HOW SMOOTH IS YOUR WAVELET? WAVELET REGULARITY VIA THERMODYNAMIC FORMALISM

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Abstract. A popular wavelet reference [W] states that "in theoretical and practical studies, the notion of (wavelet) regularity has been increasing in importance." Not surprisingly, the study of wavelet regularity is currently a major topic of investigation. Smoother wavelets provide sharper frequency resolution of functions. Also, the iterative algorithms to construct wavelets converge faster for smoother wavelets. The main goals of this paper are to extend, refine, and unify the thermodynamic approach to the regularity of wavelets and to devise a faster algorithm for estimating regularity.

We present an algorithm for computing the Sobolev regularity of wavelets and prove that it converges with super-exponential speed. As an application we construct new examples of wavelets that are smoother than the Daubechies wavelets and have the same support. We establish smooth dependence of the regularity for wavelet families, and we derive a variational formula for the regularity. We also show a general relation between the asymptotic regularity of wavelet families and maximal measures for the doubling map. Finally, we describe how these results generalize to higher dimensional wavelets.

0. Introduction

While the Fourier transform is useful for analyzing stationary functions, it is much less useful for analyzing non-stationary cases, where the frequency content evolves over time. In many applications one needs to estimate the frequency content of a nonstationary function locally in time, for example, to determine when a transient event occurred. This might arise from a sudden computer fan failure or from a pop on a music compact disk. The usual Fourier transform does not provide simultaneous time and frequency localization of a function.

The windowed or short-time Fourier transform does provide simultaneous time and frequency localization. However, since it uses a fixed time window width, the same window width is used over the entire frequency domain. In applications, a fixed window width is frequently unnecessarily large for a signal having strong high frequency components and unnecessarily small for a signal having strong low frequency components.

In contrast, the wavelet transform provides a decomposition of a function into components from different scales whose degree of localization is connected to the size of the scale

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window. This is achieved by integer translations and dyadic dilations of a single function: the wavelet.

The special class of orthogonal wavelets are those for which the translations and dilations of a fixed function, say, ψ form an orthonormal basis of $L^2(\mathbb{R})$. Examples include Haar wavelet, Shannon wavelet, Meyer wavelet, Battle-Lemari wavelets, Daubechies wavelets, and Coiffman wavelets. A standard way to construct an orthogonal wavelet is to solve the dilation equation

$$\phi(x) = \sqrt{2} \sum_{k=-\infty}^{\infty} c_k \phi(2x - k), \tag{0.1}$$

with the normalization $\sum_k c_k = 1$. Very few solutions of the dilation equation for wavelets are known to have closed form expressions. This is one reason why it is difficult to determine the regularity of wavelets.[1] Provided the solution ϕ satisfies some additional conditions, the function ψ defined by

$$\psi(x) = \sum_{n=-\infty}^{\infty} (-1)^k c_{1-k} \phi(2x - k),$$

is an orthogonal wavelet, i.e., the set $\{2^{j/2}\psi(2^j x - n) : j, n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$. Henceforth, the term wavelet will mean orthogonal wavelet.

Daubechies [Da1] made the breakthrough of constructing smoother compactly supported wavelets. For these wavelets, the wavelet coefficients $\{c_k\}$ satisfy simple recursion relations, and thus can be quickly computed. However, the Daubechies wavelets necessarily possess only finite smoothness.[2] Smoother wavelets provide sharper frequency resolution of functions. Also, the regularity of the wavelet determines the speed of convergence of the cascade algorithm [Da1], which is another standard method to construct the wavelet from equation (0.1). Thus knowing the smoothness or regularity of wavelets has important practical and theoretical consequences.

The main goals of this paper are to extend, refine, and unify the thermodynamic approach to the regularity of wavelets and to devise a faster algorithm for estimating regularity.

In Section 1 we recall the construction of wavelets using multiresolution analysis. We then discuss how the L^p-Sobolev regularity of wavelets is related to the thermodynamic pressure of the doubling map $E_2 : [0, 2\pi) \to [0, 2\pi)$. The latter is the crucial link between wavelets and dynamical systems.

Previous authors have noted that for compactly supported wavelets, the transfer operator preserves a finite dimensional subspace and is thus represented by a $d \times d$ matrix. Providing the matrix is small, its maximal eigenvalue, which is related to the pressure, can be explicitly computed, and thus the L^p-Sobolev regularity can be determined by matrix algebra.

[1] We learned in [Da2] (see references) that this dilation equation arises in other areas, including subdivision schemes for computer aided design, where the goal is the fast generation of smooth curves and surface.

[2] Frequently, increasing the support of the function leads to increased smoothness.

Cohen and Daubechies identified the L^p-Sobolev regularity in terms of the smallest zero of the Fredholm-Ruelle determinant for the transfer operator acting on a certain Hilbert space of analytic functions [CD1, Da2]. Their analysis leads to an algorithm for the Sobolev regularity of wavelets having analytic filters that converges with exponential speed. In Section 2, we study the transfer operator acting on the Bergman space of analytic functions and effect a more refined analysis. This leads to a more efficient algorithm which *converges with super-exponential speed* to the L^p-Sobolev regularity of the wavelet. The algorithm involves computing certain orbital averages over the periodic points of the doubling map E_2.

In Section 3 we illustrate, in a number of examples, the advantage of this method over the earlier Cohen-Daubechies method. Running on a fast desktop computer, this algorithm provides a highly accurate approximation for the Sobolev regularity in less than two minutes.

A feature of this thermodynamic approach is that it works equally well for compactly supported and non-compactly supported wavelets (e.g., Butterworth filters). In practice, for wavelets with large support, the distinction may become immaterial, and the use of our algorithm which applies to arbitrary wavelets may prove quite useful.

In Section 4, we use this improved algorithm to study the parameter dependence of the L^p-Sobolev regularity in several smooth families of wavelets. In particular, we construct new examples of wavelets having the same support as Daubechies wavelets, but with higher regularity.

In Section 5, we show that the L^p-Sobolev regularity depends smoothly on the wavelet, and we provide, via thermodynamic formalism, a variational formula. This problem has been considered experimentally by Daubechies [Da1] and Ojanen [Oj1] for different parameterized families. Our thermodynamic approach, based on properties of pressure, gives a sound theoretical foundation for this analysis.

In Section 6, we study the asymptotic behavior of the L^p-Sobolev regularity of wavelet families. We establish a general relation between this asymptotic regularity and maximal measures for the doubling map, extending work of Cohen and Conze [CC1]. The study of maximal measures is currently an active research area in dynamical systems.

In Section 7, we describe the natural generalization of these results to arbitrary dimensions.

Finally, in Section 8, we briefly describe the natural generalization of these results to wavelets whose filters are not analytic.

1. Wavelets: construction and regularity

Constructing Wavelets. We begin by recalling the construction of wavelets via multiresolution analysis. A comprehensive reference for wavelets is [Da1], which also contains an extensive bibliography.

The Fourier transform of the scaling equation (0.1) satisfies

$$\widehat{\phi}(\xi) = m(\xi/2)\widehat{\phi}(\xi/2), \quad \text{where} \quad m(\xi) = \frac{1}{\sqrt{2}} \sum_k c_k e^{-ik\xi}. \tag{1.1}$$

The function m is sometimes called the *filter* defined by the scaling equation. Iterating this expression, and assuming that $m(0) = 1$, one obtains that

$$\widehat{\phi}(\xi) = \prod_{j=1}^{\infty} m(2^{-j}\xi). \tag{1.2}$$

Conditions (1)–(3) below allow one to invert this Fourier transform and obtain the scaling function ϕ. The wavelet ψ is then explicitly given by

$$\psi(t) = \sqrt{2} \sum_{k=-\infty}^{\infty} d_k \phi(2t - k), \quad \text{where} \quad d_k = (-1)^{k-1}\bar{c}_{-1-k}, \tag{1.3}$$

and thus the family

$$\psi_{j,k} = \left\{ 2^{j/2}\psi(2^j t - k) \right\}, j, k \in \mathbb{Z}$$

forms an orthonormal basis of $L^2(\mathbb{R})$.[3]

A major result in the theory is the following.

Theorem 1.1 [Da1, p.186]. *Consider an analytic 2π-periodic function m with Fourier expansion $m(\xi) = \sum_n c_n e^{in\xi}$ satisfying the following conditions:*

(1) $m(0) = 1$;
(2) $|m(\xi)|^2 + |m(\xi + \pi)|^2 = 1$; *and*
(3) $|m(\xi)|^2$ *has no zeros in the interval* $[-\frac{\pi}{3}, \frac{\pi}{3}]$.

Then the function $\widehat{\phi}$ defined in (1.2) is the Fourier transform of a function $\phi \in L^2(\mathbb{R})$ and the function ψ defined by (1.3) defines a wavelet.

Condition (1) is equivalent to $\widehat{\phi}(0) \neq 0$, which is necessary to ensure the completeness in the multiresolution defined by ϕ. Condition (1) is also necessary for the infinite product (1.2) to converge. Condition (2) is necessary for the wavelet to be orthogonal, and condition (3), called the "Cohen condition", is a sufficient condition for the wavelet to be orthogonal [Da1, Cor. 6.3.2].

If one wishes to construct compactly supported wavelets, then one needs the additional assumption:

(4) There exists $N \geq 1$ with $c_n = 0$ for $|n| \geq N$.

If $c_k \neq 0$ for $k \in [N_1, N_2]$, then the support of ϕ is also contained in $[N_1, N_2]$, and the support of ψ is contained in $[(N_1 - N_2 + 1)/2, (N_2 - N_1 + 1)/2]$.

It is frequently useful to work with the function $q(\xi) = |m(\xi)|^2$. This function is positive, analytic, and even. Condition (1) is equivalent to $q(0) = 1$ and (2) is equivalent to $q(\xi) + q(\xi + \pi) = 1$. A simple exercise shows that (1) is equivalent to $\sum_{k\in\mathbb{Z}} c_k = 1/2$ and (2) is equivalent to $q(\xi) = 1/2 + \sum_{k\in\mathbb{Z}} c_{2k+1} \cos((2k+1)\xi)$. Thus the set $\mathcal{K}$ of positive even

[3]Engineers often call the family $\{c_k, d_k\}$ a quadratic mirror filter (QMF).

analytic functions satisfying both (1) and (2) is a convex subset of the positive analytic functions.

Example (Daubechies wavelet family).

This family of continuous and compactly supported wavelets ϕ_N^{Daub} $\{N \in \mathbb{N}\}$, is obtained by choosing the filter m such that

$$|m(\xi)|^2 = \cos^{2N}(\xi/2)P_N(\sin^2(\xi/2)), \quad \text{where} \quad P_N(y) = \sum_{k=0}^{N-1} \binom{N-1+k}{k} y^k.$$

The wavelets in this family have no known closed form expression.

Note that $|m(\xi)|^2$ vanishes to order $2N$ at $\xi = \pi$. More generally, a trigonometric polynomial m with $|m(\xi)|^2 = \cos^{2N}\left(\frac{\xi}{2}\right)u(\xi)$ satisfies (2) only if $u(\xi) = Q(\sin^2(\xi/2))$, where $Q(y) = P_N(y) + y^N R(1/2 - y)$, and R is an odd polynomial, chosen such that $Q(y) \geq 0$ for $0 \leq y \leq 1$ [Da1, 171].

Wavelet regularity. Since the Fourier transform is such a useful tool for constructing wavelets, and since wavelets (and other solutions of the dilation equation) rarely have closed form expressions, it seems useful to measure the regularity of wavelets using the Fourier transform. For $p \geq 1$, the *L^p-Sobolov regularity of ϕ* is defined by

$$s_p(f) = \sup\left\{s : (1 + |\xi|^p)^s |\widehat{f}(\xi)|^p \in L^1(\mathbb{R})\right\}.$$

The most commonly studied case is $p = 2$, where if $s_2(f) > 1/2$, then $f \in C^{s_2 - 1/2 - \epsilon}$.

One can also easily see that

$$s_p(f) - s_q(f) \leq \frac{1}{p} - \frac{1}{q}, \text{ for } 0 < p < q.$$

Thermodynamic Approach to Wavelet Regularity. We now discuss a thermodynamic formalism approach to wavelet regularity. We assume that $|m(\xi)|^2$ has a maximal zero of order M at $\pm\pi$ and write

$$|m(\xi)|^2 = \cos^{2M}(\xi/2)r(\xi), \tag{1.1}$$

where $r \in C^\omega([0, 2\pi))$ is an analytic function with no zeros at $\pm\pi$. We further assume that $r > 0$, and thus $\log r \in C^\omega([0, 2\pi))$.

Given a function $g : [0, 2\pi) \to \mathbb{R}$ and the doubling map $E_2 : [0, 2\pi) \to [0, 2\pi)$ defined by $E_2(x) = 2x (\mathrm{mod}\ 2\pi)$, the pressure of g is defined by

$$P(g) = \sup\left\{h(\mu) + \int_0^{2\pi} g\, d\mu : \mu \text{ a } E_2\text{-invariant probability measure}\right\},$$

where $h(\mu)$ is the measure theoretic entropy of μ. If g is Hölder continuous, then this supremum is realized by a unique E_2-invariant measure μ_g, called the equilibrium state for g.

The following theorem expresses the relationship between the pressure and the Sobolev regularity $s_p(\phi)$, and provides the crucial link between wavelets and dynamical systems.

Theorem 1.2. *The L^p-Sobolov regularity satisfies the expression*

$$s_p(\phi) = M - \frac{P(p\log r)}{p\log 2}.$$

Proof. The proof for $p = 2$ is due to Cohen and Daubechies [CD1] and the general case is due to Eirola [Ei1] and Villemoes [Vi1]. See also [He1, He2]. This result was originally formulated in terms of the spectra radius of the transfer operator on continuous functions; but this is precisely the pressure (see Proposition 2.1). □

See Appendix I for a proof of $\geq$ in arbitrary dimensions.

2. Determinants and transfer operators

Given $h : [0, 2\pi) \to \mathbb{R}$ we define the semi-norm

$$|h|_\beta = \sup_{x\neq y} \frac{|h(x) - h(y)|}{|x-y|^\beta},$$

and the Hölder norm $||h||_\beta = |h|_\infty + |h|_\beta$, where $0 < \beta < 1$.

Given $g \in C^\beta[0, 2\pi)$, the transfer operator $L_g : C^\beta([0, 2\pi) \to C^\beta([0, 2\pi)$ is the *positive* linear operator defined by

$$L_g k(x) = \exp\left(g\left(\frac{x}{2}\right)\right) k\left(\frac{x}{2}\right) + \exp\left(g\left(\frac{x}{2}+\pi\right)\right) k\left(\frac{x}{2}+\pi\right).$$

The following properties of pressure are well known [Ru1, PP1].

Proposition 2.1.
1. *The spectral radius of L_g is $\exp(P(g))$;*
2. *The pressure $P(g)$ has an analytic dependence on $g \in C^\beta([0, 2\pi))$; and*
3. *The derivative $D_g P(f) = \int_0^{2\pi} f d\mu_g$.*

The transfer operator $L_g\colon C^\beta([0, 2\pi)) \to C^\beta([0, 2\pi))$ is not trace class on this class of functions. However, following Jenkinson and Pollicott [JP1, JP2], we consider the transfer operator acting on the Bergman space $A_2(\Delta)$ of square-integrable analytic functions on an open unit disk $\Delta \subset \mathbb{C}$ containing $[0, 2\pi)$. In Appendix I we show that for analytic functions g, the transfer operator $L_g\colon A_2(\Delta) \to A_2(\Delta)$ is trace class and L_g has a well-defined *Ruelle-Fredholm determinant.* For small z this can be defined by $\det(I - zL_g) = \exp(\mathrm{tr}(\log(I - zL_g)))$ [Si]. We can write $\mathrm{tr}(\log(I - zL_g)) = \sum_{k=1}^\infty (z^n/n)\mathrm{tr}(L_g^k)$, and each operator L_g^k is the sum of weighted composition operators having sharp-trace (in the sense of Atiyah and Bott)

$$\mathrm{tr}(L_g^k) = \sum_{E_2^k x = x} \frac{\exp(S_k g(x))}{1 - 2^{-k}}.$$

The Ruelle-Fredholm determinant for L_g can be expressed as

$$\det(I - zL_g) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{E_2^n x = x} \frac{\exp(S_n g(x))}{1 - 2^{-n}} \right), \tag{2.1}$$

and has the "usual determinant property" $\det(I - zL_g) = \prod_{n=1}^{\infty} (1 - z\lambda_n)$, where λ_n denotes the n-th eigenvalue of L_g. Note that this determinant is formally quite similar to the Ruelle zeta function, and has an additional factor of the form $1/(1 - 2^{-n})$ in the inner sum. More generally, L_g need not be trace class for expression (2.1) to be well defined for small z (see Appendix III).

For our applications to wavelet regularity, we consider $g = p \log r$ (see (1.1)). To compute the determinant, one needs to compute the orbital averages of the function g over all the periodic points of E_2. Every root of unity of order $2^n - 1$ is a periodic point for E_2, and there are exactly $2^n - 1$ such roots of unity, hence there are exactly $2^n - 1$ periodic points of period n for E_2. A routine calculation yields

$$d_p(z) := \det(I - zL_{plogr}) = 1 + \sum_{n=1}^{\infty} b_n z^n, \qquad \text{where}$$

$$b_n = \sum_{n_1 + \ldots + n_k = n} (-1)^k \frac{a^p_{n_1} \cdots a^p_{n_k}}{n_1! \cdots n_k!} \quad \text{and} \quad a^p_n = \left(\frac{1}{1 - 2^{-n}} \right) \sum_{k=0}^{2^n - 1} \left[\prod_{j=0}^{n-1} r\left(\frac{2^j k}{2^n - 1} \right) \right]^p .$$

The next theorem summarizes some important properties of this determinant $d_p(z)$.

Theorem 2.2. *Assume* $\log r \in C^\omega[0, 2\pi)$. *Then*

(1) *The function $d_p(z)$ can be extended to an entire function of z and a real-analytic function of p;*
(2) *The reciprocal of the exponential of pressure $z_p := \exp(-P(p \log r))$ is the smallest zero for $d_p(z)$;*
(3) *The Taylor coefficients b_n of $d_p(z)$ decay to zero at a* **super-exponential rate**, *i.e., there exists $0 < A < 1$ such that $b_n = O(A^{n^2})$;*
(4) *Let $d_{p,N}(z) = 1 + \sum_{n=1}^{N} b_n z^n$ be the truncation of the Taylor series to N terms and consider smallest zero $d_{p,N}(z_p^N) = 0$. Then $z_p^N \to z_p$ at a* **super-exponential rate**, *i.e., there exists $0 < B < 1$ such that $|z_p - z_p^N| = O(B^{N^2})$.*

In fact, in Theorem 2.2 (3) and (4) we can choose A and B arbitrarily close to $1/2$.

Parts (1) and (2) follow from [Ru2] and (4) follows from (3) via the argument principle. The essential idea in proving (3) is that one can also express the determinant

$$d_p(z) = 1 + \sum_{n=1}^{\infty} b_n z^n = \prod_{n=1}^{\infty} (1 - z\lambda_n),$$

where λ_n denotes the n-th eigenvalue of $L_{p \log r}$ [Sim]. It follows that the coefficient $b_n = \sum_{i_1 < \cdots < i_n} \lambda_{i_1} \cdots \lambda_{i_n}$. To show that the Taylor coefficients have super exponential decay, it suffices to show a suitable estimate on the eigenvalues. We present a proof in Appendix I.

Part (4) of this theorem provides an algorithm for computing the Sobolev regularity which converges with super-exponential speed in N. The calculation of $d_{p,N}(z)$ for $N = 12$ takes less than two minutes on a desktop computer.

For our applications to wavelets, we shall take either $g = \log |m|^2$ or $g = \log r$. In fact, both functions lead to equivalent results. The following proposition relates the zeros of $\det(1 - zL_{\log r})$ and $\det(1 - zL_{|m|^2})$.[4]

Proposition 2.3. *The* determinant $\det(1 - zL_{\log |m|^2})$ *has zeros at*

(1) $1, 2, 4, \dots, 2^{M-1}$*; and*
(2) $2^M z$*, where* z *is a zero for* $\det(1 - zL_{\log r})$.

This was observed by Cohen and Daubechies [CD1] and we include a short proof in Appendix II.

3. Examples: Computation of Sobolev regularity using the determinant

In this section we shall apply Theorem 2.2 to estimate the Sobolev exponent $s_p(\phi)$ of certain interesting examples.

3.1 Daubechies wavelet family.

Example 1: ($N = 3$ and $p = 2$). In this case $d_2(z)$ is a polynomial of degree 5 with rational coefficients:

$$d_2(z) = 1 - 2z - 57.1875z^2 - 74.53125z^3 + 210.09375z^4 - 91.125z^5,$$

which has its first zero at $z_0 = 0.111\ldots$. By Theorem 1.2 we deduce the precise value $s_2(\psi_3^{\text{Daub}}) = 3 + \log(0.1111\ldots)/2\log 2 = 1.415037\ldots$.[5]

The following table illustrates the remarkable speed of approximation of this algorithm by computing the first few approximations to $s_2(\psi_3^{\text{Daub}})$. The level k approximation to the determinant is obtained by summing over all periodic points with period less than or equal to k. Level 6 approximations take less than a minute on a desktop computer.

[4] In our method it is desirable to factor out the zeros and work with r because this reduces the amount of computation and may increase the efficiency of estimating the determinant by decreasing $C > 0$

[5] This compares with the numerical estimate of Cohen-Daubechies of 1.414947, which is accurate to two decimal places.

N	Smallest root of d_2	Approx to $s_2(\psi_3^{\mathrm{Daub}})$
1	0.50000000000000000	2.5000000000000000
2	0.11590073269851241	1.4454807959406297
3	0.10935186260699924	1.4035248496765980
4	0.11120565746767627	1.4156510452962938
5	0.11111111111111109	1.4150374992788437
6	0.11111111111111111	1.4150374992788437

Example 2: ($N = 2$ and $p = 4$). In this case $d_4(z)$ is a polynomial of degree 5 with rational cofficients:

$$d_4(z) = 1 - 2z - 50.75z^2 - 76.25z^3 + 115z^4 - 32z^5,$$

which has its first zero at $z_0 = 0.115146\ldots$. By Theorem 2.1 we deduce that $s_4(\psi_2^{\mathrm{Daub}}) = 3 + \log(0.115146\ldots)/2\log 2 = 1.22038\ldots$.[6]

Example 3: ($N = 4$ and $p = 2$). In this case $d_2(z)$ is a polynomial of degree 7 with rational coefficients

$$d_2(z) = 1 - 2z - 433z^2 - 996z^3 + 23572z^4 - 32288z^5 - 68627z^6 - 24414z^7,$$

which has its first zero at $z_0 = 0.045788\ldots$. By Theorem 2.1 we deduce that $s_2(\psi_4^{\mathrm{Daub}}) = 4 + \log(0.045788\ldots)/2\log 2 = 1.77557\ldots$.[7]

Based on the same approximation scheme, the following table contains our approximations to $s_2(\psi_N^{\mathrm{daub}})$ for $N = 1, 2, \ldots, 10$, and compares them with the values obtained by Cohen and Daubechies in [CD1].

N	new approx to $s_2(\psi_4^{\mathrm{Daub}})$	previous approx to $s_2(\psi_4^{\mathrm{Daub}})$
1	0.500000	0.338856
2	1.000000	0.999820
3	1.415037	1.414947
4	1.775565	1.775305
5	2.096786	2.096541
6	2.388374	2.388060
7	2.658660	2.658569
8	2.914722	2.914556
9	3.161667	3.161380
10	3.402723	3.402546

[6] This compares with the estimate of Cohen-Daubechies of $1.220150\ldots$, which is accurate to three decimal places.

[7] This compares with the estimate of Cohen-Daubechies of 1.775305, which is accurate to three decimal places.

Remark. In examples $1-3$ the determinant is a rational function. This is an immediate consequence of the fact that the transfer operator preserves the finite dimensional subspace of functions spanned by

$$\{1, \cos(\xi), \cos(2\xi), \cdots, \cos(N\xi)\}.$$

This fact has been observed by several authors, c.f., [Oj1].

3.2 Butterworth filter family.

The Butterworth filter is the filter type with flattest pass band and which allows a moderate group delay [OS1]. These filters form an important family of non-compactly supported wavelets and are defined by

$$|m(\xi)|^2 = \frac{\cos^{2N}(\xi/2)}{\sin^{2N}(\xi/2) + \cos^{2N}(\xi/2)}, N \in \mathbb{N}.$$

We suppress the index n for notational convenience. It is easy to check that $|m(\xi)|^2$ satisfies conditions $(1)-(3)$ on page 4.

Example 4*: (Butterworth filter with N=2, p=2.)* For this case the determinant $d_2(z)$ is a power series of the form

$$d_2(z) = 1 - 2.\, z - 2.08\, z^2 - 0.479053\, z^3 - 0.0124521\, z^4 - 0.000086246\, z^5 - 2.06875\, 10^{-9}\, z^6 - 3.2482\, 10^{-13}\, z^7 - 7.38964\, 10^{-13}\, z^8 + 4.24431\, 10^{-12}\, z^9 - 2.94676\, 10^{-11}\, z^{10} + \cdots,$$

which has its first zero at $z_0 = 0.356702\ldots$, and thus by Theorem 1.2 we deduce that $s_2(\psi_2^{\text{Butt}}) = 2 + \log(0.356702\ldots)/2\log 2 = 1.25497\ldots$.

The following table contains the first few approximations of $s_2(\psi_2^{\text{Butt}})$:

N	Root	Approx to $s_2(\psi_2^{\text{Butt}})$
1	0.500000000000000	1.0000000000000000
2	1.256072268804003	1.2689757055091953
3	0.356757200613385	1.2565072268804003
4	0.356702225093496	1.2563960602082473
5	0.356702089349239	1.2563957856969186
6	0.356702089348077	1.2563957856945698
7	0.356702089348077	1.2563957856945698

We can similarly compute $s_2(\psi_N^{\text{Butt}})$ for $N = 1, 2, \ldots, 10$ and compare these with the values in [CD1].

N	new approx to $s_2(\psi_N^{\mathrm{Butt}})$	previous approx to $s_2(\psi_N^{\mathrm{Butt}})$
1	0.500000	0.338856
2	1.256211	1.256395
3	2.044109	2.044117
4	2.843768	2.843771
5	3.648646	3.648817
6	4.456118	4.456252
7	5.264533	5.264586
8	6.072947	6.073034
9	6.881125	6.881173
10	7.688598	7.688776

4. Compactly supported wavelets smoother than the Daubechies family

We now consider smooth finite dimensional families of wavelets and attempt to find the maximally regular wavelets in the family. Several authors have used this approach to find compactly supported wavelets that are more regular than the Daubechies wavelets with the same support. The most straightforward approach to the problem of maximizing the regularity of compactly supported wavelets in parameterized families reduces to the problem of maximizing the maximal eigenvalue of an $N \times N$ matrix. However, except for very small N, there is no algebraic expression for the eigenvalue. In this section, we illustrate the advantages of our fast converging algorithm in seeking more regular wavelets.

Example 1. Ojanen [Oj1] experimentally studied the regularity of wavelets corresponding to filters $|m(\xi)|^2$ of width N and which vanish to order M at π. In the case that $M \geq N-1$ he found that the Daubechies polynomials (where $M = N$) are apparently the smoothest. For $N = 2$ and $M = 1$, say, one can define the family of filters

$$|m(\xi)|^2 = \frac{1}{2}(1+\cos\xi)\underbrace{(1+8a\cos^2\xi - 8a\cos\xi)}_{:=r(\xi)}. \tag{4.1}$$

The transfer operator preserves the subspace spanned by $\{1, \cos(\xi), \cos(2\xi)\}$ and thus can be represented by a 3×3 matrix whose maximal eigenvalue is $\exp P = 1 + \sqrt{1+16a}$. In the special case when $a = -1/16$ one obtains the filter corresponding to ψ_2^{Daub}, for which $r(\pi) = 0$. One can easily show that this wavelet is the smoothest in this family. *However, for the broader class $M \leq N-2$, this is no longer the case.*

Example 2. We consider the two-parameter family of *putative wavelets* corresponding to filters

$$|m(x)|^2 = \frac{1}{2} + a\cos(x) + b\cos(3x) + \left(\frac{1}{2} - a - b\right)\cos(5x). \tag{4.2}$$

We say putative, because all the conditions in Theorem 1.1 need to be verified for all parameter values. The Daubechies wavelet ψ_3^{Daub} corresponds to parameters $a = 75/128$ and $b = -25/256$.

Here $|m(x)|^2$ has a double zero at $x = \pi$ and thus one can always take out (at least) a factor of $(1 + \cos(\xi))/2$. However, it is always necessary to check "by hand" that for each particular choice of a and b expression (4.2) is positive and the Cohen condition holds.

It is useful to write (4.2) in the form $r(\xi)(1 + \cos(\xi))/2$, where

$$\begin{aligned} r(\xi) &= 1 - 4\,(-1 + 2\,a + 4\,b)\,\cos(\xi) + 4\,(-1 + 2\,a + 4\,b)\,\cos^2(\xi) \\ &\quad + 16\,(-1 + 2\,a + 2\,b)\,\cos^3(\xi) - 16\,(-1 + 2\,a + 2\,b)\,\cos^4(\xi). \end{aligned}$$

Notice this trigonometric polynomial has the form $r(\xi) = 1 - \alpha\cos(\xi) + \alpha\cos^2(\xi) + \beta\cos^3(\xi) - \beta\cos^4(\xi)$.

(A) The transfer matrix approach

We consider the finite dimensional space spanned by $\{1, \cos(\xi), \cos^2(\xi), \cos^3(\xi), \cos^4(\xi)\}$ and write the action of the transfer operator associated to weights of the general form

$$r(\xi) = 1 - \alpha\cos(\xi) + \alpha\cos^2(\xi) + \beta\cos^3(\xi) - \beta\cos^4(\xi).$$

A routine calculation yields that the transfer operator on this space has the following matrix representation:

$$L_{\log r} = L_{\log r}(\alpha, \beta) = \begin{pmatrix} 2 + \alpha - \frac{\beta}{2} & \alpha - \beta & -\frac{\beta}{2} & 0 & 0 \\ -\alpha + \frac{\beta}{2} & -\alpha + \beta & \frac{\beta}{2} & 0 & 0 \\ 1 + \frac{\alpha}{2} - \frac{\beta}{4} & 1 + \alpha - \frac{3\beta}{4} & \frac{\alpha}{2} - \frac{3\beta}{4} & -\frac{\beta}{4} & 0 \\ -\frac{\alpha}{2} + \frac{\beta}{4} & -\alpha + \frac{3\beta}{4} & -\frac{\alpha}{2} + \frac{3\beta}{4} & \frac{\beta}{4} & 0 \\ \frac{1}{2} + \frac{\alpha}{4} - \frac{\beta}{8} & 1 + \frac{3\alpha}{4} - \frac{\beta}{2} & \frac{1}{2} + \frac{3\alpha}{4} - \frac{3\beta}{4} & \frac{\alpha}{4} - \frac{\beta}{2} & -\frac{\beta}{8} \end{pmatrix}.$$

The matrix corresponding to ψ_3^{Daub} is

$$L_{\log r}(-7/8, -3/8) = \begin{pmatrix} \frac{21}{16} & -\frac{1}{2} & \frac{3}{16} & 0 & 0 \\ \frac{11}{16} & \frac{1}{2} & -\frac{3}{16} & 0 & 0 \\ \frac{21}{32} & \frac{13}{32} & -\frac{5}{32} & \frac{3}{32} & 0 \\ \frac{11}{32} & \frac{19}{32} & \frac{5}{32} & -\frac{3}{32} & 0 \\ \frac{21}{64} & \frac{17}{32} & \frac{1}{8} & -\frac{1}{32} & \frac{3}{64} \end{pmatrix},$$

with eigenvalues 1, 9/16, $\pm 1/4$ and 3/64. Then $\det(I - zL_{\log r})$, has a zero at $z = 16/9$. By Theorem 1.2 we again recover the Sobolev regularity of ψ_3^{Daub}: $s_2(\psi_3^{\mathrm{Daub}}) = 1 - (\log(9/16)/(2\log 2) = 1.41504\ldots$.

By varying the parameters α and β, and always checking that expression (4.2) is positive and satisfies the Cohen condition, one can search for wavelets having maximal L^p–Sobolev regularity, and, in particular, having greater regularity than the Daubechies wavelet. The two required conditions make this a difficult constrained optimization problem, which is even difficult numerically. In contrast with Example 1, there appears to be no simple algebraic expression for the eigenvalues for the matrix corresponding to $L_{\log r}$; one must resort to empirical searches using different choices of α and β.

However, it is more illuminating to implement this computation using determinants.

(B) The determinant approach

If one computes the determinant $\det(1 - zL_{\log|m|^2_{\mathrm{Daub}}})$ associated with the Daubechies wavelet ψ_3^{Daub} using our determinant algorithm, one obtains the power series expansion

$$\det(1 - zL_{\log|m|^2_{\mathrm{Daub}}}) = 1 - 2\,z + 1.31904\,z^2 - 0.35310\,z^3 + 0.03319\,z^4 + 0.00120\,z^5 - 0.000351\,z^6 + \cdots .$$

We observe that:

(1) There are "trivial" zeros for $\det(1 - zL_{\log|m|^2_{\mathrm{Daub}}})$ at the values $z = 1, 2, 4, 8$;

(2) There is a zero at $z = 7.07337\ldots.$, which by Proposition 2.3 corresponds to a zero of $\det(1 - zL_{\log r_{\mathrm{Daub}}})$ at $7.07337/2^6 = 0.111\ldots.$ This agrees with our previous calculation in Section 3.

By Theorem 1.2, if we change the filter $|m|^2$ to decrease the pressure, then the Sobolev regularity of the associated wavelet increases.

One can easily check that if one replaces $a = 75/128$ by $75/128 - 0.00068$, say, and $b = -25/256$ by $-25/256 + 0.00002$, say, then the associated filter $|m(\xi)^2|$ remains positive, and by Theorems 1.2 and 2.1, the zero of the determinant increases from 7.1111 to 8.51989. This reflects an increase in the L^2-Sobolev regularity from 1.415 to 1.545.

Thus, this two-parameter family contains wavelets having the same support as ψ_3^{Daub} with greater L^2-Sobolev regularity. The same idea can be applied for other N.

Remarks: Other approaches.

(a) Daubechies [Da1, p.242] considered the one-parameter family of filters

$$|m(\xi)|^2 = \cos^{2(N-1)}(\xi/2)(P_{N-1}(\sin^2(\xi/2)) + a\cos^{2N}(\xi/2))\cos(\xi),$$

where $a \in \mathbb{R}$. For definiteness, we discuss the case $N = 3$. For different values of the parameter a we calculate the first zero of the determinant, and thus the Sobolev regularity of the associated wavelet using Theorem 2.2. The following table shows that this zero is monotone increasing as a increases to 3. The zero of the determinant is $0.40344\ldots$ when $a = 3$, which corresponds to L^2 Sobolev regularity $1.34443\ldots.$ It is not necessary to study the case $a > 3$ since the expansion for $|m|^2$ is no longer positive. To see this, we can expand in terms of $\xi - \pi$ as

$$\begin{aligned}|m(\xi)|^2 &= \cos^4(\xi/2)(P_2(\sin^2(\xi/2)) + a\cos^4(\xi/2))\cos(\xi)\\
&= \cos^4(\xi/2 - \pi/2)(1 + 2\sin^2(\xi/2 - \pi/2) + a\sin^6(\xi/2 - \pi/2)\cos(\xi - \pi))\\
&= \sin^4(\xi/2)(1 + 2\cos^2(\xi/2) - a\cos^6(\xi/2)\cos(\xi))\\
&= 1 + 2(1 - (\xi/2)^2 + \ldots)^2 - a(1 - (\xi/2)^2 + \ldots)^6(1 - \xi^2 + \ldots)\\
&= (3 - a) - (1 + 5a/2)\xi^2 + \ldots .\end{aligned}$$

It is clear this expression is negative values for $a > 3$.

We see that the parameter value $a = 3$ corresponding to the maximally regular wavelet lies on the boundary of an allowable parameter interval, and thus could not be detected using variational methods.

a	Zero of determinant	Sobolev regularity
-1.0	0.230	0.939
-0.5	0.239	0.967
0.0	0.250	1.000
0.5	0.262	1.030
1.0	0.276	1.071
1.5	0.294	1.116
2.0	0.317	1.710
2.5	0.350	1.242
2.7	0.367	1.276
2.8	0.377	1.296
2.9	0.389	1.318
3.0	0.403	1.344

(b) Daubechies [Da3] also considered the one-parameter family of functions

$$m(\xi) = \left(\frac{1+e^{i\xi}}{2}\right)^2 Q(\xi),$$

where $|Q(\xi)|^2 = P(\cos\xi)$ and $P(x) = 2 - x + \frac{a}{4}(1-x)^2$, with the parameter a ichosen such that $P(x) > 0$ for $0 \le x < 1$. The value $a = 3$ corresponds to ψ_2^{Daub}, and the associated polynomial has a zero at $x = -1$. She reparameterizes this wavelet family using the zero at $x = -1 + \delta$, and obtains, numerically, a wavelet having the same support as ψ_2^{Daub} but with higher regularity.

5. Perturbation theory and variational formulae

In this section we observe that the analyticity of pressure implies that the L^p Sobolev regularity depends smoothly on the wavelet. It thus seems natural to search for critical points (maximally smooth wavelets) in parametrized families of wavelets by explicitly calculating the derivative of the Sobolev regularity functional. We caution that the example in the previous section shows that maximally regular wavelets can lie on the boundary of allowable parameter intervals.

We now restrict $|m(\xi)|^2$ to the subset of analytic functions $\mathcal{B}$ satisfying conditions (1) – (3) and let Δr_0 be a tangent vector to this space. It follows from remarks in Section 1 that Δr_0 is represented by an analytic function of the form

$$\Delta r_0 = \frac{1}{2} + \sum_{n=0}^{\infty} d_{2n+1}\cos((2n+1)\xi), \qquad \text{such that} \qquad \sum_{n=0}^{\infty} d_n = \frac{1}{2}.$$

We have the following result.

Proposition 5.1.

(1) *For every $p \geq 1$, the regularity mapping $\mathcal{K} \to \mathbb{R}$ defined by $|m(\xi)|^2 \mapsto s_p(\phi)$ is analytic;*

(2) *Let*

$$|m^\lambda(\xi)|^2 = \cos^{2M}(\xi/2)\,(r_0(\xi) + \lambda\Delta r_0(\xi) + \dots)$$

be a smooth parametrized family of filters. Then the change in the Sobolev regularity is given by

$$s_p(\phi^\lambda) = s_p(\phi^0) + \left(\frac{1}{p\log 2}\int \frac{\Delta r_0}{r_0}d\mu\right)\lambda + \dots\,,$$

where μ is the unique equilibrium state for $p\log r_0$.

Proof. The analyticity of the Sobolev regularity follows from Proposition 2.1(2), the analyticity of the pressure, and the implicit function theorem. Since $\log r^\lambda = \log(r^0 + \Delta r_0 + \dots) = \log r^0 + \frac{\Delta r_0}{r_0}\lambda + \dots$ and $D_g P(\cdot) = \int \cdot\, d\mu_g$, this follows from the chain rule. □

If we restrict to finite dimensional spaces we can consider formulae for the second (and higher) derivatives of the pressure, which translate into formulae for the next term in the expansion of $s_p(\phi_\lambda)$.

Also observe that we can estimate the derivatives of the Sobolev exponent of a family $s_p(\phi_\lambda)$ using the series for $d_p(\phi_\lambda, z)$. More precisely, by the implicit function theorem

$$\frac{ds_p(\phi^\lambda)}{d\lambda} = \frac{d(\det(\phi^\lambda, z))}{d\lambda}\Bigg/\frac{d(\det(\phi^\lambda, z))}{dz} = \frac{d(\det(\phi^\lambda, z))}{d\lambda}\Bigg/\frac{d(\det(\phi^\lambda, z))}{dz} + O(2^{-n^2})$$

which is possible to compute for periodic orbit data. We can similarly compute the higher derivatives

$$\left.\frac{d^k s_p(\phi^\lambda)}{d\lambda^k}\right|_{\lambda=\lambda_0}, \text{ for } k \geq 1$$

using Mathematica, say. We can then expand the power series for the Sobolev function:

$$s_p(\phi^\lambda) = s_p(\phi^{\lambda_0}) + (\lambda - \lambda_0)\left.\frac{ds_p(\phi^\lambda)}{d\lambda}\right|_{\lambda=\lambda_0} + \frac{(\lambda-\lambda_0)^2}{2}\left.\frac{d^2 s_p(\phi^\lambda)}{d\lambda^2}\right|_{\lambda=\lambda_0} + \dots\,.$$

6. Asymptotic Sobolev regularity

The use of the variational principle to characterize pressure (and thus the Sobolev regularity) leads one naturally to the idea of maximizing measures. More precisely, one can associate to g:

$$Q(g) = \sup\left\{\int_0^{2\pi} g\,d\mu, \quad \mu \text{ an } E_2\text{-invariant Borel probability measure}\right\}$$

Unlike pressure, which has analytic dependence on the function, the quantity $Q(g)$ is less regular, but we have the following trivial bound.

Lemma 6.1. *For any continuous function* $g\colon [0,2\pi) \to \mathbb{R}$,

$$Q(g) \leq P(g) \leq Q(g) + \log 2.$$

Proof. Since $0 \leq h(\mu) \leq \log 2$, this follows from the definitions. □

Proposition 6.2. *Let* $\phi_n, n \in \mathbb{N}$ *be a family of wavelets with filters*

$$|m(\xi)|^2 = \cos^{2M_n}(\xi/2) r_n(\xi),$$

where $M_n \geq 0$. *Then the asymptotic* L^p*-Sobolev regularity of the family is given by*

$$\lim_{n\to\infty} \frac{s_p(\phi_n)}{n} = \lim_{n\to\infty} \frac{M_n}{n} - \lim_{n\to\infty} \frac{Q(\log r_n)}{n \log 2}.$$

Proof. Using Theorem 1.2, Lemma 6.1, and (1.1), one can easily show

$$Q(p \log r_n) \leq p \log 2 (M_n - s_p(\phi_n)) \leq Q(p \log r_n) + \log 2.$$

The proposition immediately follows after dividing by n and taking the limit as $n \to \infty$. □

We now apply Proposition 6.2 to compute the L^p-Sobolev regularity of the Daubechies family of wavelets for arbitrary $p \geq 2$. Cohen and Conze [CC1] first proved this result for $p = 2$.

Proposition 6.3. *Let* $p \geq 2$. *The asymptotic* L^p*-Sobolev regularity of the Daubechies family is given by*

$$\lim_{N\to\infty} \frac{s_p(\psi_N^{\mathrm{Daub}})}{N} = 1 - \frac{\log 3}{2 \log 2}.$$

Proof. If one applies Proposition 6.2 to the Daubechies filter coefficients (see Section I), one immediately obtains

$$\lim_{N\to\infty} \frac{s_p(\phi_N^{\mathrm{Daub}})}{N} = 1 - \left(\lim_{N\to\infty} \frac{Q(\log P_N(\sin^2(\frac{y}{2})))}{N \log 2} \right).$$

Cohen and Conze [CC1, p. 362] show that

$$Q(\log_2 P_N(\sin^2(\frac{y}{2}))) = \frac{\log P_N(3/4)}{2 \log 2},$$

and the sup is attained for the period two periodic orbit $\{1/3, 2/3\}$. Thus

$$\lim_{N\to\infty} \frac{s_p(\phi_N^{\mathrm{Daub}})}{N} = 1 - \frac{1}{2 \log 2} \lim_{N\to\infty} \frac{\log P_N(3/4)}{N}.$$

The exponentially dominant term in $P_N(3/4)$ is easily seen to be $\binom{2N-1}{N-1}(\frac{3}{4})^{N-1}$, and a routine application of Stirling's formula yields

$$\lim_{N\to\infty} \frac{\log P_N(3/4)}{N} = \log 3.$$

The proposition immediately follows. □

Remark. As briefly discussed in Section 3.2, the Butterworth filter family ψ_N^{Butt}, $N \in \mathbb{N}$ is the family of non-compactly supported wavelets defined by the filter

$$|m(\xi)|^2 = \frac{\cos^{2N}(\xi/2)}{\sin^{2N}(\xi/2) + \cos^{2N}(\xi/2)}.$$

Fan and Sun [FS1] computed the asymptotic regularity for this wavelet family; they showed that

$$\lim_{N\to+\infty} \frac{s_p(\psi_N^{\mathrm{Butt}})}{N} = \frac{\log 3}{\log 2}.$$

This asymptotic linear growth of regularity is similar to that for the Daubechies wavelet family.

7. Higher dimensions

The dilation equation (0.1) has a natural generalization to d-dimensions of the form:

$$\phi(x) = \det(D) \sum_{\underline{k}\in\mathbb{Z}^d} c_{\underline{k}} \phi(D\underline{x} - \underline{k}), \tag{7.1}$$

where $D \in GL(d,\mathbb{Z})$ is a $d \times d$ matrix with all eigenvalues having modulus strictly greater than 1. We are interested in solutions $\phi \in L^2(\mathbb{R}^d)$. We consider the expanding map $T : [0,2\pi)^d \to [0,2\pi)^d$ by $T(\underline{x}) = D\underline{x}$ (mod 1). By taking d-dimensional Fourier transformations, (7.1) gives rise to the equation

$$\widehat{\phi}(\underline{\xi}) = m(D^{-1}\underline{\xi})\widehat{\phi}(D^{-1}\underline{\xi}), \quad \text{where} \quad m(\underline{\xi}) = \sum_{\underline{k}\in\mathbb{Z}^d} c_{\underline{k}} e^{-i\langle \underline{k},\underline{\xi}\rangle}.$$

Thus by iterating this identity we can write $\widehat{\phi}(\underline{\xi}) = \prod_{n=1}^{\infty} m(D^{-n}\underline{\xi})$.

For $g \in C^\beta([0,2\pi)^d)$, one can define a transfer operator $L_g : C^\beta([0,2\pi)^d) \to C^\beta([0,2\pi)^d)$ by

$$L_g k(\underline{x}) = \sum_{T\underline{y}=\underline{x}} \exp\big(g(\underline{y})\big)\, k(\underline{y}),$$

where the sum is over the $\det(D)$ preimages of $\underline{x}$.

We can consider a Banach space of periodic analytic functions $\mathcal{A}$ in a uniform neighbourhood of the torus $[0,\pi)^d$ and the associated Bergman space of functions. We have that for $g \in \mathcal{A}$ the operator $L_g : \mathcal{A} \to \mathcal{A}$ and its iterates L_g^k are trace class operators for each $k \geq 1$. We can compute

$$\mathrm{tr}(L_g^k) = \sum_{T^n \underline{x} = \underline{x}} \frac{|\det(D)|^n \exp(S_n g(\underline{x}))}{|\det(D^n - I)|},$$

where $S_n g(\underline{x}) = g(\underline{x}) + g(T\underline{x}) + \cdots + g(T^{n-1}\underline{x})$.

Assume that we can write $m(\underline{x}) = \left(\frac{a(D\underline{x})}{|\det(D)| a(\underline{x})}\right)^M r(\underline{x})$, where $r(\underline{x}) > 0$.[8] The Ruelle-Fredholm determinant of $L_{p \log r}$ takes the form:

$$\det(I - L_{p\log r}) = \exp\left(- \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{T^n \underline{x} = \underline{x}} \frac{|\det(D)|^n \exp(p S_n \log r(\underline{x}))}{|\det(D^n - I)|} \right).$$

The statement of Theorem 1.2 has a partial generalization to d-dimensions. We define the L^p-Sobolev regularity of p to be

$$s_p(\phi) := \sup\{ s > 0 : |\widehat{\phi}(\underline{\xi})|^p (1 + ||\underline{\xi}||^p)^s \in \mathrm{Ł}^1(\mathbb{R}^d)\}.$$

Proposition 7.1. *The L^p-Sobolev regularity satisfies the inequality*

$$s_p(\phi) \geq \frac{M \det(D)}{\log \lambda} - \frac{P(p \log r)}{p \log \lambda},$$

where $\lambda > 1$ is smallest modulus of an eigenvalue of D.

Proof. We can write $|\widehat{\phi}(\underline{\xi})| \leq C |\det(D)|^{Mn} \prod_{i=0}^n r(D^{-i}\underline{\xi})$, for some $C > 0$ and all $\underline{\xi} \in D^n([0,2\pi)^d)$. However, we can identify $\int_{T^n D} \prod_{i=0}^n r(D^{-i}\underline{\xi})^p d\underline{\xi} = \int_D L_{p\log r}^n 1(\underline{\xi}) d\xi$. Moreover, since $L_{p \log r}$ has maximal eigenvalue $e^{P(p\log r)}$, we can bound this by $C' e^{nP(p\log r)}$, for some $C' > 0$. In particular,

$$\begin{aligned}\int_{\mathbb{R}^d} |\widehat{\phi}(\underline{\xi})|^p (1 + ||\underline{\xi}||^p)^s d\xi &\leq \sum_{n=1}^{\infty} \lambda^{psn} \int_{T^n D - T^{n-1} D} |\widehat{\phi}(\underline{\xi})|^p d\xi \\ &\leq CC' \sum_{n=1}^{\infty} \lambda^{psn} |\det(D)|^{Mpn} e^{nP(p\log r)}\end{aligned}$$

In particular, the right hand side converges if $\lambda^{ps} |\det(D)|^{Mp} e^{P(p \log r)} < 1$, from which the result follows. □

[8]In particular, if $T^n \underline{x} = \underline{x}$ we have that $\prod_{i=0}^{n-1} |m(T^i \underline{x})|^2 = 2^{-n} \prod_{i=0}^{n-1} g(T^i \underline{x})$.

In certain cases, for example if D has eigenvalues of the same modulus and some other technical assumptions, then this inequality becomes an equality [CD1].

As in Section 2, a routine calculation gives that

$$d_p(z) := \det(I - zL_{p\log r}) = 1 + \sum_{n=1}^{\infty} b_n z^n, \text{ where}$$

$$b_n = \sum_{n_1+\cdots+n_k} (-1)^k \frac{a_{n_1}^p \cdots a_{n_1}^p}{n_1! \cdots n_k!} \quad \text{and} \quad a_n^p = \sum_{T^n\underline{x}=\underline{x}} \frac{|\det(D)|^n \sum_{T^n\underline{x}=\underline{x}} \left(\prod_{i=0}^n r(\underline{x})\right)^p}{|\det(D^n - I)|}.$$

The simplest case is that of diagonal matrices, but this reduces easily to tensor products $\Phi(x_1, \ldots, x_d) = \phi_1(x_1)\cdots\phi_d(x_d)$ of the one dimensional case. The next easiest example illustrates the "non-separable" case:

Example. For $d = 2$ we let $D = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, which has eigenvalues $\lambda = \pm\sqrt{2}$.

In the d-dimensional setting we have the following version of Theorem 2.2:

Theorem 7.2. *Assume that* $\log r \in \mathcal{A}$. *Then*

(1) *The function* $d_p(z)$ *can be extended to an entire function of* z *and a real analytic function of* p*;*
(2) *The exponential of pressure* $z_p := \exp(-P(p\log r))$ *is the smallest zero for* $d_p(z)$*;*
(3) *The Taylor coefficients* b_n *of* $d_p(z)$ *decay to zero at a super-exponential rate, i.e., there exists* $0 < A < 1$ *such that* $b_n = O(A^{n^{1+1/d}})$*;*
(4) *Let* $d_{p,N}(z) = \sum_{n=1}^N b_n z^n$ *be the truncation of the Taylor series to* N *terms and consider smallest zeros* $d_{p,N}(z_p^N) = 0$. *Then* $z_p^N \to z_p$ *at a super-exponential rate, i.e., there exists* $0 < B < 1$ *such that* $|z_p - z_p^N| = O(B^{N^{1+1/d}})$.

Furthermore, we can choose A *and* B *arbitrarily close to* λ^{-1}.

The main difference for $d \geq 2$ is that the exponent N^2 is replaced by $N^{1+1/d}$ in parts 3 and 4.

8: Generalizations to non real analytic filters m

We now consider the more general setting where the filter m is not analytic, and we necessarily obtain weaker results. Recall that the filter $m(\xi) = (1/\sqrt{2})\sum_k c_k e^{-ik\xi}$, and assume there exists $\beta > 0$ such that $|c_n| = O(|n|^{-\beta})$. In this case we can only expect that the determinant algorithm for the Sobolev regularity converges with exponential speed. Recall $d_p(z) = \det(I - zL_{plogr}) = 1 + \sum_{n=1}^{\infty} b_n z^n$.

Theorem 8.1.

(1) *Let* $0 < \beta < 1$. *If* $|c_n| = O\left(\frac{1}{|n|^\beta}\right)$, *then* $b_n = O(2^{-n\beta(1-\epsilon)})$ *for any* $\epsilon > 0$*;*
(2) *Let* $k \geq 1$. *If* $|c_n| = O\left(\frac{1}{|n|^k}\right)$, *then* $b_n = O(2^{-nk(1-\epsilon)})$ *for any* $\epsilon > 0$.

This follows from a standard analysis of transfer operators, and in particular, studying their essential spectral radii. The radius of convergence of $d_p(z)$ is the reciprocal of the essential spectral radius of the operators. In the case that $0 < \beta < 1$, this theorem follows from the Hölder theory of transfer operators [PP1]. But for $k > 1$ this requires a slightly different analysis [Ta1]. The necessary results are summerized in the following proposition.

Proposition 8.2.

(1) *Assume $c_n = O(|n|^{-\beta})$. The transfer operator L_f has essential spectral radius $(1/2)^\alpha e^{P(f)}$ (and the function $d(z)$ in analytic in a disk $|z| < 2^\beta$).*

(2) *Assume $c_n = O(|n|^{-k})$. The transfer operator L_f has essential spectral radius $(1/2)^k e^{P(f)}$ (and the function $d(z)$ in analytic in a disk $|z| < 2^k$).*

The first part is proved in [Po1], [PP2]. The second part is proved in [Ta1], [Ru2][9]

As in Section 2, the speed of decay of the coefficients b_n immediately translate into the corresponding speed of convergence for the main algorithm.

Appendix I: Proof of Theorem 2.1.

Given a bounded linear operator $L : H \to H$ on a Hilbert space H, its i^{th} *approximation number* (or *singular value*)[10] $s_i(L)$ is defined as

$$s_i(L) = \inf\{||L - K|| : \mathrm{rank}(K) \le i - 1\},$$

where K is a bounded linear operator on H.

Let $\Delta_r \subset \mathbb{C}$ denote the open disk of radius r centered at the origin in the complex plane. The phase space $[0, 2\pi)$ for $T = E_2$ is contained in $\Delta_{2\pi+\epsilon}$ for any $\epsilon > 0$, and the two inverse branches $T_1(x) = \frac{1}{2}x$ and $T_2(x) = \frac{1}{2}x + \frac{1}{2}$ have analytic extensions to $\Delta_{2\pi+\epsilon}$ satisfying $T_1(\Delta_{2\pi+\epsilon}) \cup T_2(\Delta_{2\pi+\epsilon}) \subset \Delta_{2\pi+\frac{\epsilon}{2}}$. Thus T_1 and T_2 are strict contractions of $\Delta_{2\pi+\epsilon}$ onto $\Delta_{2\pi+\frac{\epsilon}{2}}$ with contraction ratio $\theta = (2\pi + \epsilon)/(2\pi + 2\epsilon) < 1$. We can choose $\epsilon > 0$ arbitrarily large (and thus θ arbitrarily close to 1/2).

Let $A_2(\Delta_r)$ denote the Bergman Hilbert space of analytic functions on Δ_r with inner product $\langle f, g\rangle := \int_\Delta f(z)\,\overline{g(z)}\,dx\,dy$ [Ha1].

Lemma A.1. *The approximation numbers of the transfer operator $L_{p\log r} : A_2(\Delta_{2\pi}) \to A_2(\Delta_{2\pi})$ satisfy*

$$s_j(L_{p\log r}) \le \frac{||L_{p\log r}||_{A_2(\Delta_{2\pi})}}{1-\theta}\theta^j,$$

for all $j \ge 1$. This implies that the operator $L_{p\log r}$ is of trace class.

Proof. Let $g \in A_2(\Delta_{2\pi+\epsilon})$ and write $L_{p\log r}g = \sum_{k=-\infty}^{\infty} l_k(g)p_k$, where $p_k(z) = z^k$. We can easily check that $||p_k||_{A_2(\Delta_{2\pi})} = \sqrt{\frac{\pi}{k+1}}(2\pi)^{k+1}$ and $||p_k||_{A_2(\Delta_{2\pi+\epsilon})} = \sqrt{\frac{\pi}{k+1}}(2\pi +$

[9] At least when $\beta \in \mathbb{N}$.

[10] We use these terms interchangably because one can show that $s_k(L) = \sqrt{\lambda_k(L^*L)}$.

$\epsilon)^{k+1}$. The functions $\{p_k\}_{k\in\mathbb{Z}}$ form a complete orthogonal family for $A_2(\Delta_{2\pi+\epsilon})$, and so $\langle L_{p\log r}g, p_k\rangle_{A_2(\Delta_{2\pi+\epsilon})} = l_k(g)||p_k||^2_{A_2(\Delta_{2\pi+\epsilon})}$. The Cauchy-Schwarz inequality implies that

$$|l_k(g)| \leq ||L_{p\log r}g||_{A_2(\Delta_{2\pi+\epsilon})}\,||p_k||^{-1}_{A_2(\Delta_{2\pi+\epsilon})}.$$

We define the rank-j projection operator by $L^{(j)}_{p\log r}(g) = \sum_{k=0}^{j-1} l_k(g)p_k$. For any $g \in A_2(\Delta_\theta)$ we can estimate

$$||\left(L_{p\log r} - L^{(j)}_{p\log r}\right)(g)||_{A_2(\Delta_{2\pi})} \leq ||L_{p\log r}g||_{A_2(\Delta_{2\pi})}\sum_{k=j}^{\infty}\theta^{k+1}.$$

It follows that

$$||L_{p\log r} - L^{(j)}_{p\log r}||_{A_2(\Delta_{2\pi})} \leq \frac{||L_{p\log r}||_{A_2(\Delta_{2\pi})}}{1-\theta}\theta^{j+1},$$

and so

$$s_{j+1}(L_{p\log r}) \leq \frac{||L_{p\log r}||_{A_2(\Delta_{2\pi})}}{1-\theta}\theta^{j+1},$$

and the result follows. □

We now show that the coefficients of the power series of the determinant decay to zero with super-exponential speed.

Lemma A.2. *If one writes* $\det(I - zL_{p\log r}) = 1 + \sum_{m=1}^{\infty} c_m z^m$, *then*

$$|c_m| \leq B\left(\frac{|L_{p\log r}||_{A_2(\Delta_{2\pi})}}{1-\theta}\right)^m \theta^{m(m+1)/2},$$

where $B = \prod_{m=1}^{\infty}(1-\theta^m)^{-1} < \infty$.

Proof. By [Si1, Lemma 3.3], the coefficients c_n in the power series expansion of the determinant have the form $c_m = \sum_{i_1<\ldots<i_m} \lambda_{i_1}\cdots\lambda_{i_m}$, the summation is over all m-tuples $(i_1, \ldots, i_m)$ of positive integers satisfying $i_1 < \ldots < i_m$. Since $|\lambda_i| \leq s_i$ we can bound

$$\begin{aligned}
|c_m| &= \left|\sum_{i_1<\ldots<i_m}\lambda_{i_1}\cdots\lambda_{i_m}\right| \leq \sum_{i_1<\ldots<i_m} s_{i_1}(L_{p\log r})\cdots s_{i_m}(L_{p\log r}) \\
&\leq \left(\frac{||L_{p\log r}||_{A_2(\Delta_{2\pi})}}{1-\theta}\right)^m \sum_{i_1<\ldots<i_m}\theta^{i_1+\ldots+i_m} \\
&= \left(\frac{||\ L_{p\log r}||_{A_2(\Delta_{2\pi})}}{1-\theta}\right)^m \frac{\theta^{m(m+1)/2}}{(1-\theta)(1-\theta^2)\cdots(1-\theta^m)} \\
&\leq B\left(\frac{||lL_{p\log r}||_{A_2(\Delta_{2\pi})}}{1-\theta}\right)^m \theta^{m(m+1)/2},
\end{aligned}$$

for some $B > 0$. $\square$

We finish by showing that Theorem 2.1 follows from Lemma 2.2. The coefficients of $\det(I - zL_s) = 1 + \sum_{n=1}^{\infty} b_n z^n$ are given by Cauchy's Theorem:

$$|b_n| \leq \frac{1}{r^n} \sup_{|z|=r} |\det(I - zL_s)|, \text{ for any } r > 0.$$

Using Weyl's inequality we can deduce that if $|z| = r$ then

$$\begin{aligned} |\det(I - zL_s)| &\leq \prod_{j=1}^{\infty} (1 + |z|\lambda_j) \leq \prod_{j=1}^{\infty} (1 + |z| s_j) \\ &\leq \left(1 + B \sum_{m=1}^{\infty} (r\alpha)^m \theta^{\frac{m(m+1)}{2}}\right) \end{aligned}$$

where $\alpha = ||L_s||_{\mathcal{A}_2(\Delta_{2\pi})}$. If we choose $r = r(n)$ appropriately then we can get the bounds given in the theorem.

Appendix II: Proof of Proposition 2.3

The periodic orbits of period n for E_2 are precisely the sets $\{2\pi\, 2^i k/(2^n - 1) : i = 0, 1, \ldots, n-1\}$, where $k = 0, \ldots, 2^n - 1$. We can now consider the factorization (1.1) and write

$$\begin{aligned} Z_n :=& \frac{1}{1 - 2^{-n}} \sum_{T^n x = x} \prod_{i=0}^{n-1} |m(2^i x)|^2 \\ =& \frac{1}{1 - 2^{-n}} \sum_{k=0}^{2^n - 1} \prod_{i=0}^{n-1} \left| m\left(2\pi \frac{2^i k}{2^n - 1}\right)\right|^2 \\ =& \frac{1}{1 - 2^{-n}} \sum_{k=0}^{2^n - 1} \left(\prod_{i=0}^{n-1} \cos^M \left(2\pi \frac{2^i k}{2(2^n - 1)}\right)\right) \left(\prod_{i=0}^{n-1} r\left(2\pi \frac{2^i k}{2(2^n - 1)}\right)\right). \end{aligned}$$

However, since $\cos\theta = \frac{1}{2}\sin(2\theta)/\sin\theta$, we can write

$$\prod_{i=0}^{n-1} \cos^M \left(2\pi \frac{2^i k}{2(2^n - 1)}\right) = \prod_{i=0}^{n-1} \cos^M \left(2\pi \frac{2^i k}{2(2^n - 1)}\right) = \begin{cases} 2^{-Mn} & \text{if } k \neq 0 \\ 1 & \text{if } k = 0 \end{cases}.$$

Thus we can write

$$Z_n = \frac{1}{1 - 2^{-n}} \left(\sum_{k=0}^{2^n - 1} \frac{1}{2^{nM}} \prod_{i=0}^{n-1} r\left(2\pi \frac{2^i k}{2(2^n - 1)}\right) + 1 - \frac{1}{2^{nM}}\right).$$

Since $(1-2^{-nM})/(1-2^{-n}) = \sum_{j=0}^{M-1} 2^{-jn}$, we can write

$$\det(1 - zL_{p\log|m|^2}) = \exp\left(\sum_{n=0}^{\infty}\sum_{j=0}^{M-1}(z2^{-j})^n\right)\det\left(1 - \frac{z}{2^M}L_{p\log r}\right)$$
$$= \prod_{j=0}^{M-1}\left(1 - z2^{-j}\right)\det\left(1 - \frac{z}{2^M}L_{p\log r}\right).$$

□

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