The dressed nonrelativistic electron in a magnetic field

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Abstract

We consider a nonrelativistic electron interacting with a classical magnetic field pointing along the x_3-axis and with a quantized electromagnetic field. When the interaction between the electron and photons is turned off, the electronic system is assumed to have a ground state of finite multiplicity. Because of the translation invariance along the x_3-axis, we consider the reduced Hamiltonian associated with the total momentum along the x_3-axis and, after introducing an ultraviolet cutoff and an infrared regularization, we prove that the reduced Hamiltonian has a ground state if the coupling constant and the total momentum along the x_3-axis are sufficiently small. Finally we determine the absolutely continuous spectrum of the reduced Hamiltonian.

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Contents

1 Introduction

We consider a nonrelativistic electron in $\mathbb{R}^3$ of charge e and mass m interacting with a magnetic field pointing along the x_3-axis and with photons. The magnetic field takes the form $(0,0,b(x_1,x_2))$ with $b(x_1,x_2)=\frac{\partial a_2}{\partial x_1}(x_1,x_2)-\frac{\partial a_1}{\partial x_2}(x_1,x_2)$ where $a(x_1,x_2)$ is a vector potentiel. The associated Pauli Hamiltonian in Coulomb gauge is formally given by

$$\begin{aligned} H =& \frac{1}{2m}(p-ea(x')-eA(x))^2-\frac{e}{2m}b(x')\sigma_3\otimes 1+V(x')\otimes 1 \\ &+1\otimes H_{ph}-\frac{e}{2m}\sigma\cdot B(x)\ . \end{aligned} \tag{1.1}$$

Here the unit are such that $\bar{h}=c=1$, $p=-i\nabla_x$, $x=(x_1,x_2,x_3)$ together with $x'=(x_1,x_2)$, $\sigma=(\sigma_1,\sigma_2,\sigma_3)$ is the 3-component vector of the Pauli matrices, $V(x')$ is an electric potential depending only on the transverse variables. The quantized electromagnetic field is formally given by

$$A(x)=\frac{1}{2\pi}\sum_{\mu=1,2}\int d^3k\left(\frac{1}{|k|^{1/2}}\epsilon_\mu(k)e^{-ik\cdot x}a^\star_\mu(k)+\frac{1}{|k|^{1/2}}\epsilon_\mu(k)e^{ik\cdot x}a_\mu(k)\right)\ , \tag{1.2}$$

$$\begin{aligned} B(x)=\frac{i}{2\pi}\sum_{\mu=1,2}\int d^3k\,\Big\{&-|k|^{1/2}\left(\frac{k}{|k|}\wedge\epsilon_\mu(k)\right)e^{-ik\cdot x}a^\star_\mu(k) \\ &+|k|^{1/2}\left(\frac{k}{|k|}\wedge\epsilon_\mu(k)\right)e^{ik\cdot x}a_\mu(k)\Big\} \end{aligned} \tag{1.3}$$

where $\epsilon_\mu(k)$ are the real polarization vectors satisfying $\epsilon_\mu(k)\cdot\epsilon_{\mu'}(k)=\delta_{\mu\mu'}$, $k\cdot\epsilon_\mu(k)=0$; $a_\mu(k)$ and $a^\star_\mu(k)$ are the usual annihilation and creation operators acting in the Fock space

$$\mathcal{F}:=\oplus_{n=0}^{\infty}L^2(\mathbb{R}^3,\mathbb{C}^2)^{\otimes^n_s}$$

where $L^2(\mathbb{R}^3,\mathbb{C}^2)^{\otimes^0_s}=\mathbb{C}$ and $L^2(\mathbb{R}^3,\mathbb{C}^2)^{\otimes^n_s}$ is the symmetric n-tensor power of $L^2(\mathbb{R}^3,\mathbb{C}^2)$ appropriate for Bose-Einstein statistics. The annihilation and creation operator obey the canonical commutation relations ($a^\sharp=a^\star$ or a)

$$[a^\sharp_\mu(k),a^\sharp_{\mu'}(k')]=0\quad\text{et}\quad[a_\mu(k),a^\star_{\mu'}(k')]=\delta_{\mu\mu'}\delta(k-k')\ . \tag{1.4}$$

Finally the Hamiltonian for the photons is given by

$$H_{ph}=\sum_{\mu=1,2}\int|k|a^\star_\mu(k)a_\mu(k)d^3k\ . \tag{1.5}$$

The Hilbert space associated with H is then

$$\mathcal{H}=L^2(\mathbb{R}^3,\mathbb{C}^2)\otimes\mathcal{F}\simeq L^2(\mathbb{R}^3,\mathbb{C}^2\otimes\mathcal{F})\ .$$

As it stands, the Hamiltonian H cannot be defined as a self-adjoint operator in $\mathcal{H}$ and we need to introduce cutoff functions, both in $A(x)$ and in $B(x)$, which will satisfy appropriate hypothesis in order to get a self adjoint operator in $\mathcal{H}$.

This operator, still denoted by H, commutes with the third component, denoted by P_3, of the total momentum of the system (cf. [1]). We have $P_3 = p_3 \otimes 1 + 1 \otimes \mathrm{d}\Gamma(k_3)$ where $\mathrm{d}\Gamma(k_3)$ is the second quantized operator associated to the multiplication operator by the third component of k in $L^2(\mathbb{R}^3, \mathbb{C}^2)$. The spectrum of P_3 is the real line. In turns out that H admits a decomposition over the spectrum of P_3 as a direct integral

$$H \simeq \int_{\mathbb{R}}^{\oplus} H(P_3) dP_3 \tag{1.6}$$

on

$$\mathcal{H} \simeq \int_{\mathbb{R}}^{\oplus} L^2(\mathbb{R}^3, \mathbb{C}^2 \otimes \mathcal{F}) dP_3 \simeq \int_{\mathbb{R}}^{\oplus} L^2(\mathbb{R}^3, \mathbb{C}^2) \otimes \mathcal{F} dP_3 \ .$$

The reduced operator $H(P_3)$ will be explicitely computed and the aim of this article is to initiate the spectral analysis of $H(P_3)$ when $|P_3|$ is small.

In the free case, i.e. when $b = V = 0$, a similar problem has been studied in [2] by T. Chen who considers a freely propagating nonrelativistic spinless charged particle interacting with the quantized electromagnetic field. T. Chen proves that the reduced Hamiltonian associated to the total momentum P has a unique ground state when P is sufficiently small by applying the renormalization group method introduced in [3]. In the case of the one-particle sector of Nelson's model, similar result has been obtained first by J. Fröhlich (see [4], [5]) and more recently by A. Pizzo (see [6], [7]) and J.S. Møller [8].

When $V = 0$ and $b \neq 0$, the electronic part has an infinitely degenerated ground state and we face a difficult mathematical problem: the perturbation of an eigenvalue of infinite multiplicity at the bottom of the essential spectrum. To overcome this difficulty we essentially add an electrostatic potential V in order to obtain an electronic Hamiltonian having a ground state of finite multiplicity (cf. section 2.1 for the precise hypothesis on b and V). Of course we still face the problem of perturbing an eigenvalue at the bottom of the essential spectrum but we give a simple proof for the existence of a ground state for $H(P_3)$ with an ultraviolet cutoff and an infrared regularization. The proof borrows ideas both from [4] (see also [5, 6, 7, 9]) where the Hamiltonian is invariant by translation and [10] (see also [11, 12, 13, 14, 15, 16, 17, 18] where the electronic part is confined).

Following [2], we can conjecture that $H(P_3)$ without infrared regularization has no ground state in $\mathcal{H}$.

The same proof also works for any free atom or positive ion interacting with the quantized electromagnetic field.

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2 Definition of the model and selfadjointness

The Hamiltonian H can be written as

$$H = H_0 + H_I \tag{2.1}$$

where

$$H_0 = \left\{ \frac{1}{2m} p_3^2 + \frac{1}{2m} \sum_{j=1,2} (p_j - e a_j(x'))^2 - \frac{e}{2m} b(x') \sigma_3 + V(x') \right\} \otimes 1 + 1 \otimes H_{ph} \tag{2.2}$$

and H_I describes the interaction between the electron in the magnetic field $b(x')$ with the photons. A basic tool is now to describe the spectral properties of the Pauli operator in $L^2(\mathbb{R}^2, \mathbb{C}^2)$ that we are dealing with.

2.1 The Pauli operator with magnetic fields

Let $h(b, V)$ be the following operator in $L^2(\mathbb{R}^2, \mathbb{C}^2)$

$$h(b, V) = \frac{1}{2m} \sum_{j=1,2} (p_j - e a_j(x_1, x_2))^2 - \frac{e}{2m} b(x_1, x_2) \sigma_3 + V(x_1, x_2) \ . \tag{2.3}$$

As in [19] the a_j's are real functions in $C^1(\mathbb{R}^2)$ such that $b(x_1, x_2) = \frac{\partial}{\partial x_1} a_2(x_1, x_2) - \frac{\partial}{\partial x_2} a_1(x_1, x_2)$ and we suppose that

Hyp. 2.1. *b and V sastisfy*

(i) $b \in C^1(\mathbb{R}^2)$ and

$$1/C \leq b(x_1, x_2) \leq C \ \textit{and} \ |\nabla b(x_1, x_2)| \leq C$$

for some $C > 1$.

(ii) V is a real function in $L^\infty(\mathbb{R}^2)$ and

$$V(x_1, x_2) \to 0 \ \textit{as} \ |(x_1, x_2)| \to +\infty \ .$$

We then have

Proposition 2.2. *Suppose that hypothesis 2.1 is satisfied, then the operator $h(b, V)$ with domain*

$$D(b, V) = \left\{ u \in L^2(\mathbb{R}^2, \mathbb{C}^2) \mid h(b, V) u \in L^2(\mathbb{R}^2, \mathbb{C}^2) \right\}$$

is self-adjoint in $L^2(\mathbb{R}^2, \mathbb{C}^2)$.

Furthermore $h(b, V)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2, \mathbb{C}^2)$.

The proof can be found in [20] (see also [21] and [22]).

According to [23] (see also [24, 25, 26]) $\frac{1}{2m}\sum_{j=1,2}(p_j - ea_j(x_1,x_2))^2 - \frac{e}{2m}b(x_1,x_2)\sigma_3$ has zero as an eigenvalue of infinite multiplicity. Zero is also the bottom of its spectrum. By adding $V(x_1,x_2)$, the operator $h(b,V)$ may have eigenvalues of finite multiplicity accumulating at zero. In fact there exist b and V satisfying hypothesis 2.1 such that $h(b,V)$ has strictly negative eigenvalues of finite multiplicity (see [19, 26]).

We now suppose that

Hyp. 2.3. *b and V are such that the bottom of the spectrum of $h(b,V)$ is a strictly negative isolated eigenvalue of finite multiplicity.*

Notice that hypothesis (2.1),(2.3) allow to choose a constant magnetic field but, in this case, V cannot be identically zero.

In fact in what follows we only use that : $h(b,V)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2,\mathbb{C}^2)$ and $\inf\sigma(b,V)$ is an isolated eigenvalue with finite multiplicity.

2.2 The model

We now introduce the formal Hamiltonian in the Fock space associated to (1.1). As usual we will consider the charge e in front of the quantized electromagnetic field $A(x)$ as a parameter and from now on we will denote it by g.

We introduce $\rho(k)$ a cutoff function associated with an ultraviolet cutoff and an infrared regularization the precise regularization assumption verified by ρ will be given in each theorem but ρ will always satisfy (2.7) below.

The associated quantized electromagnetic field is then given by ($j = 1,2,3$)

$$\begin{aligned} A_j(x,\rho) = \frac{1}{2\pi}\sum_{\mu=1,2}\int d^3k\,\Bigg(&\frac{\rho(k)}{|k|^{1/2}}\epsilon_\mu(k)_j e^{-ik\cdot x}a_\mu^\star(k) \\ &+\frac{\bar{\rho(k)}}{|k|^{1/2}}\epsilon_\mu(k)_j e^{ik\cdot x}a_\mu(k)\Bigg)\ , \end{aligned} \tag{2.4}$$

$$\begin{aligned} B_j(x,\rho) = \frac{i}{2\pi}\sum_{\mu=1,2}\int d^3k\,\Bigg(&-|k|^{1/2}\rho(k)\left(\frac{k}{|k|}\wedge\epsilon_\mu(k)\right)_j e^{-ik\cdot x}a_\mu^\star(k) \\ &+|k|^{1/2}\bar{\rho(k)}\left(\frac{k}{|k|}\wedge\epsilon_\mu(k)\right)_j e^{ik\cdot x}a_\mu(k)\Bigg)\ . \end{aligned} \tag{2.5}$$

The interaction Hamiltonian (cf. (2.1)) reads

$$\begin{aligned} H_I = &-\frac{g}{m}\sum_{j=1,2}\left\{A_j(x,\rho)(p_j - ea_j(x')) + (p_j - ea_j(x'))A_j(x,\rho)\right\} \\ &-\frac{g}{m}\left\{A_3(x,\rho)p_3 + p_3A_3(x,\rho)\right\} \\ &-\frac{g}{2m}\sigma\cdot B(x,\rho) + \frac{g^2}{2m}A(x,\rho)\cdot A(x,\rho)\ . \end{aligned}$$

Noticing that $k \cdot \epsilon_\mu(k) = 0$, H_I can be rewritten as

$$
\begin{aligned}
H_I = & -\frac{g}{m} A_3(x,\rho) p_3 - \frac{g}{m} \sum_{j=1,2} A_j(x,\rho)(p_j - e a_j(x')) \\
& - \frac{g}{2m} \sigma \cdot B(x,\rho) + \frac{g^2}{2m} : A(x,\rho) \cdot A(x,\rho) :
\end{aligned}
\tag{2.6}
$$

where we also substitute the Wick normal ordering $: A(x,\rho) \cdot A(x,\rho) :$ for $A(x,\rho) \cdot A(x,\rho)$. This last substitution changes the Hamiltonian by a constant as it follows from the canonical commutation relations.

Let $\mathcal{F}_{0,fin}$ be the set of $(\psi_n)_{n\geq 0} \in \mathcal{F}$ such that ψ_n is in the Schwartz space for every n and $\psi_n = 0$ for all but finitely many n. Suppose that

$$
\int_{|k|\leq 1} \frac{|\rho(k)|^2}{|k|^2} d^3k < \infty \quad \text{and} \quad \int_{|k|\geq 1} |k||\rho(k)|^2 d^3k < \infty \ . \tag{2.7}
$$

Then our model is described by the operator $H = H_0 + H_I$ given by (2.2), (2.6), and this operator is well defined on $C_0^\infty(\mathbb{R}^3, \mathbb{C}^2) \otimes \mathcal{F}_{0,fin}$.

2.3 Selfadjointness

In $L^2(\mathbb{R}^3, \mathbb{C}^2) \otimes \mathcal{F}$, the operator H_0 given by (2.2) is essentially self adjoint on $C_0^\infty(\mathbb{R}^3, \mathbb{C}^2) \otimes \mathcal{F}_{0,fin}$ (see [27]). Its self-adjoint extension is still denoted by H_0. The interaction operator H_I (see (2.6)) is a symmetric operator on $C_0^\infty(\mathbb{R}^3, \mathbb{C}^2) \otimes \mathcal{F}_{0,fin}$. We are going to prove that H_I is relatively bounded with respect to H_0 to apply the Kato-Rellich theorem

Theorem 2.4. *Assume* (2.7) *and*

$$
\frac{6|g|}{\pi\sqrt{2m}} \left(\int \frac{|\rho(k)|^2}{|k|^2} d^3k \right)^{1/2} + \frac{g^2}{\pi^2 m} \int \frac{|\rho(k)|^2}{|k|^2} d^3k < \frac{1}{2} \ .
$$

Then H is a self-adjoint operator in $\mathcal{H}$ with domain $D(H) = D(H_0)$ and H is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3, \mathbb{C}^2) \otimes \mathcal{F}_{0,fin}$.

Proof. To begin with we recall the following well known estimates (cf. [28])

$$
\|a_\mu(g(.,x))\psi\| \leq \left(\int \frac{|g(x,k)|^2}{|k|} d^3k \right)^{1/2} \|(I \otimes H_{ph}^{1/2})\psi\| \tag{2.8}
$$

and

$$
\begin{aligned}
\|a_\mu^*(g(.,x))\psi\| \leq & \left(\int \frac{|g(x,k)|^2}{|k|} d^3k \right)^{1/2} \|(I \otimes H_{ph}^{1/2})\psi\| \\
& + \left(\int |g(x,k)|^2 d^3k \right)^{1/2} \|\psi\| \ .
\end{aligned}
\tag{2.9}
$$

We get for $\psi \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^2) \otimes \mathcal{F}_{0,fin}$

$$\frac{|g|}{m}\|A_3(x,\rho)p_3\psi\| \leq \frac{4|g|}{\pi\sqrt{2m}}\left(\int \frac{|\rho(k)|^2}{|k|^2}d^3k\right)^{1/2} \\ \|(I\otimes H_{ph}^{1/2})(\frac{p_3}{\sqrt{2m}}\otimes I)\psi\| \\ +\frac{2|g|}{\pi\sqrt{2m}}\left(\int \frac{|\rho(k)|^2}{|k|}d^3k\right)^{1/2}\|(\frac{p_3}{\sqrt{2m}}\otimes I)\psi\| \ .$$

Denoting $e(b,V) := \inf\sigma(h(b,V))$, notice that

$$\|(I\otimes H_{ph}^{1/2})(\frac{p_3}{\sqrt{2m}}\otimes I)\psi\| \leq \frac{1}{2}\left(\|(I\otimes H_{ph})\psi\| + \|(\frac{p_3}{\sqrt{2m}}\otimes I)\psi\|\right) \\ \leq \|(H_0 - e(b,V))\psi\|$$

and, for every $\epsilon > 0$,

$$\|(\frac{p_3}{\sqrt{2m}}\otimes I)\psi\| \leq \sqrt{\frac{\epsilon}{2}}\|(H_0 - e(b,V))\psi\| + \frac{1}{\sqrt{2\epsilon}}\|\psi\|$$

to obtain that

$$\frac{|g|}{m}\|A_3(x,\rho)p_3\psi\| \leq \frac{4|g|}{\pi\sqrt{2m}}\left(\int \frac{|\rho(k)|^2}{|k|^2}d^3k\right)^{1/2}\|(H_0 - e(b,V))\psi\| \\ +\frac{2|g|}{\pi\sqrt{2m}}\left(\int \frac{|\rho(k)|^2}{|k|}d^3k\right)^{1/2}\left(\sqrt{\frac{\epsilon}{2}}\|(H_0 - e(b,V))\psi\| + \frac{1}{\sqrt{2\epsilon}}\|\psi\|\right) \ . \tag{2.10}$$

Therefore $\frac{g}{m}A_3(x,\rho)p_3$ is relatively bounded with respect to H_0 with relative bound $\frac{4|g|}{\pi\sqrt{2m}}\left(\int \frac{|\rho(k)|^2}{|k|^2}d^3k\right)^{1/2}$. Similarly we verify that for $j=1,2$, $\frac{g}{m}A_j(x,\rho)(p_j - ea_j(x'))$ is also relatively bounded with respect to H_0 with the same relative bound, and that $\frac{|g|}{2m}\sigma\cdot B(x,\rho)$ is relatively bounded with respect to H_0 with zero relative bound.

It remains to estimate the quadratic terms associated with $\frac{g^2}{2m}$ $: A(x,\rho)\cdot A(x,\rho) :$. Let us recall the following estimates (cf. [29]):

$$\begin{aligned}
\|a_\mu(f)a_\lambda(f)\psi\| \le{}& \left(\int \frac{|\rho(k)|^2}{|k|^2} d^3k\right) \|(H_{ph}+1)\psi\| \\
&+ K \left(\int \frac{|\rho(k)|^2}{|k|^2} d^3k\right)^{1/2} \left(\int |\rho(k)|^2 d^3k\right)^{1/2} \|(H_{ph}+1)^{1/2}\psi\| \,, \\
\|a^*_\mu(f)a_\lambda(f)\psi\| \le{}& \left(\int \frac{|\rho(k)|^2}{|k|^2} d^3k\right) \|(H_{ph}+1)\psi\| \\
&+ \left(K \left(\int \frac{|\rho(k)|^2}{|k|^2} d^3k\right)^{1/2} \left(\int |\rho(k)|^2 d^3k\right)^{1/2} \right. \\
&+ \left. \left(\frac{|\rho(k)|^2}{|k|^2} d^3k\right)^{1/2} \left(\frac{|\rho(k)|^2}{|k|} d^3k\right)^{1/2} \right) \|(H_{ph}+1)^{1/2}\psi\| \,, \\
\|a^*_\mu(f)a^*_\lambda(f)\psi\| \le{}& \left(\int \frac{|\rho(k)|^2}{|k|^2} d^3k\right) \|(H_{ph}+1)\psi\| \\
&+ \left(K \left(\int \frac{|\rho(k)|^2}{|k|^2} d^3k\right)^{1/2} \left(\int |\rho(k)|^2 d^3k\right)^{1/2} \right. \\
&+ \left. \left(\frac{|\rho(k)|^2}{|k|^2} d^3k\right)^{1/2} \left(\frac{|\rho(k)|^2}{|k|} d^3k\right)^{1/2} \right) \|(H_{ph}+1)^{1/2}\psi\| \\
&+ \left(\left(\int \frac{|\rho(k)|^2}{|k|^2} d^3k\right)^{1/2} \left(\int |k||\rho(k)|^2 d^3k\right)^{1/2} + \int \frac{|\rho(k)|^2}{|k|} d^3k \right) \|\psi\|
\end{aligned}$$

where $K = \frac{1}{\pi}\int_0^\infty \frac{\sqrt{\lambda}}{(1+\lambda)^2}$.

As $(H_{ph}+1)^{1/2}$ is relatively bounded with respect to $H_{ph}+1$ (and thus to H_0) with a zero relative bound, we deduce that the relative bound of $\frac{g^2}{2m} :A(x,\rho)\cdot A(x,\rho):$ with respect to H_0 is given by

$$16 \frac{g^2}{2m} \frac{1}{4\pi^2} \int \frac{|\rho(k)|^2}{|k|^2} d^3k \,.$$

Finally we get that H_I is relatively bounded with respect to H_0 with the relative bound

$$\frac{12|g|}{\pi\sqrt{2m}} \left(\int \frac{|\rho(k)|^2}{|k|^2} d^3k\right)^{1/2} + \frac{2g^2}{\pi^2 m} \int \frac{|\rho(k)|^2}{|k|^2} d^3k$$

and hence theorem 2.4 is a consequence of the Kato-Rellich theorem. □

2.4 The reduced Hamiltonian

The operator H is invariant by translation in the x_3-direction. Thus, denoting by P_3 the total momentum in the x_3-direction ($P_3 = p_3 \otimes 1 + 1 \otimes \mathrm{d}\Gamma(k_3)$), H

has a direct integral representation in a spectral representation P_3, i.e.

$$H \simeq \int_{\mathbb{R}}^{\oplus} H(P_3) dP_3 \tag{2.11}$$

To compute $H(P_3)$ we proceed as in [4] (see also [1, 29] and[18]). Let Π be the unitary map from $\mathcal{F}$ to $L^2(\mathbb{R}) \otimes L^2(\mathbb{R}^2, \mathbb{C}^2) \otimes \mathcal{F}$ defined by

$$(\Pi\phi)_n(P_3, x', k_1, \dots, k_n) = \hat{\phi}_n(P_3 - \sum_{i=1}^{n} k_{i,3}, x', k_1, \dots, k_n)$$

where the hat stands for the partial Fourier transform in x_3. One easily verifies that, on $C_0^\infty(\mathbb{R}^3, \mathbb{C}^2) \otimes \mathcal{F}_{0,fin}$,

$$(\Pi A_j(x', x_3, \rho)\Pi^* = A_j(x', 0, \rho) \ .$$

Therefore, for $\psi \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^2) \otimes \mathcal{F}_{0,fin}$,

$$(\Pi H \Pi^* \psi)(P_3, \cdot) = H(P_3)\psi(P_3, \cdot)$$

where the reduced Hamiltonian $H(P_3)$ is given by

$$H(P_3) = H_0(P_3) + H_I(P_3) \tag{2.12}$$

with

$$H_0(P_3) = h(b, V) \otimes 1 + 1 \otimes \left\{ \frac{1}{2m}(P_3 - \mathrm{d}\Gamma(k_3))^2 + H_{ph} \right\} \tag{2.13}$$

and

$$\begin{aligned} H_I(P_3) = &-\frac{g}{2m}\sigma \cdot B(x', 0, \rho) - \frac{g}{m} \sum_{j=1,2} A_j(x', 0, \rho)(p_j - ea_j(x')) \\ &- \frac{g}{m} A_3(x', 0, \rho)(P_3 - \mathrm{d}\Gamma(k_3)) + \frac{g^2}{2m} : A(x', 0, \rho) \cdot A(x', 0, \rho) : \end{aligned} \tag{2.14}$$

For every P_3, $H(P_3)$ is now an operator in $L^2(\mathbb{R}^2, \mathbb{C}^2) \otimes \mathcal{F}$. We want to show that this operator has a self adjoint representation such that (2.11) be satisfied.

The operator $\frac{1}{2m}(P_3 - \mathrm{d}\Gamma(k_3))^2 + H_{ph}$ is essentially self-adjoint on $\mathcal{F}_{0,fin}$. Therefore, for every $P_3 \in \mathbb{R}$, $H_0(P_3)$ is essentially self-adjoint in $C_0^\infty(\mathbb{R}^2, \mathbb{C}^2) \otimes \mathcal{F}_{0,fin}$. We still denote by $H_0(P_3)$ its self-adjoint extension. On the other hand $H_I(P_3)$ is a symmetric operator on $C_0^\infty(\mathbb{R}^2, \mathbb{C}^2) \otimes \mathcal{F}_{0,fin}$ and we want to prove that it is relatively bounded with respect to $H_0(P_3)$. For this we follow closely the lines of section 2.3 and we only focus on the estimates of the new terms. For $\psi \in C_0^\infty(\mathbb{R}^2, \mathbb{C}^2) \otimes \mathcal{F}_{0,fin}$ we have

$$\begin{aligned} \frac{|g|}{m} \|A_3(x', 0, \rho)(P_3 - d\Gamma(k_3))\psi\| \leq \frac{4|g|}{\pi\sqrt{2m}} \left(\int \frac{|\rho(k)|^2}{|k|^2} d^3k \right)^{1/2} \\ \|(I \otimes H_{ph}^{1/2})(I \otimes \frac{P_3 - d\Gamma(k_3)}{\sqrt{2m}})\psi\| \\ + \frac{2|g|}{\pi\sqrt{2m}} \left(\int \frac{|\rho(k)|^2}{|k|} d^3k \right)^{1/2} \|(I \otimes \frac{P_3 - d\Gamma(k_3)}{\sqrt{2m}})\psi\| \ . \end{aligned}$$

For every component ψ_n of $\psi \in \mathcal{F}_{0,fin}$ associated with n photons the operator $(I \otimes H_{ph}^{1/2})(I \otimes \frac{P_3 - d\Gamma(k_3)}{\sqrt{2m}})$ is the multiplication operator by the function $(\sum_{i=1}^{n} \omega(k_i))^{1/2} \frac{P_3 - \sum_{i=1}^{n} k_{i,3}}{\sqrt{2m}}$. We then get

$$\begin{aligned}\|(I \otimes H_{ph}^{1/2})(I \otimes \frac{P_3 - d\Gamma(k_3)}{\sqrt{2m}})\psi\| &\leq \frac{1}{\sqrt{2}}\|(I \otimes \left\{H_{ph} + \frac{1}{2m}(P_3 - d\Gamma(k_3))^2\right\}\psi\| \\ &\leq \frac{1}{\sqrt{2}}\|(H_0(P_3) - e(b,V))\psi\|\end{aligned}$$

and, for any $\epsilon > 0$,

$$\|I \otimes \frac{P_3 - d\Gamma(k_3)}{\sqrt{2m}})\psi\| \leq \sqrt{\frac{\epsilon}{2}}\|(H_0(P_3) - e(b,V))\psi\| + \frac{1}{\sqrt{2\epsilon}}\|\psi\| \ .$$

Therefore

$$\begin{aligned}\frac{|g|}{m}\|A_3(x', 0, \rho)(P_3 - d\Gamma(k_3))\psi\| \leq \frac{2|g|}{\pi\sqrt{m}} &\left(\int \frac{|\rho(k)|^2}{|k|^2} d^3k\right)^{1/2} \|(H_0(P_3) - e(b,V))\psi\| \\ + \sqrt{\frac{\epsilon}{m}}\frac{|g|}{\pi} &\left(\int \frac{|\rho(k)|^2}{|k|} d^3k\right)^{1/2} \|(H_0(P_3) - e(b,V))\psi\| \\ + \frac{1}{\sqrt{\epsilon m}}\frac{|g|}{\pi} &\left(\int \frac{|\rho(k)|^2}{|k|} d^3k\right)^{1/2} \|\psi\| \ .\end{aligned}$$

∎

And thus, as in section 2.3, we obtain that, for any $\eta > 0$, there exists a finite constant a_η such that

$$\|H_I(P_3)\psi\| \leq |g|(b + \eta)\|H_0(P_3)\psi\| + |g|a_\eta\|\psi\| \tag{2.15}$$

with

$$b = \frac{12}{\pi\sqrt{2m}}\left(\int \frac{|\rho(k)|^2}{|k|^2} d^3k\right)^{1/2} + \frac{2g}{\pi^2 m}\int \frac{|\rho(k)|^2}{|k|^2} d^3k \ .$$

Therefore we have

Theorem 2.5. *Assume* (2.7) *and*

$$\frac{6|g|}{\pi\sqrt{2m}}\left(\int \frac{|\rho(k)|^2}{|k|^2} d^3k\right)^{1/2} + \frac{g^2}{\pi^2 m}\int \frac{|\rho(k)|^2}{|k|^2} d^3k < \frac{1}{2} \ . \tag{2.16}$$

Then, for every $P_3 \in \mathbb{R}$, $H(P_3)$ is a self-adjoint operator in $L^2(\mathbb{R}^2, \mathbb{C}^2) \otimes \mathcal{F}$ with domain $D(H(P_3)) = D(H_0(P_3))$ and $H(P_3)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2, \mathbb{C}^2) \otimes \mathcal{F}_{0,fin}$.

Further we get

Corollary 2.6. *We have*

$$\Pi H \Pi^* = \int_{\mathbb{R}}^{\oplus} H(P_3) dP_3 \ .$$

The proof of corollary 2.6 follows by mimicking [29].

3 Main results

For a bounded below self-adjoint operator A with a ground state, $m(A)$ will denote the multiplicity of $\inf\sigma(A)$. Our main result is the following theorem which states that, for P_3 and g sufficiently small, $H(P_3)$ has a ground state:

Theorem 3.1. *Assume that the cutoff function satisfies*

$$\int_{|k|\le 1} \frac{|\rho(k)|^2}{|k|^3} d^3k < \infty \quad \text{and} \quad \int_{|k|\ge 1} |k||\rho(k)|^2 d^3k < \infty \ . \tag{3.1}$$

Then there exist $P > 0$ and $g_0 > 0$ such that for $|P_3| \le P$ and $|g| \le g_0$, $H(P_3)$ has a ground state. Furthermore $m(H(P_3)) \le m(h(b,V))$. In particular, if $e(b,V)$ is a simple eigenvalue of $h(b,V)$, then $\inf\sigma(H(P_3))$ *is a simple eigenvalue too.*

The proof of this theorem is given in the next section. Notice that the regularization condition (3.1) does not allow $\rho(k) = 1$ near the origin.

The existence of a ground state has several consequences. The first one is the existence of asymptotic Fock representations for the CCR:

For $f \in L^2(\mathbb{R}^3, \mathbb{C}^2)$, we define on $D(H_0(P_3))$ the operators

$$a^\sharp_{\mu,t}(f) := e^{itH(P_3)} e^{-itH_0(P_3)} a^\sharp_\mu(f) e^{itH_0(P_3)} e^{-itH(P_3)} \ .$$

Let Q be a closed null set such that the polarization vectors $\epsilon_\mu(k)$ are C^∞ on $\mathbb{R}^3 \setminus Q$ for $\mu = 1, 2$. We have

Corollary 3.2. *Suppose that the hypothesis of theorem 3.1 are satisfied. Then, for $f \in C_0^\infty(\mathbb{R}^3 \setminus Q)$ and for every $\Psi \in D(H_0(P_3))$ the strong limits of $a^\sharp_{\mu,t}(f)$ exist:*

$$\lim_{t\to\pm\infty} a^\sharp_{\mu,t}(f)\Psi =: a^\sharp_{\mu,\pm}(f)\Psi \ .$$

The $a^\sharp_{\mu,\pm}$'s satisfy the CCR and, if $\Phi(P_3)$ is a ground state for $H(P_3)$, we have for $f \in C_0^\infty(\mathbb{R}^3 \setminus Q)$ and $\mu = 1, 2$

$$a_{\mu,\pm}(f)\Phi(P_3) = 0 \ .$$

We then deduce the following corollary

Corollary 3.3. *Under the hypothesis of theorem 3.1, the absolutely continuous spectrum of $H(P_3)$ equals to* $[\inf\sigma(H(P_3)), +\infty)$.

The proofs of these two corollary follow by mimicking [12, 15].

Our last application concerns the renormalized mass and magnetic moment of the electron.

From now on we assume that the ground state of $h(b,V)$ is simple in such a way that the ground state of $H(P_3)$ is also simple. To state our result we need the notations and results of theorem 4.1: let $E_\sigma(P_3)$ be the ground energy of the hamiltonian with infrared cutoff $H_\sigma(P_3)$, $E_\sigma(P_3)$ is a simple and isolated eigenvalue of $H_\sigma(P_3)$ and therefore we deduce from the standart Kato's perturbation theory that $E_\sigma(P_3)$ is a regular function of P_3. We then define the renormalized mass of the electron by

$$m^\star := \liminf_{\sigma \to 0} m^\star_\sigma \tag{3.2}$$

where

$$(m^\star_\sigma)^{-1} = \partial^2_{P_3} E_\sigma(0) \ .$$

Let $\mathbf{g}_{\mathrm{el}}$ be the magnetic moment of the dressed electron. We then have

Corollary 3.4. *Under the hypothesis of theorem 3.1 and assuming $h(b,V)$ as a simple ground state, we have*

$$m^\star \geq m$$

i.e. the renormalized mass of the dressed electron in a magnetic field is larger than the bare mass of the electron. It then follows that $\mathbf{g}_{\mathrm{el}} \geq 2$.

Proof. Since the ground state of $H_\sigma(P_3)$ is non degenarate, $E_\sigma(P_3)$ and $\Phi_\sigma(P_3)$ are smooth function of the parameter P_3 and we easily obtain differentiating the relation $H_\sigma(P_3)\Phi_\sigma(P_3) = E_\sigma(P_3)\Phi_\sigma(P_3)$ the following formulas

$$\partial_{P_3} E_\sigma(P_3) = \langle \Phi_\sigma(P_3), (\partial_{P_3} H_\sigma(P_3))\Phi_\sigma(P_3)\rangle$$

and

$$\begin{aligned}\partial^2_{P_3} E_\sigma(P_3) =& \langle \Phi_\sigma(P_3), (\partial^2_{P_3} H_\sigma(P_3))\Phi_\sigma(P_3)\rangle \\ &-2\langle \partial_{P_3}\Phi_\sigma(P_3), (H_\sigma(P_3) - E_\sigma(P_3))\partial_{P_3}\Phi_\sigma(P_3)\rangle \ .\end{aligned} \tag{3.3}$$

As $\partial_{P_3} H_\sigma(P_3) = 1/m$ and $H_\sigma(P_3) - E_\sigma(P_3) \geq 0$ we obtain $m^\star_\sigma \geq m$ for all σ and thus $m^\star \geq m$.

The fact that $\mathbf{g}_{\mathrm{el}} \geq 2$ follows from $m^\star \geq m$ as in [30, 31, 32]. □

4 Proof of the main theorem

To begin with we introduce an infrared (regularized) cutoff in the interaction Hamiltonian $H_I(P_3)$. Precisely, for $\sigma > 0$, let ρ_σ be a C^∞ regularization of ρ such that

(i) $\rho_\sigma(k) = 0$ for $|k| \leq \sigma$

(ii) $\lim_{\sigma \to 0} \int \frac{|\rho_\sigma(k) - \rho(k)|^2}{|k|^j} d^3k = 0$ for $j = 1, 2$.

We define $H_{I,\sigma}(P_3)$ as the operator obtained from (2.14) by substituting $\rho_\sigma(k)$ for $\rho(k)$. We then introduce

$$H_\sigma(P_3) = H_0(P_3) + H_{I,\sigma}(P_3)$$

and we set $E_\sigma(P_3) := \inf \sigma(H_\sigma(P_3))$. Theorem 3.1 is a simple consequence of the following result (see [28])

Theorem 4.1. *There exist $g_0 > 0$ and $P > 0$ such that, for every g satisfying $|g| \leq g_0$ and for every P_3 satisfying $|P_3| \leq P$, the following properties hold:*

(i) For every $\Psi \in D(H_0(P_3))$ we have $H_\sigma(P_3)\Psi \to_{\sigma\to 0} H(P_3)\Psi$

(ii) For every $\sigma \in (0,1)$, $H_\sigma(P_3)$ has a normalized ground state $\Phi_\sigma(P_3)$ and $E_\sigma(P_3)$ is an isolated eigenvalue of $H_\sigma(P_3)$.

(iii) Fix $\lambda \in (e(b,V),0)$. For every $\sigma > 0$, we have

$$\langle \Phi_\sigma(P_3), P_{(-\infty,\lambda]} \otimes P_{\Omega_{ph}} \ \Phi_\sigma(P_3)\rangle \geq 1 - \delta_g(\lambda)$$

where $\delta_g(\lambda)$ tends to zero when g tends to zero and $\delta_g(\lambda) < 1$ for $|g| \leq g_0$.

In the last item, $P_{(-\infty,\lambda]}$ is the spectral projection on $(-\infty,\lambda]$ associated to $h(b,V)$ and $P_{\Omega_{ph}}$ is the orthogonal projection on Ω_{ph}, the vacuum state in $\mathcal{F}$.

Theorem 3.1 is easily deduced from theorem 4.1 as follows. Let Φ_σ be as in theorem 4.1 (ii). Since $\|\Phi_\sigma\| = 1$, there exits a sequence $(\sigma_k)_{k\geq 1}$ converging to zero and such that $(\Phi_{\sigma_k}(P_3))_{k\geq 1}$ converges weakly to a state $\Phi(P_3)$. On the other hand, since $P_{(-\infty,\lambda]} \otimes P_{\Omega_{ph}}$ is finite rank for $\lambda \in (e(b,V),0)$, it follows from (iii) that for $|g| \leq g_0$,

$$\langle \Phi(P_3), P_{(-\infty,\lambda]} \otimes P_{\Omega_{ph}} \ \Phi(P_3)\rangle \geq 1 - \delta_g(\lambda)$$

which implies $\Phi(P_3) \neq 0$. Then we deduce from (i) that $\Phi(P_3)$ is a ground state for $H(P_3)$.

The result concerning the multiplicity of the ground state is an easy consequence of corollary 3.4 in [15].

So it remains to prove theorem 4.1. The assertion (i) is easily verified in section 4.1 below. The second assertion is proved in the appendix. Actually the proof of (ii) is lenghty but straightforward since with the infrared cutoff we have a control of the photon's number in term of the energy. The real difficult part is the third one which allows to remove the infrared cutoff. The fondamental lemma in the proof of (iii) is lemma 4.3 which states that, for g and P_3 small enough, the fundamental energy $E_\sigma(P_3 - k_3)$ may be smaller than $E_\sigma(P_3)$ but the difference is controlled by $-\frac{3}{4}|k|$. This estimate, proved in section 4.2, is essential to control the number of photons in the ground state of $H(P_3)$ via a pull through formula (cf. section 4.3).

4.1 Proof of (i) of theorem 4.1

Let $\tilde{\rho}_\sigma := \rho - \rho_\sigma$, we have

$$\begin{aligned}
H(P_3) - H_\sigma(P_3) &= H_I(P_3) - H_{I,\sigma}(P_3) \\
&= -\frac{g}{2m}\sigma \cdot B(x',0,\tilde{\rho}_\sigma) - \frac{g}{m}\sum_{j=1,2} A_j(x',0,\tilde{\rho}_\sigma)(p_j - ea_j(x')) \\
&- \frac{g}{m}A_3(x',0,\tilde{\rho}_\sigma)(P_3 - \mathrm{d}\Gamma(k_3)) + \frac{g^2}{2m} : A(x',0,\tilde{\rho}_\sigma)\cdot A(x',0,\rho) : \\
&+ \frac{g^2}{2m} : A(x',0,\rho_\sigma)\cdot A(x',0,\tilde{\rho}_\sigma) :
\end{aligned}$$

Therefore, as by hypothesis $\lim_{\sigma\to 0}\int \frac{|\tilde{\rho}_\sigma(k)|^2}{|k|^j} d^3k = 0$ for $j = 0, 1, 2$, we deduce from the estimates of sections 2.3, 2.4 and from the Lebesgue's theorem that for every $\Psi \in D(H_0(P_3))$,

$$(H(P_3) - H_\sigma(P_3))\Psi \to_{\sigma\to 0} 0 \ .$$

$\square$

4.2 Fundamental estimates

In this section we give two lemmas that allow to control the function

$$P_3 \mapsto E_\sigma(P_3) \ .$$

Let $g_1 > 0$ such that (2.16) is satisfied for $|g| \leq g_1$.

Lemma 4.2. *There exists a finite constant $C > 0$ which does not depend on $\sigma \in [0, 1]$ and $P_3 \in \mathbb{R}$ such that*

$$e(b, V) - |g|C \leq E_\sigma(P_3) \leq e(b, V) + \frac{P_3^2}{2m} \tag{4.1}$$

for every $\sigma \in [0, 1]$, $P_3 \in \mathbb{R}$ and $|g| \leq g_1$.

Proof. Let $\phi(b, V)$ be the normalized ground state of $h(b, V)$. Since $\langle a_\mu(k)\Omega_{ph}, \Omega_{ph}\rangle = \langle \Omega_{ph}, a^*_\mu(k)\Omega_{ph}\rangle = 0$, we have

$$\begin{aligned}\langle H_\sigma(P_3)\phi(b, V) \otimes \Omega_{ph}, \phi(b, V) \otimes \Omega_{ph}\rangle &= \langle H_0(P_3)\phi(b, V) \otimes \Omega_{ph}, \phi(b, V) \otimes \Omega_{ph}\rangle \\ &= e(b, V) + \frac{P_3^2}{2m}\end{aligned}$$

and thus

$$\begin{aligned}E_\sigma(P_3) &:= \inf\{\langle H_\sigma(P_3)\Phi, \Phi\rangle \mid \Phi \in D(H_0(P_3)),\ \|\Phi\| = 1\} \\ &\leq e(b, V) + \frac{P_3^2}{2m} \ .\end{aligned}$$

On the other hand, let $\tilde{H}_\sigma$ be the following operator in $L^2(\mathbb{R}^2, \mathbb{C}^2) \otimes \mathcal{F}$:

$$\tilde{H}_\sigma = \tilde{H}_0 + \tilde{H}_{I,\sigma}$$

with

$$\tilde{H}_0 = h(b, V) \otimes I + I \otimes H_{ph}$$

and

$$\begin{aligned}\tilde{H}_{I,\sigma} = &-\frac{g}{2m}\sigma \cdot B(x', 0, \rho_\sigma) \\ &- \frac{g}{2m}\sum_{j=1,2} (A_j(x', 0, \rho_\sigma)(p_j - ea_j(x')) + (p_j - ea_j(x'))A_j(x', 0, \rho_\sigma)) \\ &+ \frac{g^2}{2m}\sum_{j=1,2} : A_j(x', 0, \rho_\sigma) \cdot A_j(x', 0, \rho_\sigma) :\end{aligned}$$

As in section 2.4 one easily checks that, for $|g| \leq g_1$, $\tilde{H}_\sigma$ is a self-adjoint operator in $L^2(\mathbb{R}^2, \mathbb{C}^2) \otimes \mathcal{F}$ with domain $D(H_0(P_3))$. Furthermore, on $D(H_0(P_3))$

$$H_\sigma(P_3) = \tilde{H}_\sigma + \frac{1}{2m}(P_3 - d\Gamma(k_3) - gA_3(x', 0, \rho_\sigma))^2 - C(g, \sigma)$$

where, in order to take into account the Wick normal ordering,

$$C(g, \sigma) := \frac{g^2}{2m} \frac{1}{(2\pi)^2} \int \frac{|\rho_\sigma|^2}{|k|} \left(\sum_{\mu=1,2} \epsilon_\mu(k)_3^2 \right) d^3k \ .$$

Hence,

$$\inf \sigma(\tilde{H}_\sigma) \leq E_\sigma(P_3) + C(g, \sigma) \tag{4.2}$$

for every $P_3 \in \mathbb{R}$. By (2.15) which also holds when ρ is replaced by ρ_σ, we get that there exist two constants $b > 0$, $a > 0$ that does not depend on $\sigma \in [0, 1]$ and $g \in [-g_1, g_1]$ and satisfying $bg_1 < 1$ such that for $\Phi \in D(\tilde{H}_0)$

$$\|\tilde{H}_{I,\sigma}\Phi\| \leq |g|(b\|\tilde{H}_0\Phi\| + a\|\Phi\|) \ .$$

Therefore, since $\inf \sigma(\tilde{H}_0) = e(b, V)$, we obtain as a consequence of the Kato-Rellich theorem,

$$\inf \sigma(\tilde{H}_\sigma) \geq e(b, V) - \max \left(\frac{a|g|}{1 - b|g|} \ , \ a|g| + b|g||e(b, V)| \right) \ . \tag{4.3}$$

Combining (4.2) and (4.3) we deduce the announced lower bound for $E_\sigma(P_3)$ with

$$C = \max \left(\frac{a}{1 - bg_1} \ , \ a + b|e(b, V)| \right) + \frac{g_1}{2m} \frac{1}{(2\pi)^2} \int \frac{|\rho|^2}{|k|} d^3k \ .$$

□

Lemma 4.3. *There exist $0 < g_2 \leq g_1$ and $\alpha > 0$ such that*

$$E_\sigma(P_3 - k_3) - E_\sigma(P_3) \geq -\frac{3}{4}|k| \tag{4.4}$$

uniformly for $k \in \mathbb{R}$, $\sigma \in [0, 1]$, $|g| \leq g_2$ and $|P_3| \leq \alpha$.

Remark that in this lemma we do not assume that $H_\sigma(P_3)$ has a ground state (i.e. we do not assume that $E_\sigma(P_3)$ is an eigenvalue of $H_\sigma(P_3)$) and actually we will use (4.4) in appendix A where we prove the existence of a ground state for the Hamiltonian with infrared cutoff.

Proof. First we remark that, if (4.4) is proved for $H_\sigma(P_3)+c$ for some constant c, it will hold also for $H_\sigma(P_3)$. Thus, in what follows, we suppose that $e(b, V) = 0$.

The proof decomposes in two steps. In the first one, we consider the large values of $|k_3|$ (namely $|k_3| \geq m/2$) while, in the second one, we consider the small values of $|k_3|$ (namely $|k_3| \leq m/2$).

From (4.1), we deduce that, uniformly for $\sigma \in [0,1]$ and $|g| \le g_1$, we have for all k and P_3

$$E_\sigma(P_3 - k_3) - E_\sigma(P_3) \ge -\frac{P_3^2}{2m} - C|g|$$

and thus assuming $|P_3| \le \frac{\sqrt{3}}{4}m$ and $|g| \le \frac{3m}{16C}$, (4.4) holds true for $|k_3| \ge m/2$.

Now we suppose $|k_3| \le m/2$. As $E_\sigma(P_3 - k_3)$ belongs to the spectrum of $H_\sigma(P_3 - k_3)$ there exists a sequence $(\psi_j)_{j\ge 1}$ in $D(H_\sigma(P_3 - k_3))$ $(= D(H_0))$ such that $\|\psi_j\| = 1$ and

$$\lim_{j\to\infty} H_\sigma(P_3 - k_3)\psi_j - E_\sigma(P_3 - k_3)\psi_j = 0 \ .$$

We then have for every j

$$\begin{aligned} \langle H_\sigma(P_3 - k_3)\psi_j, \psi_j\rangle &= \langle H_\sigma(P_3)\psi_j, \psi_j\rangle + \frac{k_3^2}{2m} - \frac{k_3}{m}\langle (P_3 - d\Gamma(k_3))\psi_j, \psi_j\rangle \\ &\quad + \frac{2gk_3}{m}\langle A_3(x', 0, \rho_\sigma)\psi_j, \psi_j\rangle \\ &\ge E_\sigma(P_3) + \frac{k_3^2}{2m} - \frac{|k_3|}{m}|\langle (P_3 - d\Gamma(k_3))\psi_j, \psi_j\rangle| \\ &\quad - \frac{2|g||k_3|}{m}|\langle A_3(x', 0, \rho_\sigma)\psi_j, \psi_j\rangle| \ . \end{aligned} \tag{4.5}$$

In what follows C will denote any positive constant which does not depend on $P_3 \in \mathbb{R}$, $k_3 \in \mathbb{R}$, $g \in [-g_1, g_1]$, $\sigma \in [0,1]$ and $j \ge 1$. We have

$$\begin{aligned} |\langle (P_3 - d\Gamma(k_3))\psi_j, \psi_j\rangle| &\le |k_3| + |\langle (P_3 - k_3 - d\Gamma(k_3))\psi_j, \psi_j\rangle| \\ &\le |k_3| + \|(P_3 - k_3 - d\Gamma(k_3))\psi_j\| \\ &\le |k_3| + \|(P_3 - k_3 - d\Gamma(k_3))^2\psi_j\|^{1/2} \\ &\le |k_3| + \sqrt{2m}\|H_0(P_3 - k_3)\psi_j\|^{1/2} \ . \end{aligned} \tag{4.6}$$

On the other hand, we get from (2.8) and (2.9),

$$\begin{aligned} |\langle A_3(x', 0, \rho_\sigma\psi_j, \psi_j\rangle| &\le C(\|H_{ph}^{1/2}\psi_j\| + 1) \\ &\le C(\|H_0(P_3 - k_3)\psi_j\|^{1/2} + 1) \ . \end{aligned} \tag{4.7}$$

Now, given $\epsilon > 0$, let J be such that

$$\|H_\sigma(P_3 - k_3)\psi_j - E_\sigma(P_3 - k_3)\psi_j\| \le \epsilon \tag{4.8}$$

for every $j \ge J$ (notice that J depends on ϵ but also on σ and $P_3 - k_3$).

Inserting (4.6) and (4.7) in (4.5) we obtain for $j \ge J$

$$\begin{aligned} E_\sigma(P_3 - k_3) - E_\sigma(P_3) &\ge -\epsilon - \frac{k_3^2}{2m} - |k_3|\sqrt{2/m}\|H_0(P_3 - k_3)\psi_j\|^{1/2} \\ &\quad - |k_3|C|g|(\|H_0(P_3 - k_3)\psi_j\|^{1/2} + 1) \end{aligned} \tag{4.9}$$

and it remains to estimate $\|H_0(P_3 - k_3)\psi_j\|$.

Writing

$$H_0(P_3-k_3)\psi_j = (H_\sigma(P_3-k_3)-E_\sigma(P_3-k_3))\psi_j+E_\sigma(P_3-k_3)\psi_j-H_{I,\sigma}(P_3-k_3)\psi_j$$

we get for $j \geq J$

$$\|H_0(P_3 - k_3)\psi_j\| \leq \epsilon + |E_\sigma(P_3 - k_3)| + \|H_{I,\sigma}(P_3 - k_3)\psi_j\| \ .$$

Using (2.15) there exists $C > 0$ such that

$$\|H_{I,\sigma}(P_3)\phi\| \leq |g|C(\|H_0(P_3)\phi\| + 1)$$

for every $P_3 \in \mathbb{R}$ and $\phi \in D(H_0(P_3))$. Thus, choosing $g_1' \leq g_1$ such that $g_1'C \leq 1/2$, we get

$$\|H_0(P_3 - k_3)\psi_j\| \leq 2\epsilon + 2|E_\sigma(P_3 - k_3)| + 2|g|C \tag{4.10}$$

for $j \geq J$ and $|g| \leq g_1'$. Inserting this last inequality in (4.9) we obtain

$$\begin{aligned} E_\sigma(P_3 - k_3) - E_\sigma(P_3) \geq -\epsilon - \frac{k_3^2}{2m} - |k_3|\sqrt{1/m}(\epsilon + |E_\sigma(P_3 - k_3)| + |g|C)^{1/2} \\ - |k_3|C|g|((2\epsilon + 2|E_\sigma(P_3 - k_3)| + 2|g|C)^{1/2} + 1) \end{aligned}$$

for every $\epsilon > 0$. Hence

$$\begin{aligned} E_\sigma(P_3 - k_3) - E_\sigma(P_3) \geq -|k_3| \Big\{ \frac{|k_3|}{2m} - \sqrt{1/m}(|E_\sigma(P_3 - k_3)| + |g|C)^{1/2} \\ -C|g|((2|E_\sigma(P_3 - k_3)| + 2|g|C)^{1/2} + 1)\Big\} \end{aligned} \tag{4.11}$$

for every k_3 and P_3 in $\mathbb{R}$. Finally we use (4.1) to get for $|k_3| \leq m/2$,

$$|E_\sigma(P_3 - k_3)| \leq C|g| + \frac{P_3^2}{2m} + \frac{|P_3|}{2} + \frac{m}{8}$$

and therefore there exit $\alpha > 0$ and $g_2 \leq g_1'$ such that for $|P_3| \leq \alpha$, $|k_3| \leq m/2$ and $|g| \leq g_2$,

$$E_\sigma(P_3 - k_3) - E_\sigma(P_3) \geq -\frac{3}{4}|k_3| \ .$$

□

4.3 Proof of (iii) of theorem 4.1

In this section we assume that assertion (ii) of theorem 4.1 is already proved (see appendix A). Thus let $\Phi_\sigma(P_3)$ denote a normalized ground state of $H_\sigma(P_3)$, i.e.

$$H_\sigma(P_3)\Phi_\sigma(P_3) = E_\sigma(P_3)\Phi_\sigma(P_3) \ .$$

The main problem in proving (iii) of theorem 4.1 is to controll the number of photons in the ground state $\Phi_\sigma(P_3)$ uniformly with respect to σ. The operator number of photons N_{ph} is given by

$$N_{ph} := \sum_{j=1,2} \int_{\mathbb{R}^3} d^3k \ a_\mu^*(k) a_\mu(k)$$

and we set

$$G(k) := |k|^{1/2} |\rho(k)| + \frac{|\rho(k)|}{|k|^{1/2}} \ .$$

Lemma 4.4. *There exists a constant C independant of g and σ such that*

$$\|(I \otimes N_{ph}^{1/2}) \Phi_\sigma(P_3)\| \leq C|g| \left(\int \frac{|G(k)|^2}{|k|^2} d^3k \right)^{1/2} \tag{4.12}$$

for every $\sigma \in (0,1]$, $|g| \leq g_2$ and $|P_3| \leq \alpha$ (g_2 and α are introduced in lemma 4.3).

Proof. One easily verifies that one has the following "pull through" formula

$$a_\mu(k) H_\sigma(P_3) = H_\sigma(P_3 - k_3) a_\mu(k) + \omega(k) a_\mu(k) + v_\mu(k) \tag{4.13}$$

with

$$\begin{aligned} v_\mu(k) = & \frac{ig}{2\pi m} |k|^{1/2} \rho_\sigma(k) e^{-ik' \cdot x'} \left(\frac{k}{|k|} \wedge \epsilon_\mu(k) \right) \\ & - \sum_{j=1,2} \frac{g}{2\pi m} \frac{\rho_\sigma(k)}{|k|^{1/2}} e^{-ik' \cdot x'} \epsilon_\mu(k)_j (p_j - e a_j(x')) \\ & - \frac{g}{2\pi m} \frac{\rho_\sigma(k)}{|k|^{1/2}} e^{-ik' \cdot x'} \epsilon_\mu(k)_3 (P_3 - d\Gamma(k_3)) \\ & + \frac{g^2}{2\pi m} \frac{\rho_\sigma(k)}{|k|^{1/2}} e^{-ik' \cdot x'} \epsilon_\mu(k) \cdot A(x', 0, \rho_\sigma) \ . \end{aligned}$$

Applying (4.13) to $\Phi_\sigma(P_3)$, we obtain

$$\begin{aligned} 0 = & (H_\sigma(P_3 - k_3) - E_\sigma(P_3) + \omega(k)) a_\mu(k) \Phi_\sigma(P_3) + v_\mu(k) \Phi_\sigma(P_3) \\ = & (H_\sigma(P_3 - k_3) - E_\sigma(P_3 - k_3) + E_\sigma(P_3 - k_3) - E_\sigma(P_3) + \omega(k)) a_\mu(k) \Phi_\sigma(P_3) \\ & + v_\mu(k) \Phi_\sigma(P_3) \end{aligned}$$

and thus, as $H_\sigma(P_3 - k_3) - E_\sigma(P_3 - k_3) \geq 0$, we get using (4.4),

$$\begin{aligned} \|a_\mu(k) \Phi_\sigma(P_3)\| & \leq \frac{1}{|E_\sigma(P_3 - k_3) - E_\sigma(P_3) + \omega(k)|} \|v_\mu(k) \Phi_\sigma(P_3)\| \\ & \leq \frac{4}{|k|} \|v_\mu(k) \Phi_\sigma(P_3)\| \end{aligned} \tag{4.14}$$

for $|P_3| \leq \alpha$ and $|g| \leq g_2$. Using estimates from section 2.4 (and similarly as (2.15)) we show that there exits a constant $C > 0$ such that

$$\|v_\mu(k)\Phi_\sigma(P_3)\| \leq C|g|G(k)(\|H_0(P_3) - e(b,V))\Phi_\sigma(P_3)\| + 1) \ . \tag{4.15}$$

Now, similarly as (4.10), we have for $|g| \leq g_2$

$$\|H_0(P_3)\Phi_\sigma(P_3)\| \leq 2|E_\sigma(P_3)| + 2Cg \ .$$

By lemma 4.2 $|E_\sigma(P_3)| \leq C|g| + \frac{P_3^2}{2m}$ and therefore we deduce from (4.14) that

$$\|a_\mu(k)\Phi_\sigma(P_3)\| \leq C|g|\frac{G(k)}{|k|}$$

where the constant C is uniform with respect to $|P_3| \leq \alpha$, $\sigma \in (0,1]$ and $|g| \leq g_2$.

Thus lemma 4.4 follows from this last inequality and from

$$\|(I \otimes N_{ph}^{1/2})\Phi_\sigma(P_3)\|^2 = \sum_{\mu=1,2} \int d^3k \|(I \otimes a_\mu(k))\Phi_\sigma(P_3)\|^2 \ .$$

Let us remark that the above proof is a little bit formal since we do not check that $\Phi_\sigma(P_3)$ belongs to the domain of the different operators involved in the pull through formula (4.13). But by mimicking [15] one easily gets a rigourous proof. We omit the details. □

Recall that we denote by $P_{(.]}$ the spectral measure of $h(b,V)$ and by $P_{\Omega_{ph}}$ the orthogonal projection on Ω_{ph}. We have the following

Lemma 4.5. *Fix $\lambda \in (e(b,V),0)$. There exists $\delta_g(\lambda) > 0$ such that $\delta_g(\lambda) \to 0$ when $g \to 0$ and*

$$\langle P_{[\lambda,\infty)} \otimes P_{\Omega_{ph}}\Phi_\sigma(P_3)\ ,\ \Phi_\sigma(P_3)\rangle \leq \delta_g(\lambda) \tag{4.16}$$

for every $\sigma \in (0,1]$, $|P_3| \leq \alpha$ and $|g| \leq g_2$.

Proof. Since $P_{\Omega_{ph}}H_{ph} = 0$ and $P_{\Omega_{ph}}(P_3 - d\Gamma(k_3))^2 = P_3^2 P_{\Omega_{ph}}$ we get

$$\begin{aligned}(P_{[\lambda,\infty)} \otimes P_{\Omega_{ph}})(H_\sigma(P_3) - E_\sigma(P_3)) &= P_{[\lambda,\infty)}(h(b,V) \otimes I) \otimes P_{\Omega_{ph}} \\ &+ (\frac{P_3^2}{2m} - E_\sigma(P_3))P_{[\lambda,\infty)} \otimes P_{\Omega_{ph}} + P_{[\lambda,\infty)} \otimes P_{\Omega_{ph}}H_{I,\sigma}(P_3) \ .\end{aligned}$$

Applying this last equality to $\Phi_\sigma(P_3)$ we obtain

$$\begin{aligned}0 &= P_{[\lambda,\infty)}(h(b,V) \otimes I) \otimes P_{\Omega_{ph}}\Phi_\sigma(P_3) \\ &+ (\frac{P_3^2}{2m} - E_\sigma(P_3))P_{[\lambda,\infty)} \otimes P_{\Omega_{ph}}\Phi_\sigma(P_3) \\ &+ P_{[\lambda,\infty)} \otimes P_{\Omega_{ph}}H_{I,\sigma}(P_3)\Phi_\sigma(P_3) \ .\end{aligned} \tag{4.17}$$

Since $h(b,V)P_{[\lambda,\infty)} \geq \lambda P_{[\lambda,\infty)}$ we obtain from (4.17) and lemma 4.2

$$\langle P_{[\lambda,\infty)} \otimes P_{\Omega_{ph}} \Phi_\sigma(P_3) \ , \ \Phi_\sigma(P_3) \rangle \leq \\ \frac{-1}{\lambda - e(b,V)} \langle (P_{[\lambda,\infty)} \otimes P_{\Omega_{ph}}) H_{I,\sigma}(P_3) \Phi_\sigma(P_3) \ , \ \Phi_\sigma(P_3) \rangle$$

for every $\sigma \in (0,1]$. The lemma then follows from (2.15) (which is also valid for $H_{I,\sigma}(P_3)$). □

We are now able to conclude the proof of (iii) of theorem 4.1. We have

$$\langle P_{(-\infty,\lambda]} \otimes P_{\Omega_{ph}} \Phi_\sigma(P_3) \ , \ \Phi_\sigma(P_3) \rangle = 1 - \langle P_{[\lambda,\infty)} \otimes P_{\Omega_{ph}} \Phi_\sigma(P_3) \ , \ \Phi_\sigma(P_3) \rangle \\ - \langle P_{[\lambda,\infty)} \otimes P^{\perp}_{\Omega_{ph}} \Phi_\sigma(P_3) \ , \ \Phi_\sigma(P_3) \rangle \ .$$

Now it suffices to remark that the second term in the right hand side of this equality is estimated by lemma 4.5 and, noticing that $P^{\perp}_{\Omega_{ph}} \leq N_{ph}$, the third term is estimated by lemma 4.4. □

A Exitences of a ground state for the Hamiltonian with infrared cutoff

In this appendix we prove the assertion (ii) of theorem 4.1 : for σ and P_3 small enough, the Hamiltonian with infrared cutoff has a ground state. This result is not surprising but the complete proof is long. Actually it follows by mimicking [9, 18, 33] (see also [8]) and, here, we only give a sketch of the proof.

In this appendix we are faced with ths lack of smoothness of the $\epsilon_\mu(k)$'s which define vector fields on sheres $|k|$ =cst (see [17, 34]). It suffices to consider one example. From now on suppose that

$$\epsilon_1(k) = \frac{1}{\sqrt{k_1^2 + k_2^2}}(k_1, -k_2, 0) \quad \text{and} \quad \epsilon_2(k) = \frac{k}{|k|} \wedge \epsilon_1(k) \ .$$

The functions $\epsilon_\mu(k)$, $\mu = 1, 2$, are smooth only on $\mathbb{R}^3 \setminus \{(0,0,k_3) \mid k_3 \in \mathbb{R}\}$. Nevertheless, in our case, we can overcome easily this problem choosing the regularization ρ_σ of ρ as a C^∞ function whose support does not intersect the line $\{(0,0,k_3) \mid k_3 \in \mathbb{R}\}$. From now on we suppose that it is the case.

Let $\omega_{\mathrm{mod}}(k)$ be the modified dispersion relation as defined in ([18], section 5, hypothesis 3), i.e. : $\omega_{\mathrm{mod}}(k)$ is a smooth function satisfying

(i) $\omega_{\mathrm{mod}}(k) \geq \max(|k|, \frac{\sigma}{2})$ for all $k \in \mathbb{R}^3$, $\omega_{\mathrm{mod}}(k) = |k|$ for $|k| \geq \sigma$.

(ii) $|\nabla \omega_{\mathrm{mod}}(k)| \leq 1$ for all $k \in \mathbb{R}^3$, and $\nabla \omega_{\mathrm{mod}}(k) \neq 0$ unless $k = 0$.

(iii) $\omega_{\mathrm{mod}}(k_1 + k_2) \leq \omega_{\mathrm{mod}}(k_1) + \omega_{\mathrm{mod}}(k_2)$ for all $k_1, k_2 \in \mathbb{R}^3$.

We set

$$H_{ph,\mathrm{mod}} = \sum_{\mu=1,2} \int \omega_{\mathrm{mod}}(k) a^\star_\mu(k) a_\mu(k) d^3k$$

and

$$H_{\mathrm{mod},\sigma}(P_3) = h(b,V) \otimes I + I \otimes \left\{ \frac{1}{2m}(P_3 - d\Gamma(k_3))^2 + H_{ph,\mathrm{mod}} \right\} + H_{I,\sigma}(P_3) \ .$$

Theorem 2.5, with the same assumtion (2.16), is still valid for $H_{\mathrm{mod},\sigma}(P_3)$. Set $E_{\mathrm{mod},\sigma}(P_3) := \inf \sigma(H_{\mathrm{mod},\sigma}(P_3))$. Then $E_{\mathrm{mod},\sigma}(P_3)$ still satisfies lemma 4.3 and (4.4) for the same constants g_2 and α. Moreover, according to ([18]; thm 3), $E_\sigma(P_3) = E_{\mathrm{mod},\sigma}(P_3)$ for $|P_3| \leq \alpha$ and $|g| \leq g_2$ and $E_\sigma(P_3)$ is an eigenvalue of $H_\sigma(P_3)$ if and only if $E_{\mathrm{mod},\sigma}(P_3)$ is an eigenvalue of $H_{\mathrm{mod},\sigma}(P_3)$. Thus in order to prove that $H_\sigma(P_3)$ has a ground state it suffices to prove that $E_{\mathrm{mod},\sigma}(P_3) < \inf \sigma_{\mathrm{ess}}(H_{\mathrm{mod},\sigma}(P_3))$. The proof is by contradiction, so we suppose that $E_{\mathrm{mod},\sigma}(P_3) = \inf \sigma_{\mathrm{ess}}(H_{\mathrm{mod},\sigma}(P_3))$ and we set

$$\lambda = E_{\mathrm{mod},\sigma}(P_3) = \inf \sigma_{\mathrm{ess}}(H_{\mathrm{mod},\sigma}(P_3)) \ . \tag{A.1}$$

We now observe that $E_{\mathrm{mod},\sigma}(P_3)$ satisfies (4.1) for $|g| \leq g_2$.
Let $\delta := \mathrm{dist}\,(e(v,V)\ ,\ \sigma(h(b,V)) \setminus \{e(b,V)\}) > 0$. According to (4.1) there exist $0 < \beta \leq \alpha$ and $0 < g_3 \leq g_2$ such that

$$\begin{aligned} &\lambda \leq e(b,V) + \frac{\delta}{3} \quad \mathrm{for} |P_3| \leq \beta \\ &C|g| \leq \frac{\delta}{12} \quad \mathrm{for} |g| \leq g_3 \end{aligned} \tag{A.2}$$

where C is the constant in (4.1).

Let Δ be an interval such that $\lambda \in \Delta$ and $\sup \Delta < e(b,V) + \frac{\delta}{2}$. Thus

$$e(b,V) + \frac{2\delta}{3} - \sup \Delta - C|g| \geq \frac{\delta}{12}$$

for $|P_3| \leq \beta$ and $|g| \leq g_3$ and we introduce $\eta > 0$ such that

$$\eta^2 < e(b,V) + \frac{2\delta}{3} - \sup \Delta - C|g| \ .$$

Then, along the same lines as in the proof of theorem II.1 in [28], one easily shows that it exists M_Δ such that for any $|P_3| \leq \beta$ and any $|g| \leq g_3$

$$\|(e^{\eta|x'|} \otimes I)\chi_\Delta(H_{\mathrm{mod},\sigma}(P_3))\| \leq M_\Delta \ . \tag{A.3}$$

Since we assume $\lambda \in \sigma_{\mathrm{ess}}(H_{\mathrm{mod},\sigma}(P_3))$ there exits a sequence $(\phi_n)_{n\geq 1}$, with $\|\phi_n\| = 1$, such that

$$\begin{aligned} &\phi_n \in \mathrm{Ran}\chi_\Delta(H_{\mathrm{mod},\sigma}(P_3)) \ , \\ &(H_{\mathrm{mod},\sigma}(P_3) - \lambda)\phi_n \to_{n\to 0} 0, \\ &w - \lim_{n\to 0} \phi_n = 0 \end{aligned}$$

and therefore

$$\lambda = \lim_{n\to\infty} \langle \phi_n, H_{\mathrm{mod},\sigma}(P_3)\phi_n \rangle \ . \tag{A.4}$$

Notice that, as for (A.3), one easily shows that for any $|P_3| \le \beta$, any $|g| \le g_3$ and any $n \ge 1$

$$\begin{aligned} &\|(e^{\frac{\eta}{2}|x'|} \otimes I)\chi_\Delta(H_{\mathrm{mod},\sigma}(P_3))\nabla_{x'}\phi_n\| \le M'_\Delta \\ &\|(e^{\frac{\eta}{2}|x'|} \otimes I)\chi_\Delta(H_{\mathrm{mod},\sigma}(P_3))d\Gamma(k_3)\phi_n\| \le M'_\Delta \end{aligned} \tag{A.5}$$

where M'_Δ is a finite constant.

Now, in order to estimate $\langle \phi_n, H_{\mathrm{mod},\sigma}(P_3)\phi_n\rangle$ from below, we need to localize the photons. Let $j_0, j_\infty \in C^\infty(\mathbb{R}^3)$ be real valued functions with $j_0^2 + j_\infty^2 = 1$, $j_0(y) = 1$ for $|y| \le 1$ and $j_0(y) = 0$ for $|y| \ge 2$. Given $R > 0$, we set $j_{\cdot,R}(y) = j_\cdot(\frac{y}{R})$ and let $j_R = (j_{0,R}, j_{\infty,R})$. Here $y = \frac{1}{i}\nabla_k$ and j_R is an operator from $\mathcal{F} \otimes \mathcal{F}$ to $\mathcal{F}$.

Then let $\check{\Gamma}(j_R)$, $d\check{\Gamma}(j_R, \underline{\omega_{\mathrm{mod}}} j_R - j_R\omega_{\mathrm{mod}})$ be the operators from $L^2(\mathbb{R}^3, \mathbb{C}^2) \otimes \mathcal{F} \otimes \mathcal{F}$ to $L^2(\mathbb{R}^3, \mathbb{C}^2) \otimes \mathcal{F}$ as defined in sections 2.13 and 2.14 of [33] (see also section 2.6 of [18]). Here $\underline{\omega_{\mathrm{mod}}} := (\omega_{\mathrm{mod}}, \omega_{\mathrm{mod}})$. Roughly speaking, $\check{\Gamma}(j_R)$ separates the set photons between photons localized arround the electron and photons that escape to infinity (when $R \to \infty$).

Set

$$\begin{aligned} G_{l,\mu}(x', \rho_\sigma) &= \frac{1}{2\pi}\frac{\rho_\sigma}{|k|^{1/2}}e^{-ikx'}\epsilon_\mu(k)_l \ , \quad l = 1,2,3, \\ H_\mu(x', \rho_\sigma) &= -\frac{i}{2\pi}|k|^{1/2}\rho_\sigma(k)\sigma.(\frac{k}{|k|} \wedge \epsilon_\mu(k))e^{-ikx'} \ , \\ \Phi_\mu(h) &= \frac{1}{\sqrt{2}}(a_\mu(h) + a^\star_\mu(h)) \ . \end{aligned}$$

Let $\check{H}_{\mathrm{mod},\sigma}(P_3)$ be the following operator in $L^2(\mathbb{R}^3, \mathbb{C}^2) \otimes \mathcal{F} \otimes \mathcal{F}$:

$$\begin{aligned} &\check{H}_{\mathrm{mod},\sigma}(P_3) = h(b,V) \otimes I \otimes I + I \otimes H_{\mathrm{mod},ph} \otimes I + I \otimes I \otimes H_{\mathrm{mod},ph} \\ &+\frac{1}{2m}\ I \otimes (P_3 - I \otimes d\Gamma(k_3) \otimes I - I \otimes I \otimes d\Gamma(k_3))^2 \\ &-\frac{g}{2m}\sqrt{2}\sum_{\mu=1,2}\Phi_\mu(H_\mu(x', \rho_\sigma)) \otimes I \\ &-\frac{g}{m}\sqrt{2}\sum_{\mu=1,2}\sum_{j=1,2}(p_j - ea_j(x'))\Phi_\mu(G_{j,\mu}(x', \rho_\sigma)) \otimes I \\ &-\frac{g}{m}\sqrt{2}(P_3 - I \otimes d\Gamma(k_3) \otimes I - I \otimes I \otimes d\Gamma(k_3))\left(\sum_{\mu=1,2}\Phi_\mu(G_{3,\mu}(x', \rho_\sigma)) \otimes I\right) \\ &+\frac{g^2}{2m}\sum_{l=1,2,3}\left(\sum_{\mu=1,2}\Phi_\mu(G_{l,\mu}(x', \rho_\sigma))\right)^2 \otimes I \ . \end{aligned}$$

Again, assuming (2.16), $\check{H}_{\mathrm{mod},\sigma}(P_3)$ is a self-adjoint operator in $L^2(\mathbb{R}^3, \mathbb{C}^2) \otimes \mathcal{F} \otimes \mathcal{F}$ for $|g| \le g_1$. We remark that

$$\check{H}_{\mathrm{mod},\sigma}(P_3) \ge E_{\mathrm{mod},\sigma}(P_3) + I \otimes I \otimes H_{\mathrm{mod},ph} \tag{A.6}$$

and thus $\langle\phi, \check{H}_{\mathrm{mod},\sigma}(P_3)\phi\rangle \geq E_{\mathrm{mod},\sigma}(P_3) + \frac{\sigma}{2}$ if the state ϕ has a component along the delocalized photons. Actually we are going to prove that, since λ is in the essential spectrum, $\check{\Gamma}(j_R)\phi_n$ has a non vanishing component along the delocalized photons and thus, in view of (A.4), we will obtain a contradiction with (A.1).

Finally set for $l = 1, 2, 3$

$$T_{l,R}(G)(x') = \sum_{\mu=1,2} (\Phi_\mu((j_{0,R} - 1)G_{l,\mu}(x', \rho_\sigma)) \otimes 1 + 1 \otimes \Phi_\mu(j_{\infty,R}G_{l,\mu}(x', \rho_\sigma)))$$

and

$$T_R(H)(x') = \sum_{\mu=1,2} (\Phi_\mu((j_{0,R} - 1)H_\mu(x', \rho_\sigma)) \otimes 1 + 1 \otimes \Phi_\mu(j_{\infty,R}H_\mu(x', \rho_\sigma))) \ .$$

By using [33] (sections 2.13 and 2.14) and [18] (sections 2.5 and 2.6) we obtain that $\check{H}_{\mathrm{mod},\sigma}(P_3)$ and $H_{\mathrm{mod},\sigma}(P_3)$ are almost conjugated by $\check{\Gamma}(j_R)$, namely

$$\begin{aligned}
&\check{\Gamma}(j_R)H_{\mathrm{mod},\sigma}(P_3) - \check{H}_{\mathrm{mod},\sigma}(P_3)\check{\Gamma}(j_R) = -\frac{g}{2m}\sqrt{2}T_R(H)(x')\check{\Gamma}(j_R) \\
&\quad -\frac{g}{2m}\sqrt{2}(P_3 - I \otimes d\Gamma(k_3) \otimes I - I \otimes I \otimes d\Gamma(k_3))T_{3R}(G)(x')\check{\Gamma}(j_R) \\
&\quad -\frac{g}{m}\sqrt{2}\sum_{j=1,2}(p_j - ea_j(x'))T_{jR}(G)(x')\check{\Gamma}(j_R) - d\check{\Gamma}(j_R, \underline{\omega_{\mathrm{mod}}}j_R - j_R\omega_{\mathrm{mod}}) \\
&\quad -d\check{\Gamma}(j_R, \underline{k_3}j_R - j_Rk_3)\left(\frac{1}{2m}(P_3 - d\Gamma(k_3)) + \sqrt{2}\sum_{\mu=1,2}\Phi_\mu(G_{3,\mu}(x', \rho_\sigma))\right) \\
&\quad -\frac{1}{2m}(P_3 - I \otimes d\Gamma(k_3) \otimes I - I \otimes I \otimes d\Gamma(k_3))d\check{\Gamma}(j_R, \underline{k_3}j_R - j_Rk_3) \\
&\quad +\frac{g^2}{2m}\Bigg[\sum_{l=1,2,3}\sum_{\mu=1,2}\sum_{\mu'=1,2} \\
&\qquad \{a_\mu(j_{0,R}G_{l,\mu}(x', \rho_\sigma)) \otimes 1 + 1 \otimes a_\mu(j_{\infty,R}G_{l,\mu}(x', \rho_\sigma)) \\
&\quad +a^\star_\mu(j_{0,R}G_{l,\mu}(x', \rho_\sigma)) \otimes 1 + 1 \otimes a^\star_\mu(j_{\infty,R}G_{l,\mu}(x', \rho_\sigma))\} \\
&\qquad \{a_{\mu'}(j_{0,R}G_{l,\mu'}(x', \rho_\sigma)) \otimes 1 + 1 \otimes a_{\mu'}(j_{\infty,R}G_{l,\mu'}(x', \rho_\sigma)) \\
&\quad +a^\star_{\mu'}(j_{0,R}G_{l,\mu'}(x', \rho_\sigma)) \otimes 1 + 1 \otimes a^\star_{\mu'}(j_{\infty,R}G_{l,\mu'}(x', \rho_\sigma))\} \\
&\quad -\sum_{l=1,2,3}\left(\sum_{\mu=1,2}\Phi_\mu(G_{l,\mu}(x', \rho_\sigma))\right)^2 \otimes I\Bigg]\check{\Gamma}(j_R) \ .
\end{aligned} \tag{A.7}$$

Since ρ_σ is a C_0^∞ function, one has for $\gamma > 0$

$$(1 - \Delta_{k'})^\gamma \frac{\rho_\sigma(k)}{|k|^{1/2}}\epsilon_\mu(k)_l \ \in \ L^2(\mathbb{R}^3) \quad l = 1, 2, 3, \ \mu = 1, 2, \ \sigma > 0$$

$$(1 - \Delta_{k'})^\gamma \rho_\sigma(k)|k|^{1/2}\sigma \cdot \left(\frac{k}{|k|} \wedge \epsilon_\mu(k)\right) \ \in \ L^2(\mathbb{R}^3) \quad \mu = 1, 2, \ \sigma > 0 \ .$$

Here $k' = (k_1, k_2)$.

We then prove, as in ([18] lemma 9), that both

$$e^{-\frac{\eta}{2}|x'|}T_{l,R}(G)(x')(I + I \otimes N_{ph} \otimes I + I \otimes I \otimes N_{ph})^{-1/2}$$

and

$$e^{-\frac{\eta}{2}|x'|}T_R(H)(x')(I + I \otimes N_{ph} \otimes I + I \otimes I \otimes N_{ph})^{-1/2}$$

tend to zero in $L^2(\mathbb{R}^3, \mathbb{C}^2) \otimes \mathcal{F} \otimes \mathcal{F}$ when $R \to \infty$. Therefore it follows from (A.3) and (A.7) that

$$\langle \phi_n, H_{\mathrm{mod},\sigma}(P_3)\phi_n\rangle = \langle \phi_n, \check{\Gamma}(j_R)^\star \check{H}_{\mathrm{mod},\sigma}(P_3)\check{\Gamma}(j_R)\phi_n\rangle + o(R^0) \tag{A.8}$$

uniformly in n (cf. [18] and [33]).

Denoting by P_{Ω_∞} the orthogonal projection on the vacuum of delocalized photons, we have using (A.6)

$$\begin{aligned}\langle \phi_n, \check{\Gamma}(j_R)^\star \check{H}_{\mathrm{mod},\sigma}(P_3)\check{\Gamma}(j_R)\phi_n\rangle &\geq E_{\mathrm{mod},\sigma}(P_3) + \frac{\sigma}{2} \\ &\quad - \frac{\sigma}{2}\langle \phi_n, \check{\Gamma}(j_R)^\star (I \otimes I \otimes P_{\Omega_\infty})\check{\Gamma}(j_R)\phi_n\rangle\ .\end{aligned}$$

On the other hand we verify

$$\check{\Gamma}(j_R)^\star (I \otimes I \otimes P_{\Omega_\infty})\check{\Gamma}(j_R) = \Gamma(j_{0,R}^2)\ .$$

Then, by using lemma 4.2 for $E_{\mathrm{mod},\sigma}(P_3)$ and the compactness of $\chi_\Delta(H_{\mathrm{mod},\sigma}(P_3))e^{-\eta|x'|}\Gamma(j_{0,R}^2)(H_{\mathrm{mod},\sigma}(P_3) + i)^{-1}$ (see [33] lemma 34 or [18] lemma 36 and [20] theorem 2.6), we deduce from (A.8), letting $n \to \infty$,

$$\lambda \geq E_{\mathrm{mod},\sigma}(P_3) + \frac{\sigma}{2} + o(R^0)\ .$$

Letting $R \to \infty$ we get a contradiction with (A.1) and thus assertion (ii) of theorem 4.1 is proved. □

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