On the controllability of conservative systems

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Abstract

We consider a weak form of controllability for system that have a conserved quantity and satisfy a condition of Hörmander type. It is shown that such systems are approximately controllable under a weak growth condition for the conserved quantity. The proof of the result combines analytic tools with probabilistic arguments. A counterexample is given that shows that the growth condition is essential for the result to hold. Applications of the result to ergodicity questions for systems arising from non-equilibrium statistical mechanics and to the controllability of Galerkin approximations to the Euler equations are also given.

1 Introduction

We are interested in the controllability of systems of the form

$$\dot{z} = f(z) + u(t) \;, \quad z(t) \in \mathbf{R}^N \;, \tag{1.1}$$

where $f\colon \mathbf{R}^N \to \mathbf{R}^N$ is a smooth vector field and the control u is only allowed to take values in a given linear subspace E of $\mathbf{R}^N$. Our aim is to study the possibility, given a starting point $z_0 \in \mathbf{R}^N$ and a final point $z_1 \in \mathbf{R}^N$, of finding a control u taking values in a (possibly small) subspace of $\mathbf{R}^N$ that allows to 'stir' the solution of (1.1) from z_0 into a small neighbourhood of z_1.

We show that if this control system satisfies a Hörmander-type condition and f has some additional geometric structure (it should have a conserved quantity that satisfies a certain growth condition at infinity), then (1.1) is approximately controllable in the following sense. For every initial condition z_0 and every open set $A \subset \mathbf{R}^N$ there exists a time T and a control $u \in \mathcal{C}^\infty([0,T], E)$ such that the solution $z(T)$ of (1.1) at time T lies in A. In the case when f is a polynomial vector field of odd degree, it is well-known (see for example [JK85] and references therein) that Hörmander's condition is both necessary and sufficient for the system (1.1) to be controllable. As the example in Section 4.2 below shows, this is not the case in general.

We actually provide an example which satisfies all of the conditions of the main theorem of the present article, except for the growth condition of the conserved quantity H. This example turns out not to be controllable in the above sense, thus showing that the growth condition on H encodes global geometric information on f that complements the local geometric information given by the Hörmander condition and is essential to the result. We also discuss applications of our result to the controllability of the Euler equations and to an ergodicity problem coming from non-equilibrium statistical mechanics.

2 Setting

We consider the controllability of systems that have a smooth conserved quantity called H, *i.e.* we assume throughout this paper that

$$\langle \nabla H(z), f(z)\rangle = 0\;, \qquad \forall z \in \mathbf{R}^N\;. \tag{2.1}$$

We assume that the state space $\mathbf{R}^N$ splits in a natural way as $E \oplus E^\perp$ and we denote its elements by (x, y). We use this notation to emphasise the fact that even though our prime interest is Hamiltonian systems, the splitting under consideration is not necessarily the standard splitting in position and momentum variables. Typically, x would consist only of part of the momentum variables and y would consist of all the other variables of the system. We therefore allow for the case where $\dim E \neq \dim E^\perp$ which is actually the most interesting case in our situation (see the examples below).

Our first assumption is that the flow generated by f preserves the Lebesgue measure and that it has a conserved quantity H:

Assumption 2.1 *The vector field f is divergence-free and there exists a smooth function $H\colon \mathbf{R}^N \to \mathbf{R}$ such that (2.1) holds.*

Our second assumption ensures that H grows sufficiently fast at infinity:

Assumption 2.2 *The level sets of H are compact, the function $e^{-H(x,y)}$ is integrable, and there exist constants $C_1, C_2 \in \mathbf{R}$ such that*

$$\Delta_x H(x, y) - \|\nabla_x H(x, y)\|^2 \le C_1 + C_2 H(x, y)\;, \tag{2.2}$$

for every $(x, y) \in \mathbf{R}^N$.

Before we state our last assumption, we introduce some additional terminology. Given a vector field f on $\mathbf{R}^N$, we identify it with the corresponding differential operator $\sum_i f_i(x)\partial_{x_i}$. With this identification, the Lie bracket between two vector fields is simply the vector field corresponding to the commutator of the two differential operators. The Lie algebra generated by a family of smooth vector fields is the smallest subspace of the space of all vector fields that is closed under the Lie bracket operation.

We also introduce an extended phase space $\mathbf{R}^{N+1}$, which includes time as an extra dimension, and extend f to a vector field $\tilde f$ on $\mathbf{R}^{N+1}$ by setting $\tilde f(x, t) =$

$(f(x), 1)$. Our last assumption then essentially says that the differential operator $\partial_t + \mathcal{L}^*$ (with $\mathcal{L}^*$ defined as in (3.3) below) is hypoelliptic, see [Hör85].

Assumption 2.3 *Given a basis $\{e_1, \ldots, e_n\}$ of E, the Lie algebra generated by $\{\tilde{f}, e_i, \ldots, e_n\}$ spans $\mathbf{R}^{N+1}$ at every point.*

Note that the statement of Assumption 2.3 is actually independent of the particular choice of the basis of E. With these notations, the main result of this article is:

Theorem 2.4 *If assumptions 2.1, 2.2 and 2.3 hold, then for every initial condition $z_0 \in \mathbf{R}^N$ and every open set $A \subset \mathbf{R}^N$ there exists a time T and a control $u \in \mathcal{C}^\infty([0,T], E)$ such that the solution $z(T)$ of (1.1) at time T lies in A.*

Remark 2.5 We believe that condition (2.2) in Assumption 2.2 is not essential to our result, but it appears as a technical condition in the proof.

3 Proof of the main result

The main idea in the proof of Theorem 2.4 is to consider the following stochastic differential equation on $\mathbf{R}^n$:

$$d\xi(t) = f_x(\xi, \eta)\, dt - \nabla_x H(\xi, \eta)\, dt + \sqrt{2} dw(t)\;, \tag{3.1a}$$

$$d\eta(t) = f_y(\xi, \eta)\, dt\;. \tag{3.1b}$$

Denoting by Z the value of $\int e^{-H(x,y)}\, dx\, dy$, we start our study of (3.1) with the following elementary result:

Lemma 3.1 *Under the above assumptions, (3.1) has a unique global strong solution for all times. Furthermore, $\mu_H(dx, dy) = Z^{-1} e^{-H(x,y)}\, dx\, dy$ is an invariant probability measure for (3.1).*

Proof. The existence of a unique local strong solution is a standard result for SDEs with smooth coefficients [Øks03]. In order to show that the solution can be continued for all times, note that Itô's formula yields

$$\begin{aligned} H(\xi(t), \eta(t)) = H(\xi_0, \eta_0) &+ \int_0^t (\Delta_x H(\xi(s), \eta(s)) - \|\nabla_x H(\xi(s), \eta(s))\|^2)\, ds \\ &+ \sqrt{2} \int_0^t \langle \nabla_x H(\xi(s), \eta(s)), dw(s)\rangle\;, \end{aligned}$$

so that, at least on a formal level,

$$\mathbf{E} H(\xi(t), \eta(t)) \le H(\xi_0, \eta_0)\, e^{C_2 t} + \frac{C_1}{C_2}(e^{C_2 t} - 1)\;, \tag{3.2}$$

for every $t \geq 0$. The validity of (3.2) (*i.e.* the finiteness of the expectations involved) follows from standard approximation arguments.

That μ_H is an invariant probability measure for (3.1) follows immediately from the fact that

$$\mathcal{L}^* e^{-H(x,y)} = 0 \;,$$

where $\mathcal{L}^*$ is the adjoint of the generator of the semigroup generated by (3.1),

$$\mathcal{L}^* \varrho = -\nabla(f \cdot \varrho) + \nabla_x(\nabla_x H(x,y)\varrho(x,y)) + \Delta_x \varrho \;. \tag{3.3}$$

See *e.g.* [Has80, Øks03, RY99] for more details. □

Note furthermore that the hypoellipticity assumption 2.3 implies that the transition probabilities corresponding to the solutions of (3.1) have a density with respect to Lebesgue measure that is smooth in all of its arguments (including time). This is an immediate consequence of the fact that the transition probabilities are a solution (in the sense of distributions) to the equation

$$(\partial_t + \mathcal{L}^*) p_t(z, z') = 0 \;,$$

and that $\partial_t + \mathcal{L}^*$ is hypoelliptic by assumption (see [Hör85]).

The result stated in Theorem 2.4 is now an almost immediate conclusion of the following two facts:

1. The measure μ_H satisfies $\mu_H(A) > 0$ for every open set $A \subset \mathbf{R}^N$.
2. The measure μ_H is the *only* invariant probability measure for (3.1) and is therefore ergodic.

While the first claim is obvious, the second claim requires some more explanation. Surprisingly, it will turn out to be an almost immediate consequence of the first claim, once we realise that the hypoellipticity assumption 2.3 implies the following.

Lemma 3.2 *Fix a time $t > 0$ and denote by $\mathcal{P}_t(z, \cdot\,)$ the transition probabilities corresponding to (3.1). Then, they are continuous in the total variation topology. In particular, for every $z \in \mathbf{R}^N$, there exists $\delta_z > 0$ such that*

$$\|\mathcal{P}_t(z, \cdot\,) - \mathcal{P}_t(z', \cdot\,)\|_{\mathrm{TV}} < \tfrac{1}{2} \;,$$

for every z' such that $|z' - z| < \delta_z$. Here, $\|\cdot\|_{\mathrm{TV}}$ denotes the total variation distance between probability measures.

Proof. This is an immediate consequence of the smoothness of the transition probabilities. □

Recall that the topological support $\operatorname{supp}\mu$ of a probability measure μ is the smallest closed set of full μ-measure. Equivalently, it is characterised as the set of points z such that every neighbourhood of z has positive μ-measure. As an immediate consequence of Lemma 3.2 and the fact that distinct ergodic invariant measures are mutually singular we have

Corollary 3.3 *For every $z \in \mathbf{R}^N$ there exists δ_z such that at most one ergodic invariant probability measure μ_z for (3.1) satisfies* $\operatorname{supp}\mu_z \cap \mathcal{B}(z,\delta_z) \neq \phi$. *Here $\mathcal{B}(z,\delta)$ denotes the open ball of radius δ centred in z.*

It follows from Corollary 3.3 and the σ-compactness of $\mathbf{R}^N$ that the set of all ergodic invariant measures for (3.1) is countable. Denote this set by $\{\mu_i\}_{i\geq 0}$ and define $S_i = \operatorname{supp}\mu_i$. An important property of the S_i's which follows immediately from Corollary 3.3 is

Corollary 3.4 *Every compact region of $\mathbf{R}^N$ intersects at most finitely many of the S_i's.*

Since every invariant measure for (3.1) is a convex combination of ergodic invariant measures, there exist weights p_i with $p_i \geq 0$ and $\sum p_i = 1$ such that $\mu_H = \sum_i p_i\mu_i$. It follows from Corollary 3.4 that

$$\operatorname{supp}\mu_H = \bigcup\{S_i \,|\, p_i \neq 0\}\;.$$

Since the S_i's are disjoint closed sets satisfying Corollary 3.4, the only way in which they can cover $\mathbf{R}^N$ is by having only one ergodic invariant measure with support $\mathbf{R}^N$. We have thus shown that μ_H is the only invariant probability measure for (3.1) and as a consequence is ergodic.

Let us now turn to the

Proof of Theorem 2.4. Fix an arbitrary open set $A \subset \mathbf{R}^N$ and denote by B the set of all points z_0 in $\mathbf{R}^N$ such that there exists a time T and a smooth control $u \in \mathcal{C}^\infty([0,T],E)$ such that the solution of (1.1) with initial condition z_0 satisfies $z(T) \in A$. Our aim is to show that $B = \mathbf{R}^N$.

Consider now the solution of (3.1) with initial condition $(\xi_0,\eta_0) = z \in \mathbf{R}^N$ and define the stopping time $T_z = \inf\{t > 0 \,|\, (\xi(t),\eta(t)) \in A\}$. It follows immediately from Birkhoff's ergodic theorem and the fact that $\mu_H(A) > 0$ that the set

$$B_0 = \{z \in \mathbf{R}^N \,|\, \mathbf{P}(T_z < \infty) = 1\}$$

satisfies $\mu_H(B_0) = 1$, so that B_0 is dense in $\mathbf{R}^N$. Furthermore, a consequence of Lemma 3.2 is that if $\mathcal{B}(z,\delta_z) \cap B_0 \neq \phi$ then $\mathbf{P}(T_z < \infty) \geq \frac{1}{2}$. Combining these two statements shows that for every $z \in \mathbf{R}^N$ there exists a time $t \geq 0$ such that $\mathcal{P}_t(z,A) > 0$. The support theorem [SV72], combined with the fact that the control problem associated to (3.1) is equivalent to (1.1) (since $\nabla_x H$ takes values in E) allows us to conclude that $B = \mathbf{R}^N$. □

4 Examples, counterexamples, and applications

In this section, we present a number of examples which explore to which extent the conditions formulated in this paper are necessary to the result. We also present a few applications in which our result may prove to be useful.

4.1 Dropping the hypoellipticity assumption

It is clear from [JK85] that Assumption 2.3 is crucial for any result of the type of Theorem 2.4, as can be seen from the following very easy example. Consider the Hamiltonian

$$H = \frac{p_1^2 + p_2^2}{2} + \frac{q_1^2 + q_2^2}{2} ,$$

and define f as the corresponding Hamiltonian vector field. If we define E to be the linear subspace of $\mathbf{R}^4$ corresponding to the variable p_1, it is clear that all of our assumptions are satisfied, except for Assumption 2.3. However, $q_2^2 + p_2^2$ is an integral of motion of this system that cannot be perturbed by acting on the variable p_1, so that the conclusion of Theorem 2.4 does not hold.

4.2 Dropping the conservative structure or the growth condition on H

Consider the control system in $\mathbf{R}^2$ given by

$$\dot{x} = u(t) - x , \qquad \dot{y} = g(y + g(x)) - y\varphi(y) , \tag{4.1}$$

where $g\colon \mathbf{R} \to [-1, 1]$ is an odd function with $g'(x) > 0$ for all x and such that $\lim_{x\to\pm\infty} g(x) = \pm 1$. The function $\varphi\colon \mathbf{R} \to [0, 1]$ is a smooth function such that $\varphi(y) = 0$ for $|y| \leq 2$ and $\varphi(y) = 1$ for $|y| \geq 3$.

One can see immediately that, whatever the values of u and x are, one has $\dot{y} > 0$ for $y \in (1, 2)$ and $\dot{y} < 0$ for $y \in (-2, -1)$. This shows that the conclusions of Theorem 2.4 cannot hold in this situation. However, the system (4.1) is perfectly hypoelliptic since the commutator between ∂_x and $-x\partial_x + g(y+g(x))\partial_y - y\varphi(y)\partial_y$ is given by

$$-\partial_x + g'(y + g(x))g'(x)\partial_y .$$

This vector field always has a non-zero component in the y-direction because of the assumption that g' is strictly positive.

Note that the orbits of (4.1) in the absence of control stay bounded for all times and that the corresponding diffusion process has global strong solutions and a smooth invariant probability measure. (The above arguments actually show that it has at least two distinct ergodic smooth invariant probability measures.) The missing point in this argument of course is the fact that we have no explicit expression for any of these invariant measures and therefore no *a priori* knowledge about their support.

One may argue on the other hand that it is possible to find a Hamiltonian function $H(x, y)$ such that $\partial_x H(x, y) = g(y+g(x)) - y\varphi(y)$ and that the control system

$$\dot{x} = -\partial_y H(x, y) + u(t) , \qquad \dot{y} = \partial_x H(x, y) ,$$

is equivalent (from the point of view taken in this paper) to (4.1). Actually, this Hamiltonian even satisfies (2.2) since $\Delta_x H$ is bounded, so that Assumption 2.1, Assumption 2.3, and (2.2) are satisfied. This example shows that even though the condition that $e^{-H(x,y)}$ be integrable may look at first sight like an artificial condition whose only goal is to make the probabilistic argument work, it is essential for Theorem 2.4 to hold.

4.3 Controllability in finite time

Let us show by a simple example that, unlike in the polynomial case studied in [JK85], it is not reasonable in general, under the conditions of Section 2, to expect the existence of a time T independent of z and A such that (1.1) can be driven from z to A in time T. Consider the function $H(x,y) = \sqrt{1+x^2+y^2}$ and the control system

$$\dot x = -\partial_y H(x,y) + u(t)\;, \qquad \dot y = \partial_x H(x,y)\;. \tag{4.2}$$

It is a straightforward exercise to check that all the conditions from the previous section are satisfied, so that Theorem 2.4 applies.

It is equally straightforward to check that $|\partial_x H(x,y)| \le 1$ for every value of x and y. Therefore, whatever control u is used to stir (4.2) from $z = (x,y)$ into A, it will always require a time T larger than $\inf_{(x',y')\in A} |y - y'|$.

4.4 The Euler equations

The three-dimensional Euler equations on the torus are given by

$$\dot u(x,t) = -(u(x,t)\cdot\nabla_x)u(x,t) - \nabla p(x,t)\;, \qquad \operatorname{div} u(x,t) = 0\;, \tag{4.3}$$

where $x \in \mathbf{T}^3$. (Note that the algebraic condition on the divergence of u determines p in a unique way.) If we assume that $\int u_0(x)\,dx = 0$ and expand this equation in Fourier modes, we obtain

$$du_k = -i \sum_{\substack{h,\ell\in\mathbf{Z}^3\setminus\{0\}\\ h+\ell=k}} (k\cdot u_h)\Big(u_\ell - \frac{k\cdot u_\ell}{|k|^2}k\Big)$$

subject to the algebraic conditions $k\cdot u_k = 0$ and $u_{-k} = \bar u_k$. Here, the index k takes values in $\mathbf{Z}^3 \setminus \{0\}$ (which reflects the fact that $\int u(x,t)\,dx = 0$ for all times) and $u_k \in \mathbf{R}^3$. Furthermore, the dot $\cdot$ denotes the usual scalar product in $\mathbf{R}^3$.

Since we are only interested in finite-dimensional systems, we fix a (large) value $N_\star$ and impose $u_k = 0$ for every k such that one of its components is larger than $N_\star$ in absolute value.

It is a straightforward exercise to check that the total energy $\sum |u_k|^2$ is a conserved quantity for this system and that the right-hand side of (4.3) is divergence-free. This allows to recover immediately a weak form of the controllability results obtained in [Rom04, Section 6]. The same argument allows to show that the finite-dimensional Galerkin approximations for the two-dimensional Euler equations are controllable under the conditions presented in [HM04].

4.5 Non-equilibrium statistical mechanics

The articles [EPR99a, EPR99b, EH00] considered a mechanical system coupled to heat baths at different temperatures T_i. This situation can be modelled by the following system of SDEs:

$$dq_j = \partial_{p_j} H_S\,dt - \sum_{i=1}^M (\partial_{p_j} F_i) r_i\,dt\;, \qquad j = 1,\ldots,N\;,$$

$$
\begin{aligned}
dp_j &= -\partial_{q_j} H_S\, dt + \sum_{i=1}^{M} (\partial_{q_j} F_i) r_i\, dt\;, \\
dr_i &= -\gamma_i r_i\, dt + \gamma_i \lambda_i^2 F_i(p,q)\, dt - \sqrt{2\gamma_i T_i}\, dw_i(t)\;, \qquad i = 1,\ldots,M\;,
\end{aligned}
\tag{4.4}
$$

The interpretation of this equation is that a Hamiltonian system with N degrees of freedom described by $H_S(p,q)$ is coupled to M heat baths with internal states r_i that are maintained at temperatures T_i. The functions $F_i(p,q)$ and the constants γ_i and λ_i describe coupling between the Hamiltonian system and the ith heat bath, as well as the relaxation times of the heat baths.

The control problem corresponding to this system is given by

$$
\begin{aligned}
\dot q_j &= \partial_{p_j} H_S - \sum_{i=1}^{M} (\partial_{p_j} F_i) r_i\;, \qquad & j &= 1,\ldots,N\;, \\
\dot p_j &= -\partial_{q_j} H_S + \sum_{i=1}^{M} (\partial_{q_j} F_i) r_i\;, \\
\dot r_i &= u_i(t)\;, & i &= 1,\ldots,M\;,
\end{aligned}
$$

which is of the form (1.1) with conserved quantity

$$
H(p,q,r) = H_S(p,q) + \sum_{i=1}^{M} \Bigl(\frac{r_i^2}{2\lambda_i^2} - r_i F_i(p,q) \Bigr)\;.
$$

(Note that one could actually add any function of r to H and it would still be a conserved quantity.) Provided that H_S grows sufficiently fast at infinity, it is easy to check that all of the assumptions of Section 2 are satisfied, except for Assumption 2.3 which has to be checked on a case by case basis. It can for example be checked for the chain of anharmonic oscillators studied in the abovementioned works, provided that the nearest-neighbour coupling potential does not contain any infinitely degenerate point.

Note that if all the T_i appearing in (4.4) are equal, then this equation is similar to (3.1) and one can check that $e^{-H(p,q,r)/T}\, dp\, dq\, dr$ is its invariant probability measure. However, the mere *existence* of an invariant probability measure for (4.4) with arbitrary temperatures is an open problem in general. (In the case of a chain of oscillators it was shown in [EPR99a, EH00, RBT02] to exist, provided that the nearest-neighbour interaction dominates the on-site potential at high energies.)

The controllability result shown in this article, combined with the support theorem [SV72] and the smoothness of transition probabilities for (4.4) immediately implies the following:

Theorem 4.1 *If (4.4) satisfies Assumption 2.3, the level sets of H_S are compact, and $e^{-H_S(p,q)}$ is integrable, then (4.4) can have at most one invariant probability measure.*

Proof. The conditions of Theorem 2.4 are satisfied by assumption (note that one can always add to H a function of r that grows sufficiently fast at infinity, thus ensuring that Assumption 2.2 holds). This immediately implies that every invariant probability measure for (4.4) has the whole phase space as its support. The fact that every invariant measure has a smooth density allows to conclude by the same argument used to show that the measure μ_H is the only invariant measure for (3.1). □

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