Wandering domains and random walks in Gevrey near-integrable systems

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Abstract. We construct examples of Gevrey non-analytic perturbations of an integrable Hamiltonian system which give rise to an open set of unstable orbits and to a special kind of symbolic dynamics. We find an open ball in the phase space, which is transported by the Hamiltonian flow from $-\infty$ to $+\infty$ along one coordinate axis, at a speed that we estimate with respect to the size of the perturbation. Taking advantage of the hyperbolic features of this unstable system, particularly the splitting of invariant manifolds, we can also embed a random walk along this axis into the near-integrable dynamics.

Contents

1. *Introduction*

1.1. Michael Herman came to see us in October 1999. He had read our joint work with P. Lochak devoted to the study of the splitting of invariant manifolds for near-integrable Hamiltonians [**LMS03**], which was motivated by the question of the optimality of the "stability exponents" in Nekhoroshev's theorem in connection with the speed of Arnold diffusion. He had observed that an analogue of Nekhoroshev's Theorem would hold if "Gevrey class" Hamiltonians were considered instead of analytic ones and that the size of the splitting and the speed at which instability could develop would be easier to evaluate in the Gevrey framework.

He thus proposed to collaborate with us on Nekhoroshev's theory for Gevrey quasi-convex Hamiltonians and started to explain a number of ideas he had already thought of. Several discussions took place in the office of one of us, and we sometimes saw him forget his cane and stand up to improvise an explanation at the blackboard, in order to elucidate a point that seemed obscure to us.

Before we were able to contribute in any significant fashion to the research project, he produced a thick set of notes [**He99**], which he distributed to a few close colleagues and to us. Here was the plan of that manuscript:

(**I**) Examples of instabilities in Hamiltonian systems and their speed
(**N**) Nekhoroshev's estimate for Gevrey classes (in the convex case)
(**A**) Appendix on Gevrey classes.

It contained a first stability result (**N**), analogous to Nekhoroshev's, and a method (**I**) to construct examples of unstable Hamiltonian systems in the Gevrey category, but also in other non-quasianalytic ultradifferentiable classes or in the C^k category. The method of (**I**) consisted in reasoning at the level of exact-symplectic mappings (passing from discrete dynamical systems to Hamiltonian flows at the end by a standard suspension procedure) and coupling the so-called "standard map", suitably rescaled, with a well chosen system possessing a periodic orbit of large period. The first "accelerator mode" of the standard map would yield a wandering point of the total system, drifting from $-\infty$ to $+\infty$ along one coordinate axis, provided the coupling function would be adapted to the periodic orbit. It was essential to have compact-supported functions at one's disposal at that stage. The appendix (**A**) was devoted to technical estimates, which were used, for instance, to correctly choose the parameters, so as to make the Gevrey distance to integrability arbitrarily small (and to compare it to the speed of drift).

Both the stability and the instability results needed to be improved and we worked together to increase the stability time and to lower the drifting time, in view of making them match. The stability time could indeed be characterized by a "stability exponent" a (the action variables would remain confined for $|t| \leq \exp(\mathrm{const}(\frac{1}{\varepsilon})^a)$, where ε measures the distance of the system to integrability), but the first result was not as satisfactory as in the analytic case, where Lochak, Neishtadt and Pöschel had succeeded in obtaining a as large as $\frac{1}{2N}$ for quasi-convex N-degree-of-freedom Hamiltonians. On the other hand, the method for designing examples left a lot of leeway and could probably result in a smaller "instability exponent" a^* than the one obtained in the manuscript [**He99**].

A few months later we obtained better estimates for the stability time. Replacing the strategy of part (**N**) of [**He99**], which relied on the approximation of Gevrey Hamiltonians by analytic ones (as described in (**A**)) and the classical analytic normal form, we completely rewrote the normal form in the Gevrey framework and ended up with a stability exponent $a = \frac{1}{2N\alpha}$, where α denotes the Gevrey index. We could thus recover the analytic result in the particular case where $\alpha = 1$, which exactly corresponds to analytic Hamiltonians. The passage from the Gevrey normal form to the stability result was performed by using Lochak's periodic orbit method; we thus obtained, as in the analytic case, a larger exponent $a = \frac{1}{2(N-m)\alpha}$ for the solutions passing close to any m-fold resonant surface. But the instability exponent remained at least two times larger than that in the examples we had at that time.

Michael Herman gave several seminars on the work in progress and described the results so far obtained at the Rome conference in September 2000. There he told one of us that he maybe had an idea to gain a factor 2 in the exponent for the examples of instability. We should have discussed that matter some weeks later in Paris. . .

We never knew what idea Michael Herman had had. In the months following his death, we went on studying his method of producing unstable systems, trying to make it as conceptual and powerful as possible, and began to think of the writing of an article that would gather all of this together. We finally realized that we could get stronger examples of instability, which established the optimality of the exponents $\frac{1}{2(N-m)\alpha}$ for $2 \le m < N$, by using the periodic orbits of a scaled pendulum in the coupling method. This gave rise to the article [**MS03**], which was completed at the end of 2001 (see [**Sa03**] for a survey of that long paper).

1.2. The present article is a continuation of [**MS03**]: we obtain new instability examples as Gevrey perturbations of the completely integrable Hamiltonian $h(r) = \frac{1}{2}(r_1^2 + \cdots + r_{N-1}^2) + r_N$ on the annulus $\mathbb{T}^N \times \mathbb{R}^N$. In fact, our method could be applied as well to yield the same kind of unstable behaviour in perturbations of

$$h_\epsilon = \frac{1}{2}(\epsilon_1 r_1^2 + \cdots + \epsilon_{N-1} r_{N-1}^2) + r_N,$$

with arbitrary $\epsilon_1, \ldots, \epsilon_{N-1} \in \{-1, +1\}$; the quasi-convexity of h indeed does not play any role in our constructions (but it plays a role in the exponential stability we are fighting against). What matters in our constructions is rather the product structure (the time-1 map Φ^{h_ϵ} can be written as an uncoupled product of N maps of $\mathbb{T} \times \mathbb{R}$).

The first new instability result is the existence of near-integrable systems which possess wandering *open sets*, in place of the wandering points we previously obtained in [**MS03**]. Namely, we shall be able to construct a sequence $(\mathcal{H}_j)_{j\ge 0}$ of Gevrey perturbations of $h(r)$, such that there exist for each $j \ge 0$ an open set O_j satisfying $\left(\Phi^{\mathcal{H}_j}\right)^k(O_j) \cap O_j = \emptyset$ for each integer $k \in \mathbb{Z}$ (where $\Phi^{\mathcal{H}_j}$ denotes the time-1 map for $\mathcal{H}_j$). Indeed, these open sets O_j drift from $-\infty$ to $+\infty$ along the r_1-axis (denoting by r_1 the first of the action variables). Such a phenomenon was obtained in [**He99**], but with C^k systems only, and with the further restriction that $k \leq N-3$. Here we shall reach any value of k and even the Gevrey category. The upshot is an extension to the multidimensional case of the transport phenomenon which, to our knowledge, was observed until now only for two-dimensional maps (see [**Mei92**] for a survey on transport theory). We can furthermore estimate the speed of transport as a function of the size of the perturbation. Just as is the case of drifting points the speed is still given by an exponential, although the corresponding exponent is only $\frac{1}{2(N-m)(\alpha-1)}$, thus no optimality can be claimed here.

Another interesting feature of that construction can be noticed when considering the complete orbit of the open domain O_j under the continuous Hamiltonian flow. This is an open connected invariant set, of positive measure, which is contained in the complement of the KAM set of the perturbed Hamiltonian $\mathcal{H}_j$. The system is therefore non-ergodic in the complement of the KAM set.

Our second result concerns the existence of symbolic dynamics for near-integrable systems. Here we continue to investigate the relations between our method and the so-called Arnold mechanism of instability and construct a new sequence $(\mathcal{H}_j)_{j\geq 0}$ of Gevrey perturbations of h such that the time-one map $\Phi^{\mathcal{H}_j}$ possesses a two-dimensional normally hyperbolic invariant annulus, the stable and unstable manifolds of which intersect along two homoclinic two-dimensional annuli. This (non-generic) phenomenon enables us to obtain explicit examples of a situation described by Moeckel [**Moe02**]. The main consequence is that we can replace the drift along the r_1-axis by a *random walk*: the orbits described in the drift result correspond to a bi-infinite sequence of upward jumps, whereas in the second result any sequence of upward and downward jumps can be realized by some orbit which, so to speak, materializes the symbolic dynamics. Moreover, these jumps occur at moments which are integer multiples of a certain large integer q_j, with a step which is $\pm 1/q_j$ each time, and we can estimate q_j quite precisely in terms of the size of the perturbation (of course it has to be exponentially large). Our coding by symbolic dynamics is thus more precise than in the standard constructions.

This last construction has interesting byproducts concerning the topic of lower bounds for the splitting of invariant manifolds, a subject which was also lightly touched on in [**He99**]. We showed in [**MS03**] that our construction could produce a chain of partially hyperbolic tori with tangent heteroclinic connections from one torus to its successor, and that the homoclinic splitting could be perfectly controlled, at least in the drifting direction. Here we shall go farther: we shall be able to arrange things so as to control all the entries of the splitting matrix. This is because we use, instead of the pendulum of [**MS03**], a perturbed pendulum that possesses transverse homoclinic orbits.

Another byproduct of the random walk construction is the existence of near-integrable Hamiltonian systems on $(\mathbb{T}\times\mathbb{R})^N$, which admit an orbit whose projection onto the first factor $\mathbb{T}\times\mathbb{R}$ is *dense* (in the first angle variable as well as in the first action).

We shall seize the opportunity of this article to indicate Herman's construction of C^k systems with a wandering domain for $k \leq N-3$. Another work should be devoted to the systematic study of finite-time stability in the differentiable category and in ultradifferentiable classes. Here we shall content ourselves with an estimation of the speed of drift in a C^k variant of our unstable system.

We wish to mention that compact-supported functions were already used in the context of Hamiltonian perturbations and Arnold diffusion in [**Do86**] and [**FM01**].

2. *Statement of the main results*

2.1. *Notations.* Let $N \geq 3$. For $R > 0$ we denote by $\overline{B}_R$ the closed ball of radius R in $\mathbb{R}^N$ with center at the origin. As in [**MS03**], we shall work with the Gevrey spaces defined by

$$G^{\alpha,\Lambda}(K) = \{\, \varphi \in C^\infty(K) \mid \|\varphi\|_{\alpha,\Lambda,K} < \infty \,\},$$

with real numbers $\alpha \geq 1$, $\Lambda > 0$, compact sets of the form $K = \mathbb{T}^N \times \overline{B}_R$ and Gevrey norms

$$\|\varphi\|_{\alpha,\Lambda,K} = \sum_{\ell \in \mathbb{N}^{2N}} \frac{\Lambda^{|\ell|\alpha}}{\ell!^\alpha} \|\partial^\ell \varphi\|_{C^0(K)} \tag{1}$$

(we shall sometimes omit K in the indices, when there is no risk of confusion). We have used the following notation for multi-indices of derivation:

$$|\ell| = \ell_1 + \cdots + \ell_{2N}, \quad \ell! = \ell_1! \dots \ell_{2N}!, \quad \partial^\ell = \partial_{x_1}^{\ell_1} \dots \partial_{x_{2N}}^{\ell_{2N}}$$

and $(x_1, \dots, x_{2N}) = (\theta_1, \dots, \theta_N, r_1, \dots, r_N)$. But since our aim is to describe dynamics in non-compact parts of the phase space $\mathbb{T}^N \times \mathbb{R}^N$, we shall require a new definition.

Definition. If $\alpha \geq 1$ and $\Lambda > 0$, we set

$$\mathcal{K}_\nu = \mathbb{T}^N \times \overline{B}_{R_\nu}, \qquad R_\nu = 3^{\nu\alpha}, \qquad \Lambda_\nu = 3^{-\nu+1}\Lambda, \qquad \nu \in \mathbb{N}^*, \tag{2}$$

and we define $\mathcal{G}^{\alpha,\Lambda}(\mathbb{T}^N \times \mathbb{R}^N)$ to be the complete metric space obtained by endowing the intersection $\bigcap_{\nu \geq 1} G^{\alpha,\Lambda_\nu}(\mathcal{K}_\nu)$ with the distance

$$d_{\alpha,\Lambda}(\varphi,\psi) = \sum_{\nu \geq 1} 2^{-\nu} \min(\|\varphi - \psi\|_{\alpha,\Lambda_\nu,\mathcal{K}_\nu}, 1).$$

2.2. *A drift result for an open set.* If H is a Hamiltonian function generating a complete vector field, we shall denote by $\Phi^{\tau H}$ the time-τ map for $\tau \in \mathbb{R}$. We denote by $\mathcal{T}$ the translation of step 1 in the r_1-direction and introduce a transformation $\mathcal{R}$, which preserves r_1 and commutes with $\mathcal{T}$:

$$\begin{aligned} \mathcal{T}(\theta_1, r_1, \theta_2, r_2, \dots, \theta_N, r_N) &= (\theta_1, r_1 + 1, \theta_2, r_2, \dots, \theta_N, r_N), \\ \mathcal{R}(\theta_1, r_1, \theta_2, r_2, \dots, \theta_N, r_N) &= (\theta_1 + \theta_N, r_1, \theta_2, r_2, \dots, \theta_N, r_N). \end{aligned} \tag{3}$$

THEOREM 2.1. *Let $\alpha > 1$, $\Lambda > 0$, $N \geq 3$, $h(r) = \frac{1}{2}(r_1^2 + \cdots + r_{N-1}^2) + r_N$, $m \in \{2, \dots, N-1\}$ and*

$$a^* = \frac{1}{2(N-m)(\alpha-1)}.$$

There exists a sequence $(\mathcal{H}_j)_{j\geq 0}$ of functions converging to h in $\mathcal{G}^{\alpha,\Lambda}(\mathbb{T}^N \times \mathbb{R}^N)$, such that, for each $j \geq 0$, the Hamiltonian system generated by $\mathcal{H}_j$ is complete and admits a wandering open set $\mathcal{U}_j$ that is biasymptotic to infinity.

More precisely, the sets $\Phi^{\ell\mathcal{H}_j}(\mathcal{U}_j)$, $\ell \in \mathbb{Z}$, are mutually disjoint and there exists a positive integer τ_j such that

$$\Phi^{\ell\tau_j\mathcal{H}_j}(\mathcal{U}_j) = \mathcal{T}^\ell\mathcal{R}^\ell(\mathcal{U}_j), \qquad \ell \in \mathbb{Z}. \tag{4}$$

The time τ_j required to translate by 1 the r_1-projection of $\mathcal{U}_j$ is related to $\varepsilon_j = d_{\alpha,\Lambda}(h, \mathcal{H}_j)$ by inequalities of the form

$$\frac{C_1}{\varepsilon_j^2}\exp(C_1\Big(\frac{1}{\varepsilon_j}\Big)^{a^*}) \leq \tau_j \leq \frac{C_2}{\varepsilon_j^2}\exp(C_2\Big(\frac{1}{\varepsilon_j}\Big)^{a^*}), \qquad j \geq 0,$$

where the positive constants $C_1 < C_2$ depend only on α, Λ and $N - m$.

The domain $\mathcal{U}_j$ is located close to the m-fold resonant surface $\mathcal{S} = \{ r_1 = r_2 = \cdots = r_m = 0 \}$:

$$\operatorname{dist}(x, \mathcal{S}) \leq 3\sqrt{\varepsilon_j}, \qquad x \in \mathcal{U}_j.$$

The proof of Theorem 2.1 is contained in Sections 3–5.

Observe that the role of the transformation $\mathcal{R}$ is merely to describe the slight deformation undergone by the domain under the time-τ_j map of the Hamiltonian, whereas the most important feature of the dynamics is the drift described by the translation $\mathcal{T}$.

Of course, the novelty with respect to [**MS03**] is mainly the obtention of wandering domains instead of wandering points. Still, we have tried to estimate the speed of drift through an "instability exponent" a^* like we did in [**MS03**] and, if the factor $\frac{1}{N-m}$ reflects well the proximity of a resonance of multiplicity m, the factor $\frac{1}{\alpha-1}$ is not so satisfactory. We recall indeed that the optimal exponent is $\frac{1}{2(N-m)\alpha}$ for individual solutions and we do not know whether it can be attained with open sets; see Remark 4.2 below.

It is for the sake of clarity and to facilitate the comparison with the classical Nekhoroshev Theorem that we have formulated our drift result using autonomous Hamiltonians only, but the Hamiltonians $\mathcal{H}_j$ can be reduced to non-autonomous time-periodic Hamiltonians

$$H_j = \frac{1}{2}(r_1^2 + \cdots + r_{N-1}^2) + f_j(\theta_1, \dots, \theta_{N-1}, r_1, \dots, r_{N-1}, t),$$

(see formula (43) below), which are themselves obtained by "suspending" discrete dynamical systems $\underline{\Psi}_j$ possessing wandering domains $\underline{\mathcal{D}}_j$ in $\mathbb{T}^{N-1} \times \mathbb{R}^{N-1}$ (see Section 5). The sets $\underline{\mathcal{D}}_j$ will be polydiscs $\mathcal{A}_1 \times \cdots \times \mathcal{A}_{N-1}$, the Shilov boundaries of which are Lagrangian tori. Choosing functions f_j that are bounded on $\mathbb{T}^{N-1} \times \mathbb{R}^{N-1} \times \mathbb{T}$ and that vanish identically for $t \in [0, \frac{1}{4}]$, we shall be able to take

$$\mathcal{U}_j = \{(\hat{\theta} + \theta_N\hat{r}, \hat{r}, \theta_N, r_N);\ (\hat{\theta}, \hat{r}) \in \underline{\mathcal{D}}_j,\ \theta_N \in [0, \tfrac{1}{4}],\ r_N \in \mathbb{R}\}. \tag{5}$$

We shall also mention in passing results concerning the C^k category, using the semi-norms

$$\|\varphi\|_{C^k(K)} = \sum_{|\ell|\le k} \frac{1}{\ell!}\|\partial^\ell \varphi\|_{C^0(K)}, \tag{6}$$

with the notation of (1) above (this way, $\|\varphi\psi\|_{C^k(K)} \le \|\varphi\|_{C^k(K)}\|\psi\|_{C^k(K)}$), and the corresponding distance when dealing with C^k functions on a non-compact set.

2.3. *Embedding of a random walk.* Leaving the question of the transport of open sets aside, we shall be able to find near-integrable systems exhibiting instability in another striking way.

Let π_1 denote the projection onto the r_1-axis:

$$\pi_1(\theta_1, r_1, \theta_2, r_2, \dots, \theta_N, r_N) = r_1.$$

THEOREM 2.2. *Let $\alpha > 1$, $\Lambda > 0$, $N \ge 3$ and $h(r) = \frac{1}{2}(r_1^2 + \cdots + r_{N-1}^2 + r_N)$. There exist a sequence $(\mathcal{H}_j)_{j\ge 0}$ of functions converging to h in $\mathcal{G}^{\alpha,\Lambda}(\mathbb{R}^N \times \mathbb{T}^N)$ and a sequence (q_j) of positive integers such that, for each $j \ge 0$, the Hamiltonian system generated by $\mathcal{H}_j$ is complete and its time-q_j map contains the random walk of step $\frac{1}{q_j}$ along the r_1-axis in the following sense:*

For each $\kappa \in \{-1, +1\}^{\mathbb{Z}}$, there exists $x \in \mathbb{T}^N \times \mathbb{R}^N$ such that

$$\pi_1\big(\Phi^{\ell q_j \mathcal{H}_j}(x)\big) = \pi_1\big(\Phi^{(\ell-1) q_j \mathcal{H}_j}(x)\big) + \frac{\kappa_\ell}{q_j}, \qquad \ell \in \mathbb{Z}.$$

Moreover, q_j is related to $\varepsilon_j = d_{\alpha,\Lambda}(h, \mathcal{H}_j)$ by inequalities of the form

$$\frac{C_1}{\varepsilon_j^2}\exp(C_1\Big(\frac{1}{\varepsilon_j}\Big)^{a^*}) \le q_j \le \frac{C_2}{\varepsilon_j^2}\exp(C_2\Big(\frac{1}{\varepsilon_j}\Big)^{a^*}), \qquad j \ge 0,$$

where $a^* = \frac{1}{2(N-2)(\alpha-1)}$ and the positive constants $C_1 < C_2$ depend only on α, Λ and N.

The proof is given in Section 6.3.

In fact, we shall see that the time-q_j map admits as a subsystem the random walk, defined as usual as a skew-product over the two-sided Bernoulli shif $\flat$:

$$P(r_1, \kappa) = (r_1 + \frac{\kappa_1}{q_j}, \flat(\kappa)), \qquad r_1 \in \frac{1}{q_j}\mathbb{Z}, \quad \kappa \in \{-1, +1\}^{\mathbb{Z}}.$$

In that result, we can think of the addition of $\pm\frac{1}{q_j}$ as small upward or downward jumps, a bi-infinite sequence of which can be realized in any prescribed order: one can always find solutions which oscillate in the prescribed manner exactly at instants that are multiples of q_j. Theorem 2.1 corresponds to the case of the constant sequence $\kappa_\ell = +1$ with $\tau_j = q_j^2$, except that Theorem 2.2 does not deal with open sets of solutions.

Still, one can improve slightly the conclusion of Theorem 2.2, replacing the initial condition x by a set $\mathcal{V}_j$ that is defined as in (5) but with a $(N-2)$-polydisc $\mathcal{B}_1 \times \{x_2\} \times \mathcal{B}_3 \times \cdots \times \mathcal{B}_{N-1}$ instead of the $(N-1)$-polydisc $\underline{\mathcal{D}}_j = \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_{N-1}$. This yields oscillating submanifolds $\mathcal{V}_j$ of codimension 2.

In addition to the proof of Theorem 2.2 (which corresponds to a particular case of the situation considered in [**Moe02**]), Section 6 also contains a study of the splitting of the invariant manifolds associated with partially hyperbolic circles and annuli, in the spirit of [**LMS03**] and [**MS03**]. These hyperbolic objects are the traditional features of Arnold's mechanism.

Moreover, we indicate in Remark 6.2 a variant of the construction yielding an orbit of $\Phi^{q_j \mathcal{H}_j}$ in $(\mathbb{T}\times\mathbb{R})^N$ whose projection onto the first factor $\mathbb{T}\times\mathbb{R}$ is dense.

2.4. *Overview of the method.* For the convenience of the reader, we include a heuristic description of our constructions.

Since we shall follow closely [**MS03**], it is worth recalling the main features of the method which was introduced there to obtain drifting points. This method deals with discrete dynamical systems of the annulus $\mathbb{A}^n = \mathbb{T}^n \times \mathbb{R}^n$ which are obtained as perturbations of $\Phi^{h_0} : (\theta, r) \mapsto (\theta + r, r)$. Here $n = N - 1$, $h_0(r) = \frac{1}{2}(r_1^2 + \cdots + r_n^2)$, and a suspension procedure is used later to recover continuous Hamiltonian systems in N degrees of freedom. We split the annulus $\mathbb{A}^n$ and the unperturbed map into two factors: $\Phi^{h_0} = \Phi^{\frac{1}{2}r_1^2} \times \Phi^{\frac{1}{2}r_2^2+\cdots+\frac{1}{2}r_n^2} : \mathbb{A}\times\mathbb{A}^{n-1} \to \mathbb{A}\times\mathbb{A}^{n-1}$. Our main parameter is a large integer q. The first idea is to consider a q-periodic point $a \in \mathbb{A}^{n-1}$ on the second factor and to try to define a "coupling diffeomorphism" Φ^{u_q} on the product $\mathbb{A}\times\mathbb{A}^{n-1}$ so that $\Phi^{u_q}\circ\Phi^{h_0}$ have a wandering point which drifts along the r_1-axis, with the further requirement that $u_q \to 0$ when $q \to \infty$ (in a suitable Gevrey function space).

On the first factor $\mathbb{A}$, the interesting part of the dynamics is localized on the union $\mathbf{C}_q$ of the circles $C_{k/q} = \{(\theta_1, r_1) \in \mathbb{A},\ r_1 = k/q\}$, $k \in \mathbb{Z}$. Each of these circles is invariant under $\Phi^{\frac{1}{2}r_1^2}$ and supports q-periodic dynamics, even if q is not the minimal period. On the second factor we only consider the orbit $O(a) = \{a_{(s)},\ 0 \le s \le q-1\}$ of $a = a_{(0)}$ under $\Phi^{\frac{1}{2}r_2^2+\cdots+\frac{1}{2}r_n^2}$. The coupling diffeomorphism Φ^{u_q} is chosen so as to satisfy the "synchronization conditions"

$$\Phi^{u_q}((0,r_1),a) = ((0, r_1 + 1/q), a),$$
$$\Phi^{u_q}((\theta_1,r_1),a_{(s)}) = ((\theta_1, r_1), a_{(s)}), \qquad 1 \le s \le q-1,$$

for all $(\theta_1, r_1) \in \mathbb{A}$. Due to the q-periodicity of Φ^{h_0} on $\mathbf{C}_q \times O(a)$, one sees that the point $((0,0),a)$ is wandering for $\Phi^{u_q}\circ\Phi^{h_0}$, and satisfies in particular

$$[\Phi^{u_q}\circ\Phi^{h_0}]^{\ell q}((0,0),a) = ((0,\ell/q),a), \qquad \ell \in \mathbb{Z},$$

while the first components of the other iterates move around the circles of the family $\mathbf{C}_q$. So q^2 iterations of the coupled diffeomorphism $\Phi^{u_q}\circ\Phi^{h_0}$ make the point $((0,0),a)$ drift along the r_1-axis over an interval of length 1. This is all we need in order to estimate our instability time.

As is easily checked, a simple way to get the synchronization is to choose u_q of the form $u_q((\theta_1,r_1),x') = \frac{1}{q}U(\theta_1)\,g^{(q)}(x')$ for $((\theta_1,r_1),x') \in \mathbb{A}\times\mathbb{A}^{n-1}$, where $U'(0) = -1$ and

$$g^{(q)}(a) = 1, \qquad dg^{(q)}(a) = 0,$$
$$g^{(q)}(a_{(s)}) = 0, \quad dg^{(q)}(a_{(s)}) = 0, \qquad 1 \le s \le q-1.$$

The size of the function u_q is seen to be of the order of $\|g^{(q)}\|/q$. The main difficulty of the construction is to ensure the condition $\varepsilon = \|u_q\| \to 0$ as $q \to \infty$: by compactness the distance between the initial point a and its nearest iterate tends to 0 as $q \to \infty$, and the values of $g^{(q)}$ on a and this iterate differ by 1; so any Gevrey norm of $g^{(q)}$ will tend to ∞ when $q \to \infty$.

One can convince oneself that the construction is not possible with the second factor kept equal to $\Phi^{\frac{1}{2}(r_2^2+\cdots+r_n^2)}$: the periodic points are equidistributed on periodic tori, and the distance between two of them is just too short. For this reason we add a perturbation to the initial Hamiltonian h_0, splitting the dynamics on the second factor $\mathbb{A}^{n-1} = \mathbb{A} \times \mathbb{A}^{n-2}$ into two parts: the first one is the time-1 map of a pendulum suitably rescaled, $\Phi^{\frac{1}{2}r_2^2 - \frac{1}{N^2}\cos(2\pi\theta_2)}$, with a new large parameter N, and the second part is still the integrable twist map $\Phi^{\frac{1}{2}(r_3^2+\cdots+r_n^2)}$ on $\mathbb{A}^{n-2}$. The main property of this system is that, due to the presence of the pendulum and its separatrix, one can find q-periodic points $a = a^{(q)}$, with arbitrarily large q, whose distance to the rest of their orbit is of the order of $1/N$. The introduction of such a pendulum component in [**MS03**] was one of the main innovations with respect to [**He99**]. In the present article too this will be crucial.

When applied to this system, the previous method leads to a function $g^{(q)}$ whose Gevrey-α norm is exponentially large with respect to N, but this can be compensated by choosing the parameter q large enough, namely $q = O(\exp(\mathrm{const}\, N^{1/2\alpha(n-1)}))$. This way we keep the Gevrey norms of u_q and $v_N = -\frac{1}{N^2}\cos(2\pi\theta_2)$ of the same order $\varepsilon = 1/N^2$ and we obtain the connection between the instability time τ (for a drift of order one) and the size ε of the perturbations u_q and v_N:

$$\tau = q^2 = O\big(\exp\big(\mathrm{const}\left(\frac{1}{\varepsilon}\right)^{\frac{1}{2\alpha(n-1)}}\big)\big).$$

We now come to the necessary modifications from [**MS03**] to the present article, in order to obtain drifting open sets instead of drifting points. The global structure of our new examples will be very close to that we have just described. In particular, we shall still split the annulus $\mathbb{A}^n$ into the same three factors, and the drift will occur along the action axis of the first annulus. Loosely speaking, we shall try to keep the same drifting point as above but modify the various functions in order to produce nondegenerate elliptic dynamics in the neighborhood of its projections on the three factors. We shall then apply Moser's invariant curve theorem and obtain in each factor an open set centred on the projection, having the same global behaviour. The product of these open sets will give us our drifting domain.

As for the random walk, the main modification concerns the pendulum factor, on which we shall use a suitable perturbation of the pendulum map, for which both upper and lower separatrices intersect transversely. We shall get two homoclinic points, and prove the existence of a horseshoe on which the map is conjugate to a Bernouilli shift on two symbols, one for the upper point and one for the lower point. These two symbols will correspond to two distinct zones in the pendulum space, and a suitable version of the coupling lemma will enable us to generate positive or negative jumps of the orbits along the first action axis, in any prescribed order.

Since these jumps occur at prescribed instants and have prescribed length, we can think of the motion on the first action axis as a random walk. The most technical work in that part will be to estimate the number of iterates which is necessary to get the symbolic dynamics.

3. *A method for constructing unstable mappings*

3.1. *Discrete version of the Gevrey unstable system.* Let $\mathbb{A} = \mathbb{T} \times \mathbb{R}$ denote the annulus. As already mentioned, we shall work with mappings rather than with flows. Theorem 2.1 will follow from the construction of near-integrable discrete dynamical systems in $\mathbb{A}^n \approx T^*\mathbb{T}^n = \mathbb{T}^n \times \mathbb{R}^n$, with $n \geq 2$. These mappings Ψ_j will be compositions of three time-1 maps of Hamiltonian systems and will admit nearly doubly resonant wandering domains $\mathcal{D}_j$.

The case $m = 2$ will be obtained by "suspension" of Proposition 3.1 below (see Section 5.2), whereas in the case $m \geq 3$ we shall insert $m-2$ extra degrees of freedom before the suspension procedure, obtaining the intermediate system $\underline{\Psi}_j$ (see Section 5.1). The relation between dimensions in Theorem 2.1 and Proposition 3.1 is thus $N = n + (m-2) + 1$.

We shall use systematically the notation Φ^H introduced before the statement of Theorem 2.1. For instance, $h_0 = \frac{1}{2}(r_1^2 + \cdots + r_n^2)$ gives rise to the standard twist map of $\mathbb{A}^n$,

$$\Phi^{h_0}(\theta, r) = (\theta + r, r),$$

whereas a function $u(\theta)$ which does not depend on the action variables r_i yields $\Phi^u(\theta, r) = (\theta, r - \nabla u(\theta))$. For each $\delta > 0$, we shall use the notation T_δ for the translation of step δ in the r_1-direction:

$$T_\delta(\theta_1, r_1, \theta_2, r_2, \ldots, \theta_n, r_n) = (\theta_1, r_1 + \delta, \theta_2, r_2, \ldots, \theta_n, r_n). \tag{7}$$

PROPOSITION 3.1. *Let $\alpha > 1$, $L > 0$, $n \geq 3$ and*

$$a = \frac{1}{2(n-1)(\alpha-1)}.$$

There exist sequences (u_j), (v_j), (w_j) of smooth functions, which have compact supports contained in $(\mathbb{T} \times [0,3])^n$ and belong to $G^{\alpha,L}\big((\mathbb{T} \times [0,3])^n\big)$, and a sequence (q_j) of integers such that

$$\varepsilon_j = \max(\|u_j\|_{\alpha,L}, \|v_j\|_{\alpha,L}, \|w_j\|_{\alpha,L}) \xrightarrow[j\to\infty]{} 0, \qquad q_j \xrightarrow[j\to\infty]{} \infty$$

and, for each j, the system

$$\Psi_j = \Phi^{u_j} \circ \Phi^{\frac{1}{2}(r_1^2+\cdots+r_n^2)+v_j} \circ \Phi^{w_j}$$

admits a wandering open set $\mathcal{D}_j$ such that

$$(\Psi_j)^{\ell q_j}(\mathcal{D}_j) = T_{\frac{\ell}{q_j}}(\mathcal{D}_j), \qquad \ell \in \mathbb{Z}. \tag{8}$$

The number $\tau_j = q_j^2$ of iterates required to translate by 1 its r_1-projection satisfies inequalities of the form

$$\frac{C_1}{\varepsilon_j^2}\, \exp\Big(C_1\Big(\frac{1}{\varepsilon_j}\Big)^a\Big) \leq \tau_j \leq \frac{C_2}{\varepsilon_j^2}\, \exp\Big(C_2\Big(\frac{1}{\varepsilon_j}\Big)^a\Big), \tag{9}$$

where C_1 and C_2 are positive numbers which do not depend on j.

Moreover, $\operatorname{dist}(x, \{\, r_1 = r_2 = 0 \,\}) \leq 3\sqrt{\varepsilon_j}$ *for all* $x \in \mathcal{D}_j$*.*

The proof of this proposition will occupy us until the end of Section 4.

In fact, we shall have $v_j = \frac{1}{N_j^2} V(\theta_2)$, with $V(\theta_2) = -1 - \cos(2\pi\theta_2)$ and integers N_j directly related to ε_j:

$$\|u_j\|_{\alpha,L}, \|w_j\|_{\alpha,L} \leq \varepsilon_j = \frac{1}{N_j^2} \|V\|_{\alpha,L}.$$

The functions w_j will be sums of non-interacting potentials of the form

$$w_j = w_2^{(j)}(\theta_2, r_2) + w_3^{(j)}(\theta_3) + \cdots + w_n^{(j)}(\theta_n)$$

(and the functions u_j will not depend on the actions r). Our unstable mappings may thus be written $\Psi_j = \Phi^{u_j} \circ \big(\Phi^{\frac{1}{2} r_1^2} \times G\big)$, with

$$G = \big(\Phi^{\frac{1}{2} r_2^2 + v_j} \circ \Phi^{w_2^{(j)}}\big) \times \big(\Phi^{\frac{1}{2} r_3^2} \circ \Phi^{w_3^{(j)}}\big) \times \cdots \times \big(\Phi^{\frac{1}{2} r_n^2} \circ \Phi^{w_n^{(j)}}\big).$$

Correspondingly, our wandering domain $\mathcal{D}_j$ will be a polydisc $\mathcal{B}_{q_j} \times \mathcal{A}_2^{(j)} \times \cdots \times \mathcal{A}_n^{(j)}$, the Shilov boundary of which is the Lagrangian torus $\partial\mathcal{B}_{q_j} \times \partial\mathcal{A}_2^{(j)} \times \cdots \times \partial\mathcal{A}_n^{(j)}$.

There will also be a preliminary C^k version of this proposition (Proposition 3.3 in Section 3.4), directly inspired by [**He99**], which may be used as an introduction to Section 4, and a more elaborate result in Section 5.3 which is a more exact analogue of Proposition 3.1 and Theorem 2.1 in finite differentiability.

3.2. *Wandering domains for standard maps.* As in [**MS03**], our starting point is the knowledge of unstable orbits for a certain map of $\mathbb{A}$, which is not close to integrable (it would have more to do with an anti-integrable limit) but which we shall be able to embed into some iterate of a near-integrable map of $\mathbb{A}^n$.

Given a smooth function U on $\mathbb{T}$ and a positive integer q, we define the "standard map"

$$\psi_{q,U} = \Phi^{\frac{1}{q} U} \circ \big(\Phi^{\frac{1}{2} r_1^2}\big)^q \,:\, \mathbb{A} \to \mathbb{A}.$$

When there is no risk of confusion, we shall sometimes omit the index U or even both indices and denote by ψ_q or ψ this map. Thus

$$\psi(\theta_1, r_1) = \big(\theta_1 + q r_1, r_1 - \tfrac{1}{q} U'(\theta_1 + q r_1)\big).$$

As noticed in [**MS03**], if $U'(0) = -1$, the origin is a drifting point:

$$\psi^\ell(0,0) = \Big(0, \frac{\ell}{q}\Big), \qquad \ell \in \mathbb{Z}.$$

This point goes from $r_1 = 0$ to $r_1 = 1$ in q iterations and its orbit is biasymptotic to infinity (this is known as the "first accelerator mode of the standard map" in the literature).

The function $U(\theta_1) = -\frac{1}{2\pi} \sin(2\pi\theta_1)$ was used in [**MS03**], but we shall be led to choose a different function by the next proposition, which is suggested by [**He99**]:

Proposition 3.2. *Consider the map $\psi_{q,U}$ with an integer $q \geq 1$ and a function $U \in C^\infty(\mathbb{T})$ the derivative of which admits a Taylor expansion at the origin of the form*

$$U'(\theta_1) = -1 + \beta_1\theta_1 + \beta_3\theta_1^3 + O(\theta_1^4), \qquad 0 < \beta_1 < 2,\ \beta_3 \neq 0. \tag{10}$$

Then there exists a neighbourhood $\mathcal{B}_*$ of $(0,0)$ in $\mathbb{A}$ (which depends only on the function U'), contained in the region $\{\,|r_*| \leq \frac{1}{4}\,\}$, such that the domain

$$\mathcal{B}_q = \big\{\big(\theta_*, \frac{r_*}{q}\big)\big\}_{(\theta_*,r_*)\in\mathcal{B}_*} \subset \mathbb{A}$$

is wandering for $\psi_{q,U}$. More precisely, the iterates of $\mathcal{B}_q$ by $\psi_{q,U}$ are mutually disjoint and can be obtained from $\mathcal{B}_q$ by translation of step $\frac{1}{q}$ in the r_1-direction:

$$(\psi_{q,U})^\ell(\mathcal{B}_q) = \{(\theta_1, \tfrac{\ell}{q} + r_1)\}_{(\theta_1,r_1)\in\mathcal{B}_q}, \qquad \ell \in \mathbb{Z}.$$

The proposition follows from the following technical lemma, whose greater generality will be used in Section 4.3 and whose proof is deferred to the appendix.

Lemma 3.1. *Consider the mapping $f : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ defined by the formula*

$$f(X,Y) = (X + A'(Y - B'(X)), Y - B'(X)),$$

that is $f = \Phi^A \circ \Phi^B$, where $A(Y)$ and $B(X)$ are smooth functions defined in real intervals containing 0 the derivatives of which admit Taylor expansions of the form

$$A'(Y) = A_1Y + A_2Y^2 + A_3Y^3 + O(Y^4), \qquad A_1 \in \mathbb{R}^*,\ A_2, A_3 \in \mathbb{R},$$

and

$$B'(X) = \mu(X + bX^3) + O(X^4), \qquad \mu \in \mathbb{R}^*,\ b \in \mathbb{R}, \quad \text{with } 0 < \mu A_1 < 2.$$

Then the origin is an elliptic fixed point of f, its eigenvalue $\lambda = \mathrm{e}^{i\gamma_0}$ is determined by

$$\cos\gamma_0 = 1 - \frac{A_1\mu}{2}, \quad -\frac{\pi}{2} < \gamma_0 < 0,$$

and its first Birkhoff invariant can be written $\gamma_1 = \Gamma + b\Gamma'$ with

$$\Gamma = \frac{i\mu^3}{\lambda - \bar\lambda}\Big(3A_3 + \frac{2\omega\mu A_2^2}{\lambda - \bar\lambda}\Big), \qquad \Gamma' = \frac{3i\mu^2A_1^2}{\lambda - \bar\lambda}$$

where $\omega = \frac{2\lambda^{3/2}+3\lambda^{1/2}+3\bar\lambda^{1/2}+2\bar\lambda^{3/2}}{\lambda^{3/2}-\bar\lambda^{3/2}} \in i\mathbb{R}$ (with the convention $\lambda^{1/2} = \mathrm{e}^{i\gamma_0/2}$).

The absence of quadratic term in the Taylor expansion of $B'(X)$ (or in that of $U'(\theta_1)$ in (10)) is just intended to simplify the calculation of γ_1 which can be found in the appendix.

As a corollary, it is easy to choose μ and b so as to be able to apply Moser's theorem of stability of non-degenerate elliptic fixed points ([**Mos62**], [**SM71**, §§31–34]): as soon as the twist condition $\gamma_1 \neq 0$ is satisfied, every neighbourhood of the origin contains an f-invariant neighbourhood of this point, because invariant curves ("KAM circles") accumulate the origin. Such an invariant neighbourhood will sometimes be called an "elliptic island" in the sequel; it is in fact *completely* invariant (*i.e.* it coincides with its image by f).

Lemma 3.1 implies Proposition 3.2: Assume (10) is satisfied. The idea is simply to pass to the quotient by the translation $r_1 \mapsto r_1 + \frac{1}{q}$, so that the origin appear as a stable elliptic fixed point of an area-preserving map of the 2-torus: an invariant neighbourhood for the quotient map will lift to a wandering domain $\mathcal{B}_q$ the iterates of which are obtained by the translation of step $\frac{1}{q}$. But, to be able to keep track of the dependence upon q of this domain, we prefer to begin with the scaling

$$(\theta_*, r_*) = \sigma(\theta_1, r_1), \qquad \theta_* = \theta_1, \quad r_* = qr_1,$$

which conjugates ψ_q and ψ_1. Recalling the definition of ψ_1,

$$\psi_1(\theta_*, r_*) = (\theta_* + r_*, r_* - U'(\theta_* + r_*)),$$

we observe that we can pass to the quotient by the integer translations along the r_1-direction: ψ_1 induces a transformation F of $\mathbb{T} \times \mathbb{T}$, which admits the origin as a fixed point and which can be written

$$F(X, Y) = (X + Y, Y + B'(X + Y)), \qquad B(X) = -X - U(X),$$

in local coordinates (X, Y) near the origin of $\mathbb{T} \times \mathbb{T}$. Setting $A(Y) = -\frac{1}{2}Y^2$, we have $F = \Phi^{-B(Y)} \circ \Phi^{-A(X)}$, thus we can apply Lemma 3.1 to $f = F^{-1}$, with $\mu = -\beta_1$ and $b = \beta_3/\beta_1$.

The origin is thus an elliptic fixed point of F, and it is stable because we have $\Gamma = 0$ and $\Gamma' \neq 0$ in this particular case. By Moser's theorem, we get an f-invariant domain containing the origin (surrounded by a KAM circle), which lifts to a wandering domain $\mathcal{B}_*$ of ψ_1, which in turn gives rise to a wandering domain $\mathcal{B}_q$ by the scaling σ. □

In the sequel we shall fix some analytic 1-periodic function U satisfying (10), so as to have at our disposal wandering domains $\mathcal{B}_q$ for the maps $\psi_{q,U}$. The next section indicates a method to construct, starting from $\psi_{q,U}$, a map Ψ of $\mathbb{A}^n$ which is close to integrable when q is large (notice that, when $q \to \infty$, $\psi_{q,U}$ does not tend to any integrable map!). In fact, in the notation of Proposition 3.1, q_j will be exponentially large with respect to ε_j and, as a consequence, the domain $\mathcal{B}_q$ will be exponentially thin in the r_1-direction.

3.3. *The coupling lemma and the strategy for using it.* We now give a slight refinement of the "coupling lemma" which was one of the key ideas of [**MS03**] (Lemma 2.1 in that paper):

LEMMA 3.2. *Let $m, m', q \geq 1$. Suppose we are given two diffeomorphisms, $F : \mathbb{A}^m \to \mathbb{A}^m$ and $G : \mathbb{A}^{m'} \to \mathbb{A}^{m'}$, and two Hamiltonian functions $f : \mathbb{A}^m \to \mathbb{R}$ and $g : \mathbb{A}^{m'} \to \mathbb{R}$ which generate complete vector fields and define time-1 maps Φ^f and Φ^g. Suppose moreover that $\mathcal{A} \subset \mathbb{A}^{m'}$ is completely G^q-invariant (i.e. $\mathcal{A} =$*

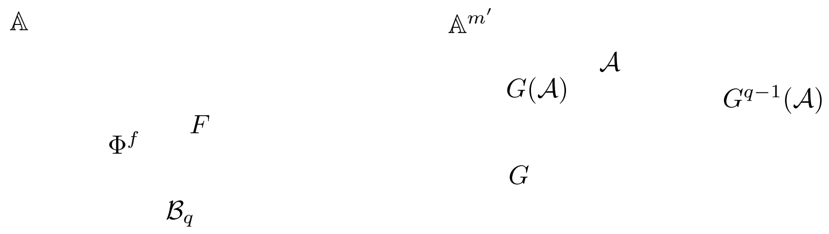

FIGURE 1. Use of the coupling lemma to make $\mathcal{B}_q \times \mathcal{A}$ drift.

$G^q(\mathcal{A})$) and that, for all $x' \in \mathcal{A}$,

$$g(x') = 1, \qquad dg(x') = 0, \qquad g(G^s(x')) = 0, \quad dg(G^s(x')) = 0, \qquad 1 \le s \le q-1. \tag{11}$$

Then $f \otimes g$ generates a complete Hamiltonian vector field and the mapping

$$\Psi = \Phi^{f\otimes g} \circ (F \times G) \ : \ \mathbb{A}^{m+m'} \ \to \ \mathbb{A}^{m+m'}$$

satisfies

$$\Psi^{\ell q}(x,x') = (\psi^{\ell}(x), G^{\ell q}(x')), \qquad x \in \mathbb{A}^m,\ x' \in \mathcal{A},\ \ell \in \mathbb{Z}, \tag{12}$$

with $\psi = \Phi^f \circ F^q$.

We have denoted by $f \otimes g$ the function $(x,x') \mapsto f(x)g(x')$, and by $F \times G$ the mapping $(x,x') \mapsto (F(x), G(x'))$.

Proof. The proof is an obvious adaptation of that of Lemma 2.1 of [**MS03**], where it was already checked that $X_{f\otimes g}$ is complete and that

$$\Phi^{f\otimes g}(x,x') = (\Phi^{g(x')f}(x), \Phi^{f(x)g}(x')), \qquad (x,x') \in \mathbb{A}^{m+m'}. \tag{13}$$

Let $x \in \mathbb{A}^m$ and $x' \in \mathcal{A}$. It is sufficient to prove the desired identity for $\ell = 1$; indeed, it will then be possible to iterate it backwards or forwards thanks to the G^q-invariance of $\mathcal{A}$.

The points $(F^s(x), G^s(x'))$, $1 \le s \le q-1$, are fixed points of $\Phi^{f\otimes g}$ because of (11) and (13). Thus

$$\Psi^s(x,x') = (F^s(x), G^s(x')), \qquad 0 \le s \le q-1.$$

But for the q^{th} iteration, (11) and (13) yield

$$\Psi^q(x,x') = \Phi^{f\otimes g}(F^q(x), G^q(x')) = (\Phi^f(F^q(x)), G^q(x')).$$

□

To construct near-integrable mappings with wandering domains, it is thus sufficient to apply this coupling lemma with $\Phi^f \circ F^q = \psi_{q,U}$, $q \ge 1$ and U like in Section 3.2, that is with

$$f = \tfrac{1}{q} U(\theta_1), \quad F = \Phi^{\frac12 r_1^2},$$

and to find G, $\mathcal{A}$ and g in such a way that the hypotheses of the lemma be fulfilled: we shall obtain a wandering domain $\mathcal{B}_q \times \mathcal{A}$ as illustrated on Figure 1, and the map Ψ will be near-integrable if $\frac{1}{q}\|g\|$ is small and G is close to integrable, in the C^k or Gevrey-α topology. Since the dynamics of $\psi_{q,U}$ will then appear as a subsystem of Ψ^q, the time needed for transporting $\mathcal{B}_q \times \mathcal{A}$ from $r_1 = 0$ to $r_1 = 1$ will be q^2.

The choice of G with a globally q-periodic domain $\mathcal{A}$ is a new feature of the present work with respect to [**MS03**] and constitutes the essential part of the next sections.

As for the choice of g, we shall use "bump functions" of one variable†, the existence of which in the Gevrey case is alluded to at the end of Section A.1 of [**MS03**], and this is precisely the point where the estimates differ from what we could do for unstable points. Let us indicate now the technical statement:

LEMMA 3.3. *Let $\alpha > 1$, $\Lambda > 0$. There exists $c > 0$ such that, for each real $p > 2$, the space $G^{\alpha,\Lambda}(\mathbb{T})$, contains a function $\eta_{p,\Lambda}$ which satisfies*

$$\eta_{p,\Lambda}(\theta) = \left| \begin{array}{ll} 1 & \textit{if } -\frac{1}{2p} \le x \le \frac{1}{2p}, \\[2mm] 0 & \textit{if } -\frac{1}{2} \le x \le -\frac{1}{p} \textit{ or } \frac{1}{p} \le x \le \frac{1}{2}, \end{array} \right. \tag{14}$$

and

$$\|\eta_{p,\Lambda}\|_{\alpha,\Lambda} \le \exp\big(c\, p^{\frac{1}{\alpha-1}}\big). \tag{15}$$

REMARK 3.1. *We shall also use non-periodic bump functions $\tilde{\eta}_{p,\Lambda}$ which have compact supports contained in $[-\frac{1}{p}, \frac{1}{p}]$ and are defined by*

$$\tilde{\eta}_{p,\Lambda}(x) = \left| \begin{array}{ll} \eta_{p,\Lambda}(x) & \textit{if } -\frac{1}{2} \le x \le \frac{1}{2}, \\ 0 & \textit{if not.} \end{array} \right.$$

REMARK 3.2. *Analogously, if $k \ge 0$, the space $C^k(\mathbb{T})$ contains functions which satisfy (14), but the C^k norm grows more slowly with p; in fact, we can have a C^∞ function η_p such that*

$$\|\eta_p^{(\ell)}\|_{C^0(\mathbb{T})} \le c p^\ell, \qquad \ell \in \mathbb{N}, \tag{16}$$

where $c > 0$ does not depend on any parameter, simply by taking $\eta_p(\theta) = \eta_1(p\theta)$ for $-\frac{1}{2} \le \theta \le \frac{1}{2}$, where $\eta_1(x)$ is a fixed C^∞ function which vanishes outside $[-1, 1]$ and is equal to 1 identically in $[-\frac{1}{2}, \frac{1}{2}]$.

Proof of Lemma 3.3. Let $\beta > 0$ be determined by $\alpha = 1 + \frac{1}{\beta}$; in view of the proof of Lemma A.3 of [**MS03**], the function f defined by

$$f(x) = \left| \begin{array}{ll} 0 & \text{if } x \le 0, \\ \exp(-\frac{\lambda}{x^\beta}) & \text{otherwise} \end{array} \right.$$

† In this article, by "bump function" we mean a function which vanishes identically outside a given interval I and whose value is 1 at each point of a given subinterval of I.

is known to belong to $G^{\alpha,\Lambda}(\mathbb{R})$ for λ large enough with respect to α and Λ. We define

$$\varphi_p(x) = f\big(\frac{1}{4p} + x\big) f\big(\frac{1}{4p} - x\big), \quad \Phi_p(x) = \int_{-\infty}^{x} \varphi_p(x')\,dx'.$$

Observe that φ_p vanishes outside $[-\frac{1}{4p}, \frac{1}{4p}]$, hence

$$\Phi_p(x) = \left| \begin{array}{ll} 0 & \text{if } x \leq -\frac{1}{4p}, \\ K_p = \int_{-\frac{1}{4p}}^{\frac{1}{4p}} \varphi_p(x')\,dx' & \text{if } x \geq \frac{1}{4p}. \end{array} \right.$$

It follows that the function

$$\eta_{p,\Lambda}(\theta) = \frac{1}{K_p^2} \Phi_p\big(\frac{3}{4p} + \theta\big) \Phi_p\big(\frac{3}{4p} - \theta\big), \qquad -\tfrac{1}{2} \leq \theta \leq \tfrac{1}{2}$$

(extended by 1-periodicity) takes the desired values on the intervals specified in the statement of the lemma.

As for the Gevrey norms, clearly $\|\varphi_p\|_{\alpha,\Lambda} \leq \|f\|_{\alpha,\Lambda}^2$ (because of the property of Banach algebra) and the inequality $\|\Phi_p\|_{\alpha,\Lambda} \leq \|\Phi_p\|_{C^0} + \Lambda^\alpha \|\varphi_p\|_{\alpha,\Lambda}$ shows that $\|\Phi_p\|_{\alpha,\Lambda}$ is bounded independently of p. Since f is monotonic non-decreasing, we have $\varphi_p(x) \geq f(\frac{1}{8p})^2$ for $|x| \leq \frac{1}{8p}$, thus $K_p \geq \frac{1}{4p}\exp(-2\lambda(8p)^\beta)$ and the conclusion follows. □

In the sequel, having fixed $\alpha > 1$ and $L > 0$, we shall define close to integrable maps of the form

$$\Psi = \Phi^{\frac{1}{q} U \otimes g} \circ (\Phi^{\frac{1}{2} r_1^2} \times G), \qquad G = \Phi^{\frac{1}{2}(r_2^2 + \cdots + r_n^2) + v} \circ \Phi^w,$$

with $\varepsilon = \max(\frac{1}{q}\|U \otimes g\|_{\alpha,L}, \|v\|_{\alpha,L}, \|w\|_{\alpha,L})$ arbitrarily small, in such a way that G admits a globally q-periodic domain $\mathcal{A}$ and g "separates" $\mathcal{A}$ from its iterates by G in the sense of (11).

It is clear that $\|g\|_{\alpha,L}$ will depend crucially on

$$d = \min_{1 \leq s \leq q-1} \operatorname{dist}(\mathcal{A}, G^s(\mathcal{A})).$$

We shall indeed resort to Lemma 3.3 to define g (adapting it to take advantage of the fact that g can depend on several variables) and this will yield a Gevrey-α norm of the order of $\exp(\text{const}\, d^{-\frac{1}{\alpha-1}})$.

But we shall need to take q large to make $\frac{1}{q}\|g\|_{\alpha,L}$ as small as ε (anyway, we know in advance that the time of drift $\tau = q^2$ needs to be large). This will tend to diminish d and thus to increase $\|g\|_{\alpha,L}$. Hence we shall need to take q even larger, to compensate this growth. However, it is a priori not obvious to prevent this increase of q from diminishing in turn the distance d, thus increasing again the norm of g. We shall see in Section 3.4 a C^k example where this method leads to the limitation $k \leq n-2$.

Faced with such an inflation of norms, we shall use in Section 4 an idea which was already an important feature of the unstable system of [**MS03**]. We shall choose a system G which is $\frac{1}{N^2}$-close to integrable, involving a rescaled pendulum (the integer N will play the role of the scaling parameter), and impose the condition

$$\tfrac{1}{q}\|g\|_{\alpha,L},\ \|w\|_{\alpha,L} \le \|v\|_{\alpha,L} = \frac{1}{N^2}.$$

We shall manage to have $q = NM$, with M extremely large, but d almost independent of M. More precisely, d will be larger than $(\frac{1}{N})^{\frac{1}{n-1}}$ independently of the choice of M, thus Lemma 3.3 will provide us with a function g of norm $\|g\|_{\alpha,L} \le \exp(\mathrm{const}\, N^{\frac{1}{\gamma}})$, where $\gamma = (n-1)(\alpha-1)$, in view of which we shall choose M of the order of $N \exp(\mathrm{const}\, N^{\frac{1}{\gamma}})$ (formula (33) below). This way, we shall obtain a time of drift $\tau \sim N^4 \exp(\mathrm{const}\, N^{\frac{1}{\gamma}})$, where $N = \varepsilon^{-1/2}$.

3.4. *Herman's examples of C^k unstable systems.* Following [**He99**], we now illustrate our strategy with a first construction, which is very simple but works only for $k \le n-2$.

We fix a smooth function S on $\mathbb{T}$ such that

$$S(\theta_*) = \frac{1}{2}\theta_*^2, \qquad -\frac{1}{4} \le \theta_* \le \frac{1}{4}. \tag{17}$$

Proposition 3.3. *Suppose $n \ge 2$ and $0 \le k \le n-2$. There exist sequences (u_j) and (v_j) of smooth functions, which converge to 0 in $C^k(\mathbb{T}^n)$, and an increasing sequence (q_j) of integers such that, for each j, the system*

$$\Psi_j = \Phi^{u_j} \circ \Phi^{\frac{1}{2}(r_1^2+\cdots+r_n^2)+v_j}$$

admits a wandering open set $\mathcal{D}_j$ whose iterates $(\Psi_j)^{\ell q_j}(\mathcal{D}_j)$ are mutually disjoint and satisfy

$$(\Psi_j)^{\ell q_j}(\mathcal{D}_j) = T_{\frac{\ell}{q_j}}(\mathcal{D}_j), \qquad \ell \in \mathbb{Z}.$$

Proof. Let $(p_j)_{j\ge 0}$ be the sequence of prime numbers. We set

$$q_j = p_j p_{j-1} \dots p_{j-n+2}, \qquad j \ge n-1,$$

and consider $G = \Phi^{\frac{1}{2}(r_2^2+\cdots+r_n^2)+v_j}$, with

$$v_j(\theta) = \Big(\frac{1}{p_j}\Big)^2 S(\theta_2) + \Big(\frac{1}{p_{j-1}}\Big)^2 S(\theta_3) + \ldots + \Big(\frac{1}{p_{j-n+2}}\Big)^2 S(\theta_n).$$

The system G is obviously decoupled and can be written

$$G = G_2 \times \cdots \times G_n, \qquad G_i = \Phi^{\frac{1}{2}r_i^2 + (\frac{1}{p_{j-i+2}})^2 S(\theta_i)}, \qquad 2 \le i \le n.$$

Moreover, in the region $\mathcal{R}_i = \{\, r_i^2 + (\frac{1}{p_{j-i+2}})^2\theta_i^2 < \frac{1}{16}(\frac{1}{p_{j-i+2}})^2 \,\}$, the map G_i is periodic with minimal period p_{j-i+2}, since the scaling

$$(\theta_*, r_*) = \sigma_i(\theta_i, r_i) = (\theta_i, p_{j-i+2} r_i)$$

conjugates it with the time-$\frac{1}{p_{j-i+2}}$ map of the Hamiltonian $h_*(\theta_*, r_*) = \frac{1}{2}r_*^2 + S(\theta_*)$ which coincides with the normalized harmonic oscillator $\frac{1}{2}(r_*^2 + \theta_*^2)$ in the region $\mathcal{R}_* = \{\, r_*^2 + \theta_*^2 < \frac{1}{16} \,\}$. Consequently, all the points of $\mathcal{R}_2 \times \cdots \times \mathcal{R}_n$ (except those for which $(\theta_i, r_i) = (0,0)$ for some i) are G-periodic with minimal period q_j (the periods p_{j-i+2} have no non-trivial common divisor because we have used prime numbers).

$\frac{2\pi\delta}{p}$

$\mathcal{A}_*^{(\delta)}$

0 $\frac{1}{8}$ $\frac{1}{4}$ θ_*

FIGURE 2. Iteration of $\mathcal{A}_*^{(\delta)}$ under $\Phi^{\frac{1}{p}h_*}$.

LEMMA 3.4. *Given a real* $\delta \in]0, 1/6\pi]$ *and an odd integer* p *not smaller than* 3, *the domain*

$$\mathcal{A}_*^{(\delta)} = \{\, \tfrac{1}{8} < \sqrt{\theta_*^2 + r_*^2} < \tfrac{1}{4},\ |\theta_*| < (\tan \tfrac{2\pi\delta}{p}) r_* \,\}$$

satisfies $\mathcal{A}_*^{(\delta)} = \Phi^{h_*}(\mathcal{A}_*^{(\delta)}) \subset \{\, |\theta_*| \le \frac{\pi\delta}{2p} \,\}$ *and*

$$\Phi^{\frac{s}{p}h_*}(\mathcal{A}_*^{(\delta)}) \subset \{\, \tfrac{\pi\delta}{p} \le |\theta_*| \le \tfrac{1}{4} \,\}, \qquad 1 \le s \le p-1.$$

Proof of Lemma 3.4. Each point in $\mathcal{A}_*^{(\delta)}$ is Φ^{h_*}-periodic with minimal period 1 because $\mathcal{A}_*^{(\delta)}$ is contained in $\mathcal{R}_*$. Elementary trigonometry yields

$$\mathcal{A}_*^{(\delta)} \subset \{\, |\theta_*| < \tfrac{1}{4} \sin \tfrac{2\pi\delta}{p} \,\}, \quad \Phi^{\frac{s}{p}h_*}(\mathcal{A}_*^{(\delta)}) \subset \{\, \tfrac{1}{8} \sin \tfrac{\pi(1-2\delta)}{p} < |\theta_*| < \tfrac{1}{4} \,\},$$

for $1 \le s \le p-1$ (see Figure 2). We conclude by observing that $\frac{2x}{\pi} < \sin x < x$ for $0 < x < \pi/2$ and that δ is small enough to yield the desired inequalities. □

We thus choose

$$\mathcal{A} = \sigma_2^{-1}\mathcal{A}_*^{(\delta)} \times \cdots \times \sigma_n^{-1}\mathcal{A}_*^{(\delta)}, \qquad \delta = \tfrac{1}{6\pi},$$

and, using the bump functions furnished by Remark 3.2,

$$g^{(j)}(\theta_2, \ldots, \theta_n) = \eta_{6p_j}(\theta_2) \ldots \eta_{6p_{j-n+2}}(\theta_n).$$

We can apply Lemma 3.2 to

$$\Psi_j = \Phi^{u_j} \circ \Phi^{\frac{1}{2}(r_1^2 + \cdots + r_n^2) + v_j}, \qquad u_j = \frac{1}{q_j} U \otimes g^{(j)}.$$

According to formula (12), the dynamics of $\psi = \psi_{q_j,U}$ is thus embedded into the $(\Psi_j)^{q_j}$-invariant set $\mathbb{A} \times \mathcal{A}$. In particular, by Proposition 3.2, we get a wandering polydisc $\mathcal{D}_j = \mathcal{B}_{q_j} \times \mathcal{A}$ with the kind of orbit we wanted.

It only remains for us to check that $\varepsilon_j = \max(\|u_j\|_{C^k(\mathbb{T}^n)}, \|v_j\|_{C^k(\mathbb{T}^n)})$ tends to 0 when $j \to \infty$. With our definition (6) of C^k-norms, we have $\|v_j\|_{C^k(\mathbb{T}^n)} = \big(\big(\frac{1}{p_j}\big)^2 + \cdots + \big(\frac{1}{p_{j-n+2}}\big)^2\big)\|S\|_{C^k(\mathbb{T})}$ and, using (16),

$$\begin{aligned}
\|u_j\|_{C^k(\mathbb{T}^n)} &= \frac{1}{q_j}\sum_{|\ell|\le k}\frac{1}{\ell!}\|U^{(\ell_1)}\|_{C^0(\mathbb{T})}\|\eta_{6p_j}^{(\ell_2)}\|_{C^0(\mathbb{T})}\cdots\|\eta_{6p_{j-n+2}}^{(\ell_n)}\|_{C^0(\mathbb{T})}\\
&\le \frac{c^{n-1}}{q_j}\|U\|_{C^k(\mathbb{T})}\sum_{\ell_2+\cdots+\ell_n\le k}\frac{1}{\ell_2!\ldots\ell_n!}(6p_j)^{\ell_2}\ldots(6p_{j-n+2})^{\ell_n}\\
&\le \mathrm{e}^6 c^{n-1}\|U\|_{C^k(\mathbb{T})}\frac{(p_j+\cdots+p_{j-n+2})^k}{p_j\cdots p_{j-n+2}}.
\end{aligned}$$

The conclusion follows from the Prime Number Theorem which ensures that, for j large enough, all the numbers p_{j-i+2}, $2 \le i \le n$, lie in the interval $[\frac{1}{2}p_j, p_j]$ (this was the interest of choosing successive prime numbers). It is only to ensure the C^k-convergence to 0 of (u_j) that we imposed $k \le n-2$.

This ends the proof of Proposition 3.3. □

It is easy to check that the number $\tau_j = q_j^2$ of iterates required to translate by 1 the r_1-projection of $\mathcal{D}_j$ satisfies inequalities of the form

$$\tau_j \le \left| \begin{array}{ll} C\big(\frac{1}{\varepsilon_j}\big)^{n-1} & \text{if } 0 \le k \le n-3,\\ C\big(\frac{1}{\varepsilon_j}\big)^{2(n-1)} & \text{if } k = n-2. \end{array}\right. \tag{18}$$

We shall indicate later (Section 5.3) a more elaborate construction which yields better results, without any bound imposed upon k.

As for the size of the polydisc $\mathcal{D}_j$, when $k \le n-3$ we obtain a diameter $\approx (\sqrt{\varepsilon_j})^{n-1}$ for its (θ_1, r_1)-projection and $\sqrt{\varepsilon_j}$ for the other canonical projections; when $k = n-2$ we find respectively ε_j^{n-1} and ε_j.

4. *Examples of Gevrey unstable systems*

We now move on to the proof of Proposition 3.1, applying the strategy described at the end of Section 3.3. We thus fix $\alpha > 1$ and $L > 0$.

Let $(p_j)_{j\ge0}$ be the sequence of prime numbers (inverses of primes will now be used as action variables to produce periodic points of the integrable twist map). We define

$$N_j = p_j N'_j \quad \text{where } N'_j = 1 \text{ if } n = 2, \text{ and } N'_j = p_{j-(n-3)}p_{j-(n-4)}\cdots p_j \text{ if } n \ge 3, \tag{19}$$

and

$$v_j(\theta) = \frac{1}{N_j^2}V(\theta_2), \qquad V(\theta_2) = -1 - \cos(2\pi\theta_2). \tag{20}$$

We shall first define a small function of the form

$$w_j(\theta, r) = w_2^{(j)}(\theta_2, r_2) + w_3^{(j)}(\theta_3) + \cdots + w_n^{(j)}(\theta_n),$$

and it is only in Section 4.4 that we shall introduce the other ingredients $g^{(j)}$, M_j and $u_j = \frac{1}{N_j M_j} U \otimes g^{(j)}$ (with U like in Section 3.2). In the notation of Lemma 3.2, this means that we shall have $G = G_2 \times G_3 \times \cdots \times G_n$, with

$$G_2 = \Phi^{\frac{1}{2}r_2^2 + \frac{1}{N_j^2}V(\theta_2)} \circ \Phi^{w_2^{(j)}}, \qquad G_i = \Phi^{\frac{1}{2}r_i^2} \circ \Phi^{w_i^{(j)}(\theta_i)}, \qquad 3 \le i \le n.$$

We shall define $G_3 \times \cdots \times G_n$ so as to have a globally N'_j-periodic domain $\mathcal{A}_3^{(j)} \times \cdots \times \mathcal{A}_n^{(j)}$ and the role of $\mathcal{A}$ in Lemma 3.2 will be played by a domain

$$\mathcal{A} = \mathcal{A}_2^{(j)} \times \mathcal{A}_3^{(j)} \times \cdots \times \mathcal{A}_n^{(j)},$$

where $\mathcal{A}_2^{(j)}$ is globally $N_j M_j$-periodic for the perturbed pendulum G_2. Finally, setting $q_j = N_j M_j$, we shall have a wandering domain $\mathcal{D}_j = \mathcal{B}_{q_j} \times \mathcal{A}$ for the mapping $\Psi_j = \Phi^{\frac{1}{q_j} U \otimes g^{(j)}} \times (\Phi^{\frac{1}{2}r_1^2} \times G)$ (of course G is reduced to G_2 and $\mathcal{A}$ to $\mathcal{A}_2^{(j)}$ when $n = 2$).

4.1. *Choice of the map $G_3 \times \cdots \times G_n$.* The present section is concerned with the case $n \ge 3$ only. If $p \in \mathbb{N}^*$, the point $(0, \frac{1}{p})$ is a parabolic fixed point of the p^{th} iterate of the standard twist map $\Phi^{\frac{1}{2}r^2} : \mathbb{A} \to \mathbb{A}$; we shall make use of the bump functions of one variable defined in Lemma 3.3, with the given values of α and L, to perturb $\Phi^{\frac{1}{2}r^2}$ and create ellipticity.

LEMMA 4.1. *Let $p \in \mathbb{N}^*$ and*

$$\beta_{p,\mu}(\theta) = \mu\big(\tfrac{\theta^2}{2} + \tfrac{\theta^4}{4}\big)\eta_{2p,L}(\theta), \qquad 0 < \mu \le \tfrac{1}{p}.$$

The point $(0, \frac{1}{p}) \in \mathbb{A}$ is a stable elliptic fixed point of the p^{th} iterate of $G_{p,\mu} = \Phi^{\frac{1}{2}r^2} \circ \Phi^{\beta_{p,\mu}}$, which is contained in an elliptic island $\mathcal{A}_{p,\mu}$ satisfying $\mathcal{A}_{p,\mu} = (G_{p,\mu})^p(\mathcal{A}_{p,\mu}) \subset \{ -\frac{1}{4p} \le \theta \le \frac{1}{4p} \}$ and

$$(G_{p,\mu})^s(\mathcal{A}_{p,\mu}) \subset \{ \tfrac{3}{4p} \le \theta \le 1 - \tfrac{3}{4p} \}, \qquad 1 \le s \le p-1.$$

Proof. Let $\mathcal{N}$ denote the neighbourhood $\{ -\delta \le \theta \le \delta,\ |r - \frac{1}{p}| \le \delta \}$ of $x_p = (0, \frac{1}{p})$, with

$$\delta = \frac{1}{4p(3p+4)},$$

and consider the iterates of $\mathcal{N}$ by $G_{p,\mu}$. Since the functions $\beta_{p,\mu}$ and $\mu(\frac{\theta^2}{2} + \frac{\theta^4}{4})$ coincide on $\mathcal{N}$, with $\mu|\theta + \theta^3| \le 2\delta$ on that set, we have

$$\mathcal{N}' = \Phi^{\beta_{p,\mu}}(\mathcal{N}) \subset \{ -\delta \le \theta \le \delta,\ |r - \frac{1}{p}| \le 3\delta \},$$

and the inclusions

$$\big(\Phi^{\frac{1}{2}r^2}\big)^s(\mathcal{N}') \subset \Sigma = \{ \tfrac{3}{4p} \le \theta \le 1 - \tfrac{3}{4p} \}, \qquad 1 \le s \le p-1$$

follow easily from our choice of δ.

Consequently, since $\Phi^{\beta_{p,\mu}}$ boils down to identity on Σ, the mappings $(G_{p,\mu})^s$ and $(\Phi^{\frac{1}{2}r^2})^s \circ \Phi^{\beta_{p,\mu}}$ coincide on $\mathcal{N}$ for each $s \in \{1, \ldots, p\}$.

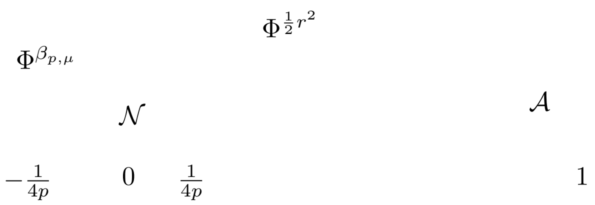

FIGURE 3. Elliptic island of $G_{p,\mu} = \Phi^{\frac{1}{2}r^2} \circ \Phi^{\beta_{p,\mu}}$.

Applying Lemma 3.1 to $(G_{p,\mu})^p = (\Phi^{\frac{1}{2}r^2})^p \circ \Phi^{\beta_{p,\mu}}$ near its elliptic fixed point x_p, we get the twist condition $\gamma_1 \neq 0$ (because $\Gamma = 0$ and $\gamma_1 = \Gamma' \neq 0$ in this case). By Moser's theorem, we obtain an elliptic island $\mathcal{A}_{p,\mu} \subset \mathcal{N}$, the orbit of which obviously satisfies the required properties (see Figure 3). □

We choose

$$w_i^{(j)} = \beta_{p_{j-i+3},\mu_{i,j}}, \quad G_i = \Phi^{\frac{1}{2}r_i^2} \circ \Phi^{w_i^{(j)}}, \quad \mathcal{A}_i^{(j)} = \mathcal{A}_{p_{j-i+3},\mu_{i,j}}, \qquad 3 \le i \le n, \tag{21}$$

with $\mu_{i,j} = \big(\max\big(p_{j-i+3},\, (n-2)N_j^2 \|\beta_{p_{j-i+3},1}\|_{\alpha,L}\big)\big)^{-1}$, so that

$$\sum_{i=3}^{n} \|w_i^{(j)}\|_{\alpha,L} \le \frac{1}{N_j^2}.$$

Observe that, since the p_{j-i+3}'s are mutually prime, the domain $\hat{\mathcal{A}}^{(j)} = \mathcal{A}_3^{(j)} \times \cdots \times \mathcal{A}_n^{(j)}$ is globally periodic for the system $\hat{G} = G_3 \times \cdots \times G_n$ with minimal period N_j' and

$$\hat{\mathcal{A}}^{(j)} \subset \bigcap_{i=3}^{n} \{ -\tfrac{1}{4p_{j-i+3}} \le \theta_i \le \tfrac{1}{4p_{j-i+3}} \} \subset \mathbb{A}^{n-2}, \tag{22}$$

$$\hat{G}^s(\hat{\mathcal{A}}^{(j)}) \subset \bigcup_{i=3}^{n} \{ \tfrac{3}{4p_{j-i+3}} \le \theta_i \le 1 - \tfrac{3}{4p_{j-i+3}} \}, \qquad 1 \le s \le N_j' - 1. \tag{23}$$

4.2. *Time-energy coordinates for the simple pendulum.* The present section and the next one are devoted to the obtention of a perturbed pendulum with an elliptic island of large period, which will be our system G_2. We begin with elementary facts on the pendulum, to prepare the ground for the choice of the perturbation $w_2^{(j)}$. For the sake of clarity, we shall omit the indices j.

Let us introduce the notation

$$P_N(\theta_2, r_2) = \frac{1}{2}r_2^2 + \frac{1}{N^2}V(\theta_2), \quad P_*(\theta_*, r_*) = \frac{1}{2}r_*^2 + V(\theta_*).$$

The mapping ϕ^{P_N} is the time-1 map of the Hamiltonian flow generated by P_N, which is obtained from the flow of the normalized pendulum P_* by rescaling time and action:

$$\Phi^{tP_N} = \sigma^{-1} \circ \Phi^{\frac{t}{N}P_*} \circ \sigma, \quad \text{where } (\theta_*, r_*) = \sigma(\theta_2, r_2) = (\theta_2, Nr_2). \tag{24}$$

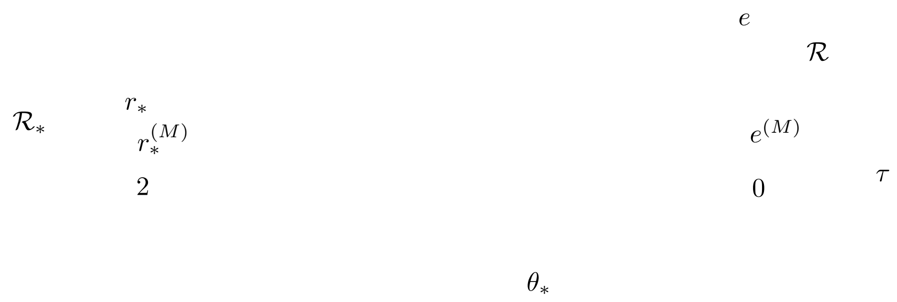

FIGURE 4. Straightening of the pendulum flow in $\mathcal{R}_*$.

Let $M \in \mathbb{N}^*$ (in fact, M will be chosen equal to the integer M_j defined in equation (33) below). We shall be interested in a neighbourhood of the point $(0, r_*^{(M)})$ that is determined by the conditions

$$\Phi^{MP_*}(0, r_*^{(M)}) = (0, r_*^{(M)}), \quad r_*^{(M)} > 2,$$

i.e. $(0, r_*^{(M)})$ is the intersection of $\{\theta_* = 0\}$ with the unique orbit of P_* which has period M and is located above the upper separatrix $r_* = |\cos \pi\theta_*|$.

The mapping $(\Phi^{P_*})^M$ can be described near its fixed point $(0, r_*^{(M)})$ as follows. Using $\{\theta_* = 0\}$ as a reference section, we define symplectic flow-box coordinates (τ, e) for the normalized pendulum, say, in the region

$$\mathcal{R}_* = \{\, |\theta_*| \le \tfrac{1}{4} \text{ and } r_* \ge 1 \,\}.$$

Since $P_*(0, r_*) = \frac{1}{2} r_*^2 - 2$, this amounts to considering the canonical change of coordinates

$$(\theta_*, r_*) = \Phi^{\tau P_*}\Big(0, \sqrt{2(e+2)}\Big) \in \mathcal{R}_* \quad \Leftrightarrow \quad (\tau, e) = \big(\tau(\theta_*, r_*), e(\theta_*, r_*)\big) \in \mathcal{R}$$

(see Figure 4). In particular, the functions e and P_* coincide (they are nothing but the energy function), while the time is given by the incomplete elliptic integral

$$\tau(\theta_*, r_*) = \int_0^{\theta_*} \frac{d\theta}{\sqrt{2(e(\theta_*, r_*) - V(\theta))}}, \qquad (\theta_*, r_*) \in \mathcal{R}_*. \tag{25}$$

In the region $\{e > 0\}$ (*i.e.* above the separatrices), the period of motion is given as a (decreasing) function of energy by the formula

$$T_*(e) = \int_0^1 \frac{d\theta}{\sqrt{2(e - V(\theta))}}. \tag{26}$$

Thus, $T_*(e^{(M)}) = M$ with the notation $e^{(M)} = e(0, r_*^{(M)})$ (observe that $T_*(e) < 1/\sqrt{2e}$, hence $e^{(M)} < \frac{1}{2M^2}$).

In the coordinates (τ, e), the flow of P_* is straightened: in the domain $\mathcal{R}$,

$$\Phi^{tP_*} : \ (\tau, e) \mapsto (\tau + t, e), \qquad |t| \text{ small enough},$$

whereas one can check that the mapping $(\Phi^{P_*})^M$ takes the form

$$(\Phi^{P_*})^M : \ (\tau, e) \mapsto (\tau + A'_*(e), e), \qquad A'_*(e) = M - T_*(e),$$

for (τ, e) in a neighbourhood of $(0, e^{(M)})$ of the form $\{\, |\tau| < \delta_*,\ |e - e^{(M)}| < \rho_*^{(M)} \,\}$, where $\delta_* \le 1$ can be chosen independent of M and $\rho_*^{(M)} \le \frac{1}{2} e^{(M)}$ must be chosen† small enough to ensure the return of the orbit to $\mathcal{R}$ within a time M. In other words, locally, $(\Phi^{P_*})^M$ must be viewed as generated by a Hamiltonian $A_*(e)$ explicitly computable in terms of the function T_*.

Correspondingly, setting $q = NM$, we obtain a description of Φ^{P_N} and $\left(\Phi^{P_N}\right)^q$ in a new system of local symplectic coordinates centred at the q-periodic point $(0, \frac{1}{N} r_*^{(M)})$ as follows.

We consider the change of coordinates

$$(X, Y) \ \mapsto \ (\theta_2, r_2) = \Phi^{NXP_N}\left(0, \tfrac{1}{N}\sqrt{\left(r_*^{(M)}\right)^2 + 2NY}\right), \tag{27}$$

for the inverse of which we have the formulae

$$X = \tau(\theta_2, Nr_2), \quad Y = \frac{e(\theta_2, Nr_2) - e^{(M)}}{N}, \qquad (\theta_2, r_2) \in \sigma^{-1}\mathcal{R}_*. \tag{28}$$

This transformation is symplectic since $dX \wedge dY = \frac{1}{N} d\tau \wedge de = \frac{1}{N} d\theta_* \wedge dr_* = d\theta_2 \wedge dr_2$. We end up with the formulae

$$\Phi^{P_N} : \ (X, Y) \mapsto (X + \tfrac{1}{N}, Y) \tag{29}$$

(in the domain corresponding to $(\theta_2, r_2) \in \sigma^{-1}\mathcal{R}_* \cap \Phi^{-P_N}(\sigma^{-1}\mathcal{R}_*)$) and

$$\left(\Phi^{P_N}\right)^q : \ (X, Y) \mapsto (X + A'(Y), Y), \qquad A'(Y) = M - T_*(e^{(M)} + NY) \tag{30}$$

for $|X| < \delta_*$ and $|Y| < \frac{1}{N}\rho_*^{(M)}$.

The following estimates will be used in the next section:

LEMMA 4.2. *Let $(\theta_*, r_*) \in \mathcal{R}_*$ admit time-energy coordinates (τ, e).*

- *If $1 \le r_* \le 3$, then $|\tau| \le |\theta_*| \le \sqrt{11}\,|\tau|$.*
- *If $e > 0$, then $|\theta_*| \ge \frac{2|\tau|}{1 + 4\pi^2\tau^2}$.*
- *If $0 < e < 2e^{(M)}$, then $|\theta_*| \le (2 + M^{-2})|\tau|$.*

† For instance, one can fix δ_* small enough to guarantee that $[-2\delta_*, 2\delta_*] \times \{e^{(M)}\} \subset \mathcal{R}$ for all $M \ge 1$ and choose $\rho_*^{(M)} = \min(\frac{1}{2}e^{(M)}, \delta_*/|T'_*(\frac{1}{2}e^{(M)})|)$. The mean value theorem yields indeed $|T_*(e^{(M)} \pm \rho_*^{(M)}) - T_*(e^{(M)})| \le \delta_*$ because $|T'_*|$ is decreasing.

Proof. We can assume $\tau \geq 0$ thanks to reversibility.

The first statement is obtained by checking that, because of the conservation of energy, $1 \leq \dot{\theta}_* = r_* \leq \sqrt{11}$ in the time-interval $[0, \tau]$.

The second statement is obtained by comparison between the angular component $\tau \mapsto \theta_*^{(e)}(\tau)$ of the solution obtained by fixing the energy to the positive value e and that of the separatrix solution $\tau \mapsto \theta_*^{(0)}(\tau) = \frac{1}{\pi} \arctan(\sinh 2\pi\tau)$: we have $\theta_*^{(e)}(\tau) \geq \theta_*^{(0)}(\tau) \geq \frac{2\tau}{1+4\pi^2\tau^2}$.

The last inequality is an easy consequence of formula (25) with $e - V(\theta) \leq e + 2$ and $e < 2e^{(M)} < M^{-2}$. □

4.3. *Elliptic islands of large period for a perturbed pendulum.* The point $x_2 = (0, \frac{1}{N} r_*^{(M)})$, which corresponds to the origin in the coordinates (X, Y), is Φ^{P_N}-periodic with period $q = NM$, but it is in fact a *parabolic* fixed point of $\left(\Phi^{P_N}\right)^q$, in view of (30). To introduce some ellipticity, we shall compose Φ^{P_N} by a close-to-identity map which leaves x_2 fixed. We shall imitate here the method used in Section 4.1, but the picture is slightly distorted because it is the pendulum that we perturb, instead of the standard twist map; this is why we use the symplectic transformation $(\theta_2, r_2) \mapsto (X, Y)$ defined in $\sigma^{-1}\mathcal{R}_*$ by $X = \tau \circ \sigma(\theta_2, r_2)$ and $Y = (e \circ \sigma(\theta_2, r_2) - e^{(M)})/N$ (according to formulae (27) and (28); see formula (24) for the definition of σ).

Let $\mathcal{K} = [-\frac{1}{4}, \frac{1}{4}] \times [1, 3] \subset \mathcal{R}_*$ (thus $\sigma^{-1}\mathcal{K} = [-\frac{1}{4}, \frac{1}{4}] \times [\frac{1}{N}, \frac{3}{N}]$). Since the function τ defined by (25) is analytic in $\mathcal{R}_*$, its derivatives satisfy inequalities of the form $\|\partial^k \tau\|_{C^0(\mathcal{K})} \leq C\, \lambda^{|k|}$, from which we deduce that

$$\tau \circ \sigma \in G^{\alpha,L}(\sigma^{-1}\mathcal{K}), \quad \text{with } \|\tau \circ \sigma\|_{\alpha,L} \leq 2^{1-\alpha} \Lambda_N^\alpha,$$

where Λ_N is defined by $\Lambda_N^\alpha = 2^{\alpha-1} C \sum \frac{1}{k!^{\alpha-1}} (\lambda L^\alpha N)^{|k|}$. We shall use the bump functions defined in Remark 3.1. Let us consider

$$\xi_N(X) = \left(\tfrac{X^2}{2} + \tfrac{X^4}{4}\right) \tilde{\eta}_{2N,\Lambda_N}(X).$$

Proposition A.1 from [**MS03**] can be applied: the function $\xi_N \circ (\tau \circ \sigma)$ belongs to $G^{\alpha,L}(\sigma^{-1}\mathcal{K})$ (and has its Gevrey-(α, L) norm bounded by $\|\xi_N\|_{\alpha,\Lambda_N}$). Moreover, this function vanishes for $|\tau| \geq \frac{1}{N}$ and the first statement of Lemma 4.2 shows that, for $N \geq 14$,

$$|\tau| \leq \tfrac{1}{N} \text{ and } \tfrac{1}{N} \leq r_2 \leq \tfrac{3}{N} \quad \Rightarrow \quad (\theta_2, r_2) \in \sigma^{-1}\mathcal{R}_*,$$

thus the formula

$$W_N(\theta_2, r_2) = \tilde{\eta}_{N,L}(r_2 - \tfrac{2}{N})\, \xi_N\big(\tau(\theta_2, N r_2)\big)$$

defines a function $W_N \in G^{\alpha,L}(\mathbb{T} \times [0, 3])$ which has its support contained in $\sigma^{-1}\mathcal{K} \subset \sigma^{-1}\mathcal{R}_*$. Moreover, inside $\sigma^{-1}\mathcal{R}_*$, the functions W_N and $\frac{X^2}{2} + \frac{X^4}{4}$ coincide in a neighbourhood of x_2, whereas W_N vanishes identically for $|X| \geq \frac{1}{2N}$.

LEMMA 4.3. *Assume $N \geq \max(\frac{5}{4\delta_*}, 14)$. There exists a positive number $\mu_{N,M}$ which depends only on N and M such that, if $0 < \mu \leq \mu_{N,M}$, the point $x_2 = (0, \frac{1}{N} r_*^{(M)})$ is a stable elliptic fixed point of the q^{th} iterate of*

$$G_2 = \Phi^{\frac{1}{2} r_2^2 + \frac{1}{N^2} V(\theta_2)} \circ \Phi^{\mu W_N},$$

contained in an elliptic island $\mathcal{A}_2$ which satisfies

$$\mathcal{A}_2 = (G_2)^q(\mathcal{A}_2) \subset \{ -\tfrac{1}{4N} \leq \theta_2 \leq \tfrac{1}{4N},\ \tfrac{2}{N} \leq r_2 \leq \tfrac{3}{N} \}.$$

Moreover, if $N = pN'$ with integers $p \geq 6$ and $N' \geq 1$,

$$(G_2)^{\ell N'}(\mathcal{A}_2) \subset \{ \tfrac{3}{4p} \leq \theta_2 \leq 1 - \tfrac{3}{4p} \}, \qquad 1 \leq \ell \leq p - 1.$$

Proof. Let $w(\theta_2, r_2) = \mu W_N(\theta_2, r_2)$ and $B(X) = \mu\big(\frac{X^2}{2} + \frac{X^4}{4}\big)$: these functions can be identified in the region $\{ |X| < \frac{1}{4N} \}$ inside $\sigma^{-1}\mathcal{R}_*$. On the other hand, if we define the function A by $A(0) = 0$ and $A'(Y) = M - T_*(e^{(M)} + NY)$, equation (30) allows us to identify $\big(\Phi^{P_N}\big)^q$ and $\Phi^{A(Y)}$ in the region $\{ |X| < \delta_*, |Y| < \frac{1}{N}\rho_*^{(M)} \}$ inside $\sigma^{-1}\mathcal{R}_*$.

We first determine a neighbourhood of x_2 where $(G_2)^q$ can be written $\Phi^A \circ \Phi^B$ when using the coordinates (X, Y). We shall assume $0 < \mu \leq 1$.

Let

$$\mathcal{N} = \{ |X| < \delta, |Y| < \delta \} \subset \sigma^{-1}\mathcal{R}_*, \qquad \delta = \tfrac{1}{3N} \min(\rho_*^{(M)}, \tfrac{1}{4N|T_*'(\frac{1}{2}e^{(M)})|}).$$

We have $\delta \leq \frac{1}{4N}$ (and even $\delta < \frac{1}{12N}$ because $\rho_*^{(M)} < \frac{1}{4}$), therefore the maps Φ^w and Φ^B can be identified in $\mathcal{N}$ and, since $|X + X^3| \leq 2\delta$ in $\mathcal{N}$,

$$\mathcal{N}' = \Phi^w(\mathcal{N}) \subset \{ |X| < \delta, |Y| < 3\delta \}.$$

With a slight abuse of notation, we can consider that G_2 coincides with $\Phi^{P_N} \circ \Phi^B$ on $\mathcal{N}$.

In view of equation (29), we have $G_2(\mathcal{N}) = \Phi^{P_N}(\mathcal{N}') \subset \{ \frac{1}{N} - \delta < X < \frac{1}{N} + \delta, |Y| \leq 3\delta \}$ and, recalling that $\delta \leq \frac{1}{4N}$, $w = 0$ on $G_2(\mathcal{N})$. By an easy induction, we obtain $(G_2)^s = \big(\Phi^{P_N}\big)^s \circ \Phi^B$ on $\mathcal{N}$ for $s \geq 2$ and s small enough to ensure $w = 0$ on $(G_2)^{s-1}(\mathcal{N}) = \big(\Phi^{P_N}\big)^{s-1}(\mathcal{N}')$.

Let us check that we can reach the value $s = q$. Writing $\big(\Phi^{P_N}\big)^{q-1}(\mathcal{N}') = \big(\Phi^{P_N}\big)^q\big(\Phi^{-P_N}(\mathcal{N}')\big)$, with

$$\Phi^{-P_N}(\mathcal{N}') \subset \{ -\frac{1}{N} - \delta < X < -\frac{1}{N} + \delta, |Y| \leq 3\delta \},$$

we see that our choice of δ ensures $3\delta \leq \frac{1}{N}\rho_*^{(M)}$, while $\frac{1}{N} + \delta \leq \delta_*$ since N is large enough, therefore $\big(\Phi^{P_N}\big)^q$ and Φ^A coincide on $\Phi^{-P_N}(\mathcal{N}')$. Moreover, in that domain, the mean value theorem yields $|A'(Y)| \leq 3N\delta |T_*'(\frac{1}{2}e^{(M)})|$ (because $e^{(M)} + NY \geq \frac{1}{2}e^{(M)}$ and $|T_*'|$ is decreasing), hence

$$X + A'(Y) < -\frac{1}{N} + \delta + \frac{1}{4N} < -\frac{1}{2N},$$

thus $w = 0$ on $\Phi^{P_N}(\mathcal{N}'), (\Phi^{P_N})^2(\mathcal{N}'), \dots, (\Phi^{P_N})^{q-1}(\mathcal{N}')$. We have thus proved our claim.

Having at our disposal this neighbourhood $\mathcal{N}$ of x_2, in which the q^{th} iterate of G_2 can be written $\Phi^A \circ \Phi^B$, we now apply Lemma 3.1.

The Taylor formula gives $A'(Y) = A_1 Y + A_2 Y^2 + A_3 Y^3 + O(Y^4)$ with $A_1 = -NT_*'(e^{(M)}) > 0$, $A_2 = -\frac{1}{2}N^2 T_*''(e^{(M)})$ and $A_3 = -\frac{1}{6}N^3 T_*'''(e^{(M)})$, while $B'(X) = \mu(X + X^3)$. We thus get ellipticity as soon as $\mu < \frac{2}{N|T_*'(e^{(M)})|}$ and we can check that the first Birkhoff invariant $\gamma_1 = \Gamma + \Gamma'$ is negative for μ small enough. For this, since $\Gamma' < 0$, it is sufficient to observe that

$$\lim_{\mu\to 0} \frac{|\gamma_0|\Gamma}{\mu^3} = \frac{10A_2^2 - 9A_1A_3}{6A_1}$$

(this is obtained by a straightforward asymptotic analysis of γ_0, λ, ω and Γ) and that the last quantity is negative, since the Cauchy-Schwarz inequality yields

$$\left(\int_0^1 \frac{d\theta}{\big(e - V(\theta)\big)^{5/2}}\right)^2 < \int_0^1 \frac{d\theta}{\big(e - V(\theta)\big)^{3/2}} \int_0^1 \frac{d\theta}{\big(e - V(\theta)\big)^{7/2}}.$$

The twist condition $\gamma_1 \neq 0$ being fulfilled, Moser's stability theorem provides us with the desired q-periodic elliptic island $\mathcal{A}_2 \subset \mathcal{N}$, the orbit of which is well enough located in view of the above description of the first q iterates of $\mathcal{N}$ under G_2 and Lemma 4.2 (on the one hand, $\mathcal{A}_2 \subset \mathcal{N} \subset \{\, |\theta_2| \le (2 + M^{-2})\delta,\ \frac{2}{N} \le r_2 \le \frac{3}{N}\,\}$; on the other hand, information on the location of $(G_2)^{\ell N'}(\mathcal{A}_2)$ is obtained by observing that a point in that set—unless it falls outside $\sigma^{-1}\mathcal{R}_*$—has coordinates (X, Y) with $|X \pm \frac{1}{p}| \le \delta$ when $\ell = 1$ or $p-1$, hence $|\theta_2| \ge \frac{3}{4p}$ because $1 + 4\pi^2 X^2$ is close enough to 1, and the same is true *a fortiori* for the intermediate values of ℓ). $\square$

For the sequel, we thus choose

$$w_2^{(j)} = \mu_j W_{N_j}, \qquad \mu_j = \min\Big(\frac{1}{N_j^2 \|W_{N_j}\|_{\alpha,L}}, \mu_{N_j,M_j}\Big), \tag{31}$$

with M_j as in (33) below. This way,

$$\|w_2^{(j)}\|_{\alpha,L} \le \frac{1}{N_j^2}$$

and $G_2 = \Phi^{\frac{1}{2}r_2^2 + \frac{1}{N_j^2}V(\theta_2)} \circ \Phi^{w_2^{(j)}(\theta_2, r_2)}$ admits a q_j-periodic domain $\mathcal{A}_2^{(j)}$ as described in Lemma 4.3, with $N = N_j$, $p = p_j$, $N' = N_j'$ and $q_j = N_j M_j$.

Remark 4.1. *We did not try to study the size of $\mathcal{A}_2^{(j)}$ (nor that of $\hat{\mathcal{A}}^{(j)}$ in Section 4.1). This size is probably exponentially small, because $\|W_{N_j}\|_{\alpha,L}$ is exponentially large, but it seems to us that it would be larger (although still exponentially small) if one would use a function like $\nu_j \eta_{N_j,L}(\theta_2)$, with ν_j well chosen, instead of $w_2^{(j)}$; however, checking that such a function is sufficient to create an elliptic island would require more complicated calculations than those of the proof of Lemma 3.1.*

4.4. *Choice of the function $g^{(j)}$ and end of the proof of Proposition 3.1.* Putting together the conclusions of Sections 4.1 and 4.3, we obtain a function

$$w_j = w_2^{(j)}(\theta_2, r_2) + w_3^{(j)}(\theta_3) + \cdots + w_n^{(j)}(\theta_n)$$

satisfying $\|w^{(j)}\|_{\alpha,L} \leq \frac{2}{N_j^2}$ and such that $G = \Phi^{\frac{1}{2}(r_2^2+\cdots+r_n^2)+v_j(\theta_2)} \circ \Phi^{w_j}$ admits a periodic domain $\mathcal{A} = \mathcal{A}_2^{(j)} \times \hat{\mathcal{A}}^{(j)}$ of minimal period q_j.

Let us define the function

$$g^{(j)} = \eta_{2p_j,L}(\theta_2)\hat{g}^{(j)}(\theta_3, \ldots, \theta_n), \qquad \hat{g}^{(j)} = \eta_{2p_{j,L}} \otimes \cdots \otimes \eta_{2p_{j-n+3,L}}, \tag{32}$$

and consider the map

$$\Psi_j = \Phi^{u_j} \circ \big(\Phi^{\frac{1}{2}r_1^2} \times G\big), \qquad u_j = \frac{1}{N_j M_j} U \otimes g^{(j)},$$

with U like in Section 3.2 and M_j as follows:

Lemma 4.4. *Let us denote by* $[\,.\,]$ *the integer part of a real number and use the same* $c > 0$ *(which depends only on* α *and* L*) as in Lemma 3.3. There exist an integer* J*, which depends only on* n*, and positive real numbers* $c_1 < c_2$*, which depend only on* n, α, L*, such that, with the choice*

$$M_j = \big[N_j \|U\|_{\alpha,L} \exp\big((n-1)c(2p_j)^{\frac{1}{\alpha-1}}\big) + 1\big], \tag{33}$$

the numbers $\varepsilon_j = \max(\|u_j\|_{\alpha,L}, \|v_j\|_{\alpha,L}, \|w_j\|_{\alpha,L})$ *and* $q_j = N_j M_j$ *satisfy*

$$\varepsilon_j = \frac{\|V\|_{\alpha,L}}{N_j^2}, \quad c_1 N_j^2 \exp\big(c_1 N_j^{\frac{1}{(n-1)(\alpha-1)}}\big) \leq q_j \leq c_2 N_j^2 \exp\big(c_2 N_j^{\frac{1}{(n-1)(\alpha-1)}}\big) \tag{34}$$

for all $j \geq J$*.*

Proof. In view of Lemma 3.3,

$$\|g^{(j)}\|_{\alpha,L} \leq \exp\big((n-1)c(2p_j)^{\frac{1}{\alpha-1}}\big), \tag{35}$$

and the definition of M_j yields

$$\|u_j\|_{\alpha,L} \leq \frac{\|U\|_{\alpha,L}}{N_j M_j} \exp\big((n-1)c(2p_j)^{\frac{1}{\alpha-1}}\big) \leq \frac{1}{N_j^2}.$$

Since $\|V\|_{\alpha,L} > 2$, the number ε_j thus coincides with $\|v_j\|_{\alpha,L}$.

On the other hand, if $n \geq 3$, we can use the Prime Number Theorem to ensure $N_j' \in [2^{-(n-2)}p_j^{n-2}, p_j^{n-2}]$ for $j \geq J$, hence $N_j^{\frac{1}{n-1}} \leq p_j \leq 2N_j^{\frac{1}{n-1}}$, and the conclusion follows. □

Lemma 3.2 can be applied to this situation, with $F = \Phi^{\frac{1}{2}r_1^2}$ and $f = \frac{1}{q_j}U$. Our function $g^{(j)}$ satisfies indeed the requirement (11), as is easily checked by distinguishing the cases where N_j' divides s and the cases where it does not. According to formula (12), the dynamics of $\psi = \psi_{q_j,U}$ is thus embedded into the $(\Psi_j)^{q_j}$-invariant set $\mathbb{A} \times \mathcal{A}$. In particular, by Proposition 3.2, we get a wandering domain $\mathcal{D}_j = \mathcal{B}_{q_j} \times \mathcal{A}$ which satisfies the desired properties.

We end the proof of Proposition 3.1 by renumbering our sequences, replacing j by $J + j$.

Remark 4.2. *Observe that the instability exponent $a = \frac{1}{2(n-1)(\alpha-1)}$ which is obtained at the end stems from inequality (35), which has dictated our choice of M_j and thus of q_j. This inequality reflects the necessity of "separating" through the function $g^{(j)}$ the set $\mathcal{A}$ from its iterates which lie at a distance $\approx p_j \approx N_j^{\frac{-1}{n-1}}$. It is the use of Lemma 3.3 which has introduced the factor $\frac{1}{\alpha-1}$. This is to be compared with Lemma 2.4 of [**MS03**], where we had managed to introduce the optimal factor $\frac{1}{\alpha}$ instead of $\frac{1}{\alpha-1}$ because compact-supported functions were used somewhat differently in that paper.*

In fact, we do not know whether it is possible to get the same instability exponent for wandering points and for wandering polydiscs. The difficulty is that the condition that the function $g^{(j)}$ be identically equal to 1 on $\mathcal{A}$ imposed by the coupling lemma is much more demanding when $\mathcal{A}$ is not reduced to one point. It can be shown that the exponent $\frac{1}{\alpha-1}$ in Lemma 3.3 is optimal.

5. *Proof of Theorem 2.1 and a C^k variant*

Let $\alpha > 1$, $\Lambda > 0$, $N \geq 3$ and $m \in \{2, \dots, N-1\}$. We shall apply Proposition 3.1 with

$$n = N - m + 1, \qquad L = \Lambda\Big(1 + (\Lambda^\alpha + 3^\alpha + \frac{1}{2})\|\varphi\|_{\alpha,\Lambda}\Big)^{\frac{1}{\alpha}}, \qquad \varphi(t) = \frac{\eta_{8,\Lambda}(t)}{\int_{\mathbb{T}} \eta_{8,\Lambda}}. \quad (36)$$

We set $h_0(r) = \frac{1}{2}(r_1^2 + \cdots + r_{N-1}^2)$.

5.1. *Nearly m-resonant wandering domains.*

We suppose in this section that $m \geq 3$ and we wish to obtain nearly m-resonant domains instead of the nearly doubly resonant domains of Proposition 3.1. To that end, we add $m-2$ degrees of freedom and consider

$$\underline{G}^{(j)} = \Phi^{\frac{1}{2}(r_{n+1}+\cdots+r_{n+m-2}) + \frac{1}{N_j^2}(S(\theta_{n+1})+\cdots+S(\theta_{n+m-2}))},$$

with the same function S as in Section 3.4 and the same N_j as in Section 4.

We make use of the system $\underline{G}^{(j)}$ quite in the same way as in Section 3.4: since

$$\underline{G}^{(j)} = \underline{G}^{(j)}_{n+1} \times \ldots \times \underline{G}^{(j)}_{n+m-2}, \qquad \underline{G}^{(j)}_i = \Phi^{\frac{1}{2}r_i^2 + \frac{1}{N_j^2}S(\theta_i)}, \qquad n+1 \leq i \leq n+m-2$$

and the scaling $(\theta_*, r_*) = \sigma_i^{(j)}(\theta_i, r_i) = (\theta_i, N_j r_i)$ conjugates $\underline{G}^{(j)}_i$ with the time-$\frac{1}{N_j}$ map of the Hamiltonian $h_*(\theta_*, r_*) = \frac{1}{2}r_*^2 + S(\theta_*)$ which is reduced to the normalized harmonic oscillator in $\mathcal{R}_* = \{\, \theta_*^2 + r_*^2 < \frac{1}{16} \,\}$, we have a $\underline{G}^{(j)}$-periodic domain

$$\underline{\mathcal{A}}^{(j)} = \big(\sigma^{(j)}_{n+1}\big)^{-1}\mathcal{R}_* \times \ldots \times (\sigma^{(j)}_{n+m-2})^{-1}\mathcal{R}_*$$

with minimal period N_j.

Since N_j divides q_j, the domain $\underline{\mathcal{D}}_j = \mathcal{D}_j \times \underline{\mathcal{A}}^{(j)}$ is wandering for

$$\underline{\Psi}_j = \Psi_j \times \underline{G}^{(j)} = \Phi^{u_j} \circ \Phi^{h_0(r) + \underline{v}_j} \circ \Phi^{w_j},$$

with

$$\underline{v}_j = \frac{1}{N_j^2}\big(V(\theta_2) + S(\theta_{n+1}) + \cdots + S(\theta_{n+m-2})\big).$$

Moreover,

$$(\underline{\Psi}_j)^{\ell q_j}(\underline{\mathcal{D}}_j) = T_{\frac{\ell}{q_j}}(\underline{\mathcal{D}}_j), \qquad \ell \in \mathbb{Z} \tag{37}$$

and $\operatorname{dist}(x, \{ r_1 = r_2 = 0 \text{ and } r_{n+1} = \ldots = r_{n+m-2} \}) \le 3\sqrt{\underline{\varepsilon}_j}$ for all $x \in \mathcal{D}_j$, where $\underline{\varepsilon}_j = \frac{1}{N_j^2}(\|V\|_{\alpha,L} + (m-2)\|S\|_{\alpha,L})$ is the new small parameter.

Since this new parameter is only slightly larger than the ε_j of Section 4 (up to a multiplicative constant, the true small parameter is in fact $1/N_j^2$), we can also find new constants $C_1 < C_2$ such that inequalities (9) hold with $\underline{\varepsilon}_j$ in place of ε_j.

5.2. *Suspension.* Our aim is now to pass from the discrete dynamical system $\underline{\Psi}_j$, defined on $\mathbb{A}^{N-1}$, to a near-integrable Hamiltonian flow. We shall first define a non-autonomous time-periodic Hamiltonian $H_j(\theta, r, t)$, where $(\theta, r) \in \mathbb{A}^{N-1}$ and $t \in \mathbb{T}$, the return map of which coincides with $\underline{\Psi}_j$ for the section $\{ t \equiv 0 \} \approx \mathbb{A}^{N-1}$.

If $H(\theta, r, t)$ is a non-autonomous Hamiltonian function defined on $\mathbb{A}^{N-1} \times \mathbb{T}$, we extend the notation Φ^H by considering the time-1 map of the vector field $\tilde{X}_H$ of the *extended phase space* $\mathbb{A}^{N-1} \times \mathbb{T}$,

$$\tilde{X}_H \left| \begin{array}{rcl} \dot{\theta} & = & \partial_r H(\theta, r, t) \\ \dot{r} & = & -\partial_\theta H(\theta, r, t) \\ \dot{t} & = & 1. \end{array} \right.$$

Thus Φ^H is a mapping of $\mathbb{A}^{N-1} \times \mathbb{T}$, the last component of which is always trivial.

We shall obtain the desired Hamiltonian $H_j(\theta, r, t)$ by applying the following lemma with $u = u_j$, $v = \underline{v}_j$ and $w = w_j$.

Lemma 5.1. *Assume Λ and L are related by (36) and set*

$$K_1 = \frac{1}{\int_{\mathbb{T}} \eta_{8,\Lambda}}, \qquad K_2 = \frac{\|\eta_{8,\Lambda}\|_{\alpha,\Lambda}}{\int_{\mathbb{T}} \eta_{8,\Lambda}}.$$

Let $u, v, w : \mathbb{A}^{N-1} \to \mathbb{R}$ be smooth functions with support $\subset (\mathbb{T} \times [0,3])^{N-1}$, which belong to $G^{\alpha,L}\big((\mathbb{T} \times [0,3])^{N-1}\big)$ and have norms $\le 1/K_2$, and

$$\Psi = \Phi^u \circ \Phi^{h_0 + v} \circ \Phi^w, \qquad h_0(r) = \frac{1}{2}(r_1^2 + \cdots + r_{N-1}^2).$$

There exists a non-autonomous time-periodic Hamiltonian function H belonging to $\mathcal{G}^{\alpha,\Lambda}(\mathbb{A}^{N-1} \times \mathbb{T})$, such that $H(\theta, r, t) = h_0(r)$ if $(\theta, r, t) \in \mathbb{A}^{N-1} \times [0, \frac{1}{4}]$,

$$K_1 \max(\|u\|_{C^0}, \|v\|_{C^0}, \|w\|_{C^0}) \le d_{\alpha,\Lambda}(H, h_0) \le K_2 \max(\|u\|_{\alpha,L}, \|v\|_{\alpha,L}, \|w\|_{\alpha,L}) \tag{38}$$

and the time-1 map Φ^H for the corresponding autonomous vector field $\tilde{X}_H$ of $\mathbb{A}^{N-1} \times \mathbb{T}$ satisfies

$$\Phi^H(x, 0) = (\Psi(x), 0), \qquad x \in \mathbb{A}^{N-1}. \tag{39}$$

Proof. We shall obtain $H = h_0(r) + f(\theta, r, t)$ by adapting the suspension procedure of [**MS03**], Section 2.4.1.

Let us fix three periodic functions φ_1, φ_2 and φ_3 depending only on t, each one of total mass 1, such that the first one has support $\subset [\frac{3}{4}, 1]$, the second one $\subset [\frac{1}{2}, \frac{3}{4}]$ and the third one $\subset [\frac{1}{4}, \frac{1}{2}]$. For instance, we may take $\varphi_i(t) = \varphi(t - \frac{9-2i}{8})$ with φ as in (36). A simple way of fulfilling (39) is to use $H = H^*$ defined by

$$H^* = u \otimes \varphi_1 + (h_0 + v) \otimes \varphi_2 + w \otimes \varphi_3.$$

But the limit of H^* as $u, v, w \to 0$ is $h_0 \otimes \varphi_2$ rather than h_0. We thus apply the same modification as in [**MS03**], using $\tilde{\varphi}_2(t) = \int_0^t (\varphi_2(t') - 1)\, dt'$ to define an exact-symplectic time-periodic transformation $x^* = \mathcal{F}_t(x)$ by the formula

$$\mathcal{F}_t(\theta, r) = (\theta + \tilde{\varphi}_2(t) r, r), \qquad (\theta, r) \in \mathbb{A}^{N-1}$$

(notice that $\tilde{\varphi}_2$ is periodic because $\int_{\mathbb{T}} \varphi_2 = 1$). The conjugate vector field in $\mathbb{A}^{N-1} \times \mathbb{T}$ is generated by the Hamiltonian

$$H = h_0(r) + f(\theta, r, t), \qquad f(x, t) = \varphi_1(t) u \circ \mathcal{F}_t(x) + \varphi_2(t) v \circ \mathcal{F}_t(x) + \varphi_3(t) w \circ \mathcal{F}_t(x)$$

and still satisfies (39) because $\mathcal{F}_t$ is reduced to identity for $t = 0$. Our final formula is thus

$$f(\theta, r, t) = \varphi_1(t) u(\theta + \tilde{\varphi}_2(t) r, r) + \varphi_2(t) v(\theta + \tilde{\varphi}_2(t) r, r) + \varphi_3(t) w(\theta + \tilde{\varphi}_2(t) r, r) \quad (40)$$

(one can notice that $\tilde{\varphi}_2(t) = -t$ for $t \in [0, \frac{1}{2}]$ and $\tilde{\varphi}_2(t) = 1 - t$ for $t \in [\frac{3}{4}, 1]$).

In order to check (38), let us estimate norms on $\mathcal{K}_\nu = \mathbb{T}^N \times \overline{B}_{R_\nu}$ for $\nu \geq 1$. We are in fact dealing with functions which depend on one angular variable (as φ and the φ_i's) or on $2(N-1)$ variables (as u, v, w); in all cases, their support can be viewed as a subset of $\mathcal{K}_\nu$ (because $R_\nu \geq 3$).

On the one hand, since φ_1, φ_2 and φ_3 have disjoint supports and attain the value $K_1 = 1/\int_{\mathbb{T}} \eta_{8,\Lambda}$,

$$K_1 \max(\|u\|_{C^0}, \|v\|_{C^0}, \|w\|_{C^0}) \leq \|f\|_{C^0(\mathcal{K}_\nu)} \leq \|f\|_{\alpha, \Lambda_\nu, \mathcal{K}_\nu}.$$

On the other hand, we can apply the result on composition contained in Remark A.1 of the appendix of [**MS03**] to get the inequality

$$\|f\|_{\alpha, \Lambda_\nu, \mathcal{K}_\nu} \leq \|\varphi\|_{\alpha, \Lambda_\nu} \max(\|u\|_{\alpha, L}, \|v\|_{\alpha, L}, \|w\|_{\alpha, L}).$$

It is here that we use the definition of L in (36), to ensure the inequality

$$(\Lambda_\nu^\alpha + R_\nu) \|\tilde{\varphi}_2\|_{\alpha, \Lambda_\nu} - R_\nu \|\tilde{\varphi}_2\|_{C^0(\mathbb{T})} \leq L^\alpha - \Lambda_\nu^\alpha$$

which is required for applying the composition result (we also use the inequality $(\Lambda_\nu^\alpha + R_\nu) \|\tilde{\varphi}_2\|_{\alpha, \Lambda_\nu} - R_\nu \|\tilde{\varphi}_2\|_{C^0(\mathbb{T})} \leq (\Lambda_\nu^\alpha + R_\nu) \Lambda_\nu^\alpha \|\varphi_2\|_{\alpha, \Lambda_\nu} + \frac{1}{2} \Lambda_\nu^\alpha \|\varphi_2\|_{C^0(\mathbb{T})}$—see [**MS03**], Section 2.4.1—and the fact that, according to (2), $R_\nu \Lambda_\nu^\alpha$ is bounded by $3^\alpha \Lambda^\alpha$ independently of ν while $\Lambda_\nu \leq \Lambda$).

We end up with

$$K_1 \max(\|u\|_{C^0}, \|v\|_{C^0}, \|w\|_{C^0}) \leq \|f\|_{\alpha, \Lambda_\nu, \mathcal{K}_\nu} \leq K_2 \max(\|u\|_{\alpha, L}, \|v\|_{\alpha, L}, \|w\|_{\alpha, L}),$$

with $K_2 = \|\varphi\|_{\alpha, \Lambda} = K_1 \|\eta_{8,\Lambda}\|_{\alpha, \Lambda}$, whence the conclusion follows, since the right-hand side is ≤ 1. □

Applying the lemma to the map $\underline{\Psi}_j$ defined in Section 5.1, we obtain a non-autonomous Hamiltonian H_j. Let us consider the "tube of solutions" generated by the flow of H_j from $\underline{\mathcal{D}}_j \times \{0\}$ in the time-interval $[0, \frac{1}{4}]$:

$$\tilde{\underline{\mathcal{D}}}_j = \{\Phi^{tH_j}(x,0);\, x \in \underline{\mathcal{D}}_j,\, t \in [0, \tfrac{1}{4}]\} = \{(\theta + tr, r, t);\, (\theta, r) \in \underline{\mathcal{D}}_j,\, t \in [0, \tfrac{1}{4}]\}, \quad (41)$$

where the last identity is due to the fact that H_j is reduced to $h_0(r)$ for $t \in [0, \frac{1}{4}]$.

We shall now check that

$$\Phi^{\ell\tau_j H_j}(\tilde{\underline{\mathcal{D}}}_j) = \mathcal{T}^\ell \mathcal{R}^\ell(\tilde{\underline{\mathcal{D}}}_j), \qquad \ell \in \mathbb{Z}, \quad (42)$$

where $\tau_j = q_j^2$ and $\mathcal{T}$ and $\mathcal{R}$ are commuting transformations defined as in (3), except for the absence of the last component:

$$\begin{aligned} \mathcal{T}(\theta_1, r_1, \dots, \theta_{N-1}, r_{N-1}, t) &= (\theta_1, r_1 + 1, \dots, \theta_{N-1}, r_{N-1}, t), \\ \mathcal{R}(\theta_1, r_1, \dots, \theta_{N-1}, r_{N-1}, t) &= (\theta_1 + t, r_1, \dots, \theta_{N-1}, r_{N-1}, t). \end{aligned}$$

Notice the relation $\mathcal{T}(x,t) = (T_1(x), t)$, where T_1 is the translation defined just above the statement of Proposition 3.1. We observe that

$$\mathcal{R}^{-\ell}\mathcal{T}^{-\ell}\Phi^{sh_0}(x,0) = \Phi^{sh_0}\mathcal{T}^{-\ell}(x,0), \qquad x \in \mathbb{A}^{N-1},\, \ell \in \mathbb{Z},\, s \in \mathbb{R},$$

where the action of Φ^{sh_0} is extended to $\mathbb{A}^{N-1} \times \mathbb{T}$ in the obvious way: $\Phi^{sh_0}(\theta, r, t) = (\theta + sr, r, t + s)$. This action coincides with that of Φ^{sH_j} on the points $(x,t) \in \mathbb{A}^{N-1} \times \mathbb{T}$ such that $[t, t+s] \subset [0, \frac{1}{4}]$. Thus, if $(y,t) \in \tilde{\underline{\mathcal{D}}}_j$, we can write $(y,t) = \Phi^{tH_j}(x,0) = \Phi^{th_0}(x,0)$ for some $x \in \underline{\mathcal{D}}_j$ and $t \in [0, \frac{1}{4}]$, and

$$\Phi^{\ell\tau_j H_j}(y,t) = \Phi^{(t+\ell\tau_j)H_j}(x,0) = \Phi^{tH_j}((\underline{\Psi}_j)^{\ell\tau_j}(x), 0)$$

by virtue of (39), whence

$$\mathcal{R}^{-\ell}\mathcal{T}^{-\ell}\Phi^{\ell\tau_j H_j}(y,t) = \mathcal{R}^{-\ell}\mathcal{T}^{-\ell}\Phi^{th_0}((\underline{\Psi}_j)^{\ell\tau_j}(x), 0) = \Phi^{th_0}(T_1^{-\ell}(\underline{\Psi}_j)^{\ell\tau_j}(x), 0).$$

We conclude that

$$\mathcal{R}^{-\ell}\mathcal{T}^{-\ell}\Phi^{\ell\tau_j H_j}(\tilde{\underline{\mathcal{D}}}_j) = \bigcup_{t \in [0, \frac{1}{4}]} \Phi^{th_0}(T_1^{-\ell}(\underline{\Psi}_j)^{\ell\tau_j}(\underline{\mathcal{D}}_j) \times \{0\}) = \bigcup_{t \in [0, \frac{1}{4}]} \Phi^{th_0}(\underline{\mathcal{D}}_j \times \{0\})$$

by (37), which yields $\tilde{\underline{\mathcal{D}}}_j$. Thus (42) is proved.

Finally, we pass to the autonomous Hamiltonian system generated by

$$\mathcal{H}_j(\theta, r, \theta_N, r_N) = r_N + H_j(\theta, r, \theta_N), \qquad (\theta, r) \in \mathbb{A}^{N-1},\, (\theta_N, r_N) \in \mathbb{A} \quad (43)$$

and take $\mathcal{U}_j = \tilde{\underline{\mathcal{D}}}_j \times \mathbb{R}$ as wandering domain for $\Phi^{\mathcal{H}_j}$. This ends the proof of Theorem 2.1.

REMARK 5.1. *One could also take for $\mathcal{U}_j$ the tube of solutions generated by $\mathcal{H}_j$ from $\underline{\mathcal{D}}_j \times \{0\} \times I$ during the time-interval $[0, \frac{1}{4}]$, where I is some real interval. We would still obtain a wandering domain, but relation (4) would persist for the $N-1$ first components only: the last component r_N does indeed drift so as to maintain $\frac{1}{2}r_1^2 + r_N$ bounded.*

5.3. *Variants in the C^k category.* The above construction can be adapted with little effort to the case of finite differentiability.

Let $k \geq 0$. If we do not care about the size of the wandering domain, we can use the coupling lemma with the same systems $\psi = \Phi^{\frac{1}{q_j}U} \circ \big(\Phi^{\frac{1}{2}r_1^2}\big)^{q_j}$ and $G = G_2 \times G_3 \times \cdots \times G_{n-1}$ as in Section 4, except that we modify the tuning of the parameters: $q_j = N_j M_j$ with a new value

$$M_j = \big[(\mathrm{e}^2 c)^{n-1} \|U\|_{C^k} N_j p_j^k + 1\big],$$

where c is the constant appearing in Remark 3.2. Indeed, choosing now

$$g^{(j)} = \eta_{2p_j} \otimes \hat{g}^{(j)}, \qquad \hat{g}^{(j)} = \eta_{2p_j} \otimes \cdots \eta_{2p_{j-n+3}}$$

and observing that

$$\|g^{(j)}\|_{C^k} \leq c^{n-1} \sum_{|\ell| \leq k} \frac{1}{\ell!} (2p_j)^{\ell_1} (2p_j)^{\ell_2} \dots (2p_{j-n+3})^{\ell_{n-1}} \leq (c\mathrm{e}^2)^{n-1} p_j^k,$$

we get $\|\frac{1}{q_j} U \otimes g^{(j)}\|_{C^k} \leq \frac{1}{N_j^2}$, hence the small parameter ε_j is still $1/N_j^2$ (up to some multiplicative constant). We therefore obtain a C^k version of Proposition 3.1 with

$$C_1 N_j^{2(\frac{k}{n-1}+2)} \leq \tau_j = q_j^2 \leq C_2 N_j^{2(\frac{k}{n-1}+2)}$$

for some positive constants $C_1 < C_2$. (We would probably obtain larger wandering domains by replacing the various Gevrey functions $\eta_{p,L}$ occurring in the construction of G by their C^k counterpart.)

Inserting $m-2$ extra degrees of freedom like we did in Section 5.1 and applying the suspension procedure of Section 5.2 with minor changes, we end up with a C^k version of Theorem 2.1 where the drifting time τ_j satisfies inequalities of the form

$$C_1 \Big(\frac{1}{\varepsilon_j}\Big)^{a^*} \leq \tau_j \leq C_2 \Big(\frac{1}{\varepsilon_j}\Big)^{a^*}, \qquad a^* = \frac{k}{N-m} + 2.$$

Observe that "C^k version" refers here to the topology, in which our unstable systems tend to the standard integrable one, and to the way ε_j, *i.e.* the distance to integrability, is measured, but all the functions we manipulate are in fact C^∞ (as in Proposition 3.3).

This instability result is to be compared with the C^k analogue of Nekhoroshev's Theorem in the quasi-convex case. As one would expect, the exponentially long stability times that are available in the analytic and Gevrey categories must replaced by shorter times, which are given by some positive power of the inverse of the small parameter, in the C^k category. More precisely, Theorem A from [**MS03**] and its addendum can be transposed as follows, for any $k \geq 2$: the hypotheses "$h, H \in G^{\alpha,L}(\mathbb{T}^N \times \overline{B}_R)$" being replaced by "$h, H \in C^k(\mathbb{T}^N \times \overline{B}_R)$" and ε being defined using $\|H - h\|_{C^k}$ instead of $\|H - h\|_{\alpha,L}$, the conclusion is that the confinement property $\|r(t) - r_0\| \leq C_2 \varepsilon^b$ holds at least for $|t| \leq C_1(\frac{1}{\varepsilon})^a$, with

$$a = \frac{k}{N}, \quad b = \frac{1}{2N}.$$

And for solutions starting at a distance $\varepsilon^{1/2}$ of a m-fold resonant surface (defined by some resonant submodule of codimension $d = N - m$ of $\mathbb{Z}^N$), the exponents improve:

$$a = \frac{k}{N-m}, \quad b = \frac{1}{2(N-m)}.$$

We do not give here any detail of the proof of the C^k stability result, preferring to devote another paper to the systematic study of finite-time stability in the differentiable category and in ultradifferentiable classes.

6. *Splitting, symbolic dynamics and random walk*

The aim of this section is to investigate new dynamical behaviours, closely related to the preceding examples. Many possibilities are clearly left open, but we shall content ourselves with the case where the pendulum map has transverse homoclinic points instead of elliptic islands; as we shall see, the functions involved in the construction are almost the same as in the previous examples. The existence of such homoclinic points has several new consequences, which will enable us to extend the results of [**MS03**].

Regarding the splitting problem, we shall prove the existence of a one-parameter family of hyperbolic tori, the invariant manifolds of which split along at least two orthogonal directions, and we shall also completely describe the structure of the splitting matrix of these manifolds at their homoclinic points.

Concerning the search for unstable orbits, we shall take advantage of the existence of a horseshoe associated with the homoclinic point (Birkhoff-Smale-Alexeiev theorem), and shall be able to construct oscillating orbits: these are orbits whose r_1-projection can take any prescribed sequence of values chosen in a given set. The drifting orbits biasymptotic to infinity of [**MS03**] now appear as a particular case of this new construction, but we also get much more general ones, whose complete description will be given in Proposition 6.4.

6.1. *Transverse homoclinic points for the perturbed pendulum.*

The construction of our homoclinic points will be very similar to that of the elliptic islands in Section 4.3. The main difference is that we shall make use of a perturbation centred on the upper point of the separatrix of the pendulum map, instead of the M-periodic point that we considered in Section 4.3. We shall keep the notation of Sections 4.2 and 4.3, in particular $P_N(\theta_2, r_2) = \frac{1}{2}r_2^2 + \frac{1}{N^2}V(\theta_2)$ with $V(\theta_2) = -1 - \cos(2\pi\theta_2)$, and we assume $N \geq 3$.

We start again with a straightening symplectic transformation $(\theta_2, r_2) \mapsto (X, Y)$ defined in $\sigma^{-1}(\mathcal{R}_*) = \{\, |\theta_2| < \frac{1}{4},\ r_2 > \frac{1}{N}\,\}$, which differs from that defined in Section 4.2 only by its central point. Namely, we set

$$X = \tau \circ \sigma(\theta_2, r_2), \quad Y = e \circ \sigma(\theta_2, r_2)/N,$$

so $\{Y = 0\}$ corresponds to the upper separatrix of the pendulum map Φ^{P_N}, and the upper point on that separatrix has $(0, 0)$ as new coordinates. We introduce a

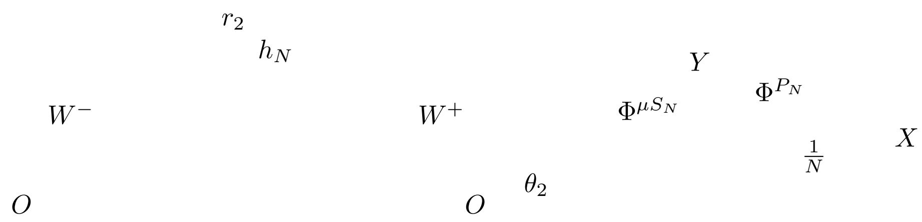

FIGURE 5. Left: Invariant manifolds of $G_2 = \Phi^{P_N} \circ \Phi^{\mu S_N}$. Right: Action of $\Phi^{\mu S_N}$ and Φ^{P_N}.

compact-supported function

$$S_N(\theta_2, r_2) = \tilde{\eta}_{N,L}(r_2 - \tfrac{2}{N})\big(\tfrac{X^2}{2}\,\tilde{\eta}_{4N,\Lambda_N}(X)\big); \tag{44}$$

whose effect will be to create a local distorsion of the separatrix around its upper point. As in Section 4.3, the function S_N belongs to $G^{\alpha,L}(\mathbb{T}\times[0,3])$ and satisfies

$$\|S_N\|_{\alpha,L} \le \exp(c\,N^{\frac{1}{\alpha-1}})$$

where c is a positive real number depending only on α. Moreover, one checks that the support of S_N is contained in $R_N = [-\frac{3}{4N}, \frac{3}{4N}] \times [\frac{1}{N}, \frac{3}{N}]$.

A map of the form $\Phi^{\mu S_N}$, for any real number μ, will be referred to as a *splitting map.* The right part of Figure 5 shows, in the coordinates (X, Y), the effect of $\Phi^{\mu S_N}$ on the axis $X = 0$, which will be the main feature for the creation of the splitting of the invariant manifolds in the next lemma.

LEMMA 6.1. *The point $O = (1/2, 0)$ is a hyperbolic fixed point for the map*

$$G_2 = \Phi^{P_N} \circ \Phi^{\mu S_N}.$$

Its stable and unstable manifolds $W^{\pm}(O, G_2)$ have a transverse intersection at the homoclinic point $h_N = (0, 2/N)$. At this point, the angle Θ of these two manifolds, measured in the (θ_2, r_2) coordinates, satisfies $\tan\Theta = -\frac{\mu}{4}$.

Moreover, there exists a connected neighbourhood Σ_N^+ of h_N in the intersection $R_N \cap W^+(O, G_2)$ such that the restriction of P_N to Σ_N^+ is a bijection onto the energy segment $[-\mu/8N, \mu/8N]$.

Proof. The assertion on O is plain: the map $\Phi^{\mu S_N}$ is reduced to identity outside the set $\{|\theta_2| \le 1/N\}$. Consider the fundamental domain Δ centred on the point h_N for the invariant manifolds $W^{\pm}(O, \Phi^{P_N})$, that is the segment of the upper separatrix comprised between the points $\Phi^{-\frac{1}{2}P_N}(h_N)$ and $\Phi^{-\frac{1}{2}P_N}(h_N)$, and denote by Δ^- and Δ^+ its negative and positive iterates under Φ^{P_N}. Observe that, due again to the form of $\Phi^{\mu S_N}$, $\Delta^- \subset W^-(O, G_2)$ and $\Delta^+ \subset W^+(O, G_2)$. Therefore, the direct and inverse images of these segments under G_2 satisfy:

$$G_2(\Delta^-) \subset W^-(O, G_2), \quad G_2^{-1}(\Delta^+) \subset W^+(O, G_2).$$

To determine these two images, is is convenient to make use of the straightening coordinates (X, Y), in which the map Φ^{P_N} is just the translation of step $1/N$

along the X-axis. The segments Δ, $\Delta^{\pm}$ are the subsets the X-axis defined by $\Delta = [-1/2N, 1/2N]$, $\Delta^+ = [1/2N, 3/2N]$ and $\Delta^- = -\Delta^+$, while the support of $\Phi^{\mu S_N}$ is contained in $\{|X| \le 1/4N\}$, from which one deduces

$$G_2(\Delta^-) = \Delta, \quad G_2^{-1}(\Delta^+) = \Phi^{-\mu S_N}(\Delta).$$

Therefore, in a neighbourhood of h_N, the unstable manifold coincides with the separatrix of the pendulum map, while the stable one is the preimage of that separatrix by the splitting map (as illustrated on the left part of Figure 5). The second assertion comes from the diagonal form of the derivative of the change $(X, Y) \mapsto (\theta_2, r_2)$ at the point $(0, 0)$.

Finally, the segment Σ_N^+ will be the rectilinear part of $G_2^{-1}(\Delta^+)$, whose projection on the Y-axis is the segment $[-\mu/8N, \mu/8N]$. By definition of the straightening coordinates, this proves the last part of the lemma. □

In the following, we shall denote by Σ_N^- the neighbourhood of the point h_N in $W^-(O, G_2)$ whose projection on the X-axis is the interval $[-1/4N, 1/4N]$.

6.2. *Splitting of the invariant manifolds.* In this section we examine the consequences of the existence the previous homoclinic point concerning the coupling of G_2 to other degrees of freedom. Our main purpose will be to describe the splitting of the invariant manifolds of the n-dimensional tori which already appeared in the example of [**MS03**]. In the present example, we want to describe not only the longitudinal splitting but also the transverse one (see the precise meaning below). We also would like to keep the structure of the system as close as possible to that of [**MS03**], but in order to determine a complete splitting matrix we shall have to use, in addition to the new map G_2, the coupling functions $g_i^{(j)}$ introduced in Section 4.4. Nevertheless, as the construction is very much like that of [**MS03**], we shall content ourselves with detailed statements and short proofs.

We assume $n \ge 3$ in the rest of this section, the case $n = 2$ being analogous and simpler. As in the preceding sections, we shall use the sequences (N_j) and (N'_j) defined in (19). We introduce a new sequence of maps (Ψ_j):

$$\Psi_j = \Phi^{\frac{1}{q_j} U \otimes g^{(j)}} \circ \left(\Phi^{\frac{1}{2} r_1^2} \times \left(\Phi^{\frac{1}{2} r_2^2 + \frac{1}{N_j^2} V(\theta_2)} \circ \Phi^{\mu_j S_{N_j}}\right) \times \Phi^{\frac{1}{2}(r_3^2 + \cdots + r_n^2)}\right), \tag{45}$$

with $U(\theta_1) = \frac{1}{2\pi} \sin(2\pi\theta_1)$, $V(\theta_2) = -1 - \cos(2\pi\theta_2)$, the same function $g^{(j)}$ as in (32), a function S_{N_j} defined by (44), the same $q_j = M_j N_j$ as in Lemma 4.4 and

$$\mu_j = \frac{1}{N_j^2 \|S_{N_j}\|_{\alpha,L}}.$$

Notice that $\mu_j \ge \exp(-c N_j^{\frac{1}{\alpha-1}})$, while $\|\frac{1}{q_j} U \otimes g^{(j)}\|_{\alpha,L}, \|\mu_j S_{N_j}\|_{\alpha,L} < \varepsilon_j = \frac{\|V\|_{\alpha,L}}{N_j^2}$.

We finally introduce the following neighbourhood of the origin in the torus $\mathbb{T}^{n-2}$:

$$\hat{B}_j = \{(\theta_3, \ldots, \theta_n) \in \mathbb{T}^{n-2} \mid |\theta_3| < \tfrac{1}{2p_j}, \ldots, |\theta_n| < \tfrac{1}{2p_{j-n+3}}\},$$

so $g_3 \otimes \cdots \otimes g_n \equiv 1$ on $\hat{B}_j$, and we set

$$\hat{r}^{(j)} = \Big(\frac{1}{p_j}, \dots, \frac{1}{p_{j+n-3}}\Big).$$

Our first statement depicts the various hyperbolic objects of Ψ_j, together with their invariant manifolds. As in [**MS03**], one first notices that the $(2n-2)$-dimensional annulus $\mathbb{A} \times \{(1/2, 0)\} \times \mathbb{A}^{n-2}$ is invariant under Ψ_j and normally hyperbolic in $\mathbb{A}^n$. Moreover, for each $r_1^0 \in \mathbb{R}$ and $\hat{r}^0 \in \mathbb{R}^{n-2}$, the $(n-1)$-dimensional torus $\mathcal{T}_{r_1^0,\hat{r}^0} = C_{r_1^0} \times \{(1/2, 0)\} \times T_{\hat{r}^0}$, where $C_{r_1^0} = \mathbb{T} \times \{\hat{r}_1^0\} \subset \mathbb{A}$ and $T_{\hat{r}^0} = \mathbb{T}^{n-2} \times \{\hat{r}^0\} \subset \mathbb{A}^{n-2}$, is invariant and partially hyperbolic, with n-dimensional invariant manifolds.

PROPOSITION 6.1. *Let $\mathcal{S}$ be the surface of equation $\theta_2 = 0$ in $\mathbb{A}^n$.*
1. *For each $\hat{\theta} \in \hat{B}_j$, the 2-dimensional annulus $V_{\hat{\theta}}^{(j)} = \mathbb{A} \times \{(1/2, 0)\} \times \{(\hat{\theta}, \hat{r}^{(j)})\}$ is invariant under $\Psi_j^{N'_j}$ and partially hyperbolic in $\mathbb{A}^n$. Its 3-dimensional stable and unstable manifolds $W^{\pm}(V_{\hat{\theta}}^{(j)}, \Psi_j^{N'_j})$ satisfy*

$$\mathbb{A}\times\Sigma_{N_j}^{+}\times\{(\hat{\theta},\hat{r}^{(j)})\} \subset W^{+}(V_{\hat{\theta}}^{(j)}, \Psi_j^{N'_j}), \qquad \mathbb{A}\times\Sigma_{N_j}^{-}\times\{(\hat{\theta},\hat{r}^{(j)})\} \subset W^{-}(V_{\hat{\theta}}^{(j)}, \Psi_j^{N'_j}).$$

They admit a 2-dimensional homoclinic annulus inside $\mathcal{S}$:

$$\mathbb{A} \times \{h_{N_j}\} \times \{(\hat{\theta}, \hat{r}^{(j)})j\} \subset W^{+}(V_{\hat{\theta}}^{(j)}, \Psi_j^{N'_j}) \cap W^{-}(V_{\hat{\theta}}^{(j)}, \Psi_j^{N'_j}) \cap \mathcal{S}.$$

2. *For each $r_1^0 \in \mathbb{R}$ and each $\hat{\theta} \in \hat{B}_j$, the circle $\mathcal{C}_{r_1^0,\hat{\theta}}^{(j)} = C_{r_1^0} \times \{(1/2, 0)\} \times \{(\hat{\theta}, \hat{r}^{(j)})\}$, where $C_{r_1^0} = \mathbb{T} \times \{r_1^0\}$, is invariant under $\Psi_j^{N'_j}$ and partially hyperbolic. Its 2-dimensional stable and unstable manifolds $W^{\pm}(\mathcal{C}_{r_1^0,\hat{\theta}}^{(j)}, \Psi_j^{N'_j})$ satisfy*

$$\begin{aligned} &C_{r_1^0} \times \Sigma_{N_j}^{+} \times \{(\hat{\theta}, \hat{r}^{(j)})\} \subset W^{+}(\mathcal{C}_{r_1^0,\hat{\theta}}^{(j)}, \Psi_j^{N'_j}), \\ &\Phi^{\frac{1}{q_j}U}(C_{r_1^0}) \times \Sigma_{N_j}^{-} \times \{(\hat{\theta}, \hat{r}^{(j)})\} \subset W^{-}(\mathcal{C}_{r_1^0,\hat{\theta}}^{(j)}, \Psi_j^{N'_j}). \end{aligned}$$

Proof. Recall that the support of S_{N_j} is contained in $R_{N_j} = [-\frac{3}{4N_j}, \frac{3}{4N_j}] \times [\frac{1}{N_j}, \frac{3}{N_j}]$. One just has to remark that the $(N'_j)^{\text{th}}$ iterate of R_{N_j} by $G_2^{(j)} = \Phi^{P_{N_j}} \circ \Phi^{\mu_j S_{N_j}}$ does not intersect the support $\{|\theta_2| \le \frac{1}{p_j}\}$ of the function $\eta_{p_j,L}$; the proof is then completely analogous to that of the corresponding statement of [**MS03**]. □

We can now pass to the splitting problem. As a consequence of the previous proposition, one sees that for each $\hat{\theta} \in B$, the circle $\mathcal{C}_{r_1^0,\hat{\theta}}^{(j)}$ has two homoclinic orbits $\Gamma^{\pm}$, whose intersection with $\mathcal{S}$ are the points

$$\omega^{\pm}(\hat{\theta}) = \Big((\pm 1/4, r_1^0), (0, 2/N_j), (\hat{\theta}, \hat{r}^{(j)})\Big).$$

One then observes the following inclusion, relating the invariant manifolds of the circles $\mathcal{C}_{r_1^0,\hat{\theta}}^{(j)}$ to those of the invariant tori $\mathcal{T}_{r_1^0,\hat{r}^{(j)}}$: for each $r_1^0 \in \mathbb{R}$,

$$W^{\pm}(\mathcal{C}_{r_1^0,\hat{\theta}}^{(j)}, \Psi_j^{N'_j}) = W^{\pm}(\mathcal{T}_{r_1^0,\hat{r}^{(j)}}, \psi_j) \cap (\mathbb{A} \times \mathbb{A} \times \{(\hat{\theta}, \hat{r}^{(j)})\}).$$

We refer to [**MS03**], §2.5.4, for a proof in an essentially analogous context. As a consequence, one sees that each torus $\mathcal{T}_{r_1^0,\hat{r}^{(j)}}$ possesses an open $(n-2)$-dimensional homoclinic manifold, formed by the union of the orbits of the homoclinic points $\omega(\hat{\theta})$, $\hat{\theta} \in \hat{B}_j$. This remark enables us to give an explicit description of the invariant manifolds as graphs of suitable functions, in a neighbourhood of one of the homoclinic points $\omega(\hat{\theta})$, and thus to fully determine the splitting matrix.

PROPOSITION 6.2. *For each $\hat{\theta} \in \hat{B}_j$, the splitting matrix $\mathbf{S}$ of the invariant manifolds of the torus $\mathcal{T}_{r_1^0,\hat{r}^{(j)}}$ at the homoclinic point $\omega(\hat{\theta})$, relative to the coordinates θ and r, is diagonal, and its diagonal elements are $[\frac{2\pi}{q_j}, -\frac{\mu_j}{4}, 0, \dots, 0]$.*

Proof. Note that both segments $\Sigma_N^{\pm}$ have the same projection on the θ_2-axis, that we shall denote by J_N. Let $O_j = \mathbb{T} \times J_{N_j} \times \hat{B}_j \subset \mathbb{T}^n$, and let $w_2^{\pm} J_{N_j} \to \mathbb{R}$ be the functions whose graphs are the segments $\Sigma_{N_j}^{\pm}$. The manifolds $W^{\pm}(\mathcal{T}_{r_1^0,\hat{r}^{(j)}}, \psi_j)$ are, over the open set O_j, the graphs of the functions $w^{\pm} : O_j \to \mathbb{R}^n$ defined by:

$$w^+(\theta) = (r_1^0, w_2^+(\theta_2), \hat{r}^{(j)}), \qquad w^-(\theta) = (\Phi^{\frac{U}{q_1}}(\theta_0, r_1^0), w_2^-(\theta_2), \hat{r}^{(j)}),$$

and the splitting matrix has components $\mathbf{S}_{ij} = \partial_i(w_j^+ - w_j^-)(\omega(\hat{\theta}))$. □

Observe that, in the above result, the first diagonal element is exponential small as indicated in (34); when related to the small parameter ε_j, it is thus characterized by the exponent $\frac{1}{2(n-1)(\alpha-1)}$, whereas the second diagonal element is potentially much smaller (one can modify the construction so as to make it of the order of magnitude of $\exp(-\operatorname{const}(\frac{1}{\varepsilon_j})^{\frac{1}{2(\alpha-1)}}))$.

REMARK 6.1. *The above contruction is only an example, chosen among many others, for which the splitting matrix is completely depictable. For instance, one can produce an example of nondegenerate splitting matrix by choosing a new function $g^{(j)} = g_2^{(j)} \otimes g_3 \otimes \cdots \otimes g_n$, with*

$$g_2^{(j)} = \eta_{N_j,L}, \quad g_3 = \cdots = g_n = U,$$

a new parameter $q_j = N_j^2 \|\eta_{N_j,L}\|_{\alpha,L} \|U\|_{\alpha,L}^{n-1}$ and the same $\mu_j = (N_j^2 \|S_{N_j}\|_{\alpha,L})^{-1}$. The $(2n-2)$-dimensional annulus $\mathbb{A} \times \{(1/2, 0)\} \times \mathbb{A}^{n-2}$ and the $(n-1)$-dimensional tori $\mathcal{T}_{r_1^0,\hat{r}^0}$ are still invariant under Ψ_j, for all $\hat{r}^0 = (r_3^0, \dots, r_n^0)$. The torus $\mathcal{T}_{r_1^0,\hat{r}^0}$ has now 2^{n-1} homoclinic orbits, whose intersection with $\mathcal{S}$ are the points

$$\big((\pm 1/4, r_1^0), (0, 2/N_j), (\pm 1/4, r_3^0), \dots, (\pm 1/4, r_n^0)\big).$$

At these points, the splitting matrix is still diagonal, but the diagonal elements are now $[\pm\frac{2\pi}{q_j}, -\frac{\mu_j}{4}, \pm\frac{2\pi}{q_j}, \dots, \pm\frac{2\pi}{q_j}]$.

6.3. *Horseshoe and fibred dynamics over the shift.* We now wish to produce oscillating orbits instead of drifting ones and to prove Theorem 2.2. To generate positive and negative jumps of the r_1-variable, it is necessary to have two zones in the pendulum space at our disposal, one for each sign; then, to produce a random walk, it suffices to find orbits visiting these two zones in any prescribed order, and to make use of an adapted version of the coupling lemma. Our starting point will be the creation of two transverse homoclinic points for the perturbed pendulum, which will enable us to construct adapted symbolic dynamics by the classical horseshoe method, the two aforementioned zones being two suitable neighbourhoods of the homoclinic points.

But to make sure that a coupling perturbation of the form which we already used in the previous sections is still relevant in the present context, we shall have in addition to control the minimal number of iterates required to produce a horseshoe. And this will cause new difficulties, since we shall have to take care of the fact that not only the splitting angle, but also the Lyapounov exponents of the fixed point go to 0 when $j \to \infty$, the latter even being exponentially small with respect to the perturbation. To overcome that difficulty without introducing cumbersome estimates, we shall slightly modify our pendulum map in order to facilitate the local analysis in the neighbourhood of the hyperbolic point.

We introduce a new function V, which differs from the previous one mainly by the fact that it is exactly reduced to its quadratic part in the neighbourhood of its maximum. Namely, we choose in $G^{\alpha,L}(\mathbb{T})$ a function V taking its values in $[-2,0]$ and satisfying the conditions:

$$V(0) = -2, \qquad V(\tfrac{1}{2}) = 0, \qquad V(\theta) = -\tfrac{1}{2}\left|\theta - \tfrac{1}{2}\right|^2 \quad \text{if } \left|\theta - \tfrac{1}{2}\right| < \tfrac{1}{8} \tag{46}$$

(such a function is easily constructed, *e.g.* using Gevrey partitions of unity). Our new functions P_* and P_N have the same expressions as above:

$$P_*(\theta_*, r_*) = \frac{1}{2}r_*^2 + V(\theta_*), \quad P_N(\theta_2, r_2) = \frac{1}{2}r_2^2 + \frac{1}{N^2}V(\theta_2),$$

but they now involve the new function V. Note in particular that the relation $P_* \circ \sigma = N^2\, P_N$ and the conjugacy equation $\Phi^{P_N} = \sigma^{-1} \circ \Phi^{\frac{1}{N}P_*} \circ \sigma$ still remain valid.

The point $O = (\frac{1}{2}, 0)$ is still a hyperbolic fixed point for the pendulum-like maps Φ^{P_*} and Φ^{P_N}, the separatrices of which are the curves of equations $r_* = \pm\sqrt{-2V(\theta_*)}$ and $r_2 = \pm\frac{1}{N}\sqrt{-2V(\theta_2)}$ respectively; we shall denote by $h_*^{(\pm 1)} = (0, \pm 2)$ and $h_N^{(\pm 1)} = (0, \pm\frac{2}{N})$ the upper and lower points on them.

To create transverse intersections of the invariant manifolds of O at the points $h_N^{(\pm 1)}$, we shall also modify our previous splitting function. We now set:

$$S_N(\theta_2, r_2) = \Big(\tilde{\eta}_{N,L}(r_2 - \tfrac{2}{N}) + \tilde{\eta}_{N,L}(r_2 + \tfrac{2}{N})\Big)\Big(\tfrac{X^2}{2}\, \tilde{\eta}_{4N,\Lambda_N}(X)\Big), \tag{47}$$

so the behaviour of S_N is exactly the same as that of the previous function in the neighbourhood of the upper point and we now have an analogous effect in the neighbourhood of the lower one. We define our new "perturbed pendulum" map as:

$$G_2^{(j)} = \Phi^{\frac{1}{2}r_2^2 + \frac{1}{N_j^2}V(\theta_2)} \circ \Phi^{\mu_j S_{N_j}}, \tag{48}$$

with the functions V and S_{N_j} just defined by (46) and (47), and with the same N_j as always (defined in (19)) and $\mu_j = 1/(N_j^2 \|S_{N_j}\|_{\alpha,L})$. Note that we still have

$\mu_j \geq \exp(-c\,N_j^{\frac{1}{\alpha-1}})$ for a given positive constant c, because the supports of the $\tilde{\eta}$ functions involved in the first term of the definition of S_N are disjoint.

Note also that $O = (1/2, 0)$ is still a hyperbolic fixed point for $G_2^{(j)}$, and now admits the two homoclinic points $h_{N_j}^{(\pm 1)} = (0, \pm\frac{2}{N_j})$.

Finally, we set $q_j = 2N_jM_j$ with M_j as in Lemma 4.4, so q_j is now even but still satisfies inequalities of the form (34). As in the previous sections, q_j is chosen in order to control the size of the coupling perturbation $\frac{1}{q_j}U \otimes g^{(j)}$, which will be defined later in the construction (equation (49)): we shall still have $\|\frac{1}{q_j}U \otimes g^{(j)}\|_{\alpha,L}, \|\mu_j S_{N_j}\|_{\alpha,L} \leq \varepsilon_j := \frac{\|V\|_{\alpha,L}}{N_j^2}$.

The following lemma provides us with the necessary symbolic dynamics for the q_j^{th} power of the function $G_2^{(j)}$.

Proposition 6.3. *Let $\mathcal{B}_j^{(+1)} = [-\frac{1}{N_j}, \frac{1}{N_j}] \times [\frac{1}{N_j}, \frac{3}{N_j}]$ and $\mathcal{B}_j^{(-1)} = [-\frac{1}{N_j}, \frac{1}{N_j}] \times [-\frac{3}{N_j}, -\frac{1}{N_j}]$. There exist a compact set $\mathcal{K}_j \subset \mathcal{B}_j^{(+1)} \cup \mathcal{B}_j^{(-1)}$, invariant under $\big(G_2^{(j)}\big)^{q_j}$, and a homeomorphism ρ_j from $\mathcal{K}_j$ to the space $\{-1,+1\}^{\mathbb{Z}}$ conjugating the restriction $\big(G_2^{(j)}\big)^{q_j}_{|\mathcal{K}_j}$ with the shift $\flat$ on $\{-1,+1\}^{\mathbb{Z}}$.*

Moreover, given $x \in \mathcal{K}_j$, one has $\rho_j(x) = (\kappa_\ell)_{\ell\in\mathbb{Z}}$ if and only if $\big(G_2^{(j)}\big)^{\ell q_j}(x) \in \mathcal{B}_j^{\kappa_\ell}$ for each $\ell \in \mathbb{Z}$.

The shift $\flat$ is defined here by $\flat((\kappa_\ell)_{\ell\in\mathbb{Z}}) = (\kappa'_\ell)_{\ell\in\mathbb{Z}}$, with $\kappa'_\ell = \kappa_{\ell+1}$. The proof of Proposition 6.3 will be the main part of the present section. To obtain $\mathcal{K}_j$, we shall first construct an invariant set in the neighbourhood of the fixed point, using the classical horseshoe method; then the composition by a suitable iterate of the map $G_2^{(j)}$ will yield the desired localization inside the boxes $\mathcal{B}_j^{(\pm 1)}$. We shall closely follow Moser's simple presentation of the horseshoe theorem for two-dimensional systems ([**Mos73**], Chapter III). Our main difficulty here will be to prove that the minimal number of iterates required to produce a horseshoe for the system $G_2^{(j)}$ is smaller that the number q_j. To overcome the difficulty caused by the j-dependence of the map $G_2^{(j)}$, we shall take advantage, as far as possible, of the conjugacy between the pendulum and normalized pendulum maps $\Phi^{P_{N_j}}$ and Φ^{P_*}, and perform our constructions simultaneously for the two maps.

We shall need some additional definitions based on Moser's presentation. We fix a coordinate system (z_h, z_v) in the plane $\mathbb{R}^2$. Given a real number $\beta > 0$, we consider the rectangle $R = \{|z_h| \leq \beta,\ |z_v| \leq \beta\}$ and a fixed positive number ζ. We define a ζ-*horizontal curve* in R as the graph of a ζ-Lipschitz map $z_h \mapsto z_v = \gamma_h(z_h)$ defined on the interval $|z_h| \leq \beta$, and a ζ-*vertical curve* as the graph of a ζ-Lipschitz map $z_v \mapsto z_u = \gamma_v(z_v)$ defined on the interval $|z_v| \leq \beta$. A ζ-*horizontal strip* is a domain of the form:

$$\mathcal{S}_h = \{(z_h, z_v) \mid \gamma_h^d(z_h) \leq z_v \leq \gamma_h^u(z_h)\},$$

where γ_h^d, γ_h^u are two ζ-horizontal curves such that $\gamma_h^d(z_h) < \gamma_h^u(z_h)$ if $|z_h| \leq \beta$. One has an analogous definition for the vertical strips $\mathcal{S}_v$.

The horseshoe method for producing symbolic dynamics in our problem first requires to find a rectangle R together with two vertical strips $V^{(\pm 1)}$, such that the images $H^{(\pm 1)} = \big(G_2^{(j)}\big)^{q_j}(V^{(\pm 1)})$ intersect R along two horizontal strips. Then one has to check some cone conditions ensuring the existence of an invariant set and the conjugacy of the dynamics to a shift. The proof will necessitate several steps, the first of which is the definition of natural coordinate systems in suitable neighbourhoods of the fixed point and the separatrices for Φ^{P_*} and $\Phi^{P_{N_j}}$.

We first take into account the particular form of the function V and introduce linearizing symplectic coordinates in the neighbourhood $\mathcal{V}_* = \{\big|\theta_* - \frac{1}{2}\big| < \frac{1}{8}\}$ of the hyperbolic fixed point O. Writing $\bar\theta_*$ for $\theta_* - \frac{1}{2}$, we set:

$$u_* = \tfrac{1}{\sqrt{2}}(r_* + \bar\theta_*), \qquad s_* = \tfrac{1}{\sqrt{2}}(r_* - \bar\theta_*);$$

so $ds_* \wedge du_* = dr_* \wedge d\bar\theta_*$ and $P_*(\theta_*, r_*) = \frac{1}{2}r_*^2 - \frac{1}{2}\bar\theta_*^2 = u_* s_*$ in $\mathcal{V}_*$. As a consequence, the map Φ^{P_*} is linear inside $\mathcal{V}_*$:

$$\Phi^{P_*}(u_*, s_*) = (\mathrm{e}\, u_*,\ \mathrm{e}^{-1}\, s_*).$$

We introduce the ball

$$\mathcal{B}_*^0 = \{|u_*| \le \mathrm{e}^{-3},\ |s_*| \le \mathrm{e}^{-3}\} \subset \mathcal{V}_*,$$

which will serve as a reference neighbourhood for O; note that the extremal values of P_* on $\mathcal{B}_*^0$ are $\pm\mathrm{e}^{-6}$. For each $\beta \in]0, \mathrm{e}^{-6}]$, we define a tubular neighbourhood of the separatrices by $V_*(\beta) = \{|P_*| \le \beta\}$; remark that $V_*(\beta)$ is moreover invariant under the flow Φ^{tP_*}, and that $\mathcal{B}_0^* \subset V_*(\mathrm{e}^{-6})$.

In the subset $V_*(\beta) \cap \{0 < \theta_* < 1,\ r_* > 0\}$ we introduce the coordinate system (τ_*, e_*), where τ_* is defined as in (25) (beware of the change of V) and e_* is the value of P_*; note that $e_* = u_* s_*$ in the common part $V_*(\beta) \cap \mathcal{B}_*^0$.

We now get the analogous neighbourhoods for the map Φ^{P_N}, taking care of the various transformation formulae (for the sake of clarity, we omit the subscript j in this paragraph). Writing $\bar\theta_2 = \theta_2 - \frac{1}{2}$, the symplectic linearizing coordinates (u, s) for Φ^{P_N} are defined in $\mathcal{V} = \{\big|\bar\theta_2\big| < \frac{1}{8}\}$ by:

$$u = \tfrac{1}{\sqrt{2}}(\sqrt{N} r_2 + \tfrac{1}{\sqrt{N}}\bar\theta_2), \qquad s = \tfrac{1}{\sqrt{2}}(\sqrt{N} r_2 - \tfrac{1}{\sqrt{N}}\bar\theta_2);$$

and the (u_*, s_*)-coordinates of a point (θ_*, r_*) are related to the (u, s)-coordinates of $\sigma^{-1}(\theta_*, r_*)$ by $u = \frac{u_*}{\sqrt{N}}$, $s = \frac{s_*}{\sqrt{N}}$. In $\mathcal{V}$ we get the simple expression $P_N(\theta_2, r_2) = \frac{1}{2}r_2^2 - \frac{1}{N^2}\theta_2^2 = \frac{1}{N}us$ for the Hamiltonian, which yields a time-one map of the form

$$\Phi^{P_*}(u_*, s_*) = (\mathrm{e}^{\frac{1}{N}}\, u_*,\ \mathrm{e}^{-\frac{1}{N}}\, s_*).$$

We finally introduce the neighbourhood $\mathcal{B}^0 = \sigma^{-1}(\mathcal{B}_*^0) = \{|u| \le \frac{\mathrm{e}^{-3}}{\sqrt{N}},\ |s| \le \frac{\mathrm{e}^{-3}}{\sqrt{N}}\}$.

We shall construct our vertical and horizontal strips in the rectangle $\mathcal{B}^0$, which we endow with the coordinates $(z_h = u,\ z_v = s)$. Let us state a first auxillary lemma, to gather the necessary information relative to the strips.

LEMMA 6.2. *There exists an integer j_0 and a sequence $(\zeta_j)_{j \ge j_0}$ of positive numbers, tending to* 0 *when $j \to \infty$, such that, for each $j \ge j_0$, there exists in $\mathcal{B}^0$ two ζ_j-vertical strips $V_j^{(\pm 1)}$, whose images $H_j^{(\pm 1)} = \big(G_2^{(j)}\big)^{q_j}(V_j^{(\pm 1)})$ are two ζ_j-horizontal strips in $\mathcal{B}^0$.*

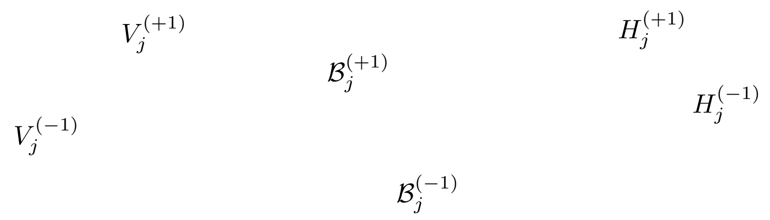

FIGURE 6. Construction of the symbolic dynamics for $G_2^{(j)} = \Phi^{P_{N_j}} \circ \Phi^{\mu_j S_{N_j}}$.

Proof. We shall first construct the vertical and horizontal strips $V_j^{(+1)}$ and $H_j^{(+1)}$. To preserve a certain symmetry, we find it useful to choose a small neighbourhood Δ_j of the upper homoclinic point $h_{N_j}^{(+1)}$ and examine its iterates of order $+\frac{q_j}{2}$ and $-\frac{q_j}{2}$ by the map $G_2^{(j)}$. We shall first select inside Δ_j a subset whose $\frac{q_j}{2}$-backward iterate will give us the desired vertical strip $V_j^{(+1)}$, and then prove that the forward iterate $\big(G_2^{(j)}\big)^{q_j}(V_j^{(+1)})$ is a horizontal strip in $\mathcal{B}^0$. The other strips $H_j^{(-1)}$ and $V_j^{(-1)}$ will be obtained by the same process, using the lower homoclinic point $h_{N_j}^{(-1)}$ instead of the upper one.

It would of course be easier to perform the constructions for the normalized map Φ^{P_*} only, but we shall need some information relative to the j-dependent map $\Phi^{P_{N_j}}$, because the splitting map does not possess a simple expression in the (θ_*, r_*) coordinates. So we shall mainly work in the (θ_*, r_*) system for defining intermediate subsets, but at the same time we consider their (θ_2, r_2)-analogues, obtained by composition by the map σ^{-1}.

For each integer $m \geq 1$, we define a neighbourhood $\Delta_*(m)$ of h_* by

$$\Delta_*(m) = \{(\theta_*, r_*) \mid |\tau_*(\theta_*, r_*)| \leq \tfrac{1}{m},\ |e_*| \leq \mathrm{e}^{-10}\},$$

and provide it with the coordinates (τ_*, e_*). Let ν be the smallest integer for which $\Phi^{\nu P_*}(h_*)$ belongs to the interior of $\mathcal{B}_*^0$. One can fix m_0 large enough so as to ensure the inclusion $\Phi^{\nu P_*}(\Delta_*(m)) \subset \mathcal{B}_*^0$ when $m \geq m_0$. Let j_0 be such that $N_{j_0} \geq m_0$. In the following, for $j \geq j_0$, we shall consider the normalized domains $\Delta_{*j} = \Delta_*(N_j)$ and their images $\Delta_j = \sigma^{-1}\Delta_{*j}$. The final horizontal and vertical strips we are searching for will be obtained as the $\frac{q_j}{2}$-forward and $\frac{q_j}{2}$-backward iterates of a certain subdomain of Δ_j, which will be constructed by means of two successive reductions of Δ_j. More precisely, the final subdomain will be nearly a parallelogram in the (θ_2, r_2) coordinates, the boundary of which is composed of nearly horizontal upper and lower sides, and right and left sides nearly rectilinear with slope of the order of the splitting μ_j. The horizontal part of the boundary will be determined in order that the $\frac{q_j}{2}$-forward iterate be a horizontal strip contained in $\mathcal{B}^0$, while the right and left part will be determined in order that the $\frac{q_j}{2}$-backward iterate be a vertical strip contained in $\mathcal{B}^0$.

Let us perform the first reduction. We recall that $M_j = \frac{q_j}{2N_j}$ is integer and denote by $\mathcal{D}_{*j}$ the set of points $x_* \in \Delta_{*j}$ such that the iterate $\Phi^{kP_*}(x_*)$ belongs to the set

$$(V(\mathrm{e}^{-10}) \cap \{\tfrac{1}{2} < \theta_* < 1\}) \cup \mathcal{B}^0_*$$

for each $k \in \{1, \dots, M_j\}$. Remark that

$$\Phi^{\nu P_*}(\mathcal{D}_{*j}) = \Phi^{\nu P_*}(\Delta_{*_j}) \cap \{|u_*| \le \mathrm{e}^{-M_j+\nu-3}\} \subset \{\mathrm{e}^{-3} \le s_* \le \mathrm{e}^{-5}\}$$

and observe that the map $\Phi^{\nu P_*}$ sends the τ_*-axis on the s_*-axis. Let

$$\xi_j = 2\mathrm{e}^{-M_j+\nu-3}.$$

Simple estimates based on the mean value theorem, applied to the derivative of $\Phi^{\nu P_*}$, show that, for j large enough, $\mathcal{D}_{*j}$ is a ξ_j-horizontal strip in $R_{*j} = \{|\tau_*| \le \frac{1}{N_j},\ |e_*| \le 1\}$, relatively to the (τ_*, e_*) coordinates, and that the image $H_* = \Phi^{M_j P_*}(\mathcal{D}_{*j})$ is a horizontal strip in the rectangle $\mathcal{B}_0$, relatively to the (u_*, s_*) coordinates. In the following we assume that j_0 is large enough to ensure that these two properties hold when $j \ge j_0$. Therefore, for $j \ge j_0$, the set $\mathcal{D}_j = \sigma^{-1}\mathcal{D}_{*j}$ is also a $\frac{\xi_j}{N_j}$-horizontal strip in the rectangle $R_j = \sigma^{-1}(R_{*j})$, relatively to the (u_2, s_2)-coordinates, and the image $H_j = \Phi^{\frac{q_j}{2}P_{N_j}}(\mathcal{D}_j)$ is a horizontal strip in $\mathcal{B}^0$, the Lipschitz factor of which we do not have to make precise. This ends the first reduction of the initial domain.

The second reduction and the construction of the vertical strip $V_j^{(+1)}$ will be performed simultaneously: we shall now examine the behaviour of the backward iterate $\big(G_2^{(j)}\big)^{-q_j}(H_j)$, and prove that some part of its intersection with $\mathcal{B}^0$ is a vertical strip in $\mathcal{B}^0$, that we shall choose as $V_j^{(+1)}$; the corresponding part in $\mathcal{D}_j$ will be the reduced domain. This is where the splitting map comes into play. Indeed, remark that

$$\big(G_2^{(j)}\big)^{-q_j}(H_j) = \Phi^{-\frac{q_j}{2}P_{N_j}} \circ \Phi^{-\mu_j S_{N_j}} \circ \Phi^{-\frac{q_j}{2}P_{N_j}}(H_j) = \Phi^{-\frac{q_j}{2}P_{N_j}}\big(\Phi^{-\mu_j S_{N_j}}(\mathcal{D}_j)\big),$$

so we first have to describe the set $\mathcal{E}_j = \Phi^{-\mu_j S_{N_j}}(\mathcal{D}_j)$. Equivalently, we shall describe the image $\mathcal{E}_{*j} = \sigma(\mathcal{E}_j)$. This is in fact another strip, for which we shall come back to the (τ_*, e_*)-coordinates in the neighbourhood of the upper point h_*. More precisely, we shall limit ourselves to the intersection of $\mathcal{E}_{*j}$ with the rectangular domain $R_{*j} = \{|\tau_*| \le \frac{1}{N_j},\ |e_*| \le 1\}$. We claim that, if j_0 is large enough and $j \ge j_0$, the intersection $\mathcal{E}_{*j} \cap R_{*j}$ is a strip contained between the graphs of two decreasing functions $\gamma_d < \gamma_u$ defined on the interval $\{|\tau_*| \le \frac{1}{N_j}\}$, which furthermore satisfy

$$\gamma_d\Big(-\frac{1}{8N_j}\Big) \ge \frac{\mu_j}{16}, \quad \gamma_u\Big(\frac{1}{8N_j}\Big) \le -\frac{\mu_j}{16},$$
$$-2\mu_j N_j \le \gamma_u'(\tau_*), \gamma_d'(\tau_*) \le -\frac{\mu_j N_j}{2}, \qquad |\tau_*| \le \frac{1}{N_j}.$$

The verification is easy, using Lemma 6.1, since $\mathcal{D}_{*j}$ is a ξ_j-horizontal strip and the ratio ξ_j/μ_j goes to 0 when j tends to ∞.

Now we can take up the description of $\big(G_2^{(j)}\big)^{-q_j}(H_j) = \Phi^{-\frac{q_j}{2}P_{N_j}}(\mathcal{E}_j)$, which will lead to the construction of the vertical strip $V_j^{(+1)}$. We shall proceed as for the first reduction of domain and consider the points x_* in $\mathcal{E}_{*j}$ whose backward iterates $\Phi^{-kP_*}(x_*)$ belong to the set $(V(\mathrm{e}^{-10}) \cap \{0 < \theta_* < \frac{1}{2}\}) \cup \mathcal{B}_*^0$ for $k \in \{1, \ldots, M_j\}$. The problem is now to determine the intersection

$$\Phi^{-\nu P_*}(\mathcal{E}_{*j}) \cap \{\, |s_*| \le \mathrm{e}^{-M_j+\nu-3}\,\}.$$

We claim that this intersection is a horizontal strip $\mathcal{I}_{*j}$, relatively to the (u_*, s_*)-coordinates in $\mathcal{B}_*^0$, limited by the graphs of two functions $\varphi_d < \varphi_u$, the derivatives of which satisfy

$$\max\{\|\varphi_d'\|_{C^0}, \|\varphi_u'\|_{C^0}\} \le \frac{c_1}{\mu_j},$$

the positive number c_1 being independent of j. The proof of that assertion is straightforward and left to the reader.

We can now conclude our description of the interesting part of $\big(G_2^{(j)}\big)^{-q_j}(H_j)$. It only remains for us to examine the backward iterate $\Phi^{-(M_j-\nu)P_*}(\mathcal{I}_{*j})$, which is easy since, due to the linear character of the map Φ^{P_*} inside $\mathcal{B}_*^0$, our last assertion directly implies that $\Phi^{-(M_j-\nu)P_*}(\mathcal{I}_{*j})$ is a vertical strip in the rectangle $\mathcal{B}_*^0$, with Lipschitz factor $\zeta_j = \frac{c_1}{\mu_j}\exp(-2(M_j - \nu))$ tending to 0 when j tends to ∞. We denote by $V_{*j}^{(+1)}$ that vertical strip, and we finally get the desired vertical strip in $\mathcal{B}^0$ by setting $V_j^{(+1)} = \sigma^{-1}(V_{*j}^{(+1)})$, which is also a ζ_j-vertical strip in $\mathcal{B}^0$.

As for the reduction of domain, we now set $\mathcal{P}_{*j} = \Phi^{-M_jP_*}(V_{*j}^{(+1)})$. This is a small neighbourhood of the point h_*, contained in the strip $\mathcal{E}_{*j}$, which has nearly the shape of a parallelogram in the (τ_*, e_*)-coordinates. More precisely, $\mathcal{P}_{*j}$ is obtained as the intersection of $\mathcal{E}_{*j}$ with the horizontal strip $\mathcal{S}_{*j}$ limited by the images by $\Phi^{\nu P_*}$ of the two segments $s_* = \pm\mathrm{e}^{-M_j+\nu-3}$ in $\mathcal{B}_0^*$. As above, it is not difficult to see that these images are ξ_j-Lipschitz graphs of functions from the τ_*-axis to the e_*-axis. Hence the boundary of $\mathcal{P}_{*j}$ is composed of two nearly horizontal parts, the intersections of these two graphs with $\mathcal{E}_{*j}$, and of right and left parts which are the intersections of the right and left parts of the boundary of $\mathcal{E}_{*j}$ with the horizontal strip $\mathcal{S}_{*j}$. These parts are graphs of functions, the slope of which is approximately $-\mu_jN_j$ in the (θ_*, r_*) variables. Finally, we end up with the corresponding parallelogram $\mathcal{P}_j = \sigma^{-1}(\mathcal{P}_{*j})$ in the (θ_2, r_2) variables, the boundary of which has obvious description.

To conclude the proof, it only remains for us to go forward and show that the image $\big(G_2^{(j)}\big)^{q_j}(V_j^{(+1)})$ is indeed a horizontal strip in $\mathcal{B}^0$. Just as above, we have to take care of the occurrence of the splitting map in the chain of iterates. We remark that

$$\big(G_2^{(j)}\big)^{q_j}(V_j^{(+1)}) = \Phi^{\frac{q_j}{2}P_{N_j}} \circ \Phi^{\mu_jS_{N_j}} \circ \Phi^{\frac{q_j}{2}P_{N_j}}(V_j^{(+1)}) = \Phi^{\frac{q_j}{2}P_{N_j}} \circ \Phi^{\mu_jS_{N_j}}(\mathcal{P}_j)$$

The domain $\mathcal{L}_j = \Phi^{\mu_jS_{N_j}}(\mathcal{P}_j)$ is still nearly a parallelogram, the boundary of which is the image of the previous one: the right and left parts of its boundary are now the images by the splitting map of the horizontal parts of the boundary of $\mathcal{P}_j$, while the other two ones are nearly horizontal. For the final step, we go back to the (τ_*, e_*)-coordinates and consider $\mathcal{L}_{*j} = \sigma(\Phi^{\mu_j S_{N_j}}(\mathcal{P}_j))$. The slope of the right and left parts of its boundary has a lower bound of the form $c_2\mu_j N_j$, and this is enough to ensure that $\Phi^{q_j/2N_j}(\mathcal{L}_{*j})$ is a ζ_j-horizontal strip in $\mathcal{B}^0_*$. The final strip $H_j^{(+1)}$ is therefore a ζ_j-horizontal strip in $\mathcal{B}^0$, and we are done. This ends the proof of Lemma 6.2. □

As indicated at the beginning of the proof, we proceed in a similar way to get two other strips $V^{(-1)}$ and $H^{(-1)}$ inside $\mathcal{B}^0$, with the same Lipschitz factor. Our last task will be to consider the usual cone conditions inside the vertical and horizontal strips. Following Moser, given $\delta > 0$, we now define two trivial sector bundles $\mathcal{C}_h(\delta)$ and $\mathcal{C}_v(\delta)$ in the tangent space $T\mathcal{B}^0 = \mathcal{B}^0 \times \mathbb{R}^2$, whose fibres over each point are respectively the sectors $C_h(\delta) = \{(\omega_u, \omega_s) \mid |\omega_s| \leq \delta\,|\omega_u|\}$ and $C_v(\delta) = \{(\omega_u, \omega_s) \mid |\omega_u| \leq \delta\,|\omega_s|\}$. The following and last auxillary lemma, whose proof is straightforward, depicts the behaviour of these bundles under positive and negative iteration by $G_2^{(j)}$.

Lemma 6.3. *There exists $j_1 \geq j_0$ and a sequence $(\delta_j)_{j\geq j_1}$ of positive numbers, converging to 0, such that the sector bundle $\mathcal{C}_h(\delta_j)$, restricted to $V^{(+1)} \cup V^{(-1)}$, is invariant under $\big(G_2^{(j)}\big)^{q_j}$, and that the sector bundle $\mathcal{C}_v(\delta_j)$, restricted to $H^{(+1)} \cup H^{(-1)}$, is invariant under $\big(G_2^{(j)}\big)^{-q_j}$. Moreover, for each point $z \in V^{(+1)} \cup V^{(-1)}$ and each $(\omega_u, \omega_s) \in C_h(\delta_j)$, the image $(\omega'_u, \omega'_s) = T_z\big(G_2^{(j)}\big)^{q_j}(\omega_u, \omega_s)$ satisfies the dilatation condition $|\omega'_u| \geq \delta_j^{-1}\,|\omega_u|$, and for each point $z \in H^{(+1)} \cup H^{(-1)}$ and each $(\omega_u, \omega_s) \in C_v(\delta_j)$, the image $(\omega'_u, \omega'_s) = T_x\big(G_2^{(j)}\big)^{q_j}(\omega_u, \omega_s)$ satisfies the contraction condition $|\omega'_s| \leq \delta_j\,|\omega_s|$.*

The horseshoe theorem applies (see [**Mos73**]) and yields a compact invariant set $\bar{\mathcal{K}}_j$ contained in the union $H_j^{(-1)} \cup H_j^{(+1)}$, and a homeomorphism $\bar{\rho}_j$ from $\bar{\mathcal{K}}_j$ to $\{-1, +1\}^{\mathbb{Z}}$ satisfying the conjugacy equation $\big(G_2^{(j)}\big)^{q_j}_{|\bar{\mathcal{K}}_j} = \bar{\rho}_j \circ \flat \circ \bar{\rho}_j^{-1}$. The coding sequence $\bar{\rho}_j(x) = (\kappa_\ell)$ of a point $x \in \bar{\mathcal{K}}_j$ is determined by the sequence of visits of its iterates in the strips $H^{(+1)}$ and $H^{(-1)}$: by construction

$$(G_2^{(j)})^{\ell q_j}(x) \in H^{(\kappa_\ell)}, \qquad \ell \in \mathbb{Z}.$$

The last step of our construction will be to send the previous invariant set $\bar{\mathcal{K}}_j$ into the two neighbourhoods $\mathcal{B}_j^{(\pm 1)}$ of the upper and lower homoclinic points. For this, one just has to consider the image

$$\mathcal{K}_j = (G_2^{(j)})^{\frac{q_j}{2}}(\bar{\mathcal{K}}_j) \subset (G_2^{(j)})^{\frac{q_j}{2}}\big(H^{(+1)} \cup H^{(-1)}\big),$$

and to set $\rho_j = \bar{\rho}_j \circ (G_2^{(j)})^{-\frac{q_j}{2}}$. Since $(G_2^{(j)})^{\frac{q_j}{2}}\big(H^{(\pm 1)}\big) \subset \mathcal{B}_j^{(\pm 1)}$, the conclusion of Proposition 6.3 follows. □

We now consider a suitable composed diffeomorphism Ψ_j, for which we shall prove the existence of skew-product dynamics, fibred over the shift map on two symbols. This construction may be seen as a perturbative example of the situation described in [**Moe02**]. We have to modify the coupling function, in order to take advantage of our two disjoint zones in the pendulum space and choose the sign of the drift. We define a function

$$g^{(j)} = g_2^{(j)}(\theta_2, r_2)\hat{g}^{(j)}(\theta_3, \dots, \theta_n), \qquad \hat{g}^{(j)} = \eta_{2p_{j,L}} \otimes \cdots \otimes \eta_{2p_{j-n+3,L}}, \tag{49}$$

i.e. a function of the same form as in (32), but with

$$g_2^{(j)}(\theta_2, r_2) = \tilde{\eta}_{N_j,L}(r_2 - \tfrac{2}{N_j})\, \eta_{2p_j,L}(\theta_2) - \tilde{\eta}_{N_j,L}(r_2 + \tfrac{2}{N_j})\, \eta_{2p_j,L}(\theta_2), \tag{50}$$

and we consider the map

$$\Psi_j = \Phi^{\frac{1}{q_j}U \otimes g^{(j)}} \circ \left(\Phi^{\frac{1}{2}r_1^2} \times G_2^{(j)} \times \Phi^{\frac{1}{2}(r_3^2 + \cdots + r_n^2)}\right), \tag{51}$$

where $G_2^{(j)} = \Phi^{P_{N_j}} \circ \Phi^{\mu_j S_{N_j}} = \Phi^{\frac{1}{2}r_2^2 + \frac{1}{N_j^2}V(\theta_2)} \circ \Phi^{\mu_j S_{N_j}}$ is the map which has occupied us until now in the present section.

PROPOSITION 6.4. *For each $\hat{x}^{(j)} \in \hat{B}_j \times \{\hat{r}^{(j)}\}$, the set $\mathcal{I}_j = \mathbb{A} \times \mathcal{K}_j \times \{\hat{x}^{(j)}\}$ is invariant under the map $\Psi_j^{q_j}$. After identification of $\mathcal{I}_j$ with $\mathbb{A} \times \mathcal{K}_j$, the restriction to $\mathcal{I}_j$ of $\Psi_j^{q_j}$ satisfies the conjugacy equation*

$$(\Psi_j)^{q_j}_{|\mathcal{I}_j} = (\mathrm{Id} \times \rho_j)^{-1} \circ \mathcal{P}_j \circ (\mathrm{Id} \times \rho_j)$$

where the map $\mathcal{P}_j : \mathbb{A} \times \{-1,1\}^{\mathbb{Z}} \to \mathbb{A} \times \{-1,1\}^{\mathbb{Z}}$ is defined by

$$\mathcal{P}_j(x_1, \kappa) = (\Phi^{\frac{\kappa_1}{q_j}U} \circ (\Phi^{\frac{1}{2}r_1^2})^{q_j}(x_1), \flat(\kappa)) \quad \text{for } x_1 \in \mathbb{A} \text{ and } \kappa = (\kappa_\ell)_{\ell \in \mathbb{Z}} \in \{-1,1\}^{\mathbb{Z}}.$$

Proof. We are in a situation comparable to that of Lemma 3.2, with $f = \frac{1}{q_j}U$, $g = g^{(j)}$, $F = \Phi^{\frac{1}{2}r_1^2}$ and $G = G_2^{(j)} \times \hat{G}$, where $\hat{G} = \Phi^{\frac{1}{2}(r_3^2 + \cdots + r_n^2)}$. We first observe that $\Phi^{N_j' P_{N_j}}(\mathcal{K}_j)$ lies outside the support of $G_2^{(j)}$. Let $(x_1, x_2) \in \mathbb{A} \times \mathcal{K}_j$ and $x = (x_1, x_2, \hat{x}^{(j)})$. In view of (13), we readily compute $\Psi_j^{q_j - 1}(x) = \big(F^{q_j-1}(x_1), (G_2^{(j)})^{q_j-1}(x_2), \hat{G}^{q_j-1}(\hat{x}^{(j)})\big)$, hence

$$\begin{aligned}
\Psi_j^{q_j}(x) &= \Big(\Phi^{\frac{1}{q_j}g_2^{(j)}((G_2^{(j)})^{q_j}(x_2))U}(F^{q_j}(x_1)), (G_2^{(j)})^{q_j}(x_2), \hat{x}^{(j)}\Big) \\
&= \Big(\Phi^{\frac{\kappa_1}{q_j}U} \circ F^{q_j}(x_1), \big(G_2^{(j)}\big)^{q_j}(x_2), \hat{x}^{(j)}\Big),
\end{aligned}$$

the last equality stemming from the definition of $g_2^{(j)}$ and that of the coding sequence for x_2. This proves the proposition. □

As a consequence, we can now obtain the random walk behaviour, by an appropriate choice of the initial point on the first factor. As in (7), we denote by T_δ the translation of step $\delta > 0$ along the first action axis in $\mathbb{A}^n$.

COROLLARY 6.1. *Let $\mathcal{G}^{(j)}$ be the diffeomorphism of $\mathbb{A}^n$ defined by $\mathcal{G}^{(j)}(x_1, x_2, \hat{x}) = (x_1, G_2^{(j)}(x_2), \hat{x})$. For all $\kappa = (\kappa_\ell)_{\ell \in \mathbb{Z}} \in \{-1,1\}^{\mathbb{Z}}$, there exists a point $x \in \mathbb{A}^n$ such that*

$$\Psi_j^{\ell q_j}(x) = T_{\frac{\kappa_\ell}{q_j}} \circ \mathcal{G}^{(j)}(\Psi_j^{(\ell-1)q_j}(x)), \qquad \ell \in \mathbb{Z}.$$

In particular, the sequence $(r_1^{(\ell)})_{\ell \in \mathbb{Z}}$ formed by the r_1-coordinates of the points $\Psi_j^{\ell q_j}(x)$ satisfies

$$r_1^{(\ell)} = r_1^{(\ell-1)} + \frac{\kappa_\ell}{q_j}.$$

Proof. The assertion results of an iterated application of Proposition 6.4, using a point of the form $x = ((0,0), x_2^{(j)}, \hat{x}^{(j)})$, where $x_2^{(j)} = \rho_j^{-1}(\kappa)$ and $\hat{x}^{(j)}$ is any point of $\hat{B}_j \times \{\hat{r}^{(j)}\}$. □

Theorem 2.2 is deduced from this result, with $N = n+1$, by the same suspension procedure as in Section 5.2. In view of (51), we can indeed apply Lemma 5.1 with $u = \frac{1}{q_j} U \otimes g^{(j)}$, $v = \frac{1}{N_j^2} V(\theta_2)$ and $w = \mu_j S_{N_j}(\theta_2)$; we get a non-autonomous time-periodic Hamiltonian $H_j(\theta, r, t)$, from which we obtain the desired Hamiltonian of $\mathbb{T}^{n+1} \times \mathbb{R}^{n+1}$ as

$$\mathcal{H}_j(\theta, \theta_{n+1}, r, r_{n+1}) = r_{n+1} + H_j(\theta, r, \theta_{n+1}),$$

for $(\theta, \theta_{n+1}) \in \mathbb{T} \times \mathbb{T}^n$ and $(r, r_{n+1}) \in \mathbb{R} \times \mathbb{R}^n$.

REMARK 6.2. *The ergodicity result contained in § 4 of [**Moe02**] can also to some little extent be adapted to our situation, at the price of a slight modification of the map Ψ_j.*

According to Proposition 6.4, we can indeed view the restriction $(\Psi_j)^{q_j}_{|\mathcal{I}_j}$ as a skew-product

$$\mathcal{P}_j(x_1, \kappa) = (\mathcal{F}_{\kappa_1}(x_1), \flat(\kappa)), \qquad \mathcal{F}_{\kappa_1} = \Phi^{\frac{\kappa_1}{q_j} U} \circ \Phi^{\frac{1}{2} q_j r_1^2},$$

and consider $\mathcal{F}_{-1}$ and $\mathcal{F}_{+1}$ as two maps randomly iterated on the annulus $\mathbb{A}$.

*If we change the definition of $g_2^{(j)}$ in (50) by retaining the first term only, the new map Ψ_j defined by (51) now corresponds to the same $\mathcal{F}_{+1} = \Phi^{\frac{1}{q_j} U} \circ \Phi^{\frac{1}{2} q_j r_1^2}$ but to $\mathcal{F}_{-1} = \Phi^{\frac{1}{2} q_j r_1^2}$. We then recover a particular case of the situation of Theorem 4 of [**Moe02**], except that we have not restricted ourselves to a finite-measure part of the annulus. By the same arguments as Moeckel, we can prove the ergodicity of the Lebesgue measure $\lambda_{\mathbb{A}}$ for the pair of maps $\{\mathcal{F}_{-1}, \mathcal{F}_{+1}\}$. This is sufficient to get a dense orbit for the skew-product $\mathcal{P}_j$, and thus an orbit of $\Psi_j^{q_j}$ whose first projection onto $\mathbb{A}$ is dense (indeed, the non-exitence of non-trivial Borel subsets of $\mathbb{A}$ simultaneously invariant by $\mathcal{F}_{-1}$ and $\mathcal{F}_{+1}$ implies that for any two non-void open subsets U, U' of $\mathbb{A}$, there exist $r \geq 0$ and $\kappa_1, \ldots, \kappa_r \in \{-1, +1\}$ such that $\mathcal{F}_{\kappa_r} \circ \cdots \circ \mathcal{F}_{\kappa_1}(U) \cap U' \neq \emptyset$; hence, for any two non-void subsets E, E' of $\mathbb{A} \times \{-1, +1\}^{\mathbb{Z}}$, there exists $n \geq 0$ such that $\mathcal{P}_j^n(E) \cap E' \neq \emptyset$, since each of these open sets contains a "rectangle" of the form $U \times C$ where C is a "symmetric cylinder" $\{\kappa \in \{-1, +1\}^{\mathbb{Z}} \mid \kappa_{-m} = \alpha_{-m}, \ldots, \kappa_m = \alpha_m\}$ for some finite sequence $\alpha_{-m}, \ldots, \alpha_m$ and iterates of such rectangles are easy to compute; topological transitivity follows by standard arguments). However, we were not able to use this to obtain an ergodic measure for $\Psi_j^{q_j}$, the support of which would have been $\mathcal{I}_j = \mathbb{A} \times \mathcal{K}_j \times \{\hat{x}^{(j)}\}$, simply because $\mathbb{A}$ has infinite Lebesgue measure and we cannot invoke Kakutani's theorem as Moeckel does in the finite measure case.*

A. *Appendix: proof of Lemma 3.1*

A.1. We suppose

$$A'(Y) = A_1 Y + A_2 Y^2 + A_3 Y^3 + O(Y^4), \quad B'(X) = \mu(X + bX^3) + O(X^4),$$

with real coefficients such that $0 < \mu A_1 < 2$, and we wish to study locally $f = \Phi^A \circ \Phi^B$, defined by the formula

$$f(X,Y) = (X_1, Y_1), \qquad X_1 = X + A'(Y - B'(X)), \quad Y_1 = Y - B'(X).$$

The origin is a fixed point, at which the linear part is given by the matrix

$$Df(0) = \begin{pmatrix} 1-\mu A_1 & A_1 \\ -\mu & 1 \end{pmatrix}.$$

We thus obtain $\lambda = \mathrm{e}^{i\gamma_0}$ (and $\bar\lambda = \mathrm{e}^{-i\gamma_0}$) as eigenvalue, with γ_0 determined by

$$\cos\gamma_0 = 1 - \frac{A_1\mu}{2}, \qquad -\frac{\pi}{2} < \gamma_0 < 0,$$

as indicated in the statement of the lemma. Notice that $\frac{\lambda-\bar\lambda}{i} = 2\sin\gamma_0 < 0$ (it is understood in that statement that $Df(0)$ is rotation of angle γ_0 rather than $-\gamma_0$) and that our assumptions ensure that none of the numbers $\lambda, \lambda^2, \lambda^3, \lambda^4$ coincide with 1.

The general theory guarantees the existence of a symplectic change of coordinates which puts f in Birkhoff normal form at order 2:

$$\tilde X_1 = \tilde X\cos\alpha - \tilde Y\sin\alpha, \quad \tilde Y_1 = \tilde X\sin\alpha + \tilde Y\cos\alpha, \qquad \alpha = \gamma_0 + \gamma_1\tfrac{\tilde X^2+\tilde Y^2}{2} + \dots.$$

Our aim is to compute the number γ_1 (which is uniquely determined).

We shall follow the procedure described in [**SM71**, §§31–34], according to which we need not care about the symplectic character of the tranformations and can content ourselves with searching for complex coordinates (ξ,η) in which f takes the form

$$\xi_1 = \lambda\xi(1 + i\gamma_1\xi\eta + \dots), \quad \eta_1 = \bar\lambda\eta(1 - i\gamma_1\xi\eta + \dots). \tag{52}$$

A.2. We first perform a complex linear change of coordinates

$$x = \tfrac{-1}{\lambda-\bar\lambda}(X + \tfrac{\bar\lambda-1}{\mu}Y), \quad y = \tfrac{1}{\lambda-\bar\lambda}(X + \tfrac{\lambda-1}{\mu}Y),$$

to diagonalize $Df(0)$: in these coordinates, f becomes

$$\begin{aligned} x_1 &= p(x,y) = \lambda x + p_2(x,y) + p_3(x,y) + \dots, \\ y_1 &= q(x,y) = \bar\lambda y + q_2(x,y) + q_3(x,y) + \dots, \end{aligned} \tag{53}$$

with $q(x,y) = \bar p(y,x)$ because of the realness of f. The computation of γ_1 requires the knowledge of all the coefficients of the quadratic part

$$p_2(x,y) = C_2x^2 + D_2xy + E_2y^2, \quad q_2(x,y) = \bar E_2x^2 + \bar D_2xy + \bar C_2y^2,$$

and of the coefficient D_3 of the cubic part

$$p_3(x,y) = C_3x^3 + D_3x^2y + E_3xy^2 + F_3y^3, \quad q_3(x,y) = \bar F_3x^3 + \bar E_3x^2y + \bar D_3xy^2 + \bar C_3y^3.$$

We leave it to the reader to check that the inverse formulae for the linear change of coordinates are

$$X = (1-\lambda)x + (1-\bar\lambda)y, \quad Y = \mu(x+y)$$

and that

$$C_2 = \Omega\lambda^2, \quad D_2 = 2\Omega, \quad E_2 = \Omega\bar{\lambda}^2, \quad D_3 = -\frac{3\lambda\mu^2}{\lambda-\bar{\lambda}}(A_1^2 b + A_3\mu) \qquad \Omega = -\frac{A_2\mu^2}{\lambda-\bar{\lambda}}. \tag{54}$$

A.3. We can now write the conjugation equation to be satisfied by a tangent-to-identity transformation

$$x = \phi(\xi,\eta) = \xi + \phi_2(\xi,\eta) + \phi_3(\xi,\eta) + \dots, \quad y = \psi(\xi,\eta) = \eta + \psi_2(\xi,\eta) + \psi_3(\xi,\eta) + \dots$$

to pass from (53) to (52):

$$\phi(\lambda\xi(1 + i\gamma_1\xi\eta + \dots), \bar{\lambda}\eta(1 - i\gamma_1\xi\eta + \dots)) = \\ \lambda\phi(\xi,\eta) + p_2(\phi(\xi,\eta),\psi(\xi,\eta)) + p_3(\phi(\xi,\eta),\psi(\xi,\eta)) + \dots,$$

with $\psi(\xi,\eta) = \bar{\phi}(\eta,\xi)$ because of the realness condition. The quadratic part of the transformation is easily obtained by solving $\phi_2(\lambda\xi,\bar{\lambda}\eta) - \lambda\phi_2(\xi,\eta) = p_2(\xi,\eta)$:

$$\phi_2(\xi,\eta) = \frac{C_2}{\lambda^2-\lambda}\xi^2 + \frac{D_2}{1-\lambda}\xi\eta + \frac{E_2}{\bar{\lambda}^2-\lambda}\eta^2$$

and $\psi_2(\xi,\eta) = \bar{\phi}_2(\eta,\xi)$.

The coefficient γ_1 is determined by examining the coefficient of $\xi^2\eta$:

$$i\lambda\gamma_1\xi^2\eta + \phi_3(\lambda\xi,\bar{\lambda}\eta) - \lambda\phi_3(\xi,\eta) - p_3(\xi,\eta) = \\ \text{cubic part of } \big[p_2(\xi + \phi_2(\xi,\eta), \eta + \psi_2(\xi,\eta))\big],$$

thus

$$i\lambda\gamma_1 = D_3 + \frac{2-\bar{\lambda}}{1-\lambda}C_2D_2 + \frac{1}{1-\bar{\lambda}}D_2\bar{D}_2 + \frac{2\bar{\lambda}^2}{1-\bar{\lambda}^3}E_2\bar{E}_2.$$

This corresponds to the formula given by [**Io79**], p. 30.

In view of (54), since $\Omega \in i\mathbb{R}$, we can write

$$i\lambda\gamma_1 - D_3 = -2\Omega^2\lambda\left(\frac{(2-\bar{\lambda})\lambda}{\lambda-1} + \frac{2}{\lambda-1} + \frac{1}{\lambda^3-1}\right) = -2\Omega^2\lambda\omega,$$

whence the conclusion follows.

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