + v)}\, |f|\right)_{\mathcal{V}_0}. \tag{3.18}$$

Proof. Recall that the abosulute value of matrix elements of $\widetilde{A}(p)$ is same as that of $\widetilde{A}(0)$ By the Trotter-Kato product formula, we have

$$\left(f, e^{-\beta(E-\widetilde{A}(p)+v)}f\right)_{\mathcal{V}_0} = \lim_{N\to\infty}\left(f, \left[e^{-\frac{\beta}{N}(E+v)}(1+\frac{\widetilde{\beta A}(p)}{N}\right]^N f\right)_{\mathcal{V}_0}$$

$$\leq \lim_{N\to\infty}\left(|f|, \left[e^{-\frac{\beta}{N}(E+v)}(1+\frac{\beta\widetilde{A}(0)}{N})\right]^N |f|\right)_{\mathcal{V}_0}$$

$$= \left(|f|, e^{-\beta(E-\widetilde{A}(0)+v)}|f|\right)_{\mathcal{V}_0}. \tag{3.19}$$

End of Proof.

Proposition 3.6 *If $E = E(p_0)$ for some $p_0 \neq 0$, there exists a diagonal unitary W on $l^2(\mathcal{V}_0)$ such that*

$$W\widetilde{A}(p)W^* = \widetilde{A}(p-p_0) \tag{3.20}$$

in a neighborhood of p_0

Proof. Suppose that $E = E(0) = E(p_0)$ and f is the unit eigenvector for the largest eigenvalue of $\widetilde{A}(p) - v$:

$$(\widetilde{A}(p) - v)f = Ef, \quad \|f\| = 1.$$

By the diamagnetic inequality,

$$1 = \left(f, e^{-\beta(E-\widetilde{A}(p_0)+v)}f\right)_{\mathcal{V}_0} \leq \left(|f|, e^{-\beta(E-\widetilde{A}(0)+v)}|f|\right)_{\mathcal{V}_0} \leq 1 \tag{3.21}$$

Thus by differentiating (3.21),

$$(\widetilde{A}(0) - v)|f| = E|f|.$$

It turns out that $|f|$ is the Perron Frobenius eigenvector of $\widetilde{A}(0) - v$ and all the components of $|f|$ are positive. Now we define δ_a ($a \in \Gamma_0$) and a diagonal unitary V via the following equations:

$$f_a = e^{i\delta_a}|f_a|, \quad W = \mathrm{diag}(e^{-i\delta_a}).$$

By definition,$f = W^*|f|$, $WvW^* = v$ and

$$(W\widetilde{A}(p_0)W^* - v)|f| = E|f|. \tag{3.22}$$

We claim that

$$W\widetilde{A}(p)W^* = \widetilde{A}(p-p_0). \tag{3.23}$$

First, due to (3.8),

$$[W\widetilde{A}(p)W^*]_{ij} = e^{i(\theta_{ij}(p)+\delta_i-\delta_j)}[\widetilde{A}(0)]_{ij}. \tag{3.24}$$

Taking the real part of (3.22), we have

$$(B(p_0) - v)\,|f| = E\,|f|\,, \tag{3.25}$$

where

$$B(p) \equiv \mathrm{Re}(W\widetilde{A}(p_0)W^*).$$

The matrix elements of $B(p)$ is given by

$$[B(p)]_{ij} = \cos(\theta_{ij}(p) + \delta_i - \delta_j)[\widetilde{A}(0)]_{ij}.$$

Consider the following inner product:

$$(|f|\,, (B(p_0) - v)\,|f|)_{l^2(\mathcal{V}_0)} = E\,(|f|\,, |f|)_{l^2(\mathcal{V}_0)} = E \tag{3.26}$$

As $|f|$ is the Perron Frobenius eigenvector of $\widetilde{A}(p_0) - v$, we have two identities:

$$\sum_{ij} \Big(\cos(\theta_{ij}(p_0) + \delta_i - \delta_j)[\widetilde{A}(0)]_{ij} - \delta_{ij}\Big)\,|f|_i\,|f|_j = E, \tag{3.27}$$

$$\sum_{ij} \Big([\widetilde{A}(0)]_{ij} - \delta_{ij}\Big)\,|f|_i\,|f|_j = E \tag{3.28}$$

As these two identities must be valid simultaneously and all the components of $|f|$ are non-vanishing,

$$\cos(\theta_{ij}(p_0) + \delta_i - \delta_j) = 1.$$

This implies $\sin(\theta_{ij}(p_0) + \delta_i - \delta_j) = 0$ and

$$e^{\sqrt{-1}(\theta_{ij}(p)+\delta_i-\delta_j)} = e^{\sqrt{-1}(\theta_{ij}(p-p_0)+(\theta_{ij}(p_0)+\delta_i-\delta_j))} = e^{\sqrt{-1}(\theta_{ij}(p-p_0))} \tag{3.29}$$

The equations (eqn:z761) and (3.29) suggest the claim of Proposition3.6.
End of Proof.

The following is a corollary of Proposition3.6.

Corollary 3.7 *(i) The number of p_0 satisfying $E = E(p_0)$ is finite.*
(ii) The integral (3.11) is finite if and only if the following integral is finite in a neighborhood of $p = 0$:

$$\int \frac{1}{E - E(p)} dp \tag{3.30}$$

Remark 3.8 *(i) For Schrödinger operators with periodic potential on Euclidean spaces,* $E = E(p)$ *holds only at the origin* $p = 0$ *See [9].*

As $\widetilde{A}(p) - v$ *can be interpreted as a Schrödinger operators with a magnetic field on the finite graph* Γ_0*, degenracy* $E = E(p)$ *can happen even if* $p \neq 0$*. Such examples are presented explicitly by Yusuke Higuchi and Tomoyuki Shirai in [8].*
(ii) The above Proposition 3.6 and Corollary 3.7 are valid for any dimension ν*.*

Next we compare $E - E(p)$ when v is the degree of the periodic lattice (i.e. when $h = d - A = -\triangle_\Gamma$) and other v. Now E^d, $E^d(p)$ stand for E and $E(p)$ when v is the degree d and E^v, $E^v(p)$ for those of other periodic potential v. By definition, $E^d = 0$. We also denote $h(v) = E - A_\Gamma + v$ and $h_l(v) = E^{(l)} - A_{\Gamma_l^{(p)}} + v$.

Lemma 3.9 *Let* $\Omega = \Omega(i)(i \in \mathcal{V})$ *and* $\Omega^{(l)} = \Omega^{(l)}(i)$ *be the positive periodic ground states for* $h(v)$ *and* $h_l(v)$ *satisfying the following normalization condition:*

$$(\Omega, \Omega)_{l^2(\Gamma_0)} = \sum_{i \in \mathcal{V}_0} |\Omega(i)|^2 = |\mathcal{V}_0|, \quad (\Omega^{(l)}, \Omega^{(l)})_{l^2(\mathcal{V}_0)} = |\mathcal{V}_0|$$

Then,

$$(f\Omega, h(v) f\Omega)_{l^2(\mathcal{V})} = \sum_{(i,j) \in \mathcal{E}} |f(i) - f(j)|^2 \, \Omega(i)\Omega(j) \tag{3.31}$$

for any f *in* $l^2(\mathcal{V})$ *and*

$$(f\Omega^{(l)}, h_l(v) f\Omega^{(l)})_{l^2(\mathcal{V}_l)} = \sum_{(i,j) \in \mathcal{E}_l^{(p)}} |f(i) - f(j)|^2 \, \Omega^{(l)}(i)\Omega^{(l)}(j) \tag{3.32}$$

for any f *in* $l^2(\mathcal{V}_0)$*.*
If f *satisfies the twisted boundary condition with quasi-momentum* p*, we have*

$$(f\Omega, (E - A_\Gamma(p) + v) f\Omega)_{l^2(\mathcal{V}_0)} = \sum_{(i,j) \in \mathcal{E}_0^{(p)}} \left| f(i) - e^{i\theta_{ij}(p)} f(j) \right|^2 \Omega(i)\Omega(j) \tag{3.33}$$

where $h^{(0)}(v)$*, and* $\mathcal{E}_0^{(p)}$ *are obtained by* $A_{\Gamma_0} - v$ *and* $\mathcal{E}_0$ *with the periodic boundary condition.* $\theta_{ij}(p)$ *is defined in (3.6) when* i *and* j *are connected by an edge not belonging to* $\mathcal{E}_0$ *and when* i *and* j *are connected by an edge in* $\mathcal{E}_0$*, we set* $\theta_{ij}(p) = 0$*.*

Proof. To derive (3.33) , divide (3.32) by the volume of Γ_l and take the limit of l to the infinity. Thus we have only to show the identity (3.31) which can be obtained by direct calculation as follows. (c.f. [9].)

First we consider the case of $d = v$. In this case $\Omega(i) = 1$ and by definition,

$$(f\Omega, (d - A_\Gamma)f\Omega)_{l^2(\mathcal{V})} = \sum_{(i,j)\in\mathcal{E}} \left(|f(i)|^2 + |f(j)|^2 - \overline{f}(j)f(i) - \overline{f}(i)f(j)\right)$$

$$= \sum_{(i,j)\in\mathcal{E}} |f(i) - f(j)|^2 . \qquad (3.34)$$

Next we set $w = v - d$. Then ,$-(d - A_\Gamma)\Omega = (E + w)\Omega$. Using (3.34),

$$\begin{aligned}
& (f\Omega, (E + d - A_\Gamma + w)f\Omega)_{l^2(\mathcal{V})} \\
= & (f\Omega, (d - A_\Gamma)f\Omega)_{l^2(\mathcal{V})} - (f\Omega, f(d - A_\Gamma)\Omega)_{l^2(\mathcal{V})} \\
= & \sum_{(i,j)\in\mathcal{E}} |f(i)\Omega(i) - f(j)\Omega(j)|^2 \\
& - \sum_{(i,j)\in\mathcal{E}} \left\{|f(i)|^2 \left(|\Omega(i)|^2 - \Omega(i)\Omega(j)\right) + |f(j)|^2 \left(|\Omega(j)|^2 - \Omega(i)\Omega(j))\right)\right\} \\
= & \sum_{(i,j)\in\mathcal{E}} \left\{|f(i)|^2 + |f(j)|^2 - \left(\overline{f}(i)f(j) + \overline{f}(j)f(i)\right)\right\} \Omega(i)\Omega(j) \\
= & \sum_{(i,j)\in\mathcal{E}} |f(i) - f(j)|^2 \Omega(i)\Omega(j). \qquad (3.35)
\end{aligned}$$

End of Proof.
Proof of Theorem1.8. By Lemma 3.9 , we have

$$\begin{aligned}
E^v(0) - E^v(p) &= \inf_{f\in\mathfrak{H}_p} \frac{\sum_{(i,j)\in\mathcal{E}_0^{(p)}} |f(i) - f(j)|^2 \Omega(i)\Omega(j)}{\sum_{i\in\mathcal{V}_0} |f(i)|^2 (\Omega(i))^2}, \\
E^d(0) - E^d(p) = -E^d(p) &= \inf_{f\in\mathfrak{H}_p} \frac{\sum_{(i,j)\in\mathcal{E}_0^{(p)}} |f(i) - f(j)|^2}{\sum_{i\in\mathcal{V}_0} |f(i)|^2}. \qquad (3.36)
\end{aligned}$$

Set

$$M = \frac{\sup_i \Omega(i)}{\inf_i \Omega(i)}.$$

Due to (3.36) , we obtain

$$M^{-1}(E^d(0) - E^d(p)) \leq E^v(0) - E^v(p) \leq M(E^d(0) - E^d(p)). \qquad (3.37)$$

This inequality shows that finiteness of the integral (3.30) in a neighborhood of $p = 0$ for the discrete Laplacian is equivalent to that for the Schrödinger operator h with a periodic potential v. This completes our proof of Theorem1.8. **End of Proof.**

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