Propagation of singularities for Schrödinger equations on the Euclidean space with a scattering metric

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Abstract

Given a scattering metric on the Euclidean space. We consider the Schrödinger equation corresponding to the metric, and study the propagation of singularities for the solution in terms of the *homogeneous wavefront set*. We also prove that the notion of the homogeneous wavefront set is essentially equivalent to that of the quadratic scattering wavefront set introduced by J. Wunsch [21]. One of the main results in [21] follows on the Euclidean space with a weaker, almost optimal condition on the potential.

1 Introduction

We embed the Euclidean space $\mathbb{R}^n$ into the half sphere S^n_+ using the stereographic projection following Melrose [13]:

$$\mathrm{SP} : \mathbb{R}^n \to S^n_+ = \left\{ w \in \mathbb{R}^{n+1} \,|\, |w| = 1, w_n \geq 0 \right\}, \quad z \mapsto \frac{1}{\sqrt{1+|z|^2}}\,(z,1)\,.$$

$X = S^n_+$ is regarded as the Euclidean space with boundary S^{n-1} at infinity, and $x = z^{-1}$ for $z \in \mathbb{R}^n \setminus \{0\}$ defines a boundary defining function of X near ∂X. Consider a *scattering metric* g on X. Scattering metric is a Riemannian metric in the interior X° that has, near the boundary, an expression

$$g = \frac{dx^2}{x^4} + \frac{h}{x^2}.$$

Here h is a 2-cotensor on X and, when restricted to ∂X, defines a Riemannian metric on ∂X. Under these setting we have the Schrödinger operator

$$H = -\frac{1}{2\sqrt{g}} \sum_{i,j=1}^{n} \partial_i g^{ij} \sqrt{g} \partial_j + V,$$

where V is a potential function, $\left(g^{ij}\right)$ is an inverse matrix to $g=(g_{ij})$ and $g=\det g$ (an abuse of notation). We assume that V is a smooth real-valued function on $X^\circ=\mathbb{R}^n$ with the following growth property. Take some $\nu<2$. Then, for any coordinates (x,y) of X near ∂X with y the coordinates of $U\subset\partial X$ and any compact set $K\subset U$, we have the estimates

$$\begin{aligned}\left|\partial_x^j V(z)\right| &\le C_{Kj0}\langle z\rangle^{\nu+j},\\ \left|\partial_x^j\partial_y^\alpha V(z)\right| &\le C_{Kj\alpha}\langle z\rangle^{\nu+j-1} \text{ for } |\alpha|\ge 1\end{aligned}$$

uniformly in $z\in\mathbb{R}^n$ with $y(z)\in K$. The condition above allows the potential to grow in any subquadratic rate in the radial direction. When differentiated in the spherical components, the growth at infinity gets to be weaker, which implies that the variation in the spherical components is slightly weakened. This is a modification of the symbol class $S\left(\langle z\rangle^\nu,\langle z\rangle^{-2}dz^2\right)$ with $\nu<2$ in Hörmander's notation [8]. Using the formulae compiled in appendix A, one can easily see that the set of functions satisfying the above condition contains $S\left(\langle z\rangle^{\nu-1},\langle z\rangle^{-2}dz^2\right)$ and is contained in $S\left(\langle z\rangle^\nu,\langle z\rangle^{-2}dz^2\right)$. We have to write the condition in the coordinates (x,y), the polar coordinates to exploit the information from the scattering metric g that is characterized in the polar coordinates.

H is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$ with respect to the inner product

$$(u,v)_{L^2\left(\mathbb{R}^n,\sqrt{g}dz\right)}=\int_{\mathbb{R}^n}u(z)\overline{v(z)}\sqrt{g(z)}dz.$$

Here note that $\sqrt{g}$ in the standard coordinates is bounded from above and below by positive constants, and thus there is a natural isomorphism $L^2\left(\mathbb{R}^n,\sqrt{g}dz\right)\cong L^2(\mathbb{R}^n)$, which will be seen later. Hence for any initial state $u_0\in L^2\left(\mathbb{R}^n,\sqrt{g}dx\right)$ we have the solution $u_t=e^{-itH}u$ to the time-dependent Schrödinger equation

$$i\frac{d}{dt}u_t=Hu_t.$$

We want to characterize the wavefront set for u_{t_0} in terms of *homogeneous wavefront set* for u_0:

Definition 1.1 *Let $u\in\mathcal{S}'(\mathbb{R}^n)$ and $(z_0,\zeta_0)\in T^*\mathbb{R}^n\cong\mathbb{R}^{2n}$. We denote*

$$(z_0,\zeta_0)\notin\mathrm{WF}(u),$$

if $\zeta_0\ne 0$ and there exists $\varphi\in C_0^\infty\left(\mathbb{R}^{2n}\right)$ such that $\varphi(z_0,\zeta_0)\ne 0$ and that

$$\|\varphi^w(z,hD_z)u(z)\|_{L^2}=O\left(h^\infty\right). \tag{1.1}$$

(1.1) is the same as

$$\left\|\int e^{i(z-w)\xi}\varphi\left(\frac{z+w}{2},h\zeta\right)u(w)dw\,đ\zeta\right\|_{L^2}=O\left(h^N\right) \text{ for any } N>0$$

with $đ\zeta:=(2\pi)^{-n}d\zeta$. The wavefront set $\mathrm{WF}(u)\subset\mathbb{R}^{2n}$ of u is the complement of the set of such (z_0,ζ_0)'s.

We also denote

$$(z_0, \zeta_0) \notin \mathrm{HWF}(u),$$

if $(z_0, \zeta_0) \neq (0,0)$ *and there exists* $\varphi \in C_0^\infty\left(\mathbb{R}^{2n}\right)$ *such that* $\varphi(z_0, \zeta_0) \neq 0$ *and that*

$$\|\varphi^w(hz, hD_z)u(z)\|_{L^2} = O\left(h^\infty\right).$$

The homogeneous wavefront set $\mathrm{HWF}(u) \subset \mathbb{R}^{2n}$ *of* u *is the complement of the set of such* (z_0, ζ_0)*'s.*

We also consider the Hamilton equation

$$\begin{aligned} \dot{z}(t; z_0, \zeta_0) &= \frac{\partial p}{\partial \zeta}(z(t; z_0, \zeta_0), \zeta(t; z_0, \zeta_0)), \\ \dot{\zeta}(t; z_0, \zeta_0) &= -\frac{\partial p}{\partial z}(z(t; z_0, \zeta_0), \zeta(t; z_0, \zeta_0)) \end{aligned} \tag{1.2}$$

with the initial value $(z(0; z_0, \zeta_0), \zeta(0; z_0, \zeta_0)) = (z_0, \zeta_0) \in T^*X^\circ$, where the Hamiltonian p is the free kinetic energy:

$$p(z, \zeta) = \frac{1}{2}\sum_{i,j=1}^n g^{ij}(z)\zeta_i\zeta_j.$$

Definition 1.2 *We say* (z_0, ζ_0) *is forward (respectively, backward) non-trapping if the solution* $(z(t; z_0, \zeta_0), \zeta(t; z_0, \zeta_0))$ *to the Hamilton equation (1.2) satisfies*

$$\lim_{t\to+\infty} |x(z(t; z_0, \zeta_0))| = 0 \ \textit{(respectively, } \lim_{t\to-\infty} |x(z(t; z_0, \zeta_0))| = 0 \ \textit{)},$$

where x *is a boundary defining function.*

If (z_0, ζ_0) is forward (respectively, backward) non-trapping, then the trajectory has a forward (respectively, backward) limit direction

$$\omega_\pm = \omega_\pm(z_0, \zeta_0) := \pm \lim_{t\to\pm\infty} \frac{z(t; z_0, \zeta_0)}{|z(t; z_0, \zeta_0)|} \text{ (respectively).}$$

The sign is adopted to indicate the direction of the momentum, not of the position.

Theorem 1.3 *Let* $u_0 \in L^2\left(\mathbb{R}^n\right)$*, and assume that* $(z_0, \zeta_0) \in T^*\mathbb{R}^n$ *is backward non-trapping. If there exists a* $t_0 > 0$ *such that* $(-t_0\omega_-, \omega_-) \notin \mathrm{HWF}(u_0)$*, then* $(z_0, \zeta_0) \notin \mathrm{WF}(u_{t_0})$*.*

The next proposition is from [15].

Proposition 1.4 *If* $u \in \mathcal{S}'\left(\mathbb{R}^n\right)$ *decays rapidly in a conic neighborhood of* $z_0 \in \mathbb{R}^n \setminus \{0\}$*, then* $(z_0, \zeta_0) \notin \mathrm{HWF}(u)$ *for any* $\zeta_0 \in \mathbb{R}^n$*.*

Theorem 1.3 and Proposition 1.4 result in

Corollary 1.5 *Let $u_0 \in L^2(\mathbb{R}^n)$ and $(z_0, \zeta_0) \in T^*\mathbb{R}^n$ be backward non-trapping. If u_0 decays rapidly in a conic neighborhood of $-\omega_-$, then $(z_0, \zeta_0) \notin \mathrm{WF}(u_t)$ for any $t > 0$.*

If the metric g is asymptotically flat, this corollary is known as microlocal smoothing property of Craig-Kappeler-Strauss for the Schrödinger equation [1].

Wunsch [21] introduced the notion of the quadratic scattering (qsc) wavefront set $\mathrm{WF}_{\mathrm{qsc}}(u)$ after Melrose [13] to study the propagation of singularities. $\mathrm{WF}_{\mathrm{qsc}}(u)$ is a subset of

$$C_{\mathrm{qsc}}X = \partial\left({}^{\mathrm{qsc}}\bar{T}^*X\right) \cong \left(\mathbb{R}^n \times S^{n-1}\right) \cup \left(S^{n-1} \times S^{n-1}\right) \cup \left(S^{n-1} \times \mathbb{R}^n\right),$$

if $X = S^n_+ \supset \mathbb{R}^n$. The intersection $\mathrm{WF}_{\mathrm{qsc}}(u) \cap \left(\mathbb{R}^n \times S^{n-1}\right)$ corresponds to $\mathrm{WF}(u)$, and $\mathrm{WF}_{\mathrm{qsc}}(u) \cap \left(S^{n-1} \times \mathbb{R}^n\right)$ is regarded as a blow-up of the scattering (sc) wavefront set in its corner, where the information on the wavefront sets of u and $\mathcal{F}u$ is mixed up. The next theorem implies that $\mathrm{WF}_{\mathrm{qsc}}(u) \cap \left(S^{n-1} \times \mathbb{R}^n\right)$ is essentially equivalent to $\mathrm{HWF}(u)$.

Theorem 1.6 *Define $\Psi : \mathbb{R}^n \setminus \{0\} \to \mathrm{GL}(n; \mathbb{R})$ by*

$$\Psi(z) = \left(\delta_{ij} + \frac{z^i z^j}{|z|^2}\right)_{ij}.$$

Then the equality

$$\begin{aligned} &\left\{(z, \Psi(z)\zeta) \in \mathbb{R}^{2n} \middle| (z, \zeta) \in \mathrm{HWF}(u) \setminus (\{0\} \times \mathbb{R}^n)\right\} \\ &= \left\{(tz, t\zeta) \in \mathbb{R}^{2n} \middle| (z, \zeta) \in \mathrm{WF}_{\mathrm{qsc}}(u) \cap \left(S^{n-1} \times \mathbb{R}^n\right), t > 0\right\} \end{aligned}$$

holds.

We can also interpret that the homogeneous wavefront set is a blow-down of the qsc wavefront set in its wavefront set part $\mathrm{WF}_{\mathrm{qsc}}(u) \cap \left(\mathbb{R}^n \times S^{n-1}\right)$.

If we note that for $t > 0$

$$(-t\omega_-, \omega_-) \in \mathrm{HWF}(u_t) \iff \left(-\omega_-, \frac{\omega_-}{2t}\right) \in \mathrm{WF}_{\mathrm{qsc}}(u_t),$$

then one of the main results in [21] follows from Theorem 1.3 under a weaker condition on the potential on the Euclidean space.

We refer to the papers [6], [7] by Hassel and Wunsch for the sophisticated results on the Schrödinger propagator on scattering manifolds. In particular [7] is very useful as an introductory paper to [6].

The homogeneous wavefront set was originally adopted by Nakamura [15] for characterizing the singularity and the growth property simultaneously. Theorem 1.3 generalizes one of the results in [15] to the scattering metric. Though the proof of the theorem is based on Nakamura's argument, the class of the pseudodifferential operators gets to be even worse and we have to use the polar coordinates for more precise estimates. We also have to prepare the pseudodifferential calculus suitable for our purpose. The proof of Theorem 1.6 is just a straightforward application of this calculus.

The microlocal smoothing property has been studied also in the analytic category [12], [17], [18], [19]. Robbiano and Zuily [17], [18], [19] used the *analytic quadratic wavefront set*, an FBI-transform-based analogue to the qsc wavefront set. On the other hand, Martinez, Nakamura and Sordoni [12] succeeded to generalize the results in [17], [18] using the *analytic homogeneous wavefront set*.

In this article the potential has the subquadratic growth in radial direction and is dealt with as a perturbation to the free Laplacian. The potential with the quadratic growth is not just a perturbation any more, which would be seen from the proof of Theorem 1.3. The case of potential with a quadratic growth are studied in [5], [22], [23], [24]. Microlocal smoothing property is completely different under the existence of the quadratic potential term. Hence our setting is almost optimal.

See works by Doi [2],[3], [4] in the case that the trajectory is trapped .

In Section 2 we study the free classical trajectories on general scattering manifolds. We will show the existence of the global solution to the Hamilton equation. In particular the non-trapped trajectory asymptotically approaches the straight line near the infinity, the boundary ∂X, and collide with a point on ∂X. The methods applied here are rather elementary.

We adjust the theory of the pseudodifferential operators for our purpose in Section 3. We consider two kinds of parameters, t, the time, and h, the semiclassical parameter. The class $S_\Omega(m)$ is defined as the set of symbols such that

$$\left|\partial_z^\alpha \partial_\zeta^\beta a(z,\zeta;t,h)\right| \leq C_{\alpha\beta} m(z,\zeta;t,h)$$

uniformly in $t \in \Omega$ and $t \in (0,1]$. The theory of the semiclassical analysis would demand the operators of the form $a(z,hD_z;t,h)$, however, then the corresponding class in our argument would be $S_{[-t_0,0]}\left(1,\langle z\rangle^{-2}dz^2+\langle z\rangle^2 d\zeta\right)$ or $S_{[-t_0,0]}\left(1,\left\langle h^{-1}t\right\rangle^{-2}dz^2+\left\langle h^{-1}t\right\rangle^2 d\zeta^2\right)$ and the theory doesn't work. We also need the operators of such form as $a(hz,hD_z;t,h)$ and their compositions with other classes. In this point of view we give up the composition formula from the general theory and check each time if the calculus work, that is, we do not present the asymptotic expansion formula for the composition in *strict* classes, but demonstrate a simple method to check the asymptotic expansion in rather *loose* classes. We also give the inequality of Gårding type.

In Section 4 we give the proof of Theorem 1.3. Observing the equality

$$\begin{aligned}
&\langle F(0,h)u_{t_0},u_{t_0}\rangle_{L^2\left(\mathbb{R}^n,\sqrt{g}dz\right)}\\
&=\langle F(-t_0,h)u_0,u_0\rangle_{L^2\left(\mathbb{R}^n,\sqrt{g}dz\right)}+\int_{-t_0}^0 \langle \delta F(t,h)u_t,u_t\rangle_{L^2\left(\mathbb{R}^n,\sqrt{g}dz\right)}\,dt,
\end{aligned}\tag{1.3}$$

where

$$\delta F(t,h)=\frac{\partial}{\partial t}F(t,h)+i\left[H,F(t,h)\right],$$

we will construct the symbol $\varphi(z,\zeta;t,h)$ of $F(t,h)$ with appropriate properties. The support $\operatorname{supp}\varphi(\cdot,\cdot;t,h)$ as $h\to 0$ moves towards some direction in (z,ζ)-space, so that the left-hand side and the fist term of right-hand side in (1.3) get to be of the form in Definition 1.1. We also require that the Heisenberg derivative δF is almost non-positive in semiclassical sense, which corresponds to that the Lagrange derivative $\frac{D}{Dt}\varphi$ is non-positive.

Theorem 1.6 is proved in Section 5. Here the calculus in Section 3 works well.

We often use the coordinate change between the standard coordinates z and the polar coodinates (x, y) on the Euclidean space in Section 4. We gather the formulae in Appendix A.

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2 Classical Flows

In this section we study properties of classical trajectories on general scattering manifolds. Let X be a compact manifold with boundary and x a boundary defining function. Given a scattering metric g, we can write near the boundary ∂X

$$\begin{aligned} g &= \frac{dx^2}{x^4} + \frac{h(x,y,dx,dy)}{x^2} \\ &= \frac{dx^2}{x^4} + \frac{1}{x^2}\left[h_0 dx^2 + \sum_{i=1}^{n-1} h_i \left(dx dy^i + dy^i dx\right) + \sum_{i,j=1}^{n-1} h_{ij} dy^i dy^j\right] \end{aligned}$$

with y local coordinates of the boundary ∂X and h_i, h_{ij} depending smoothly on (x, y). Since the inverse matrix to

$$(g_{ij}) = \frac{1}{x^2}\begin{pmatrix} x^{-2} + h_0 & {}^t\left(h_i\right)_{i\geq 1} \\ \left(h_i\right)_{i\geq 1} & \left(h_{ij}\right) \end{pmatrix} \tag{2.1}$$

is given using the Cramer's formula by

$$x^2\begin{pmatrix} x^2 + x^4\varphi & {}^t\left(x^2\varphi^i\right) \\ \left(x^2\varphi^i\right) & \left(h^{ij} + x^2\varphi^{ij}\right) \end{pmatrix}, \tag{2.2}$$

where φ, φ^i, φ^{ij} depend smoothly on (x, y) and $\left(h^{ij}\right) = \left(h_{ij}\right)^{-1}$, we can write the Hamiltonian function p on T^*X° in the form

$$\begin{aligned} &p(x, y, \xi, \eta) \\ &= \frac{1}{2}\left[\left(x^4 + x^6\varphi\right)\xi^2 + 2x^4\xi\sum_{i=1}^{n-1}\varphi^i\eta_i + \sum_{i,j=1}^{n-1}\left(x^2h^{ij} + x^4\varphi^{ij}\right)\eta_i\eta_j\right]. \end{aligned} \tag{2.3}$$

Then the Hamilton equation (1.2) near the boundary is

$$\dot{x} = \left(x^4 + x^6\varphi\right)\xi + x^4\sum_{i=1}^{n-1}\varphi^i\eta_i, \tag{2.4}$$

$$\dot{y}^i = x^4\xi\varphi^i + \sum_{j=1}^{n-1}\left(x^2h^{ij} + x^4\varphi^{ij}\right)\eta_j, \tag{2.5}$$

$$\begin{aligned}\dot{\xi} =& -\frac{1}{2}\left(4x^3 + 6x^5 + x^6\frac{\partial\varphi}{\partial x}\right)\xi^2 - x^3\xi\sum_{i=1}^{n-1}\left(4\varphi^i + x\frac{\partial\varphi}{\partial x}\right)\eta_i \\ &- \frac{1}{2}x\sum_{i,j=1}^{n-1}\left(2h^{ij} + x\frac{\partial h^{ij}}{\partial x} + 4x^2\varphi^{ij} + x^3\frac{\partial\varphi^{ij}}{\partial x}\right)\eta_i\eta_j,\end{aligned} \tag{2.6}$$

$$\dot{\eta}_i = -\frac{1}{2}x^6\frac{\partial\varphi}{\partial y^i}\xi^2 - x^4\xi\sum_{j=1}^{n-1}\frac{\partial\varphi^j}{\partial y^i}\eta_j - \frac{1}{2}\sum_{j,k=1}^{n-1}\left(x^2\frac{\partial h^{jk}}{\partial y^i} + x^4\frac{\partial\varphi^{jk}}{\partial y^i}\right)\eta_j\eta_k. \tag{2.7}$$

Here the variables are omitted, that is, $x = x(t)$, $h^{ij} = h^{ij}(x(t), y(t))$ and etc. Take the initial value in T^*X°, then we can show that the solution exists globally for $t \in \mathbb{R}$. The existence of the local solution and its uniqueness is clear from the general theory of the differential equations. Before going to the proof of the existence of the global solution, we prepare a lemma, which is valid for any metric on an open manifold.

One notes that the Hamiltonian preserves along the flow:

$$p(z(t), \zeta(t)) = p(x(t), y(t), \xi(t), \eta(t)) \equiv p_0 = \text{const.}$$

Lemma 2.1 *Let $(z(t), \zeta(t))$ be a solution to (1.2) which is defined on an interval $(\alpha, \beta) \subset \mathbb{R}$ and does not extend out of this interval any more. If $(\alpha, \beta) \neq \mathbb{R}$, then the trajectory escapes from any compact sets in X° in finite time, i.e., for any compact set $K \subset X^\circ$ there exists $c \in (\alpha, \beta)$ such that*

$$z(t) \notin K, \quad c < \forall t < \beta \quad \text{or} \quad a < \forall t < c,$$

when $\beta < +\infty$ or $\alpha > -\infty$ respectively.

Proof. If $(z(t), \zeta(t))$ satisfies (1.2), so does $(z(-t), -\zeta(-t))$, and thus the situation reduces to the case $\beta < +\infty$. Note that the trajectory is bound to the equienergy surface

$$S_{p_0} := \{(z, \zeta) \in T^*X^\circ \,|\, p(z, \zeta) = p_0\}.$$

Thus the Hamilton vector field

$$H_p := \frac{\partial p}{\partial \zeta}\frac{\partial}{\partial x} - \frac{\partial p}{\partial x}\frac{\partial}{\partial \zeta} \in \Gamma\left(T^*X^\circ; T\left(T^*X^\circ\right)\right),$$

is tangent to S_{p_0}, which implies that H_p is regarded as a flow on S_{p_0}.

Assume the opposite of the conclusion, that is, there exists a compact set $K \subset X^\circ$ and a sequence $\{t_n\} \subset (\alpha, \beta)$ such that

$$z(t_n) \in K \text{ and } \lim_{n\to\infty} t_n = \beta.$$

Consider a compact set

$$S_{p_0}\big|_K = \{(z,\zeta) \in S_{p_0} \mid z \in K\},$$

a restriction of a sphere bundle on X° to K. Using the compactness, we can cover $S_{p_0}\big|_K$ with a finite covering

$$\mathfrak{U} = \left\{ \left(U_j, \{\varphi_j(t,\cdot)\}_{|t|<\varepsilon_j} \right) \right\}_{j=1}^{N}, \quad S_{p_0}\big|_K \subset \bigcup_{j=1}^{N} U_j,$$

where $\mathfrak{U}$ is so-called a local 1-parameter group of local transformations. In other words, for each $|t| < \varepsilon_j$,

$$\varphi_j(t,\cdot) : U_j \to S_{p_0}$$

is a diffeomorphism from $U_j \subset S_{p_0}$ into S_{p_0}, and, for each $z \in U_j$,

$$\varphi_j(\cdot, z) : (-\varepsilon_j, \varepsilon_j) \to S_{p_0}$$

is a curve along H_p with $\varphi_j(0,z) = z$. Then, putting $\varepsilon = \min_j \varepsilon_j > 0$, for any $(z,\zeta) \in S_{p_0}\big|_K$, we can solve the Hamilton equation with an initial value (z,ζ), at least, for time $t \in (-\varepsilon, \varepsilon)$. Now take t_n such that $\beta - t_n < \varepsilon/2$, then the trajectory can be extended out of (α, β), which contradicts the assumption. $\square$

Proposition 2.2 *Let X be a scattering manifold and $(z_0, \zeta_0) \in T^*X^\circ$. Then the Hamilton equation (1.2) has a unique solution $(z(t; z_0, \zeta_0), \zeta(t; z_0, \zeta_0))$ defined for all $t \in (-\infty, +\infty)$.*

Proof. We assume that the solution $(z(t), \zeta(t))$ is defined on $(\alpha, \beta) \subset \mathbb{R}$ with $\beta < +\infty$ and does not extend out of this interval. We derive a contradiction by computing

$$L = \lim_{\varepsilon \downarrow 0} \int_0^{\beta-\varepsilon} \sqrt{g\left(\dot{z}(t), \dot{z}(t)\right)} dt,$$

the length of the trajectory in two ways. We can assume that the solution is written in the form $(x(t), y(t), \xi(t), \eta(t))$ by translating t and exchanging the initial value. (The coordinates (x, y) might not be defined far from the boundary.) Then in view of Lemma 2.1 $x(t)$ satisfies

$$\lim_{t \uparrow \beta} x(t) = 0.$$

Since

$$
\begin{aligned}
2p_0 &= \left(1 - x + x^2\varphi\right) x^4 \xi^2 + \sum_{i,j=1}^{n-1} \left(h^{ij} - x\varphi^i\varphi^j + x^2\varphi^{ij}\right) x^2 \eta_i \eta_j \\
&\quad + x^3 \left(x\xi + \sum_{i=1}^{n-1} \varphi^i \eta_i\right)^2 \qquad (2.8) \\
&\geq \left(1 - x + x^2\varphi\right) x^4 \xi^2 + \sum_{i,j=1}^{n-1} \left(h^{ij} - x\varphi^i\varphi^j + x^2\varphi^{ij}\right) x^2 \eta_i \eta_j,
\end{aligned}
$$

considering orders of x, we obtain the estimates

$$
\left|x(t)^2 \xi(t)\right| < C, \quad |x(t)\eta(t)| < C \qquad (2.9)
$$

for $t \in [0, \beta)$ with a large constant C. Here we used the facts that φ's are smooth for $x \geq 0$ and that h^{ij} is positive definite near the boundary. We also refer to that, though the coordinates y might not remain the same for all t, the estimate is valid since we can cover the boundary with a finite number of charts. This kind of argument will be used below without mentioned. Then it follows from (2.4) and (2.5) that

$$
|\dot{x}| < Cx^2, \quad \left|\dot{y}^i\right| < Cx,
$$

and we have

$$
\begin{aligned}
\sqrt{g\left(\dot{z}(t), \dot{z}(t)\right)} &= \left(\frac{\dot{x}^2}{x^4} + \frac{1}{x^2}\left[h_0 \dot{x}^2 + \sum_{i=1}^{n-1} h_i \left(\dot{x}\dot{y}^i + \dot{y}^i \dot{x}\right) + \sum_{i,j=1}^{n-1} h_{ij} \dot{y}^i \dot{y}^j\right]\right)^{\frac{1}{2}} \\
&\geq \frac{|\dot{x}|}{x^2} - \frac{1}{x}\left|h_0 \dot{x}^2 + \sum_{i=1}^{n-1} h_i \left(\dot{x}\dot{y}^i + \dot{y}^i \dot{x}\right) + \sum_{i,j=1}^{n-1} h_{ij} \dot{y}^i \dot{y}^j\right|^{\frac{1}{2}} \\
&\geq \frac{|\dot{x}|}{x^2} - C'.
\end{aligned}
$$

Then

$$
\begin{aligned}
L &\geq \lim_{\varepsilon \downarrow 0} \left|\int_0^{\beta - \varepsilon} \frac{\dot{x}}{x^2} dt\right| - C'\beta \\
&= \lim_{\varepsilon \downarrow 0} \left|\int_{x(0)}^{x(\beta - \varepsilon)} \frac{1}{x^2} dx\right| - C'\beta \\
&= \lim_{\varepsilon \downarrow 0} \frac{1}{x(\beta - \varepsilon)} - \frac{1}{x(0)} - C'\beta \\
&= +\infty.
\end{aligned}
$$

On the other hand, since

$$\begin{aligned} g\left(\dot{z}(t), \dot{z}(t)\right) &= \sum_{i,j} g_{ij}(z(t))\dot{z}^i(t)\dot{z}^j(t) \\ &= \sum_{i,j} g_{ij}(z(t)) \sum_k g^{ik}(z(t))\zeta_k(t) \sum_l g^{jl}(z(t))\zeta_l(t) \\ &= 2p_0, \end{aligned}$$

we have

$$L = \sqrt{2p_0}\beta.$$

These are the contradiction and the proof is completed. □

Provided that the initial value is non-trapping, the more information about the trajectory can be extracted. It suffices to study the only forward non-trapping case, since, if (z_0, ζ_0) is backward non-trapping, then $(z_0, -\zeta_0)$ is forward non-trapping. Thus we assume (z_0, ζ_0) is forward non-trapping. Taking $T_0 > 0$ large enough, the equations (2.4-2.7) with respect to the coordinates (x, y) are valid for $t \geq T_0$.We obtain from (2.8) the estimates (2.9) for $t \geq 0$. It follows from the equations (2.4), (2.6), (2.7) and the estimates (2.9) that

$$\begin{aligned} \dot{x} =& x^4\xi + O\left(x^3\right), \\ \ddot{x} =& \left[4x^3\dot{x} + 6x^5\dot{x}\varphi + x^6\left(\dot{x}\partial_x\varphi + \dot{y}\partial_y\varphi\right)\right]\xi + \left(x^4 + x^6\varphi\right)\dot{\xi} \\ &+ 4x^3\dot{x}\sum \varphi^i\eta_i + x^4\sum\left[\left(\dot{x}\partial_x\varphi^i + \dot{y}\partial_y\varphi^i\right)\eta_i + \varphi^i\dot{\eta}_i\right] \\ =& \left[4x^3\left(x^4\xi + O\left(x^3\right)\right) + 6x^5 O\left(x^2\right)\varphi + x^6\left(O\left(x^2\right)\partial_x\varphi + O\left(x\right)\partial_y\varphi\right)\right]\xi \\ &+ \left(x^4 + x^6\varphi\right)\left[-2x^3\xi^2 - x\sum h^{ij}\eta_i\eta_j + O\left(1\right)\right] \\ &+ 4x^3 O\left(x^2\right)\sum\varphi^i\eta_i + x^4\sum\left[\left(O\left(x^2\right)\partial_x\varphi + O\left(x\right)\partial_y\varphi\right)\eta_i + \varphi^i O(1)\right] \\ =& 2x^7\xi^2 - x^5\sum h^{ij}\eta_i\eta_j + O\left(x^4\right), \end{aligned}$$

where $O\left(x^N\right)$ is a C^∞ function in $t \in \mathbb{R}$ whose absolute value is estimated from above by some constant times x^N when $t \geq 0$. Hence

$$\begin{aligned} \frac{d^2}{dt^2}\left(\frac{1}{x^2}\right) &= 6\frac{\dot{x}^2}{x^4} - 2\frac{\ddot{x}}{x^3} \\ &= 2x^4\xi^2 + 2x^2\sum h^{ij}\eta_i\eta_j + O(x) \\ &= 4p_0 + O(x). \end{aligned} \tag{2.10}$$

Taking T_0 larger if necessary, we have

$$3p_0 \leq \frac{d^2}{dt^2}\left(\frac{1}{x^2}\right) \leq 5p_0 \quad \text{for} \quad t \geq T_0$$

and so,

$$\frac{3}{2}p_0\left(t-T_0\right)^2 \le \frac{1}{x^2}+Ct+C' \le \frac{5}{2}p_0\left(t-T_0\right)^2.$$

Therefore, by taking T_0 larger if necessary, there is some $C>0$ such that

$$C^{-1}t^2 \le \frac{1}{x^2} \le Ct^2 \quad \text{for} \quad t \ge T_0$$

and thus

$$C^{-1}t^{-2} \le x^2 \le Ct^{-2} \quad \text{for} \quad t \ge T_0.$$

Considering this estimate, (2.10) becomes

$$\frac{d^2}{dt^2}\left(\frac{1}{x^2}\right) = 4p_0 + O\left(t^{-1}\right),$$

where $O\left(t^{-1}\right)$ is a C^∞ function in $t \in \mathbb{R}$ and its absolute value is estimated from above by a constant times t^{-1} for $t \ge T_0$. Thus we have

$$\frac{1}{x^2} = 2p_0t^2 + O\left(t\log t\right).$$

Moreover, using (2.3), (2.4) and (2.6), we have

$$\begin{aligned}\frac{d}{dt}\left(x\xi\right) &= -x^4\xi^2 - x^2\sum h^{ij}\eta_i\eta_j + O(x)\\ &= -2p_0 + O\left(t^{-1}\right),\end{aligned}$$

and thus

$$\begin{aligned}\xi &= \frac{1}{x}\left(-2p_0t + O(\log t)\right)\\ &= -\left(2p_0\right)^{\frac{3}{2}}t^2 + O(t\log t).\end{aligned}$$

We obtain

$$x^2\xi = -\sqrt{2p_0} + O\left(\frac{\log t}{t}\right).$$

Combining this with the inequality (2.8), we have

$$x\eta = O\left(\frac{\log t}{t}\right), \tag{2.11}$$

and then, from (2.5) and (2.7) the estimates

$$\dot{y}^i = O\left(\frac{\log t}{t^2}\right), \quad \dot{\eta}^i = O\left(\left(\frac{\log t}{t}\right)^2\right)$$

are obtained. In particular, with an appropriate choice of the coordinates y, the limits

$$y_+^i = \lim_{t\to\infty} y^i(t), \quad \eta_+^i = \lim_{t\to\infty} \eta^i(t)$$

exist. Note that we can improve the estimates, for example, (2.11) is improved to be

$$x\eta = O\left(t^{-1}\right)$$

by the fact that η is bounded. We don't do so, however, since we won't need it later.

Proposition 2.3 *If $(z_0, \zeta_0) \in T^*X^\circ$ is forward non-trapping,*

$$x(t; z_0, \zeta_0) = \left(2p_0 t^2 + O\left(t \log t\right)\right)^{-\frac{1}{2}},$$
$$\xi(t; z_0, \zeta_0) = -\left(2p_0\right)^{\frac{3}{2}} t^2 + O(t \log t)$$

as $t \to +\infty$. Moreover

$$z_+(z_0, \zeta_0) := \lim_{t\to+\infty} z(t; z_0, \zeta_0) \in \partial X$$

exists, and, with an appropriate choice of coordinates y of the boundary,

$$y_+(z_0, \zeta_0) := \lim_{t\to+\infty} y(t; z_0, \zeta_0), \quad \eta_+(z_0, \zeta_0) := \lim_{t\to+\infty} \eta(t; z_0, \zeta_0)$$

exist.

We now apply the above results to the backward non-trapped trajectory on the Euclidean space $\mathbb{R}^n \subset X = S_+^n$ with a scattering metric. If (z_0, ζ_0) is backward non-trapping, then $(z_0, -\zeta_0)$ is forward non-trapping and

$$\left(z\left(t; z_0, \zeta_0\right), \zeta\left(t; z_0, \zeta_0\right)\right) = \left(z(-t; z_0, -\zeta), -\zeta\left(-t; z_0, -\zeta_0\right)\right)$$

hold. Since

$$z_-(z_0, \zeta_0) := \lim_{t\to-\infty} z\left(t; z_0, \zeta_0\right) = z_+(z_0, -\zeta)$$

exists as a point on ∂X, taking T_0 large enough and exchanging the standard coordinate axes if necessary, we can assume $z^n(t; z_0, \zeta_0) \geq \varepsilon > 0$ for all $t < -T_0$. This in particular allow us to take the coordinates $(x, y_{(+n)})$, which is defined in Appendix A, near the trajectory for $t < -T_0$. This choice of the coordinates will be used in Section 4. As

$$y_-(z_0, \zeta_0) := \lim_{t\to-\infty} y(t; z_0, \zeta_0), \quad \eta_-(z_0, \zeta_0) := \lim_{t\to-\infty} \eta(t; z_0, \zeta_0)$$

exists, $\omega_-(z_0,\zeta_0)$ exists. Moreover, since we have

$$x(t;z_0,\zeta_0) = \left(2p_0t^2 + O\left(|t|\log|t|\right)\right)^{-\frac{1}{2}},$$
$$\xi(t;z_0,\zeta_0) = (2p_0)^{\frac{3}{2}}\, t^2 + O\left(|t|\log|t|\right)$$

as $t \to -\infty$, it follows, using the formula (A.7), that

$$\zeta_-(z_0,\zeta_0) := \lim_{t\to-\infty} \zeta(t;z_0,\zeta_0) = \sqrt{2p_0}\,\omega_-(z_0,\zeta_0).$$

Thus $(-t_0\omega_-,\omega_-)$ in Theorem 1.3 can be replaced by $(-t_0\zeta_-,\zeta_-)$.

3 Pseudodifferential Calculus

For the proof of Theorem 1.3 and 1.6 we need the pseudodifferential calculus. The symbol classes we consider here are $S_\Omega(m)$ following Martinez [11], and $S_\Omega(\tilde{m},\tilde{g})$ following Hörmander [8], both added the parameters $t \in \Omega$ and $h \in (0,1]$. We use the class $S_\Omega(m)$ as theoretical foundation.

Definition 3.1 *Let Ω be the set of ordinary parameters. Here 'ordinary' means that we also consider the distinguished parameter $h \in (0,1]$, what is called semi-classical parameter other than $t \in \Omega$. A positive measurable function $m(\cdot,\cdot;t,h)$ on $\mathbb{R}^{2n}$ parameterized by $(t,h) \in \Omega \times (0,1]$ is an order function if there are constants $N_0 \in \mathbb{R}$ and $C_0 > 0$ such that*

$$m(z,\zeta;t,h) \le C_0 \left\langle z - z_1; \zeta - \zeta_1\right\rangle^{N_0} m(z_1,\zeta_1;t,h) \tag{3.1}$$

uniformly in $(z,\zeta),(z_1,\zeta_1) \in \mathbb{R}^{2n}$ and $(t,h) \in \Omega \times (0,1]$, where

$$\langle z;\zeta\rangle := \left(1 + |z|^2 + |\zeta|^2\right)^{\frac{1}{2}}.$$

A C^∞ function $a(\cdot,\cdot;t,h)$ on $\mathbb{R}^{2n}$ parameterized by $(t,h) \in \Omega \times (0,1]$ is in the symbol class $S_\Omega(m)$ if and only if for any $\alpha,\beta \in \mathbb{Z}^n_+$ there exists a constant $C_{\alpha\beta}$ such that

$$\left|\partial_z^\alpha \partial_\zeta^\beta a(z,\zeta;t,h)\right| \le C_{\alpha\beta} m(z,\zeta;t,h)$$

uniformly in $(z,\zeta,t,h) \in \mathbb{R}^{2n} \times \Omega \times (0,1]$.

The semiclassical parameter h is important when it is small, and so the symbol may not be defined for h near 1.

The class $S_\Omega(\tilde{m},\tilde{g})$ is defined following Hörmander, where the weight function $\tilde{m}$ and the metric $\tilde{g}$ depend on parameters with the uniformity in them assumed. We use $S_\Omega(\tilde{m},\tilde{g})$ just for the notational simplification and the precise definition is not needed.

There will often appear the class $S_{[-t_0,0]}(m,\tilde{g}_1)$, where m does not depend on $(z,\zeta) \in \mathbb{R}^{2n}$ and $\tilde{g}_1$ is given by

$$\tilde{g}_1 = \left\langle h^{-1}t\right\rangle^{-2} dz^2 + h^2 \left\langle h^{-1}t\right\rangle^2 d\zeta^2.$$

Note that, since $h\left\langle h^{-1}t\right\rangle \leq \langle t_0\rangle$, we have $S_{[-t_0,0]}\left(m,\tilde{g}_1\right) \subset S_{[-t_0,0]}\left(m\right)$.

If $a, a_j \in S_\Omega(m)$, $j = 0, 1, \ldots$, satisfies

$$a - \sum_{j=0}^{N} a_j \in S_\Omega\left(h^{k_{N+1}}m\right) \quad \text{for} \quad N = 0, 1, \ldots$$

with

$$\lim_{N\to\infty} k_N = +\infty,$$

we write

$$a \sim \sum_{j=0}^{\infty} a_j.$$

If one defines

$$S_\Omega\left(h^\infty m\right) = \bigcap_{N\geq 0} S_\Omega\left(h^N m\right),$$

it is easy to verify the asymptotic sum determines a modulo $S_\Omega\left(h^\infty m\right)$, that is, if $a', a_j \in S_\Omega(m)$, $j = 0, 1, \ldots$, satisfies the same condition as above, then

$$a - a' \in S_\Omega\left(h^\infty m\right).$$

Let $a \in S_\Omega(m)$ and $K_{t,h} \subset \mathbb{R}^{2n}$. If there is an $a' \in S_\Omega(m)$ such that

$$\operatorname{supp} a'\left(\cdot,\cdot;t,h\right) \subset K_{t,h} \quad \text{and} \quad a - a' \in S_\Omega\left(h^\infty m\right),$$

we say a is supported in $K_{t,h}$ modulo $S_\Omega\left(h^\infty m\right)$.

Proposition 3.2 *Suppose $a_j \in S_\Omega\left(h^{k_j}m\right)$, $j = 0, 1, 2, \ldots$, and $k_j \to +\infty$ as $j \to +\infty$, then there exists $a \in S_\Omega\left(m\right)$ such that*

$$a \sim \sum_{j=0}^{\infty} a_j.$$

In particular, if all $a_j\left(\cdot,\cdot;t,h\right)$ are supported in $K_{t,h} \subset \mathbb{R}^{2n}$, we can choose $a\left(\cdot,\cdot;t,h\right)$ with the support in $K_{t,h}$.

Proof. Replacing a_j by

$$a'_j = \sum_{j\leq k_l<j+1} a_l \in S_\Omega\left(h^j m\right),$$

we can assume $k_j = j$. Take a cut-off function $\chi \in C_0^\infty(\mathbb{R})$ with

$$\chi(\lambda) = \begin{cases} 1, & \text{if } |\lambda| \leq 1, \\ 0, & \text{if } |\lambda| \geq 2, \end{cases}$$

and increasing constants C_j such that

$$\sup_{\substack{|\alpha|+|\beta|\leq j \\ (z,\zeta,t,h)\in\mathbb{R}^{2n}\times\Omega\times(0,1]}} h^{-j}m(z,\zeta;t,h)^{-1}\left|\partial_z^\alpha\partial_\zeta^\beta a_j(z,\zeta;t,h)\right| \leq C_j$$

and

$$\lim_{j\to\infty} C_j = +\infty.$$

Then for $|\alpha|+|\beta|\leq j$ and $(z,\zeta,t,h)\in\mathbb{R}^{2n}\times\Omega\times(0,1]$, we have

$$\begin{aligned} h\left(1-\chi\left(\frac{1}{C_j h}\right)\right)\left|\partial_z^\alpha\partial_\zeta^\beta a_j(z,\zeta;t,h)\right| &\leq \frac{1}{C_j}C_j h^j m(z,\zeta;t,h) \\ &\leq h^j m(z,\zeta;t,h), \end{aligned}$$

from which it follows that

$$\left(1-\chi\left(\frac{1}{C_j h}\right)\right)\left|\partial_z^\alpha\partial_\zeta^\beta a_j(z,\zeta;t,h)\right| \leq h^{j-1}m(z,\zeta;t,h).$$

We now define

$$a(z,\zeta;t,h) = \sum_{j=0}^{\infty}\left(1-\chi\left(\frac{1}{C_j h}\right)\right)a_j(z,\zeta;t,h),$$

and check this a has the properties of the proposition.

First note that the series converges and $a(\cdot,\cdot;t,h)$ is in $C^\infty(\mathbb{R}^{2n})$ since for each $h\in(0,1]$ the sum is the finite sum. In particular, if all $a_j(\cdot,\cdot;t,h)$ are supported in $K_{t,h}$, so is $a(\cdot,\cdot;t,h)$. For any $\alpha,\beta\in\mathbb{Z}_+^n$, if we take $j_0=|\alpha|+|\beta|$, we have

$$\begin{aligned} \left|\partial_z^\alpha\partial_\zeta^\beta a(z,\zeta;t,h)\right| &\leq \left|\sum_{j=0}^{j_0-1}\left(1-\chi\left(\frac{1}{C_j h}\right)\right)\partial_z^\alpha\partial_\zeta^\beta a_j(z,\zeta;t,h)\right| \\ &\quad + \left|\sum_{j=j_0}^{\infty}\left(1-\chi\left(\frac{1}{C_j h}\right)\right)\partial_z^\alpha\partial_\zeta^\beta a_j(z,\zeta;t,h)\right| \\ &\leq \sum_{j=0}^{j_0-1}\left|\partial_z^\alpha\partial_\zeta^\beta a_j(z,\zeta;t,h)\right| + \sum_{j=j_0}^{\infty} h^{j-1}m(z,\zeta;t,h) \end{aligned}$$

uniformly in $(z,\zeta,t,h)\in\mathbb{R}^{2n}\times\Omega\times(0,1]$. Therefore $a\in S_\Omega(m)$.

Take any $k \geq 0$ and $\alpha, \beta \in \mathbb{Z}_+^n$ and set $j_0 = \max\{k+1, |\alpha| + |\beta|\}$. Then

$$
\begin{aligned}
&\left| \partial_z^\alpha \partial_\zeta^\beta \left(a(z,\zeta;t,h) - \sum_{j=0}^{k} a_j(z,\zeta;t,h) \right) \right| \\
\leq &\left| \sum_{j=0}^{k} \chi\left(\frac{1}{C_j h}\right) \partial_z^\alpha \partial_\zeta^\beta a_j(z,\zeta;t,h) \right| \\
&+ \left| \sum_{j=k+1}^{j_0} \left(1 - \chi\left(\frac{1}{C_j h}\right)\right) \partial_z^\alpha \partial_\zeta^\beta a_j(z,\zeta;t,h) \right| \\
&+ \left| \sum_{j=j_0+1}^{\infty} \left(1 - \chi\left(\frac{1}{C_j h}\right)\right) \partial_z^\alpha \partial_\zeta^\beta a_j(z,\zeta;t,h) \right| \\
\leq &\left| \sum_{j=0}^{k} \chi\left(\frac{1}{C_j h}\right) \partial_z^\alpha \partial_\zeta^\beta a_j(z,\zeta;t,h) \right| \\
&+ \left| \sum_{j=k+1}^{j_0} \left(1 - \chi\left(\frac{1}{C_j h}\right)\right) \partial_z^\alpha \partial_\zeta^\beta a_j(z,\zeta;t,h) \right| + \sum_{j=j_0+1}^{\infty} h^{j-1} m(z,\zeta;t,h).
\end{aligned}
$$

In the last formula the first term is zero for small h, and the second and the third terms are estimated from above by a constant times $h^{k+1} m(z,\zeta;t,h)$. The proof is completed. $\square$

Let g be a scattering metric on $X = S_+^n \supset \mathbb{R}^n$ and denote $\det g$ by g.

Definition 3.3 *Let $a \in S_\Omega(m)$ and $u \in \mathcal{S}(\mathbb{R}^n)$. We define the pseudodifferential operators*

$$
a^w(z, D_z; t, h) u(z) = \int e^{i(z-w)\zeta} a\left(\frac{z+w}{2}, \zeta; t, h\right) u(w) dw đ\zeta, \tag{3.2}
$$

$$
a^{gw}(z, D_z; t, h) u(z) = \frac{1}{\sqrt[4]{g(z)}} a^w(z, D_z; t, h) \sqrt[4]{g(z)} u(z). \tag{3.3}
$$

If an operator $A(t,h)$ with parameters $(t,h) \in \Omega \times (0,1]$ can be written in the form (3.2) with some $a \in S_\Omega(m)$, we say a is the Weyl symbol of $A(t,h)$ and denote it by $\sigma^w(A)$. Similarly if $A(t,h)$ is of the form (3.3), we say a is the g-Weyl symbol of $A(t,h)$ and denote it by $\sigma^{gw}(A)$.

Proposition 3.4 *Let $a \in S_\Omega(m)$. $a^{gw}(z, D_z; t, h)$ defines a continuous operator*

$$
a^{gw}(z, D_z; t, h) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)
$$

for each $(t,h) \in \Omega \times (0,1]$.

Proof. First we prove the multiplications by $\sqrt[4]{g}$ and $\frac{1}{\sqrt[4]{g}}$ define continuous operators $\mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$. It is sufficient to show that the derivatives of g are bounded, and that g is estimated from below by a positive constant . Using the expression (2.1) we can write

$$\sqrt{g(z)}|dz| = \frac{1}{x^{n+1}}\sqrt{\det\left(h_{ij}\left(x,y\right)\right) + xr_1\left(x,y\right)}|dxdy|$$

with r_1 depending smoothly on (x, y). Thus we have from the formulae (A.5) and (A.6)

$$\begin{aligned} g(z) &= \Big[\det\left(h_{ij}\left(x,y\right)\right) + xr_1\left(x,y\right)\Big] r_2\left(y\right) \\ &= \left[\det\left(h_{ij}\left(\frac{1}{|z|},\frac{z}{|z|}\right)\right) + \frac{1}{|z|}r_1\left(\frac{1}{|z|},\frac{z}{|z|}\right)\right] r_2\left(\frac{z}{|z|}\right), \end{aligned} \tag{3.4}$$

where r_2 is smooth in y. Then we easily see the properties we want. Now we have only to see the continuity of $a^w(z, D_z; t, h)$. We denote the seminorms on $\mathcal{S}(\mathbb{R}^n)$ by

$$|u|_k := \sup_{\substack{z\in\mathbb{R}^n,\\ l+|\alpha|\le k}} \left|\langle z\rangle^l \partial_z^\alpha u(z)\right|.$$

For any $l \geq 0$ and $\alpha \in \mathbb{Z}_+^n$, we have

$$\begin{aligned} &\left|\langle z\rangle^l\partial_z^\alpha a^w(z,D_z;t,h)u(z)\right| \\ &= \left|\langle z\rangle^l\partial_z^\alpha\int e^{i(z-w)\zeta}\langle\zeta\rangle^{-N}\langle D_w\rangle^N\left[a\left(\frac{z+w}{2},\zeta;t,h\right)u(w)\right]dwđ\zeta\right| \\ &= \left|\int e^{i(z-w)\zeta}\langle D_\zeta\rangle^l\partial_w^\alpha\langle D_w\rangle^N\left[\langle\zeta\rangle^{-N}a\left(\frac{z+w}{2},\zeta;t,h\right)u(w)\right]dwđ\zeta\right| \\ &\le C\,|u|_{N+|\alpha|+N_0+n+1}\int\langle\zeta\rangle^{-N}m\left(\frac{z+w}{2},\zeta;t,h\right)\langle w\rangle^{-N_0-n-1}dwđ\zeta \\ &\le C\,|u|_{N+|\alpha|+N_0+n+1}\,m\left(0,0;t,h\right)\langle z\rangle^{N_0}\int\langle\zeta\rangle^{N_0-N}\langle w\rangle^{-n-1}dwđ\zeta \\ &\le Cm\left(0,0;t,h\right)|u|_{N+|\alpha|+N_0+n+1}\,\langle z\rangle^{N_0}, \end{aligned}$$

where N is any number larger than $n + N_0$, and C, which might be different from line to line, is a constant depending only on l, α and N. Then we have

$$\left|\langle z\rangle^{l-N_0}\partial_z^\alpha a^w(z,D_z;t,h)u(z)\right| \le Cm\left(0,0;t,h\right)|u|_{N+|\alpha|+N_0+n+1}$$

for any $l \geq 0$ and $\alpha \in \mathbb{Z}_+^n$, which concludes the proof. $\square$

Proposition 3.5 *The formal adjoint operator to $a^{gw}(z, D_z; t, h)$ on $\mathcal{S}(\mathbb{R}^n)$ with respect to the inner product $(\cdot,\cdot)_{L^2\left(\mathbb{R}^n,\sqrt{g}dz\right)}$ is given by $\bar{a}^{gw}(z, D_z; t, h)$, that is,*

$$\langle a^{gw}(z,D_z;t,h)u(z),v(z)\rangle_{L^2\left(\mathbb{R}^n,\sqrt{g}dz\right)} = \langle u(z),\bar{a}^w(z,D_z;t,h)v(z)\rangle_{L^2\left(\mathbb{R}^n,\sqrt{g}dz\right)}$$

for all $u, v \in \mathcal{S}(\mathbb{R}^n)$, where $\bar{a}(z,\zeta;t,h) = \overline{a(z,\zeta;t,h)}$. In particular, if a is real-valued, then $a^{gw}(z, D_z; t, h)$ is formally self-adjoint.

Proof. Obvious. □

Proposition 3.6 *Let $a \in S_\Omega(m)$ and $b \in S_\Omega(m')$. Then the composite operator*

$$a^{gw}(z, D_z; t, h) \circ b^{gw}(z, D_z; t, h)$$

is written in the form

$$c^{gw}(z, D_z; t, h) = a^{gw}(z, D_z; t, h) \circ b^{gw}(z, D_z; t, h)$$

with the symbol $c \in S_\Omega(mm')$ given by

$$c(z, \zeta; t, h) = e^{\frac{i}{2}\left(D_z D_{\zeta_1} - D_\zeta D_{z_1}\right)} a(z, \zeta; t, h) b\left(z_1, \zeta_1; t, h\right)\Big|_{z_1 = z, \zeta_1 = \zeta}, \tag{3.5}$$

which would be denoted by $a\sharp b(z, \zeta; t, h)$. We can also write

$$\begin{aligned} a\sharp b(z, \zeta; t, h) = &\sum_{j=0}^{N-1} \frac{1}{j!} \left\{\frac{i}{2}\left(\partial_z \partial_{\zeta_1} - \partial_\zeta \partial_{z_1}\right)\right\}^j a(z, \zeta; t, h) b(z_1, \zeta_1; t, h)\Big|_{z_1 = z, \zeta_1 = \zeta} \\ &+ R_N a(z, \zeta; t, h) b(z_1, \zeta_1; t, h)\Big|_{z_1 = z, \zeta_1 = \zeta} \end{aligned} \tag{3.6}$$

with

$$R_N = \int_0^1 \frac{(1-\tau)^{N-1}}{(N-1)!} e^{\frac{i}{2}\tau\left(D_z D_{\zeta_1} - D_\zeta D_{z_1}\right)} d\tau \left\{\frac{i}{2}\left(\partial_z \partial_{\zeta_1} - \partial_\zeta \partial_{z_1}\right)\right\}^N.$$

If there is a sequence $k_0, k_1, \ldots$ such that

$$R_j ab\big|_{z_1 = z, \zeta_1 = \zeta} \in S\left(h^{k_j} mm'\right) \quad \text{and} \quad \lim_{j \to \infty} k_j = \infty,$$

$a\sharp b$ can be expanded into the asymptotic sum

$$a\sharp b \sim \sum_{j=0}^{\infty} \frac{1}{j!} \left\{\frac{i}{2}\left(\partial_z \partial_{\zeta_1} - \partial_\zeta \partial_{z_1}\right)\right\}^j ab\Big|_{z_1 = z, \zeta_1 = \zeta}, \tag{3.7}$$

where we omitted writing variables. This convention would be the same in the following.

Proof. Since

$$\begin{aligned} &a^{gw}(z, D_z; t, h) \circ b^{gw}(z, D_z; t, h) \\ &= \frac{1}{\sqrt[4]{g(z)}} \circ a^w(z, D_z; t, h) \circ b^w(z, D_z; t, h) \circ \sqrt[4]{g(z)}, \end{aligned}$$

the formula (3.5) follows from the composition formula in, e.g., [11]. We now prove (3.6). First consider the case where $a, b \in \mathcal{S}\left(\mathbb{R}^{2n}\right)$ for each (t, h). Then using the Taylor expansion

$$e^{\frac{i}{2}(z^*\zeta_1^*-\zeta^* z_1^*)}=\sum_{j=0}^{N-1}\frac{1}{j!}\left\{\frac{i}{2}\left(z^*\zeta_1^*-\zeta^* z_1^*\right)\right\}^j$$
$$+\left\{\frac{i}{2}\left(z^*\zeta_1^*-\zeta^* z_1^*\right)\right\}^N\int_0^1\frac{(1-\tau)^{N-1}}{(N-1)!}e^{\frac{i}{2}\tau(z^*\zeta_1^*-\zeta^* z_1^*)}d\tau$$

and partial integration, it is easy to verify (3.6). Since all terms in the right-hand side of (3.6) are continuous in $(a,b)\in\mathcal{S}'\times\mathcal{S}'$, we have (3.6) for any $a\in S_\Omega(m)$ and $b\in S_\Omega(m')$. The rest part of the proposition is obvious. □

We represented the remainder terms using the operator R_j followed by the restriction of the variables. This is because the restriction means the loss of information. We want to exploit from $R_j ab$ as much information as possible before the restriction. It is shown below that R_j preserves the symbol class $S_\Omega(mm')$, but, before that, we give a corollary to Proposition 3.6.

Corollary 3.7 *Let $a\in S_\Omega(m)$ and $b\in S_\Omega(m')$, then we have*

$$a\sharp b-b\sharp a=\frac{1}{i}\{a,b\}+R_3\left(ab-ba\right)\big|_{z_1=z,\zeta_1=\zeta},$$

where $\{\cdot,\cdot\}$ is the Poisson bracket defined by

$$\{a,b\}:=\frac{\partial a}{\partial\zeta}\frac{\partial b}{\partial z}-\frac{\partial a}{\partial z}\frac{\partial b}{\partial\zeta}.$$

In particular, if there are k_j's for $a\sharp b$ and $b\sharp a$ as in Proposition 3.6, then we have

$$a\sharp b-b\sharp a-\frac{1}{i}\{a,b\}\in S_\Omega\left(h^{k_3}mm'\right).$$

Proof. From Proposition 3.6, it follows

$$\begin{aligned}a\sharp b=&ab+\frac{1}{2i}\left(-\partial_z a\partial_\zeta b+\partial_\zeta a\partial_z b\right)\\&-\frac{1}{8}\left(\partial_z\partial_{\zeta_1}-\partial_\zeta\partial_{z_1}\right)^2 ab\big|_{z_1=z,\zeta_1=\zeta}+R_3 ab\big|_{z_1=z,\zeta_1=\zeta},\\b\sharp a=&ba+\frac{1}{2i}\left(-\partial_z b\partial_\zeta a+\partial_\zeta b\partial_z a\right)\\&-\frac{1}{8}\left(\partial_z\partial_{\zeta_1}-\partial_\zeta\partial_{z_1}\right)^2 ba\big|_{z_1=z,\zeta_1=\zeta}+R_3 ba\big|_{z_1=z,\zeta_1=\zeta}.\end{aligned}$$

One observes that

$$\begin{aligned}&\left(\partial_z\partial_{\zeta_1}-\partial_\zeta\partial_{z_1}\right)^2 a(z,\zeta;t,h)b(z_1,\zeta_1;t,h)\big|_{z_1=z,\zeta_1=\zeta}\\&=\left(\partial_{z_1}\partial_\zeta-\partial_{\zeta_1}\partial_z\right)^2 b(z_1,\zeta_1;t,h)a(z,\zeta;t,h)\big|_{z_1=z,\zeta_1=\zeta}\\&=\left(\partial_z\partial_{\zeta_1}-\partial_\zeta\partial_{z_1}\right)^2 b(z,\zeta;t,h)a(z_1,\zeta_1;t,h)\big|_{z_1=z,\zeta_1=\zeta}.\end{aligned}$$

Then the corollary follows. □

Lemma 3.8 *Let $a \in S_\Omega(m)$ and $p \in C^\infty\left(\mathbb{R}^{2n}\right)$. Suppose that, for any $\alpha, \beta \in \mathbb{Z}_+^n$, there exist constants $C_{\alpha\beta} > 0$ and $n_{\alpha\beta}$ such that*

$$\left|\partial_z^\alpha \partial_\zeta^\beta p(z,\zeta)\right| \le C_{\alpha\beta} \langle z;\zeta\rangle^{n_{\alpha\beta}}. \tag{3.8}$$

Then $p(D_z, D_\zeta)\, a \in S_\Omega(m)$.

Proof. Set $Z = (z,\zeta)$. One has to check the estimate for $\partial_Z^\alpha p(D_Z)\, a$, but, since ∂_Z^α and $p(D_Z)$ commutes, it suffices to check the estimate only for $p(D_Z)\, a$. By the definition we have

$$\begin{aligned} p(D_Z)a(Z;t,h) &= \int e^{i(Z-Z')Z^*} p(Z^*)\, a(Z';t,h)\, dZ' đZ^* \\ &= \int e^{i(Z-Z')Z^*} \frac{\langle D_{Z^*}\rangle^N p(Z^*)}{\langle Z^*\rangle^M} \langle D_{Z'}\rangle^M \frac{a(Z';t,h)}{\langle Z-Z'\rangle^N} dZ' đZ^*. \end{aligned}$$

One applies the inequalities (3.1) and (3.8) to the above and obtains

$$|p(D_Z)a(Z;t,h)| \le C_{NM} m(Z;t,h) \int \langle Z^*\rangle^{n_N - M} \langle Z-Z'\rangle^{N_0 - N} dZ' đZ^*.$$

Taking $N > N_0 + 2n$ and $M > n_N + 2n$, and changing the variable $Z' \to Z'' = Z' - Z$, the integral in the right-hand side converges to a constant independent of Z. □

The operators of the form $p(D_z, D_\zeta)$ often appear, for instance, in compositions and changes of the quantization. Proposition 3.6 guarantees the composition for the symbols in $S_\Omega(m)$ and $S_\Omega(m')$, however, this doesn't guarantee the asymptotic expansion. We use Lemma 3.8 to check the expansion. We demonstrate the procedure in the proof of the following proposition, which implies that the subclass $S_{[-t_0,0]}(m,\tilde g_1) \subset S_\Omega(m)$ makes an algebra. Recall that $\tilde g_1$ is given by

$$\tilde g_1 = \langle h^{-1}t\rangle^{-2} dz^2 + h^2 \langle h^{-1}t\rangle^2 d\zeta^2.$$

Proposition 3.9 *Let $a \in S_{[-t_0,0]}(m,\tilde g_1)$ and $b \in S_{[-t_0,0]}(m',\tilde g_1)$. Since*

$$S_{[-t_0,0]}(m,\tilde g_1) \subset S_{[-t_0,0]}(m) \quad \text{and} \quad S_{[-t_0,0]}(m',\tilde g_1) \subset S_{[-t_0,0]}(m'),$$

Proposition 3.6 can be applied to a and b, and we can consider the composite symbol $a\sharp b \in S_{[-t_0,0]}(mm')$. Then $a\sharp b \in S_{[-t_0,0]}(mm',\tilde g_1)$ and the asymptotic expansion (3.7) is valid.

Proof. Note that

$$\partial_{(z,z_1)}^\alpha \partial_{(\zeta,\zeta_1)}^\beta a(z,\zeta;t,h) b(z_1,\zeta_1;t,h) \in S_{[-t_0,0]}\left(h^{|\beta|}\langle h^{-1}t\rangle^{-|\alpha|+|\beta|} mm'\right)$$

as a function of $(z,z_1,\zeta,\zeta_1,t,h)$. Then, apply Lemma 3.8 with $p(z^*,z_1^*,\zeta^*,\zeta_1^*) = e^{\frac{i}{2}(z^*\zeta_1^* - \zeta^* z_1^*)}$, and we see

$$\partial_{(z,z_1)}^\alpha \partial_{(\zeta,\zeta_1)}^\beta e^{\frac{i}{2}\left(D_z D_{\zeta_1} - D_\zeta D_{z_1}\right)} ab \in S_{[-t_0,0]}\left(h^{|\beta|}\langle h^{-1}t\rangle^{-|\alpha|+|\beta|} mm'\right),$$

which implies, through the restriction of variables, $a\sharp b \in S_{[-t_0,0]}\left(mm', \tilde{g}_1\right)$. To see the asymptotic expansion we estimate the remainder term. Since

$$\begin{aligned}&\left\{\frac{i}{2}\left(\partial_z\partial_{\zeta_1} - \partial_\zeta\partial_{z_1}\right)\right\}^N ab\\ &\in S_{[-t_0,0]}\left(h^N mm', \langle h^{-1}t\rangle^{-2}\left(dz^2+dz_1^2\right) + h^2\langle h^{-1}t\rangle^{-2}\left(d\zeta^2+d\zeta_1^2\right)\right),\end{aligned}$$

we can check similarly to the above argument with

$$p(z^*, z_1^*, \zeta^*, \zeta_1^*) = \frac{1}{(N-1)!}\int_0^1 (1-\tau)^{N-1} e^{\frac{i}{2}\tau(z^*\zeta_1^* - \zeta^* z_1^*)} d\tau$$

that

$$R_N ab\big|_{z_1=z,\zeta_1=\zeta} \in S_{[-t_0,0]}\left(h^N mm', \tilde{g}_1\right) \subset S\left(h^N mm'\right).$$

Then the asymptotic expansion follows. □

We will often use this argument in Section 4 to see what class the composite symbol belongs to, when we composite the symbols in $S_\Omega(m, \tilde{g})$ and $S_\Omega(m', \tilde{g}')$ with different $\tilde{g}$ and $\tilde{g}'$. Then Lemma 3.8 will be very useful for estimating the remainder terms.

Theorem 3.10 *Let $a \in S_\Omega(m)$ with an order function $m = m(t,h)$ which is independent of (z,ζ). Then $a^{gw}(z, D_z; t, h)$ extends to a bounded operator on $L^2\left(\mathbb{R}^n, \sqrt{g}dz\right)$, and there exist constants C_n and M_n depending only on n such that*

$$\|a^{gw}(z, D_z; t, h)\|_{\mathcal{L}\left(L^2\left(\mathbb{R}^n,\sqrt{g}dz\right)\right)} \le C_n \sum_{|\alpha|\le M_n} \left\|\partial^\alpha_{(z,\zeta)} a(\cdot,\cdot;t,h)\right\|_{L^\infty(\mathbb{R}^{2n})}$$

for each $(t,h) \in \Omega \times (0,1]$.

For the proof of the theorem, see any textbook for pseudodifferential operators, e.g., [11].

The next theorem is the sharp Gårding inequality revised for our purpose:

Theorem 3.11 *Suppose that $a \in S_{[-t_0,0]}(m, \tilde{g}_1)$ with m independent of (z,ζ) satisfies*

$$a(z,\zeta;t,h) \ge 0$$

for all $(z,\zeta,t,h) \in \mathbb{R}^{2n} \times [-t_0,0] \times (0,1]$. Then there exists $r \in S_{[-t_0,0]}(hm, \tilde{g}_1)$ such that

$$a^{gw}(z, D_z; t, h) \ge -r^{gw}(z, D_z; t, h),$$

that is, we have for any $u \in \mathcal{S}(\mathbb{R}^n)$

$$\langle a^{gw}(z, D_z; t, h)u, u\rangle_{L^2\left(\mathbb{R}^n,\sqrt{g}dz\right)} \ge -\langle r^{gw}(z, D_z; t, h)u, u\rangle_{L^2\left(\mathbb{R}^n,\sqrt{g}dz\right)}.$$

Moreover, for any $N > 0$, there exists $\tilde{r}_N \in S_{[-t_0,0]}(hm, \tilde{g}_1)$ such that

$$\operatorname{supp}\tilde{r}_N(\cdot,\cdot;t,h) \subset \operatorname{supp} a(\cdot,\cdot;t,h) \quad \forall (t,h) \in [-t_0,0]\times(0,1]$$

and

$$r - \tilde{r}_N \in S_{[-t_0,0]}\left(h^{\frac{N}{2}} m, \tilde{g}_1\right).$$

This implies in particular that r *is supported in* $\operatorname{supp} a(\cdot,\cdot;t,h)$ *modulo* $S\left(h^\infty m\right)$.

Proof. Put

$$\begin{aligned}\tilde{a}(z,\zeta;t,h) &= \frac{1}{\pi^n}\int e^{-z_1^2-\zeta_1^2} a\left(z+pz_1,\zeta+q\zeta_1;t,h\right) dz_1 d\zeta_1\\ &= \frac{1}{\pi^n}\int e^{-q^2(z_1-z)^2-p^2(\zeta_1-\zeta)^2} a\left(z_1,\zeta_1;t,h\right) dz_1 d\zeta_1,\end{aligned}$$

where $p = h^{\frac{1}{2}}\left\langle h^{-1}t\right\rangle$ and $q = h^{-\frac{1}{2}}\left\langle h^{-1}t\right\rangle^{-1}$. By the Taylor expansion,

$$\begin{aligned}a\left(z+pz_1,\zeta+q\zeta_1;t,h\right) =& a(z,\zeta;t,h)\\ &+ p\sum_{j=1}^n\int_0^1 z_1^j\partial_{z^j} a\left(z+\tau pz_1,\zeta+\tau q\zeta_1;t,h\right) d\tau\\ &+ q\sum_{j=1}^n\int_0^1 (\zeta_1)_j\partial_{\zeta_j} a\left(z+\tau pz_1,\zeta+\tau q\zeta_1;t,h\right) d\tau.\end{aligned}$$

Then, using partial integration, we have

$$\tilde{a}(z,\zeta;t,h) = a(z,\zeta;t,h) + r(z,\zeta;t,h)$$

with

$$\begin{aligned}r(z,\zeta;t,h) =& \frac{1}{2}p^2\iint_0^1 e^{-z_1^2-\zeta_1^2}\tau\triangle_z a\left(z+\tau pz_1,\zeta+\tau q\zeta_1;t,h\right) d\tau dz_1 d\zeta_1\\ &+ \frac{1}{2}q^2\iint_0^1 e^{-z_1^2-\zeta_1^2}\tau\triangle_\zeta a\left(z+\tau pz_1,\zeta+\tau q\zeta_1;t,h\right) d\tau dz_1 d\zeta_1.\end{aligned}$$

One can easily see that $r \in S_{[-t_0,0]}\left(hm,\tilde{g}_1\right)$.

We show the positivity of $\tilde{a}^{gw}(z,D_z;t,h)$ as an operator. Since we have for $u \in C_0^\infty(\mathbb{R}^n)$

$$\langle\tilde{a}^{gw}(z,D_z;t,h)u,u\rangle_{L^2\left(\mathbb{R}^n,\sqrt{g}dz\right)} = \langle\tilde{a}^w(z,D_z;t,h)\sqrt[4]{g}u,\sqrt[4]{g}u\rangle_{L^2(\mathbb{R}^n,dz)},$$

by replacing u with $\frac{1}{\sqrt[4]{g}}v$, it suffices to prove the positivity of $\tilde{a}^w(z,D_z;t,h)$ with respect to the inner product $\langle\cdot,\cdot\rangle_{L^2(\mathbb{R}^n,dz)}$. For $u \in C_0^\infty(\mathbb{R}^n)$

$$\begin{aligned}&\pi^n\left\langle\tilde{a}^w(z,D_z;t,h)u,u\right\rangle_{L^2(\mathbb{R}^n,dz)}\\ &= \int\left[\int e^{i(z-w)\zeta}\left\{\int e^{-q^2\left(z_1-\frac{z+w}{2}\right)^2-p^2(\zeta_1-\zeta)^2}\right.\right.\\ &\qquad\qquad\left.\left. a\left(z_1,\zeta_1;t,h\right) dz_1 d\zeta_1\right\} u(w) dw đ\zeta\right]\overline{u(z)} dz\\ &= \int\left\{\int e^{i(z-w)\zeta} e^{-q^2\left(z_1-\frac{z+w}{2}\right)^2-p^2(\zeta_1-\zeta)^2}\right.\\ &\qquad\qquad\left. a\left(z_1,\zeta_1;t,h\right) u(w)\overline{u(z)} dw đ\zeta dz\right\} dz_1 d\zeta_1,\end{aligned}$$

where the change of the order of integrals is verified, for example, by that, using partial integration, the both of the above integrals are equal to the absolutely convergent integral

$$\int e^{i(z-w)\zeta}\langle\zeta\rangle^{-N}\langle D_w\rangle^{N}\left[e^{-q^2\left(z_1-\frac{z+w}{2}\right)^2-p^2(\zeta_1-\zeta)^2}u(w)\right]$$
$$a\left(z_1,\zeta_1;t,h\right)\overline{u(z)}dz_1d\zeta_1dw\bar{d}\zeta dz.$$

By the change of variables $\zeta\to\zeta_2=\zeta_1-\zeta$,

$$\begin{aligned}
&\pi^n\left\langle\tilde{a}^w(z,D_z;t,h)u,u\right\rangle_{L^2(\mathbb{R}^n,dz)}\\
&=\int\left\{\int e^{-i(z-w)\zeta_2}e^{-p^2\zeta_2^2}\bar{d}\zeta_2\right\}e^{-q^2\left(z_1-\frac{z+w}{2}\right)^2+i(z-w)\zeta_1}\\
&\qquad\qquad a(z_1,\zeta_1;t,h)u(w)\overline{u(z)}dwdzdz_1d\zeta_1\\
&=\pi^{\frac{n}{2}}q^n\int e^{-\frac{1}{4}q^2(z-w)^2-q^2\left(z_1-\frac{z+w}{2}\right)^2+i(z-w)\zeta_1}\\
&\qquad\qquad a(z_1,\zeta_1;t,h)u(w)\overline{u(z)}dwdzdz_1d\zeta_1\\
&=\pi^{\frac{n}{2}}q^n\int\left|\int e^{-\frac{1}{2}q^2w^2+q^2z_1w-iw\zeta_1}u(w)dw\right|^2e^{-q^2z_1^2}a(z_1,\zeta_1;t,h)dz_1\bar{d}\zeta_1\\
&\geq 0.
\end{aligned}$$

Thus we have

$$\langle\tilde{a}^{gw}(z,D_z;t,h)u,u\rangle_{L^2\left(\mathbb{R}^n,\sqrt{g}dz\right)}\geq 0$$

for any $u\in\mathcal{S}\left(\mathbb{R}^n\right)$. Then

$$\begin{aligned}
&\langle a^{gw}(z,D_z;t,h)u,u\rangle_{L^2\left(\mathbb{R}^n,\sqrt{g}dz\right)}\\
&=\langle(a-\tilde{a})^{gw}\left(z,D_z;t,h\right)u,u\rangle_{L^2\left(\mathbb{R}^n,\sqrt{g}dz\right)}+\langle\tilde{a}^{gw}(z,D_z;t,h)u,u\rangle_{L^2\left(\mathbb{R}^n,\sqrt{g}dz\right)}\\
&\geq-\langle r^{gw}(z,D_z;t,h)u,u\rangle_{L^2\left(\mathbb{R}^n,\sqrt{g}dz\right)}.
\end{aligned}$$

The required r is obtained.

Now let us prove the second part of the theorem. By the Taylor expansion

$$\begin{aligned}
&a\left(z+pz_1,\zeta+q\zeta_1;t,h\right)\\
&=\sum_{|\alpha|+|\beta|<N}\frac{p^{|\alpha|-|\beta|}}{\alpha!\beta!}z_1^\alpha\zeta_1^\beta\partial_z^\alpha\partial_\zeta^\beta a(z,\zeta;t,h)\\
&\quad+\sum_{|\alpha|+|\beta|=N}\frac{Np^{|\alpha|-|\beta|}}{\alpha!\beta!}z_1^\alpha\zeta_1^\beta\int_0^1(1-\tau)^{N-1}\partial_z^\alpha\partial_\zeta^\beta a\left(z+\tau pz_1,\zeta+\tau q\zeta_1;t,h\right)d\tau,
\end{aligned}$$

and then, corresponding to this, r is expanded to be

$$r(z,\zeta;t,h)=\sum_{j=1}^{N-1}r_j(z,\zeta;t,h)+\tilde{r}_N(z,\zeta;t,h)$$

with

$$r_j(z,\zeta;t,h) = \sum_{|\alpha|+|\beta|=j} \frac{p^{|\alpha|-|\beta|}}{\pi^n \alpha!\beta!} \partial_z^\alpha \partial_\zeta^\beta a(z,\zeta;t,h) \int e^{-z_1^2-\zeta_1^2} z_1^\alpha \zeta_1^\beta dz_1 d\zeta_1,$$

$$\tilde{r}_N(z,\zeta;t,h) = \sum_{|\alpha|+|\beta|=N} \frac{Np^{|\alpha|-|\beta|}}{\pi^n \alpha!\beta!} \iint_0^1 (1-\tau)^{N-1} e^{-z_1^2-\zeta_1^2} z_1^\alpha \zeta_1^\beta$$

$$\partial_z^\alpha \partial_\zeta^\beta a\left(z+\tau p z_1, \zeta+\tau q \zeta_1; t,h\right) d\tau dz_1 d\zeta_1.$$

One can easily check that this $\tilde{r}_N$ satisfies the properties of the theorem. Hence

$$r \sim \sum_{|\alpha|+|\beta|\geq 1} \frac{p^{|\alpha|-|\beta|}}{\pi^n \alpha!\beta!} \partial_z^\alpha \partial_\zeta^\beta a \int e^{-z_1^2-\zeta_1^2} z_1^\alpha \zeta_1^\beta dz_1 d\zeta_1.$$

Each term in the right-hand side is supported in $\operatorname{supp} a\left(\cdot,\cdot;t,h\right)$, and so, by Proposition 3.2, we can choose $r' \in S_{[-t_0,0]}\left(hm\right)$ such that

$$r' \sim \sum_{|\alpha|+|\beta|\geq 1} \frac{p^{|\alpha|-|\beta|}}{\pi^n \alpha!\beta!} \partial_z^\alpha \partial_\zeta^\beta a \int e^{-z_1^2-\zeta_1^2} z_1^\alpha \zeta_1^\beta dz_1 d\zeta_1$$

and $\operatorname{supp} r'\left(\cdot,\cdot;t,h\right) \subset \operatorname{supp} a\left(\cdot,\cdot;t,h\right)$. From the uniqueness of the asymptotic sum it follows that $r - r' \in S\left(h^\infty m\right)$, which shows that r is supported in $\operatorname{supp} a\left(\cdot,\cdot;t,h\right)$ modulo $S\left(h^\infty m\right)$. □

We give the g-Weyl symbol of H_0:

Proposition 3.12 *The g-Weyl symbol of*

$$H_0 = -\frac{1}{2\sqrt{g}} \sum_{i,j=1}^n \partial_i g^{ij} \sqrt{g} \partial_j$$

is given by

$$\sigma^{gw}(H_0)(z,\zeta) = e^{-\frac{i}{2}D_z D_\zeta} k(z,\zeta),$$

where $k(z,\zeta)$ is equal to the polynomial

$$\frac{1}{2}\sum_{j,k}\left[g^{jk}\zeta_j\zeta_k - i\partial_j g^{jk}\zeta_k + \frac{1}{4g}\left(g^{jk}\partial_j\partial_k g - \frac{3}{4}g^{jk}\partial_j g\partial_k g + \partial_j g^{jk}\partial_k g\right)\right].$$

Then it follows

$$\sigma^{gw}(H_0) \in S\left(\langle\zeta\rangle^2, \langle z\rangle^{-2} dz^2 + \langle\zeta\rangle^{-2} d\zeta^2\right), \tag{3.9}$$

$$\sigma^{gw}(H_0) - \frac{1}{2}\sum_{j,k} g^{jk}\zeta_j\zeta_k \in S\left(\langle\zeta\rangle \langle x\rangle^{-1}, \langle z\rangle^{-2} dz^2 + \langle\zeta\rangle^{-2} d\zeta^2\right). \tag{3.10}$$

Proof. By an easy computation

$$\sqrt[4]{g} \circ H_0 \circ \frac{1}{\sqrt[4]{g}} = k(z, D_z),$$

and then the g-Weyl symbol $\sigma^{gw}(H_0)$ is given by the formula for the change of the quantization, i.e., $\sigma^{gw}(H_0)(z,\zeta) = e^{-\frac{i}{2}D_z D_\zeta} k(z,\zeta)$. See, e.g., [11]. If we prove

$$k \in S\left(\langle\zeta\rangle^2, \langle z\rangle^{-2} dz^2 + \langle\zeta\rangle^{-2} d\zeta^2\right), \tag{3.11}$$

then, using Lemma 3.8, we obtain (3.9) and (3.10) similarly to the proof of Proposition 3.9. We write down $\det g$ and g^{ij}'s with respect to the coordinates z. The expression of $\det g$ in z has already been seen in (3.4). For the expression of g^{ij} we use (2.2). Substituting (A.2) and (A.3) into the equality

$$\sum_{i,j=1}^{n} g^{ij}(z)\partial_{z^i} \otimes \partial_{z^j} = \left(x^4 + x^6\varphi\right)\partial_x \otimes \partial_x + x^4 \sum_{i=1}^{n-1} \varphi^i \left(\partial_x \otimes \partial_{y^i} + \partial_{y^i} \otimes \partial_x\right)$$
$$+ x^2 \sum_{i,j=1}^{n-1} \left(h^{ij} + x^2\varphi^{ij}\right)\partial_{y^i} \otimes \partial_{y^j},$$

we see that $g^{ij}(z)$ is a C^∞ function in (x,y). Then it follows

$$\det g,\ g^{ij} \in S\left(1, \langle z\rangle^{-2} dz^2\right),$$

which implies (3.11). $\square$

4 Proof of Theorem 1.3

Let $u_0 \in L^2(\mathbb{R}^n)$, $(z_0,\zeta_0) \in T^*\mathbb{R}^n$ and $t_0 > 0$ as in Theorem 1.3. We often use the notation

$$\Delta f = f(z,\zeta) - f(z(t;z_0,\zeta_0), \zeta(t;z_0,\zeta_0)),$$

where f is any function on $T^*\mathbb{R}^n$. Take large $T_0 > 0$ as in Section 2 so that

$$c^{-1}\langle t\rangle^{-1} \le x(t;z_0,\zeta_0) \le c\langle t\rangle^{-1} \quad \forall t \le -T_0$$

for some $c > 0$. Given small $\delta > 0$ and large $C > 0$, take any δ_0 with $0 < 4\delta_0 < \delta$. We fix large $T_1 > 0$ such that

$$T_1 > \max\left\{T_0, \frac{C}{\delta_0}, t_0\right\} + 1.$$

Choosing a C^∞ function χ on $[0,+\infty)$ such that

$$\chi(\lambda) = \begin{cases} 1, & \text{if } \lambda < \frac{1}{2}, \\ 0, & \text{if } \lambda > 1, \end{cases} \qquad \text{and} \quad \frac{d}{d\lambda}\chi(\lambda) \le 0 \quad \forall \lambda \ge 0,$$

we define $\psi_{-1} : (-\infty, -T_1 + 1] \times T^*\mathbb{R}^n \to \mathbb{R}$ by

$$\begin{aligned}\psi_{-1}(t,z,\zeta) &= \chi\left(\frac{\left|\Delta\left(x^{-1}\right)\right|}{-4\delta_0 t}\right)\chi\left(\frac{|\Delta y|}{\delta_0 + Ct^{-1}}\right)\chi\left(\frac{\left|\Delta\left(x^2\xi\right)\right|}{\delta_0 + Ct^{-1}}\right)\chi\left(\frac{|\Delta\eta|}{\delta_0 + Ct^{-1}}\right)\\ &= \chi_1\chi_2\chi_3\chi_4,\end{aligned}$$

where each χ_i is the corresponding factor. Here we note that in general the subtractions such as $\Delta\eta = \eta - \eta(t; z_0, \zeta_0)$ are senseless since points in the base space are different and trivializations might be different. But, as noted right after Proposition 2.3, $z(t; z_0, \zeta_0)$ has the limit $z_-(z_0, \zeta_0) \in \partial X$ as $t \to -\infty$, and thus, by exchanging coordinate axes and taking T_0 larger if necessary, we can assume only the local coordinates $(x, y_{(+n)})$ are being taken when defining ψ_{-1}. Then the well-definedness of ψ_{-1} follows by the zero-extension. We modify ψ_{-1} to be defined for all $t \le 0$ by solving the transport equation

$$\frac{D}{Dt}\psi_0(t,z,\zeta) = \alpha(t)\frac{D}{Dt}\psi_{-1}(t,z,\zeta), \tag{4.1}$$

with the initial condition

$$\psi_0(-T_1, z, \zeta) = \psi_{-1}(-T_1, z, \zeta),$$

where α is in $C^\infty((-\infty, 0])$ and satisfies

$$\alpha(t) = \begin{cases} 1, & \text{if } t \le -T_1, \\ 0, & \text{if } t \ge -T_1 + 1. \end{cases}$$

Here $\frac{D}{Dt}$ is the Lagrange derivative defined by

$$\frac{D}{Dt} := \frac{\partial}{\partial t} + \frac{\partial p}{\partial \zeta}\frac{\partial}{\partial z} - \frac{\partial p}{\partial z}\frac{\partial}{\partial \zeta},$$

where p is the Hamiltonian. The transport equation is easily solved using classical trajectories. Indeed, substituting *any* trajectory, i.e., $(z(t; z, \zeta), \zeta(t; z, \zeta))$ to (4.1), we see that the equation gets to be

$$\frac{d}{dt}\psi_0(t, z(t; z, \zeta), \zeta(t; z, \zeta)) = \alpha(t)\frac{d}{dt}\psi_{-1}(t, z(t; z, \zeta), \zeta(t; z, \zeta)),$$

and so

$$\begin{aligned}\psi_0(t, z(t; z, \zeta), \zeta(t; z, \zeta)) = &\int_{-T_1}^{t} \alpha(s)\frac{d}{ds}\psi_{-1}(s, z(s; z, \zeta), \zeta(s; z, \zeta))ds \\ &+ \psi_{-1}(-T_1, z(-T_1; z, \zeta), \zeta(-T_1; z, \zeta)).\end{aligned} \tag{4.2}$$

If we use the partial integration, this is rewritten by

$$\begin{aligned}\psi_0(t, z(t; z, \zeta), \zeta(t; z, \zeta)) = &\int_{t}^{-T_1} \frac{d\alpha}{ds}(s)\psi_{-1}(s, z(s; z, \zeta), \zeta(s; z, \zeta))ds \\ &+ \alpha(t)\psi_{-1}(t, z(t; z, \zeta), \zeta(t; z, \zeta)).\end{aligned} \tag{4.3}$$

Since, by Proposition 2.2, for any point $(z,\zeta) \in T^*\mathbb{R}^n$ and any $t \in \mathbb{R}$, there is a trajectory that hits (z,ζ) at the time t, ψ_0 is defined on all of $\mathbb{R}_- \times T^*\mathbb{R}^n$ and in the class of C^∞. One notes that, for $t \leq -T_1$,

$$\psi_0(t,z,\zeta) = \psi_{-1}(t,z,\zeta) \tag{4.4}$$

follows. We now clarify properties of ψ_0:

Lemma 4.1 *ψ_0 satisfies the following:*

1. *We have*
$$\psi_0(t,z,\zeta) \geq 0 \quad \textit{for all} \quad (t,z,\zeta) \in \mathbb{R}_- \times T^*\mathbb{R}^n,$$
and
$$\psi_0(t, z(t;z_0,\zeta_0), \zeta(t;z_0,\zeta_0)) = 1 \quad \textit{for all} \quad t \leq 0.$$

2. *For $t \leq -T_1$, $\psi_0(t,\cdot,\cdot)$ is supported in*
$$\left\{(z,\zeta) \in T^*\mathbb{R}^n \middle| \left|\Delta x^{-1}\right| < -4\delta_0 t, |\Delta y| < \delta_0, \left|\Delta x^2 \xi\right| < \delta_0, |\Delta\eta| < \delta_0 \right\}.$$

3. *If one takes sufficiently small $\delta > 0$ and large $C > 0$ in the construction of ψ_0, the inequality*
$$\frac{D}{Dt}\psi_0(t,z,\zeta) \leq 0 \quad \textit{for all} \quad (t,z,\zeta) \in \mathbb{R}_- \times T^*\mathbb{R}^n$$
holds.

4. *$\psi_0(t,z,\zeta)$ satisfies the estimates*
$$\left|\partial_z^\alpha \partial_\zeta^\beta \partial_t^n \psi_0(t,z,\zeta)\right| \leq C_{\alpha\beta n} \langle t \rangle^{-|\alpha|+|\beta|-n},$$
that is, $\partial_t^n \psi_0 \in S_{\mathbb{R}_-}\left(\langle t \rangle^{-n}, \langle t \rangle^{-2} dz^2 + \langle t \rangle^2 d\zeta^2\right)$.

Proof. *1.* The positivity for $t \leq -T_1$ follows from (4.4). See (4.3) for $t \geq -T_1$. $\psi_0(t, z(t;z_0,\zeta_0), \zeta(t;z_0,\zeta_0)) = 1$ is also easy to be seen by substituting the trajectory $(z(t;z_0,\zeta_0), \zeta(t;z_0,\zeta_0))$ to (4.2) and (4.3).

2. Obvious from (4.4) and the construction of ψ_{-1}.

3. Note that

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{\partial p}{\partial \zeta}\frac{\partial}{\partial z} - \frac{\partial p}{\partial z}\frac{\partial}{\partial \zeta} = \frac{\partial}{\partial t} + \frac{\partial p}{\partial \xi}\frac{\partial}{\partial x} + \frac{\partial p}{\partial \eta}\frac{\partial}{\partial y} - \frac{\partial p}{\partial x}\frac{\partial}{\partial \xi} - \frac{\partial p}{\partial y}\frac{\partial}{\partial \eta},$$

and that

$$\begin{aligned}\frac{D}{Dt}\psi_0 &= \alpha \frac{D}{Dt}\psi_{-1} \\ &= \alpha\left(\frac{D\chi_1}{Dt}\chi_2\chi_3\chi_4 + \chi_1\frac{D\chi_2}{Dt}\chi_3\chi_4 + \chi_1\chi_2\frac{D\chi_3}{Dt}\chi_4 + \chi_1\chi_2\chi_3\frac{D\chi_4}{Dt}\right).\end{aligned}$$

Let us compute the differentiations concretely. We first get

$$\frac{D\chi_1}{Dt} = \frac{1}{4\delta_0 t}\left[\frac{\left|\Delta x^{-1}\right|}{t} + \frac{\Delta x^{-1}}{|\Delta x^{-1}|}\Delta\left(\frac{1}{x^2}\frac{\partial p}{\partial \xi}\right)\right]\chi'\left(\frac{\left|\Delta x^{-1}\right|}{-4\delta_0 t}\right).$$

We have on $\operatorname{supp}\alpha\frac{D\chi_1}{Dt}\chi_2\chi_3\chi_4$

$$\frac{1}{2} \leq \frac{\left|\Delta x^{-1}\right|}{-4\delta_0 t} \leq 1$$

and

$$\left|\Delta\left(\frac{1}{x^2}\frac{\partial p}{\partial \xi}\right)\right| \leq \left|\Delta x^2 \xi\right| + \left|\Delta\left(x^4\xi\varphi + x^2\sum \varphi^i \eta_i\right)\right| \leq \delta_0 + O\left(t^{-2}\right).$$

Thus, taking $C > 0$ larger if necessary, which makes T_1 larger, we obtain

$$\alpha\frac{D\chi_1}{Dt}\chi_2\chi_3\chi_4 \leq \frac{\alpha(t)}{2\delta_0 t}\left(-2\delta_0 + \delta_0 + O\left(t^{-2}\right)\right)\chi'\left(\frac{\left|\Delta x^{-1}\right|}{-4\delta_0 t}\right)\chi_2\chi_3\chi_4 \leq 0.$$

Similarly, by an easy computation,

$$\begin{aligned}
\frac{D\chi_2}{Dt} &= \frac{1}{\delta_0 + Ct^{-1}}\left[Ct^{-2}\frac{|\Delta y|}{\delta_0 + Ct^{-1}} + \frac{\Delta y}{|\Delta y|}\Delta\frac{\partial p}{\partial \eta}\right]\chi'\left(\frac{|\Delta y|}{\delta_0 + Ct^{-1}}\right)\\
\frac{D\chi_3}{Dt} &= \frac{1}{\delta_0 + Ct^{-1}}\left[Ct^{-2}\frac{\left|\Delta x^2\xi\right|}{\delta_0 + Ct^{-1}} + \frac{\Delta x^2\xi}{|\Delta x^2\xi|}\Delta\left(2x\xi\frac{\partial p}{\partial \xi} - x^2\frac{\partial p}{\partial x}\right)\right]\\
&\qquad\qquad\qquad\qquad\qquad\qquad \cdot\chi'\left(\frac{\left|\Delta x^2\xi\right|}{\delta_0 + Ct^{-1}}\right)\\
\frac{D\chi_4}{Dt} &= \frac{1}{\delta_0 + Ct^{-1}}\left[Ct^{-2}\frac{|\Delta \eta|}{\delta_0 + Ct^{-1}} - \frac{\Delta \eta}{|\Delta \eta|}\Delta\frac{\partial p}{\partial y}\right]\chi'\left(\frac{|\Delta \eta|}{\delta_0 + Ct^{-1}}\right).
\end{aligned}$$

Considering the supports, we have

$$\begin{aligned}
\alpha\chi_1\frac{D\chi_2}{Dt}\chi_3\chi_4 &\leq \frac{\alpha}{\delta_0 + Ct^{-1}}\left[\frac{C}{2}t^{-2} - O\left(t^{-2}\right)\right]\chi_1\chi'\left(\frac{|\Delta y|}{\delta_0 + Ct^{-1}}\right)\chi_3\chi_4,\\
\alpha\chi_1\chi_2\frac{D\chi_3}{Dt}\chi_4 &\leq \frac{\alpha}{\delta_0 + Ct^{-1}}\left[\frac{C}{2}t^{-2} - O\left(t^{-2}\right)\right]\chi_1\chi_2\chi'\left(\frac{\left|\Delta x^2\xi\right|}{\delta_0 + Ct^{-1}}\right)\chi_4,\\
\alpha\chi_1\chi_2\chi_3\frac{D\chi_4}{Dt} &\leq \frac{\alpha}{\delta_0 + Ct^{-1}}\left[\frac{C}{2}t^{-2} - O\left(t^{-2}\right)\right]\chi_1\chi_2\chi_3\chi'\left(\frac{|\Delta \eta|}{\delta_0 + Ct^{-1}}\right).
\end{aligned}$$

Thus, taking C large enough, the nonpositivity of the Lagrange derivative of ψ_0 follows.

4. Since ψ_0 is C^∞ in $(t, z, \zeta) \in \mathbb{R}_- \times T^*\mathbb{R}^n$ and, for each $t \in \mathbb{R}_-$, $\psi_0(t, \cdot, \cdot)$ has compact support, one can find constants $C_{\alpha\beta n}$ as in the lemma for $(t, z, \zeta) \in$

$[-T_1, 0] \times T^*\mathbb{R}^n$. For $t \leq -T_1$ one can differentiate the function on the right-hand side of (4.4) concretely using formulae (A.1) and (A.10). Then, noting that on the support of ψ_0 there exists a constant $\tilde{c}$ such that

$$\tilde{c}^{-1} \langle t \rangle^{-1} \leq x \leq \tilde{c} \langle t \rangle^{-1}$$

for $t \leq -T_1$, we can easily find constants $C_{\alpha\beta n}$ of the lemma. $\square$

Put

$$\tilde{\psi}_0 (z, \zeta; t, h) = \psi_0 \left(h^{-1} t, z, h\zeta\right),$$

and restrict $t \in \mathbb{R}_-$ to $[-t_0, 0]$. Then by Lemma 4.1 $\tilde{\psi}_0 \in S_{[-t_0,0]} (1, \tilde{g}_1)$.

We consider the operator

$$F_0(t, h) = \tilde{\psi}_0^{gw}(z, D_z; t, h) \circ \tilde{\psi}_0^{gw}(z, D_z; t, h).$$

By Proposition 3.9 we can write

$$F_0(t, h) = \varphi_0^{gw}(z, D_z; t, h)$$

with $\varphi_0 \in S_{[-t_0,0]} (1, \tilde{g}_1)$. Note that $F_0(t, h)$ extends to be a bounded operator on $L^2 \left(\mathbb{R}^n, \sqrt{g} dz\right)$ with operator norm uniformly bounded in (t, h). Also note that $F_0(t, h)$, as an operator on $L^2 \left(\mathbb{R}^n, \sqrt{g} dz\right)$, is differentiable in $t \in [-t_0, 0]$, because, by Theorem 3.10, for fixed $(t, h) \in [-t_0, 0] \times (0, 1]$,

$$\begin{aligned}
&\Bigg\| \frac{1}{\varepsilon} \left[\psi_0^{gw}(h^{-1}(t+\varepsilon), z, hD_z) - \psi_0^{gw}(h^{-1}t, z, hD_z)\right] \\
&\qquad\qquad\qquad - h^{-1} \left(\frac{\partial \psi_0}{\partial t}\right)^{gw} (h^{-1}t, z, hD_z)\Bigg\|_{\mathcal{L}\left(L^2\left(\mathbb{R}^n, \sqrt{g}dz\right)\right)} \\
&\leq C_n \sum_{|\alpha| \leq M_n} \Bigg\| \partial^\alpha_{(z,\zeta)} \Bigg(\frac{1}{\varepsilon} \left[\psi_0(h^{-1}(t+\varepsilon), \cdot, h\cdot) - \psi_0(h^{-1}t, \cdot, h\cdot)\right] \\
&\qquad\qquad\qquad\qquad - h^{-1} \frac{\partial \psi_0}{\partial t}(h^{-1}t, \cdot, h\cdot)\Bigg)\Bigg\|_{L^\infty} \\
&\leq C_n \varepsilon h^{-2} \sum_{|\alpha| \leq M_n} \left\| \partial^\alpha_{(z,\zeta)} \int_0^1 (1-\tau) \frac{\partial^2 \psi_0}{\partial t^2}(h^{-1}(t+\varepsilon\tau), \cdot, h\cdot) d\tau \right\|_{L^\infty} \\
&\leq C_n \varepsilon h^{-2} \sum_{|\alpha| \leq M_n} C_\alpha,
\end{aligned}$$

and this means the differentiability of $\psi_0^{gw}(h^{-1}t, z, hD_z)$ in $t \in [-t_0, 0]$. Moreover we see, from this inequality, that $\frac{\partial}{\partial t} \tilde{\psi}_0^{gw}(z, D_z; t, h)$ is also a pseudodifferential operator with the symbol given by

$$\frac{\partial \tilde{\psi}_0}{\partial t}(z, \zeta; t, h) = h^{-1} \frac{\partial \psi_0}{\partial t}(h^{-1}t, z, h\zeta) \in S_{[-t_0,0]} \left(h^{-1} \left\langle h^{-1} t\right\rangle^{-1}, \tilde{g}_1\right).$$

Lemma 4.2 *There exists* $r_0 \in S_{[-t_0,0]}\left(\langle h^{-1}t\rangle^{\max\{-1,\nu-2\}}, \tilde{g}_1\right)$ *such that*

$$\frac{\partial}{\partial t}F_0(t,h) + i[H, F_0(t,h)] \le r_0(z, D_z; t, h),$$

and that r_0 *has the support in* $\operatorname{supp}\tilde{\psi}_0$ *modulo* $S_{[-t_0,0]}(h^\infty)$.

Proof. We compute the *principal* g-Weyl symbol of $\frac{\partial}{\partial t}F_0(t,h) + i[H, F_0]$ and apply the sharp Gårding inequality. Divide the operator into three parts:

$$\frac{\partial}{\partial t}F_0(t,h) + i[H, F_0(t,h)] = \frac{\partial}{\partial t}F_0(t,h) + i[H_0, F_0(t,h)] + i[V, F_0(t,h)].$$

Step 1. From Proposition 3.9 it follows that

$$\sigma^{gw}\left(\frac{\partial}{\partial t}F_0\right) = \frac{\partial\tilde{\psi}_0}{\partial t}\sharp\tilde{\psi}_0 + \tilde{\psi}_0\sharp\frac{\partial\tilde{\psi}_0}{\partial t} = 2\tilde{\psi}_0\frac{\partial\tilde{\psi}_0}{\partial t} + r_{0,1},$$

where the remainder term $r_{0,1} \in S_{[-t_0,0]}\left(\langle h^{-1}t\rangle^{-1}, \tilde{g}_1\right)$. Note that

$$\operatorname{supp} r_{0,1} \subset \operatorname{supp}\tilde{\psi}_0 \mod S_{[-t_0,0]}(h^\infty),$$

since each term in the asymptotic expansion of $\sigma^{gw}\left(\frac{\partial}{\partial t}F_0\right)$ is supported in $\operatorname{supp}\tilde{\psi}_0$.

Step 2. Next we compute the symbol of $i[H_0, F_0]$. We write

$$i[H_0, F_0] = i\left[H_0, \tilde{\psi}_0^{gw}\right]\tilde{\psi}_0^{gw} + i\tilde{\psi}_0^{gw}\left[H_0, \tilde{\psi}_0^{gw}\right].$$

We use Proposition 3.6 to composite the operators H_0 and $\tilde{\psi}_0^{gw}$, but Proposition 3.9 cannot be applied directly to estimate the remainder term, since H_0 does not belong to the class $S_{[-t_0,0]}(m, \tilde{g}_1)$. We have to repeat the modified procedure of the proof of Proposition 3.9 to estimate the remainder term. Using Corollary 3.7,

$$\sigma^{gw}\left(i\left[H_0, \tilde{\psi}_0^{gw}\right]\right) = \left\{\sigma^{gw}(H_0), \tilde{\psi}_0\right\} + r_{0,2}$$

with

$$r_{0,2} = R_3\sigma^{gw}(H_0)\tilde{\psi}_0\big|_{z_1=z,\zeta_1=\zeta} - R_3\tilde{\psi}_0\sigma^{gw}(H_0)\big|_{z_1=z,\zeta_1=\zeta}.$$

We claim that $r_{0,2} \in S_{[-t_0,0]}(h, \tilde{g}_1)$. We prove the claim only for the first term $R_3\sigma^{gw}(H_0)\tilde{\psi}_0\big|_{z_1=z,\zeta_1=\zeta}$, as the claim for the other follows similarly. It suffices to estimate

$$\begin{aligned}\partial_z^\alpha\partial_{z_1}^{\alpha_1}\partial_\zeta^\beta\partial_{\zeta_1}^{\beta_1}R_3\sigma^{gw}(H_0)\tilde{\psi}_0 = \frac{1}{2!}\int_0^1(1-\tau)^2 e^{\frac{i}{2}\tau\left(D_zD_{\zeta_1}-D_\zeta D_{z_1}\right)}d\tau \\ \cdot\,\partial_z^\alpha\partial_{z_1}^{\alpha_1}\partial_\zeta^\beta\partial_{\zeta_1}^{\beta_1}\left\{\frac{i}{2}\left(\partial_z\partial_{\zeta_1} - \partial_\zeta\partial_{z_1}\right)\right\}^3\sigma^{gw}(H_0)\tilde{\psi}_0.\end{aligned} \tag{4.5}$$

Since $\sigma^{gw}(H_0)$ is a polynomial in ζ of degree 2, the derivations in (4.5) with respect to ζ act on $\sigma^{gw}(H_0)$ at most twice, so we have

$$\begin{aligned}&\partial_z^{\alpha}\partial_{z_1}^{\alpha_1}\partial_\zeta^{\beta}\partial_{\zeta_1}^{\beta_1}\left\{\frac{i}{2}\left(\partial_z\partial_{\zeta_1}-\partial_\zeta\partial_{z_1}\right)\right\}^3\sigma^{gw}(H_0)\tilde{\psi}_0\\&\in\bigoplus_{k+|\beta|\le 2}S_{[-t_0,0]}\left(\langle z\rangle^{k-|\alpha|-3}\langle\zeta\rangle^{2-k-|\beta|}h^{3-k+|\beta_1|}\left\langle h^{-1}t\right\rangle^{3-2k+|\beta_1|-|\alpha_1|}\right)\end{aligned}$$

as a function of $(z,z_1,\zeta,\zeta_1,t,h)$. On the support of the left-hand side there is $c>0$ such that

$$1\le\langle h\zeta_1\rangle\le c,\quad c^{-1}\left\langle h^{-1}t\right\rangle\le\langle z_1\rangle\le c\left\langle h^{-1}t\right\rangle,$$

so we can rewrite

$$\begin{aligned}&\partial_z^{\alpha}\partial_{z_1}^{\alpha_1}\partial_\zeta^{\beta}\partial_{\zeta_1}^{\beta_1}\left\{\frac{i}{2}\left(\partial_z\partial_{\zeta_1}-\partial_\zeta\partial_{z_1}\right)\right\}^3\sigma^{gw}(H_0)\tilde{\psi}_0\\&\in\bigoplus_{k+|\beta|\le 2}S_{[-t_0,0]}\left(\left(\frac{\langle z\rangle}{\langle z_1\rangle}\right)^{k-|\alpha|-3}\left(\frac{\langle h\zeta\rangle}{\langle h\zeta_1\rangle}\right)^{2-k-|\beta|}\right.\\&\qquad\qquad\left.h^{1+|\beta|+|\beta_1|}\left\langle h^{-1}t\right\rangle^{-k+|\beta_1|-|\alpha|-|\alpha_1|}\right).\end{aligned}$$

Then the application of Lemma 3.8 followed by the substitution $z_1=z,\zeta_1=\zeta$ means that

$$\partial_z^{\alpha}\partial_{z_1}^{\alpha_1}\partial_\zeta^{\beta}\partial_{\zeta_1}^{\beta_1}R_3\sigma^{gw}(H_0)\tilde{\psi}_0\big|_{z_1=z,\zeta_1=\zeta}\in S_{[-t_0,0]}\left(h^{1+|\beta|+|\beta_1|}\left\langle h^{-1}t\right\rangle^{|\beta_1|-|\alpha|-|\alpha_1|}\right),$$

from which it follows $R_3\sigma^{gw}(H_0)\tilde{\psi}_0\big|_{z_1=z,\zeta_1=\zeta}\in S_{[-t_0,0]}(h,\tilde{g}_1)$. Therefore we see the claim.

Hence we obtain

$$\begin{aligned}\sigma^{gw}\left(i\left[H_0,F_0\right]\right)=&\left\{p,\tilde{\psi}_0\right\}\sharp\tilde{\psi}_0+\tilde{\psi}_0\sharp\left\{p,\tilde{\psi}_0\right\}+\left\{\sigma^{gw}(H_0)-p,\tilde{\psi}_0\right\}\sharp\tilde{\psi}_0\\&+\tilde{\psi}_0\sharp\left\{\sigma^{gw}(H_0)-p,\tilde{\psi}_0\right\}+r_{0,2}\sharp\tilde{\psi}_0+\tilde{\psi}_0\sharp r_{0,2}.\end{aligned}$$

The support of $\tilde{\psi}_0$ and Proposition 3.12 taken into consideration, we have

$$\left\{\sigma^{gw}(H_0)-p,\tilde{\psi}_0\right\}\in S_{[-t_0,0]}\left(\left\langle h^{-1}t\right\rangle^{-1},\tilde{g}_1\right).$$

Similarly we have $\left\{p,\tilde{\psi}_0\right\}\in S_{[-t_0,0]}\left(h^{-1},\tilde{g}_1\right)$, however, this estimate is not sufficient. To improve it we compute $\left\{p,\tilde{\psi}_0\right\}$ in the coordinates (x,y), that is, we use the representation

$$\left\{p,\tilde{\psi}_0\right\}=\frac{\partial p}{\partial\xi}\frac{\partial\tilde{\psi}_0}{\partial x}+\frac{\partial p}{\partial\eta}\frac{\partial\tilde{\psi}_0}{\partial y}-\frac{\partial p}{\partial x}\frac{\partial\tilde{\psi}_0}{\partial\xi}-\frac{\partial p}{\partial y}\frac{\partial\tilde{\psi}_0}{\partial\eta}.$$

Then $\left\{p, \tilde{\psi}_0\right\} \in S_{[-t_0,0]}\left(h^{-1}\left\langle h^{-1}t\right\rangle^{-1}, \tilde{g}_1\right)$ follows. (cf. Proof of Lemma 4.1.)

Now that all the symbols are in the class with metric $\tilde{g}_1$, Proposition 3.9 can be applied. We have

$$\sigma^{gw}\left(i\left[H_0, F_0\right]\right) = 2\tilde{\psi}_0\left\{p, \tilde{\psi}_0\right\} + r_{0,3}$$

with $r_{0,3} \in S_{[-t_0,0]}\left(\left\langle h^{-1}t\right\rangle, \tilde{g}_1\right)$ satisfying

$$\operatorname{supp} r_{0,3} \subset \operatorname{supp} \tilde{\psi}_0 \mod S_{[-t_0,0]}\left(h^{\infty}\right).$$

Step 3. Let us go to the third part $\sigma^{gw}\left(i\left[V, F_0\right]\right)$. Since

$$i\left[V, F_0\right] = i\left[V, \tilde{\psi}_0^{gw}\right]\tilde{\psi}_0^{gw} + i\tilde{\psi}_0^{gw}\left[V, \tilde{\psi}_0^{gw}\right],$$

we first compute $\sigma^{gw}\left(i\left[V, \tilde{\psi}_0^{gw}\right]\right)$. As $V \in S\left(\langle z\rangle^{\nu}, \langle z\rangle^{-2}dz^2\right)$, Proposition 3.6 is applicable and we have

$$\sigma^{gw}\left(i\left[V, \tilde{\psi}_0^{gw}\right]\right) = \left\{V, \tilde{\psi}_0\right\} + r_{0,4}$$

with

$$r_{0,4} = R_3 V\tilde{\psi}_0\big|_{z_1=z,\zeta_1=\zeta} - R_3\tilde{\psi}_0 V\big|_{z_1=z,\zeta_1=\zeta}$$

Exactly the same way as in the step 2 shows that

$$r_{0,4} \in S_{[-t_0,0]}\left(h^3\left\langle h^{-1}t\right\rangle^{\nu}, \tilde{g}_1\right)$$

Also as in the step 2, we have

$$\left\{V, \tilde{\psi}_0\right\} = -\frac{\partial V}{\partial x}\frac{\partial\tilde{\psi}_0}{\partial\xi} - \frac{\partial V}{\partial y}\frac{\partial\tilde{\psi}_0}{\partial\eta},$$

and, if we combine this with the assumption on V, we get

$$\left\{V, \tilde{\psi}_0\right\} \in S_{[-t_0,0]}\left(h\left\langle h^{-1}t\right\rangle^{\nu-1}, \tilde{g}_1\right),$$

where the support of $\tilde{\psi}_0$ is considered. Now that

$$\sigma^{gw}\left(i\left[V, F_0\right]\right) = \left\{V, \tilde{\psi}_0\right\}\sharp\tilde{\psi}_0 + \tilde{\psi}_0\sharp\left\{V, \tilde{\psi}_0\right\} + r_{0,4}\sharp\tilde{\psi}_0 + \tilde{\psi}_0\sharp r_{0,4}$$

and Proposition 3.9 can be applied, we obtain

$$\sigma^{gw}\left(i\left[V, F_0\right]\right) = 2\tilde{\psi}_0\left\{V, \tilde{\psi}_0\right\} + r_{0,5}$$

with $r_{0,5} \in S_{[-t_0,0]}\left(h^2\left\langle h^{-1}t\right\rangle^{\nu-1}, \tilde{g}_1\right)$ satisfying

$$\operatorname{supp} r_{0,5} \subset \operatorname{supp} \tilde{\psi}_0 \mod S_{[-t_0,0]}\left(h^{\infty}\right).$$

Step 4. Summing up the steps 1-3, we can write

$$\sigma^{gw}\left(\frac{\partial}{\partial t}F_0(t,h)+i\left[H,F_0\right]\right)=2\tilde{\psi}_0\frac{D}{Dt}\tilde{\psi}_0+2\tilde{\psi}_0\left[V,\tilde{\psi}_0\right]+r_{0,1}+r_{0,3}+r_{0,5},$$

with $2\tilde{\psi}_0\frac{D}{Dt}\tilde{\psi}_0\in S_{[-t_0,0]}\left(h^{-1}\left\langle h^{-1}t\right\rangle^{-1},\tilde{g}_1\right)$. Then, since $2\tilde{\psi}_0\frac{D}{Dt}\tilde{\psi}_0$ is nonpositive, a symbol $r_{0,6}\in S_{[-t_0,0]}\left(\left\langle h^{-1}t\right\rangle^{-1},\tilde{g}_1\right)$ is found by the sharp Gårding inequality such that

$$2\left(\tilde{\psi}_0\frac{D}{Dt}\tilde{\psi}_0\right)^{gw}(z,D_z;t,h)\le r_{0,6}^{gw}(z,D_z;t,h),$$

and thus combining $2\tilde{\psi}_0\left[V,\tilde{\psi}_0\right]$ and $r_{0,j}$'s, we obtain r_0 that is wanted. $\square$

Let t_0, T_0, C, δ, δ_0 and T_1 be as so far, and take an increasing sequence

$$0<\delta_0<\delta_1<\delta_2<\cdots<\frac{\delta}{4}.$$

Using these δ_j, C and T_1, we construct ψ_j similarly to ψ_0, that is, we put

$$\psi_{-1}(t,z,\zeta)=\chi\left(\frac{\left|\Delta\left(x^{-1}\right)\right|}{-4\delta_j t}\right)\chi\left(\frac{|\Delta y|}{\delta_j+Ct^{-1}}\right)\chi\left(\frac{\left|\Delta\left(x^2\xi\right)\right|}{\delta_j+Ct^{-1}}\right)\chi\left(\frac{|\Delta\eta|}{\delta_j+Ct^{-1}}\right),$$

and solve the equation

$$\frac{D}{Dt}\psi_j(t,z,\zeta)=\alpha(t)\frac{D}{Dt}\psi_{-1}(t,z,\zeta).$$

Then we define

$$\tilde{\psi}_j(z,\zeta;t,h)=\psi_j\left(h^{-1}t,z,h\zeta\right).$$

$\tilde{\psi}_1$ is bounded from below by a positive constant on $\operatorname{supp}\tilde{\psi}_0$. Indeed, when $t\le -T_1$, we have an expression using χ_i's, and when $t\ge -T_1$, we have only to observe the construction of the solution to the transport equation. Then it follows that $\operatorname{supp}\tilde{\psi}_1$ is in the interior of $\operatorname{supp}\tilde{\psi}_0$. Thus, if we decompose $r_0=r_0'+r_0''$ with

$$\operatorname{supp}r_0'\subset\operatorname{supp}\tilde{\psi}_0,\quad r_0''\in S_{[-t_0,0]}\left(h^{\infty}\right),$$

we have for large $C_1>0$

$$r_0'\left(z,\zeta;t,h\right)\le C_1\tilde{\psi}_1\left(z,\zeta;t,h\right).$$

Put

$$\begin{aligned}F_1(t,h)&=\varphi_1^{gw}(z,D_z;t,h),\\ \varphi_1&=-C_1t\tilde{\psi}_1\in S_{[-t_0,0]}\left(t,\tilde{g}_1\right).\end{aligned}$$

Let us consider the operator $\frac{\partial}{\partial t}F_1 + i\left[H, F_1\right]$ and iterate the argument similar to, or even easier than, the proof of Lemma 4.2. Each part is to be

$$\begin{aligned}
\sigma^{gw}\left(\frac{\partial}{\partial t}F_1\right) &= -C_1\tilde{\psi}_1 - +C_1 t\frac{\partial\tilde{\psi}_1}{\partial t}, \\
\sigma^{gw}\left(i\left[H_0, F_1\right]\right) &= -C_1 t\left\{p, \tilde{\psi}_1\right\} - C_1 t r_{1,2}, \\
\sigma^{gw}\left(i\left[V, F_1\right]\right) &= -C_1 t\left\{V, \tilde{\psi}_1\right\} - C_1 t r_{1,4},
\end{aligned}$$

where $r_{1,2}$ and $r_{1,4}$ correspond to $r_{0,2}$ and $r_{0,4}$ in the proof of Lemma 4.2, respectively. To sum up, we can write

$$\begin{aligned}
&\sigma^{gw}\left(\frac{\partial}{\partial t}F_1 + i\left[H, F_1\right]\right) \\
&= -C_1 t\frac{D\tilde{\psi}_1}{Dt} - C_1\tilde{\psi}_1 + r_0' - r_0 + r_0'' - C_1 t\left(\left\{V, \tilde{\psi}_1\right\} + r_{1,2} + r_{1,4}\right).
\end{aligned}$$

Since

$$-C_1 t\frac{D\tilde{\psi}_1}{Dt} - C_1\tilde{\psi}_1 + r_0' \in S_{[-t_0,0]}\left(1, \tilde{g}_1\right)$$

and

$$-C_1 t\frac{D\tilde{\psi}_1}{Dt} - C_1\tilde{\psi}_1 + r_0' \leq 0,$$

we can find by the sharp Gårding inequality $r_{1,7} \in S_{[-t_0,0]}\left(h, \tilde{g}_1\right)$ with the support in $\operatorname{supp}\tilde{\psi}_1$ modulo $S_{[-t_0,0]}\left(h^\infty\right)$ such that

$$\left[-C_1 t\frac{D\tilde{\psi}_1}{Dt} - C_1\tilde{\psi}_1 + r_0'\right]^{gw}(z, D_z; t, h) \leq r_{1,7}^{gw}(z, D_z; t, h).$$

Noting

$$r_0'' - C_1 t\left(\left\{V, \tilde{\psi}_1\right\} + r_{1,2} + r_{1,4}\right) \in S_{[-t_0,0]}\left(h^{\min\{1,2-\nu\}}, \tilde{g}_1\right),$$

$r_{1,7}$ with it makes a symbol $r_1 \in S_{[-t_0,0]}\left(h^{\min\{1,2-\nu\}}, \tilde{g}_1\right)$ that has the support in $\operatorname{supp}\tilde{\psi}_1$ modulo $S_{[-t_0,0]}\left(h^\infty\right)$ and satisfies

$$\frac{\partial}{\partial t}F_1(t,h) + i\left[H, F_1(t,h)\right] \leq r_1^{gw}(z, D_z; t, h) - r_0^{gw}(z, D_z; t, h).$$

Thus

$$\frac{\partial}{\partial t}\left(F_0(t,h) + F_1(t,h)\right) + i\left[H, F_0(t,h) + F_1(t,h)\right] \leq r_1^{gw}(z, D_z; t, h).$$

We repeat this procedure to get $F_j(t,h) = \varphi_j^{gw}(z, D_z; t, h)$ for $j = 1, 2, \ldots$. Suppose $\varphi_1, \ldots, \varphi_k$ is given such that

$$\frac{\partial}{\partial t}\sum_{j=0}^{k} F_j(t,h) + i\left[H, \sum_{j=0}^{k} F_j(t,h)\right] \le r_k^{gw}(z, D_z; t, h), \tag{4.6}$$

where $r_k \in S_{[-t_0,0]}\left(h^{k\min\{1,2-\nu\}}, \tilde{g}_1\right)$ has a decomposition $r_k = r_k' + r_k''$ such that

$$\operatorname{supp} r_k' \subset \operatorname{supp} \tilde{\psi}_k, \quad r_k'' \in S_{[-t_0,0]}\left(h^\infty\right).$$

Then one finds $C_{k+1} > 0$ such that

$$r_k'(z,\zeta;t,h) \le C_{k+1} h^{k\min\{1,2-\nu\}} \tilde{\psi}_{k+1}(h^{-1}t, x, h\zeta).$$

Put

$$F_{k+1}(t,h) = \varphi^{gw}(z, D_z; t, h), \quad \varphi_{k+1}(z,\zeta;t,h) = -C_{k+1} h^{k\min\{1,2-\nu\}} t \tilde{\psi}_{k+1}.$$

There exists $r_{k+1} \in S_{[-t_0,0]}\left(h^{(k+1)\min\{1,2-\nu\}}, \tilde{g}_1\right)$ with the support contained in $\operatorname{supp} \tilde{\psi}_{k+1}$ modulo $S_{[-t_0,0]}\left(h^\infty\right)$ satisfying

$$\frac{\partial}{\partial t} F_{k+1} + i\left[H, F_{k+1}\right] \le r_{k+1}^{gw}(z, D_z; t, h) - r_k^{gw}(z, D_z; t, h),$$

so that

$$\frac{\partial}{\partial t}\sum_{j=0}^{k+1} F_j(t,h) + i\left[H, \sum_{j=0}^{k+1} F_j(t,h)\right] \le r_{k+1}^{gw}(z, D_z; t, h).$$

φ_{k+1} is constructed.

Lemma 4.3 *There exists a pseudodifferential operator $F(t,h)$ with the symbol $\varphi \in S_{[-t_0,0]}(1)$ such that*

1. *$F(t,h)$ is differentiable in $t \in [-t_0, 0]$ and*
$$F(0,h) = F_0(0,h) = \psi_0^{gw}(0, z, hD_z)^2. \tag{4.7}$$
2. *For any $\varepsilon > 0$, choose small $\delta > 0$, then the support of $\varphi(z,\zeta;-t_0,h)$ is contained in*
$$\left\{(z,\zeta) \in T^*\mathbb{R}^n \;\middle|\; \left|z + \zeta_- h^{-1} t_0\right| < \varepsilon h^{-1} t_0, \left|\zeta - h^{-1}\zeta_-\right| < \varepsilon h^{-1}\right\},$$
modulo $S\left(h^\infty\right)$.
3. *The Heisenberg derivative of $F(t,h)$ satisfies*
$$\delta F(t,h) := \frac{\partial}{\partial t} F(t,h) + i\left[H, F(t,h)\right] \le R(t),$$
where $R(t)$ is an L^2-bounded operator with $\sup_{-t_0 \le t \le 0} \|R(t)\| = O(h^\infty)$.

Proof. Since $\varphi_j \in S_{[-t_0,0]}\left(h^{(j-1)\min\{1,2-\nu\}}\right)$ for $j=0,1,2,\ldots$, the asymptotic sum

$$\varphi \sim \varphi_0 + \sum_{j=1}^{\infty} \varphi_j$$

exists by Proposition 3.2. Here, in the definition of φ, we take the asymptotic sum $\sum_{j=1}^{\infty} \varphi_j$ according to the proof of Proposition 3.2, and define φ by φ_0 added to the sum. Set

$$F(t,h) = \varphi^{gw}(z, D_z; t, h).$$

1. φ is defined by the locally finite sum with respect to $h>0$. Then $F(t,h)$ is differentiable in t for each h from the argument right before the Lemma 4.2. (4.7) is a consequence of the definition of φ.

2. Since by a formula in Appendix A

$$\begin{aligned}\left|z+\zeta_- h^{-1}t_0\right| &\leq \left|z - z\left(-h^{-1}t_0; z_0, \zeta_0\right)\right| + \left|z\left(-h^{-1}t_0; z_0, \zeta_0\right) + \zeta_- h^{-1}t_0\right| \\ &\leq \left|\frac{1}{x} - \frac{1}{x(-h^{-1}t_0; z_0, \zeta_0)}\right| |y| + \frac{\left|y - y(-h^{-1}t_0; z_0, \zeta_0)\right|}{x(-h^{-1}t_0; z_0, \zeta_0)} \\ &\quad + \left|z\left(-h^{-1}t_0; z_0, \zeta_0\right) + \zeta_- h^{-1}t_0\right|,\end{aligned}$$

taking $\delta>0$ small enough, we have the bound $\varepsilon h^{-1}t_0$ from above for the right-hand side on the support of $\varphi(z,\zeta; t_0, h)$. The second inequality is similarly obtained.

3. The conclusion follows from (4.6) and Theorem 3.10. □

Proof of Theorem 1.3. We have

$$\begin{aligned}&\langle F(0,h)u_{t_0}, u_{t_0}\rangle_{L^2\left(\mathbb{R}^n, \sqrt{g}dz\right)} \\ &= \langle F(-t_0,h)u_0, u_0\rangle_{L^2\left(\mathbb{R}^n, \sqrt{g}dz\right)} + \int_{-t_0}^{0} \langle \delta F(t,h)u_t, u_t\rangle_{L^2\left(\mathbb{R}^n, \sqrt{g}dz\right)}\, dt \\ &\leq \langle F(-t_0,h)u_0, u_0\rangle_{L^2\left(\mathbb{R}^n, \sqrt{g}dz\right)} + t_0 \sup_{0\leq t\leq t_0} \|R(t,h)\|.\end{aligned}$$

The second term in the last formula is $O(h^\infty)$. Thus we have only to check $\|F(-t_0,h)u_0\|_{L^2\left(\mathbb{R}^n,\sqrt{g}dz\right)} = O(h^\infty)$. By the assumption, there exists a compactly supported C^∞ function $\tilde{\varphi}$ on $\mathbb{R}^{2n}$ such that $\tilde{\varphi}=1$ near $(-t_0\zeta_-, \zeta_-)$ and

$$\left\|\sqrt[4]{g\left(z\right)}\tilde{\varphi}^{gw}\left(hz, hD_z\right)u_0\right\|_{L^2(\mathbb{R}^n)} = O\left(h^\infty\right).$$

For, if

$$\left\|\tilde{\varphi}^{w}\left(hz, hD_z\right)u_0\right\| = O\left(h^\infty\right),$$

then, in the right-hand side of

$$\begin{aligned}\tilde{\varphi}^{gw}\left(hz, hD_z\right)u_0 =& \tilde{\varphi}^{gw}\left(hz, hD_z\right)\tilde{\varphi}^{w}\left(hz, hD_z\right)u_0 \\ &+ \frac{1}{\sqrt[4]{g(z)}}\tilde{\varphi}^{w}\left(hz, hD_z\right)\sqrt[4]{g(z)}\left(1-\tilde{\varphi}^{w}\left(hz, hD_z\right)\right)u_0,\end{aligned}$$

the first term is $O(h^\infty)$ from the assumption, and the second term is also $O(h^\infty)$ from the variation of Proposition 3.9 for $S\left(m, dz^2+h^2d\zeta^2\right)$, which can be seen easily. By Lemma 4.3 2, choosing $\varepsilon>0$ small, we suppose that

$$\operatorname{supp}\varphi(\cdot,\cdot;t_0,h)\cap\operatorname{supp}(1-\tilde{\varphi}(h\cdot,h\cdot))=\varnothing$$

when $h>0$ is small. Then

$$\begin{aligned}F(-t_0,h)u_0=&\varphi^{gw}(z,D_z;-t_0,h)u_0(x)\\ =&\varphi^{gw}(z,D_z;-t_0,h)\circ\tilde{\varphi}^{gw}(hz,hD_z)u_0(z)\\ &+\varphi^{gw}(z,D_z;-t_0,h)\circ(1-\tilde{\varphi}^{gw}(hz,hD_z))\,u_0(z).\end{aligned}$$

The first term is $O(h^\infty)$ by the assumption, and the second term is also $O(h^\infty)$. For we can apply Proposition 3.9 to $\varphi^w(z,D_z;t_0,h)\circ(1-\tilde{\varphi}^w(hz,hD_z))$, since $\tilde{\varphi}(h\cdot,h\cdot)\in S(1,\tilde{g}_1)$, and all terms in its asymptotic expansion vanish. Thus the theorem is proved. □

5 Proof of Theorem 1.6

Let $X={}^qX=S^n_+$ and defined the mapping $q:X\to{}^qX$ by

$$q=q(z)=\left(2+|z|^2\right)^{\frac{1}{2}}z.$$

q gives a bijection between X and ${}^qX$ with the inverse

$$z=(1+\langle q\rangle)^{-\frac{1}{2}}q.$$

Since $\langle q\rangle^{-1}=\langle z\rangle^{-2}$, ${}^qX$ is thought to be X whose C^∞ structure near the boundary is generated by new boundary defining function $\langle z\rangle^{-2}$. q is C^∞ mapping, but q^{-1} is not. However, $q^*:\dot{C}^\infty({}^qX)\to\dot{C}^\infty(X)$ is bijective and extends to $C^{-\infty}({}^qX)\to C^{-\infty}(X)$ bijectively.

Let $u\in\mathcal{S}'(\mathbb{R}^n)=C^{-\infty}(X)$, and we first assume

$$(z_0,\zeta_0)\in\left(S^{n-1}\times\mathbb{R}^n\right)\setminus\mathrm{WF}_{\mathrm{qsc}}(u),$$

which is equivalent to

$$(z_0,\zeta_0)\in\left(S^{n-1}\times\mathbb{R}^n\right)\setminus\mathrm{WF}_{\mathrm{sc}}\left((q^*)^{-1}u\right).$$

Then there exists $\varphi\in C_0^\infty\left(\mathbb{R}^{2n}\right)$ such that $\varphi(z_0,\zeta_0)\neq 0$ and

$$\left\|\varphi(hq,D_q)(q^*)^{-1}u\right\|=O(h^\infty),\tag{5.1}$$

where $\varphi(hq,D_q)$ is the standard [left] quantization of $\varphi(hq,\tau)$.

If we introduce the variables $q = q(z) = \left(2+|z|^2\right)^{\frac{1}{2}} z$ and $p = q(w) = \left(2+|w|^2\right)^{\frac{1}{2}} w$ in the right-hand side of (5.1), we get

$$\begin{aligned}\left\|\varphi\left(hq, D_q\right)\left(q^*\right)^{-1} u\right\|^2 =& 8\int\left|\int e^{i(q(z)-q(w))\tau}\langle z\rangle\left(2+|z|^2\right)^{\frac{n-2}{4}}\right.\\ &\left.\cdot\langle w\rangle^2\left(2+|w|^2\right)^{\frac{n-2}{2}}\varphi\left(hq(z),\tau\right)u(w)dw\bar{d}\tau\right|^2 dz.\end{aligned} \tag{5.2}$$

We have

$$z^i - w^i = (1+\langle q\rangle)^{-\frac{1}{2}} q^i - (1+\langle p\rangle)^{-\frac{1}{2}} p^i = \sum_{j=1}^{n}\Phi_{ij}(z,w)\left(q^j-p^j\right)$$

with

$$\begin{aligned}&\Phi_{ij}(z,w)\\ &=\delta_{ij}\int_0^1\frac{dt}{\left(1+\langle p+t\left(q-p\right)\rangle\right)^{\frac{1}{2}}}-\frac{1}{2}\int_0^1\frac{\left(p^i+t\left(q^i-p^i\right)\right)\left(p^j+t\left(q^j-p^j\right)\right)}{\left(1+\langle p+t\left(q-p\right)\rangle\right)^{\frac{3}{2}}\langle p+t\left(q-p\right)\rangle}dt.\end{aligned}$$

Since for $\tau\in\mathbb{R}^n$ with $|\tau|=1$

$$\begin{aligned}&\sum_{i,j=1}^{n}\Phi_{ij}(z,w)\tau_i\tau_j\\ &=\int_0^1\frac{dt}{\left(1+\langle p+t\left(q-p\right)\rangle\right)^{\frac{1}{2}}}-\frac{1}{2}\int_0^1\frac{\left[\sum_{i=1}^{n}\left(p^i+t\left(q^i-p^i\right)\right)\tau_i\right]^2}{\left(1+\langle p+t\left(q-p\right)\rangle\right)^{\frac{3}{2}}\langle p+t\left(q-p\right)\rangle}dt,\end{aligned}$$

we have

$$\frac{1}{2}\int_0^1\frac{dt}{\left(1+\langle p+t\left(q-p\right)\rangle\right)^{\frac{1}{2}}}\le\sum_{i,j=1}^{n}\Phi_{ij}(z,w)\tau_i\tau_j\le\int_0^1\frac{dt}{\left(1+\langle p+t\left(q-p\right)\rangle\right)^{\frac{1}{2}}}.$$

This particularly means that $\Phi(z,w)$ is nondegenerate, so that we can change the variables $\tau\to\zeta=\Phi(z,w)^{-1}\tau$ in (5.2). Then we have (5.2) rewritten by

$$\left\|\varphi\left(hq, D_q\right)\left(q^*\right)^{-1} u\right\|^2 = 8\int\left|\int e^{i(z-w)\zeta}\tilde{\varphi}(z,w,\zeta;h)u(w)dw\bar{d}\zeta\right|^2 dz$$

with

$$\begin{aligned}\tilde{\varphi}\left(z,w,\zeta;h\right)=&\langle z\rangle\left(2+|z|^2\right)^{\frac{n-2}{4}}\langle w\rangle^2\left(2+|w|^2\right)^{\frac{n-2}{2}}\det\Phi(z,w)\\ &\cdot\varphi\left(h\left(2+|z|^2\right)^{\frac{1}{2}}z,\Phi(z,w)\zeta\right).\end{aligned}$$

We are going to find out the class to which $\tilde{\varphi}$ belongs and apply the argument established in Section 3. Our first observation is

Lemma 5.1

$$\tilde{\varphi} \in S\left(\frac{\langle z\rangle^{\frac{n}{2}}\langle w\rangle^{n}}{\langle z;w\rangle^{n}}, \frac{dz^2}{\langle z\rangle^2}+\frac{dw^2}{\langle w\rangle^2}+\frac{d\zeta^2}{\langle z;w\rangle^2}\right).$$

Proof. We first claim that there is $C_n > 0$ such that

$$\langle z;w\rangle^{-n} \leq \int_0^1 \frac{dt}{\langle p+t(q-p)\rangle^{\frac{n}{2}}} \leq C_n \langle z;w\rangle^{-n}$$

for each positive odd integer n. Indeed the first inequality follows from

$$\begin{aligned}
\langle p+t(q-p)\rangle^{-\frac{n}{2}} &\geq \left(1+\max\left\{|p|^2, |q|^2\right\}\right)^{-\frac{n}{4}} \\
&\geq \left(1+|p|^2+|q|^2\right)^{-\frac{n}{4}} \\
&\geq \left(1+|w|^2+|z|^2\right)^{-\frac{n}{2}}.
\end{aligned}$$

For the second inequality we consider the four cases:

(i) $|q-p| \leq \frac{1}{2}|q|$, (ii) $|q-p| \leq \frac{1}{2}|p|$,

(iii) $|q-p| \geq \frac{1}{4}(|q|+|p|) \geq 1$, (iv) $\frac{1}{4}(|q|+|p|) \leq 1$.

In the case (i), noting

$$\frac{1}{2}|q| \leq |p+t(q-p)| \leq \frac{3}{2}|q|,$$

we obtain

$$\begin{aligned}
\langle p+t(q-p)\rangle^{-\frac{n}{2}} &\leq \left(1+\frac{1}{4}|q|^2\right)^{-\frac{n}{4}} \\
&\leq \left(1+\frac{1}{13}\left(|q|^2+|p|^2\right)\right)^{-\frac{n}{4}} \\
&\leq C_n\left(1+|z|^2+|w|^2\right)^{-\frac{n}{2}}.
\end{aligned}$$

The case (ii) is dealt with in exactly the same way as the case (i). The case (iv) is a very obvious one, and so the case (iii) is left. By direct computation we get

$$\begin{aligned}\int_0^1 \frac{dt}{\langle p+t\,(q-p)\rangle^{\frac{n}{2}}} &\leq \int_0^1 \frac{dt}{\left(1+(|p|-t\,|q-p|)^2\right)^{\frac{n}{4}}} \\ &= \frac{1}{|q-p|}\int_{|p|-|q-p|}^{|p|} \frac{dt}{(1+t^2)^{\frac{n}{4}}} \\ &\leq \frac{1}{|q-p|}\int_{-\frac{1}{2}|q-p|}^{\frac{1}{2}|q-p|} \frac{dt}{(1+t^2)^{\frac{n}{4}}} \\ &\leq \frac{C_n}{\langle q-p\rangle^{\frac{n}{2}}}.\end{aligned}$$

With the condition (iii) this shows the claimed inequality. Thus the claim is verified.

We then see

$$\Phi_{ij}(z,w) \in S\left(\langle z;w\rangle^{-1}; \langle z;w\rangle^{-2}\,dz^2 + \langle z;w\rangle^{-2}\,dw^2\right),$$

and as a polynomial in $\Phi_{ij}(z,w)$ of degree n

$$\det\Phi(z,w) \in S\left(\langle z;w\rangle^{-n}; \langle z;w\rangle^{-2}\,dz^2 + \langle z;w\rangle^{-2}\,dw^2\right).$$

On the support of $\tilde{\varphi}$ we have the estimates

$$C^{-1}h^{-\frac{1}{2}} \leq |z| \leq Ch^{-\frac{1}{2}}, \quad |\zeta| \leq C\,\langle z;w\rangle\,.$$

Then the lemma follows. $\square$

Put

$$\psi(z,\zeta;h) = e^{-iD_\theta D_\zeta}\tilde{\varphi}\left(z+\frac{1}{2}\theta, z-\frac{1}{2}\theta,\zeta;h\right)\Big|_{\theta=0},$$

then

$$\psi^w(z,D_z;h) = \tilde{\varphi}\left(z,z',D_z;h\right).$$

Lemma 5.1 in particular means

$$\tilde{\varphi}\left(z+\frac{1}{2}\theta, z-\frac{1}{2}\theta,\zeta;h\right) \in S\left(h^{-\frac{n}{4}}\right),$$

and so with Lemma 3.8 it implies

$$e^{-iD_\theta D_\zeta}\tilde{\varphi}\left(z+\frac{1}{2}\theta, z-\frac{1}{2}\theta,\zeta;h\right) \in S\left(h^{-\frac{n}{4}}\right).$$

Therefore we have $\psi \in S\left(h^{-\frac{n}{4}}\right)$. The more precise estimate for ψ can be obtained. Indeed, we have

$$\psi \in S\left(h^{-\frac{n}{4}}, hdz^2 + hd\zeta^2\right),$$

which is shown by the similar type of argument used in *Step 2* of the proof of Lemma 4.2, combined with Lemma 5.1 and 3.8. Also we can estimate the remainder term:

$$\psi(z,\zeta;h)-\tilde{\varphi}\left(z,z,\zeta;h\right)\in S\left(h^{-\frac{n}{4}+1},hdz^2+hd\zeta^2\right). \tag{5.3}$$

Put for $z\neq 0$

$$\Psi(z)=\left(\delta_{ij}-\frac{z^iz^j}{2|z|^2}\right)_{ij}^{-1}=\left(\delta_{ij}+\frac{z^iz^j}{|z|^2}\right)_{ij}.$$

We want to show the *ellipticity* of $\psi\left(h^{-\frac{1}{2}}z,h^{-\frac{1}{2}}\zeta;h\right)$ at $(z_0,\Psi(z_0)\zeta_0)$. Note

$$\tilde{\varphi}\left(z,z,\zeta;h\right)=\langle z\rangle^3\left(2+|z|^2\right)^{\frac{3n-6}{4}}\det\Phi(z,z)\varphi\left(h\left(2+|z|^2\right)^{\frac{1}{2}}z,\Phi(z,z)\zeta\right)$$

with

$$\Phi_{ij}(z,z)=\delta_{ij}\frac{1}{(1+\langle q\rangle)^{\frac{1}{2}}}-\frac{1}{2}\frac{q^iq^j}{(1+\langle q\rangle)^{\frac{3}{2}}\langle q\rangle}.$$

It is easy to see

$$\left\langle h^{-\frac{1}{2}}z_0\right\rangle^3\left(2+\left|h^{-\frac{1}{2}}z_0\right|^2\right)^{\frac{3n-6}{4}}\det\Phi(h^{-\frac{1}{2}}z_0,h^{-\frac{1}{2}}z_0)\geq Ch^{-\frac{n}{4}}.$$

So we have to show the uniform positivity of

$$\varphi\left(h\left(2+\left|h^{-\frac{1}{2}}z_0\right|^2\right)^{\frac{1}{2}}h^{-\frac{1}{2}}z_0,\Phi(h^{-\frac{1}{2}}z_0,h^{-\frac{1}{2}}z_0)h^{-\frac{1}{2}}\Psi\left(z_0\right)\zeta_0\right).$$

As h to 0,

$$h\left(2+\left|h^{-\frac{1}{2}}z_0\right|^2\right)^{\frac{1}{2}}h^{-\frac{1}{2}}z_0\to z_0$$

and

$$\Phi(h^{-\frac{1}{2}}z_0,h^{-\frac{1}{2}}z_0)h^{-\frac{1}{2}}\Psi\left(z_0\right)\zeta_0\to\zeta_0.$$

Hence

$$\tilde{\varphi}\left(h^{-\frac{1}{2}}z_0,h^{-\frac{1}{2}}z_0,h^{-\frac{1}{2}}\Psi\left(z_0\right)\zeta_0;h\right)\geq Ch^{-\frac{n}{4}}$$

uniformly in small $h>0$. Thus, using (5.3), $\psi\left(h^{-\frac{1}{2}}z,h^{-\frac{1}{2}}\zeta;h\right)$ is elliptic at $(z_0,\Psi(z_0)\zeta_0)$. Since

$$\|\psi^w\left(z,D_z;h\right)u\|=O\left(h^\infty\right),$$

it follows that

$$(z_0, \Psi(z_0)\zeta_0) \notin \mathrm{HWF}(u). \tag{5.4}$$

Conversely suppose (5.4) with $z_0 \neq 0$. We can assume $|z_0| = 1$. We just reverse the procedure above. Choose $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\varphi(z_0, \zeta_0) \neq 0$, and put

$$\begin{aligned}
\tilde{\varphi}(z, w, \zeta; h) = \langle z \rangle \left(2 + |z|^2\right)^{\frac{n-2}{4}} \langle w \rangle^2 \left(2 + |w|^2\right)^{\frac{n-2}{2}} \det \Phi(z, w) \\
\cdot \varphi\left(h\left(2 + |z|^2\right)^{\frac{1}{2}} z, \Phi(z, w)\zeta\right), \\
\psi(z, \zeta; h) = e^{-iD_\theta D_\zeta} \tilde{\varphi}\left(z + \frac{1}{2}\theta, z - \frac{1}{2}\theta, \zeta; h\right)\Big|_{\theta=0}.
\end{aligned}$$

$\psi(z, \zeta; h)$ has support in that of $\tilde{\varphi}(z, z, \zeta; h)$ modulo $S(h^\infty)$, so that if we choose φ whose support is sufficiently small, we obtain

$$\left\|\varphi(hq, D_q)(q^*)^{-1} u\right\| = 8\left\|\psi^w(z, D_z; h)u\right\| = O(h^\infty).$$

Then $(z_0, \zeta_0) \in \left(S^{n-1} \times \mathbb{R}^n\right) \setminus \mathrm{WF}_{\mathrm{qsc}}(u)$ follows.

A Formulae for Coordinate Transformation

For a point $z = (z^1, \ldots, z^n) \in \mathbb{R}^n \subset X$, $z \neq 0$ we set

$$x = \frac{1}{|z|}, \quad \omega = \left(\omega^1, \ldots, \omega^n\right) = \frac{z}{|z|}.$$

Since $z \neq 0$, there exists non-zero ω^k, and so, when $\pm\omega^k > 0$, we can get rid of ω^k to make local coordinates $\left(x, y_{(\pm k)}\right) = \left(x, y^1_{(\pm k)}, \ldots, y^{n-1}_{(\pm k)}\right)$ of $X (\supset \mathbb{R}^n)$ near the boundary respectively:

$$y^j_{(\pm k)} = \begin{cases} \omega^j, & \text{for } 1 \leq j \leq k-1, \\ \omega^{j+1}, & \text{for } k \leq j \leq n-1. \end{cases}$$

We denote $y_{(\pm k)}$ simply by y if there is no confusion. We introduce local coordinates (z, ζ) and (x, y, ξ, η) of the cotangent bundle T^*X corresponding to z and (x, y) respectively. Now we write down formulae for the coordinate change between the above coordinates that will be needed later. We consider only the case where $x^n > 0$, i.e., $y = y_{(+n)}$, which is enough for the purpose of this paper. Introducing a notation

$$y^n = \sqrt{1 - (y^1)^2 - \cdots - (y^{n-1})^2},$$

we have

$$x = \frac{1}{|z|}, \quad y^i = \frac{z^i}{|z|} \quad (i = 1, \ldots, n-1); \quad z^i = \frac{y^i}{x} \quad (i = 1, \ldots, n),$$

and thus

$$\begin{aligned}\partial_{z^i} &= -\frac{z^i}{|z|^3}\partial_x + \sum_{j=1}^{n-1}\left(\frac{\delta_i^j}{|z|} - \frac{z^i z^j}{|z|^3}\right)\partial_{y^j} \\ &= -x^2 y^i \partial_x + x\sum_{j=1}^{n-1}\left(\delta_i^j - y^i y^j\right)\partial_{y^j} \quad (i = 1, \dots, n),\end{aligned} \tag{A.1}$$

$$\partial_x = -\frac{1}{x^2}\sum_{i=1}^{n} y^i \partial_{z^i} = -|z|\sum_{i=1}^{n} z^i \partial_{z^i}, \tag{A.2}$$

$$\partial_{y^i} = \frac{1}{x}\partial_{z^i} - \frac{1}{x}\frac{y^i}{y^n}\partial_{z^n} = |z|\partial_{z^i} - |z|\frac{z^i}{z^n}\partial_{z^n} \quad (i = 1, \dots, n-1), \tag{A.3}$$

$$dz^i = -\frac{y^i}{x^2}dx + \frac{1}{x}dy^i = -|z|z^i dx + |z|dy^i \quad (i = 1, \dots, n), \tag{A.4}$$

$$dx = -\sum_{i=1}^{n}\frac{z^i}{|z|^3}dz^i = -x^2\sum_{i=1}^{n} y^i dz^i, \tag{A.5}$$

$$dy^i = \sum_{j=1}^{n}\left(\frac{\delta_j^i}{|z|} - \frac{z^i z^j}{|z|^3}\right)dz^j = x\sum_{j=1}^{n}\left(\delta_j^i - y^i y^j\right)dz^j. \tag{A.6}$$

Then for the same point in $T^*\left(\mathbb{R}^n \setminus \{0\}\right) \subset T^*X$:

$$\sum_{i=1}^{n}\zeta_i dz^i = \xi dx + \sum_{i=1}^{n-1}\eta_i dy^i,$$

we obtain

$$\begin{aligned}\zeta_i &= -\frac{z^i}{|z|^3}\xi + \sum_{j=1}^{n-1}\left(\frac{\delta_i^j}{|z|} - \frac{z^i z^j}{|z|^3}\right)\eta_j \\ &= -x^2 y^i \xi + x\sum_{j=1}^{n-1}\left(\delta_i^j - y^i y^j\right)\eta_j \quad (i = 1, \dots, n)\end{aligned} \tag{A.7}$$

$$\xi = -\frac{1}{x^2}\sum_{i=1}^{n} y^i \zeta_i = -|z|\sum_{i=1}^{n} z^i \zeta_i, \tag{A.8}$$

$$\eta_i = \frac{1}{x}\zeta_i - \frac{1}{x}\frac{y^i}{y^n}\zeta_n = |z|\zeta_i - |z|\frac{z^i}{z^n}\zeta_n \quad (i = 1, \dots, n-1), \tag{A.9}$$

and on the tangent space to the cotangent bundle,

$$\partial_{\zeta_i} = \begin{cases} -\frac{y^i}{x^2}\partial_\xi + \frac{1}{x}\partial_{\eta_i} = -|z|z^i\partial_\xi + |z|\partial_{\eta_i}, & \text{if } i \neq n, \\ -\frac{y^n}{x^2}\partial_\xi - \sum_{j=1}^{n-1}\frac{y^j}{xy^n}\partial_{\eta_j} = |z|z^n\partial_\xi - \sum_{j=1}^{n-1}\frac{|z|z^j}{z^n}\partial_{\eta_j}, & \text{if } i = n, \end{cases} \tag{A.10}$$

$$\partial_\xi = -x^2\sum_{i=1}^n y^i\partial_{\zeta_i} = -\sum_{i=1}^n \frac{z^i}{|z|^3}\partial_{\zeta_i}, \tag{A.11}$$

$$\partial_{\eta_i} = x\sum_{j=1}^n\left(\delta_i^j - y^iy^j\right)\partial_{\zeta_j} = \sum_{j=1}^n\left(\frac{\delta_i^j}{|z|} - \frac{z^iz^j}{|z|^3}\right)\partial_{\zeta_j}. \tag{A.12}$$

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