ON THE RELLICH INEQUALITY WITH MAGNETIC POTENTIALS

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1. Introduction

Rellich's inequality is that

$$\frac{n^2(n-4)^2}{16}\int_{\mathbb{R}^n}\frac{|f(\mathbf{x})|^2}{|\mathbf{x}|^4}d\mathbf{x}\leq\int_{\mathbb{R}^n}|\Delta f(\mathbf{x})|^2d\mathbf{x} \tag{1.1}$$

for all $f\in C_0^\infty(\mathbb{R}^n\setminus\{0\})$ and $n\neq 2$, where the constant $n^2(n-4)^2/16$ is sharp. It first appeared in print in [9] but it had been given in 1953 in lectures at New York University which were published posthumously in [10]. In [2] an extension of Rellich's inequality to higher-order derivatives was derived through the application of a technique introduced in [8]. This technique, and Bennett's development, were the basis of the method used by Davies and Hinz in [4] in their comprehensive study of Rellich-type inequalities and their higher-order counterparts in $L^p(\Omega)$, for any $p\in[1,\infty)$ and Ω a complete Riemannian manifold with smooth boundary. The papers [2] and [4] were influenced by the work of Schmincke in [12]. In particular, Davies and Hinz obtain sharp constants C for inequalities of the form

$$\int_{\mathbb{R}^n}\frac{|f(\mathbf{x})|^p}{|\mathbf{x}|^\beta}d\mathbf{x}\leq C\int_{\mathbb{R}^n}\frac{|\Delta^m f(\mathbf{x})|^p}{|\mathbf{x}|^\alpha}d\mathbf{x} \tag{1.2}$$

for suitable values of α,β,m,n,p and $f\in C_0^\infty(\mathbb{R}^n\setminus\{0\})$.

In Rellich's original inequality there are two cases of special interest, and these have their counterparts in the work of Davies and Hinz. Firstly, when $n=2$ the inequality still holds but only for functions $f\in C_0^\infty(\mathbb{R}^2\setminus\{0\})$ which also satisfy

$$\int_0^{2\pi}f(r,\theta)\cos\theta d\theta=\int_0^{2\pi}f(r,\theta)\sin\theta d\theta=0.$$

Secondly, when $n=4$ there is no non-trivial inequality. Our primary concern will be with these special cases, but our general results have

Date: March 26, 2004.

The first author gratefully acknowledges the hospitality and support of the Mathematics Department at UAB where much of this work was done.

wider implications. Our approach has been influenced by the Laptev-Weidl inequality in [7] in which, *inter alia*, the Hardy-type inequality

$$\int_{\mathbb{R}^2} \frac{|f(\mathbf{x})|^2}{|\mathbf{x}|^2} d\mathbf{x} \leq C \int_{\mathbb{R}^2} |\nabla_{\mathbf{A}} f(\mathbf{x})|^2 d\mathbf{x} \tag{1.3}$$

is established for $f \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$, where $\nabla_{\mathbf{A}}$ is the *magnetic gradient*

$$\nabla_{\mathbf{A}} = \nabla - i\mathbf{A},$$

$\mathbf{A}$ is of the form (in polar co-ordinates)

$$\mathbf{A}(r,\theta) = \frac{\Psi(\theta)}{r}(-\sin\theta, \cos\theta), \quad \Psi \in L^\infty(0, 2\pi) \tag{1.4}$$

and the magnetic flux $\tilde{\Psi} := \frac{1}{2\pi}\int_0^{2\pi} \Psi(\theta) d\theta \notin \mathbb{Z}$. The magnetic field curl$\mathbf{A} = 0$ in $\mathbb{R}^2 \setminus \{0\}$ and is of Aharonov-Bohm type. The optimal constant in (1.3) is $C = \text{dist}(\tilde{\Psi}, \mathbb{Z})^{-2}$. Recall that when there is no magnetic potential $\mathbf{A}$ (or, equivalently, the magnetic flux $\tilde{\Psi} \in \mathbb{Z}$ by gauge invariance) there is no valid Hardy inequality. A method to establish an L^p-version of (1.3) is developed in [3] and this has also influenced the present work.

The L^2-version of our result is of the form

$$C \int_{\mathbb{R}^n} \frac{|f(\mathbf{x})|^2}{|\mathbf{x}|^{\alpha+4}} d\mathbf{x} \leq \int_{\mathbb{R}^n} \frac{|\Delta_{\mathbf{A}} f(\mathbf{x})|^2}{|\mathbf{x}|^\alpha} d\mathbf{x}, \tag{1.5}$$

where $\Delta_{\mathbf{A}} = \nabla_{\mathbf{A}} \cdot \nabla_{\mathbf{A}}$ and $\mathbf{A}$ is a vector field in Poincaré (or transverse) gauge (i.e. $\mathbf{A}(\mathbf{x}) \cdot \mathbf{x} = 0$) which is an exact 1-form in the simply-connected domain $\mathbb{R}^n \setminus \mathcal{L}_n$, where in appropriate polar co-ordinates $\mathbf{x} = (r, \theta_1, \cdots, \theta_{n-1})$, $\mathcal{L}_n$ is the co-ordinate axis

$$\mathcal{L}_n = \{\mathbf{x} : r \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-2} = 0\}.$$

As in the Laptev-Weidl inequality, our constant depends on the distance of the magnetic flux $\tilde{\Psi}$ to the set of integers $\mathbb{Z}$.

The L^p-version of our result takes a different form which emphasises the dependence of the inequalities on the behaviour of the functions f in a transverse direction. For instance, when $n = 2$, we prove that

$$\|f/|\cdot|^2\|_{L^p(\mathbb{R}^2)} \leq B_p \|\frac{1}{r^2}\left(i\frac{\partial}{\partial\theta} + \Psi(\theta)\right) f\|_{L^p(\mathbb{R}^2)}, \tag{1.6}$$

where $B_p \leq C_p/d^2(\cos(\pi d))^{1/p'}$, $d := \text{dist}(\tilde{\Psi}, \mathbb{Z}) \neq 0, 1/2$ and there are similar inequalities for $n \geq 3$. Such an inequality clearly does not hold when $\Psi = 0$ since radial functions are excluded. When $p = 2$ there is a constant C depending on d such that

$$\|\frac{1}{r^2}\left(i\frac{\partial}{\partial\theta} + \Psi(\theta)\right) f\|_{L^2(\mathbb{R}^2)} \leq C\|\Delta_{\mathbf{A}}^2 f\|_{L^2(\mathbb{R}^2)}. \tag{1.7}$$

We also investigate an analogue of (1.7) for $n = 3$.

2. The main result in $L^2(\mathbb{R}^n)$.

Throughout this paper, polar coordinates in $\mathbb{R}^n$ will be denoted by (r,ω) with $r := |\mathbf{x}|$, $\omega = \mathbf{x}/|\mathbf{x}|$ for $\mathbf{x} \in \mathbb{R}^n$. The unit ball in $\mathbb{R}^n$ is denoted by $\mathbb{S}^{n-1}$ and $\mathbb{R}_+ := (0,\infty)$. We shall also denote the $L^2(\mathbb{R}^n)$ norm by $\|\cdot\|$.

Theorem 1. *Let Λ_ω be a non-negative self-adjoint operator in $L^2(\mathbb{S}^{n-1};d\omega)$ whose spectrum is discrete and consists of eigenvalues $\lambda_m, m \in \mathcal{I}$, where $\mathcal{I}$ is a countable index set. Let*

$$L_r := -\frac{\partial^2}{\partial r^2} - \frac{n-1}{r}\frac{\partial}{\partial r} \tag{2.1}$$

and define the operator $D := L_r + \frac{1}{r^2}\Lambda_\omega$ on the domain

$$\begin{aligned}\mathcal{D}_0 := \{f : & f \in C_0^\infty(\mathbb{R}^n \setminus \{0\}), f(r,\cdot) \in \mathcal{D}(\Lambda_\omega) \\ & \text{for } 0 < r < \infty, Df \in L^2(\mathbb{R}^n)\}\end{aligned} \tag{2.2}$$

in $L^2(\mathbb{R}^n)$, where $\mathcal{D}(\Lambda_\omega)$ denotes the domain of Λ_ω. Then, for all $f \in \mathcal{D}_0$ and $\alpha \geq -1$ such that $|\cdot|^{-\alpha/2}Df \in L^2(\mathbb{R}^n)$, we have that $|\cdot|^{-2-\alpha/2}f \in L^2(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} \frac{|Df(\mathbf{x})|^2}{|\mathbf{x}|^\alpha} d\mathbf{x} \geq C(n,\alpha) \int_{\mathbb{R}^n} \frac{|f(\mathbf{x})|^2}{|\mathbf{x}|^{\alpha+4}} d\mathbf{x} \tag{2.3}$$

where

$$C(n,\alpha) = \inf_{m\in\mathcal{I}} \big\{\lambda_m + \big(\frac{n+\alpha}{2}\big)\big(\frac{n-\alpha-4}{2}\big)\big\}^2. \tag{2.4}$$

Proof. Since the spectrum of Λ_ω is assumed to be discrete, its normalised eigenvectors $u_m, m \in \mathcal{I}$ form an orthonormal basis of $L^2(\mathbb{S}^{n-1};d\omega)$: we assume that the eigenvalues are repeated according to multiplicity. For $f \in \mathcal{D}_0$, set

$$F_m(r) := \int_{\mathbb{S}^{n-1}} f(r,\omega)\overline{u_m(\omega)}d\omega. \tag{2.5}$$

Then $F_m \in C_0^\infty(\mathbb{R}_+)$ and on using Parseval's identity we obtain

$$\begin{aligned}\sum_{m\in\mathcal{I}} \|L_r F_m\|^2_{L^2(\mathbb{R}_+,r^{n-1}dr)} &= \int_0^\infty \sum_{m\in\mathcal{I}} |L_r F_m(r)|^2 r^{n-1} dr \\ &= \|L_r f\|^2 < \infty,\end{aligned} \tag{2.6}$$

with

$$L_r F_m(r) = \int_{\mathbb{S}^{n-1}} L_r f(r,\omega)\overline{u_m(\omega)}d\omega.$$

Also, $L_r f, Df \in L^2(\mathbb{R}^n)$ imply that $|\cdot|^{-2}\Lambda_\omega f \in L^2(\mathbb{R}^n)$ and

$$\begin{aligned}\sum_{m\in\mathcal{I}} \lambda_m^2 \int_0^\infty |F_m(r)|^2 r^{n-5} dr &= \int_0^\infty \sum_{m\in\mathcal{I}} \lambda_m^2 |F_m(r)|^2 r^{n-5} dr \\ &= \||\cdot|^{-2}\Lambda_\omega f\|^2 < \infty.\end{aligned} \tag{2.7}$$

In fact if $|\cdot|^{-\alpha/2}Df \in L^2(\mathbb{R}^n)$ then $|\cdot|^{-2-\alpha/2}\Lambda_\omega f \in L^2(\mathbb{R}^n)$ and

$$\sum_{m\in\mathcal{I}}\int_0^\infty |F_m(r)|^2 r^{n-\alpha-5}dr = \||\cdot|^{-2-\alpha/2}f\|^2 < \infty, \tag{2.8}$$

$$\sum_{m\in\mathcal{I}}\int_0^\infty |L_rF_m(r)|^2 r^{n-\alpha-1}dr = \||\cdot|^{-\alpha/2}L_rf\|^2 < \infty, \tag{2.9}$$

$$\sum_{m\in\mathcal{I}}\lambda_m^2\int_0^\infty |F_m(r)|^2 r^{n-\alpha-5}dr = \||\cdot|^{-2-\alpha/2}\Lambda_\omega f\|^2 < \infty. \tag{2.10}$$

To prove the theorem we start with

$$\int_{\mathbb{R}^n}|Df|^2\frac{d\mathbf{x}}{|\mathbf{x}|^\alpha} = \int_{\mathbb{R}^n}|L_rf|^2\frac{d\mathbf{x}}{|\mathbf{x}|^\alpha} + 2\Re e[\int_{\mathbb{R}^n}L_rf\overline{\Lambda_\omega f}\frac{d\mathbf{x}}{|\mathbf{x}|^{\alpha+2}}] + \int_{\mathbb{R}^n}|\Lambda_\omega f|^2\frac{d\mathbf{x}}{|\mathbf{x}|^{\alpha+4}}. \tag{2.11}$$

On integration by parts and using the Hardy inequality (see [5], II(9))

$$\int_0^\infty |F'(r)|^2 r^{t+2}dr \geq \left(\frac{t+1}{2}\right)^2\int_0^\infty |F(r)|^2 r^t dr, \tag{2.12}$$

we obtain

$$\begin{aligned}
&\int_0^\infty |L_rF_m(r)|^2 r^{n-\alpha-1}dr\\
&=\int_0^\infty (|F_m''(r)|^2 + \frac{2(n-1)}{r}\Re e[F_m''\overline{F_m'}]\\
&\quad + \frac{(n-1)^2}{r^2}|F_m'(r)|^2)r^{n-\alpha-1}dr\\
&\geq(\frac{n-\alpha-2}{2})^2\int_0^\infty |F_m'(r)|^2 r^{n-\alpha-3}dr\\
&\quad - (n-1)(n-\alpha-2)\int_0^\infty |F_m'(r)|^2 r^{n-\alpha-3}dr\\
&\quad + (n-1)^2\int_0^\infty |F_m'(r)|^2 r^{n-\alpha-3}dr\\
&=[(n-1)(\alpha+1) + (\frac{n-\alpha-2}{2})^2]\int_0^\infty |F_m'(r)|^2 r^{n-\alpha-3}dr\\
&\geq[(n-1)(\alpha+1) + (\frac{n-\alpha-2}{2})^2](\frac{n-\alpha-4}{2})^2\\
&\quad \times\int_0^\infty |F_m(r)|^2 r^{n-\alpha-5}dr.
\end{aligned}$$

Thus, from (2.8) and (2.9)

$$\int_{\mathbb{R}^n} \frac{|L_r f|^2}{|\mathbf{x}|^\alpha} d\mathbf{x} \geq \big\{(n-1)(\alpha+1) + \big(\frac{n-\alpha-2}{2}\big)^2\big\}\big(\frac{n-\alpha-4}{2}\big)^2 \times \int_{\mathbb{R}^n} \frac{|f|^2}{|\mathbf{x}|^{\alpha+4}} d\mathbf{x}. \tag{2.13}$$

Since

$$\int_{\mathbb{S}^{n-1}} L_r f(r,\omega)\overline{u_m(\omega)} d\omega = L_r F_m(r)$$

and

$$\int_{\mathbb{S}^{n-1}} \Lambda_\omega f(r,\omega)\overline{u_m(\omega)} d\omega = \lambda_m F_m(r),$$

it follows from Parseval's identity that in (2.11)

$$\int_{\mathbb{R}^n} \frac{L_r f\overline{\Lambda_\omega f}}{|\mathbf{x}|^{\alpha+2}} d\mathbf{x} = \int_0^\infty \sum_{m\in\mathcal{I}} \lambda_m \overline{F_m(r)} L_r F_m(r) r^{n-\alpha-3} dr. \tag{2.14}$$

Integration by parts yields

$$\begin{aligned}
&2\Re e\big[\int_0^\infty \overline{F_m(r)} L_r F_m(r) r^{n-\alpha-3} dr\big] \\
&= 2\Re e\big[\int_0^\infty F_m'\big(\overline{F_m'} r^{n-\alpha-3} + (n-\alpha-3)\overline{F_m} r^{n-\alpha-4}\big) dr\big] \\
&\quad + (n-1)(n-\alpha-4)\int_0^\infty |F_m|^2 r^{n-\alpha-5} dr \\
&= 2\int_0^\infty |F_m'|^2 r^{n-\alpha-3} dr \\
&\quad + \big\{-(n-\alpha-3)+(n-1)\big\}(n-\alpha-4)\big\} \\
&\quad \times \int_0^\infty |F_m|^2 r^{n-\alpha-5} dr \\
&\geq \big\{2\big(\frac{n-\alpha-4}{2}\big)^2 + (n-\alpha-4)(\alpha+2)\big\}\int_0^\infty |F_m|^2 r^{n-\alpha-5} dr
\end{aligned}$$

(by(2.12)),

$$= \frac{1}{2}(n-\alpha-4)(n+\alpha)\int_0^\infty |F_m|^2 r^{n-\alpha-5} dr.$$

This gives in (2.14)

$$2\Re e\big[\int_{\mathbb{R}^n} \frac{L_r f\overline{\Lambda_\omega f}}{|\mathbf{x}|^{\alpha+2}} d\mathbf{x}\big] \geq \frac{1}{2}(n-\alpha-4)(n+\alpha) \times \sum_{m\in\mathcal{I}} \lambda_m \int_0^\infty |F_m(r)|^2 r^{n-\alpha-5} dr. \tag{2.15}$$

Finally in (2.11), by (2.9),

$$\int_{\mathbb{R}^n} \frac{|\Lambda_\omega f|^2}{|\mathbf{x}|^{\alpha+4}} d\mathbf{x} = \sum_{m\in\mathcal{I}} \lambda_m^2 \int_0^\infty |F_m(r)|^2 r^{n-\alpha-5} dr. \tag{2.16}$$

The theorem follows on substituting (2.13), (2.15) and (2.16) in (2.11) and noting (2.8). □

Corollary 1. *Suppose that $n \geq 4$ or, when $n \leq 3$, that $C(n,0) > 0$. Then, there exists a positive constant $C(n)$ such that in the notation of Theorem 1*

$$\|| \cdot |^{-2} \Lambda_\omega f\| \leq C(n) \|Df\|. \tag{2.17}$$

Proof. From (2.13) and (2.15) with $\alpha = 0$, we have

$$\int_{\mathbb{R}^n} |L_r f|^2 d\mathbf{x} \geq \big[\frac{n(n-4)}{4}\big]^2 \int_{\mathbb{R}^n} \frac{|f(\mathbf{x})|^2}{|\mathbf{x}|^4} d\mathbf{x}$$

and

$$\begin{aligned} 2\Re e\big[\int_{\mathbb{R}^n} L_r f \overline{\Lambda_\omega f} \frac{d\mathbf{x}}{|\mathbf{x}|^2}\big] &\geq \frac{1}{2} n(n-4) \int_{\mathbb{R}^n} \big(|\mathbf{x}|^{-2}\Lambda_\omega f\big)\big(|\mathbf{x}|^{-2}\overline{f}\big) d\mathbf{x} \\ &\geq -\frac{1}{4} n[n-4]_- \{\epsilon \|| \cdot |^{-2}\Lambda_\omega f\|^2 + \frac{1}{\epsilon} \|| \cdot |^{-2} f\|^2\}, \end{aligned}$$

where $\varepsilon > 0$ is arbitrary and $[a]_- := \max\{0, -a\}$. Hence, from (2.11) with $\alpha = 0$,

$$\begin{aligned} \|Df\|^2 \geq & \frac{1}{4}[\frac{n(n-4)}{4}]^2 \|| \cdot |^{-2} f\|^2 \\ & - \frac{1}{4} n[n-4]_- \{\epsilon \|| \cdot |^{-2}\Lambda_\omega f\|^2 + \frac{1}{\epsilon} \|| \cdot |^{-2} f\|^2\} + \|| \cdot |^{-2}\Lambda_\omega f\|^2 \\ = & \{\frac{1}{4}[\frac{n(n-4)}{4}]^2 - \frac{1}{4\epsilon} n[n-4]_-\} \|| \cdot |^{-2} f\|^2 \\ & + \{1 - \frac{\epsilon}{4} n[n-4]_-\} \|| \cdot |^{-2}\Lambda_\omega f\|^2. \end{aligned}$$

When $n \geq 4$, this gives

$$\|Df\|^2 \geq \ \tfrac{1}{4}[\tfrac{n(n-4)}{4}]^2 \|| \cdot |^{-2} f\|^2 + \|| \cdot |^{-2}\Lambda_\omega f\|^2$$

whence (2.11) with $C(n) = 1$. For $n \leq 3$, we set $\epsilon = \frac{4}{n[n-4]_-}\delta$, $\delta \in (0,1)$, and use Theorem 1 to get

$$\begin{aligned} (1-\delta)\|| \cdot |^{-2}\Lambda_\omega f\|^2 \leq & \ \|Df\|^2 + [\tfrac{n(n-4)}{4}]^2 (\tfrac{1}{\delta} - \tfrac{1}{4}) \|| \cdot |^{-2} f\|^2 \\ \leq & \ \{1 + [\tfrac{n(n-4)}{4}]^2 C(n,0)^{-1} (\tfrac{1}{\delta} - \tfrac{1}{4})\} \|Df\|^2 \end{aligned}$$

which again yields (2.17). □

Remark 1. *There is a valid inequality (2.3) (i.e. $C(n,\alpha) > 0$) for all $n > \alpha + 4$.*

Remark 2. *It follows from the proof of Theorem 1 that for $f \in \mathcal{D}_0$,*

$$Df(r,\omega) = \sum_{m\in\mathcal{I}} \big(L_r + r^{-2}\lambda_m\big)F_m(r)u_m(\omega)$$

in the $L^2(\mathbb{S}^{n-1})$ sense and $F_m \in C_0^\infty(\mathbb{R}_+)$. Hence, on identifying $L^2(\mathbb{R}^n)$ with $L^2(\mathbb{R}_+; r^{n-1}dr) \bigotimes L^2(\mathbb{S}^{n-1}; d\omega)$, we have that

$$\mathcal{D}_0 = \bigoplus_{m\in\mathcal{I}} \big\{C_0^\infty(\mathbb{R}_+) \bigotimes \{u_m\}\big\}$$

and

$$D = \bigoplus_{m\in\mathcal{I}} \big\{\big(L_r + \lambda_m r^{-2}\big) \bigotimes I_m,$$

where $\bigoplus$ denotes orthogonal sum and I_m is the identity on $\{u_m\}$.

The map $J : F \mapsto r^{(n-1)/2}F : L^2(\mathbb{R}_+; r^{n-1}dr) \to L^2(\mathbb{R}_+)$is an isometry and if $D_m := L_r + \lambda_m r^{-2}$ on $C_0^\infty(\mathbb{R}_+)$, then $D_m = J^{-1}T_mJ$, where T_m also has domain $C_0^\infty(\mathbb{R}_+)$ and

$$T_m g = -\frac{d^2g}{dr^2} + r^{-2}\big[\lambda_m + \frac{(n-1)(n-3)}{4}\big]g.$$

By Hardy's inequality, for $g \in C_0^\infty(\mathbb{R}_+)$,

$$-\int_0^\infty \overline{g}g''dr = \int_0^\infty |g'|^2dr \geq \frac{1}{4}\int_0^\infty r^{-2}|g|^2dr$$

and so

$$\int_0^\infty \overline{g}T_m g dr \geq \big[\lambda_m + \big(\frac{n-2}{2}\big)^2\big]\frac{1}{4}\int_0^\infty r^{-2}|g|^2dr.$$

Thus T_m and D_m are non-negative.

3. Applications of Theorem 1

We shall apply Theorem 1 to the operator $D = -\Delta_{\mathbf{A}}$, the magnetic Laplacian associated with a magnetic potential $\mathbf{A}$ of Aharonov-Bohm type when $n = 2$ and which has analogous characteristics for other values of n. Although the analysis continues to be effective for unrestricted values of n, we shall concentrate on the cases $n = 2, 4$, which are the singular cases when $D = -\Delta$ and (2.3) is the standard Rellich inequality. To handle the case $n = 4$, we shall also need to discuss the case $n = 3$.

3.1. **The case $n = 2$.** A magnetic field $B = \mathrm{curl}\mathbf{A}$ of Aharonov-Bohm type vanishes in $\mathbb{R}^2\backslash\{0\}$ and in the simply connected domain $\mathbb{R}^2\backslash[0,\infty)$ the 1-form associated with $\mathbf{A}$ is exact. Hence there exists a potential P such that $\mathbf{A} = \nabla P$. If we assume, as we can without loss of generality, that $\mathbf{A}\cdot\mathbf{x} = 0$ (Poincaré gauge) (see [13], Section 8.4.2), then in polar coordinates $\mathbf{x} = r(\cos\theta, \sin\theta)$,

$$\mathbf{A}(r,\theta) := \frac{1}{r}\Psi(\theta)\mathbf{e}_\theta, \quad \Psi \in L^\infty(\mathbb{S}^1), \qquad \Psi(0) = \Psi(2\pi),$$

where $\mathbf{e}_\theta := (-\sin\theta, \cos\theta)$. We have, with $\mathbf{e}_r := (\cos\theta, \sin\theta)$,

$$\nabla = \mathbf{e}_r \frac{\partial}{\partial_r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial\theta}$$

and define

$$\nabla_{\mathbf{A}} := \nabla - i\mathbf{A} = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\left(\frac{\partial}{\partial\theta} - i\Psi(\theta)\right) \tag{3.1}$$

and

$$-\Delta_{\mathbf{A}} = -\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(i\frac{\partial}{\partial\theta} + \Psi(\theta)\right)^2. \tag{3.2}$$

Thus, in the notation of Theorem 1, $\Lambda_\omega = \Lambda_\theta$ is the non-negative self-adjoint operator in $L^2(0, 2\pi)$ defined by $\Lambda_\theta = K_\theta^2$, where

$$K_\theta u(\theta) = i\frac{du}{d\theta} + \Psi(\theta)u(\theta) \tag{3.3}$$

with domain

$$\big\{u : u \in C^1[0, 2\pi], u(0) = u(2\pi)\big\}.$$

It is readily seen that K_θ has eigenvalues $m + \widetilde{\Psi}, m \in \mathbb{Z}$, where $\widetilde{\Psi} = (1/2\pi)\int_0^{2\pi}\Psi(\theta)d\theta$ is the magnetic flux, and the corresponding normalised eigenfunctions are

$$u_m(\theta) = \frac{1}{\sqrt{2\pi}}\exp[-i(\theta[m + \tilde{\Psi}] - \int_0^\theta \Psi(\eta)d\eta)] \tag{3.4}$$

which constitute an orthonormal basis of $L^2(\mathbb{S}^1)$.

For $m \in \mathbb{Z}, U : f \mapsto e^{-im\theta}f$ is unitary on $L^2(\mathbb{R}^2)$ and

$$U^{-1}\nabla_{\mathbf{A}}U = \nabla_{\widetilde{\mathbf{A}}},$$

where $\widetilde{\mathbf{A}} = \frac{(\Psi+m)}{r}\mathbf{e}_\theta$. Since $\mathrm{curl}\mathbf{A} = \mathrm{curl}\widetilde{\mathbf{A}}$, the operators $\nabla_{\mathbf{A}}, \nabla_{\widetilde{\mathbf{A}}}$, and also $\Delta_{\mathbf{A}}, \Delta_{\widetilde{\mathbf{A}}}$ are gauge equivalent. Furthermore the associated magnetic fluxes differ by m. Therefore we may assume that $\tilde{\Psi} \in [0, 1)$.

Since $\mathcal{D}_0 = C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ we have from Theorem 1

Corollary 2. *For all $f \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ and $\alpha \geq -1$,*

$$\int_{\mathbb{R}^2} |\Delta_{\mathbf{A}}f(x)|^2 \frac{dx}{|x|^\alpha} \geq C(2, \alpha)\int_{\mathbb{R}^2} |f(x)|^2 \frac{dx}{|x|^{\alpha+4}}, \tag{3.5}$$

where

$$C(2, \alpha) = \inf_{m\in\mathbb{Z}}\{(m + \tilde{\Psi})^2 - (\frac{\alpha+2}{2})^2\}^2. \tag{3.6}$$

If $\tilde{\Psi} \notin \mathbb{Z}$ ($\tilde{\Psi} \in [0, 1)$ without loss of generality), we have

$$\begin{aligned} C(2, 0) = &\ \min\{(\tilde{\Psi}^2 - 1)^2, \tilde{\Psi}^2(\tilde{\Psi} - 2)^2\} \\ = &\ \begin{cases} (\tilde{\Psi}^2 - 1)^2 & \text{if } \ \tilde{\Psi} \in [\frac{1}{2}, 1), \\ \tilde{\Psi}^2(\tilde{\Psi} - 2)^2 & \text{if } \ \tilde{\Psi} \in [0, \frac{1}{2}). \end{cases} \end{aligned} \tag{3.7}$$

Remark 3. *If $\tilde{\Psi} \in \mathbb{Z}$, then $C(2,0) = 0$. However, if $F_1 = F_{-1} = 0$ (see (2.5)), the infimum in (3.6) is over $m \in \mathbb{Z} \setminus \{-1, 1\}$ and this gives $C(2,0) = 1$. Hence, Rellich's result in ([10], p.100) for $n = 2$ is recovered.*

3.2. **The case $n = 3$.** In spherical polar coordinates, we define the orthonormal vectors

$$\begin{aligned} \mathbf{e}_0 &:= \tfrac{\mathbf{x}}{|\mathbf{x}|} = (\cos\theta_1, \sin\theta_1\cos\theta_2, \sin\theta_1\sin\theta_2), \\ \mathbf{e}_1 &:= (-\sin\theta_1, \cos\theta_1\cos\theta_2, \cos\theta_1\sin\theta_2), \\ \mathbf{e}_2 &:= (0, -\sin\theta_2, \cos\theta_2), \end{aligned}$$

where $r = |\mathbf{x}| \in (0,\infty)$, $\theta_1 \in (0,\pi)$, and $\theta_2 \in (0, 2\pi)$. We now take

$$\mathbf{A} := \frac{1}{r\sin\theta_1}\Psi(\theta_2)\mathbf{e}_2, \qquad \Psi \in L^\infty(0,2\pi), \qquad \Psi(0) = \Psi(2\pi),$$

on $\mathbb{R}^3 \setminus \mathcal{L}_3$, where $\mathcal{L}_3 = \{(r,\theta_1,\theta_2) : r\sin\theta_1 = 0\}$. It is in Poincaré gauge and $\mathrm{curl}\mathbf{A} = 0$ in $\mathbb{R}^3 \setminus \mathcal{L}_3$.

Then

$$\nabla_{\mathbf{A}} = \nabla - i\mathbf{A} = \mathbf{e}_0\frac{\partial}{\partial r} + \mathbf{e}_1\frac{1}{r}\frac{\partial}{\partial\theta_1} + \mathbf{e}_2\frac{1}{r\sin\theta_1}[\frac{\partial}{\partial\theta_2} - i\Psi(\theta_2)] \tag{3.8}$$

and

$$-\Delta_{\mathbf{A}} = -\frac{\partial^2}{\partial r^2} - \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Lambda(\theta_1,\theta_2) \tag{3.9}$$

where

$$\Lambda(\theta_1,\theta_2) = -\frac{\partial^2}{\partial\theta_1^2} - \cot\theta_1\frac{\partial}{\partial\theta_1} + \frac{1}{\sin^2\theta_1}K_{\theta_2}^2 \tag{3.10}$$

and

$$K_{\theta_2} = i\frac{\partial}{\partial\theta_2} + \Psi(\theta_2). \tag{3.11}$$

As above for (3.3), the self-adjoint operator K_{θ_2} in $L^2(\mathbb{S}^1)$ has eigenvalues $k + \tilde{\Psi}$, $k \in \mathbb{Z}$, with $\tilde{\Psi} := \frac{1}{2\pi}\int_0^{2\pi}\Psi(\theta_2)d\theta_2$, and eigenvectors $u_k(\theta_2)$ which form an orthonormal basis of $L^2(\mathbb{S}^1)$. Identifying $L^2(\mathbb{S}^2)$ with $\bigoplus_{k\in\mathbb{Z}}\left(L^2(0,\pi;\sin\theta_1 d\theta_1)\bigotimes\{u_k\}\right)$ we shall take the operator Λ_ω of Theorem 1 to be

$$\Lambda_\omega = \bigoplus_{k\in\mathbb{Z}}\left(\Lambda_k(\theta_1)\bigotimes I_k\right), \tag{3.12}$$

where I_k denotes the identity on $\{u_k\}$ and $\Lambda_k(\theta_1)$ is a self-adjoint realisation of the operator $\Lambda_k^0(\theta_1)$ defined on $C_0^\infty(0,\pi)$ by

$$\Lambda_k^0(\theta_1)u = \big(-\frac{d^2}{d\theta_1^2} - \cot\theta_1\frac{d}{d\theta_1} + \frac{(k+\tilde{\Psi})^2}{\sin^2\theta_1}\big)u. \tag{3.13}$$

Before we are able to apply Theorem 1 we must first make a suitable choice of the operators $\Lambda_k(\theta_1)$ for all $k \in \mathbb{Z}$ and determine their eigenvalues. The information required is contained in the next two lemmas.

Lemma 1. *For $\mu \in [0, \infty)$, the associated Legendre equation*

$$\frac{d^2u}{d\theta^2} + \cot\theta \frac{du}{d\theta} + \left(\lambda - \frac{\mu^2}{\sin^2\theta}\right) u = 0, \quad \lambda \in \mathbb{C}, \tag{3.14}$$

is in the limit-circle case at 0 and π if $\mu \in [0, 1)$ and in the limit-point case at 0 and π otherwise.

Let

$$\mathcal{D}_\mu := \big\{u : u, \sin\theta \frac{du}{d\theta} \in AC_{loc}(0, \pi), u, L_\mu u \in L^2(0, \pi; \sin\theta d\theta)\big\}, \tag{3.15}$$

where

$$L_\mu := -\frac{d^2}{d\theta^2} - \cot\theta \frac{d}{d\theta} + \frac{\mu^2}{\sin^2\theta}$$

and denote the restriction of L_μ to $C_0^\infty(0, \pi)$ by Λ_μ^0. Then Λ_μ^0 is non-negative. It is essentially-self-adjoint if and only if $\mu \in [1, \infty)$ and for $\mu \in [0, 1)$ its Friedrichs extension Λ_μ is the realisation of L_μ on the following domains:

- *if $\mu \in (0, 1)$,*

$$\mathcal{D}(\Lambda_\mu) = \big\{u : u \in \mathcal{D}_\mu, \sin^\mu\theta \; u(\theta) \to 0 \; \text{as} \; \theta \to 0, \pi\big\}; \tag{3.16}$$

- *if $\mu = 0$,*

$$\mathcal{D}(\Lambda_\mu) = \big\{u : u \in \mathcal{D}_\mu, u(\theta)/\ln(\cot\theta) \to 0 \quad \text{as} \; \theta \to 0, \pi\big\}; \tag{3.17}$$

- *if $\mu \in [1, \infty)$, Λ_μ is the closure of Λ_μ^0 and*

$$\mathcal{D}(\Lambda_\mu) = \mathcal{D}_\mu. \tag{3.18}$$

For $\mu \in [0, 1)$, $\Lambda_\mu \geq \mu(\mu + 1)$.

Proof. On substituting $x = \cos\theta$, (3.14) becomes

$$\tau_\mu u := -\frac{d}{dx}\big\{(1 - x^2)\frac{du}{dx}\big\} + \frac{\mu^2}{1 - x^2} u = \lambda u, \quad x \in (-1, 1). \tag{3.19}$$

Denote the restriction of τ_μ to $C_0^\infty(-1, 1)$ by T_μ^0. Clearly $T_\mu^0 \geq 0$.

Define the functions

$$\begin{aligned} a(x) &= (1 - x^2)^{\mu/2}, \\ b(x) &= a(x)h(x), \quad h(x) = |\int_0^x (1 - t^2)^{-1-\mu} dt|. \end{aligned} \tag{3.20}$$

Note that

$$h(x) = \begin{cases} = & \frac{1}{2}\left|\ln\left(\frac{1+x}{1-x}\right)\right| \quad \text{for} \quad \mu = 0, \\ \approx & (1 - x^2)^{-\mu} \quad \text{for} \quad \mu > 0, \end{cases} \tag{3.21}$$

where $f \approx g$ means that $f(x)/g(x)$ is bounded above and below by positive constants. We also have

$$\begin{aligned} & a(x)/b(x) \to 0 \text{ as } x \to \pm 1; \\ & \int_0^1 \frac{dt}{(1-t^2)a^2(t)} = \int_{-1}^0 \frac{dt}{(1-t^2)a^2(t)} = \infty; \\ & \int_\delta^1 \frac{dt}{(1-t^2)b^2(t)} < \infty, \quad |\int_{-1}^{-\delta} \frac{dt}{(1-t^2)b^2(t)}| < \infty, \end{aligned}$$

for any $\delta > 0$. Furthermore

$$\begin{aligned} \tau_\mu a &= \mu(\mu+1)a, \\ \tau_\mu b &= \mu(\mu+1)b. \end{aligned} \tag{3.22}$$

Hence a, b are respectively *principal* and *non-principal* solutions of $\tau_\mu u = \mu(\mu+1)u$. For $\mu \in [0,1), a, b \in L^2(-1,1)$ and so (3.19) is in the limit-circle case at -1 and $+1$. If $\mu \geq 1, b$ is neither in $L^2(-1,0)$ nor $L^2(0,1)$ and hence (3.19) is limit-point at both the end points ± 1. Thus T_μ^0 is essentially-self-adjoint for $\mu \geq 1$. To characterise the Friedrichs extension T_μ of T_μ^0 when $\mu \in [0,1)$ we apply Rosenberger's Theorem 3 in [11] to get

$$\mathcal{D}(T_\mu) = \big\{u : u \in \mathcal{D}_\mu, \lim_{x \to -1} u(x)/b(x) = \lim_{x \to 1} u(x)/b(x) = 0\big\}, \tag{3.23}$$

where

$$\mathcal{D}_\mu = \big\{u : u, (1-x^2)u' \in AC_{loc}(-1,1), u, \tau_\mu u \in L^2(-1,1)\big\}.$$

For $\mu \geq 1$ we have seen that T_μ^0 is essentially-self-adjoint and so its unique self-adjoint extension has the domain of its adjoint, namely $\mathcal{D}_\mu$.

To prove that $T_\mu \geq \mu(\mu+1)$ when $\mu \in [0,1)$, we appeal to results of Kalf in [6]. In Remark 3 he gives the following characterisation of T_μ:

$$\begin{aligned} \mathcal{D}(T_\mu) = \big\{u :& u \in \mathcal{D}_\mu, \int^x (1-t^2)b^2(t)\big|\big(\frac{u}{b}\big)'\big|^2 dt < \infty \text{ at } \pm 1 \\ & \text{and } \lim_{x \to \pm 1} |u(x)/b(x)| = 0\big\}. \end{aligned} \tag{3.24}$$

On using Jacobi's factorisation identity (5) in [6], it follows from (3.22) (cf [6] (6)) that for $u \in \mathcal{D}_\mu$ and $-1 < X < Y < 1$

$$\begin{aligned} \int_X^Y \overline{u}\tau_\mu u dx =& [u,b](x)\overline{u(x)}/b(x)\big|_X^Y \\ & + \int_X^Y \Big\{(1-x^2)b^2\big|\big(\frac{u}{b}\big)'\big|^2 + \mu(\mu+1)|u|^2\Big\} dx, \end{aligned} \tag{3.25}$$

where

$$[u,b](x) := (1-x^2)\{u(x)b'(x) - u'(x)b(x)\}.$$

Since $b \in \mathcal{D}_\mu$ for $\mu \in [0,1)$ we know from Green's identity

$$\int_X^Y \big(u\tau_\mu b - b\tau_\mu u\big)dx = [u,b](x)\big|_X^Y$$

that $\lim_{x\to\pm 1}[u,b](x)$ exists. Hence, from (3.23) and (3.24), for $u \in \mathcal{D}(T_\mu)$,

$$\begin{aligned}\int_{-1}^1 \overline{u}T_\mu u dx &= \int_{-1}^1 \big\{(1-x^2)b^2\big|\big(\frac{u}{b}\big)'\big|^2 + \mu(\mu+1)|u|^2\big\}dx \\ &\geq \mu(\mu+1)\|u\|^2. \end{aligned} \tag{3.26}$$

The lemma follows on reverting to the θ-co-ordinates. □

Lemma 2. *The operator T_μ in Lemma 1 has a discrete spectrum consisting of eigenvalues*

$$\nu_j(\mu) = \big(j-\mu\big)\big(j+1-\mu\big), \quad j \in \mathbb{Z}\setminus\{0\}. \tag{3.27}$$

Proof. From Bateman [1] (§3.4, p.143), the associated Legendre polynomials $P_\nu^\mu(x)$ and $P_\nu^\mu(-x)$ are linearly independent solutions of (3.19) with $\lambda = \nu(\nu+1) \in \mathbb{C}\setminus\mathbb{R}$ and as $x \to \pm 1$,

$$P_\nu^\mu(x) \approx (x \mp 1)^{-\mu/2};$$

see Bateman [1], pp.163-64. We now use an argument similar to that of Titchmarsh in [14], §§4.5, 4.6. Let $\phi_0(x) := P_\nu^\mu(x)$ and

$$\begin{aligned}\theta(x,\lambda) &= \tfrac{\phi_0(x)+\phi_0(-x)}{2\phi_0(0)} \\ \phi(x,\lambda) &= \tfrac{\phi_0(-x)-\phi_0(x)}{2\phi_0'(0)}.\end{aligned} \tag{3.28}$$

Note that $\phi_0(0) \neq 0$ and $\phi_0'(0) \neq 0$ if $\nu+\mu \notin \mathbb{Z}$ from [1], p 145, (20) and (23). We therefore have

$$\theta(0,\lambda) = 1,\ \theta'(0,\lambda) = 0;\ \phi(0,\lambda) = 0,\ \phi'(0,\lambda) = -1,$$

so that θ and ϕ satisfy the initial conditions (2.1.4) in [14] with $\alpha = 0$.

In view of (3.23), a Titchmarsh-Weyl function for the interval $(0,1)$ is determined as

$$m_2(\lambda) = \lim_{X\to 1} l_2(\lambda, X)$$

where

$$\lim_{X\to 1}\big\{\big[\theta(X,\lambda) + l_2(\lambda,X)\phi(X,\lambda)\big]/b(X)\big\} = 0. \tag{3.29}$$

If $\lambda = \nu(\nu+1) \notin \mathbb{R}$, then $\lim_{X\to 1}\phi(X,\lambda)/b(X) \neq 0$ since otherwise ϕ would be an eigenfunction of the self-adjoint operator generated by τ_μ on the interval $[0,1)$ with boundary conditions $u(0) = 0$, $\lim_{x\to 1} u(x)/b(x) = 0$. Hence (3.29) yields

$$m_2(\lambda) = -\lim_{X\to 1}\big\{\theta(X,\lambda)/\phi(X,\lambda)\big\}. \tag{3.30}$$

Of course, this corresponds to a point on the limit-circle when $\mu \in [0,1)$ but is unique otherwise. Hence, on substituting (3.28),

$$m_2(\lambda) = \lim_{b\to 1-} \left\{ \frac{\phi_0(b)+\phi_0(-b)}{\phi_0(b)-\phi_0(-b)} \right\} \frac{\phi_0'(0)}{\phi_0(0)}. \tag{3.31}$$

From Bateman [1], p.163, since $\mu \notin \mathbb{N}$, as $x \to 1$

$$\begin{aligned} \phi_0(x) &\approx 2^{\mu/2}(1-x)^{-\mu/2}/\Gamma(1-\mu), \\ \phi_0(-x) &\approx -2^{\mu/2}\pi^{-1}(1-x)^{-\mu/2}\sin(\pi\nu)\Gamma(\mu). \end{aligned}$$

Thus, on recalling that $\Gamma(\mu)\Gamma(1-\mu) = \pi \csc(\pi\mu)$ (see [1], p.3),

$$m_2(\lambda) = K(\nu,\mu)\frac{\phi_0'(0)}{\phi_0(0)}, \qquad \lambda = \nu(\nu+1) \in \mathbb{C}\setminus\mathbb{R}$$

where

$$K(\nu,\mu) = \tan\frac{\pi}{2}(\mu-\nu)\cot\frac{\pi}{2}(\mu+\nu).$$

The corresponding Titchmarsh-Weyl solution of (3.19) on $(0,1)$ is

$$\begin{aligned} \psi_2(x,\lambda) &= \theta(x,\lambda) + m_2(\lambda)\phi(x,\lambda) \\ &= \tfrac{1}{2\phi_0(0)}\{[1-K(\nu,\mu)]\phi_0(x) + [1+K(\nu,\mu)]\phi_0(-x)\}. \end{aligned} \tag{3.32}$$

In the interval $(-1,0)$, we put $x=-y$, $y\in(0,1)$, to get the Titchmarsh-Weyl solution

$$\psi_1(x,\lambda) = \theta(x,\lambda) + m_1(\lambda)\phi(x,\lambda)$$

where

$$\begin{aligned} m_1(\lambda) &= \lim_{y\to -1}\left\{\tfrac{\phi_0(y)+\phi_0(-y)}{\phi_0(-y)-\phi_0(y)}\right\}\tfrac{\phi_0'(0)}{\phi_0(0)} \\ &= -K(\nu,\mu)\tfrac{\phi_0'(0)}{\phi_0(0)} \end{aligned}$$

and so

$$\psi_1(x,\lambda) = \frac{1}{2\phi_0(0)}\left\{[1-K(\nu,\mu)]\phi_0(-x) + [1+K(\nu,\mu)]\phi_0(x)\right\}. \tag{3.33}$$

For $x<z$, the Green's function $G(x,z;\lambda)$ of (3.19) on $(-1,1)$ is given by

$$G(x,z;\lambda) = \frac{\psi_1(x,\lambda)\psi_2(z,\lambda)}{W(\psi_1,\psi_2)}$$

where

$$W(\psi_1,\psi_2) = (\psi_2\psi_1' - \psi_1\psi_2')(0) = m_2(\lambda) - m_1(\lambda) = 2K(\nu,\mu)\frac{\phi_0'(0)}{\phi_0(0)}. \tag{3.34}$$

It follows that T_μ has a discrete spectrum with eigenvalues located at the zeros of

$$X(\nu,\mu) := \sin\frac{\pi}{2}(\mu-\nu)\cos\frac{\pi}{2}(\mu+\nu)\phi_0(0)\phi_0'(0).$$

From Bateman [1], p.145 ((20) & (23)),

$$\begin{aligned}\phi_0(0) &= 2^{\mu}\pi^{-\frac{1}{2}}\cos\tfrac{\pi}{2}(\mu+\nu)\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\nu+\tfrac{1}{2}\mu)/\Gamma(1+\tfrac{1}{2}\nu-\tfrac{1}{2}\mu)\\ \phi_0'(0) &= 2^{\mu+1}\pi^{-\frac{1}{2}}\sin\tfrac{\pi}{2}(\mu+\nu)\Gamma(1+\tfrac{1}{2}\nu+\tfrac{1}{2}\mu)/\Gamma(1+\tfrac{1}{2}\nu-\tfrac{1}{2}\mu).\end{aligned}$$

The zeros of $X(\nu,\mu)$ are therefore given by

$$1+\frac{1}{2}\nu-\frac{1}{2}\mu=-m,\ m\in\mathbb{N}_0;\ \mu+\nu=2n+1,\ n\in\mathbb{Z};$$
$$\text{and}\ \ \mu-\nu=2n,\ n\in\mathbb{Z}.$$

When $1+\frac{1}{2}\nu-\frac{1}{2}\mu=-m$,

$$\lambda=\nu(\nu+1)=(2m+1-\mu)(2m+1-\mu+1),\qquad m\in\mathbb{N}_0.$$

When $\mu+\nu=2n+1$,

$$\lambda=(2n+1-\mu)(2n+1-\mu+1),\qquad n\in\mathbb{Z},$$

and $\mu-\nu=2n$ yields

$$\lambda=(2n-1-\mu)(2n-1-\mu+1),\ n\in\mathbb{Z}.$$

The lemma is therefore proved. □

Corollary 3. *For all $f\in C_0^\infty(\mathbb{R}^3\setminus\{0\})$, we have*

$$\int_{\mathbb{R}^3}|\Delta_{\mathbf{A}}f(x)|^2\frac{dx}{|x|^\alpha}\geq C(3,\alpha)\int_{\mathbb{R}^3}|f(x)|^2\frac{dx}{|x|^{\alpha+4}}\tag{3.35}$$

where

$$C(3,\alpha):=\inf_{m\in\mathbb{Z}}\{(m-\tilde{\Psi})(m-\tilde{\Psi}+1)-\frac{(3+\alpha)(1+\alpha)}{4}\}^2.\tag{3.36}$$

Proof. We may clearly suppose that $|\cdot|^{-\alpha/2}\Delta_{\mathbf{A}}f\in L^2(\mathbb{R}^3)$. Then $\Lambda_\omega f=\Lambda(\theta_1,\theta_2)f\in L^2(\mathbb{R}^3)$ and for all $r\in(0,\infty),\Lambda(\theta_2,\theta_2)f\in L^2(\mathbb{S}^2)$. If $\Lambda_k(\theta_1)$ denotes provisionally the formal operator in (3.13) and

$$F_k(r,\theta_1):=\int_0^{2\pi}f(r,\theta_1,\theta_2)\overline{u_k(\theta_2)}d\theta_2,$$

where the u_k are the functions in (3.4), then we have

$$\int_0^{2\pi}|\Lambda(\theta_1,\theta_2)f(r,\theta_1,\theta_2)|^2d\theta_2=\sum_{k\in\mathbb{Z}}|\Lambda_k(\theta_1)F_k(r,\theta_1)|^2.$$

Hence, for any $k\in\mathbb{Z}$ and $r\in(0,\infty),\Lambda_k(\theta_1)F_k(r,\theta_1)\in L^2(0,\pi;\sin\theta_1 d\theta_1)$. Since $F_k(r,\cdot)\in L^2(0,\pi;\sin\theta_1 d\theta_1)$ and the boundary conditions in Lemma 1 are satisfied, it follows that $F_k(r,\cdot)$ lies in the domain of the operator $\Lambda_k(\theta_1)$ and $f(r,\cdot)\in\mathcal{D}(\Lambda_\omega)$.

From Lemmas 1 and 2 the eigenvalues of the operator Λ_k in (3.12) are

$$\nu_j(k)=(j-|k+\tilde{\Psi}|)(j+1-|k+\tilde{\Psi}|),\quad j\in\mathbb{Z}\setminus\{0\}.\tag{3.37}$$

Denote the corresponding normalised eigenvectors by $P_{j,k}(\theta_1;\tilde{\Psi})$. Then

$$Y_{j,k;\tilde{\Psi}}(\theta_1,\theta_2) := P_{j,k}(\theta_1;\tilde{\Psi})u_k(\theta_2), \quad j,k\in\mathbb{Z}$$

are the eigenvectors of Λ_ω corresponding to the eigenvalues $\nu_j(k)$ and form an orthonormal basis of $L^2(\mathbb{S}^2)$. The corollary follows from Theorem 1. □

Remark 4. *When $\alpha = 0$ and $\tilde{\Psi} = 0$ in (3.36), $C(3,0) = \frac{9}{16}$, which is the constant obtained by Rellich* [10].

3.3. **The case $n = 4$.** In this case we define the orthonormal vectors

$$\begin{aligned}
\mathbf{e}_0 &= \tfrac{\mathbf{x}}{|\mathbf{x}|} = (\cos\theta_1, \sin\theta_1\cos\theta_2, \sin\theta_1\sin\theta_2\cos\theta_3, \sin\theta_1\sin\theta_2\sin\theta_3)\\
\mathbf{e}_1 &= (-\sin\theta_1, \cos\theta_1\cos\theta_2, \cos\theta_1\sin\theta_2\cos\theta_3, \cos\theta_1\sin\theta_2\sin\theta_3)\\
\mathbf{e}_2 &= (0, -\sin\theta_2, \cos\theta_2\cos\theta_3, \cos\theta_2\sin\theta_3)\\
\mathbf{e}_3 &= (0,0,-\sin\theta_3,\cos\theta_3),
\end{aligned}$$

where $\theta_1,\theta_2\in(0,\pi), \theta_3\in(0,2\pi)$. In this case,

$$\nabla = \mathbf{e}_0\frac{\partial}{\partial r} + \mathbf{e}_1(\frac{1}{r}\frac{\partial}{\partial\theta_1}) + \mathbf{e}_2(\frac{1}{r\sin\theta_1}\frac{\partial}{\partial\theta_2}) + \mathbf{e}_3(\frac{1}{r\sin\theta_1\sin\theta_2}\frac{\partial}{\partial\theta_3}).$$

We now take

$$\mathbf{A} := \frac{1}{r\sin\theta_1\sin\theta_2}\Psi(\theta_3)\mathbf{e}_3, \qquad \Psi\in L^\infty(0,2\pi), \qquad \Psi(0)=\Psi(2\pi), \tag{3.38}$$

in $\mathbb{R}^4\setminus\mathcal{L}_4$, where $\mathcal{L}_4 := \{\mathbf{x} = (r,\theta_1,\cdots,\theta_3) : r\sin\theta_1\sin\theta_2 = 0\}$. We then have

$$-\Delta_{\mathbf{A}} = L_r + \frac{1}{r^2}\Lambda$$

for L_r defined in (2.1), where

$$\Lambda = -\frac{\partial^2}{\partial\theta_1^2} - 2\cot\theta_1\frac{\partial}{\partial\theta_1} + \frac{1}{\sin^2\theta_1}\{-\frac{\partial^2}{\partial\theta_2^2} - \cot\theta_2\frac{\partial}{\partial\theta_2} + \frac{K_{\theta_3}^2}{\sin^2\theta_2}\} \tag{3.39}$$

with

$$K_{\theta_3} := i\frac{\partial}{\partial\theta_3} + \Psi(\theta_3).$$

On repeating the procedure in Section 3.1 and using the notation of Section 3.1, we take Λ_ω to be

$$\Lambda_\omega = \bigoplus_{j,k\in\mathbb{Z}}\left(\Lambda_{j,k}(\theta_1)\bigotimes I_{j,k}\right), \tag{3.40}$$

where $\Lambda_{j,k}(\theta_1)$ is the self-adjoint operator generated as in Section 3.1 by

$$\Lambda_{j,k}(\theta_1) = -\frac{\partial^2}{\partial\theta_1^2} - 2\cot\theta_1\frac{\partial}{\partial\theta_1} + \frac{\nu_j(k)}{\sin^2\theta_1} \tag{3.41}$$

in $L^2((0,\pi);\sin^2\theta_1\sin\theta_2 d\theta_1)$. As in the analogous situation in Section 3.1, the operator $\Lambda_{j,k}(\theta_1)$ is chosen to be the Friedrichs extension of the

operator defined on $C_0^\infty(0,\pi)$ or its unique self-adjoint extension when there is one. To apply Theorem 1 we need

Lemma 3. *The Friedrichs extension of* $\Lambda_{j,k}(\theta_1)\big|_{C_0^\infty(0,\pi)}$ *in* $L^2((0,\pi);\sin^2\theta_1 d\theta_1)$ *has eigenvalues*

$$\gamma_\ell(j,k) = (\ell - [\nu_j(k)+1/4]^{\frac{1}{2}})(\ell - [\nu_j(k)+1/4]^{\frac{1}{2}}+1) - \frac{3}{4}, \qquad \ell \in \mathbb{Z}' \tag{3.42}$$

where $\nu_j(k)$ *is given by (3.37) and* $\mathbb{Z}' := \{\ell \in \mathbb{Z} : \gamma_\ell(j,k) \geq 0\}$.

Proof. On substituting $x = \cos\theta_1$, the equation $\Lambda_{j,k}(\theta_1)u = \lambda u$ becomes

$$(1-x^2)\frac{d^2u}{dx^2} - 3x\frac{du}{dx} + (\lambda - \frac{\nu_j(k)}{1-x^2})u = 0.$$

Further, set $w = (1-x^2)^{\frac{1}{4}}u$ to obtain

$$(1-x^2)\frac{d^2w}{dx^2} - 2x\frac{dw}{dx} + (\lambda + \frac{3}{4} - \frac{\nu_j(k)+\frac{1}{4}}{1-x^2})w = 0 \tag{3.43}$$

with $L^2(-1,1)$ for the underlying Hilbert space. The problem is therefore reduced to that for (3.19) with $\lambda + \frac{3}{4}$ instead of λ and $\nu_j(k) + \frac{1}{4}$ for μ^2, and the lemma follows from Lemma 2. □

From (3.37), $\nu_j(k) = (j - |k+\tilde{\Psi}|)(j+1-|k+\tilde{\Psi}|)$, which implies that

$$\nu_j(k) + \frac{1}{4} = \left(j - |k+\tilde{\Psi}| + \frac{1}{2}\right)^2.$$

Thus, from (3.42), $\gamma_\ell(j,k) = (\ell - j + |k+\tilde{\Psi}|)^2 - 1$ if $j - |k+\tilde{\Psi}| + 1/2 \geq 0$ and $\gamma_\ell(j,k) = (\ell + j - |k+\tilde{\Psi}|+1)^2 - 1$ otherwise. These can be enumerated as

$$\lambda_m = (m+\tilde{\Psi})^2 - 1, \qquad m \in \mathbb{Z}'$$

where $\mathbb{Z}' := \{m : (m+\tilde{\Psi})^2 \geq 1\}$.

It follows by an argument similar to that in the proof of Corollary 3 that for any $f \in C_0^\infty(\mathbb{R}^4 \setminus \{0\})$ with $\Delta_{\mathbf{A}} f \in L^2(\mathbb{R}^4)$ we have $f(r,\cdot) \in \mathcal{D}(\Lambda_\omega)$. Hence, from Theorem 1,

Corollary 4. *Let* $f \in C_0^\infty(\mathbb{R}^4 \setminus \{0\})$. *Then*

$$\int_{\mathbb{R}^4} |\Delta_{\mathbf{A}} f(x)|^2 \frac{dx}{|x|^\alpha} \geq C(4,\alpha) \int_{R^4} |f(x)|^2 \frac{dx}{|x|^{\alpha+4}}$$

where

$$C(4,\alpha) := \inf_{m\in\mathbb{Z}'} \{[(m+\tilde{\Psi})^2 - 1 - \frac{\alpha(\alpha+4)}{4}]^2\}.$$

In particular, when $\alpha = 0$

$$C(4,0) = \min\{(-1+\tilde{\Psi})^2 - 1, (-2+\tilde{\Psi})^2 - 1\}^2 > 0.$$

When $\tilde{\Psi} = 0, C(4,0) = 0$ and the inequality is trivial. If $F_1 = F_2 = 0$ (see (2.5)), then the infimum is attained for $m = \pm 2$ giving $C(4,0) = 9$.

4. Rellich-type inequalities in $L^p(\mathbb{R}^n)$, $1 < p < \infty$.

We shall now apply the method used in [3] to prove the following theorem.

Theorem 2. *For all $p \in (1, \infty)$, $f \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ and*

$$d := \inf_{k \in \mathbb{Z}} \{|k - \tilde{\Psi}|\} \neq 0, \frac{1}{2},$$

where $\tilde{\Psi} = \frac{1}{2\pi} \int_0^{2\pi} \Psi(\theta) d\theta$ and $\Psi(0) = \Psi(2\pi)$, we have

$$\|f/|\cdot|^2\|_{L^p(\mathbb{R}^2)} \leq B_p \| \frac{1}{r^2} (i \frac{\partial}{\partial \theta} + \Psi(\theta))^2 f \|_{L^p(\mathbb{R}^2)} \tag{4.1}$$

where

$$B_p \leq \frac{C_p}{d^2 (\cos \pi d)^{1/p'}}, \qquad C_p := \frac{(2\pi)^{\frac{1}{p}} \pi^{1+\frac{2}{p'}}}{8}, \qquad p' = \frac{p}{p-1}.$$

When $n \geq 3$, then for all $f \in C_0^\infty(\mathbb{R}^n \setminus \mathcal{L}_n), \mathcal{L}_n = \{(r, \theta_1, \cdots, \theta_{n-1} : \sin \theta_1 \cdots \sin \theta_{n-2} = 0\}$,

$$\begin{aligned} &\|f/r^2 \sin^2 \theta_1 \cdots \sin^2 \theta_{n-2}\|_{L^p(\mathbb{R}^n)} \\ &\qquad \leq B_p \| \frac{1}{r^2 \sin^2 \theta_1 \cdots \sin^2 \theta_{n-2}} (i \frac{\partial}{\partial \theta_{n-1}} + \Psi(\theta_{n-1}))^2 f \|_{L^p(\mathbb{R}^n)}. \end{aligned} \tag{4.2}$$

Before embarking on the proof of this theorem we need some preliminaries. Let

$$u(r, \theta) = e^{-2\pi i \Phi(\theta)} f(r, \theta), \qquad \Phi(\theta) := \frac{1}{2\pi} \int_0^\theta \Psi(t) dt. \tag{4.3}$$

Then

$$\frac{\partial u}{\partial \theta} = e^{-2\pi i \Phi(\theta)} (\frac{\partial}{\partial \theta} - i\Psi(\theta)) f(r, \theta) \tag{4.4}$$

and for $f \in C^1(\mathbb{R}^2)$ and $\delta = e^{-2\pi i \tilde{\Psi}}$

$$\begin{aligned} u(r, 2\pi) &= \delta\ u(r, 0), \\ \tfrac{\partial u}{\partial \theta}(r, 2\pi) &= \delta \tfrac{\partial u}{\partial \theta}(r, 0). \end{aligned} \tag{4.5}$$

We now define

$$W_\delta^p \equiv W_\delta^p[0, 2\pi] := \{v \in C^1[0, 2\pi] : v(2\pi) = \delta v(0)\}. \tag{4.6}$$

Hence, as a function of θ, $u = u(r, \cdot)$ and $u' \equiv \frac{\partial u}{\partial \theta}(r, \cdot)$ satisfy

$$u, u' \in W_\delta^p.$$

Consequently, from Theorem 1 of [3], for $x \in [0, 2\pi]$

$$\begin{aligned} u(x) &= \int_0^{2\pi} u'(y) \overline{h_y'}(x, y) dy \\ u'(x) &= \int_0^{2\pi} u''(y) \overline{h_y'}(x, y) dy \end{aligned}$$

where $h(x,y) = (y-x)\{\chi_{_{[0,x)}}(y) - \frac{\delta}{\delta-1}\} - \frac{2\pi\delta}{(\delta-1)^2}$ in which χ denotes the characteristic function. It follows that

$$u(x) = \int_0^{2\pi} H(x,z)u''(z)dz \tag{4.7}$$

where

$$\begin{aligned} H(x,z) = & \int_0^{2\pi} \overline{h'_y(x,y)h'_z(y,z)}dy \\ = & \begin{cases} \frac{z-x}{1-\delta} + \frac{2\pi\delta}{(1-\delta)^2}, & x \le z \\ \frac{\delta(z-x)}{1-\delta} + \frac{2\pi\delta}{(1-\delta)^2}, & x > z. \end{cases} \end{aligned} \tag{4.8}$$

Hence, for all $u \in W^p_\delta$ and $x \in [0, 2\pi]$

$$|u(x)| \le \|H(x,\cdot)\|_{L^{p'}(0,2\pi)}\|u''\|_{L^p(0,2\pi)}, \tag{4.9}$$

when $1 < p < \infty$ and $p' = \infty$ if $p = 1$. We have from (4.8), for $1 < p < \infty$,

$$\int_0^x |H(x,z)|^{p'}dz = \frac{1}{|1-\delta|^{p'}}\int_0^x |x - z - \frac{2\pi}{1-\delta}|^{p'}dz =: \frac{1}{|1-\delta|^{p'}}I_1(x)$$

say. Let $\delta = e^{2\pi id}$. (Note that $\delta = e^{\pm 2\pi id}$.) Then

$$\frac{1}{1-\delta} = \frac{1}{2}[1 + i\cot(\pi d)]$$

and

$$\begin{aligned} I_1(x) = & \int_0^x [(x-z-\pi)^2 + \pi^2\cot^2(\pi d)]^{p'/2}dz \\ = & [\pi\cot(\pi d)]^{p'+1}\int_{\frac{-(x-\pi)}{\pi\cot(\pi d)}}^{\pi/(\pi\cot(\pi d))}[w^2+1]^{p'/2}dw \end{aligned}$$

on substituting $z = (x-\pi) + \pi\cot(\pi d)w$. Similarly

$$\int_x^{2\pi} |H(x,z)|^{p'}dz = \frac{1}{|1-\delta|^{p'}}I_2(x)$$

and

$$I_2(x) = \int_x^{2\pi} [(z-x-\pi)^2 + \pi^2\cot^2(\pi d)]^{p'/2}dz.$$

Also

$$\begin{aligned} \int_0^{2\pi} |H(x,z)|^{p'}dz = & \frac{[\pi\cot(\pi d)]^{p'+1}}{|1-\delta|^{p'}}\int_{-\tan(\pi d)}^{\tan(\pi d)}(w^2+1)^{p'/2}dw \\ = & \frac{[\pi\cos(\pi d)]^{p'+1}}{2^{p'-1}[\sin(\pi d)]^{2p'+1}}\int_0^{\pi d}[\sec\theta]^{p'+2}d\theta \end{aligned} \tag{4.10}$$

on substituting $w = \tan\theta$ and noting that $|1-\delta| = 2\sin(\pi d)$. In $[0, \pi d]$, $\cos\theta \ge \cos(\pi d)$ and $\sin(\pi d) \ge 2d$, since $d \le \frac{1}{2}$. Hence, if $0 < d < \frac{1}{2}$,

$$\int_0^{2\pi} |H(x,z)|^{p'}dz \le \frac{\pi^{2+p'}}{8^{p'}}\frac{1}{\cos(\pi d)d^{2p'}}. \tag{4.11}$$

The same holds if $\delta = e^{-2\pi id}$. Hence, from (4.9) we have the following lemma.

Lemma 4. *For all u for which $u, u' \in W^p_\delta$, $1 < p < \infty$, and $x \in [0, 2\pi]$, if $0 < d < \frac{1}{2}$,*

$$|u(x)| \leq A_p \frac{1}{[\cos(\pi d)]^{1/p'}} \frac{1}{d^2} \|u''\|_{L^p(0,2\pi)}, \tag{4.12}$$

where

$$A_p \leq \frac{\pi^{1+2/p'}}{8}.$$

If $p = 2$ and $d = \frac{1}{2}$, then in (4.10) the last integral becomes

$$\int_0^{\frac{\pi}{2}} \sec^4 \theta d\theta = \infty.$$

From (4.10), a more accurate version of (4.12) is

$$|u(x)| \leq \left\{ \frac{[\pi \cos(\pi d)]^{p'+1}}{2^{p'-1}[\sin(\pi d)]^{2p'+1}} \int_0^{\pi d} [\sec \theta]^{p'+2} d\theta \right\}^{1/p'} \|u''\|_{L^p(0,2\pi)}. \tag{4.13}$$

Proof. of *Theorem 2.* On reverting to the functions defined in (4.3), by Lemma 4 we have that

$$|u(r,\theta)| \leq A_p \frac{1}{[\cos(\pi d)]^{1/p'}} \frac{1}{d^2} \|\frac{\partial^2}{\partial \theta^2} u(r,\cdot)\|_{L^p(0,2\pi)}.$$

This yields (4.1) and (4.2) on noting (4.4). □

Remark 5. *If $\tilde{\Psi} \in \mathbb{Z}$, (4.1) does not hold for any $p \in (1, \infty)$ since the inequality*

$$\|\frac{u}{|\cdot|^2}\|_{L^p(\mathbb{R}^2)} \leq const. \|\frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}\|_{L^p(\mathbb{R}^2)}$$

fails for any radial function u.

Remark 6. *From Corollary 1 we have that*

$$\|\frac{1}{r^2}(i\frac{\partial}{\partial \theta} + \Psi(\theta))^2 f\|_{L^2(\mathbb{R}^2)} \leq C(2) \|\Delta_{\mathbf{A}} f\|_{L^2(\mathbb{R}^2)}$$

for all $f \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$. This fact and Theorem 2 indicate that the introduction of the vector field $\mathbf{A}$ affects the transverse rather than the radial part of the inequality (at least when $p = 2$), as for the Laptev-Weidl [7] *inequality.*

Remark 7. *For $n = 3$, it is a consequence of Theorem 3 below that*

$$\|\frac{1}{r^2 \sin^2 \theta_1} \left(i\frac{\partial}{\partial \theta_2} + \Psi(\theta_2) \right)^2 f\|_{L^2(\mathbb{R}^3)} \leq C \|\Delta_{\mathbf{A}} f\|_{L^2(\mathbb{R}^3)},$$

but only for functions $f \in C_0^\infty(\mathbb{R}^3 \setminus \{\mathcal{L}_3\})$ whose Fourier coefficients $F_m(r,\cdot)$ vanish for specified values of m. This is reminiscent of Rellich's requirement for his inequality when $n = 2$. If $\rho = r \sin \theta_1$, the distance of $x \in \mathbb{R}^3$ from the $x_3 -$ axis, $\mathcal{L}_3$, we therefore have an inequality

$$\|f/\rho^2\|_{L^2(\mathbb{R}^3)} \leq C \|\Delta_{\mathbf{A}} f\|_{L^2(\mathbb{R}^3)}$$

for a restricted set of functions $f \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$ supported away from $\mathcal{L}_3$. This is effectively a two-dimensional result as the vector field $\mathbf{A}$ is singular on $\mathcal{L}_3$ and the use of cylindrical co-ordinates would reduce the problem to one in $\mathbb{R}^2$ with $\mathbf{A}$ of Aharonov-Bohm type.

To prove the theorem we shall need

Lemma 5. *Let $u \in C_0^2(-1,1)$. Then*

$$\int_{-1}^{1} \frac{|u(x)|^2}{(1-x^2)^2} dx \leq c^2 \int_{-1}^{1} |u'(x)|^2 dx \tag{4.14}$$

$$\int_{-1}^{1} |u'(x)|^2 dx \leq c^2 \int_{-1}^{1} \left| \frac{d}{dx} \left([1-x^2] u'(x) \right) \right|^2 dx \tag{4.15}$$

where

$$\ln\left(\frac{c+1}{c-1}\right) = 2c. \tag{4.16}$$

Proof. On integration by parts,

$$\begin{aligned} \int_{-1}^{1} \frac{|u(x)|^2}{(1-x^2)^2} dx &= -2\Re e \left[\int_{-1}^{1} \overline{u} u' \left(\int_0^x \frac{dt}{(1-t^2)^2} \right) dx \right] \\ &= -2\Re e \left[\int_{-1}^{1} \frac{\overline{u(x)}}{(1-x^2)} u'(x) \varphi(x) dx \right], \end{aligned}$$

where

$$\varphi(x) := (1-x^2) \int_0^x \frac{dt}{(1-t^2)^2}.$$

Hence, by the Cauchy-Schwarz inequality,

$$\int_{-1}^{1} \frac{|u(x)|^2}{(1-x^2)^2} dx \leq c \left(\int_{-1}^{1} \frac{|u(x)|^2}{(1-x^2)^2} dx \right)^{1/2} \left(\int_{-1}^{1} |u'|^2 dx \right)^{1/2},$$

whence (4.14) with $c = 2 \sup_{x \in (-1,1)} |\varphi(x)|$. The function φ is skew-symmetric, is given by

$$\varphi(x) = \frac{x}{2} + \frac{1}{4}(1-x^2) \ln\left(\frac{1+x}{1-x}\right)$$

for $x \geq 0$ and attains its maximum value of $c/2$ at the point $x = 1/c$ satisfying (4.16). This yields (4.14).

For (4.15) we have similarly

$$\begin{aligned} \int_{-1}^{1} |u'|^2 dx &= \int |(1-x^2)u'(x)|^2 \frac{dx}{(1-x^2)^2} \\ &= -2\Re e \left[\int_{-1}^{1} (1-x^2) \overline{u'(x)} \{ \frac{d}{dx} (1-x^2) u'(x) \} \int_0^x \frac{dt}{(1-t^2)^2} dx \right] \\ &= -2\Re e \left[\int_{-1}^{1} u'(x) \{ \frac{d}{dx} (1-x^2) u'(x) \} \varphi(x) dx \right] \end{aligned}$$

and (4.15) follows as before. □

Theorem 3. *For all $f \in C_0^\infty(\mathbb{S}^2 \setminus \mathcal{L}_3)$ there exist positive constants K_1, K_2 such that*

$$\left\| \sin^{-2}\theta_1 \left(i\frac{\partial}{\partial\theta_2} + \Psi(\theta_2) \right)^2 f \right\|^2_{L^2(\mathbb{S}^2)} \leq K_1 \left\| \Lambda(\theta_1,\theta_2) f \right\|^2_{L^2(\mathbb{S}^2)} + K_2 \left\| \sin^{-2}\theta_1 f \right\|^2_{L^2(\mathbb{S}^2)}, \quad (4.17)$$

where $\Lambda(\theta_1,\theta_2)$ is given in (3.10), i.e.

$$\Lambda(\theta_1,\theta_2) = -\frac{\partial^2}{\partial^2\theta_1} - \cot\theta_1 \frac{\partial}{\partial\theta_1} + \frac{1}{\sin^2\theta_1}\left(i\frac{\partial}{\partial\theta_2} + \Psi(\theta_2) \right)^2.$$

Let $\tilde{\Psi} \in [0,1)$ and let f be such that

$$\begin{aligned} F_m(\theta_1) &:= \int_0^{2\pi} f(\theta_1,\theta_2)\overline{u_m(\theta_2)} d\theta_2 \\ &= 0 \quad \text{for } \mathrm{m} \in \mathcal{J}(\tilde{\Psi}), \end{aligned} \quad (4.18)$$

where the u_m are given by (3.4) and

$$\mathcal{J}(\tilde{\Psi}) = \begin{cases} \{-1,1\}, & \text{when } \tilde{\Psi} \in [0, 0.447566), \\ \{-1,-2,0,1\}, & \text{when } \tilde{\Psi} \in (0.447566, 0.552434), \\ \{-2,0\}, & \text{when } \tilde{\Psi} \in (0.552434, 1). \end{cases}$$

Then

$$\left\| \Lambda(\theta_1,\theta_2) f \right\|^2_{L^2(\mathbb{S}^2)} \geq C \left\| \sin^{-2}\theta_1 f \right\|^2_{L^2(\mathbb{S}^2)}, \quad (4.19)$$

where

$$C \leq \min_{\mathrm{m}\notin\mathcal{J}(\tilde{\Psi})} \left\{ \mathrm{c}^{-4} + (\mathrm{m}+\tilde{\Psi})^4 - 2(2-\mathrm{c}^{-2})(\mathrm{m}+\tilde{\Psi})^2 \right\} \quad (4.20)$$

and c is the positive solution of

$$c = \frac{1}{2}\ln\left(\frac{c+1}{c-1} \right). \quad (4.21)$$

Thus, for all $f \in C_0^\infty(\mathbb{R}^3 \setminus \mathcal{L}_3)$ satisfying (4.18) we have

$$\left\| \sin^{-2}\theta_1 \left(i\frac{\partial}{\partial\theta_2} + \Psi(\theta_2) \right)^2 f \right\|^2_{L^2(\mathbb{S}^2)} \leq K \left\| \Lambda(\theta_1,\theta_2) f \right\|^2_{L^2(\mathbb{S}^2)} \quad (4.22)$$

and so, from Corollary 1,

$$\left\| (r\sin\theta_1)^{-2} \left(i\frac{\partial}{\partial\theta_2} + \Psi(\theta_2) \right)^2 f \right\|^2_{L^2(\mathbb{R}^3)} \leq \|Df\|^2_{L^2(\mathbb{R}^3)}. \quad (4.23)$$

Proof. Set

$$M_{\theta_1} := -\frac{\partial^2}{\partial\theta_1^2} - \cot\theta_1 \frac{\partial}{\partial\theta_1}$$

and

$$\Lambda_{\theta_2} := K_{\theta_2}^2,$$

where K_{θ_2} is the self-adjoint operator defined by (3.11) in $L^2(\mathbb{S}^1)$; Λ_{θ_2} has a discrete spectrum of eigenvalues $\lambda_m = (m+\tilde{\Psi})^2$ and eigenvectors $u_m, m \in \mathbb{Z}$ which form an orthonormal basis of $L^2(0, 2\pi)$. Hence

$$\Lambda(\theta_1, \theta_2) = M_{\theta_1} + \frac{1}{\sin^2\theta_1}\Lambda_{\theta_2}.$$

Let

$$F_m(\theta_1) := \int_0^{2\pi} f(\theta_1, \theta_2)\overline{u_m(\theta_2)}d\theta_2. \tag{4.24}$$

Then, by Parseval's identity,

$$\begin{aligned}
M_{\theta_1}F_m(\theta_1) &= \int_0^{2\pi} M_{\theta_1} f(\theta_1, \theta_2)\overline{u_m(\theta_2)}d\theta_2, \\
\|M_{\theta_1} f\|^2_{L^2(\mathbb{S}^2)} &= \sum_{m\in\mathbb{Z}} \int_0^{\pi} |M_{\theta_1}F_m(\theta_1)|^2 \sin\theta_1 d\theta_1, \\
\int_{\mathbb{S}^2} (M_{\theta_1} f)\left(\frac{\Lambda_{\theta_2} f}{\sin^2\theta_1}\right) d\omega &= \sum_{m\in\mathbb{Z}} \lambda_m \int_0^{\pi} M_{\theta_1}F_m(\theta_1)\frac{\overline{F_m(\theta_1)}}{\sin^2\theta_1}\sin\theta_1 d\theta_1, \\
\int_{\mathbb{S}^2} \left|\frac{\Lambda_{\theta_2} f}{\sin^2\theta_1}\right|^2 d\omega &= \sum_{m\in\mathbb{Z}} \lambda_m^2 \int_0^{\pi} \left|\frac{F_m(\theta_1)}{\sin^2(\theta_1)}\right|^2 \sin\theta_1 d\theta_1.
\end{aligned} \tag{4.25}$$

Therefore, with $\|\cdot\|, (\cdot,\cdot)$ denoting the $L^2(\mathbb{S}^2)$ norm and inner product,

$$\begin{aligned}
\|\Lambda(\theta_1, \theta_2) f\|^2 =& \|M_{\theta_1} f\|^2 + 2\Re e(M_{\theta_1} f, \frac{\Lambda_{\theta_2}}{\sin^2\theta_1} f) + \left\|\frac{\Lambda_{\theta_2}}{\sin^2\theta_1} f\right\|^2 \\
=& \sum_{m\in\mathbb{Z}} \{ \int_0^{\pi} |M_{\theta_1}F_m(\theta_1)|^2 \sin\theta_1 d\theta_1 \\
&+ 2\lambda_m \Re e \int_0^{\pi} M_{\theta_1}F_m(\theta_1)\frac{\overline{F_m(\theta_1)}}{\sin^2(\theta_1)}\sin\theta_1 d\theta_1 \\
&+ \lambda_m^2 \int_0^{\pi} \left|\frac{F_m(\theta_1)}{\sin^2(\theta_1)}\right|^2 \sin\theta_1 d\theta_1 \}.
\end{aligned} \tag{4.26}$$

Let $x = \cos\theta_1$ and write $F_m(x)$ for $F_m(\theta_1)$. Then, on using Lemma 5 and integrating by parts, (4.26) becomes

$$\begin{aligned}
\|\Lambda(\theta_1,\theta_2)f\|^2 &= \sum_{m\in\mathbb{Z}}\{\int_{-1}^{1}\left|-\frac{d}{dx}\{(1-x^2)F_m'(x)\}\right|^2 dx \\
&\quad + 2\Re e[\lambda_m\int_{-1}^{1}\left(-\frac{d}{dx}\{(1-x^2)F_m'(x)\}\right)\frac{\overline{F_m(x)}}{(1-x^2)}dx] \\
&\quad + \lambda_m^2\int_{-1}^{1}\left|\frac{F_m(x)}{(1-x^2)}\right|^2 dx\} \\
&\geq \sum_{m\in\mathbb{Z}}\left[c^{-4}+\lambda_m^2\right]\int_{-1}^{1}\left|\frac{F_m(x)}{(1-x^2)}\right|^2 dx \\
&\quad + 2\Re e[\lambda_m\int_{-1}^{1}(1-x^2)F_m'\left\{\frac{\overline{F_m'(x)}}{(1-x^2)}+\frac{2x}{(1-x^2)^2}\overline{F_m}\right\}dx] \\
&\geq \sum_{m\in\mathbb{Z}}\{\left[c^{-4}+\lambda_m^2+\frac{2\lambda_m}{c^2}\right]\int_{-1}^{1}\left|\frac{F_m(x)}{(1-x^2)}\right|^2 dx \\
&\quad - 2\lambda_m\int_{-1}^{1}\left|\frac{F_m(x)}{(1-x^2)}\right|^2(1+x^2)dx\} \qquad (4.27)
\end{aligned}$$

by (4.15) and since

$$\begin{aligned}
\Re e\int_{-1}^{1}F_m'\overline{F_m}\frac{2x}{(1-x^2)}dx &= -\int_{-1}^{1}|F_m|^2\left[\frac{1}{(1-x^2)}+\frac{2x^2}{(1-x^2)^2}dx\right. \\
&= -\int_{-1}^{1}|F_m|^2\left(\frac{1+x^2}{(1-x^2)^2}\right)dx,
\end{aligned}$$

on integration by parts again. From (4.27) it follows that

$$\begin{aligned}
\|\Lambda(\theta_1,\theta_2)f\|^2 \geq & c^{-4}\sum_{m\in\mathbb{Z}}\int_{-1}^{1}\left|\frac{F_m}{1-x^2}\right|^2 dx+\sum_{m\in\mathbb{Z}}\lambda_m^2\int_{-1}^{1}\left|\frac{F_m}{1-x^2}\right|^2 dx \\
& - 2(2-c^{-2})\sum_{m\in\mathbb{Z}}\lambda_m\int_{-1}^{1}\left|\frac{F_m}{1-x^2}\right|^2 dx \\
= & c^{-4}\left\|\frac{f}{\sin^2\theta_1}\right\|^2+\left\|\frac{\Lambda_{\theta_2}f}{\sin^2\theta_1}\right\|^2 \\
& - 2(2-c^{-2})\left(\frac{\Lambda_{\theta_2}f}{\sin^2\theta_1},\frac{f}{\sin^2\theta_1}\right) \\
\geq & \left\{c^{-4}-\varepsilon^{-1}(2-c^{-2})\right\}\left\|\frac{f}{\sin^2\theta_1}\right\|^2 \\
& + \left\{1-\varepsilon(2-c^{-2})\right\}\left\|\frac{\Lambda_{\theta_2}f}{\sin^2\theta_1}\right\|^2
\end{aligned}$$

for any $\varepsilon > 0$. The inequality (4.17) follows on choosing ε sufficiently small.

We also have from (4.27)

$$\|\Lambda(\theta_1, \theta_2) f\|^2 \geq \sum_{m \in \mathbb{Z}} K_m \int_{-1}^{1} \left| \frac{F_m}{(1-x^2)} \right|^2 dx, \tag{4.28}$$

where

$$K_m = \left(c^{-4} + \lambda_m^2 - 2(2 - c^{-2})\lambda_m \right).$$

A calculation of the constant c gives

$$c = 1.19967864$$

to 8 decimal places. On recalling that $\lambda_m = (m + \tilde{\Psi})^2$, it follows that the K_m are positive except for the following exceptions:

- if $0 \leq \tilde{\Psi} < 0.447566, K_1, K_{-1} < 0$;
- if $0.447566 < \tilde{\Psi} < 0.552434, K_{-2}, K_{-1}, K_0, K_1 < 0$;
- if $0.552434 < \tilde{\Psi} < 1, K_0, K_{-2} < 0$.

This completes the proof. □

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