Singular Continuous Floquet Operator for Periodic Quantum Systems

Olivier Bourget[*]

Pontificia Universidad Católica, Facultad de Matemáticas

Av. Vicuña Mackenna 4860, 690 4411, Macul, Santiago, Chile

E-mail address: bourget@mat.puc.cl

Abstract

Consider the Floquet operator of a time independent quantum system, acting on a separable Hilbert space, periodically perturbed by a rank one kick: $e^{-iH_0 T} e^{-i\kappa T |\phi\rangle\langle\phi|}$ where T, κ are respectively the period and the coupling constant and H_0 is a pure point self-adjoint operator, bounded from below. Under some hypotheses on the vector ϕ, cyclic for H_0 we prove the following:

If the gaps between the eigenvalues (λ_n) are such that: $\lambda_{n+1} - \lambda_n \geq Cn^{-\gamma}$ for some $\gamma \in]0,1[$ and $C > 0$, then the Floquet operator of the perturbed system is purely singular continuous T-a.e.

If H_0 is the Hamiltonian of the one-dimensional rotator on $L^2(\mathbb{R}/T_0\mathbb{Z})$ and the ratio $2\pi T/T_0^2$ is irrational, then the Floquet operator is purely singular continuous as soon as $\kappa T \neq 0(2\pi)$

We also establish an integral formula for the family $(e^{-iH_0 T} e^{-i\kappa T |\phi\rangle\langle\phi|})_{T>0, \kappa \in \mathbb{R}}$.

1 Introduction

The spectral properties of the Floquet operator associated to a periodic time-dependent quantum system give some informations on the long time behaviour of this system [1]. In the literature related to the subject, a special attention has been paid to the study of pure point time-independent systems which are perturbed by a smooth time-periodic potential or by periodic kicks. We refer the reader to the introduction of [2] for a review on the subject.

For periodically kicked systems, the form of the Floquet operator is explicit. Moreover, if the perturbation is of rank one, then, following a non-perturbative method developped in [3], we may characterize completely the structure of its spectrum [4]. The spectral properties of the Floquet operator of these systems appear as the result of an interplay between the distribution of the eigenvalues of the unperturbed system and the choice of the rank-one perturbation.

In [4], Combescure gave a simple criteria which ensures the Floquet operator of the perturbed system to remain "generically" pure point. But at the same time, she conjectured that without this condition, the spectrum of this operator may be singular continuous

---

as soon as the eigenvalues of the unperturbed Hamiltonian are simple [1] and defined by a polynomial with some Diophantine coefficient ([4], Remark C, p. 682). She proved it when the polynomial is of degree 1 (the harmonic oscillator). In a following step, Bourget proved the conjecture for polynomials of degree greater than 3 with some irrational coefficients [2].

The aim of this article is twofold: first, we fill a gap left between [4] and [2] in proving the conjecture for pure point Hamiltonians whose eigenvalues are given by polynomials of degree 2 with some irrational coefficients (Theorem 3.4). It means in particular that the Floquet operator of a one-dimensional rotator periodically perturbed by a rank-one kick may be purely singular continuous (Corollary 4.1). Then, using a similar strategy, we prove that the spectrum of the Floquet operator remains, in fact, purely singular continuous for almost every period and for a large class of rank one perturbations, if the sequence of eigenvalues of the unperturbed Hamiltonian diverges rapidly enough (Theorems 3.5 and 3.6). In appendix, we establish an integral formula for the Floquet operators of time-independent quantum systems periodically perturbed by a rank-one kick (Theorem 8.1), in analogy to the self-adjoint case [5].

2 General hypotheses and notations

The evolution of a time-independent quantum system periodically perturbed by a rank one kick is described by the following Floquet operator [4]:

$$V_{\kappa T} = e^{-iH_0 T} e^{-iT\kappa|\phi\rangle\langle\phi|} = e^{-iH_0 T} \left(1 + (e^{-iT\kappa} - 1)|\phi\rangle\langle\phi|\right) , \quad T > 0 , \quad \kappa \in \mathbb{R} \qquad (2.1)$$

acting on a separable Hilbert space $\mathcal{H}$. H_0 is a self-adjoint operator defined on $\mathcal{H}$ and ϕ a vector of this space. The real numbers κ and T are respectively the coupling constant and the period of the system. Given all these ingredients, we may address the following question: *What are the spectral properties of the corresponding unitary operator $V_{\kappa T}$?*
Up to now, a special attention has been paid to those verifying the following hypotheses [4], [2]:

$\mathbf{H_1}$: H_0 is bounded from below, with pure point spectrum. The set of eigenvalues (counted with multiplicities) is infinite and will be written $(\lambda_m)_{m \in \mathbb{N}^*}$ where $\mathbb{N}^*$ denotes the set of positive integers. The corresponding family of orthogonal eigenprojections will be written $(P_m)_{m \in \mathbb{N}^*}$.
$\mathbf{H_2}$: The vector ϕ is cyclic for H_0 ($\|\phi\| = 1$ for simplicity).

Remark 1: These hypotheses are not so restrictive as they may appear. Indeed,

- If H_0 is pure point with a finite number of eigenvalues $(\lambda_m)_{m \in \{1,...,N\}}$, $N \in \mathbb{N}^*$, the essential spectrum of the operator $e^{-iH_0 T}$ is reduced to a finite number of eigenvalues for any $T > 0$. Therefore, from relation (2.1) and Weyl's theorem on the invariance of the essential spectrum of bounded operators under compact perturbations (e.g. [6]), it is clear that the spectrum of $V_{\kappa T}$ is pure point for any $\kappa \in \mathbb{R}$ and any $T > 0$.

---

[1] In the course of this article, we will see that this hypothesis is not necessary.

- If the vector ϕ is not cyclic, the whole family of unitary operators $(V_{\kappa T})_{\kappa \in \mathbb{R}, T>0}$ is simultaneously reduced by the orthogonal subspaces $\mathcal{H}_0$ and $\mathcal{H}_0^\perp$ (e.g. [7] paragraph 5), where:
$$\mathcal{H}_0 = \overline{\mathrm{Span}\{e^{-inH_0T}\phi; n \in \mathbb{Z}\}} \subsetneq \mathcal{H} \ .$$

  For any value of the coupling constant κ and the period $T > 0$, the spectrum of $V_{\kappa T}$ is the union of the spectra of $V_{\kappa T \restriction \mathcal{H}_0}$ and $V_{\kappa T \restriction \mathcal{H}_0^\perp}$, respectively seen as unitary operators acting on the Hilbert subspaces $\mathcal{H}_0$ and $\mathcal{H}_0^\perp$. The spectrum of $V_{\kappa T \restriction \mathcal{H}_0^\perp}$ is nothing but the spectrum of $e^{-iH_0T}{}_{\restriction \mathcal{H}_0^\perp}$, which is known by hypothesis. On the other hand, the determination of the spectral properties of $V_{\kappa T \restriction \mathcal{H}_0}$ is exactly the problem addressed initially with hypothesis $\mathbf{H_2}$, since the vector ϕ belongs to $\mathcal{H}_0$ and is cyclic w.r.t. $e^{-iH_0T}{}_{\restriction \mathcal{H}_0}$.

Remark 2: Notice that whenever $\kappa T = 0 \mod(2\pi)$: $V_{\kappa T} = e^{-iH_0T}$. Such values of κ will be excluded from our discussion in the sequel. For any fixed value of $T > 0$, we will denote by $\mathscr{Z}_T$ the set $\{\kappa \in \mathbb{R}; \kappa T = 2k\pi, k \in \mathbb{Z}\}$.

3 State of the art and main results

The existing works on the spectral properties of the family of unitary operators $(V_{\kappa T})_{\kappa \in \mathbb{R}, T>0}$ defined in section 2 let appear two types of results. On one hand, we know the spectrum remains pure point in the following case:

Theorem 3.1 *Let $T > 0$ and $(V_{\kappa T})_{\kappa \in \mathbb{R}}$ be a family of unitary operators defined by relation (2.1), hypotheses $\mathbf{H_1}$ and $\mathbf{H_2}$. If $(\|P_m\phi\|)_{m \in \mathbb{N}^*}$ belongs to $l^1(\mathbb{N}^*)$, then, the spectrum of $V_{\kappa T}$ is pure point for almost every κ with respect to the Lebesgue measure on $\mathbb{R}$.*

This result, established for the first time by Combescure [4], is related here in a slightly extended framework. In this article, we propose an alternative proof of this theorem based on the integral formula presented in section 8.

On the other hand, it is known that if $(\|P_m\phi\|)_{m \in \mathbb{N}^*}$ belongs to $l^2(\mathbb{N}^*) \setminus l^1(\mathbb{N}^*)$, the spectrum of our Floquet operator may be purely singular continuous, at least in the two following cases:

Theorem 3.2 *Let H_0 be the Hamiltonian of the harmonic oscillator with pulsation $\omega_0 > 0$. Let $T > 0$ and assume $\omega_0 T/2\pi$ is Diophantine. If there exists a unit vector ϕ, cyclic w.r.t. H_0 such that: $\exists \gamma \in]\frac{1}{2}, 1[$, $\exists c > 0$, $\exists m_0 \in \mathbb{N}^*$, $\forall m \geq m_0$,*

$$\|P_m\phi\| \geq cm^{-\gamma} \ ,$$

then, the spectrum of $V_{\kappa T}$ is purely singular continuous for all value of $\kappa \notin \mathscr{Z}_T$.

Theorem 3.3 *Let $(V_{\kappa T})_{\kappa \in \mathbb{R}, T>0}$ be the family of unitary operators defined by relation (2.1), hypotheses $\mathbf{H_1}$ and $\mathbf{H_2}$, where the eigenvalues $(\lambda_m)_{m \in \mathbb{N}^*}$ are given by: $\forall m \in \mathbb{N}^*$,*

$$\lambda_m = \sum_{k=0}^{d} p_k m^k \ \text{with} \ d \geq 3 \ .$$

3

Let $T > 0$ and suppose $p_r T/2\pi$ is irrational for an r in $\{1, \ldots, d\}$ such that $rd > 3$. If there exist $c > 0$, $m_0 \in \mathbb{N}^*$ such that for all $m \geq m_0$,

$$\|P_m \phi\| \geq c m^{-\frac{1+\beta}{2}} \quad \text{with} \quad 0 < \beta < \rho \ , \ \rho = \frac{1}{8d^2(\ln d + 1.5 \ln \ln d + 4.2)}$$

then, the spectrum of the operator $V_{\kappa T}$ is purely singular continuous, provided $\kappa \notin \mathscr{Z}_T$.

Theorem 3.2 was established by Combescure [4] for frequencies ω_0 such that $\omega_0 T/2\pi$ is Diophantine. Theorem 3.3 was proved in [2] and extends the conclusion of Theorem 3.2 to a class of Hamiltonians H_0 with increasing gaps. They are related here with slightly generalized hypotheses. We explain briefly why in sections 5 and 6.

Remark 3: As emphasized in [2], the irrationality condition on the coefficients of the polynomials which define the eigenvalues of the operator H_0 in Theorems 3.2 and 3.3 is essential. In fact, the proofs of both theorems use the fact that in these cases, the eigenvalues of $e^{-iH_0 T}$ are uniformly distributed on the unit circle. If the spectrum of $e^{-iH_0 T}$ was constituted by a finite number of eigenvalues, then the operator $V_{\kappa T}$ would be pure point, for any real κ, T and any vector ϕ (see Remark 1 or [2] Proposition 2.1).

This article exhibits other examples where the operator $V_{\kappa T}$ is purely singular continuous. It somewhat supports the idea that, by contrast with Theorem 3.1, the singular continuous spectrum appears "generically" when the sequence $(\|P_m \phi\|)_{m \in \mathbb{N}^*}$ belongs to $l^2(\mathbb{N}^*) \setminus l^1(\mathbb{N}^*)$.

First, we fill a gap between Theorems 3.2 and 3.3 by considering Hamiltonians H_0 whose eigenvalues are given by a polynomial of degree 2, whose coefficients satisfy some irrationality condition. Namely,

Theorem 3.4 *Let $(V_{\kappa T})_{\kappa \in \mathbb{R}, T > 0}$ be a family of unitary operators defined by relation (2.1), hypotheses $\mathbf{H_1}$ and $\mathbf{H_2}$, where the eigenvalues $(\lambda_m)_{m \in \mathbb{N}^*}$ are given by: $\forall m \in \mathbb{N}^*$,*

$$\lambda_m = p_2 m^2 + p_1 m + p_0$$

Let $T > 0$ and suppose $p_2 T/2\pi$ irrational. If there exist $c > 0$, $m_0 > 0$ and $\beta > 0$, such that for all $m \geq m_0$,

$$\|P_m \phi\| \geq c \, m^{-\frac{7}{12}} \ln^\beta (m+1) \ ,$$

then, $V_{\kappa T}$ is purely singular continuous, provided $\kappa \notin \mathscr{Z}_T$.

However, in most of basic models, we only know the asymptotics of the eigenvalues of the Hamiltonian H_0. Despite of this, it is possible to exhibit purely singular continuous Floquet operators $V_{\kappa T}$, if we assume the sequence of eigenvalues $(\lambda_m)_{m \in \mathbb{N}^*}$ of H_0 is increasing and, given a real positive number γ, satisfies one of the two following properties:

$\mathbf{H_a}(\gamma)$: $\exists c_1 > 0$, $\exists c_2 > 0$, $\forall m \in \mathbb{N}^*$, $\forall k \geq c_2 \, m^\gamma$, $\lambda_{m+k} - \lambda_m \geq c_1$
$\mathbf{H_b}(\gamma)$: $\exists c_1 > 0$, $\exists c_2 > 0$, $\forall m \in \mathbb{N}^*$, $\forall k \geq c_2 \, m \ln^{-\gamma}(m+1)$, $\lambda_{m+k} - \lambda_m \geq c_1$

More precisely, we have:

Theorem 3.5 *Let* $(V_{\kappa T})_{\kappa \in \mathbb{R}, T>0}$ *be a family of unitary operators defined by relation (2.1), hypotheses* $\mathbf{H_1}$ *and* $\mathbf{H_2}$*, where the eigenvalues* $(\lambda_m)_{m \in \mathbb{N}^*}$ *of* H_0 *fulfills the property* $\mathbf{H_a}(\gamma)$ *for some* $\gamma \in]0,1[$*. Assume there exist* $c > 0$*,* $\epsilon > 0$ *and* $m_0 > 0$ *such that:* $\forall m \geq m_0$,

$$\|P_m \phi\| \geq c \, \frac{\ln^{\frac{5}{4}+\epsilon}(m+1)}{m^{\frac{3-\gamma}{4}}} \ .$$

Then, $V_{\kappa T}$ *is purely singular continuous for any value of* T *in a set of complete Lebesgue measure and any* $\kappa \in \mathbb{R}^*$.

Theorem 3.6 *Let* $(V_{\kappa T})_{\kappa \in \mathbb{R}, T>0}$ *be a family of unitary operators defined by relation (2.1), hypotheses* $\mathbf{H_1}$ *and* $\mathbf{H_2}$*, where the eigenvalues* $(\lambda_m)_{m \in \mathbb{N}^*}$ *of* H_0 *fulfills the property* $\mathbf{H_b}(\gamma)$ *for some* $\gamma > 8$*. Assume there exist* $c > 0$*,* $\epsilon \in]0, \frac{\gamma-8}{6}[$ *and* $m_0 > 0$ *such that:* $\forall m \geq m_0$,

$$\|P_m \phi\| \geq c \, m^{-\frac{1}{2}} \ln^{\frac{2-\gamma}{6}+\epsilon}(m+1) \ .$$

Then, $V_{\kappa T}$ *is purely singular continuous for any value of* T *in a set of complete Lebesgue measure and any* $\kappa \in \mathbb{R}^*$.

Remark 4: If the eigenvalues $(\lambda_m)_{m \in \mathbb{N}^*}$ of the self-adjoint operator H_0 behaves as: $\forall m \in \mathbb{N}^*$, $\lambda_{m+1} - \lambda_m \geq c \, m^{-\gamma_0}$ for some γ_0 in $]0,1[$ and some positive number c, then it satisfies the property $\mathbf{H_a}(\gamma_0)$. For all $m \in \mathbb{N}^*$, for all $k \geq m^{\gamma_0}$,

$$\lambda_{m+k} - \lambda_m = \sum_{j=0}^{k-1} \lambda_{m+j+1} - \lambda_{m+j} \geq cm^{-\gamma_0} \sum_{j=0}^{k-1} \left(1 + \frac{j}{m}\right)^{-\gamma_0} \geq c_{\gamma_0} > 0 \ .$$

Similarly, a sequence of eigenvalues $(\lambda_m)_{m \in \mathbb{N}^*}$ that behaves as: $\forall m \in \mathbb{N}^*$, $\lambda_{m+1} - \lambda_m \geq cm^{-1} \ln^{\gamma_0}(m+1)$ for some positive numbers γ_0 and c, satisfies the property $\mathbf{H_b}(\gamma_0)$. The general formulation of Theorems 3.5 and 3.6 includes cases where the operator H_0 may have less regularly distributed eigenvalues.

Remark 5: The *almost everywhere* in Theorems 3.5 and 3.6 cannot be erased. Assume for example that provided hypothesis $\mathbf{H_1}$ is checked, the eigenvalues of H_0 are of finite multiplicities and given by: $\forall m \in \mathbb{N}^*$, $\lambda_m T = e.m!$. Writing $\{x\}$ to denote the fractional part of the real number x, the sequence $(\{\lambda_m T\})_{m \in \mathbb{N}^*}$ is convergent and the essential spectrum of $e^{-iH_0 T}$ is therefore reduced to a single point. It follows from Weyl's theorem that the spectrum of $V_{\kappa T}$ is pure point.

Remark 6: We may also wonder if the hypotheses on the growth rate of the sequence $(\lambda_m)_{m \in \mathbb{N}^*}$ in Theorem 3.6 may be improved. If the growth rate is too weak, the spectrum of $V_{\kappa T}$ may remain pure point for all value of $\kappa \in \mathbb{R}$ and any choice of the vector ϕ:

Proposition 3.1 *Let* $\kappa \in \mathbb{R}$*,* $T > 0$ *and* $V_{\kappa T}$ *a unitary operator defined by relation (2.1) and hypothesis* $\mathbf{H_1}$*. Assume the eigenvalues of* H_0 *are of finite multiplicities and there exists an integral number* N *such that the sequence* $(\lambda_m)_{m \geq N}$ *is increasing with the following condition :* $\exists C \geq 0$*,* $\forall m \geq N$*,* $\lambda_{m+1} - \lambda_m \leq C m^{-1} \ln^b m$ *with* $b < -1$*. Then, the operator* $V_{\kappa T}$ *is pure point.*

5

Indeed, the hypotheses of the Proposition ensure the convergence of the sequence $(\lambda_m)_{m \in \mathbb{N}^*}$ to some limit λ. The essential spectrum of e^{-iH_0T} is reduced to the single point $e^{-i\lambda T}$. The conclusion follows as above.

Before turning to the proofs of Theorems 3.4, 3.5 and 3.6, let us illustrate them by some applications.

4   Applications

As an application of Theorem 3.4, we can now exhibit periodically kicked one-dimensional rotators with purely singular continuous spectrum:

Corollary 4.1 *Let $T_0 > 0$, $T > 0$ such that $2\pi T/T_0^2$ is irrational and $\kappa \notin \mathcal{Z}_T$. Consider the Hamiltonian H_0 defined by: $H_0 = -\partial_\theta^2$ on $L^2(\mathbb{R}/T_0\mathbb{Z})$. If the Floquet operator $V_{\kappa T}$, associated to H_0 and defined by relation (2.1), hypotheses $\mathbf{H_1}$ and $\mathbf{H_2}$ is such that: $\exists \beta > 0$, $\exists m_0 > 0$, $\exists c > 0$, $\forall m \geq m_0$,*

$$\|P_m \phi\| \geq c\, m^{-\frac{7}{12}} \ln^\beta (m+1) \ ,$$

then, it is purely singular continuous.

For the proof, it is enough to note that the eigenvalues $(\lambda_m)_{m \in \mathbb{N}}$ of H_0 are given by: $\forall m \in \mathbb{N}$,

$$\lambda_m = \frac{(2\pi)^2}{T_0^2} m^2 \ ,$$

and to apply Theorem 3.4.

Now, let us consider the consequences of periodic kicks on an unperturbed self-adjoint operator H_0 of the form: $H_0 = -\partial_x^2 + V(x)$ on $L^2(\mathbb{R})$ with $V(x) \sim \beta|x|^p$, $\beta > 0$ and $p > 0$ (e.g. the one-dimensional anharmonic oscillator [8]). Using Bohr-Sommerfeld type conditions, we know that the eigenvalues $(\lambda_m)_{m \in \mathbb{N}^*}$ of this operator are such that: $\lambda_n \sim cn^{\frac{2p}{p+2}}$ for some positive c. In this case, Theorem 3.5 allows us to derive the following result:

Corollary 4.2 *Let $\kappa \in \mathbb{R}^*$, $p > 0$, $\beta > 0$ and H_0 be the Hamiltonian: $H_0 = -\partial_x^2 + V(x)$ on $L^2(\mathbb{R})$ with $V(x) \sim \beta|x|^p$. If the Floquet operator $V_{\kappa T}$, associated to H_0 and defined by relation (2.1), hypotheses $\mathbf{H_1}$ and $\mathbf{H_2}$ is such that: $\exists c > 0$, $\exists \epsilon > 0$, $\exists m_0 > 0$, $\forall m \geq m_0$,*

$$\|P_m \phi\| \geq c\, \frac{\ln^{\frac{5}{4}+\epsilon}(m+1)}{m^{\frac{3-\gamma}{4}}} \quad with \quad \gamma = 1 - \frac{4}{p+2} \ ,$$

then, it is purely singular continuous for any period T in a set of complete Lebesgue measure.

The remainder of the article is devoted to the proof Theorems 3.4, 3.5, 3.6 and the integral formula. In the next section, we present and discuss some criteria whose combination allows to prove the absence of eigenvalues in the spectrum of $V_{\kappa T}$. The proofs of

Theorems 3.4, 3.5 and 3.6 are presented as an application of these criteria in section 6. We also mention how Theorem 3.2 may be derived in a similar way. Some intermediate results of analytic number theory are gathered and postponed in section 7. Section 8 is devoted to the proof of the integral formula from which we deduce Theorem 3.1.

5   General strategy

From now, $\{a\}$ and $[a]$ will denote respectively the fractional and the integral part of any real number a: $a = [a] + \{a\}$. The fractional part of a real number belongs to the unit interval $[0, 1[$. The discrepancy $(D_N)_{N \in \mathbb{N}^*}$ of a sequence of real numbers $(x_m)_{m \in \mathbb{N}^*}$ is defined by: $\forall N \in \mathbb{N}^*$,

$$D_N = \sup_{0 \leq a < b \leq 1} \left| \frac{A([a, b[; N; (x_m))}{N} - (b - a) \right|$$

$$\text{with} \quad A([a, b[; N; (x_m)) = \#\{1 \leq m \leq N; \{x_m\} \in [a, b[\} \ .$$

This quantity measures the rate of convergence of the distribution of the sequence $(\{x_m\})_{m \in \mathbb{N}^*}$ to an ideal uniform distribution. If $\lim_{N \to +\infty} D_N = 0$ the sequence $(x_m)_{m \in \mathbb{N}^*}$ is said to be uniformly distributed $\mathrm{mod}(1)$.

5.1   Key Lemmas

We already know that the spectrum of $V_{\kappa T}$ is purely singular for all values of κ and T if the family $(V_{\kappa T})$ obeys hypothesis $\mathbf{H_1}$ (e.g. [2] section 3.1). So, it remains to justify how the hypotheses of Theorems 3.4, 3.5 and 3.6 exclude the existence of eigenvalues in the spectrum of $V_{\kappa T}$ for $T > 0$ and $\kappa T \neq 0 (2\pi)$.

Lemma 5.1 gives a necessary and sufficient condition for a point of the unit circle to belong to the point spectrum of $V_{\kappa T}$. It is a straightforward reformulation of the criterion given by Combescure ([4] corollary 2) on the basis of Simon-Wolff method [3]:

Lemma 5.1 *Let $T > 0$ and $\kappa \in \mathbb{R}$ such that $\kappa T \neq 0 \ \mathrm{mod}(2\pi)$. Consider the unitary operator $V_{\kappa T}$ as defined by relation (2.1) and satisfying the hypotheses $\mathbf{H_1}$ and $\mathbf{H_2}$. Then, the complex number e^{ix} belongs to the point spectrum of $V_{\kappa T}$ iff:*

$$B(x)^{-1} = \sum_{m \in \mathbb{N}^*} \frac{\|P_m \phi\|^2}{\sin^2 \left(\frac{x - \theta_m(T)}{2} \right)} < +\infty$$

$$\text{and} \quad \sum_{m \in \mathbb{N}^*} \|P_m \phi\|^2 \cot(\frac{x - \theta_m(T)}{2}) = \cot(\frac{\kappa T}{2})$$

where $\theta_m(T)$ is defined by: $\forall m \in \mathbb{N}^*$,

$$\theta_m(T) \equiv 2\pi \{\frac{\lambda_m T}{2\pi}\} \ . \tag{5.2}$$

The convergence (or divergence) of the series $B(x)^{-1}$ is the result of a competition between the decay rate of the numerator and the distribution of the sequence $(\theta_m(T))_{m \in \mathbb{N}^*}$ in $[0, 2\pi[$. It is enough to establish a general criterion which ensures the divergence of the series $B(x)^{-1}$ for any value of x in $[0, 2\pi[$, to prove the absence of eigenvalues in the spectrum of $V_{\kappa T}$ if $T > 0$ and $\kappa T \neq 0 (2\pi)$. This criteria is given by Lemma 5.2. Namely,

Lemma 5.2 *Let $(c_m)_{m \in \mathbb{N}^*}$ be a complex-valued sequence and $(\theta_m)_{m \in \mathbb{N}^*} \in [0, 2\pi[^{\mathbb{N}^*}$ a uniformly distributed mod(1). $(D_N)_{N \in \mathbb{N}^*}$ will denote the sequence of discrepancies associated to $(\theta_m)_{m \leq N}$. Suppose we can construct:*

- *two positive sequences $(\epsilon_m)_{m \in \mathbb{N}^*}$ and $(b_m)_{m \in \mathbb{N}^*}$ such that: $\exists m_* \in \mathbb{N}^*, \forall m \geq m_*$, $0 < b_m \leq |c_m|$, $\epsilon_{m+1} < \epsilon_m$ and $\lim_{m \to +\infty} \epsilon_m = +\infty$,*

- *a subsequence of integral numbers $(N_k)_{k \in \mathbb{N}}$, where $\mathbb{N}$ denote the set of non-negative integers, such that: $\forall k \in \mathbb{N}, D_{N_k} \leq \epsilon_{N_k}$*

which verify,

$$\lim_{k \to +\infty} N_k \epsilon_{N_k} \left(\inf_{1 \leq m \leq N_k} \frac{b_m^2}{\epsilon_m^2} \right) = +\infty \ ,$$

then, $\forall x \in [0, 2\pi[$,

$$\sum_{m \in \mathbb{N}^*} \frac{|c_m|^2}{\sin^2 \left(\frac{x - \theta_m}{2} \right)} = +\infty$$

This lemma is proven in section 5.2.

Remark 7: Lemma 5.2 relates the divergence of $B(x)^{-1}$ to the asymptotics of $(D_m)_{m \in \mathbb{N}^*}$ and $(c_m)_{m \in \mathbb{N}^*}$. However, regardless of the distribution of the sequence $(\theta_m)_{m \in \mathbb{N}^*}$, this lemma is no help for sequences $(c_m)_{m \in \mathbb{N}^*}$ such that: $\exists C > 0, \forall m \in \mathbb{N}^*$,

$$|c_m| \leq C \frac{\sqrt{\ln m}}{m} \ . \tag{5.3}$$

This limitation is related to Schmidt's theorem (e.g. [9] Theorem 2.3):

Theorem 5.1 *For any infinite sequence of real numbers, there exists $c > 0$,*

$$D_N \geq c \frac{\ln N}{N} \ .$$

Taking in account Lemma 3 of [4], it means in particular, that we have still no criteria for the divergence or the convergence of $B(x)^{-1}$ if the sequence $(c_m)_{m \in \mathbb{N}^*}$ verifies inequality (5.3) but does not belongs to $l^1(\mathbb{N}^*)$.

5.2 Proof of Lemma 5.2

Let us recall first that the uniform distribution property of the sequence $(\theta_m)_{m \in \mathbb{N}^*}$ implies:

$$\lim_{N \to +\infty} D_N = 0 \ .$$

Let us define the following families of sets: $\forall N \in \mathbb{N}^*, \forall x \in [0, 2\pi[$,

$$
\begin{aligned}
S_{1,N}(x) &= \{1 \leq m \leq N; \theta_m \in [x, x + 2\epsilon_m[\} \ , \\
S_{2,N}(x) &= \{1 \leq m \leq N; \theta_m \in [x, x + 2\epsilon_N[\} \ .
\end{aligned}
$$

For any fixed x, $(S_{1,N}(x))_{N \in \mathbb{N}^*}$ is an increasing sequence of subsets of $\mathbb{N}^*$, and since the sequence $(\epsilon_m)_{m \in \mathbb{N}^*}$ is positive, asymptotically decreasing and converges to 0: $\exists N_{\epsilon,x} > 0$, $\forall N \geq N_{\epsilon,x}$, $S_{2,N}(x) \subset S_{1,N}(x)$. Therefore, using the definition of the discrepancy of the sequence $(\theta_m)_{m \leq N}$, we get: $\forall x \in [0, 2\pi[$, $\forall N \geq N_{\epsilon,x}$, $2N\epsilon_N - ND_N \leq \#S_{2,N}(x) \leq \#S_{1,N}(x)$. In particular, since $0 < D_{N_k} < \epsilon_{N_k}$, for all $k \in \mathbb{N}$ such that $N_k \geq N_{\epsilon,x}$,

$$N_k D_{N_k} \leq N_k \epsilon_{N_k} \leq \#S_{1,N_k}(x) \ . \tag{5.4}$$

Note that the combination of this inequality and Theorem 5.1 implies:

$$\lim_{k \to +\infty} N_k D_{N_k} = \lim_{k \to +\infty} N_k \epsilon_{N_k} = \lim_{k \to +\infty} \#S_{1,N_k}(x) = +\infty \ . \tag{5.5}$$

On the other hand, following [2], we can do the following estimates: $\forall x \in [0, 2\pi[$, $\forall N \in \mathbb{N}^*$,

$$\sum_{m=1}^{N} \frac{|c_m|^2}{\sin^2\left(\frac{x-\theta_m}{2}\right)} \geq 4 \sum_{m=1}^{N} \frac{|c_m|^2}{(x-\theta_m)^2} \geq 4 \sum_{m \in S_{1,N}(x)} \frac{|c_m|^2}{(x-\theta_m)^2} \geq 4 \sum_{m \in S_{1,N}(x), m \geq m_*} \frac{|c_m|^2}{(x-\theta_m)^2}$$

$$\geq 4 \sum_{m \in S_{1,N}(x), m \geq m_*} \frac{b_m^2}{\epsilon_m^2} \ .$$

Therefore, considering the same inequality for the subsequence $(N_k)_{k \in \mathbb{N}}$ and using inequality (5.4), we obtain: $\forall x \in [0, 2\pi[$, $\forall k \in \mathbb{N}$ such that $N_k \geq N_{\epsilon,x}$,

$$\sum_{m=1}^{N_k} \frac{|c_m|^2}{\sin^2\left(\frac{x-\theta_m}{2}\right)} \geq 4([N_k\epsilon_{N_k}] - m_*) \inf_{m \in S_{1,N_k}(x)} \frac{b_m^2}{\epsilon_m^2} \geq 4([N_k\epsilon_{N_k}] - m_*) \inf_{1 \leq m \leq N_k} \frac{b_m^2}{\epsilon_m^2} \ .$$

Since the subsequence $(N_k\epsilon_{N_k})_{k \in \mathbb{N}}$ is divergent (see relation (5.5)), the conclusion follows from the hypotheses when taking the limit k to $+\infty$. $\qquad\square$

Let us now combine these Lemmas for our purposes.

6 Proofs

The proofs of Theorems 3.4, 3.5 and 3.6 will be led by means of Lemmas 5.1 and 5.2. In each case, their application requires the identification of suitable lower bounds (resp. upper bounds) on the sequence $(\|P_m\phi\|)$ (resp. (D_m)), expressed in terms of the input sequence $(b_m)_{m \in \mathbb{N}^*}$ (resp. $(\epsilon_m)_{m \in \mathbb{N}^*}$). This can be realized if we get some reasonably good estimates on some subsequence of discrepancies $(D_{N_k})_{k \in \mathbb{N}}$ associated to the sequence $(\theta_m(T))_{m \in \mathbb{N}^*}$ defined by relation (5.2). These estimates will be specified in each case.

6.1 Proof of Theorem 3.4

Assume the eigenvalues $(\lambda_m)_{m \in \mathbb{N}^*}$ of the Hamiltonian H_0 are defined by: $\lambda_m = p_2 m^2 + p_1 m + p_0$ for all $m \in \mathbb{N}^*$. Following relation (5.2) of Lemma 5.1, we write for any $m \in \mathbb{N}^*$ and any $T > 0$,

$$\theta_m(T) = 2\pi\{\frac{\lambda_m T}{2\pi}\} = 2\pi\{P_{H_0,T}(m)\} \qquad T > 0 \ ,$$

where $P_{H_0,T}$ is the polynomial defined by: $\forall x \in \mathbb{R}$,

$$P_{H_0,T}(x) = \frac{T}{2\pi}\left(p_2 x^2 + p_1 x + p_0\right).$$

If T is chosen in such a way the coefficient $p_2 T/2\pi$ is irrational, the sequence $(P_{H_0,T}(m))_{m \in \mathbb{N}^*}$ (resp. $(\theta_m(T))_{m \in \mathbb{N}^*}$) is uniformly distributed mod(1) [9]. Denoting by D_N the discrepancy of $(P_{H_0,T}(m))_{m \leq N}$, we know by Proposition 7.1 (see section 7.1), there exist a positive constant C and an infinite subsequence of integers $(N_k)_{k \in \mathbb{N}}$ such that: $\forall k \in \mathbb{N}$,

$$D_{N_k} \leq C N_k^{-\frac{1}{6}} \ln(N_k + 1) \ .$$

Now, if view of Lemma 5.2, we define the sequences $(b_m)_{m \in \mathbb{N}^*}$, $(\epsilon_m)_{m \in \mathbb{N}^*}$ and $(c_m)_{m \in \mathbb{N}^*}$ by: $\forall m \in \mathbb{N}^*$,

$$b_m = c\, m^{-\frac{7}{12}} \ln^\beta(m+1) \ , \quad \epsilon_m = C m^{-\frac{1}{6}} \ln(m+1) \ , \quad c_m = \|P_m \phi\|$$

where c is a fixed positive constant and $\beta > 0$. These sequences clearly fulfill the hypotheses of Lemma 5.2 for some m_*. In particular,

$$\lim_{N \to +\infty} N \epsilon_N \inf_{m \leq N} \frac{b_m^2}{\epsilon_m^2} = +\infty \ .$$

Therefore, by Lemma 5.2: $B(x)^{-1} = +\infty$ for all $x \in [0, 2\pi[$, which proves the result, provided the constant $\kappa \notin \mathcal{Z}_T$. $\qquad\square$

Note that Theorem 3.3 may be proven by a similar procedure. We explain briefly how in the next paragraph.

6.2 About the proof of Theorem 3.3

Assume the eigenvalues $(\lambda_m)_{m \in \mathbb{N}^*}$ of the Hamiltonian H_0 are given by a polynomial of degree d greater than 3: $\forall m \in \mathbb{N}^*$,

$$\lambda_m = \sum_{k=0}^{d} p_k m^k \ ,$$

and $T > 0$ is chosen such as $T p_r / 2\pi$ is irrational for some r in $\{1, \ldots, d\}$. It is known the sequence $(P_{H_0,T}(m))_{m \in \mathbb{N}^*}$ where: $\forall m \in \mathbb{N}^*$,

$$P_{H_0,T}(x) = \frac{T}{2\pi} \sum_{k=0}^{d} p_k x^k$$

is uniformly distributed mod(1) [9]. Define $(b_m)_{m \in \mathbb{N}^*}$ and $(c_m)_{m \in \mathbb{N}^*}$ respectively by: $\forall m \in \mathbb{N}^*$,

$$b_m = cm^{-\frac{1+\beta}{2}} \ , \quad c_m = \|P_m \phi\|$$

where c is a positive constant, β belongs to $]0, \rho[$ and ρ is defined in the statement of Theorem 3.3. Now, if we denote by D_N the discrepancy of $(P_{H_0,T}(m))_{m \leq N}$, assume the

index r mentionned above is such that $rd > 3$ and choose ϵ in $]\beta, \min(\rho, \frac{1+\beta}{2})[$ (note that $0 < \beta < \rho \ll 1$), then by Lemma 3.3 of [2], we can construct an infinite sequence of integers $(N_k)_{k\in\mathbb{N}}$ such that: $\exists C_\epsilon > 0$, $\forall k \in \mathbb{N}$,

$$D_{N_k} \leq C_\epsilon N_k^{-\epsilon} \ .$$

Therefore, it is enough to define $(\epsilon_m)_{m\in\mathbb{N}^*}$ by: $\forall m \in \mathbb{N}^*$, $\epsilon_m = C_\epsilon m^{-\epsilon}$ and the result follows by means of Lemmas 5.2 and 5.1, provided $\kappa \notin \mathcal{Z}_T$.

6.3 Proof of Theorems 3.5 and 3.6

Assume the eigenvalues $(\lambda_m)_{m\in\mathbb{N}^*}$ of the Hamiltonian H_0 obeys the hypothesis $\mathbf{H_a}(\gamma)$ (resp. $\mathbf{H_b}(\gamma)$) for some $\gamma \in]0,1[$ (resp. $\gamma > 8$). For any $T \in \mathbb{R}$, let us define the sequence $(f_{H_0,T}(m))_{m\in\mathbb{N}^*}$ by: $\forall m \in \mathbb{N}^*$,

$$f_{H_0,T}(m) = \frac{T}{2\pi}\lambda_m \ .$$

Theorem 6.1 states that such sequences are uniformly distributed mod(1) for every value of T in a set of complete Lebesgue measure. More precisely, we have:

Theorem 6.1 *Let $(x_m)_{m\in\mathbb{N}^*}$ be a non decreasing sequence of real numbers which satisfies the property $\mathbf{H_a}(\gamma)$ for some $\gamma \in]0,1[$. If $D_{N,T}$ denotes the discrepancy of the sequence $(x_m T)_{1\leq m\leq N}$ and ϵ is a fixed positive constant, then for Lebesgue almost every T :*

$$\lim_{N\to+\infty} \frac{N^{\frac{1-\gamma}{2}} D_{N,T}}{\ln^{\frac{5}{2}+\epsilon}(N+1)} = 0 \ .$$

Moreover, if $(x_m)_{m\in\mathbb{N}^*}$ satisfies the property $\mathbf{H_b}(\gamma)$ for some $\gamma > 2$ and if ϵ is a fixed positive constant, then for Lebesgue almost every T:

$$\lim_{N\to+\infty} D_{N,T} \ln^{\frac{\gamma-2}{3}-\epsilon}(N+1) = 0 \ .$$

The proof of the first part is derived in [10] p. 288 and the second part in the commentaries about [10] Theorem 5.2 (pp. 291-292). This result is an improvement of a former theorem of Erdös and Koksma [11].

It means in our context that if $D_{N,T}$ denote the discrepancy of $(f_{H_0,T}(m))_{m\leq N}$ and ϵ is chosen positive (resp. in $]0, \frac{\gamma-8}{6}[$), there exist a subset of $\mathbb{R}^+$ of complete Lebesgue measure, $\mathcal{L}_\epsilon$ and a positive constant $C_{\epsilon,\gamma}$ such that: $\forall N \in \mathbb{N}^*$,

$$D_{N,T} \leq C_{\epsilon,\gamma}\frac{\ln^{\frac{5}{2}+\epsilon}(N+1)}{N^{\frac{1-\gamma}{2}}} \quad \left(\text{resp. } D_{N,T} \leq C_{\epsilon,\gamma}\ln^{-\frac{\gamma-2}{3}+\epsilon}(N+1)\right) \ .$$

Now, define the sequences $(c_m)_{m\in\mathbb{N}^*}$, $(b_m)_{m\in\mathbb{N}^*}$ and $(\epsilon_m)_{m\in\mathbb{N}^*}$ by: $\forall m \in \mathbb{N}^*$,

$$c_m = \|P_m\phi\| \ , \ b_m = c\frac{\ln^{\frac{5}{4}+\epsilon}(m+1)}{m^{\frac{3-\gamma}{4}}} \ , \ \epsilon_m = C_{\epsilon,\gamma}\frac{\ln^{\frac{5}{2}+\epsilon}(m+1)}{m^{\frac{1-\gamma}{2}}}$$

$$\left(\text{resp. } b_m = c\, m^{-\frac{1}{2}}\ln^{-\frac{\gamma-2}{6}+\epsilon}(m+1) \ , \ \epsilon_m = C_{\epsilon,\gamma}\ln^{-\frac{\gamma-2}{3}+\epsilon}(m+1)\right) \ ,$$

11

where c is a fixed positive constant. These sequences verify the hypotheses of Lemma 5.2, which means that for all periods T in $\mathcal{L}_\epsilon$ and for all x in $[0, 2\pi[$,

$$B(x)^{-1} \equiv \sum_{m=1}^{+\infty} \frac{\|P_m \phi\|^2}{\sin^2 \left(\frac{x - \theta_m(T)}{2} \right)} = +\infty \ .$$

The last part of the proof follows directly from Lemma 5.1, provided $\kappa \in \mathbb{R}^*$. $\qquad \square$

7 Technicalities

This section is devoted to the statement and the proof of Proposition 7.1, which is used in the proof of Theorems 3.4. This proposition is an attempt to estimate the discrepancy of sequences $(x_n)_{n \leq N}$ given by some polynomials of degree 2. More precisely:

Proposition 7.1 *Let $(x_n)_{n \in \mathbb{N}^*}$ be the sequence defined by: $\forall n \in \mathbb{N}^*$, $x_n = a_2 n^2 + a_1 n + a_0$ where a_2 is irrationnal and a_1, a_0 are real. If we denote by D_N the discrepancy associated to $(x_n)_{n \leq N}$, then there exist a constant $C > 0$ and a infinite subsequence of integers $(N_k)_{k \in \mathbb{N}}$ such that: $\forall k \in \mathbb{N}$,*

$$D_{N_k} \leq C N_k^{-\frac{1}{6}} \ln(N_k + 1) \ .$$

The general strategy of the proof is similar to that of [2] Lemma 3.3, for sequences $(x_n)_{n \leq N}$ given by some polynomials of degree greater than 3. The estimates on the discrepancy are obtained from the study of the asymptotical behaviour of the corresponding exponential sum. In the present case, this will be achieved by means of the following proposition:

Proposition 7.2 *Let $(x_n)_{n \in \mathbb{N}^*}$ be a sequence of real numbers and denote by D_N the discrepancy of $(x_n)_{1 \leq n \leq N}$. Let $(u_n)_{n \in \mathbb{N}^*}$ be a positive non-decreasing sequence and define:*

$$S_N = \sup_{h \in \{1, \ldots, [u_N]\}} \left| \sum_{n=1}^{N} e^{2\pi i h x_n} \right| \ .$$

Then, $\forall N \in \mathbb{N}^*$,

$$D_N \leq \left(\frac{S_N^2}{N^2} + \frac{6}{\pi^2} \frac{1}{[u_N]} \right)^{1/3} \ .$$

Proof: LeVeque's inequality states (e.g. [9] Theorem 2.4) that for any $N \in \mathbb{N}^*$, the discrepancy D_N associated to $(x_n)_{1 \leq n \leq N}$ is such that:

$$D_N \leq \left(\frac{6}{\pi^2} \sum_{h=1}^{+\infty} \frac{1}{h^2} \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} \right|^2 \right)^{1/3} \ .$$

But: $\forall N \in \mathbb{N}^*$,

$$\sum_{h=1}^{+\infty} \frac{1}{h^2} \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} \right|^2 = \sum_{h=1}^{[u_N]} \frac{1}{h^2} \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} \right|^2 + \sum_{h=[u_N]+1}^{+\infty} \frac{1}{h^2} \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} \right|^2$$

$$\leq \left(\sum_{h=1}^{[u_N]} \frac{1}{h^2} \right) \frac{S_N^2}{N^2} + \sum_{h=[u_N]+1}^{+\infty} \frac{1}{h^2}$$

$$\leq \left(\sum_{h=1}^{+\infty} \frac{1}{h^2} \right) \frac{S_N^2}{N^2} + \int_{[u_N]}^{+\infty} \frac{dx}{x^2} = \frac{\pi^2}{6} \frac{S_N^2}{N^2} + \frac{1}{[u_N]} \ ,$$

which proves the proposition. $\qquad\qquad\square$

So, now, the problem is reduced to the study of the asymptotical behaviour of the sequence $(S_N)_{N \in \mathbb{N}^*}$ for a suitable sequence $(u_N)_{N \in \mathbb{N}^*}$. We will do it by means of Lemma 7.1:

Lemma 7.1 *Define the polynomial P by: $\forall x \in \mathbb{R}$, $P(x) = a_2 x^2 + a_1 x + a_0$, where a_1, a_0 are real numbers and a_2 is an irrational number for which the following rational approximation holds:*

$$\left| a_2 - \frac{s}{q} \right| \leq \frac{1}{q^2}$$

where s and q are relatively prime integers and $q \geq 1$. Then, there exists a positive constant C such that: $\forall m \in \mathbb{N}^*$, $\forall N \in \mathbb{N}^*$

$$\left| \sum_{n=1}^{N} e^{2i\pi m P(n)} \right|^2 \leq C \left(q + N + Nm + \frac{N^2 m}{q} \right) \max\{1, \ln q\} \ . \tag{7.6}$$

The proof of this lemma, which follows essentially that of Weyl's inequality (e.g. [12], chapter 4), is based on the two following results:

Lemma 7.2 *For every real number α and all integers N_1, N_2 such that $N_1 < N_2$,*

$$\left| \sum_{n=N_1+1}^{N_2} e^{2i\pi\alpha n} \right| \leq \min \left(\frac{1}{2\langle\alpha\rangle}, N_2 - N_1 \right) \ , \tag{7.7}$$

where $\langle\alpha\rangle = \inf(\{\alpha\}, 1 - \{\alpha\})$.

Lemma 7.3 *Let α be an irrational number, q ($q \geq 1$) and s two integral numbers that are relatively prime. If,*

$$\left| \alpha - \frac{s}{q} \right| \leq \frac{1}{q^2} \ ,$$

then, there exists a positive constant C such that for any positive integral numbers m_1, m_2,

$$\sum_{k=1}^{m_1} \min \left(m_2, \frac{1}{\langle\alpha k\rangle} \right) \leq C \left(q + m_1 + m_2 + \frac{m_1 m_2}{q} \right) \max\{1, \ln q\} \ .$$

13

The reader may consult [12] Lemmas 4.7 and 4.11 for a proof.

Proof of Lemma 7.1: First notice that for all $m \in \mathbb{N}^*$ and all $N \in \mathbb{N}^*$,

$$\left| \sum_{n=1}^{N} e^{2i\pi m P(n)} \right|^2 = \left| \sum_{|d|<N} \sum_{n \in I(d)} e^{2i\pi m \Delta_d(P)(n)} \right| , \qquad (7.8)$$

where $I(d)$ is the interval $[1-d, N-d] \cap [1, N]$ and for all $n \in I(d)$, $\Delta_d(P)(n) = P(n+d) - P(n)$ (e.g. [12] Lemma 4.12). On one hand, $\Delta_0(P) \equiv 0$ and on the other hand, if $d \neq 0$, then $\forall n \in \mathbb{Z}$, $\Delta_d(P)(n) = 2a_2 nd + a_d$, with $a_d = a_1 d + a_2 d^2$. Therefore, for any positive integer m,

$$\left| \sum_{|d|<N} \sum_{n \in I(d)} e^{2i\pi m \Delta_d(P)(n)} \right| \leq N + \left| \sum_{0<|d|<N} \sum_{n \in I(d)} e^{2i\pi m \Delta_d(P)(n)} \right|$$

$$= N + \left| \sum_{0<|d|<N} \sum_{n \in I(d)} e^{2i\pi m (2a_2 nd + a_d)} \right|$$

$$\leq N + \sum_{0<|d|<N} \left| \sum_{n=\max(1-d,1)}^{\min(N-d,N)} e^{2i\pi(2a_2 mnd)} e^{2i\pi m a_d} \right| .$$

If we note that the intervals $(I(d))_{0<|d|<N}$ are included in $[1, N]$, we get by lemma 7.2:

$$\left| \sum_{|d|<N} \sum_{n \in I(d)} e^{2i\pi m \Delta_d(P)(n)} \right| \leq N + \sum_{0<|d|<N} \min\left(N, \frac{1}{2\langle 2a_2 dm \rangle} \right)$$

$$\leq N + \sum_{1 \leq d < N} \min\left(2N, \frac{1}{\langle 2a_2 dm \rangle} \right)$$

$$\leq N + \sum_{n=1}^{2Nm} \min\left(2N, \frac{1}{\langle a_2 n \rangle} \right)$$

The last sum may be estimated by means of Lemma 7.3. It results that: $\exists C > 0, \forall m \in \mathbb{N}^*$, $\forall N \in \mathbb{N}^*$,

$$\left| \sum_{n=1}^{N} e^{2i\pi m P(n)} \right|^2 \leq N + C\left(q + N + Nm + \frac{N^2 m}{q} \right) \max\{1, \ln q\} ,$$

hence the result. $\qquad \qquad \square$

Proof of Proposition 7.1: On one hand, the continued fractions expansion of the irrational number a_2 enables us to construct two infinite sequences of integers $(s_k)_{k \in \mathbb{N}}$ and $(q_k)_{k \in \mathbb{N}}$ such that for all k in $\mathbb{N}$, $q_k \geq 1$, q_k and s_k are relatively prime and

$$\left| a_2 - \frac{s_k}{q_k} \right| \leq \frac{1}{q_k^2} \text{ (e.g. [13]).}$$

14

On the other hand, it follows from Lemmas 7.2: $\forall N \in \mathbb{N}^*$,

$$D_N \leq \left(\frac{1}{N^2} \sup_{h \in \{1,\ldots,[\sqrt{N}]\}} \left| \sum_{n=1}^{N} e^{2\pi i h x_n} \right|^2 + \frac{6}{\pi^2} \frac{1}{[\sqrt{N}]} \right)^{1/3}$$

Combining both with Lemma 7.1, we get: $\exists C > 0$, $\forall N \in \mathbb{N}^*$, $\forall k \in \mathbb{N}$,

$$D_N \leq C \left(\left(\frac{1}{N} + \frac{q_k}{N^2} + \frac{[\sqrt{N}]}{N} + \frac{[\sqrt{N}]}{q_k} \right) \max\{1, \ln q_k\} + \frac{1}{[\sqrt{N}]} \right)^{1/3} . \tag{7.9}$$

As the number a_2 is irrationnal,

$$\lim_{k \to +\infty} q_k = +\infty \ ,$$

and therefore $\max\{1, \log q_k\} = \log q_k$ for all k great enough. The result follows then by choosing the subsequence $(N_k)_{k \in \mathbb{N}}$ such as: $N_k = q_k - 1$ (with the convention $\ln 0 = -\infty$).
$\square$

8   An integral formula

This last section is devoted to the proof of an integral formula for the Floquet operators of autonomous systems periodically perturbed by a rank-one kick. As a corollary, we derive an alternative proof of Theorem 3.1.

For $a > 0$, we denote the one-dimensional torus $\mathbb{R}/a\mathbb{Z}$ by $\mathbb{T}_a$ and its Borel σ-algebra by $\mathcal{B}(\mathbb{T}_a)$. In particular, $\mathbb{T} = \mathbb{T}_{2\pi}$.

Proposition 8.1 *Let us consider a family of unitary operators $(V_{\kappa T})_{T>0, \kappa \in \mathbb{R}}$ defined by relation (2.1), where the vector ϕ is chosen normed. For any $T > 0$ and any $\kappa \in \mathbb{R}$ the spectral family of $V_{\kappa T}$ is denoted by $(E_{\kappa T}(I))_{I \in \mathcal{B}(\mathbb{T})}$. Then, the measure $\mu_{\kappa T}$ defined by: $\forall I \in \mathcal{B}(\mathbb{T})$,*

$$\mu_{\kappa T}(I) = \langle \phi, E_{\kappa T}(I)\phi \rangle$$

is a probability measure such that: for all f in $L^1(\mathbb{T}, dx)$, the function

$$\kappa \longrightarrow \int_{\mathbb{T}} f(x) d\mu_{\kappa T}(x)$$

belongs to $L^1(\mathbb{T}_{\frac{2\pi}{T}}, d\kappa)$ and,

$$\int_{\mathbb{T}_{\frac{2\pi}{T}}} \int_{\mathbb{T}} f(x) d\mu_{\kappa T}(x) d\kappa = \int_{\mathbb{T}} f(x) \frac{dx}{T} \ .$$

Once this formula established, Theorem 3.1 may be proved rapidly. Namely,

Proof of Theorem 3.1: Let $\kappa \in \mathbb{R} \setminus \mathcal{Z}_T$. From Birman-Krein's theorem we know that the spectrum of $V_{\kappa T}$ is purely singular. Since the vector ϕ is cyclic w.r.t. $e^{-iH_0 T}$, it is also

15

true for the measure $\mu_{\kappa T}$. Now, again by the cyclicity of the unit vector ϕ, it is enough to prove that the singular continuous part of the measure $\mu_{\kappa T}$, denoted $\mu_{\kappa T, sc}$, vanishes on the torus for almost every value of κ. Combescure proved ([4] Proposition 1) that the set $\mathcal{E}$, defined by

$$\mathcal{E} \equiv \{x \in \mathbb{T}; B(x) = 0\}$$

contains the support of the measure $\mu_{\kappa T, sc}$ (i.e. $\mu_{\kappa T, sc}(\mathbb{T} \setminus \mathcal{E}) = 0$). On the other hand, if $\chi_{\mathcal{E}}$ denotes the characteristic function of the set $\mathcal{E}$, it results from Proposition 8.1 that:

$$\int_{\mathbb{T}_{\frac{2\pi}{T}}} \mu_{\kappa T}(\mathcal{E}) d\kappa = \int_{\mathbb{T}_{\frac{2\pi}{T}}} \int_{\mathbb{T}} \chi_{\mathcal{E}}(\theta) d\mu_{\kappa T}(\theta) d\kappa = \int_{\mathbb{T}} \chi_{\mathcal{E}}(x) \frac{dx}{T} \ .$$

Since we know the quantity $B(x)^{-1}$ is finite for almost every x with respect to the Lebesgue measure ([4], Lemma 3) the set $\mathcal{E}$ is of Lebesgue measure 0 and the last term of the preceding equality vanishes. Therefore,

$$\int_{\mathbb{T}_{\frac{T}{2\pi}}} \mu_{\kappa T}(\mathcal{E}) d\kappa = 0 \ .$$

and $\mu_{\kappa T}(\mathcal{E}) = 0$, κ -a.e. with respect to the Lebesgue measure. The same result holds for $\mu_{\kappa T, sc}(\mathcal{E})$ and the conclusion follows: $\mu_{\kappa T, sc}(\mathbb{T}) = \mu_{\kappa T, sc}(\mathbb{T} \setminus \mathcal{E}) + \mu_{\kappa T, sc}(\mathcal{E}) = 0$, κ-a.e. $\square$

Now, let us return to the proof of Proposition 8.1. As a preliminary, we state the following lemma:

Lemma 8.1 *Under the hypotheses of Proposition 8.1, we get for all $n \in \mathbb{Z}$,*

$$\int_{\mathbb{T}_{\frac{2\pi}{T}}} \langle \phi, V_{\kappa T}^n \phi \rangle d\kappa = \frac{2\pi}{T} \delta_{n0} \ ,$$

where δ_{n0} denotes the Kronecker symbol.

Proof: If $n = 0$, the formula is immediate since ϕ is normed. Assume now that $n \neq 0$. Since,

$$\int_{\mathbb{T}_{\frac{2\pi}{T}}} \langle \phi, V_{\kappa T}^{-n} \phi \rangle d\kappa = \overline{\left\{ \int_{\mathbb{T}_{\frac{2\pi}{T}}} \langle \phi, V_{\kappa T}^n \phi \rangle d\kappa \right\}}$$

it is enough to check the equality for the positive values of n. For these values, $\langle \phi, V_{\kappa T}^n \phi \rangle$ is a polynomial of degree n at most and valuation 1 in the variable $e^{-i\kappa T}$. Let us justify it by induction on n ($n \geq 1$). If $n = 1$, we get,

$$\langle \phi, V_{\kappa T} \phi \rangle = \langle \phi, e^{-iH_0 T} e^{-i\kappa T |\phi\rangle\langle\phi|} \phi \rangle = e^{-i\kappa T} \langle \phi, e^{-iH_0 T} \phi \rangle \ .$$

Now, assume this property was proven until the n-th step. Using relation (2.1), a rapid computation by induction shows that: $\forall n \in \mathbb{N}$,

$$\langle \phi, V_{\kappa T}^{n+1} \phi \rangle = e^{-i\kappa T} \sum_{k=0}^{n} \langle \phi, e^{-i(n+1-k)H_0 T} \phi \rangle \langle \phi, V_{\kappa T}^k \phi \rangle - \sum_{k=1}^{n} \langle \phi, e^{-i(n+1-k)H_0 T} \phi \rangle \langle \phi, V_{\kappa T}^k \phi \rangle \ .$$

16

Its combination with the induction hypothesis proves the property for $\langle \phi, V_{\kappa T}^{n+1} \phi \rangle$. $\qquad\square$

Proof of Proposition 8.1: Let $(P_r)_{0 \leq r < 1}$ be the family of Poisson kernels defined by: $\forall x \in \mathbb{T}$,

$$P_r(x) = \sum_{n \in \mathbb{Z}} r^{|n|} e^{inx} = \frac{1 - r^2}{1 + r^2 - 2r \cos x} \quad .$$

Since for all f in $L^1(\mathbb{T}, dx)$ (e.g. Corollary of Theorem 11.12 in [15]),

$$f(\theta) = \lim_{r \to 1^-} \int_{\mathbb{T}} P_r(\theta - t) f(t) \frac{dt}{2\pi} \quad ,$$

it is enough to prove the formula for the family of function $(P_r(\theta - \cdot))_{0 \leq r < 1, \theta \in \mathbb{T}}$. For any $r \in [0, 1[$ and any $\theta \in \mathbb{T}$,

$$\int_{\mathbb{T}} P_r(\theta - x) \frac{dx}{2\pi} = 1 \quad .$$

On the other hand,

$$
\begin{aligned}
\int_{\mathbb{T}_{\frac{2\pi}{T}}} \int_{\mathbb{T}} P_r(\theta - x) d\mu_{\kappa T}(x) d\kappa &= \sum_{n \in \mathbb{Z}} r^{|n|} \int_{\mathbb{T}_{\frac{2\pi}{T}}} \int_{\mathbb{T}} e^{in(x-\theta)} d\mu_{\kappa T}(x) d\kappa \\
&= \sum_{n \in \mathbb{Z}} r^{|n|} \int_{\mathbb{T}_{\frac{2\pi}{T}}} e^{-in\theta} \langle \phi, V_{\kappa T}^n \phi \rangle d\kappa \quad .
\end{aligned}
$$

Now, using Lemma 8.1, the conclusion is immediate:

$$\int_{\mathbb{T}_{\frac{2\pi}{T}}} \int_{\mathbb{T}} P_r(\theta - x) d\mu_{\kappa T}(x) d\kappa = \frac{2\pi}{T} = \int_{\mathbb{T}} P_r(\theta - x) \frac{dx}{T} \quad .$$

$\qquad\square$

Acknowledgements

References

[1] V.Enβ, K. Veselic, *Bound states and propagating states for time-dependent Hamiltonians*, Ann. Inst. H. Poincaré **39**, 159-191 (1986).

[2] O. Bourget, *Singular Continuous Floquet Operators for Systems with increasing gaps*, J. Math. Anal. Appl., **276**, 28-39 (2002). *Corrigendum to: Singular Continuous Floquet Operators for Systems with increasing gaps*, J. Math. Anal. Appl., **289**, 722-723 (2004).

[3] B. Simon, T. Wolff, *Singular continuous spectrum under rank one perturbations and localization for random Hamiltonians*, Comm. Pure Appl. Math., **vol. 39**, 75-90 (1986).

[4] M. Combescure, *Spectral properties of a periodically kicked quantum Hamiltonian*, J. Stat. Phys., **vol. 59**, 679-690 (1990).

[5] B. Simon, *Spectral Analysis of Rank One Perturbations and Applications*, Lecture given at the Vancouver Summer School in Mathematical Physics, August 10-14, 1993.

[6] M. Reed, B. Simon, *Methods of Modern Mathematical Physics IV*, Academic Press, 1978.

[7] T. Kato, S.T. Kuroda, *Theory of simple scattering and eigenfunction expansions*, in *Functional analysis and related fields*, F.Browder Ed., Springer Verlag, 1970.

[8] J. Howland, *Stability of quantum oscillators*, J. Phys. A,**25**, 5177-5181 (1992).

[9] L. Kuipers, H. Niederreiter, *Uniform distribution of sequences*, Wiley Inter-science, 1974.

[10] R.C. Baker, *Slowly growing sequences and discrepancy modulo one*, Acta Arithm., **23**, 279-293 (1973).

[11] P. Erdös, J.F. Koksma, *On the Uniform Distribution modulo 1 of lacunary sequences*, Indag. Math., **Vol. 11**, 79-88 (1949). *On the Uniform Distribution modulo 1 of sequences $f(n, \theta)$*, Indag. Math., **Vol. 11**, 299-302 (1949).

[12] M. B. Nathanson, *Additive Number theory: the classical bases*, Springer.

[13] A.Y. Khinchin, *Continued Fractions*, Phoenix books, 1964.

[14] M.S. Birman, M.G. Krein, *On the theory of Wave Operators and Scattering theory*, Soviet. Math. Dokl.,**vol. 3**, 740-744 (1962).

[15] W. Rudin, *Real and Complex analysis*, Mac Graw-Hill, 1976.