Absolutely Continuous Spectra of Quantum Tree Graphs with Weak Disorder

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Version: June 3, 2005

Abstract: We consider the Laplacian on a rooted metric tree graph with branching number $K \geq 2$ and random edge lengths given by independent and identically distributed bounded variables. Our main result is the stability of the absolutely continuous spectrum for weak disorder. A useful tool in the discussion is a function which expresses a directional transmission amplitude to infinity and forms a generalization of the Weyl-Titchmarsh function to trees. The proof of the main result rests on upper bounds on the range of fluctuations of this quantity in the limit of weak disorder.

Contents

1. Introduction

A quantum graph (QG) is a metric graph with an associated Laplace-like operator acting on the L^2-space of the union of the graph edges. The spectral and dynamical properties of such operators have been of interest both because this model mimics situations realizable with quantum dots and wires, and because QGs may provide a simple setup elucidating issues which are also of relevance for Schrödinger operators and Laplacians on manifolds (see [10, 19, 11, 8] and references therein). Examples of such topics

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are the Gutzwiller trace formula and the transition associated with the spectral and dynamical localization due to disorder. The main results of this work pertain to quantum *tree graphs* whose edge lengths are randomly stretched, but remain close to a common value. The goal is to present new results concerning the persistence of absolutely continuous spectra under weak disorder. A secondary goal is to demonstrate, in the QG context, a new technique for the proof of absolutely continuous spectrum which is also effective for discrete random Schrödinger operators on trees as was proven in [2].

1.1. Random quantum trees and their spectra. A rooted metric tree graph $\mathbb{T}$ with branching number K consists, for us, of a countably infinite set of vertices, one of which is being labeled as the root, 0, and a set $\mathcal{E}$ of edges, each joining a pair of vertices, such that: *1.* the graph is edge connected, *2.* there are no closed loops, *3.* each vertex has $K+1$ edges except for the root which has only one edge. Each edge $e \in \mathcal{E}$ is assigned a positive finite length $L_e \in (0,\infty)$ and is parametrized by a variable with values in $[0, L_e]$. Thus, the union of the edges has the natural coordinates $l \in [0, L_e]$. The orientation for the latter is chosen so that l increases away from the root, and we denote by $'$ the derivative with respect to those coordinates.

Our discussion concerns the spectral properties of the Laplacian

$$-\Delta_{\mathbb{T}}\, \psi_e = -\psi_e'' \,, \tag{1.1}$$

which acts in the Hilbert space $\mathrm{L}^2(\mathbb{T}) = \bigoplus_{e\in\mathcal{E}} \mathrm{L}^2[0, L_e]$ of complex-valued square-integrable functions $\psi = \oplus_{e\in\mathcal{E}}\, \psi_e$ defined over the union of the graph edges. The Laplacian is rendered essentially self-adjoint through the imposition of boundary conditions (BC) on the functions in its domain; here we take these to be the Kirchhoff conditions at internal vertices and α-BC at the root. More precisely, the domain consists of functions such that $\psi_e \in \mathrm{H}^2[0, L_e]$ for all $e \in \mathcal{E}$ and

1. at each vertex ψ is continuous.
2. at internal vertices the net flux defined by the directional derivatives vanishes, i.e.,

$$\psi_e'(L_e) = \sum_{f\in\mathcal{N}_e^+} \psi_f'(0) \tag{1.2}$$

   where $\mathcal{N}_e^+$ is the collection of edges which are forward to e as seen from the root.
3. at the root

$$\cos(\alpha)\, \psi_0(0) - \sin(\alpha)\, \psi_0'(0) = 0 \tag{1.3}$$

   with some $\alpha \in [0, \pi)$.

An extensive discussion of other boundary conditions which yield self adjointness can be found in [7,12]. Among those is the class of symmetric BC; the adaptation of the argument to this case is discussed in Section 6.

1.2. Statement of the main result. Our discussion will focus on the absolutely continuous (AC) component of the spectrum of the Laplacian on deformed metric trees. Before presenting the main result let us note the following fact, which may, for instance, be deduced from Theorem A.2 in Appendix A.

Proposition 1.1. *The AC spectrum of $-\Delta_{\mathbb{T}}$ is independent of the boundary condition at the root, i.e., of $\alpha \in [0, \pi)$.*

For the regular tree $\mathbb{T}$ with constant edge lengths $L \in (0, \infty)$ and branching number $K \in \mathbb{N}$ one has [21, Example 6.3]

$$\sigma_{\mathrm{ac}}(-\Delta_{\mathbb{T}}) = \bigcup_{n=0}^{\infty} \left[\left(\frac{\pi n + \theta}{L} \right)^2, \left(\frac{\pi(n+1) - \theta}{L} \right)^2 \right] \tag{1.4}$$

where $\theta := \arctan \left[\left(K^{1/2} - K^{-1/2} \right) /2 \right]$. In particular, this implies that the AC spectrum of $-\Delta_{\mathbb{T}}$ has band structure if $K \geq 2$. As an aside, we note that for $K \geq 2$ there occur infinitely degenerate eigenvalues in the band gaps [21].
The main object of interest in this paper is the AC spectrum of the Laplacian on random deformations of $\mathbb{T}$.

Definition 1.1. *A* random deformation $\mathbb{T}(\lambda, \omega)$ *of the regular rooted metric tree* $\mathbb{T}$ *is a rooted metric tree graph, which has the same vertex set and neighboring relations as* $\mathbb{T}$*, but the edge lengths are given by*

$$L_e(\lambda, \omega) := L \, \exp\left(\lambda \, \omega_e\right) \tag{1.5}$$

with a collection of real-valued, independent, and identically distributed (iid) bounded random variables $\omega = \{\omega_e\}_{e \in \mathcal{E}}$*. The parameter* $\lambda \in [0,1]$ *controls the strength of the disorder and* $L > 0$ *stands for the edge length of* $\mathbb{T}$.

Our main result is

Theorem 1.1. *For a random deformation,* $\mathbb{T}(\lambda, \omega)$*, of a regular tree graph* $\mathbb{T}$ *with branching number* $K \geq 2$ *the AC spectrum of* $-\Delta_{\mathbb{T}(\lambda,\omega)}$ *is continuous at* $\lambda = 0$ *in the sense that for any interval* $I \subset \mathbb{R}$ *and almost all* ω*:*

$$\lim_{\lambda \to 0} \mathcal{L}\big(I \cap \sigma_{\mathrm{ac}}(-\Delta_{\mathbb{T}(\lambda,\omega)})\big) = \mathcal{L}\big(I \cap \sigma_{\mathrm{ac}}(-\Delta_{\mathbb{T}})\big) \tag{1.6}$$

where $\mathcal{L}(\cdot)$ *denotes the Lebesgue measure.*

Remarks 1.1. (i) As is generally known by ergodicity arguments [3,17,1], and in our case also by the 0-1 law for the sigma-algebra of events measurable at infinity, which is applicable through Theorem A.2, for almost all ω the AC spectrum of $-\Delta_{\mathbb{T}(\lambda,\omega)}$ is given by a certain non-random set.

(ii) The assumption on the distribution of $\{\omega_e\}_{e \in \mathcal{E}}$ can be relaxed: the present proof readily extends to the class of random graphs where the distribution of these variables is *stationary* under the endomorphisms of the tree $\mathbb{T}$ and *weakly correlated* in the sense of [2, Def. 1.1].

(iii) To better appreciate the continuity asserted in Theorem 1.1, one may note that the analogous statement is not expected to be true in case the disorder is restricted to be radially symmetric, i.e., $\omega_e = a_{\mathrm{dist}\{e,0\}}$ with $\{a_n\}$ a collection of iid random variables. In this case, the AC spectrum coincides with that of a one-dimensional Sturm-Liouville operator. In view of related results about Anderson localization in one dimension [3,17, 15,16] one may expect (though we are not aware of a published proof) that also here localization sets in at any non-zero level of disorder.

2. An outline of the argument

A generally useful tool for the study of the spectral and dynamical properties of any quantum graph is provided by the Green function. For tree graphs, we find it particularly useful to consider a related quantity, which is an extension of the Weyl-Titchmarsh function familiar from the context of Sturm-Liouville or Schrödinger operators on a line. Before outlining the main steps in the derivation of Theorem 1.1, we shall introduce this function and its key properties, first somewhat informally through its appearance in a scattering problem.

2.1. A scattering perspective. As noted by Miller and Derrida [14], one may obtain a scattering perspective on extended states by considering a setup in which a wire $\mathbb{W}_x$ is attached to a tree graph $\mathbb{T}$ at an interior point x of an edge. Particles of energy E and decay rate η are sent at a steady rate down this wire. In the corresponding steady state, the quantum amplitude ψ for observing a particle at a point is given by a function satisfying $(-\Delta_{\mathbb{T}\cup\mathbb{W}_x} - z)\,\psi = 0$, were $z = E + i\eta$ and $-\Delta_{\mathbb{T}\cup\mathbb{W}_x}$ is a self adjoint Laplacian on the union of the graph and the wire, defined with suitable BC for the *three* segments meeting at the point of contact. For the latter, we assume here that it will be appropriate to take the Kirchhoff conditions.

As follows from Theorem 2.1 below, on the two subgraphs $\mathbb{T}_x^+$ and $\mathbb{T}_x^-$, produced by cutting $\mathbb{T}$ at x, the above differential equation has a unique – up to a multiplicative constant – square-integrable solution ψ^+ and correspondingly ψ^-. Thus ψ takes the form:

$$\psi(y;z) \;=\; \begin{cases} e^{i\sqrt{z}(y-x)} + r(x;z)\, e^{-i\sqrt{z}(y-x)} & \text{along the wire} \\ \psi^{\pm}(y;z) & \text{along the graph} \end{cases} \tag{2.1}$$

where $r(x;z)$ is the reflection coefficient, and the three branches are linked through the Kirchhoff conditions:

$$\begin{aligned} &\psi^+(x;z) \;=\; \psi^-(x;z) \;=\; 1 + r(x;z) \\ &\frac{\partial}{\partial x}\psi^+(x;z) \;-\; \frac{\partial}{\partial x}\psi^-(x;z) \;=\; i\sqrt{z}\,\big(1 - r(x;z)\big) \end{aligned} \tag{2.2}$$

with the differentiation taken in the direction away from the root of $\mathbb{T}$. The above relations yield

$$i\sqrt{z}\;\frac{1 - r(x;z)}{1 + r(x;z)} \;=\; R^+(x;z) \;+\; R^-(x;z) \tag{2.3}$$

where $R^{\pm} = \pm\left(\partial\psi^{\pm}/\partial x\right)/\psi^{\pm}$.

From the scattering perspective the graph absorbs some of the current directed at it, i.e., conducts it to infinity, if and only if $|r(x;z)| < 1$. A simple consequence of (2.3) is the equivalence

$$|r(x;E)| < 1 \qquad \Leftrightarrow \qquad \operatorname{Im}\big(R^+(x;E) \;+\; R^-(x;E)\big) > 0\,. \tag{2.4}$$

As it turns out R also plays a direct role in the spectral theory of $-\Delta_{\mathbb{T}}$: the diagonal of its Green function is given by

$$G_{\mathbb{T}}(x,x;z) \;=\; -\big(R^+(x;z) \;+\; R^-(x;z)\big)^{-1}\;. \tag{2.5}$$

By the theorem of de la Vallée Poussin, the AC component of the spectral measure, associated with the function in (2.5), is $\pi^{-1}\,\mathrm{Im}\,G_{\mathbb{T}}(x,x;E+i0)\,dE$. Therefore, there is a relation between the occurrence of the AC spectrum, the ability of the graph to conduct current to infinity, and the non-vanishing of $\mathrm{Im}\,R^{\pm}(x;E)$.
Let us note that the reflection coefficient for the version of the above experiment in which the particles are sent towards only the forward subtree $\mathbb{T}_x^+$, is given by a version of (2.3) with only $R^+(x;z)$ on the right side, and similarly for $\mathbb{T}_x^-$.

2.2. Tree extension of the Weyl-Titchmarsh function. We shall now follow the somewhat informal introduction above with a more careful definition of the functions $R^{\pm}$. For this purpose the following statement plays an important role.

Theorem 2.1. *Let $\mathbb{G}$ be a connected metric graph with a selected "open" vertex u which has exactly one adjacent edge. Let $-\Delta_{\mathbb{G},u}$ be the symmetric Laplacian defined with self-adjoint BC on all vertices excepting the open vertex, where it is required that both $\psi(u)=0$ and $\psi'(u)=0$. Then:*

- *(i) For any $z\in\mathbb{C}^+ := \{z\in\mathbb{C} : \mathrm{Im}\,z>0\}$, the space of square-integrable solutions of $(-\Delta^*_{\mathbb{G},u}-z)\,\psi=0$, with $-\Delta^*_{\mathbb{G},u}$ the adjoint operator, is one dimensional.*
- *(ii) The solution $\psi(x;z)$ and its derivative $\psi'(x;z)$ do not vanish on any point which disconnects $\mathbb{G}$.*
- *(iii) Normalized so that $\psi(u;z)=1$, both $\psi(x;z)$ and $\psi'(x;z)$ are analytic for $z\in\mathbb{C}^+$ and all $x\in\mathbb{G}$.*

We note that $-\Delta_{\mathbb{G},u}$ is *not* self-adjoint. The proof of this theorem is given in Appendix A.

The following corollary is a relevant implication for trees. Throughout, we denote by $\psi^{\pm}(x;z|u)$ the functions described in Theorem 2.1 which correspond to the two subtrees, $\mathbb{T}_u^{\pm}$, into which $\mathbb{T}$ is split at u, with u serving as the open vertex. We fix their normalization such that $\psi^{\pm}(u;z|u)=1$.

Corollary 2.1. *Along the edges of a metric tree $\mathbb{T}$, the ratio*

$$R^{\pm}(x;z) := \pm\,\frac{1}{\psi^{\pm}(x;z|u)}\,\frac{\partial}{\partial x}\psi^{\pm}(x;z|u) \tag{2.6}$$

does not depend on u as long as x stays in $\mathbb{T}_u^{\pm}$.

Definition 2.1. *We shall refer to the above $R^{\pm}$ as the* (generalized) Weyl-Titchmarsh (WT) functions*.*

These functions have a number of properties which are used in the proof of our main result. If not obvious, their derivation is given in Appendix A.

1. (*Relation with the Green function*) The generalized WT function may be related to the diagonal elements of the Green function which is defined on $\mathbb{T}_x^+$, with the $\alpha\neq 0$ BC at x, as

$$R^+(x;z) \;=\; \cot\alpha - \frac{1}{G^{\alpha}_{\mathbb{T}_x^+}(x,x;z)}\,, \tag{2.7}$$

and similarly for R^-.

2. (*Boundary values*) The function has the Herglotz-Nevanlinna property [5]: it is analytic for $z \in \mathbb{C}^+$ with $\operatorname{Im} R^{\pm}(x;z) > 0$ when $\operatorname{Im} z > 0$. By a standard implication, for each x the limit

$$R^{\pm}(x; E + i0) := \lim_{\eta \downarrow 0} R^{\pm}(x; E + i\eta) \tag{2.8}$$

exists for Lebesgue almost every $E \in \mathbb{R}$.
3. (*Evolution along the tree*) The values $R_e^+(\cdot; z)$ at two opposite ends of an edge e are related by a Möbius transformation, which integrates the Riccati equation:

$$\frac{\partial}{\partial x} R^+(x;z) + z + R^+(x;z)^2 = 0\,. \tag{2.9}$$

Over each vertex $R^+(\cdot; z)$ is additive thanks to (1.2):

$$R_e^+\big(L_e; z\big) \;=\; \sum_{f \in \mathcal{N}_e^+} R_f^+\big(0; z\big)\,. \tag{2.10}$$

4. (*Relation with the current*) For each u, the quantity

$$\begin{aligned} J^+(x, z|u) &:= \operatorname{Im}\left[\overline{\psi^+(x; z|u)}\,\frac{\partial}{\partial x}\psi^+(x;z|u)\right] \\ &= |\psi^+(x;z|u)|^2 \,\operatorname{Im} R^+(x;z) \geq 0 \end{aligned} \tag{2.11}$$

represents a current. It is additive at the vertices and conserved along the edges for real z. For $z \in \mathbb{C}^+$ the current is decreasing in the direction away from the root:

$$\frac{\partial}{\partial x} J^+(x; z|u) = -|\psi^+(x;z|u)|^2 \,\operatorname{Im} z \;\leq\; 0\,. \tag{2.12}$$

At interior vertices the net current flux is zero.

2.3. The core of the argument. We now have the requisite tools to outline the proof of the persistence of the AC spectrum under weak disorder. A key element in our analysis is to show that for small (λ, η), the WT function $R^+(x; E + i\eta, \lambda, \omega)$ does not depend much on ω. At each point its distribution is narrowly peaked around a value which may only depend on (λ, η), and the relative location of the point within the edge. By the rules of the evolution of R^+ along an edge, which are described above, it follows that for $(\lambda, \eta) \to (0,0)$ the limit of the "typical" value of $R_e^+\big(0; z, \lambda, \omega\big)$, or more precisely any accumulation point of such, obeys a Möbius evolution whose unique periodic solution is given by the WT function of the regular tree $\mathbb{T}$. The continuity then readily follows, though some care is needed in the presentation of the argument. In this part, we employ the strategy which was presented in [2].

It should be appreciated that the asymptotic lack of dependence of $R^+(x, z; \lambda, \omega)$ on ω is not just a trivial consequence of the smallness of λ since this parameter affects an infinite number of random terms. As commented above, it is natural to expect the corresponding statement to fail when the disorder is radial, with ω given by radially symmetric but otherwise iid random variables. To streamline the notation, in various places the dependence of ψ^+ and R^+ on λ and ω will be suppressed.

The first statement establishing a reduction of fluctuations concerns $\operatorname{Im} R^{+}(x;z)$. For that the starting point is (2.11) by which $|\psi^{+}(x;z|0)|^{2} \cdot \operatorname{Im} R^{+}(x;z)$ gives the flux at x of a conserved current. The current is injected at the root and at each vertex it is split among the forward directions. It is significant that the first factor takes a common value among the different forward directions, the second factor is independently distributed, and, furthermore, it has the same distribution as the total current $\operatorname{Im} R^{+}(0;z)$. It follows that

$$\frac{\frac{1}{K}\sum_{e\in\mathcal{N}_0^+} \operatorname{Im} R_e^{+}\big(0;z,\lambda,\omega\big)}{\operatorname{Im} R_0^{+}\big(0;z,\lambda,\omega\big)} \leq \frac{\big|\psi_0^{+}\big(0;z,\lambda,\omega|0\big)\big|^{2}}{K\,\big|\psi_f^{+}\big(0;z,\lambda,\omega|0\big)\big|^{2}} \,. \tag{2.13}$$

This expresses current conservation/attrition, and for $\operatorname{Im} z = 0$ holds as equality. Here $f \in \mathcal{N}_0^{+}$ is an arbitrary edge forward to that of the root, and due to the particular normalization chosen (before Corollary 2.1) the numerator on the right side is actually one. Our argument proceeds by combining two essential observations:

1. By the Jensen inequality the expectation value of the logarithm of the left side of (2.13) is non-negative. The inequality can be strengthened to show that the above expectation value provides an upper bound on a positive quantity which expresses the relative width of the distribution of $\operatorname{Im} R_0^{+}\big(0;z,\lambda,\omega\big)$.
2. The expectation of the logarithm of the right side of (2.13) is a quantity which it is natural to regard as a Lyapunov exponent,

$$\gamma_\lambda(z) := -\mathbb{E}\left[\log\sqrt{K}\,\frac{\big|\psi_f^{+}\big(0;z,\lambda,\cdot|0\big)\big|}{\big|\psi_0^{+}\big(0;z,\lambda,\cdot|0\big)\big|}\right] , \tag{2.14}$$

For $\lambda = 0$, this Lyapunov exponent vanishes for almost every $z \in \sigma_{\mathrm{ac}}(-\Delta_{\mathbb{T}})$. Furthermore, the average of $\gamma_\lambda(E+i\eta)$ over any energy interval is continuous in (λ,η).

The above mentioned improvement of the Jensen inequality is summarized in the following statement, which is a consequence of [2, Lemma 3.1 and Lemma D.2].

Lemma 2.1. *Let $\{X_j\}_{j=1}^{K}$ be a collection of $K \geq 2$ iid positive random variables, and X a variable of the same distribution. Then for any $a \in (0,1/2]$:*

$$\mathbb{E}\left[\log\left(\frac{1}{K}\sum_{j=1}^{K} X_j\right)\right] \;\geq\; \mathbb{E}\left[\log X\right] \;+\; \frac{a^2}{4}\delta\left(X,a\right)^{2} \,. \tag{2.15}$$

where $\delta(X,a)$ is the relative a-width of X, which is defined below.

Definition 2.2. *The relative a-width of the distribution of a positive random variable X, at $a \in (0,1/2]$, is*

$$\delta(X,a) := 1 - \frac{\xi_-(X,a)}{\xi_+(X,a)} \tag{2.16}$$

with $\xi_-(X,a) = \sup\{\xi : \mathbb{P}(X<\xi) \leq a\}$ and $\xi_+(X,a) = \inf\{\xi : \mathbb{P}(X>\xi) \leq a\}$.

A number of useful rules of estimates of the the relative width of a distribution are compiled in [2, Appendix D].

We shall now turn to the two key properties of the Lyapunov exponent which were mentioned above.

3. A Lyapunov exponent and its continuity

We shall refer to $\gamma_\lambda(z)$ which is defined by (2.14) as the Lyapunov exponent of the randomly deformed tree $\mathbb{T}(\lambda,\omega)$. The following theorem collects some of its properties. Of particular relevance is that the integral of $\gamma_\lambda(E+i\eta)$ over $E\in\sigma_{\rm ac}(\mathbb{T})$ is small for small λ and η.

Theorem 3.1. *The Lyapunov exponent $\gamma_\lambda(z)$ has the following properties:*

(i) *As a function of $z\in\mathbb{C}^+$, it is positive and harmonic with $\gamma_\lambda(i\eta)/\eta\to 0$ for $\eta\to\infty$.*

(ii) *For $\lambda=0$, it vanishes on the AC spectrum: $\gamma_0(E+i0)=0$ for Lebesgue-almost all $E\in\sigma_{\rm ac}(-\Delta_{\mathbb{T}})$.*

(iii) *For any $z\in\mathbb{C}^+$, $\gamma_\lambda(z+i\eta)$ is jointly continuous in $(\lambda,\eta)\in\mathbb{R}\times[0,\infty)$.*

(iv) *For any $[a,b]\subset\sigma_{\rm ac}(-\Delta_{\mathbb{T}})$:*

$$\lim_{\substack{\lambda\to 0\\ \eta\downarrow 0}}\int_a^b\gamma_\lambda(E+i\eta)\,dE=0. \tag{3.1}$$

Proof. (i) From (2.14) and (2.6) it follows that $\gamma_\lambda(z)$ is the negative of the real part of the Herglotz-Nevanlinna function

$$w_\lambda(z):=\log\sqrt{K}+\mathbb{E}\left[\int_0^{L_0(\lambda)}R_0^+\big(l;z,\lambda\big)\,dl\right], \tag{3.2}$$

and hence it is harmonic. The positivity of $\gamma_\lambda(z)$ follows from (2.11) and the Jensen inequality, which yield

$$2\gamma_\lambda(z)\geq\mathbb{E}\left[\log\frac{J_0^+(0;z|0)}{J_0^+(L_0(\lambda,\cdot);z|0)}\right]>0 \tag{3.3}$$

due to the current loss (2.12) on every edge for $z\in\mathbb{C}^+$. The statement of asymptotics derives from (A.5) and the bound (A.2) in Appendix A.

(ii) The vanishing of γ_0 along $\sigma_{\rm ac}(-\Delta_{\mathbb{T}})$ is a consequence of the $\operatorname{Im}z\downarrow 0$ limit of (2.13) and the fact that $R_e^+(0;z,0)$ is independent of e, with $0<\operatorname{Im}R_e^+(0;E+i0,0)<\infty$ for Lebesgue-almost all $E\in\sigma_{\rm ac}(-\Delta_{\mathbb{T}})$.

(iii) From (2.14) and (A.4) together with the dominated convergence theorem, which is applicable due to (A.5) and Theorem A.1, we conclude that the continuity of $\gamma_\lambda(z+i\eta)$ follows from that of $R_0(0;z+i\eta,\lambda,\omega)$. The latter is derived using the argument in the proof of Theorem A.1(iv).

(iv) By virtue of (ii) it suffices to prove that

$$\lim_{\lambda,\eta\to 0}\int_a^b\gamma_\lambda(E+i\eta)\,dE=\int_a^b\gamma_0(E+i0)\,dE. \tag{3.4}$$

To do so, we note that the integrals in (3.4) can be associated with the (unique) Borel measure $\sigma_{(\lambda,\eta)}$ corresponding to the positive harmonic function $h_{(\lambda,\eta)}(z)=\gamma_\lambda(z+i\eta)$ (cf. (3.6) below). Since $w_\lambda(\cdot+i\eta)$ has the Herglotz-Nevanlinna property, the harmonic conjugate of $h_{(\lambda,\eta)}=-\operatorname{Re}w_\lambda(\cdot+i\eta)$ has a definite sign and hence locally integrable boundary values [5, Thm. 1.1]. Therefore, the measure $\sigma_{(\lambda,\eta)}$ is purely AC [5, Thm 3.1 & Corollary 1] for all $(\lambda,\eta)\in[0,1]^2$ and given by

$$\sigma_{(\lambda,\eta)}\,[a,b] = \int_a^b \gamma_\lambda(E+i\eta)\,dE\,. \tag{3.5}$$

The assertion thus follows from (iii) and Lemma 3.1 below. □

The last part of the preceding proof was based on the following general convergence result for sequences of harmonic functions. Recall (cf. [5,9]) that every positive harmonic function $h:\mathbb{C}^+\to(0,\infty)$ which satisfies $\lim_{\eta\to\infty} h(i\eta)/\eta = 0$ admits the representation

$$h(z) = \int_{\mathbb{R}} \frac{\operatorname{Im} z}{|E-z|^2}\;\sigma(dE) \tag{3.6}$$

with some positive Borel measure σ on $\mathbb{R}$ with $\int_{\mathbb{R}}(E^2+1)^{-1}\sigma(dE)<\infty$.

Lemma 3.1. *Let h_n, $h:\mathbb{C}^+\to(0,\infty)$ be positive harmonic functions with $\lim_{\eta\to\infty} h_n(i\eta)/\eta = 0$ and similarly for h. Suppose that for all $z\in\mathbb{C}^+$*

$$\lim_{n\to\infty} h_n(z) = h(z). \tag{3.7}$$

Then their associated Borel measures converge vaguely, $\lim_{n\to\infty}\sigma_n = \sigma$.

The proof is an immediate consequence of the representation (3.6) and [6, Prop. 4.1] (see also [17, Lemma 5.22]).

4. Fluctuation bounds

Proceeding along the lines outlined in Subsection 2.3, we shall now show that a small Lyapunov exponent $\gamma_\lambda(z)$ implies the sharpness of the distribution of both the imaginary part and the modulus of a certain linear function of $R_0^+(0;z,\lambda,\omega)$.

Theorem 4.1. *For any $\lambda\in\mathbb{R}$, $z\in\mathbb{C}^+$ and $a\in(0,1/2]$:*

$$\delta\big(\operatorname{Im} R_0^+(0;z,\lambda,\cdot),a\big)^2 \le \frac{8}{a^2}\,\gamma_\lambda(z), \tag{4.1}$$

$$\delta\Big(\Big|\cos\big(\sqrt{z}L_0(\lambda,\cdot)\big) + \frac{\sin\big(\sqrt{z}L_0(\lambda,\cdot)\big)}{\sqrt{z}} R_0^+(0;z,\lambda,\cdot)\Big|^2, a\Big)^2 \le 512\,\frac{(K+1)^2}{a^2}\,\gamma_\lambda(z)\,. \tag{4.2}$$

Proof. The derivation of (4.1) starts from the relation

$$2\gamma_\lambda(z) \ge \mathbb{E}\Big[\log\Big(\frac{1}{K}\sum_{f\in\mathcal{N}_0^+}\operatorname{Im} R_f^+\big(0;z,\lambda,\cdot\big)\Big)\Big] - \mathbb{E}\left[\log\left(\operatorname{Im} R_0^+\left(0;z,\lambda,\cdot\right)\right)\right] \tag{4.3}$$

which is obtained by taking the expectation of the logarithm of (2.13). Applying the improved Jensen inequality (2.15), and using the fact that $\operatorname{Im} R_f^+\big(0;z,\lambda,\omega\big)$ are iid for $f\in\mathcal{N}_0^+$, the right side of (4.3) is bounded from below by $a^2\,\delta\left(\operatorname{Im} R_0^+(0;z,\lambda,\cdot),a\right)^2/4$.

This implies (4.1).
The proof of (4.2) starts by observing that the quantity in its left side can be identified with the right side of (2.13):

$$\begin{aligned}&\cos\left(\sqrt{z}L_0(\lambda,\omega)\right) + \frac{\sin\left(\sqrt{z}L_0(\lambda,\omega)\right)}{\sqrt{z}}\, R_0^+(0;z,\lambda,\omega)\\&= \psi_0^+\left(L_0(\lambda,\omega);z,\lambda,\omega|0\right). \end{aligned} \tag{4.4}$$

This follows from (A.4) in Appendix A. Setting $X := J_0^+(L_0(\lambda,\cdot);z|0)/J_0^+(0;z|0)$ and using the definition of the current, the left side in (4.2) therefore equals

$$\begin{aligned}&\delta\left(\frac{\operatorname{Im} R_0^+(0;z,\lambda,\cdot)}{\sum_{f\in\mathcal{N}_0^+}\operatorname{Im} R_f^+(0;z,\lambda,\cdot)}X, a\right)\\&\leq \delta\left(\frac{\operatorname{Im} R_0^+(0;z,\lambda,\cdot)}{\sum_{f\in\mathcal{N}_0^+}\operatorname{Im} R_f^+(0;z,\lambda,\cdot)}, \frac{a}{2}\right) + \delta\left(X, \frac{a}{2}\right), \end{aligned} \tag{4.5}$$

where the inequality results from the additivity of the relative width under multiplication [2, Lemma D.1]. This additivity and the invariance under inversion [2, Lemma D.1] ensures that the first term on the right side of (4.5) is bounded from above by

$$\begin{aligned}&\delta\Big(\operatorname{Im} R_0^+(0;z,\lambda), \frac{a}{2(K+1)}\Big) + \delta\Big(\sum_{f\in\mathcal{N}_0^+}\operatorname{Im} R_f^+(0;z,\lambda), \frac{aK}{2(K+1)}\Big)\\&\leq 2\,\delta\Big(\operatorname{Im} R_0^+(0;z,\lambda), \frac{a}{2(K+1)}\Big) \leq \frac{8\sqrt{2}\,(K+1)}{a}\sqrt{\gamma_\lambda(z)}\,. \end{aligned} \tag{4.6}$$

Here the first inequality results from the rules of addition of iid random variables [2, Lemma D.1]. The second one is a consequence of (4.1). The second term in on the right side of (4.5) is bounded from above according to $\delta(X,a/2) \leq 2\sqrt{\gamma_\lambda(z)/a}$. This follows from (3.3) and the simple bound

$$\delta(X,a)^2 \leq (1-\xi_-(X,a))^2 \leq -\ln\xi_-(X,a) \leq -\frac{\mathbb{E}\left[\ln X\right]}{a}, \tag{4.7}$$

valid for all random variables X taking values in $(0,1]$. Combining the above estimates, we arrive at (4.2). □

5. Stability of the Weyl-Titchmarsh function under weak disorder

5.1. The main stability result. Our goal in this section is to show that the boundary values of the WT function are continuous at $\lambda = 0$ in a certain distributional sense as long as $E \in \sigma_{\mathrm{ac}}(-\Delta_{\mathbb{T}})$. Here the distribution refers to the joint dependence on the energy and the randomness. The result to be derived is:

Theorem 5.1. *Let $I \subset \sigma_{\mathrm{ac}}(-\Delta_{\mathbb{T}})$ be an interval. Then the WT function converges in $\mathcal{L}_I \otimes \mathbb{P}$-measure, i.e., for all $\varepsilon > 0$:*

$$\lim_{\lambda\to 0}\mathcal{L}_I \otimes \mathbb{P}\left\{\left|R_0^+(0;E+i0,\lambda,\omega) - R_0^+(0;E+i0,0)\right| > \varepsilon\right\} = 0 \tag{5.1}$$

where $\mathcal{L}_I$ denotes the Lebesgue measure on I.

The above statement will be derived in this section by proving, in Theorem 5.2 which appears below, that for all $\varepsilon > 0$ and all sequences (λ, η) converging to zero

$$\lim_{\lambda,\eta\to 0} \mathcal{L}_I \otimes \mathbb{P}\left\{\left|R_0^+(0; E+i\eta, \lambda, \omega) - R_0^+(0; E+i0, 0)\right| > \varepsilon\right\} = 0\,. \tag{5.2}$$

Before we delve into the proof of the statements which lead to Theorem 5.1 let us note that it implies our main claim.

Proof (of Theorem 1.1; assuming Thm. 5.1). Since $\sigma_{\mathrm{ac}}(-\Delta_{\mathbb{T}(\lambda,\omega)})$ coincides almost-surely with a non-random set, it suffices to show that

$$\lim_{\lambda\to 0} \mathbb{E}\left[\mathcal{L}\big(I \cap \sigma_{\mathrm{ac}}(-\Delta_{\mathbb{T}(\lambda,\cdot)})\big)\right] = \mathcal{L}\big(I\big)\,. \tag{5.3}$$

We start the proof of this relation by observing that

$$\begin{aligned}\mathcal{L}\big(I\big) &\geq \mathbb{E}\left[\mathcal{L}\big(I \cap \sigma_{\mathrm{ac}}(-\Delta_{\mathbb{T}(\lambda,\cdot)})\big)\right] \\ &\geq \mathcal{L}_I \otimes \mathbb{P}\left\{0 < \operatorname{Im} R_0^+(0; E+i0, \lambda, \omega) < \infty\right\}\end{aligned} \tag{5.4}$$

where the second inequality is due to Theorem A.2.

For any $\varepsilon > 0$ the set on the right side includes the collection of (E, ω) for which $\varepsilon < \operatorname{Im} R_0^+(0; E{+}i0, 0) < \infty$ and $\left|\operatorname{Im} R_0^+(0; E{+}i0, \lambda, \omega) - \operatorname{Im} R_0^+(0; E{+}i0, 0)\right| \leq \varepsilon$. Accordingly, the right side of (5.4) is bounded below by the difference of

$$\mathcal{L}_I \otimes \mathbb{P}\left\{\left|\operatorname{Im} R_0^+(0; E+i0, \lambda, \omega) - \operatorname{Im} R_0^+(0; E+i0, 0)\right| \leq \varepsilon\right\} \tag{5.5}$$

and

$$\mathcal{L}_I\left\{\operatorname{Im} R_0^+(0; E+i0, 0) \in [0, \varepsilon] \cup \{\infty\}\right\}. \tag{5.6}$$

As $\lambda \to 0$ the measure in (5.5) converges to $\mathcal{L}(I)$ by Theorem 5.1. Moreover, as $\varepsilon \downarrow 0$ the measure in (5.6) converges to zero. □

5.2. Convergence in measure. In order to derive Theorem 5.1, we shall consider the distribution under the measure $\mathcal{L}_I \otimes \mathbb{P}$ of the joint values of E, $\{R_e^+(0; E+i\eta, \lambda, \omega)\}_{e\in\mathcal{E}}$, and $\{L_e(\lambda, \omega)\}_{e\in\mathcal{E}}$. In the following, $L_{\max}$ stands for some uniform upper bound on $L_e(\lambda, \omega)$, which exists due to the boundedness of the random variables. The setup is similar to that employed in [2].

Definition 5.1. *Let $(\lambda, \eta) \in [0,1]^2$ and $I \subset \sigma_{\mathrm{ac}}(-\Delta_{\mathbb{T}})$. The Borel measure $\nu_{(\lambda,\eta)}$ on $I \times \mathbb{C}^{\mathcal{E}} \times [0, L_{\max}]^{\mathcal{E}}$ is the measure induced by $\mathcal{L}_I \otimes \mathbb{P}$ under the mapping*

$$(E, \omega) \mapsto \Big(E, \big\{R_e^+\big(0; E+i\eta, \lambda, \omega\big)\big\}_{e\in\mathcal{E}}, \{L_e(\lambda, \omega)\}_{e\in\mathcal{E}}\Big)\,. \tag{5.7}$$

Moreover, its E-conditional distribution on $\mathbb{C}^{\mathcal{E}} \times [0, L_{\max}]^{\mathcal{E}}$ is abbreviated by $\nu^E_{(\lambda,\eta)}$.

Remarks 5.1. (i) The above definition relies on the fact that one may identify the edge sets $\mathcal{E}$ of $\mathbb{T}(\lambda, \omega)$ corresponding to different values of λ and/or ω.

(ii) In case $(\lambda, \eta) = (0, 0)$ the measure $\nu_{(\lambda,\eta)}$ is a product of the Lebesgue measure and products of Dirac measures:

$$\nu_{(0,0)} = dE \bigotimes_{e\in\mathcal{E}} \delta_{R_0^+(0;E+i0,0)} \bigotimes_{e\in\mathcal{E}} \delta_L\,. \tag{5.8}$$

(iii) The family of finite measures $\nu_{(\lambda,\eta)}$ is tight. Indeed, the bound (A.2) in Appendix A and arguments as in [2, Prop. B.1 & Lemma B.1] show that

$$\inf_{t>0} \sup_{(\lambda,\eta)\in[0,1]^2} \nu_{(\lambda,\eta)} \left(\left|R_e\right| > t\right) = 0 \tag{5.9}$$

for all $e \in \mathcal{E}$. Accordingly, every sequence of measures $\nu_{(\lambda,\eta)}$ corresponding to $(\lambda,\eta) \to (0,0)$ has weak accumulation points.

The issue now is to show that all of the above mentioned accumulation points coincide.

Theorem 5.2. *In the sense of weak convergence:*

$$\lim_{\lambda,\eta\to 0} \nu_{(\lambda,\eta)} = \nu_{(0,0)}. \tag{5.10}$$

The proof of this theorem closely follows ideas in [2], and rests on the following two lemmas.

We first show that all accumulation points of the sequence in (5.10) are supported on points satisfying the limiting recursion relation.

Lemma 5.1. *Let ν be a (weak) accumulation point for the family of measures $\nu_{(\lambda,\eta)}$, with the parameters (λ,η) in $[0,1]\times(0,1]$ converging to $(0,0)$. Then*

(i) the limiting recursion relation

$$\Big(\cos\big(\sqrt{E}L_e\big) + \frac{\sin\big(\sqrt{E}L_e\big)}{\sqrt{E}} R_e\Big) \sum_{f\in\mathcal{N}_e^+} R_f$$
$$= \cos\big(\sqrt{E}L_e\big) R_e - \sqrt{E}\sin\big(\sqrt{E}L_e\big) \tag{5.11}$$

holds for ν-almost all $(E,R,L) \in I \times \mathbb{C}^{\mathcal{E}} \times [0,L_{\max}]^{\mathcal{E}}$.

(ii) the lengths are ν-almost surely constant, $L_e = L$ for all $e \in \mathcal{E}$.

(iii) the variables $\{R_e\}_{e\in\mathcal{E}}$ are identically distributed ν-almost surely.

(iv) for Lebesgue-almost all $E \in I$ there exist $\mathcal{I} \in [0,\infty)$ and $\mathcal{M} \in [0,\infty)$ such that for all $e \in \mathcal{E}$

$$\operatorname{Im} R_e = \mathcal{I} \quad \textit{and} \quad \Big|\cos\big(\sqrt{E}L\big) + \frac{\sin\big(\sqrt{E}L\big)}{\sqrt{E}} R_e\Big| = \mathcal{M} \tag{5.12}$$

ν^E-almost surely.

Proof. (i) The fact that the accumulation points obey the limiting recursion relation which is the $(\lambda,\eta) = (0,0)$ version of (A.3) and (2.10), is a consequence of the general principle proven in [2, Prop. 4.1].

(ii) This statement is implied by the pointwise convergence $\lim_{\lambda\to 0} L_e(\lambda,\omega) = L$.

(iii) The claim follows from the fact that all prelimit quantities are identically distributed.

(iv) We fix $e \in \mathcal{E}$. Then Theorem 4.1 and Theorem 3.1 yield

$$\lim_{\lambda,\eta\to 0} \int_I \delta\big(\operatorname{Im} R_e^+(0; E+i\eta, \lambda, \cdot), a\big)\, dE = 0 \tag{5.13}$$

$$\lim_{\lambda,\eta\to 0} \int_I \delta\Big(\Big| \cos\big(\sqrt{E+i\eta}\, L_e(\lambda,\cdot)\big) + \frac{\sin\big(\sqrt{E+i\eta}\, L_e(\lambda,\cdot)\big)}{\sqrt{E+i\eta}} R_e^+(0; E+i\eta, \lambda, \cdot)\Big|^2, a\Big)\, dE = 0 \tag{5.14}$$

for all $a \in (0, 1/2]$. By [2, Lemma D.4] this implies that both random variables in (5.12) are almost surely constant for Lebesgue-almost all $E \in I$. Since they are identically distributed for all $e \in \mathcal{E}$, the constants $\mathcal{I}$ and $\mathcal{M}$ are independent of e . □

The explicit expression (1.4) shows that $\sin\big(\sqrt{E}L\big)/\sqrt{E} \neq 0$ for all $E \in \sigma_{\mathrm{ac}}(-\Delta_{\mathbb{T}})$. Therefore, (5.12) asserts that the R_e-marginals of ν^E are supported on the intersection of a line with a circle, that is, on at most two points. Next, we show that this support contains only one point which coincides with $R_0(0, E+i0, 0)$.

Lemma 5.2. *Assume the situation of Lemma 5.1. Then for Lebesgue-almost all $E \in I$:*

(i) there exists $\Phi \in \mathbb{C}$ with $\operatorname{Im}\Phi \geq 0$ such that for all $e \in \mathcal{E}$

$$R_e = \Phi \qquad \nu^E\text{-almost surely.} \tag{5.15}$$

(ii) $\Phi = R_0(0, E+i0, 0)$.

Proof. (i) By Lemma 5.1 there exists $\Phi^{\pm} \in \mathbb{C}$ with $\operatorname{Im}\Phi^{\pm} \geq 0$ such that the R_e-marginal of ν^E is supported on $\{\Phi^+, \Phi^-\}$ for all $e \in \mathcal{E}$. Suppose that $\Phi^+ \neq \Phi^-$. Then the distribution of $\sum_{f\in\mathcal{N}_e^+} R_f$ is supported on at least three points. This follows by explicitly identifying three points $K\Phi^{\pm}$ and $(K-1)\,\Phi^+ + \Phi^-$ in the support. But this contradicts the limiting recursion relation (5.11) since then the distribution of the left side is supported contains at least three points but the distribution of the right side on at most two points in its support.

(ii) Equation (5.11) with Φ substituted for all R_e, and $L_e = L$ for all edges e, is quadratic in Φ. For Lebesgue almost all $E \in \sigma_{\mathrm{ac}}(-\Delta_{\mathbb{T}})$ this equation has a complex non-real solution, and in this case $R_0^+(0, E+i0, 0)$ is its only solution in the upper half plane. □

5.3. Section summary. Let us now note that the above lemmas imply the two theorems stated in this section:

Proof of Theorem 5.2. Lemmas 5.1 and 5.2 jointly imply Theorem 5.2. □

Proof of Theorem 5.1. As is discussed in [2], an application of Fatou's lemma yields (5.1) from (5.2). □

6. Extensions

6.1. More general vertex conditions. A variety of boundary conditions other than (1.2) lead to self-adjoint Laplacians on metric graphs [7,12]. Of those, the argument presented here can be readily extended to the class of symmetric BC. These require at each vertex:

1. for some fixed $\beta \in [0,\pi]$ the following is common to all the edges e adjacent to the vertex
$$\cos(\beta)\,\psi + \sin(\beta)\; n_e \cdot \nabla\,\psi \tag{6.1}$$
with $n_e \cdot \nabla$ the *outward* derivative,
2. for some fixed $\alpha \in [0,\pi]$ the sums over all edges adjacent to the vertex satisfies
$$\cos(\alpha)\;\sum_e \psi_e - \sin(\alpha)\;\sum_e n_e \cdot \nabla\,\psi \;=\; 0\,, \tag{6.2}$$

The symmetric class includes the Kirchhoff BC (1.2), for which $\beta = 0$ and $\alpha = \pi/2$.

Our analysis extends to the general symmetric BC through a rotation which mixes the function ψ^+ and its derivative, where ψ^+ is defined as below Theorem 2.1 with the present boundary conditions. We denote:

$$\widetilde{\psi}^+(x;z|0) \;:=\; \cot(\beta)\,\psi^+(x;z|0) + \frac{\partial}{\partial x}\psi^+(x;z|0) \;=-\frac{\psi^+(x;z|0)}{\widetilde{R}^+(x;z)}\,, \tag{6.3}$$

and correspondingly

$$\widetilde{R}^+(x;z) := -\left[\cot(\beta) + R^+(x;z)\right]^{-1}\,. \tag{6.4}$$

Under the above boundary conditions it is the function $\widetilde{\psi}^+(x;z|0)$ which takes a common value among the forward edges of any vertex. The current can be expressed in terms of the 'rotated' quantities as

$$J^+(x;z|0) \equiv \left|\psi^+(x;z|0)\right|^2 \operatorname{Im} R^+(x;z) \;=\; \left|\widetilde{\psi}^+(x;z|0)\right|^2 \operatorname{Im} \widetilde{R}^+(x;z)\,. \tag{6.5}$$

The argument, as it is outlined in Section 2.3, applies verbatim with $\psi^+(x;z|0)$ and $R^+(x;z)$ replaced by $\widetilde{\psi}^+(x;z|0)$ and $\widetilde{R}^+(x;z)$. In this context the relevant Lyapunov exponent is

$$\widetilde{\gamma}_\lambda(z) \;:=\; -\mathbb{E}\left[\log\left(\sqrt{K}\,\frac{\left|\widetilde{\psi}^+_f(0;z,\lambda,\cdot|0)\right|}{\left|\widetilde{\psi}^+_0(0;z,\lambda,\cdot|0)\right|}\right)\right]\,, \tag{6.6}$$

where f is an arbitrary edge forward to the edge emanating from the root. It follows from (6.3) that the above expression with $\widetilde{\psi}^+$ yields the same value as with ψ^+, i.e., $\widetilde{\gamma}_\lambda(z) = \gamma_\lambda(z)$, the latter being defined by (2.14).

6.2. Tree graphs with decorations . By gluing a copy of a finite metric graph $\mathbb{G}$ to every vertex of the tree $\mathbb{T}$ one obtains a metric graph $\mathbb{T} \lhd \mathbb{G}$ which is refered to as a *decorated* tree. The Laplacian $-\Delta_{\mathbb{T}\lhd\mathbb{G}}$ is rendered self-adjoint by imposing, for example, Kichhoff BC. Such decorations provide a mechanism for the creation of gaps in the spectrum [20, 13].

The strategy presented here allows to establish the stability of the AC spectrum under random deformations of a (uniformly) decorated tree even if $\mathbb{G}$ has loops. In deriving the fluctuation bounds in this case, in the sum on the left side of (2.13) one may omit the terms $\operatorname{Im} R_f^+$ which correspond to directions f into the decorating parts. These terms vanish for real z since the finite graph $\mathbb{G}$ does not conduct current to infinity.

A. Appendix: More on the Weyl-Titchmarsh function on tree graphs

This appendix is devoted to the WT functions $R^{\pm}$ on general metric tree graphs $\mathbb{T}$, presented in Definition 2.1. We start by proving Theorem 2.1 on which the definition relies. Basic properties of $R^{\pm}$ are the topic of the second subsection. The third subsection deals with the Green function on $\mathbb{T}$ and its relation to the WT functions.

A.1. Uniqueness of square-integrable solutions on graphs with a dangling end. We will now give a proof of Theorem 2.1.

Proof. (i) That there is at least one non identically vanishing function in the kernel of the operator $-\Delta^*_{\mathbb{G},u} - z$ can be seen by elongating the dangling edge beyond u thereby creating a backward extension $\mathbb{G}' \supset \mathbb{G}$. We set

$$\psi(x;z) = (-\Delta_{\mathbb{G}'} - z)^{-1} \varphi(x) \tag{A.1}$$

where φ is some non identically vanishing function compactly supported on the elongation of the edge containing u and $-\Delta_{\mathbb{G}'}$ is a *self-adjoint* Laplacian on $\mathbb{G}'$. This function (A.1) does not vanish identically on $\mathbb{G}$, since otherwise (by (A.4) below) it would be identically zero on the whole edge containing u and the support of φ.
Suppose now there is another solution which is linearly independent of $\psi(\cdot; z)$. Since the solution space on the edge adjacent to u is two dimensional, one can linearly combine them to satisfy a self-adjoint BC (cf. (1.3)) at u. Thereby one produces an eigenfunction of a self-adjoint Laplacian $-\Delta_{\mathbb{G}}$ with eigenvalue $z \in \mathbb{C}^+$. This contradicts the self-adjointness.

(ii) In fact, more generally for all $x \in \mathbb{G}$ which disconnect the graph, we have that $\cos(\alpha)\, \psi(x;z) - \sin(\alpha)\, \psi'(x;z) \neq 0$ for all $\alpha \in [0, \pi)$. Otherwise one would have found a square-integrable, non-trivial eigenfunction with eigenvalue $z \in \mathbb{C}^+$ of a restriction of $-\Delta^*_{\mathbb{G},u}$ to functions on that disconnected piece, which does not contain u. Since this Laplacian is rendered self-adjoint by imposing α-BC at x, this is a contradiction.

(iii) This is an immediate consequence of (A.1). □

A.2. Basic properties of the Weyl-Titchmarsh functions. Following are some properties of $R^+(x;z)$ which are of relevance in the main part of the paper. Similar statements apply to R^-, with proofs differing only in the notation.

Theorem A.1. *The WT function $R^+(x;z)$ has the following properties:*

(i) $R^+(x;\cdot) : \mathbb{C}^+ \to \mathbb{C}^+$ *is analytic for fixed* x.

(ii) *For each* $e \in \mathcal{E}$ *and all* $z \in \mathbb{C}^+$*:*

$$\left| R_e^+(0;z) \right| \leq \frac{2\sqrt{|z|}}{1 - \exp\left(-2L_e \operatorname{Im} \sqrt{z}\right)}, \tag{A.2}$$

and $|R_e^+(L_e;z)| \leq 2K\sqrt{|z|}\,[1 - \exp(-2L_e \operatorname{Im}\sqrt{z})]^{-1}$ *due to* (2.10).

(iii) *Along any edge* e *the function obeys the Riccati equation* (2.9)*. In particular its values are related by Möbius transformations:*

$$R_e^+\left(l;z\right) = \frac{R_e^+\left(0;z\right)\cos\left(\sqrt{z}l\right) - \sqrt{z}\sin\left(\sqrt{z}l\right)}{\cos\left(\sqrt{z}l\right) + R_e^+\left(0;z\right)\sin\left(\sqrt{z}l\right)/\sqrt{z}} \tag{A.3}$$

for all $l \in [0, L_e]$ *and* $z \in \mathbb{C}^+$.

(iv) *Equipping the space* $[L_{\min},\infty)^{\mathcal{E}}$ *with the uniform topology,* $R_0^+(0;z)$ *is a continuous function of* $\{L_e\}_{e\in\mathcal{E}} \in [L_{\min},\infty)^{\mathcal{E}}$ *for all* $z \in \mathbb{C}^+$.

Proof. (i) The first assertion follows from the analyticity of $\psi^+(x;z|0)$ and of its deriative (cf. Theorem 2.1). The Herglotz-Nevanlinna property is a consequence of (2.7).

(ii) This is an immediate consequence of (A.6) and Lemma A.1(i) below.

(iii) This assertion follows from the fact that $\psi_e^+(\cdot;z|0) \in \mathrm{H}^2[0,L_e]$ is a solution of the free Schrödinger equation $-\psi'' = z\psi$ on the interval $[0,L_e]$ which, using the boundary conditions at $l=0$, may be written as

$$\frac{\psi_e^+(l;z|0)}{\psi_e^+(0;z|0)} = \cos(\sqrt{z}l) + R_e^+(0;z)\,\frac{\sin(\sqrt{z}l)}{\sqrt{z}} \tag{A.4}$$

for all $l \in [0, L_e]$.

(iv) Suppose the metric tree $\mathbb{T}$ is finite and has only N generations, i.e., the number of edges connecting any edge to the root is at most N. In this case, the continuity of $R_0(0;z)$ follows from the explicit evolution equations (A.3) and (2.10). Lemma A.1(iii) below shows that $R_0(0;z)$ may be uniformly approximated by its values on a finite tree provided $\operatorname{Im}\sqrt{z}$ is large enough. Hence $R_0(0;z)$ is continuous for those $z \in \mathbb{C}^+$. Since $R_0(0;z)$ is analytic in $z \in \mathbb{C}^+$, this implies continuity for all $z \in \mathbb{C}^+$. □

Remark A.1. Another immediate consequence of (A.4) and its analog with 0 and L_e interchanged, is the bound

$$e^{-\sqrt{|z|}L_e}\left(1 + \frac{|R_e^+(L_e;z)|}{\sqrt{|z|}}\right)^{-1} \leq \left|\frac{\psi_e^+(L_e;z|0)}{\psi_e^+(0;z|0)}\right| \leq e^{\sqrt{|z|}L_e}\left(1 + \frac{|R_e^+(0;z)|}{\sqrt{|z|}}\right) \tag{A.5}$$

which shows that $\psi^+(\cdot;z|0)$ de- or increases at most exponentially on any edge e.

Instead of the WT function R^+, it is sometimes more convenient to consider its transform

$$m(x;z) := \frac{R^+(x;z) - i\sqrt{z}}{R^+(x;z) + i\sqrt{z}} \tag{A.6}$$

which takes values in the complex unit disk. The evolution on the edges takes a particularly simple form for m. In fact, from (A.3) and (2.10) one obtains

$$m_e(0;z) = e^{2i\sqrt{z}L_e} m_e(L_e;z), \quad \text{and} \quad m_e(L_e;z) = g\left(\sum_{f \in \mathcal{N}_e^+} \frac{1+m_f(0;z)}{1-m_f(0;z)} \right), \tag{A.7}$$

where $g(\zeta) := \frac{\zeta-1}{\zeta+1}$. The next lemma collects some facts which are used in the proof of Theorem A.1.

Lemma A.1. *Let $z \in \mathbb{C}^+$ and assume $L_e \geq L_{\min} > 0$ for all $e \in \mathcal{E}$. Then $m(x;z)$ has the following properties:*

(i) It satisfies: $|m_e(0;z)| \leq \exp\left(-2L_e \operatorname{Im}\sqrt{z}\right)$.

(ii) At the root the dependence on a particular value $m_e(L_e;z)$ is uniformly exponential in the sense that there exists a constant $c < \infty$ such that for all $\operatorname{Im}\sqrt{z}$ sufficiently large:

$$\left| \frac{\partial m_0(0;z)}{\partial m_e(L_e;z)} \right| \leq c^N \exp\left(-2NL_{\min} \operatorname{Im}\sqrt{z}\right) \tag{A.8}$$

where N is the number of vertices between the edge e to the root.

(iii) Let $m_0(0;z)$ and $\widetilde{m}_0(0;z)$ correspond to metric tree graphs $\mathbb{T}$ and $\widetilde{\mathbb{T}}$ which coincide up to the Nth generation. Then for all $\operatorname{Im}\sqrt{z}$ sufficiently large:

$$|m_0(0;z) - \widetilde{m}_0(0;z)| \leq 2K^{N+1} c^N \exp\left(-2NL_{\min} \operatorname{Im}\sqrt{z}\right). \tag{A.9}$$

Proof. (i) This is an immediate consequence of the first evolution equation (A.7).

(ii) Using the chain rule this can be traced back to a straightforward differentiation of the equations (A.7). The edge and vertex terms are subsequently bounded with the help of (i).

(iii) We expand the difference into a telescopic sum of K^{N+1} differences and use both (ii) and the fact that the values of $m(\cdot;z)$ and $\widetilde{m}(\cdot;z)$ on the K^{N+1} leaves in the Nth generation differ at most by a complex number of modulus 2. □

A.3. The Green function on a tree graph. Analogously to one dimension [4,3], the Green function of the Laplacian $-\Delta_{\mathbb{T}}$ on a metric tree graph $\mathbb{T}$ can be constructed using two non-vanishing square-integrable functions. In fact, the following lemma is straightforward.

Lemma A.2. *The Green function $G_{\mathbb{T}}(u,x;z)$ of the Laplacian $-\Delta_{\mathbb{T}}$ can be expressed as*

$$G_{\mathbb{T}}(u,x;z) = \frac{\left(\psi^+ \wedge \psi^-\right)(u,x;z|v)}{W(\psi^+,\psi^-)(u;z|v)}, \tag{A.10}$$

independently of v, as long as $v \in \mathbb{T}_u^+ \cap \mathbb{T}_x^+$. Here

$$\big(\psi^+ \wedge \psi^-\big)(u,x;z|v) := \begin{cases} \psi^+(u;z|0)\,\psi^-(x;z|v) & \textit{for} \quad x \in \mathbb{T}_u^- \\ \psi^-(u;z|v)\,\psi^+(x;z|0) & \textit{for} \quad x \in \mathbb{T}_u^+ \,. \end{cases} \tag{A.11}$$

and $W(\psi^+,\psi^-) := \psi^+\big(\partial\psi^-/\partial x\big) - \big(\partial\psi^+/\partial x\big)\psi^-$ is the Wronskian.

Remarks A.1. (i) The Wronskian is constant along any edge in $\mathbb{T}_v^-$. In particular, this implies that $W(\psi^+,\psi^-) \neq 0$, since otherwise one could linarly combine $\psi^\pm$ to a square-integrable solution of $(-\Delta_{\mathbb{T}} - z)\,\psi = 0$ on the whole tree.

(ii) The right side of (A.10) defines an integral kernel of the resolvent $(-\Delta_{\mathbb{T}} - z)^{-1}$ which is jointly continuous in (u,x).

(iii) Setting $f_v^\pm(\cdot;z|\cdot) := \psi^\pm(\cdot;z|\cdot)/\psi^\pm(v;z|\cdot)$, where v is any point on the same edge as u, the Green function (A.10) can be rewritten in terms of WT functions:

$$G_{\mathbb{T}}(u,x;z) = -\frac{\big(f_v^+ \wedge f_v^-\big)(u,x;z)}{R^+(v;z) + R^-(v;z)}. \tag{A.12}$$

In particular, for $u = x = v$, we obtain (2.5). Moreover, at the root, we obtain

$$G_{\mathbb{T}}(0,0;z) = \big(\cot\alpha - R^+(0;z)\big)^{-1}, \qquad \alpha \neq 0, \tag{A.13}$$

because $R^-(0;z) = -\cot\alpha$ due to the BC (1.3).

For a self-adjoint Sturm-Liouville, or more specifically, Schrödinger operator on the half-line, the WT function at the origin allows one to reconstruct the spectral measure and therefore contains all spectral information [4,3]. Generally, this fails to hold for operators on tree graphs. However, the AC spectrum of $-\Delta_{\mathbb{T}}$ can still be detected by the boundary value of $R^+(0;z)$:

Theorem A.2. *The AC spectrum $\sigma_{\rm ac}(-\Delta_{\mathbb{T}})$ of the Laplacian on a rooted metric tree graph $\mathbb{T}$ is concentrated on the set*

$$\big\{E \in \mathbb{R} \,:\, 0 < \operatorname{Im} R_0^+(0;E+i0) < \infty\big\}\,. \tag{A.14}$$

Proof. Pick any edge $e \in \mathcal{E}$ and let ϕ be a compactly supported function on e. A straightforward but tedious computation using (A.12) shows that for Lebesgue-almost all $E \in \mathbb{R}$ the AC density of the spectral measure associted with ϕ is given by

$$\lim_{\eta\downarrow 0} \operatorname{Im}\Big\langle \phi,\, (-\Delta_{\mathbb{T}} - E - i\eta)^{-1}\,\phi\Big\rangle = \frac{\operatorname{Im} R_e^+(0;E+i0)\; g_\phi^-(E) + \operatorname{Im} R_e^-(0;E+i0)\; g_\phi^+(E)}{|R_e^+(0;E+i0) + R_e^-(0;E+i0)|^2}, \tag{A.15}$$

for Lebesgue-almost every $E \in \mathbb{R}$, where

$$g_\phi^\pm(E) := \big|\big\langle\phi, \operatorname{Re} f^\pm\big(\cdot;E\big)\big\rangle\big|^2 + \big|\big\langle\phi, \operatorname{Im} f^\pm\big(\cdot;E\big)\big\rangle\big|^2 \tag{A.16}$$

and $f^\pm\big(\cdot;E\big)$ is the solution of the Schrödinger equation $(-\Delta_{\mathbb{T}} - E)\,f = 0$, which satisfies $\big(df_f^\pm/dx\big)\big(0;E\big) = \pm R_f^\pm(0;E+i0)$ at every edge f and is normalized to

$f_e^{\pm}(0;E) = 1$. By the current conservation (2.12) along each edge and the positivity and additivity of the current at each vertex, we have

$$\operatorname{Im} R_e^{+}(0; E + i0) \leq \left| f_0^{+}(0; E) \right|^2 \operatorname{Im} R_0^{+}(0; E + i0) \tag{A.17}$$

for Lebesgue-almost all $E \in \mathbb{R}$. Similarly, by tracing the current flow on the backward tree emanating from e and using $\operatorname{Im} R_0^{-}(0; E+i0) = 0$, we obtain for Lebesgue-almost all $E \in \mathbb{R}$

$$\operatorname{Im} R_e^{-}(0; E + i0) = \sum_f \left| f_f^{-}(0; E) \right|^2 \operatorname{Im} R_f^{+}(0; E + i0) \tag{A.18}$$

where the sum extends over all edges $f \neq e$, which have the same distance to the root as e.

From the above considerations we conclude that for any e and Lebesgue-almost all $E \in \mathbb{R}$ if $\operatorname{Im} R_0^{+}(0; E + i0) = 0$ then *1.* $\operatorname{Im} R_e^{+}(0; E + i0) = 0$, and *2.* $\operatorname{Im} R_e^{-}(0; E + i0) = 0$. But this shows that $\sigma_{\mathrm{ac}}(-\Delta_{\mathbb{T}})$ is indeed concentrated on the set in (A.14). □

Acknowledgement. We are grateful to Uzy Smilansky for stimulating discussions concerning quantum graphs. We would also like to thank for the gracious hospitality enjoyed at the Weizmann Institute (MA) and the Department of Mathematics at UC Davis (SW). This work was supported in parts by the Einstein Center for Theoretical Physics and the Minerva Center for Nonlinear Physics at the Weizmann Institute, by the US National Science Foundation, and by the Deutsche Forschungsgemeinschaft.

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