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Abstract version of the Cauchy-Kowalewski Problem

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Abstract: We consider an abstract version of the Cauchy-Kowalewski Problem with the right hand side being free from the Lipschitz type conditions and prove the existence theorem.

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1 Introduction

There are two standard existence theorems in the theory of ODE: the Cauchy-Picard existence and uniqueness theorem and the Peano existence theorem. The Cauchy-Picard theorem states that if the right hand side of ODE satisfies the Lipschitz conditions then initial value problem has unique solution. The proof of this theorem is based on the contraction mapping principle. The Peano theorem states that for existence of a solution it is sufficient to have only continuity of the right hand side. This theorem is proven by means of compactness considerations with the help of the Arzela-Ascoli theorem.

The case of initial value problem for PDE in the abstract setup has been studied by many authors and there are existence and uniqueness theorems proved under the assumptions of Lipschitz type conditions.

An abstract form of the Cauchy-Kowalewski Problem was first considered by T. Yamanaka in [8] and L. Ovsjannikov [4] in the linear case. Some another aspects of the linear Cauchy-Kowalewski Problem were exposed by J. Treves [7].

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In [2] L. Nirenberg obtained the existence and uniqueness theorem for the abstract nonlinear Cauchy-Kowalewski Problem. The proof of Nirenberg's theorem uses an iteration procedure of Newtonian type and based on ideas of the KAM theory. In Nirenberg's theorem it is assumed that the right hand side of the problem is a strong differentiable mapping.

T. Nishida [3] simplified the iteration procedure and stated that in Nirenberg's theorem it is possible to replace strong differentiability with the Lipschitz type conditions.

In [5] M. Safonov gave a proof of Nishida's theorem by constructing a suitable Banach space of functions and then using the contraction mapping principle.

In this paper we consider a topological aspect of the abstract nonlinear Cauchy-Kowalewski Problem and prove the Peano type existence theorem.

We assume that the right hand side of the equation depends on two arguments: it is bounded and continuous in the first argument (pure Peano's case) and convex in the second one. Such a setup includes quasilinear PDE as a special case. This theorem is not deduced from Nishida's result or quasilinear versions of the Cauchy-Kowalewski Problem since the Lipschitz type conditions are not applied.

The main tools we use are Browder's generalization of the Schauder fixed point theorem and a topological construction close to Safonov's one.

2 Main theorem

Let $\{(E_s, \|\cdot\|_s)\}_{0<s<1}$ be a scale of Banach spaces:

$$E_{s+\delta} \subseteq E_s, \quad \|\cdot\|_s \leq \|\cdot\|_{s+\delta}, \quad s+\delta < 1, \quad \delta > 0. \tag{1}$$

We assume that all embeddings (1) are compact. Such an assumption always holds for the scales of analytic functions.

Let $B_s(r) = \{u \in E_s \mid \|u\|_s < r\}$ be an open ball of E_s and let $\overline{B}_s(r)$ be its closure.

The main object of our study is the following Cauchy-Kowalewski problem:

$$u_t = A(t,u,u) + h(t,u), \quad u\,|_{t=0} = 0. \tag{2}$$

For some positive constants T, R, M, K the mappings

$$A : [0,T] \times \overline{B}_{s+\delta}(R) \times \overline{B}_{s+\delta}(R) \to E_s,$$
$$h : [0,T] \times \overline{B}_{s+\delta}(R) \to E_s, \quad \delta > 0, \quad s+\delta < 1$$

are continuous and if $u, v \in \overline{B}_{s+\delta}(R)$ then the inequalities hold:

$$\|A(t,u,v)\|_s \leq \frac{M\|v\|_{s+\delta}}{\delta}, \quad \|h(t,u)\|_s \leq K, \quad \delta > 0, \quad s+\delta < 1. \tag{3}$$

Let the mapping A be convex in the third argument: for all $u, v, w \in \overline{B}_{s+\delta}(R)$ and $0 \leq \lambda \leq 1$ we have

$$\|A(t,u,\lambda v + (1-\lambda)w)\|_s \leq \lambda \|A(t,u,v)\|_s + (1-\lambda)\|A(t,u,w)\|_s. \tag{4}$$

For example, if the mapping A is linear in the third argument then the above inequality holds.

Theorem 2.1. There exists such a large constant $a > 0$ that problem (2) has a solution

$$u(t) \in \bigcap_{1-s-\tau a>0} C([0,\tau], \overline{B}_s(R)).$$

Hypothesis 2.2. The condition of convexity (4) is necessary: there must be an example of such mappings A and h that satisfy all the above conditions except (4) and problem (2) does not have the solution.

This theorem does not reduce to the Nishida result [3]. Nishida's theorem uses a type of the Lipschitz condition:

$$\|f(t,u') - f(t,u'')\|_s \leq \frac{M}{\delta}\|u' - u''\|_{s+\delta}, \quad u', u'' \in B_{s+\delta}(R), \quad s+\delta < 1,$$

where f is the right hand side of the problem.

In the case under consideration we separate the arguments of the mapping A. It is bounded in the second argument and unbounded in the third one. Thus it is sufficient to have only continuity in the second argument and linearity or convexity in the third one.

If the mapping A equals to zero identically then Theorem 2.1 is a direct generalization from the finite dimensional case to the scale of Banach spaces of the Peano existence theorem.

There is no reason to expect uniqueness in Theorem 2.1: even in the case of ordinary differential equations there are systems with continuous (but not Lipschitz) right-hand sides that do not have the uniqueness.

Before starting to prove Theorem 2.1 we must build some tools.

3 Preliminary topological construction

Introduce a triangle:

$$\Delta = \{(\tau, s) \in \mathbb{R}^2 \mid \tau > 0, \quad 0 < s < 1, \quad 1 - s - \tau a > 0\}.$$

Consider a seminormed space $E = \bigcap_{(\tau,s)\in\Delta} C([0,\tau], E_s)$ with a family of norms:

$$\|u\|_{\tau,s} = \max_{0\leq t\leq\tau} \|u(t)\|_s.$$

Obviously, these norms satisfy the following inequalities:

$$\|\cdot\|_{\tau,s} \leq \|\cdot\|_{\tau+\delta,s}, \quad \|\cdot\|_{\tau,s} \leq \|\cdot\|_{\tau,s+\delta}, \quad \delta > 0. \tag{5}$$

The space E is a topological space with a basis of the topology given by the open balls:

$$B_{\tau,s}(R) = \{u \in E \mid \|u\|_{\tau,s} < R\}.$$

Definition 3.1. A set $G \subseteq E$ is said to be uniformly continuous if for all $\varepsilon > 0$ and for all $(\tau, s) \in \Delta$ there is $\delta = \delta(\varepsilon, \tau, s) > 0$ such that if $t_1, t_2 \in [0, \tau]$ and $|t_1 - t_2| < \delta$ then

$$\sup_{u \in G} \|u(t_1) - u(t_2)\|_s < \varepsilon.$$

A set $G \subseteq E$ is said to be bounded if there are such constants $M_{\tau,s}$ that for all $u \in G$ we have $\|u\|_{\tau,s} \leq M_{\tau,s}$.

Recall the Arzela-Ascoli lemma [6]:

Lemma 3.2. Let $H \subset C([0,T], X)$ be a set in the space of continuous functions with values in a Banach space X. Assume that the set H is closed, bounded, uniformly continuous and for every $t \in [0,T]$ the set $\{u(t) \in X\}$ is compact in the space X. Then the set H is compact in the space $C([0,T], X)$.

Obviously there is a similar compactness criteria for the space E.

Lemma 3.3. If a closed set $G \subseteq E$ is uniformly continuous and bounded then it is compact.

Proof 3.4. Let (τ, s) be an arbitrary point of Δ. Since the set G is bounded and uniformly continuous in the space $C([0,\tau], E_{s+\delta})$, by the Lemma 3.2 it is compact in the space $C([0,\tau], E_s)$. Thus every sequence $\{u_k\}_{k\in\mathbb{N}} \subset G$ contains a subsequence that converges with respect to the norm $\|\cdot\|_{\tau,s}$. So it remains to prove that there is a subsequence of $\{u_k\}$ that converges by all the norms $\|\cdot\|_{\tau,s}$ at once.

Consider a set $\Delta_{\mathbb{Q}} = \Delta \bigcap \mathbb{Q}^2$. This set is countable and let $\gamma : \mathbb{N} \to \Delta_{\mathbb{Q}}$ be a corresponding bijection.

Let $\{u_k^1\} \subseteq \{u_k\}$ be a subsequence that converges by the norm $\|\cdot\|_{\gamma(1)}$. By the above arguments there is a subsequence $\{u_k^2\} \subseteq \{u_k^1\}$ that converges by the norm $\|\cdot\|_{\gamma(2)}$ etc.

The diagonal sequence $\{u_k^k\}$ converges by the norms $\{\|\cdot\|_{\gamma(k)}\}_{k\in\mathbb{N}}$. Then due to inequalities (5) it converges in all the norms. □

In the conclusion we formulate a generalized version of the Schauder fixed point theorem.

Theorem 3.5 ([1]). Let W be a compact and convex subset of the seminormed space E. Then a continuous mapping $f : W \to W$ has a fixed point $\hat{u}$ i.e. $f(\hat{u}) = \hat{u}$.

4 Proof of Theorem 2.1

Problem (2) is obviously equivalent to the following one:

$$u(t) = F(u) = \int_0^t A(\xi, u(\xi), u(\xi)) + h(\xi, u(\xi))\, d\xi. \tag{6}$$

As usual, if a solution of (6) is continuous in the variable t then it is actually C^1-smooth in t.

So, we seek for a fixed point of the mapping F.

Let $S \subset E$ be a set that consists of such elements v that satisfy the following conditions:

$$\|v(t)\|_s \le R, \tag{7}$$

for all $u \in E$ such that $\|u(t)\|_s \le R$ we have

$$\|A(t, u(t), v(t))\|_s \le \frac{1}{\sqrt{1-at-s}}, \tag{8}$$

for $t_1, t_2 \in [0, \tau], \quad (\tau, s) \in \Delta$ we have

$$\|v(t_1) - v(t_2)\|_s \le \Big(K + \frac{1}{\sqrt{1-s-a\tau}}\Big)|t_1 - t_2|. \tag{9}$$

Note that the set S is nonvoid: $0 \in S$.

The set S is convex by inequality (4) and it is compact by Lemma 3.3.

Thus if we show that

$$F(S) \subseteq S \tag{10}$$

then the Proof will be conclude by applying Theorem 3.5 to the mapping F and the set S.

So let $v \in S$ and we check inclusion (10). First observing that

$$t < \frac{1}{a}$$

we verify that the mapping F preserves inequality (7):

$$\begin{aligned}\|F(v)\|_s &\le \int_0^t \|A(\xi, u, v(\xi)\|_s + \|h(\xi, v(\xi))\|_s \, d\xi \\ &\le \int_0^t \frac{1}{\sqrt{1-a\xi-s}} + K \, d\xi = \frac{2}{a}\Big(\sqrt{1-s} - \sqrt{1-at-s}\Big) + Kt \\ &\le \frac{2+K}{a}.\end{aligned} \tag{11}$$

Thus inequality (7) will be preserved by the mapping F if we take in (11) $a \ge (2+K)/R$.

To verify inequality (8) introduce a notation:

$$w(t) = A(t, v(t), v(t)) + h(t, v(t)).$$

With the help of inequalities (4), (3) and by the Jensen Inequality [6] we have

$$\begin{aligned}\Big\|A(t, u, \int_0^t w(\xi)\, d\xi)\Big\|_s &\le \frac{1}{t}\int_0^t \|A(t, u, tw(\xi))\|_s \, d\xi \\ &\le \frac{M}{t}\int_0^t \frac{\|tw(\xi)\|_{s+\delta}}{\delta}\Big|_{\delta=\frac{1-a\xi-s}{2}} d\xi \\ &\le M\int_0^t \frac{\|w(\xi)\|_{s+\delta}}{\delta}\Big|_{\delta=\frac{1-a\xi-s}{2}} d\xi.\end{aligned} \tag{12}$$

The last term of (12) is not greater than

$$\begin{aligned}
&M\int_0^t \frac{\|A(\xi,v(\xi),v(\xi))\|_{s+\delta}+\|h(\xi,v(\xi))\|_{s+\delta}}{\delta}\Big|_{\delta=\frac{1-a\xi-s}{2}}\,d\xi \\
&\quad\le M\int_0^t \Big(\frac{1}{\delta\sqrt{1-a\xi-s-\delta}}+\frac{K}{\delta}\Big)\Big|_{\delta=\frac{1-a\xi-s}{2}}\,d\xi \le \frac{C}{a\sqrt{1-at-s}},
\end{aligned} \tag{13}$$

where the positive constant C does not depend on s,t,a. Taking in (13) $a\ge C$ we obtain that mapping F preserves inequality (8).

Now we check condition (9). Assuming for definiteness $t_2\ge t_1$ and by inequality (8) we have

$$\begin{aligned}
\|F(v)(t_2)-F(v)(t_1)\|_s &\le \int_{t_1}^{t_2}\|A(\xi,v(\xi),v(\xi))\|_s+\|h(\xi,v(\xi))\|_s\,d\xi \\
&\le \int_{t_1}^{t_2}\frac{1}{\sqrt{1-a\xi-s}}+K\,d\xi \\
&\le \Big(K+\frac{1}{\sqrt{1-s-a\tau}}\Big)(t_2-t_1).
\end{aligned}$$

Theorem 2.1 is proved.

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