Spectral Shift Function for Schrödinger Operators in Constant Magnetic Fields

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Abstract: We consider the three-dimensional Schrödinger operator with constant magnetic field, perturbed by an appropriate short-range electric potential, and investigate various asymptotic properties of the corresponding spectral shift function (SSF). First, we analyse the singularities of the SSF at the Landau levels. Further, we study the strong magnetic field asymptotic behaviour of the SSF; here we distinguish between the asymptotics far from the Landau levels, and near a given Landau level. Finally, we obtain a Weyl type formula describing the high energy behaviour of the SSF.
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1 Introduction

In this survey article based on the papers [7], [10], and [8], we consider the 3D Schrödinger operator with constant magnetic field of scalar intensity $b > 0$, perturbed by an electric potential V which decays fast enough at infinity, and discuss various asymptotic properties of the corresponding spectral shift function.
More precisely, let $H_0 = H_0(b) := (i\nabla + \mathbf{A})^2 - b$ be the unperturbed operator, essentially self-adjoint on $C_0^\infty(\mathbb{R}^3)$. Here $\mathbf{A} = \left(-\frac{bx_2}{2}, \frac{bx_1}{2}, 0\right)$ is the magnetic potential which generates the constant magnetic field $\mathbf{B} = \text{curl } \mathbf{A} = (0,0,b)$, $b > 0$. It is well-known that $\sigma(H_0) = \sigma_{\rm ac}(H_0) = [0,\infty)$ (see [1]), where $\sigma(H_0)$ stands for the spectrum of H_0, and $\sigma_{\rm ac}(H_0)$ for its absolutely continuous spectrum. Moreover, the so-called Landau levels $2bq$, $q \in \mathbb{Z}_+ := \{0,1,\ldots\}$, play the role of thresholds in $\sigma(H_0)$.
For $\mathbf{x} = (x_1,x_2,x_3) \in \mathbb{R}^3$ we denote by $X_\perp = (x_1,x_2)$ the variables on the plane perpendicular to the magnetic field. Throughout the paper assume that V satisfies

$$V \not\equiv 0, \quad V \in C(\mathbb{R}^3), \quad |V(\mathbf{x})| \leq C_0 \langle X_\perp\rangle^{-m_\perp} \langle x_3\rangle^{-m_3}, \quad \mathbf{x} = (X_\perp, x_3) \in \mathbb{R}^3, \tag{1.1}$$

with $C_0 > 0$, $m_\perp > 2$, $m_3 > 1$, and $\langle x\rangle := (1+|x|^2)^{1/2}$, $x \in \mathbb{R}^d$, $d \geq 1$. Some of our results hold under a more restrictive assumption than (1.1), namely

$$V \not\equiv 0, \quad V \in C(\mathbb{R}^3), \quad |V(\mathbf{x})| \leq C_0 \langle \mathbf{x}\rangle^{-m_0}, \quad m_0 > 3, \quad \mathbf{x} \in \mathbb{R}^3. \tag{1.2}$$

Note that (1.2) implies (1.1) with any $m_3 \in (0, m_0)$ and $m_\perp = m_0 - m_3$. In particular, we can choose $m_3 \in (1, m_0 - 2)$ so that $m_\perp > 2$.
On the domain of H_0 define the operator $H = H(b) := H_0 + V$. Obviously, $\inf\sigma(H) \leq \inf\sigma(H_0) = 0$. Moreover, if (1.1) holds, then for every $E < \inf\sigma(H)$ we have $(H-E)^{-1} - (H_0-E)^{-1} \in S_1$ where S_1 denotes the trace class. Hence, there exists a unique function $\xi = \xi(\cdot; H, H_0) \in L^1(\mathbb{R}; (1+E^2)^{-1}dE)$ which vanishes identically on $(-\infty, \inf\sigma(H))$ such that *the Lifshits-Krein trace formula*

$$\text{Tr}\,(f(H) - f(H_0)) = \int_{\mathbb{R}} \xi(E; H, H_0) f'(E)\, dE$$

holds for each $f \in C_0^\infty(\mathbb{R})$ (see the original works [22], [20], the survey article [5], or Chapter 8 of the monograph [45]). The function $\xi(\cdot; H, H_0)$ is called *the spectral shift function* (SSF) for the operator pair (H, H_0). If $E < 0 = \inf \sigma(H_0)$, then the spectrum of H below E could be at most discrete, and for almost every $E < 0$ we have

$$\xi(E; H, H_0) = -N(E; H) \tag{1.3}$$

where $N(E; H)$ denotes the number of eigenvalues of H lying in the interval $(-\infty, E)$, and counted with their multiplicities. On the other hand, for almost every $E \in [0, \infty)$, the SSF $\xi(E; H, H_0)$ is related to the scattering determinant det $S(E; H, H_0)$ for the pair (H, H_0) by *the Birman-Krein formula*

$$\det S(E; H, H_0) = e^{-2\pi i \xi(E; H, H_0)}$$

(see [4] or [45, Section 8.4]). A survey of various asymptotic results concerning the SSF for numerous quantum Hamiltonians is contained in [40].
A priori, the SSF $\xi(E; H, H_0)$ is defined for almost every $E \in \mathbb{R}$. In this article we will identify this SSF with a representative of its equivalence class which is well-defined on $\mathbb{R} \setminus 2b\mathbb{Z}_+$, bounded on every compact subset of $\mathbb{R} \setminus 2b\mathbb{Z}_+$, and continuous on $\mathbb{R} \setminus (2b\mathbb{Z}_+ \cup \sigma_{\rm pp}(H))$ where $\sigma_{\rm pp}(H)$ denotes the set of the eigenvalues of H. In the case of perturbations V of definite sign this representative is described explicitly in Subsection 3.1 below; in the case of general non-sign-definite perturbations its description can be found in [7, Section 3].
In the present article we investigate the behaviour of the SSF in several asymptotic regimes:

- First, we analyse the singularities of the SSF at the Landau levels. In other words, we fix $q \in \mathbb{Z}_+$, and investigate the behaviour of $\xi(2bq + \lambda; H, H_0)$ as $\lambda \to 0$.
- Further, we study the strong-magnetic-field asymptotics of the SSF, i.e. the behaviour of the SSF as $b \to \infty$. Here we distinguish between the asymptotics far from the Landau levels, and the asymptotics near a given Landau level.
- Finally, we obtain a Weyl type formula describing the high-energy asymptotics of the SSF.

The paper is organised as follows. In Section 2 we formulate our main results, and discuss briefly on them. More precisely, in Subsection 2.1 we introduce some basic notations used throughout the paper, Subsection 2.2 contains the results on the singularities of the SSF at the Landau levels, Subsection 2.3 is devoted to the strong-magnetic-field asymptotics of the SSF, and Subsection 2.4 to its high-energy behaviour. Section 3 contains some auxiliary results. In Subsection 3.1 we describe the representation of the SSF in the case of perturbations of fixed sign, due to A. Pushnitski (see [29]), while in Subsection 3.2 we establish estimates of some auxiliary operators of Birman-Schwinger type which are used systematically in the proofs of the main results. Some of these proofs could be found in Section 4: in Subsection 4.1 we prove the results of Subsection 2.2, and in Subsection 4.2 some of the results of Subsection 2.3. Since the detailed proofs have already been published in [10] and [7], the proofs presented here are somewhat sketchy, preference being given to the main ideas rather than to the technical details.

2 Main Results

2.1 Notations and preliminaries

In this subsection we introduce our basic notations used throughout the paper.
We denote by S_∞ the class of linear compact operators acting in a given Hilbert space.
Let $T = T^* \in S_\infty$. Denote by $\mathbb{P}_I(T)$ the spectral projection of T associated with the interval $I \subset \mathbb{R}$. For $s > 0$ set

$$n_\pm(s; T) := \text{rank}\, \mathbb{P}_{(s,\infty)}(\pm T).$$

For an arbitrary (not necessarily self-adjoint) operator $T \in S_\infty$ put

$$n_*(s;T) := n_+(s^2; T^*T), \quad s > 0. \tag{2.1}$$

If $T = T^*$, then evidently

$$n_*(s;T) = n_+(s,T) + n_-(s;T), \quad s > 0. \tag{2.2}$$

Moreover, if $T_j = T_j^* \in S_\infty$, $j = 1, 2$, then the Weyl inequalities

$$n_\pm(s_1 + s_2, T_1 + T_2) \le n_\pm(s_1, T_1) + n_\pm(s_2, T_2) \tag{2.3}$$

hold for each s_1, $s_2 > 0$.

Further, we denote by S_p, $p \in (0, \infty)$, the Schatten-von Neumann class of compact operators for which the functional $\|T\|_p : = \left(p \int_0^\infty s^{p-1} n_*(s;T)\, ds\right)^{1/p}$ is finite. If $T \in S_p$, $p \in (0, \infty)$, then the following elementary inequality of Chebyshev type

$$n_*(s;T) \le s^{-p}\|T\|_p^p \tag{2.4}$$

holds for every $s > 0$. If $T = T^* \in S_p$, $p \in (0, \infty)$, then (2.2) and (2.4) imply

$$n_\pm(s;T) \le s^{-p}\|T\|_p^p, \quad s > 0. \tag{2.5}$$

2.2 Singularities of the SSF at the Landau levels

Introduce the Landau Hamiltonian

$$h(b) := \left(i\frac{\partial}{\partial x_1} - \frac{bx_2}{2}\right)^2 + \left(i\frac{\partial}{\partial x_2} + \frac{bx_1}{2}\right)^2 - b, \tag{2.6}$$

i.e. the 2D Schrödinger operator with constant scalar magnetic field $b > 0$, essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$. It is well-known that $\sigma(h(b)) = \cup_{q=0}^\infty \{2bq\}$, and each eigenvalue $2bq$, $q \in \mathbb{Z}_+$, has infinite multiplicity (see e.g. [1]). Denote by $p_q = p_q(b)$ the orthogonal projection onto the eigenspace $\mathrm{Ker}\,(h(b) - 2bq)$, $q \in \mathbb{Z}_+$.

The estimates of the SSF for energies near the Landau level $2bq$, $q \in \mathbb{Z}_+$, will be given in the terms of traces of certain functions of Toeplitz-type operators $p_q U p_q$ where $U : \mathbb{R}^2 \to \mathbb{R}$ decays in a certain sense at infinity.

Lemma 2.1. [31, Lemma 5.1], [10, Lemma 2.1] *Let $U \in L^r(\mathbb{R}^2)$, $r \ge 1$, and $q \in \mathbb{Z}_+$. Then $p_q U p_q \in S_r$.*

Assume that (1.1) holds. Set

$$W(X_\perp) := \int_{\mathbb{R}} |V(X_\perp, x_3)| dx_3, \quad X_\perp \in \mathbb{R}^2.$$

Since V satisfies (1.1), we have $W \in L^1(\mathbb{R}^2)$, and Lemma 2.1 with $U = W$ implies $p_q W p_q \in S_1$, $q \in \mathbb{Z}_+$. Evidently, $p_q W p_q \ge 0$, and it follows from $V \not\equiv 0$ and $V \in C(\mathbb{R}^2)$, that $\mathrm{rank}\, p_q W p_q = \infty$ for all $q \in \mathbb{Z}_+$ (see below Lemma 2.4). If, moreover, V satisfies (1.2), then $0 \le W(X_\perp) \le C_0' \langle X_\perp \rangle^{-m_0+1}$, $X_\perp \in \mathbb{R}^2$, with $C_0' = C_0 \int_{\mathbb{R}} \langle x \rangle^{-m_0} dx$.

In the following two theorems we assume that V has a definite sign, i.e. that either $V \le 0$ (then we will write H_- instead of H), or $V \ge 0$ (then we will write H_+ instead of H).

Theorem 2.1. (cf. [10, Theorem 3.1]) *Assume that* (1.2) *is valid, and* $\pm V \ge 0$. *Let $q \in \mathbb{Z}_+$, $b > 0$. Then the asymptotic estimates*

$$\xi(2bq - \lambda; H_+, H_0) = O(1), \tag{2.7}$$

$$-n_+((1-\varepsilon)2\sqrt{\lambda}; p_q W p_q) + O(1) \le \xi(2bq - \lambda; H_-, H_0) \le -n_+((1+\varepsilon)2\sqrt{\lambda}; p_q W p_q) + O(1), \tag{2.8}$$

hold as $\lambda \downarrow 0$ for each $\varepsilon \in (0, 1)$.

Suppose that the potential V satisfies (1.1). For $\lambda \geq 0$ define the matrix-valued function

$$\mathcal{W}_\lambda = \mathcal{W}_\lambda(X_\perp) := \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}, \quad X_\perp \in \mathbb{R}^2, \tag{2.9}$$

where

$$w_{11} := \int_{\mathbb{R}} |V(X_\perp, x_3)| \cos^2(\sqrt{\lambda} x_3) dx_3,$$
$$w_{12} = w_{21} := \int_{\mathbb{R}} |V(X_\perp, x_3)| \cos(\sqrt{\lambda} x_3) \sin(\sqrt{\lambda} x_3) dx_3,$$
$$w_{22} := \int_{\mathbb{R}} |V(X_\perp, x_3)| \sin^2(\sqrt{\lambda} x_3) dx_3.$$

It is easy to check that for $\lambda \geq 0$ and $q \in \mathbb{Z}_+$ the operator $p_q \mathcal{W}_\lambda p_q : L^2(\mathbb{R}^2)^2 \to L^2(\mathbb{R}^2)^2$ satisfies $0 \leq p_q \mathcal{W}_\lambda p_q \in S_1$, and $\operatorname{rank} p_q \mathcal{W}_\lambda p_q = \infty$.

Theorem 2.2. (cf. [10, Theorem 3.2]) *Assume that* (1.2) *is valid, and* $\pm V \geq 0$. *Let* $q \in \mathbb{Z}_+$, $b > 0$. *Then the asymptotic estimates*

$$\pm \frac{1}{\pi} \operatorname{Tr} \arctan\left(((1 \pm \varepsilon) 2\sqrt{\lambda})^{-1} p_q \mathcal{W}_\lambda p_q\right) + O(1) \leq \xi(2bq + \lambda; H_\pm, H_0) \leq$$
$$\pm \frac{1}{\pi} \operatorname{Tr} \arctan\left(((1 \mp \varepsilon) 2\sqrt{\lambda})^{-1} p_q \mathcal{W}_\lambda p_q\right) + O(1) \tag{2.10}$$

hold as $\lambda \downarrow 0$ *for each* $\varepsilon \in (0,1)$.

Relations (2.8) and (2.10) allow us to reduce the analysis of the behaviour as $\lambda \to 0$ of $\xi(2bq+\lambda; H_\pm, H_0)$, to the study of the asymptotic distribution of the eigenvalues of Toeplitz-type operators $p_q U p_q$. The following three lemmas concern the spectral asymptotics of such operators.

Lemma 2.2. [31, Theorem 2.6] *Let the function* $0 \leq U \in C^1(\mathbb{R}^2)$ *satisfy the estimates*

$$U(X_\perp) = u_0(X_\perp/|X_\perp|)|X_\perp|^{-\alpha}(1 + o(1)), \quad |X_\perp| \to \infty,$$
$$|\nabla U(X_\perp)| \leq C_1 \langle X_\perp \rangle^{-\alpha - 1}, \quad X_\perp \in \mathbb{R}^2,$$

where $\alpha > 0$, *and* u_0 *is a continuous function on* $\mathbb{S}^1$ *which is non-negative and does not vanish identically. Then for each* $q \in \mathbb{Z}_+$ *we have*

$$n_+(s; p_q U p_q) = \frac{b}{2\pi} \left|\left\{X_\perp \in \mathbb{R}^2 | U(X_\perp) > s\right\}\right| \, (1 + o(1)) = \psi_\alpha(s) \, (1 + o(1)), \quad s \downarrow 0,$$

where $|.|$ *denotes the Lebesgue measure, and*

$$\psi_\alpha(s) := s^{-2/\alpha} \frac{b}{4\pi} \int_{\mathbb{S}^1} u_0(t)^{2/\alpha} dt, \quad s > 0. \tag{2.11}$$

Lemma 2.3. [38, Theorem 2.1, Proposition 4.1] *Let* $0 \leq U \in L^\infty(\mathbb{R}^2)$. *Assume that*

$$\ln U(X_\perp) = -\mu |X_\perp|^{2\beta}(1 + o(1)), \quad |X_\perp| \to \infty,$$

for some $\beta \in (0, \infty)$, $\mu \in (0, \infty)$. *Then for each* $q \in \mathbb{Z}_+$ *we have*

$$n_+(s; p_q U p_q) = \varphi_\beta(s)(1 + o(1)), \quad s \downarrow 0,$$

where

$$\varphi_\beta(s) := \begin{cases} \frac{b}{2\mu^{1/\beta}} |\ln s|^{1/\beta} & \text{if} \quad 0 < \beta < 1, \\ \frac{1}{\ln(1 + 2\mu/b)} |\ln s| & \text{if} \quad \beta = 1, \\ \frac{\beta}{\beta - 1} (\ln |\ln s|)^{-1} |\ln s| & \text{if} \quad 1 < \beta < \infty. \end{cases} \quad s \in (0, e^{-1}). \tag{2.12}$$

Lemma 2.4. [38, Theorem 2.2, Proposition 4.1] *Let $0 \le U \in L^\infty(\mathbb{R}^2)$. Assume that the support of U is compact, and that there exists a constant $C > 0$ such that $U \ge C$ on an open non-empty subset of $\mathbb{R}^2$. Then for each $q \in \mathbb{Z}_+$ we have*

$$n_+(s; p_q U p_q) = \varphi_\infty(s)\,(1 + o(1)), \quad s \downarrow 0,$$

where

$$\varphi_\infty(s) := (\ln|\ln s|)^{-1}|\ln s|, \quad s \in (0, e^{-1}). \tag{2.13}$$

Employing Lemmas 2.2, 2.3, 2.4, we easily find that asymptotic estimates (2.8) and (2.10) entail the following

Corollary 2.1. [10, Corollaries 3.1 – 3.2] *Let* (1.2) *hold with $m_0 > 3$.*
i) Assume that the hypotheses of Lemma 2.2 hold with $U = W$ and $\alpha > 2$. Then we have

$$\xi(2bq - \lambda; H_-, H_0) = -\frac{b}{2\pi}\left|\left\{X_\perp \in \mathbb{R}^2 | W(X_\perp) > 2\sqrt{\lambda}\right\}\right|\,(1 + o(1)) =$$

$$-\psi_\alpha(2\sqrt{\lambda})\,(1 + o(1)), \quad \lambda \downarrow 0, \tag{2.14}$$

$$\xi(2bq + \lambda; H_\pm, H_0) = \pm\frac{b}{2\pi^2}\int_{\mathbb{R}^2} \arctan\left((2\sqrt{\lambda})^{-1}W(X_\perp)\right)dX_\perp\,(1 + o(1)) =$$

$$\pm\frac{1}{2\cos(\pi/\alpha)}\,\psi_\alpha(2\sqrt{\lambda})\,(1 + o(1)), \quad \lambda \downarrow 0.$$

the function ψ_α being defined in (2.11).
ii) Assume that the hypotheses of Lemma 2.3 hold with $U = W$. Then we have

$$\xi(2bq - \lambda; H_-, H_0) = -\varphi_\beta(2\sqrt{\lambda})\,(1 + o(1)), \quad \lambda \downarrow 0, \quad \beta \in (0, \infty),$$

the functions φ_β being defined in (2.12). *If, in addition, V satisfies* (1.1) *for some $m_\perp > 2$ and $m_3 > 2$, we have*

$$\xi(2bq + \lambda; H_\pm, H_0) = \pm\frac{1}{2}\,\varphi_\beta(2\sqrt{\lambda})\,(1 + o(1)), \quad \lambda \downarrow 0, \quad \beta \in (0, \infty).$$

iii) Assume that the hypotheses of Lemma 2.4 hold with $U = W$. Then we have

$$\xi(2bq - \lambda; H_-, H_0) = -\varphi_\infty(2\sqrt{\lambda})\,(1 + o(1)), \quad \lambda \downarrow 0,$$

the function φ_∞ being defined in (2.13). *If, in addition, V satisfies* (1.1) *for some $m_\perp > 2$ and $m_3 > 2$, we have*

$$\xi(2bq + \lambda; H_\pm, H_0) = \pm\frac{1}{2}\,\varphi_\infty(2\sqrt{\lambda})\,(1 + o(1)), \quad \lambda \downarrow 0,$$

the function φ_∞ being defined in (2.13).

In particular, we find that

$$\lim_{\lambda \downarrow 0}\frac{\xi(2bq - \lambda; H_-, H_0)}{\xi(2bq + \lambda; H_-, H_0)} = \frac{1}{2\cos\frac{\pi}{\alpha}} \tag{2.15}$$

if W has a power-like decay at infinity (i.e. if the assumptions of Corollary 2.1 i) hold), or

$$\lim_{\lambda \downarrow 0}\frac{\xi(2bq - \lambda; H_-, H_0)}{\xi(2bq + \lambda; H_-, H_0)} = \frac{1}{2} \tag{2.16}$$

if W decays exponentially or has a compact support (i.e. if the assumptions of Corollary 2.1 ii) - iii) are fulfilled). Relations (2.15) and (2.16) could be interpreted as analogues of the classical Levinson formulae (see e.g. [40]).

Remarks: i) Since the ranks of $p_q W p_q$ and $p_q \mathcal{W}_\lambda p_q$ are infinite, the quantities $n_+(s2\sqrt{\lambda}; p_q W p_q)$ and $\mathrm{Tr}\arctan\left((s2\sqrt{\lambda})^{-1} p_q \mathcal{W}_\lambda p_q\right)$ tend to infinity as $\lambda \downarrow 0$ for every $s > 0$. Therefore, Theorems 2.1 and 2.2 imply that the SSF $\xi(\cdot; H_\pm, H_0)$ has a singularity at each Landau level. The existence of singularities of the SSF at strictly positive energies is in sharp contrast with the non-magnetic case $b = 0$ where the SSF $\xi(E; -\Delta + V, -\Delta)$ is continuous for $E > 0$ (see e.g. [40]). The main reason for this phenomenon is the fact that the Landau levels play the role of thresholds in $\sigma(H_0)$ while the free Laplacian $-\Delta$ has no strictly positive thresholds in its spectrum.

It is conjectured that the singularity of the SSF $\xi(\cdot; H_\pm(b), H_0(b))$, $b > 0$, at a given Landau level $2bq$, $q \in \mathbb{Z}_+$, could be related to a possible accumulation of resonances and/or eigenvalues of H at $2bq$. Here it should be recalled that in the case $b = 0$ the high energy asymptotics (see [27]) and the semi-classical asymptotics (see [28]) of the derivative of the SSF for appropriate compactly supported perturbations of the Laplacian, are related by the Breit-Wigner formula to the asymptotic distribution near the real axis of the resonances defined as poles of the meromorphic continuation of the resolvent of the perturbed operator.

ii) In the case $q = 0$, when by (1.3) we have $\xi(-\lambda; H_-, H_0) = -N(-\lambda; H_-)$ for $\lambda > 0$, asymptotic relations of the type of (2.14) have been known since long ago (see [43], [42], [44], [31], [17]). An important characteristic feature of the methods used in [31], and later in [38], is the systematic use, explicit or implicit, of the connection between the spectral theory of the Schrödinger operator with constant magnetic field, and the theory of Toeplitz operators acting in holomorphic spaces of Fock-Segal-Bargmann type, and the related pseudodifferential operators with generalised anti-Wick symbols (see [12], [3], [41], [15]). Various important aspects of the interaction between these two theories have been discussed in [37] and [7, Section 9]). The Toeplitz-operator approach turned to be especially fruitful in [38] where electric potentials decaying rapidly at infinity (i.e. decaying exponentially, or having compact support) were considered (see Lemmas 2.3 - 2.4). It is shown in [11] that the precise spectral asymptotics for the Landau Hamiltonian perturbed by a compactly supported electric potential U of fixed sign recovers the logarithmic capacity of the support of U.

iii) Let us mention several other existing extensions of Lemmas 2.2 – 2.4. Lemmas 2.2 and 2.4 have been generalised to the multidimensional case where p_q is the orthogonal projection onto a given eigenspace of the Schrödinger operator with constant magnetic field of full rank, acting in $L^2(\mathbb{R}^{2d})$, $d > 1$ (see [31] and [25] respectively). Moreover, Lemma 2.4 has been generalised in [25] to a relativistic setting where p_q is an eigenprojection of the Dirac operator. Finally, in [36] Lemmas 2.2 – 2.4 have been extended to the case of the 2D Pauli operator with variable magnetic field from a certain class including the almost periodic fields with non-zero mean value (in this case the role of the Landau levels is played by the origin), and electric potentials U satisfying the assumptions of Lemmas 2.2 – 2.4. In the case of compactly supported U of definite sign, [11] contains a more precise version of the corresponding result of [36], involving again the logarithmic capacity of the support of U.

iv) To the author's best knowledge, the singularities at the Landau levels of the SSF for the 3D Schrödinger operator in constant magnetic field has been investigated for the first time in [10]. However, it is appropriate to mention here the article [19] where an axisymmetric potential $V = V(|X_\perp|, x_3)$ has been considered. It is well-known (see e.g. [1]) that in this case the operators H_0 and H are unitarily equivalent to the orthogonal sums $\sum_{m\in\mathbb{Z}} \oplus H^{(m)}$ and $\sum_{m\in\mathbb{Z}} \oplus H_0^{(m)}$ respectively, where the operators

$$H_0^{(m)} := -\frac{1}{\varrho}\frac{\partial}{\partial\varrho}\varrho\frac{\partial}{\partial\varrho} - \frac{\partial^2}{\partial x_3^2} + \left(\frac{b\varrho}{2} + \frac{m}{\varrho}\right)^2 - b, \quad H^{(m)} := H_0^{(m)} + V(\varrho, x_3), \quad m \in \mathbb{Z}, \tag{2.17}$$

are self-adjoint in $L^2(\mathbb{R}_+ \times \mathbb{R}; \varrho d\varrho dx_3)$. For a *fixed* magnetic quantum number $m \in \mathbb{Z}$ the authors of [19] studied the behaviour of the SSF $\xi(E; H^{(m)}, H_0^{(m)})$ for energies E near the Landau level $2m$ if $m > 0$, and near the origin if $m \le 0$, and deduced analogues of the classical Levinson formulae for the operator pair $\left(H^{(m)}, H_0^{(m)}\right)$. Later, the methods in [19] were developed in [23] and [24]. However, it is

not possible to recover the results our Theorem 2.1, Theorem 2.2 and/or Corollary 2.1 from the results of [19], [23], and [24] even in the case of axisymmetric V.
v) Finally, [16] contains general bounds on the SSF for appropriate pairs of magnetic Schrödinger operators. These bounds are applied in order to deduce Wegner estimates of the integrated density of states for some random alloy-type models.

2.3 Strong Magnetic Field Asymptotics of the SSF

Our first theorem in this subsection treats the asymptotics as $b \to \infty$ of $\xi(\cdot; H(b), H_0(b))$ far from the Landau levels.

Theorem 2.3. (cf. [7, Theorem 2.1]) *Let* (1.1) *hold. Assume that* $\mathcal{E} \in (0,\infty) \setminus 2\mathbb{Z}_+$, *and* $\lambda \in \mathbb{R}$. *Then*

$$\xi(\mathcal{E}b + \lambda; H(b), H_0(b)) = \frac{b^{1/2}}{4\pi^2} \sum_{l=0}^{[\mathcal{E}/2]} (\mathcal{E} - 2l)^{-1/2} \int_{\mathbb{R}^3} V(\mathbf{x}) d\mathbf{x} + O(1), \quad b \to \infty, \tag{2.18}$$

where $[\mathcal{E}/2]$ *denotes the integer part of the real number* $\mathcal{E}/2$.

The following two theorems concern the asymptotics of the SSF near a given Landau level. In order to formulate our next theorem, we introduce the following self-adjoint operators

$$\chi_0 := -d^2/dx_3^2, \quad \chi = \chi(X_\perp) := \chi_0 + V(X_\perp, .), \quad X_\perp \in \mathbb{R}^2,$$

which are defined on the Sobolev space $\mathrm{H}^2(\mathbb{R})$, and depend on the parameter $X_\perp \in \mathbb{R}^2$. If (1.1) holds, then $(\chi(X_\perp) - \lambda_0)^{-1} - (\chi_0 - \lambda_0)^{-1} \in S_1$ for each $X_\perp \in \mathbb{R}^2$ and $\lambda_0 < \inf \sigma(\chi(X_\perp))$. Hence, the SSF $\xi(.; \chi(X_\perp), \chi_0)$ is well-defined. Set $\Lambda := \min_{X_\perp \in \mathbb{R}^2} \inf \sigma(\chi(X_\perp))$. Evidently, $\Lambda \in [-C_0, 0]$. Moreover,

$$\Lambda = \lim_{b\to\infty} \inf \sigma(H(b)) \tag{2.19}$$

(see [1, Theorem 5.8]).

Proposition 2.1. (cf. [7, Proposition 2.2]) *Assume that* (1.1) *holds.*
i) For each $\lambda \in \mathbb{R} \setminus \{0\}$ *we have* $\xi(\lambda; \chi(.), \chi_0) \in L^1(\mathbb{R}^2)$.
ii) The function $(0,\infty) \ni \lambda \mapsto \int_{\mathbb{R}^2} \xi(\lambda; \chi(X_\perp), \chi_0) dX_\perp$ *is continuous, while the non-increasing function*

$$(-\infty, 0) \ni \lambda \mapsto \int_{\mathbb{R}^2} \xi(\lambda; \chi(X_\perp), \chi_0) dX_\perp = -\int_{\mathbb{R}^2} N(\lambda; \chi(X_\perp)) dX_\perp$$

(*see* (1.3)), *is continuous at the point* $\lambda < 0$ *if and only if*

$$|\{X_\perp \in \mathbb{R}^2 | \lambda \in \sigma(\chi(X_\perp))\}| = 0. \tag{2.20}$$

iii) Assume $\pm V \geq 0$. *If* $\lambda > \Lambda$, $\lambda \neq 0$, *then* $\pm \int_{\mathbb{R}^2} \xi(\lambda; \chi(X_\perp), \chi_0) dX_\perp > 0$.

Remark: Proposition 2.1 iii) in [7, Proposition 2.2]. However, it follows easily from the representation of the SSF described in Subsection 3.1 below, and the hypotheses $v \not\equiv 0$ and $V \in C(\mathbb{R}^3)$.

Theorem 2.4. (cf. [7, Theorem 2.3]) *Assume that* (1.1) *holds. Let* $q \in \mathbb{Z}_+$, $\lambda \in \mathbb{R} \setminus \{0\}$. *If* $\lambda < 0$, *suppose also that* (2.20) *holds. Then we have*

$$\lim_{b\to\infty} b^{-1} \xi(2bq + \lambda; H(b), H_0(b)) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \xi(\lambda; \chi(X_\perp), \chi_0)\, dX_\perp. \tag{2.21}$$

The proofs of Theorems 2.3 and 2.4 are contained in Subsection 4.2. We present these proofs under the additional assumption that V has a definite sign, and refer the reader to the original paper [7] for the proofs in the general case.
By Proposition 2.1 iii), if $\pm V \geq 0$, then the r.h.s. of (2.21) is different from zero if $\lambda > \Lambda$, $\lambda \neq 0$. Unfortunately, we cannot prove that the same is true for general non-sign-definite electric potentials V. On the other hand, it is obvious that for arbitrary V we have $\int_{\mathbb{R}^2} \xi(\lambda; \chi(X_\perp), \chi_0) dX_\perp = 0$ if $\lambda < \Lambda$. The last theorem of this subsection contains a more precise version of (2.21) for the case $\lambda < \Lambda$.

Theorem 2.5. (cf. [7, Theorem 2.4]) *Let* (1.1) *hold.*
i) Let $\lambda < \Lambda$. Then for sufficiently large $b > 0$ we have $\xi(\lambda; H(b), H_0(b)) = 0$.
ii) Let $q \in \mathbb{Z}_+$, $q \geq 1$, $\lambda < \Lambda$. Assume that the partial derivatives of $\langle x_3 \rangle^{m_3} V$ with respect to the variables $X_\perp \in \mathbb{R}^2$ exist, and are uniformly bounded on $\mathbb{R}^3$. Then we have

$$\lim_{b\to\infty} b^{-1/2} \xi(2bq + \lambda; H(b), H_0(b)) = \frac{1}{4\pi^2} \sum_{l=0}^{q-1} (2(q-l))^{-1/2} \int_{\mathbb{R}^3} V(\mathbf{x}) d\mathbf{x}. \tag{2.22}$$

The first part of the theorem is trivial, and follows immediately from (2.19). We omit the proof of Theorem 2.5 ii) and refer the reader to the original work [7].
Remarks: i) Relations (2.18), (2.21), and (2.22) can be unified into a single asymptotic formula. In order to see this, notice that a general result on the high-energy asymptotics of the SSF for 1D Schrödinger operators (see e.g. [40]) implies, in particular, that

$$\lim_{E\to\infty} E^{1/2} \xi(E; \chi(X_\perp), \chi_0) = \frac{1}{2\pi} \int_{\mathbb{R}} V(X_\perp, x_3)\, dx_3, \quad X_\perp \in \mathbb{R}^2.$$

Then relation (2.18) with $0 < \mathcal{E}/2 \notin \mathbb{Z}_+$, or relations (2.21) and (2.22) with $\mathcal{E} = 2q$, $q \in \mathbb{Z}_+$, entail

$$\xi(\mathcal{E}b + \lambda; H(b), H_0(b)) = \frac{b}{2\pi} \sum_{l=0}^{[\mathcal{E}/2]} \int_{\mathbb{R}^2} \xi(b(\mathcal{E} - 2l) + \lambda; \chi(X_\perp), \chi_0) dX_\perp \, (1 + o(1)), \quad b \to \infty. \tag{2.23}$$

On its turn, (2.23) can be re-written as

$$\xi(\mathcal{E}b + \lambda; H(b), H_0(b)) = \int_{\mathbb{R}} \int_{\mathbb{R}^2} \xi(\mathcal{E}b + \lambda - s; \chi(X_\perp), \chi_0) dX_\perp d\nu_b(s) \, (1 + o(1)), \ b \to \infty,$$

where $\nu_b(s) := \frac{b}{2\pi} \sum_{l=0}^{\infty} \Theta(s - 2bl)$, $s \in \mathbb{R}$, and $\Theta(s) := \begin{cases} 0 & \text{if} \quad s \leq 0, \\ 1 & \text{if} \quad s > 0, \end{cases}$ is the Heaviside function. It is well-known that ν is *the integrated density of states* for the 2D Landau Hamiltonian (see (2.6)).
ii) By (1.3) for $\lambda < 0$ we have $\xi(\lambda; H(b), H_0) = -N(\lambda; H(b))$. The asymptotics as $b \to \infty$ of the counting function $N(\lambda; H_0(b))$ with $\lambda < 0$ fixed, has been investigated in [32] under considerably less restrictive assumptions on V than in Theorems 2.3 – 2.5. The asymptotic properties as $\lambda \uparrow 0$, and as $\lambda \downarrow \Lambda$ if $\Lambda < 0$, of the asymptotic coefficient $-\frac{1}{2\pi} \int_{\mathbb{R}^2} N(\lambda; \chi(X_\perp) dX_\perp$ which appears at the r.h.s. of (2.21) in the case of a negative perturbation, have been studied in [33]. The asymptotic distribution of the discrete spectrum for the 3D magnetic Pauli and Dirac operators in strong magnetic fields has been considered in [35] and [34] respectively. The main purpose in [32], [34], and [35] was to obtain the main asymptotic term (without any remainder estimates) of the corresponding counting function of the discrete spectrum under assumptions close to the minimal ones which guarantee that the Hamiltonians are self-adjoint, and the asymptotic coefficient is well-defined. Other results which again describe the asymptotic distribution of the discrete spectrum of the Schrödinger and Dirac operator in strong magnetic fields, but contain also sharp remainder estimates, have been obtained [17], [9], and [18] under assumptions on V which, naturally, are considerably more restrictive than those in [32], [34], and [35].
iii) Generalisations of asymptotic relation (2.18) in several directions can be found in [26]. In particular,

[26, Theorem 4] implies that if $V \in \mathcal{S}(\mathbb{R}^3)$, then the SSF $\xi(\mathcal{E}b+\lambda; H(b), H_0(b))$, $\mathcal{E} \in (0,\infty)\setminus 2\mathbb{Z}_+$, $\lambda \in \mathbb{R}$, admits an asymptotic expansion of the form

$$\xi(\mathcal{E}b+\lambda; H(b), H_0(b)) \sim \sum_{j=0}^{\infty} c_j b^{\frac{1-2j}{2}}, \quad b \to \infty.$$

iv) Together with the pointwise asymptotics as $b \to \infty$ of the SSF for the pair $(H_0(b), H(b))$ (see (2.18), (2.21), or (2.22)), it also is possible to consider its *weak asymptotics*, i.e. the asymptotics of the convolution of the SSF with an arbitrary $\varphi \in C_0^\infty(\mathbb{R})$. Results of this type are contained in [6].

2.4 High energy asymptotics of the SSF

Theorem 2.6. [8, Theorem 2.1] *Assume that V satisfies* (1.1). *Then we have*

$$\lim_{E\to\infty, E\in\mathcal{O}_r} E^{-1/2}\xi(E; H, H_0) = \frac{1}{4\pi^2}\int_{\mathbb{R}^3} V(\mathbf{x})d\mathbf{x}, \quad r \in (0,b), \tag{2.24}$$

where $\mathcal{O}_r := \{E \in (0,\infty) | \mathrm{dist}(E, 2b\mathbb{Z}_+)\}$.

We omit the proof of Theorem 2.6 which is quite similar to that of Theorem 2.3, and refer the reader to the original paper [8].
Remarks: i) It is essential to avoid the Landau levels in (2.24), i.e. to suppose that $E \in \mathcal{O}_r$, $r \in (0,b)$, as $E \to \infty$, since by Theorems 2.1 - 2.2, the SSF has singularities at the Landau levels, at least in the case $\pm V \geq 0$.
ii) For $E \in \mathbb{R}$ set

$$\xi_{\mathrm{cl}}(E) := \int_{T^*\mathbb{R}^3} \left(\Theta(E - |\mathbf{p}+\mathbf{A}(\mathbf{x})|^2) - \Theta(E - |\mathbf{p}+\mathbf{A}(\mathbf{x})|^2 - V(\mathbf{x}))\right) d\mathbf{x}d\mathbf{p} =$$

$$\frac{4\pi}{3}\int_{\mathbb{R}^3}\left(E_+^{3/2} - (E - V(\mathbf{x}))_+^{3/2}\right) d\mathbf{x}$$

where Θ, as above, is the Heaviside function. Note that $\xi_{\mathrm{cl}}(E)$ is independent of the magnetic field $b \geq 0$. Evidently, under the assumptions of Theorem 2.6 we have $\lim_{E\to\infty} E^{-1/2}\xi_{\mathrm{cl}}(E) = 2\pi\int_{\mathbb{R}^3} V(\mathbf{x})d\mathbf{x}$. Hence, if $\int_{\mathbb{R}^3} V(\mathbf{x})d\mathbf{x} \neq 0$, then (2.24) is equivalent to

$$\xi(E; H, H_0) = (2\pi)^{-3}\xi_{\mathrm{cl}}(E)(1+o(1)), \quad E \to \infty, \quad E \in \mathcal{O}_r, \quad r \in (0,b).$$

iii) As far as the author is informed, the high-energy asymptotics of the SSF for 3D Schrödinger operators in constant magnetic fields was investigated for the first time in [8]. Nonetheless, in [19] the asymptotic behaviour as $E \to \infty$, $E \in \mathcal{O}_r$, of the SSF $\xi(E; H^{(m)}, H_0^{(m)})$ for the operator pair $(H^{(m)}, H_0^{(m)})$ (see (2.17)) with fixed $m \in \mathbb{Z}$ has been been investigated. It does not seem possible to deduce (2.24) from the results of [19] even in the case of axial symmetry of V.

3 Auxiliary Results

3.1 A. Pushnitski's representation of the SSF

In the first part of this subsection we summarise several results due to A. Pushnitski on the representation of the SSF for a pair of lower-bounded self-adjoint operators (see [29]).
Let $I \in \mathbb{R}$ be a Lebesgue measurable set. Set $\mu(I) := \frac{1}{\pi}\int_I \frac{dt}{1+t^2}$. Note that $\mu(\mathbb{R}) = 1$.

Lemma 3.1. [29, Lemma 2.1] *Let* $T_1 = T_1^* \in S_\infty$ *and* $T_2 = T_2^* \in S_1$. *Then*

$$\int_{\mathbb{R}} n_\pm(s_1+s_2; T_1 + t\,T_2)\, d\mu(t) \le n_\pm(s_1;T_1) + \frac{1}{\pi s_2}\|T_2\|_1, \quad s_1, s_2 > 0. \tag{3.1}$$

Let $\mathcal{H}_\pm$ and $\mathcal{H}_0$ be two lower-bounded self-adjoint operators. Assume that

$$\mathcal{V} := \pm(\mathcal{H}_\pm - \mathcal{H}_0) \ge 0. \tag{3.2}$$

Let $\lambda_0 < \inf \sigma(\mathcal{H}_\pm) \cup \sigma(\mathcal{H}_0)$. Suppose that

$$(\mathcal{H}_0 - \lambda_0)^{-\gamma} - (\mathcal{H}_0 - \lambda_0)^{-\gamma} \in S_2, \quad \gamma > 0, \tag{3.3}$$

$$\mathcal{V}^{1/2}(\mathcal{H}_0 - \lambda_0)^{-1/2} \in S_\infty, \tag{3.4}$$

$$\mathcal{V}^{1/2}(\mathcal{H}_0 - \lambda_0)^{-\gamma'} \in S_2, \quad \gamma' > 0. \tag{3.5}$$

For $z \in \mathbb{C}$ with $\operatorname{Im} z > 0$ set $\mathcal{T}(z) \colon= \mathcal{V}^{1/2}(\mathcal{H}_0 - z)^{-1}\mathcal{V}^{1/2}$.

Lemma 3.2. [29, Lemma 4.1] *Let* (3.3) – (3.5) *hold. Then for almost every* $E \in \mathbb{R}$ *the operator-norm limit* $\mathcal{T}(E + i0) := \mathrm{n} - \lim_{\delta \downarrow 0} \mathcal{T}(E + i\delta)$ *exists, and by* (3.4) *we have* $\mathcal{T}(E+i0) \in S_\infty$. *Moreover,* $0 \le \operatorname{Im} \mathcal{T}(E+i0) \in S_1$.

Theorem 3.1. [29, Theorem 1.2] *Let* (3.2) – (3.5) *hold. Then the SSF* $\xi(\cdot; \mathcal{H}_\pm, \mathcal{H}_0)$ *for the operator pair* $(\mathcal{H}_\pm, \mathcal{H}_0)$ *is well-defined, and for almost every* $E \in \mathbb{R}$ *we have*

$$\xi(E; \mathcal{H}_\pm, \mathcal{H}_0) = \pm \int_{\mathbb{R}} n_\mp(1; \operatorname{Re} \mathcal{T}(E+i0) + t \operatorname{Im} \mathcal{T}(E+i0))\, d\mu(t).$$

Remark: The representation of the SSF described in the above theorem was generalised to non-sign-definite perturbations in [14] in the case of trace-class perturbations, and in [30] in the case of relatively trace-class perturbations. These generalisations are based on the concept of the index of orthogonal projections (see [2]).
Suppose now that V satisfies (1.1), and $\pm V \ge 0$. Then relations (3.2) – (3.5) hold with $\mathcal{V} = |V|$, $\mathcal{H}_0 = H_0$, and $\gamma = \gamma' = 1$. For $z \in \mathbb{C}$, $\operatorname{Im} z > 0$, set $T(z) := |V|^{1/2}(H_0 - z)^{-1}|V|^{1/2}$. By Lemma 3.2, for almost every $E \in \mathbb{R}$ the operator-norm limit

$$T(E + i0) := \mathrm{n} - \lim_{\delta \downarrow 0} T(E + i\delta) \tag{3.6}$$

exists, and

$$0 \le \operatorname{Im} T(E + i0) \in S_1. \tag{3.7}$$

The following proposition contains a more precise version of the above statement, and provides estimates of the norm of $T(E+i0)$, and the trace-class norm of $\operatorname{Im} T(E+i0)$.

Proposition 3.1. [7, Lemma 4.2] *Assume that* (1.1) *holds, and* $E \in \mathbb{R} \setminus 2b\mathbb{Z}_+$. *Then the operator limit* (3.6) *exists, and we have*

$$\|T(E+i0)\| \le C_1 \left(\operatorname{dist}(E, 2b\mathbb{Z}_+)\right)^{-1/2} \tag{3.8}$$

with C_1 *independent of* E *and* b.
Moreover, (3.7) *holds, and if* $E < 0$ *then* $\operatorname{Im} T(E+i0) = 0$, *while for* $E \in (0,\infty) \setminus 2b\mathbb{Z}_+$ *we have*

$$\|\operatorname{Im} T(E+i0)\|_1 = \operatorname{Tr} \operatorname{Im} T(E+i0) = \frac{b}{4\pi} \sum_{l=0}^{[\frac{E}{2b}]} (E - 2bl)^{-1/2} \int_{\mathbb{R}^3} |V(\mathbf{x})| d\mathbf{x}. \tag{3.9}$$

By Lemma 3.1 and Proposition 3.1, the quantity

$$\tilde{\xi}(E; H_\pm, H_0) = \pm \int_{\mathbb{R}} n_\mp(1; \operatorname{Re} T(E+i0) + t \operatorname{Im} T(E+i0))\, d\mu(t), \quad E \in \mathbb{R} \setminus 2b\mathbb{Z}_+. \tag{3.10}$$

is well-defined for every $E \in \mathbb{R} \setminus 2b\mathbb{Z}_+$, and bounded on every compact subset of $\mathbb{R} \setminus 2b\mathbb{Z}_+$. Moreover, by [7, Proposition 2.5], $\tilde{\xi}(\cdot; H_\pm, H_0)$ is continuous on $\mathbb{R} \setminus \{2b\mathbb{Z}_+ \cup \sigma_{\rm pp}(H_\pm\}$. On the other hand, by Theorem 3.1 we have

$$\tilde{\xi}(E; H_\pm, H_0) = \xi(E; H_\pm, H_0) \tag{3.11}$$

for almost every $E \in \mathbb{R}$. As explained in the introduction, in the case of sign-definite perturbations we will identify the SSF $\xi(E; H_\pm, H_0)$ with $\tilde{\xi}(E; H_\pm, H_0)$, while in the case of non-sign-definite perturbations, we will identify it with the generalisation of $\tilde{\xi}(E; H_\pm, H_0)$ described in [7, Section 3] on the basis of the general results of [14] and [30].

Here it should be underlined that in contrast to the case $b = 0$, we cannot rule out the possibility that the operator H has infinite discrete spectrum, or eigenvalues embedded in the continuous spectrum by imposing conditions about the fast decay of the potential V at infinity. First, it is well-known that if V satisfies

$$V(\mathbf{x}) \leq -C\chi(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3, \tag{3.12}$$

where $C > 0$, and χ is the characteristic function of a non-empty open subset of $\mathbb{R}^3$, then the discrete spectrum of H is infinite (see [1, Theorem 5.1]), [38, Theorem 2.4]). Further, if V is axisymmetric and satisfies (3.12), then the operator $H^{(q)}$ defined in (2.17) with $q \geq 0$ has at least one eigenvalue in the interval $(2bq - \|V\|_{L^\infty(\mathbb{R}^3)}, 2bq)$, and hence the operator H has infinitely many eigenvalues embedded in its continuous spectrum (see [1, Theorem 5.1]). Assume now that V is axisymmetric and satisfies the estimate

$$V(X_\perp, x_3) \leq -C\chi_\perp(X_\perp)\langle x_3\rangle^{-m_3}, \quad (X_\perp, x_3) \in \mathbb{R}^3, \tag{3.13}$$

where $C > 0$, $\chi_\perp$ is the characteristic function of a non-empty open subset of $\mathbb{R}^2$, and $m_3 \in (0, 2)$ which is compatible with (1.1) if $m_3 \in (1, 2)$. Then, using the argument of the proof of [1, Theorem 5.1] and the variational principle, we can easily check that for each $q \geq 0$ the operator $H^{(q)}$ has infinitely many discrete eigenvalues which accumulate to the infimum $2bq$ of its essential spectrum. Hence, if V is axisymmetric and satisfies (3.13), then below each Landau level $2bq$, $q \in \mathbb{Z}_+$, there exists an infinite sequence of finite-multiplicity eigenvalues of H, which converges to $2bq$. Note however that the claims in [10, p. 385] and [8, p. 3457] that [1, Theorem 5.1]) implies the same phenomenon for axisymmetric non-positive potentials compactly supported in $\mathbb{R}^3$, are not justified. The challenging and interesting problem about the accumulation at a given Landau level of embedded eigenvalues and/or resonances of H will be considered in a future work.

Finally, we note that generically the only possible accumulation points of the eigenvalues of H are the Landau levels (see [1, Theorem 4.7], [13, Theorem 3.5.3 (iii)]). Further information on the location of the eigenvalues of H can be found in [7, Proposition 2.6].

3.2 Estimates for Birman-Schwinger operators

For $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^2$ denote by $\mathcal{P}_{q,b}(\mathbf{x}, \mathbf{x}')$ the integral kernel of the orthogonal projection $p_q(b)$ onto the subspace Ker $(h(b) - 2bq)$, $q \in \mathbb{Z}_+$, the Landau Hamiltonian $h(b)$ being defined in (2.6). It is well-known that

$$\mathcal{P}_{q,b}(\mathbf{x}, \mathbf{x}') = \frac{b}{2\pi} L_q\left(\frac{b|\mathbf{x} - \mathbf{x}'|^2}{2}\right) \exp\left(-\frac{b}{4}(|\mathbf{x} - \mathbf{x}'|^2 + 2i(x_1x_2' - x_1'x_2))\right) \tag{3.14}$$

(see [21] or [37, Subsection 2.3.2]) where $L_q(t) := \frac{1}{q!}e^t\frac{d^q(t^qe^{-t})}{dt^q} = \sum_{k=0}^q \binom{q}{k}\frac{(-t)^k}{k!}$, $t\in\mathbb{R}$, $q\in\mathbb{Z}_+$, are the Laguerre polynomials. Note that

$$\mathcal{P}_{q,b}(\mathbf{x},\mathbf{x}) = \frac{b}{2\pi}, \quad q\in\mathbb{Z}_+, \quad \mathbf{x}\in\mathbb{R}^2. \tag{3.15}$$

Define the orthogonal projections $P_q : L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)$, $q\in\mathbb{Z}_+$, by $P_q := p_q\otimes I_3$ where I_3 is the identity operator in $L^2(\mathbb{R}_{x_3})$.
For $z\in\mathbb{C}$ with $\operatorname{Im} z>0$, define the operator $R(z) := \left(-\frac{d^2}{dx_3^2}-z\right)^{-1}$ bounded in $L^2(\mathbb{R})$. Note that the operator $R(z)$ admits the integral kernel $\mathcal{R}_z(x_3-x_3')$ where $\mathcal{R}_z(x) = ie^{i\sqrt{z}|x|}/(2\sqrt{z})$, $x\in\mathbb{R}$, the branch of $\sqrt{z}$ being chosen so that $\operatorname{Im}\sqrt{z}>0$.
Define that the operators

$$T_q(z) := |V|^{1/2}P_q(H_0-z)^{-1}|V|^{1/2}, \quad q\in\mathbb{Z}_+,$$

bounded in $L^2(\mathbb{R}^3)$. We have $T_q(z) = |V|^{1/2}\left(p_q(b)\otimes R(z-2bq)\right)|V|^{1/2}$.
For $\lambda\in\mathbb{R}$, $\lambda\neq 0$, define $R(\lambda)$ as the operator with integral kernel $\mathcal{R}_\lambda(x_3-x_3')$ where

$$\mathcal{R}_\lambda(x) := \lim_{\delta\downarrow 0}\mathcal{R}_{\lambda+i\delta}(x) = \begin{cases} \frac{e^{-\sqrt{-\lambda}|x|}}{2\sqrt{-\lambda}} & \text{if} \quad \lambda<0, \\ \frac{ie^{i\sqrt{\lambda}|x|}}{2\sqrt{\lambda}} & \text{if} \quad \lambda>0, \end{cases} \qquad x\in\mathbb{R}. \tag{3.16}$$

Evidently, if $w_1, w_2\in L^2(\mathbb{R})$ and $\lambda\neq 0$, then $w_1R(\lambda)w_2\in S_2$. For $E\in\mathbb{R}$, $E\neq 2bq$, $q\in\mathbb{Z}_+$, set

$$T_q(E) := |V|^{1/2}\left(p_q(b)\otimes R(E-2bq)\right)|V|^{1/2}.$$

Then $\lim_{\delta\downarrow 0}\|T_q(E+i\delta)-T_q(E)\|_2 = 0$ (see [10, Proposition 4.1]).

Proposition 3.2. *Let $E\in\mathbb{R}$, $q\in\mathbb{Z}_+$, $E\neq 2bq$. Let* (1.1) *hold. Then*

$$\|T_q(E)\| \le C_2|E-2bq|^{-1/2}, \tag{3.17}$$

$$\|T_q(E)\|_2^2 \le C_2 b|E-2bq|^{-1}, \tag{3.18}$$

with C_2 independent of E, b, and q.

Proof. We have

$$T_q(E) = MG_{q,m_\perp}\otimes t(E-2bq)M \tag{3.19}$$

where M is the multiplier by the bounded function $|V(X_\perp,x_3)|^{1/2}\langle X_\perp\rangle^{m_\perp/2}\langle x_3\rangle^{m_3/2}$, $(X_\perp,x_3)\in\mathbb{R}^3$, $G_{q,m_\perp} : L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)$ is the operator with integral kernel

$$\langle X_\perp\rangle^{-m_\perp/2}\mathcal{P}_{b,q}(X_\perp,X_\perp')\langle X_\perp'\rangle^{-m_\perp/2}, \quad X_\perp, X_\perp'\in\mathbb{R}^2,$$

and $t(\lambda) : L^2(\mathbb{R})\to L^2(\mathbb{R})$, $\lambda\in\mathbb{R}\setminus\{0\}$, is the operator with integral kernel

$$\langle x_3\rangle^{-m_3/2}\mathcal{R}_\lambda(x_3-x_3')\langle x_3'\rangle^{-m_3/2}, \quad x_3, x_3'\in\mathbb{R}.$$

Then we have

$$\|T_q(E)\| \le \|M\|_\infty^2\|G_{q,m_\perp}\|\,\|t(E-2bq)\| \le \|M\|_\infty^2\|G_{q,m_\perp}\|\,\|t(E-2bq)\|_2, \tag{3.20}$$

$$\|T_q(E)\|_2 \le \|M\|_\infty^2\|G_{q,m_\perp}\|_2\,\|t(E-2bq)\|_2, \tag{3.21}$$

where $\|M\|_\infty := \|M\|_{L^\infty(\mathbb{R}^2)}$. Evidently,

$$\|G_{q,m_\perp}\| \leq 1, \tag{3.22}$$

$$\|G_{q,m_\perp}\|_2^2 \leq \operatorname{Tr} p_q \langle X_\perp\rangle^{-m_\perp} p_q = \frac{b}{2\pi}\int_{\mathbb{R}^2} \langle X_\perp\rangle^{-m_\perp} dX_\perp \tag{3.23}$$

(see (3.15)), and

$$\|t(E-2bq)\|_2^2 \leq \frac{1}{4|E-2bq|}\int_{\mathbb{R}} \langle x_3\rangle^{-m_3} dx_3 \tag{3.24}$$

(see (3.16)). Now the combination of (3.20), (3.22), and (3.24) yields (3.17), while the combination of (3.21), (3.23), and (3.24) yields (3.18). □

Remark: Using more sophisticated tools than those of the proof of Proposition 3.2, it is shown in [7] that for $E \neq 2bq$ we have not only $T_q(E) \in S_2$, but also $T_q(E) \in S_1$. We will not use this fact here.

Proposition 3.3. *Assume that* V *satisfies* (1.1). *Let* $E \in \mathbb{R} \setminus 2b\mathbb{Z}_+$, $q \in \mathbb{Z}_+$. *Then we have* $0 \leq \operatorname{Im} T_q(E) \in S_p$ *with any* $p > 2/m_\perp$. *If* $E < 2bq$, *then* $\operatorname{Im} T_q(E) = 0$. *If* $E > 2bq$, *then the estimate*

$$n_+(s; \operatorname{Im} T_q(E)) \leq C_3 \left(1 + b\,(E-2bq)^{-1/m_\perp} s^{-2/m_\perp}\right) \tag{3.25}$$

holds for each $s > 0$ *with* C_3 *independent of* s, b, *and* E. *Moreover, if* $E > 2bq$, *then we have*

$$\|\operatorname{Im} T_q(E)\|_1 = \operatorname{Tr} \operatorname{Im} T_q(E) = \frac{b}{4\pi}(E-2bq)^{-1/2}\int_{\mathbb{R}^3} |V(\mathbf{x})| d\mathbf{x}. \tag{3.26}$$

Proof. By (3.19), we have

$$\operatorname{Im} T_q(E) = M G_{p,m_\perp} \otimes \operatorname{Im} t(E-2bq) M$$

If $E < 2bq$, then $\operatorname{Im} t(E-2bq) = 0$. If $E > 2bq$, then $\operatorname{Im} t(E-2bq)$ admits the integral kernel

$$\frac{1}{2\sqrt{E-2bq}}\langle x_3\rangle^{-m_3/2} \cos\left(\sqrt{E-2bq}\,(x_3 - x_3')\right)\langle x_3'\rangle^{-m_3/2}, \quad x_3, x_3' \in \mathbb{R}.$$

Since the function $\langle X_\perp\rangle^{-m_\perp/2}$ is radially symmetric, the eigenvalues ν_k, $k \in \mathbb{N}$, of the operator $G_{p,m_\perp} \geq 0$ can be computed explicitly, and for $k \geq k_0$ we have $\nu_k \leq C_3' b^{m_\perp/2} k^{-m_\perp/2}$ with $k_0 \in \mathbb{N}$ and C_3' independent of b and E (see the proof of [7, Lemma 9.4]). Further, if $E > 2bq$, we have $\operatorname{rank} \operatorname{Im} t(E-2bq) = 2$, and the eigenvalues of $\operatorname{Im} t(E-2bq)$ are upper-bounded by $\frac{1}{2\sqrt{E-2bq}}\int_{\mathbb{R}} \langle x_3\rangle^{-m_3} dx_3$. Therefore,

$$n_+(s; \operatorname{Im} T_q(E)) \leq k_0 + 2\left(\frac{C_3'\|M\|_\infty^2 b^{m_\perp/2} s^{-1}}{2\sqrt{E-2bq}}\int_{\mathbb{R}} \langle x_3\rangle^{-m_3} dx_3\right)^{2/m_\perp}, \quad s > 0,$$

which entails immediately (3.25). Finally, if we write the trace of the operator $\operatorname{Im} T_q(E)$ as the integral of the diagonal value of its kernel, and take into account (3.15) and (3.16), we get (3.26). □

Proposition 3.4. [10, Proposition 4.2] *Let* $q \in \mathbb{Z}_+$, $\lambda \in \mathbb{R}$, $|\lambda| \in (0, b)$, *and* $\delta > 0$. *Assume that* V *satisfies* (1.1). *Then the operator series* $T_q^+(2bq + \lambda + i\delta) := \sum_{l=q+1}^\infty T_l(2bq + \lambda + i\delta)$, *and*

$$T_q^+(2bq + \lambda) := \sum_{l=q+1}^\infty T_l(2bq + \lambda) \tag{3.27}$$

converge in S_2. *Moreover,*

$$\|T_q^+(2bq + \lambda)\|_2^2 \leq \frac{C_0 b}{8\pi}\sum_{l=q+1}^\infty (2b(l-q) - \lambda)^{-3/2}\int_{\mathbb{R}^3} V(\mathbf{x}) d\mathbf{x}. \tag{3.28}$$

Finally, $\lim_{\delta\downarrow 0} \|T_q^+(2bq + \lambda + i\delta) - T_q^+(2bq + \lambda)\|_2 = 0$.

4 Proofs of the Main Results

4.1 Proofs of the results on the singularities of the SSF at the Landau levels

The first step in the proofs of both Theorems 2.1 and 2.2 is to show that we can replace the operator $T(E+i0)$ by $T_q(E)$ in the r.h.s of (3.10) when we deal with the first asymptotic term of $\tilde{\xi}(E;H_\pm,H_0)$ as the energy E approaches a given Landau level $2bq$, $q \in \mathbb{Z}_+$. More precisely, we pick $q \in \mathbb{Z}_+$ $\lambda \in \mathbb{R}$ with $|\lambda| \in (0,b)$, and set $T_q^-(2bq+\lambda) := \sum_{l=0}^{q-1} T_l(2bq+\lambda)$; if $q=0$ the sum should be set equal to zero. Evidently,

$$T(2bq+\lambda+i0) = T_q^-(2bq+\lambda) + T_q(2bq+\lambda) + T_q^+(2bq+\lambda),$$

$$\mathrm{Re}\, T(2bq+\lambda+i0) = \mathrm{Re}\, T_q^-(2bq+\lambda) + \mathrm{Re}\, T_q(2bq+\lambda) + T_q^+(2bq+\lambda),$$

$$\mathrm{Im}\, T(2bq+\lambda+i0) = \mathrm{Im}\, T_q^-(2bq+\lambda) + \mathrm{Im}\, T_q(2bq+\lambda),$$

the operator $T_q^+(2bq+\lambda)$ being defined in (3.27). Combining the Weyl inequalities (2.3), Lemma 3.1, (3.26), the Chebyshev-type estimates (2.5) with $p=2$, (3.18), and (3.28), we easily find that the asymptotic estimates

$$\int_{\mathbb{R}} n_\pm(1+\varepsilon; \mathrm{Re}\, T_q(2bq+\lambda) + t\, \mathrm{Im}\, T_q(2bq+\lambda))\, d\mu(t) + O(1) \leq$$

$$\int_{\mathbb{R}} n_\pm(1; \mathrm{Re}\, T(2bq+\lambda+i0) + t\, \mathrm{Im}\, T(E+i0))\, d\mu(t) \leq$$

$$\int_{\mathbb{R}} n_\pm(1-\varepsilon; \mathrm{Re}\, T_q(2bq+\lambda) + t\, \mathrm{Im}\, T_q(2bq+\lambda))\, d\mu(t) + O(1) \tag{4.1}$$

hold as $\lambda \to 0$ for each $\varepsilon \in (0,1)$ (see [10, Proposition 5.1] for the details).
If $\lambda > 0$, then $T_q(2bq-\lambda)$ is a self-adjoint operator with integral kernel

$$\frac{1}{2\pi}\sqrt{|V(X_\perp,x_3)|}\, \mathcal{P}_{q,b}(X_\perp,X_\perp') \int_{\mathbb{R}} \frac{e^{ip(x_3-x_3')}}{p^2+\lambda} dp\, \sqrt{|V(X_\perp',x_3')|} =$$

$$\frac{1}{2\sqrt{\lambda}}\sqrt{|V(X_\perp,x_3)|}\, \mathcal{P}_{q,b}(X_\perp,X_\perp') e^{-\sqrt{\lambda}|x_3-x_3'|}\sqrt{|V(X_\perp',x_3')|}, \quad (X_\perp,x_3),(X_\perp',x_3') \in \mathbb{R}^3,$$

In particular, $\mathrm{Im}\, T_q(2bq-\lambda) = 0$, and $\mathrm{Re}\, T_q(2bq-\lambda) = T_q(2bq-\lambda) \geq 0$. Therefore,

$$\int_{\mathbb{R}} n_\pm(s; \mathrm{Re}\, T_q(2bq-\lambda) + t\, \mathrm{Im}\, T_q(2bq-\lambda))\, d\mu(t) = n_\pm(s; T_q(2bq-\lambda)), \quad s>0, \quad \lambda>0. \tag{4.2}$$

Since $T_q(2bq-\lambda) \geq 0$, we have $n_-(s; T_q(2bq-\lambda)) = 0$ for all $s>0$ and $\lambda>0$, which combined with (3.10), (4.1), and (4.2), implies (2.7). In order to prove (2.8), we write

$$T_q(2bq-\lambda) = \mathcal{O}_q(\lambda) + \tilde{T}_q(\lambda)$$

where $\mathcal{O}_q(\lambda)$ is an operator with integral kernel

$$\frac{1}{2\sqrt{\lambda}}\sqrt{|V(X_\perp,x_3)|}\, \mathcal{P}_{q,b}(X_\perp,X_\perp')\sqrt{|V(X_\perp',x_3')|}, \quad (X_\perp,x_3),(X_\perp',x_3') \in \mathbb{R}^3,$$

and $\tilde{T}_q(\lambda) := T_q(2bq-\lambda) - \mathcal{O}_q(\lambda)$. By (1.2) we have $\mathrm{n}-\lim_{\lambda\downarrow 0} \tilde{T}_q(\lambda) = \tilde{T}_q(0)$ where $\tilde{T}_q(0)$ is a compact operator with integral kernel

$$-\frac{1}{2}\sqrt{|V(X_\perp,x_3)|}\, \mathcal{P}_{q,b}(X_\perp,X_\perp')|x_3-x_3'|\sqrt{|V(X_\perp',x_3')|}, \quad (X_\perp,x_3),(X_\perp',x_3') \in \mathbb{R}^3.$$

Hence, the Weyl inequalities easily imply that the asymptotic estimates

$$n_+(s' : \mathcal{O}_q(\lambda)) + O(1) \leq n_+(s; T_q(2bq - \lambda)) \leq n_+(s'' : \mathcal{O}_q(\lambda)) + O(1) \tag{4.3}$$

hold for every $0 < s' < s < s''$ as $\lambda \downarrow 0$. Further, define the operator $K : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^2)$ by

$$(Ku)(X_\perp) := \int_{\mathbb{R}^2} \int_{\mathbb{R}} \mathcal{P}_{q,b}(X_\perp, X'_\perp) \sqrt{|V(X'_\perp, x'_3)|} u(X'_\perp, x'_3)\, dx'_3\, dX'_\perp, \quad X_\perp \in \mathbb{R}^2,$$

where $u \in L^2(\mathbb{R}^3)$. The adjoint operator $K^* : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^3)$ is given by

$$(K^* v)(X_\perp, x_3) := \sqrt{|V(X_\perp, x_3)|} \int_{\mathbb{R}^2} \mathcal{P}_{q,b}(X_\perp, X'_\perp) v(X'_\perp)\, dX'_\perp, \quad (X_\perp, x_3) \in \mathbb{R}^3,$$

where $v \in L^2(\mathbb{R}^2)$. Obviously, $\mathcal{O}_q(\lambda) = \frac{1}{2\sqrt{\lambda}} K^* K$, $p_q W p_q = K\, K^*$. Therefore,

$$n_+(s; \mathcal{O}_q(\lambda)) = n_+(s2\sqrt{\lambda}; p_q W p_q), \quad s > 0, \quad \lambda > 0. \tag{4.4}$$

Now the combination of (3.10) with (4.1) – (4.4) entails (2.8). Thus, we are done with the proof of Theorem 2.1.
In order to complete the proof of Theorem 2.2, we recall that if $\lambda > 0$, then the operator $\operatorname{Re} T_q(2bq + \lambda)$ admits the integral kernel

$$-\frac{1}{2\sqrt{\lambda}} \sqrt{|V(X_\perp, x_3)|} \sin\left(\sqrt{\lambda}|x_3 - x'_3|\right) \mathcal{P}_{q,b}(X_\perp, X'_\perp) \sqrt{|V(X'_\perp, x'_3)|}, \quad (X_\perp, x_3), (X'_\perp, x'_3) \in \mathbb{R}^3,$$

and hence $\mathrm{n} - \lim_{\lambda \downarrow 0} \operatorname{Re} T_q(2bq + \lambda) = \tilde{T}_q(0)$. Applying the Weyl inequalities and the evident identities

$$\int_{\mathbb{R}} n_\pm(s; tT) d\mu(t) = \frac{1}{\pi} \mathrm{Tr}\ \arctan\,(s^{-1} T), \quad s > 0,$$

where $T = T^* \geq 0$, $T \in S_1$, we find that asymptotic estimates

$$\frac{1}{\pi} \mathrm{Tr}\ \arctan\,(((1+\varepsilon)s)^{-1} \mathrm{Im}\ T_q(2bq + \lambda)) + O(1) \leq \int_{\mathbb{R}} n_\pm(s; \mathrm{Re}\ T_q(2bq + \lambda) + t\ \mathrm{Im}\ T_q(2bq + \lambda)) d\mu(t) \leq$$

$$\frac{1}{\pi} \mathrm{Tr}\ \arctan\,(((1-\varepsilon)s)^{-1} \mathrm{Im}\ T_q(2bq + \lambda)) + O(1) \tag{4.5}$$

are valid as $\lambda \downarrow 0$ for each $s > 0$ and $\varepsilon \in (0, 1)$. Define the operator $\mathcal{K} : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^2)^2$ by

$$\mathcal{K} u := \mathbf{v} = (v_1, v_2) \in L^2(\mathbb{R}^2)^2, \quad u \in L^2(\mathbb{R}^3),$$

where

$$v_1(X_\perp) := \int_{\mathbb{R}^2} \int_{\mathbb{R}} \mathcal{P}_{q,b}(X_\perp, X'_\perp) \cos(\sqrt{\lambda} x'_3) \sqrt{|V(X'_\perp, x'_3)|} u(X'_\perp, x'_3)\, dx'_3\, dX'_\perp,$$

$$v_2(X_\perp) := \int_{\mathbb{R}^2} \int_{\mathbb{R}} \mathcal{P}_{q,b}(X_\perp, X'_\perp) \sin(\sqrt{\lambda} x'_3) \sqrt{|V(X'_\perp, x'_3)|} u(X'_\perp, x'_3)\, dx'_3\, dX'_\perp, \quad X_\perp \in \mathbb{R}^2.$$

Evidently, the adjoint operator $\mathcal{K}^* : L^2(\mathbb{R}^2)^2 \to L^2(\mathbb{R}^3)$ is given by

$$(\mathcal{K}^* \mathbf{v})(X_\perp, x_3) := \cos(\sqrt{\lambda} x_3) \sqrt{|V(X_\perp, x_3)|} \int_{\mathbb{R}^2} \mathcal{P}_{q,b}(X_\perp, X'_\perp) v_1(X'_\perp)\, dX'_\perp +$$

$$\sin(\sqrt{\lambda} x_3) \sqrt{|V(X_\perp, x_3)|} \int_{\mathbb{R}^2} \mathcal{P}_{q,b}(X_\perp, X'_\perp) v_2(X'_\perp)\, dX'_\perp, \quad (X_\perp, x_3) \in \mathbb{R}^3,$$

where $\mathbf{v} = (v_1, v_2) \in L^2(\mathbb{R}^2)^2$. Obviously,

$$\operatorname{Im} T_q(2bq + \lambda) = \frac{1}{2\sqrt{\lambda}} \mathcal{K}^* \mathcal{K}, \quad p_q \mathcal{W}_\lambda p_q = \mathcal{K} \mathcal{K}^*.$$

$$n_+(s; \operatorname{Im} T_q(2bq + \lambda)) = n_+(s2\sqrt{\lambda}; p_q \mathcal{W}_\lambda p_q), \quad s > 0, \quad \lambda > 0,$$

and, therefore,

$$\operatorname{Tr} \arctan\left(s^{-1} \operatorname{Im} T_q(2bq + \lambda)\right) = \operatorname{Tr} \arctan\left((s2\sqrt{\lambda})^{-1} p_q \mathcal{W}_\lambda p_q\right), \quad s > 0, \quad \lambda > 0. \tag{4.6}$$

Now the combination of (3.10), (4.1), (4.5), and (4.6) yields (2.10).

4.2 Proofs of the results on the strong-magnetic-field asymptotics of the SSF

In this subsection we prove Theorems 2.3 and 2.4 under the additional assumption that $\pm V \geq 0$. As before if $V \geq 0$ (or if $V \leq 0$), we will write H_+ and $\chi_+(X_\perp)$, $X_\perp \in \mathbb{R}^2$, (or H_- and $\chi_-(X_\perp)$) instead of H and $\chi(X_\perp)$ respectively.
First, we prove Theorem 2.3. For brevity set

$$A = A(b) = \operatorname{Re} T(\mathcal{E}b + \lambda), \quad B = B(b) = \operatorname{Im} T(\mathcal{E}b + \lambda).$$

Note that if $\mathcal{E} \in (0, \infty) \setminus 2\mathbb{Z}_+$, and $\lambda \in \mathbb{R}$, then (3.8) and (3.9) imply

$$\|A(b)\| = O(b^{-1/2}), \quad \|B(b)\| = O(b^{-1/2}), \quad \|B(b)\|_1 = O(b^{1/2}), \quad b \to \infty. \tag{4.7}$$

Assume that b so large that $\|A(b)\| < 1$. Then the operator $I - A$ is boundedly invertible, and $\lim_{b\to\infty} \|(I - A(b))^{-1}\| = 1$. By the Birman-Schwinger principle we have

$$\int_{\mathbb{R}} n_\pm(1; A + tB) d\mu(t) = \int_{\mathbb{R}} n_\pm(1; tB^{1/2}(I \mp A)^{-1} B^{1/2}) d\mu(t) =$$

$$\int_0^\infty n_+(s; B^{1/2}(I \mp A)^{-1} B^{1/2}) d\mu(s) = \frac{1}{\pi} \operatorname{Tr} \arctan\left(B^{1/2}(I \mp A)^{-1} B^{1/2}\right). \tag{4.8}$$

Further,

$$\operatorname{Tr} \arctan\left(B^{1/2}(I \pm A)^{-1} B^{1/2}\right) \leq \operatorname{Tr}\left(B^{1/2}(I \pm A)^{-1} B^{1/2}\right) = \operatorname{Tr} B \mp \operatorname{Tr}\left((I \pm A)^{-1} AB\right), \tag{4.9}$$

$$\operatorname{Tr} \arctan\left(B^{1/2}(I \pm A)^{-1} B^{1/2}\right) \geq \operatorname{Tr}\left(B^{1/2}(I \pm A)^{-1} B^{1/2}\right) - \frac{1}{3} \|B^{1/2}(I \pm A)^{-1} B^{1/2}\|_3^3 =$$

$$\operatorname{Tr} B \mp \operatorname{Tr}\left((I \pm A)^{-1} AB\right) - \frac{1}{3} \|B^{1/2}(I \pm A)^{-1} B^{1/2}\|_3^3. \tag{4.10}$$

By (4.7) we have

$$|\operatorname{Tr}\left((I \pm A)^{-1} AB\right)| \leq \|(I \pm A)^{-1} A\| \|B\|_1 = O(1), \quad b \to \infty, \tag{4.11}$$

$$\|B^{1/2}(I \pm A)^{-1} B^{1/2}\|_3^3 \leq \|B^{1/2}(I \pm A)^{-1} B^{1/2}\|^2 \, \|B^{1/2}(I \pm A)^{-1} B^{1/2}\|_1 = O(b^{-1/2}), \quad b \to \infty. \tag{4.12}$$

Putting together (4.8) – (4.12), and bearing in mind (3.10), we get

$$\xi(\mathcal{E}b + \lambda; H_\pm(b), H_0(b)) = \frac{1}{\pi} \operatorname{Tr} B(b) + O(1), \quad b \to \infty. \tag{4.13}$$

Recalling (3.9), we find that the asymptotic estimate

$$\mathrm{Tr}\ B(b) = \frac{b^{1/2}}{4\pi} \sum_{l=0}^{[\mathcal{E}/2]} (\mathcal{E} - 2l)^{-1/2} \int_{\mathbb{R}^3} |V(\mathbf{x})| d\mathbf{x} + O(b^{-1/2}) \tag{4.14}$$

holds as $b \to \infty$. Now the combination of (4.13) and (4.14) yields (2.18).
Next, we pass to the proof of Theorem 2.4 under the additional assumption that $\pm V \geq 0$. To this end we establish some auxiliary results. Introduce the operator

$$\tau(X_\perp; z) := |V(X_\perp, .)|^{1/2} (\chi_0 - z)^{-1} |V(X_\perp, .)|^{1/2},$$

defined on $L^2(\mathbb{R})$, and depending on the parameters $X_\perp \in \mathbb{R}^2$ and $z \in \mathbb{C}$ with $\mathrm{Im}\ z > 0$. The operator $\tau(X_\perp; z)$ admits the integral kernel

$$|V(X_\perp, x_3)|^{1/2} \mathcal{R}_z(x_3 - x_3') |V(X_\perp, x_3')|^{1/2}, \quad x_3, x_3' \in \mathbb{R}.$$

Evidently, $\tau(X_\perp; z) \in S_2$. For $X_\perp \in \mathbb{R}^2$, $\lambda \in \mathbb{R} \setminus \{0\}$, define the operator $\tau(X_\perp; \lambda + i0) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ as the operator with integral kernel

$$|V(X_\perp, x_3)|^{1/2} \mathcal{R}_\lambda(x_3 - x_3') |V(X_\perp, x_3')|^{1/2}, \quad x_3, x_3' \in \mathbb{R},$$

the function $\mathcal{R}_\lambda(x)$, $x \in \mathbb{R}$, being defined in (3.16). Some explicit simple calculations with the kernel of the operator $\tau(X_\perp; \lambda + i0)$ yield the following

Proposition 4.1. *Let $X_\perp \in \mathbb{R}^2$, $\lambda \in \mathbb{R} \setminus \{0\}$. Assume that* (1.1) *holds.*
i) We have $\tau(X_\perp; \lambda + i0) \in S_2$,

$$\|\tau(X_\perp; \lambda + i0)\|_2^2 \leq \frac{1}{4|\lambda|} \left(\int_{\mathbb{R}} |V(X_\perp, x_3)| dx_3 \right)^2, \tag{4.15}$$

and $\tau(X_\perp; \lambda + i\delta) \to \tau(X_\perp; \lambda + i0)$ in S_2 as $\delta \downarrow 0$, uniformly with respect to $X_\perp \in \mathbb{R}^2$.
ii) We have $\mathrm{Im}\ \tau(X_\perp; \lambda + i0) \geq 0$, *and* $\mathrm{Im}\ \tau(X_\perp; \lambda + i0) = 0$ *if $\lambda < 0$. If $\lambda > 0$, then* $\mathrm{rank}\ \mathrm{Im}\ \tau(X_\perp; \lambda + i0) = 2$, *and*

$$n_+(s; \mathrm{Im}\ \tau(X_\perp; \lambda + i0)) \leq 2\Theta \left(\frac{1}{2\sqrt{\lambda}} \int_{\mathbb{R}} |V(X_\perp, x_3)| dx_3 - s \right), \quad s > 0. \tag{4.16}$$

For $X_\perp \in \mathbb{R}^2$, $\lambda \in \mathbb{R} \setminus \{0\}$, $s > 0$, set

$$\Xi^{\pm}_{\lambda,s}(X_\perp) := \int_{\mathbb{R}} n_\pm(s; \mathrm{Re}\ \tau(X_\perp; \lambda + i0) + t\ \mathrm{Im}\ \tau(X_\perp; \lambda + i0)\, d\mu(t). \tag{4.17}$$

Corollary 4.1. *Let* (1.1) *hold. Fix $X_\perp \in \mathbb{R}^2$, $\lambda \in \mathbb{R} \setminus \{0\}$, $s > 0$. Then we have*

$$\xi(\lambda; \chi_\pm(X_\perp), \chi_0) = \pm\ \Xi^{\mp}_{\lambda,1}(X_\perp) \tag{4.18}$$

where $\xi(\cdot; \chi_\pm(X_\perp), \chi_0))$ is the representative of SSF for the operator pair $(\chi_\pm(X_\perp), \chi_0))$ which is monotonous and left-continuous for $\lambda < 0$, and continuous for $\lambda > 0$.

Proof. It suffices to apply Theorem 3.1 with $\mathcal{H}_\pm = \chi_\pm(X_\perp)$ and $\mathcal{H}_0 = \chi_0$. ☐

Corollary 4.2. *Under the assumptions of Corollary 4.1 we have*

$$\Xi^{\pm}_{\lambda,s}(\cdot) \in L^1(\mathbb{R}^2). \tag{4.19}$$

Proof. Combine Lemma 3.1 for $T_1 = \mathrm{Re}\,\tau(X_\perp; \lambda + i0))$ and $T_2 = \mathrm{Im}\,\tau(X_\perp; \lambda + i0))$, with Proposition 4.1. $\square$

Proposition 4.2. *Let $\lambda > 0$. Assume that* (1.1) *holds. Then the function $\int_{\mathbb{R}^2} \Xi^{\pm}_{\lambda,s}(X_\perp)dX_\perp$ is continuous with respect to $s > 0$.*

Proof. Fix $s > 0$. First of all we will show that for almost every $(X_\perp, t) \in \mathbb{R}^2 \times \mathbb{R}$ the functions

$$s' \mapsto n_\pm(s'; \mathrm{Re}\ \tau(X_\perp; \lambda + i0) + t\ \mathrm{Im}\ \tau(X_\perp; \lambda + i0))$$

are continuous at the point $s' = s$. Evidently, this is equivalent to

$$\pm s \notin \sigma(\mathrm{Re}\ \tau(X_\perp; \lambda + i0) + t\ \mathrm{Im}\ \tau(X_\perp; \lambda + i0)). \tag{4.20}$$

In order to prove (4.20), we will use an argument quite close to the one of the proof of [29, Lemma 4.1]. Note that the compact operator $\mathrm{Re}\ \tau(X_\perp; \lambda + i0) + t\ \mathrm{Im}\ \tau(X_\perp; \lambda + i0)$ depends linearly on t. By the Fredholm alternative the sets

$$\Omega^{\pm}(s, X_\perp, \lambda) := \{z \in \mathbb{C} \,|\, \pm\, s \in \sigma(\mathrm{Re}\ \tau(X_\perp; \lambda + i0) + z\ \mathrm{Im}\ \tau(X_\perp; \lambda + i0))\}$$

either coincide with $\mathbb{C}$, or are discrete. However, $i \in \Omega^{\pm}(s, X_\perp, \lambda)$ is equivalent to dim Ker $(\chi_0 \mp s^{-1}|V(X_\perp, .)| - \lambda) \geq 1$. On the other hand, it is well-known that the operators $\chi_0 \mp s^{-1}|V(X_\perp, .)|$ have no positive eigenvalues (see e.g. [39, Theorem XIII.58]) since (1.1) implies $\lim_{|x_3|\to\infty} |x_3|V(X_\perp, x_3) = 0$. Therefore, dim Ker $(\chi_0 \mp s^{-1}|V(X_\perp, .)| - \lambda) = 0$, $i \notin \Omega^{\pm}(s, X_\perp, \lambda)$, and the sets $\Omega^{\pm}(s, \lambda, X_\perp)$ are discrete. In particular, $|\mathbb{R} \cap \Omega^{\pm}(s, X_\perp, \lambda)| = 0$. Put

$$\tilde{\Omega}^{\pm}(s, \lambda) := \{(X_\perp, t) \in \mathbb{R}^2 \times \mathbb{R} | \pm s \in \sigma(\mathrm{Re}\ \tau(X_\perp; \lambda + i0) + t\ \mathrm{Im}\ \tau(X_\perp; \lambda + i0))\}.$$

The eigenvalues of the compact operator $\mathrm{Re}\ \tau(X_\perp; \lambda + i0) + t\ \mathrm{Im}\ \tau(X_\perp; \lambda + i0)$ are continuous, and hence measurable with respect to $(X_\perp, t) \in \mathbb{R}^2 \times \mathbb{R}$. Therefore, the sets $\tilde{\Omega}^{\pm}(s, \lambda)$ are measurable, and by the Fubini-Tonelli theorem

$$|\tilde{\Omega}^{\pm}(s, \lambda)| = \int_{\mathbb{R}^2} \int_{\mathbb{R}} \mathbf{1}_{\tilde{\Omega}^{\pm}(s,\lambda)}(X_\perp, t)dtdX_\perp = \int_{\mathbb{R}^2} |\mathbb{R} \cap \Omega^{\pm}(s, X_\perp, \lambda)|dX_\perp = 0$$

where $\mathbf{1}_{\tilde{\Omega}^{\pm}(s,\lambda)}$ denotes the characteristic function of $\Omega^{\pm}(s, \lambda)$. On the other hand,

$$\lim_{s'\to s} n_\pm(s'; \mathrm{Re}\ \tau(X_\perp; \lambda+i0)+t\ \mathrm{Im}\ \tau(X_\perp; \lambda+i0)) = n_\pm(s; \mathrm{Re}\ \tau(X_\perp; \lambda+i0)+t\ \mathrm{Im}\ \tau(X_\perp; \lambda+i0)) \tag{4.21}$$

if $(X_\perp, t) \notin \tilde{\Omega}^{\pm}(s, \lambda)$. The Weyl inequalities (2.3) and estimates (4.15) – (4.16) imply

$$n_\pm(s'; \mathrm{Re}\ \tau(X_\perp; \lambda + i0) + t\ \mathrm{Im}\ \tau(X_\perp; \lambda + i0)) \leq$$

$$\frac{1}{s'^2\lambda} \left(\int_{\mathbb{R}} |V(X_\perp, x_3)|dx_3\right)^2 + 2\Theta\left(\frac{|t|}{\sqrt{\lambda}} \int_{\mathbb{R}} |V(X_\perp, x_3)|dx_3 - s'\right). \tag{4.22}$$

Note that the r.h.s. is in $L^1(\mathbb{R}^2 \times \mathbb{R}; dX_\perp\ d\mu(t))$ for each $s' > 0$, and is a sum of two monotonous functions of $s' > 0$. Bearing in mind (4.21) – (4.22), we apply the dominated convergence theorem, and get $\lim_{s'\to s} \int_{\mathbb{R}^2} \Xi^{\pm}_{\lambda,s'}(X_\perp)dX_\perp = \int_{\mathbb{R}^2} \Xi^{\pm}_{\lambda,s}(X_\perp)dX_\perp$. $\square$

Set

$$\Phi^{\pm}_{\lambda,s}(t) := \int_{\mathbb{R}^2} n_\pm(s; \mathrm{Re}\ \tau(X_\perp; \lambda + i0) + t\ \mathrm{Im}\ \tau(X_\perp; \lambda + i0))\, dX_\perp, \quad t \in \mathbb{R}.$$

Corollary 4.3. *Assume that* (1.1) *holds. Let $\lambda > 0$, $s > 0$. Then $\lim_{s'\to s} \Phi^{\pm}_{\lambda,s'}(t) = \Phi^{\pm}_{\lambda,s}(t)$ for almost every $t \in \mathbb{R}$.*

Proof. Since the functions $\Phi^{\pm}_{\lambda,s}(t)$ are non-increasing with respect to $s > 0$, the one-sided limits $\Phi^{\pm}_{\lambda,s-0}(t) \geq \Phi^{\pm}_{\lambda,s+0}(t)$ exist. Next, Proposition 4.2 implies $\int_{\mathbb{R}^2} \Xi^{\pm}_{\lambda,s-0}(X_\perp)dX_\perp = \int_{\mathbb{R}^2} \Xi^{\pm}_{\lambda,s+0}(X_\perp)dX_\perp$. By the Fubini theorem $\int_{\mathbb{R}^2} \Xi^{\pm}_{\lambda,s}(X_\perp)dX_\perp = \int_{\mathbb{R}} \Phi^{\pm}_{\lambda,s}(t)d\mu(t)$. Hence, $\int_{\mathbb{R}} \left(\Phi^{\pm}_{\lambda,s-0}(t) - \Phi^{\pm}_{\lambda,s+0}(t)\right) d\mu(t) = 0$. Since, the functions $\Phi^{\pm}_{\lambda,s-0}(t) - \Phi^{\pm}_{\lambda,s+0}(t)$ are non-negative, we conclude that

$$\left|\left\{t \in \mathbb{R} \,\middle|\, \Phi^{\pm}_{\lambda,s-0}(t) > \Phi^{\pm}_{\lambda,s+0}(t)\right\}\right| = 0.$$

□

The following proposition contains key limiting relations used in the proof of Theorem 2.4.

Proposition 4.3. (cf. [7, Proposition 7.1]) *Let* (1.1) *hold. Then we have*

$$\lim_{b\to\infty} b^{-1}\mathrm{Tr}\ (\mathrm{Re}\ T_q(2bq+\lambda) + t\ \mathrm{Im}\ T_q(2bq+\lambda))^p =$$

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{Tr}\ (\mathrm{Re}\ \tau(X_\perp;\lambda+i0) + t\ \mathrm{Im}\ \tau(X_\perp;\lambda+i0))^p\, dX_\perp \tag{4.23}$$

for every $t \in \mathbb{R}$ *and each integer* $p \geq 2$.

Proof. Let $t \in \mathbb{R}$, $x \in \mathbb{R}$. If $\lambda > 0$, set $\tilde{\mathcal{R}}_{\lambda,t}(x) := -\frac{\sin(\sqrt{\lambda}|x|)}{2\sqrt{\lambda}} + t\,\frac{\cos(\sqrt{\lambda}\,x)}{2\sqrt{\lambda}}$. If $\lambda < 0$, then $\tilde{\mathcal{R}}_{\lambda,t}(x) = \mathcal{R}_\lambda(x) = \frac{e^{-\sqrt{-\lambda}|x|}}{2\sqrt{-\lambda}}$. We have

$$\mathrm{Tr}\ (\mathrm{Re}\ T_q(2bq+\lambda) + t\ \mathrm{Im}\ T_q(2bq+\lambda))^p =$$

$$\int_{\mathbb{R}^{2p}}\int_{\mathbb{R}^p} \Pi_{j=1}^p |V(X_{\perp,j},x_{3,j})| \Pi'^p_{j=1}\mathcal{P}_{q,b}(X_{\perp,j},X_{\perp,j+1})\tilde{\mathcal{R}}_{\lambda,t}(x_{3,j}-x_{3,j+1})\Pi_{j=1}^p dX_{\perp,j}dx_{3,j}$$

where the notation $\Pi'^p_{j=1}$ means that in the product of p factors the variables $X_{\perp,p+1}$ and $x_{3,p+1}$ should be set equal respectively to $X_{\perp,1}$ and $x_{3,1}$. Change the variables

$$X_{\perp,1} = X'_{\perp,1}, \quad X_{\perp,j} = X'_{\perp,1} + b^{-1/2}X'_{\perp,j}, \quad j = 2,\dots,p. \tag{4.24}$$

Thus we obtain

$$\mathrm{Tr}\ (\mathrm{Re}\ T_q(2bq+\lambda) + t\ \mathrm{Im}\ T_q(2bq+\lambda))^p =$$

$$b\int_{\mathbb{R}^{2p}}\int_{\mathbb{R}^p} |V(X'_{\perp,1},x_{3,1})|\Pi_{j=2}^p|V(X'_{\perp,1}+b^{-1/2}X'_{\perp,j},x_{3,j})| \times$$

$$\mathcal{P}_{q,1}(0,X'_{\perp,2})\Pi_{j=2}^{p-1}\mathcal{P}_{q,1}({X_\perp}'_{,j},{X_\perp}'_{,j+1})\mathcal{P}_{q,1}(X'_{\perp,p},0)\,\Pi'^p_{j=1}\tilde{\mathcal{R}}_{\lambda,t}(x_{3,j}-x_{3,j+1})\Pi_{j=1}^p dX'_{\perp,j}dx_{3,j}. \tag{4.25}$$

Here and in the sequel, if $p = 2$, then the product $\Pi_{j=2}^{p-1}\mathcal{P}_{q,b}({X_\perp}'_{,j},{X_\perp}'_{,j+1})$ should be set equal to one. Bearing in mind (1.1) and (3.14), and applying the dominated convergence theorem, we easily find that (4.25) entails

$$\lim_{b\to\infty} b^{-1}\mathrm{Tr}\ (\mathrm{Re}\ T_q(2bq+\lambda+i0) + t\ \mathrm{Im}\ T_q(2bq+\lambda+i0))^p =$$

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^p}\Pi_{j=1}^p|V(X_{\perp,1},x_{3,j})|\Pi'^p_{j=1}\tilde{\mathcal{R}}_{\lambda,t}(x_{3,j}-x_{3,j+1})dX_{\perp,1}\Pi_{j=1}^p dx_{3,j} \times$$

$$\int_{\mathbb{R}^{2(p-1)}}\mathcal{P}_{q,1}(0,X_{\perp,2})\Pi_{j=2}^{p-1}\mathcal{P}_{q,1}(X_{\perp,j},X_{\perp,j+1})\mathcal{P}_{q,1}(X_{\perp,p},0)\Pi_{j=2}^p dX_{\perp,j} =$$

$$\int_{\mathbb{R}^2}\mathrm{Tr}\ \left(\mathrm{Re}\ \tau(X_{\perp,1};\lambda+i0) + t\ \mathrm{Im}\ \tau(X_{\perp,1};\lambda+i0)\right)^p dX_{\perp,1} \times$$

$$\int_{\mathbb{R}^{2(p-1)}} \mathcal{P}_{q,1}(0, X_{\perp,2})\Pi_{j=2}^{p-1}\mathcal{P}_{q,1}(X_{\perp,j}, X_{\perp,j+1})\mathcal{P}_{q,1}(X_{\perp,p}, 0)\Pi_{j=2}^{p} dX_{\perp,j}.$$

In order to conclude that the above limiting relation is equivalent to (4.23), it remains to recall (3.15), and note that

$$\int_{\mathbb{R}^{2(p-1)}} \mathcal{P}_{q,1}(0, X_{\perp,2})\Pi_{j=2}^{p-1}\mathcal{P}_{q,1}(X_{\perp,j}, X_{\perp,j+1})\mathcal{P}_{q,1}(X_{\perp,p}, 0)\Pi_{j=2}^{p} dX_{\perp,j} = \mathcal{P}_{q,1}(0,0) = \frac{1}{2\pi}.$$

□

Corollary 4.4. *Assume that the assumptions of Theorem 2.4 hold. Then we have*

$$\lim_{b\to\infty} b^{-1} n_{\pm}(s; \mathrm{Re}\, T_q(2bq+\lambda) + t\, \mathrm{Im}\, T_q(2bq+\lambda)) =$$

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} n_{\pm}(s; \mathrm{Re}\, \tau(X_\perp; \lambda+i0) + t\, \mathrm{Im}\, \tau(X_\perp; \lambda+i0)) dX_\perp,$$

for each $t \in \mathbb{R}$, provided that $s > 0$ is a continuity point of the r.h.s.

Proof. It suffices to notice that norm of the operator $T_q(2bq+\lambda)$ is uniformly bounded with respect to b, and to apply a suitable version the Kac-Murdock-Szegö theorem (see e.g. [32, Lemma 3.1]) which tells us that under appropriate hypotheses the convergence of the moments of a given measure implies the convergence of the measure itself, and to take into account Proposition 4.3. □

Now we are in position to prove Theorem 2.4. By A. Pushnitski's representation of the SSF (see (3.10) and (4.18)), in order to check the validity of (2.21), it suffices to show that

$$\lim_{b\to\infty} b^{-1}\int_{\mathbb{R}} n_{\pm}(1; \mathrm{Re}\, T(2bq+\lambda+i0) + t\mathrm{Im}\, T(2bq+\lambda+i0)) d\mu(t) =$$

$$\frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}^2} n_{\pm}(1; \mathrm{Re}\, \tau(X_\perp; \lambda+i0) + t\mathrm{Im}\, \tau(X_\perp; \lambda+i0)) d\mu(t) dX_\perp. \tag{4.26}$$

Arguing as in the derivation of (4.1), we easily find that the asymptotic estimates

$$\int_{\mathbb{R}} n_{\pm}(1+\varepsilon; \mathrm{Re}\, T_q(2bq+\lambda) + t\, \mathrm{Im}\, T_q(2bq+\lambda))\, d\mu(t) + o(b) \le$$

$$\int_{\mathbb{R}} n_{\pm}(1; \mathrm{Re}\, T(2bq+\lambda+i0) + t\, \mathrm{Im}\, T(2bq+\lambda+i0))\, d\mu(t) \le$$

$$\int_{\mathbb{R}} n_{\pm}(1-\varepsilon; \mathrm{Re}\, T_q(2bq+\lambda) + t\mathrm{Im}\, T_q(2bq+\lambda))\, d\mu(t) + o(b), \tag{4.27}$$

hold as $b \to \infty$ for each $\varepsilon \in (0,1)$. Assume $\lambda > 0$. Corollary 4.3 and Corollary 4.4 imply

$$\lim_{b\to\infty} b^{-1} n_{\pm}(s; \mathrm{Re}\, T_q(2bq+\lambda) + t\, \mathrm{Im}\, T_q(2bq+\lambda)) =$$

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} n_{\pm}(s; \mathrm{Re}\, \tau(X_\perp; \lambda+i0) + t\mathrm{Im}\, \tau(X_\perp; \lambda+i0))\, dX_\perp \tag{4.28}$$

for any fixed $s > 0$, and almost every $t \in \mathbb{R}$. Further, by (2.3), (2.5) with $p = 2$, Proposition 3.2, and Proposition 3.3 we have

$$b^{-1} n_{\pm}(s; \mathrm{Re}\, T_q(2bq+\lambda+i0) + t\, \mathrm{Im}\, T_q(2bq+\lambda+i0)) \le C_4(1+|t|^{2/m_\perp}), \quad t \in \mathbb{R}, \tag{4.29}$$

with C_4 which may depend on $s > 0$, $\lambda \in \mathbb{R} \setminus \{0\}$, q and $m_\perp$ but is independent of $b \geq 1$ and t. Note that the function on the r.h.s of (4.29) is in $L^1(\mathbb{R}; d\mu)$. By (4.28) – (4.29), the dominated convergence theorem and the Fubini Theorem imply

$$\lim_{b\to\infty} b^{-1} \int_{\mathbb{R}} n_\pm(s; \mathrm{Re}\, T_q(2bq + \lambda + i0) + t\, \mathrm{Im}\, T_q(2bq + \lambda + i0)\, d\mu(t) =$$

$$\frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}} n_\pm(s; \mathrm{Re}\, \tau(X_\perp; \lambda + i0) + t\, \mathrm{Im}\, \tau(X_\perp; \lambda + i0))\, d\mu(t)\, dX_\perp, \quad s > 0. \tag{4.30}$$

Putting together (4.27) and (4.30), we find that the following estimates

$$\int_{\mathbb{R}^2} \Xi^\pm_{\lambda,1+\varepsilon}(X_\perp) dX_\perp \leq \liminf_{b\to\infty} b^{-1} \int_{\mathbb{R}} n_\pm(1; \mathrm{Re}\, T(2bq + \lambda + i0) + t\, \mathrm{Im}\, T(2bq + \lambda + i0)\, d\mu(t) \leq$$

$$\limsup_{b\to\infty} b^{-1} \int_{\mathbb{R}} n_\pm(1; \mathrm{Re}\, T(2bq + \lambda + i0) + t\, \mathrm{Im}\, T(2bq + \lambda + i0)\, d\mu(t) \leq \int_{\mathbb{R}^2} \Xi^\pm_{\lambda,1-\varepsilon}(X_\perp) dX_\perp$$

are valid for each $\varepsilon \in (0, 1)$. Letting $\varepsilon \downarrow 0$, and taking into account Proposition 4.2, we obtain (4.26), and hence (2.21), in the case $\lambda > 0$. The modifications of the argument for $\lambda < 0$ are quite obvious; in this case we essentially use assumption (2.20) guaranteeing that λ is a continuity point of the r.h.s of (2.21).

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