# LAX PAIRS FOR THE ABLOWITZ-LADIK SYSTEM VIA ORTHOGONAL POLYNOMIALS ON THE UNIT CIRCLE

IRINA NENCIU

ABSTRACT. In [14] Nenciu and Simon found that the analogue of the Toda system in the context of orthogonal polynomials on the unit circle is the defocusing Ablowitz-Ladik system. In this paper we use the CMV and extended CMV matrices defined in [5] and [13, 14], respectively, to construct Lax pair representations for this system.

## 1. INTRODUCTION

The aim of this paper is to present new results concerning the Ablowitz-Ladik (AL) system. More precisely, we use the connection between the AL system and the theory of orthogonal polynomials on the unit circle to construct Lax pairs associated to the Hamiltonians in the defocusing AL hierarchy. Our main investigation focuses on the periodic case, but these results translate to corresponding statements in the finite and infinite cases.

We briefly introduce the main players. The defocusing AL equation was defined in 1975–76 by Ablowitz and Ladik [1, 2] as a space-discretization of the cubic nonlinear Schrödinger equation. It reads:

$$(1.1) \qquad -i\dot{\alpha}_n = \rho_n^2(\alpha_{n+1} + \alpha_{n-1}) - 2\alpha_n,$$

where $\alpha = \{\alpha_n\} \subset \mathbb{D}$ is a sequence of complex numbers inside the unit disk and

$$\rho_n^2 = 1 - |\alpha_n|^2.$$

The analogy with the continuous NLS becomes transparent if we rewrite (1.1) as

$$-i\dot{\alpha}_n = \alpha_{n+1} - 2\alpha_n + \alpha_{n-1} - |\alpha_n|^2(\alpha_{n+1} + \alpha_{n-1}).$$

Here, and throughout this paper, $\dot{f}$ will denote the time derivative of the function $f$.

We note that the name "Ablowitz-Ladik equation" that we use here for (1.1) is sometimes used for a more general equation that was introduced in the same paper [1]. Moreover, (1.1) also appears in the literature under the name IDNLS (integrable discrete nonlinear Schrödinger equation). So far, the study of this equation focused mainly around the inverse scattering transform; see, for example, [3, Chapter 3] and the references therein. Other aspects of the Ablowitz-Ladik equations have been further studied, for example, in [8], [9], [10], [11], [12], [16], and [17].

We will try to understand (1.1) from a different perspective: that of the theory of orthogonal polynomials on the unit circle. We concentrate on the periodic problem as it was the first one solved, and our main results for the finite and infinite defocusing Ablowitz-Ladik systems follow from the corresponding result in the periodic case.

While more details can be found in Appendix B and we shall freely use all the notation introduced there, we present here some of the main notions and relevant results. Let $\mu$ be a probability measure on the unit circle. By applying the Gram-Schmidt procedure to $1, z, z^2, \ldots,$ one can define the monic orthogonal polynomials $\{\Phi_n\}_{n \geq 0}$. They obey a recurrence relation

$$\Phi_{n+1}(z) = z\Phi_n(z) - \bar{\alpha}_n \Phi_n^*(z)$$

for all $n \geq 0$, where

$$\Phi_n^*(z) = z^n \overline{\Phi_n(\tfrac{1}{\bar{z}})}$$

is the reversed polynomial, and $\alpha = \{\alpha_n\}_{n \geq 0}$ is the sequence of Verblunsky coefficients. (The use of the same notation as in (1.1) is not a coincidence, as we will see.) If we represent the operator of multiplication by $z$ on $L^2(d\mu)$ in an appropriate basis, we obtain a 5-diagonal unitary matrix

$$\mathcal{C} = \begin{pmatrix} \bar{\alpha}_0 & \rho_0\bar{\alpha}_1 & \rho_0\rho_1 & 0 & 0 & \ldots \\ \rho_0 & -\alpha_0\bar{\alpha}_1 & -\alpha_0\rho_1 & 0 & 0 & \ldots \\ 0 & \rho_1\bar{\alpha}_2 & -\alpha_1\bar{\alpha}_2 & \rho_2\bar{\alpha}_3 & \rho_2\rho_3 & \ldots \\ 0 & \rho_1\rho_2 & -\alpha_1\rho_2 & -\alpha_2\bar{\alpha}_3 & -\alpha_2\rho_3 & \ldots \\ 0 & 0 & 0 & \rho_3\bar{\alpha}_4 & -\alpha_3\bar{\alpha}_4 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}.$$

This matrix was first discovered by Cantero, Moral, and Velázquez [5], and is called the CMV matrix. Furthermore, if the Verblunsky coefficients are periodic, one can very naturally define the extended CMV matrix $\mathcal{E}$ (as in (B.9)) and its Floquet restrictions $\mathcal{E}_{(\beta)}$ to the periodic subspaces (see Chapter 11 of [14]).

The theory of periodic Verblunsky coefficients was first studied by Geronimus, and, more recently, by Peherstorfer and collaborators, and Golinskii and collaborators (for detailed references to their work, see [14]). Simon used the analogy with Hill's equation to fully develop the theory for periodic Verblunsky coefficients in [14, Chapter 11]. In particular, he defines the discriminant $\Delta(z)$ naturally associated to this periodic problem and finds that

**Proposition 1.1** (Simon). *Let $p$ (the period of the coefficients) be even. Let $\{\alpha_j\}_{j=0}^{p-1}$ and $\{\gamma_j\}_{j=0}^{p-1}$ be two elements of $\mathbb{D}^p$. The following are equivalent:*

(1) $\Delta(z; \{\alpha_j\}) = \Delta(z; \{\gamma_j\})$.
(2) $\prod_j(1-|\alpha_j|^2) = \prod_j(1-|\gamma_j|^2)$, *and the eigenvalues of $\mathcal{E}_{(\beta)}(\{\alpha_j\}_{j=0}^{p-1})$ and $\mathcal{E}_{(\beta)}(\{\gamma_j\}_{j=0}^{p-1})$ coincide for one $\beta \in \partial\mathbb{D}$.*
(3) *The eigenvalues of $\mathcal{E}_{(\beta)}(\{\alpha_j\}_{j=0}^{p-1})$ and $\mathcal{E}_{(\beta)}(\{\gamma_j\}_{j=0}^{p-1})$ are equal for all $\beta \in \partial\mathbb{D}$.*
(4) $\mathrm{spec}(\mathcal{E}(\{\alpha_j\}_{j=0}^{p-1})) = \mathrm{spec}(\mathcal{E}(\{\gamma_j\}_{j=0}^{p-1}))$.

When these conditions hold, we say that $\{\alpha_j\}_{j=0}^{p-1}$ and $\{\gamma_j\}_{j=0}^{p-1}$ are isospectral.

Next, we present two examples which represented a first step in establishing the connection between OPUC and the AL system. For the full computations which justify our claims, see Examples 11.1.4 and 5 in [14].

**Example 1** (Geronimus). Let $\alpha \in \mathbb{D}$ and define $\alpha_j \equiv \alpha$ for all $j \geq 0$.
The isospectral manifold in this case is a circle

$$\{\alpha = (1 - \rho^2)^{1/2}e^{i\theta} : \theta \in [0, 2\pi]\}$$

if $|\alpha| \neq 0$, and a point (or a zero-dimensional torus), $\alpha = 0$, if $|\alpha| = 0$.

**Example 2** (Akhiezer). Consider $\alpha_{2j} = \alpha$ and $\alpha_{2j+1} = \alpha'$, with $\alpha, \alpha' \in \mathbb{D}$ and $j \geq 0$, to be periodic Verblunsky coefficients with period $p = 2$. Again the discriminant is easily computable and

$$(1.2) \qquad \Delta(e^{i\theta}) = \frac{2}{\rho\rho'}[\cos(\theta) + \mathrm{Re}(\bar{\alpha}\alpha')].$$

Let $\theta_{\pm} \in [0, \pi)$ solve $\cos(\theta_{\pm}) = -\mathrm{Re}(\bar{\alpha}\alpha') \pm \rho\rho'$. Note that $|\mathrm{Re}(\bar{\alpha}\alpha')| + \rho\rho' \leq 1$, and hence there are always solutions, with $0 \leq \theta_+ < \theta_- \leq \pi$. Hence $|\Delta(e^{i\theta})| \leq 2$ if and only if $\pm\theta \in [\theta_+, \theta_-]$. We are interested in finding the set of pairs $(\alpha, \alpha') \in \mathbb{D}^2$ which lead to a given $\Delta$ of the form (1.2). This can be done explicitly, and the conclusion is that

- There are no open gaps for $|\alpha| = |\alpha'| = 0$, and so the isospectral manifold is a point (0-dimensional torus).
- There is exactly one open gap when $\alpha = \pm\alpha' \neq 0$, which leads to the isospectral manifold being a circle.
- There are two open gaps if and only if the isospectral manifold is a two-dimensional torus.

The two examples above suggest that $\mathbb{D}^p$ fibers into tori, generically of real dimension $p$, half of the real dimension of $\mathbb{D}^p$. This was proved by Simon in [14].

Recall that a Hamiltonian vector field $V_H$ is called *completely integrable* on a domain $D$ contained in a manifold of real dimension $2n$ if there exist $n$ integrals of motion $H_1 = H, H_2, \ldots, H_n$ whose gradients are linearly independent on $D$ and which Poisson commute. In this case, the Liouville-Arnold-Jost Theorem (see [4] and [7]) says that if $N = \bigcap_k H_k^{-1}(c_k)$ is compact and connected, then it is an $n$-dimensional torus. Finding the symplectic structure and the integrable system naturally associated with periodic Verblunsky coefficients and the notion of isospectrality defined above was the main purpose of the work of Nenciu and Simon [14, Ch.11, Section 11], that we shall briefly describe here.

We begin by defining the symplectic structure. We are considering the problem of periodic Verblunsky coefficients with period $p$. So we are interested in a symplectic form on $\mathbb{D}^p$, which has real dimension $2p$. Let $\underline{\alpha} = (\alpha_0, \ldots, \alpha_{p-1}) \in \mathbb{D}^p$, and let $u_j = \operatorname{Re}\alpha_j$ and $v_j = \operatorname{Im}\alpha_j$ for all $0 \leq j \leq p-1$. Then we define our symplectic form by

$$(1.3) \qquad \omega = \frac{1}{2}\sum_{j=0}^{p-1}\frac{1}{\rho_j^2}\,du_j \wedge dv_j.$$

As all of the subsequent computations will involve only the corresponding Poisson bracket, let us note that, for $f$ and $g$ functions on $\mathbb{D}^p$, we have

$$(1.4) \qquad \{f,g\} = \frac{1}{2}\sum_{j=0}^{p-1}\rho_j^2\left[\frac{\partial f}{\partial u_j}\frac{\partial g}{\partial v_j} - \frac{\partial f}{\partial v_j}\frac{\partial g}{\partial u_j}\right]$$

$$(1.5) \qquad = i\sum_{j=0}^{p-1}\rho_j^2\left[\frac{\partial f}{\partial \bar{\alpha}_j}\frac{\partial g}{\partial \alpha_j} - \frac{\partial f}{\partial \alpha_j}\frac{\partial g}{\partial \bar{\alpha}_j}\right],$$

where for $z = u + iv \in \mathbb{D}$ we use the standard notation

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right).$$

**Lemma 1.2.** *The 2-form defined by (1.3) is a symplectic form. Equivalently, the bracket (1.4) obeys the Jacobi identity and is nondegenerate.*

*Proof.* $\omega$ is a sum of 2-forms, each of which acts only on one of the variables $\alpha_j$ for $0 \leq j \leq p - 1$. But any 2-form is closed in $\mathbb{R}^2$, and hence $\omega$ is closed. It is also nondegenerate, since the function $\rho_j^{-2}$ is positive on $\mathbb{D}^p$ for each $j$. □

The first result is

**Theorem 1** (Nenciu - Simon). *With the above Poisson bracket we have*

$$(1.6) \qquad \{\Delta(z), \Delta(w)\} = 0$$

*for any $z, w \in \mathbb{C}$.*

In particular, one has

**Corollary 1.3.** *The Hamiltonian flows generated by $\Delta(z)$ for $z \in \partial\mathbb{D}$ and by $\prod_{j=0}^{p-1} \rho_j$ all commute with each other and leave $\Delta(w)$ invariant.*

Theorem 1 is proved using the expression of $\Delta$ in terms of Wall polynomials and the recurrence relations that they obey. For details, see Section 11.11 of [14] and the references therein.

Moreover, the coefficients of the monic polynomial

$$z^{p/2} \Big( \prod_{j=0}^{p-1} \rho_j \Big) \Delta(z)$$

come in complex conjugate pairs: $c_j = \bar{c}_{p-j}$. If we also take into account the fact that $c_0 = c_p = 1$ and $c_{p/2}$ is real, we get that the real and imaginary parts of the $c_j$ with $1 \leq j \leq \frac{p}{2} - 1$, together with $c_{p/2}$ and $\prod_{j=0}^{p-1} \rho_j^2$, form a set of $p$ commuting integrals. Note that the real dimension of the space $\mathbb{D}^p$ of the Verblunsky coefficients is $2p$, twice the number of the Hamiltonians described above.

The next natural question concerns finding Lax pairs associated to these commuting Hamiltonians.

**Remark 1.4.** Recall that finding a Lax pair representation

$$(1.7) \qquad \dot{L} = [L, P]$$

for an evolution equation allows one to identify the eigenvalues of $L$ as conserved quantities. This is usually done in the case when $L$ is selfadjoint, but essentially the same proof works in the case that we are interested in, when $L$ is unitary.

Indeed, let $\lambda \in S^1$ be an eigenvalue of $L$, a unitary matrix, and $\phi$ the corresponding unit eigenvector. Then

$$\lambda = (L\phi, \phi)$$

and hence

$$\dot{\lambda} = (\dot{L}\phi, \phi) + (L\dot{\phi}, \phi) + (L\phi, \dot{\phi}).$$

Note that

$$
\begin{aligned}
(L\dot{\phi}, \phi) + (L\phi, \dot{\phi}) &= (\dot{\phi}, L^*\phi) + (L\phi, \dot{\phi}) \\
&= (\dot{\phi}, \bar{\lambda}\phi) + (\lambda\phi, \dot{\phi}) \\
&= \lambda[(\dot{\phi}, \phi) + (\phi, \dot{\phi})] \\
&= \lambda(\dot{\phi, \phi}) \\
&= 0,
\end{aligned}
$$

as $(\phi, \phi) \equiv 1$. So,

$$
\begin{aligned}
\dot{\lambda} = (\dot{L}\phi, \phi) &= ([L, P]\phi, \phi) \\
&= (P\phi, L^*\phi) - (PL\phi, \phi) \\
&= (P\phi, \bar{\lambda}\phi) - (\lambda P\phi, \phi) = 0.
\end{aligned}
$$

As it turns out, the coefficients $c_j$ for which Nenciu and Simon obtained Poisson commutativity are not the Hamiltonians we will be working with here, while being closely related to them.

A consequence of Theorem 11.2.2 and formula (11.2.17) of [14] is

$$(1.8) \qquad \det(z - \mathcal{Q}_{(1)}) = z^{p/2}\Big(\prod_{j=0}^{p-1} \rho_j\Big)[\Delta(z) - 2],$$

where $\mathcal{Q}_{(1)}$ is the restriction of $\mathcal{E}$ to the space of $p$-periodic $l^\infty$ sequences (see the definition of $\mathcal{Q}_{(d)}$ in Section 2). In particular, this shows that $\prod_{j=0}^{p-1} \rho_j$ and the coefficients of $\Delta$ Poisson commute if and only if $\prod_{j=0}^{p-1} \rho_j$ and the real and imaginary parts of the traces of the first $\frac{p}{2}$ powers of $\mathcal{Q}_{(1)}$ Poisson commute. These are (essentially, see Proposition 2.5) the Hamiltonians we will be working with. They are the natural functions to consider by the fact proved above, that existence of Lax pairs implies conservation of the eigenvalues, and hence of the traces of powers of the Lax matrix.

The organization of the paper is as follows: In Section 2 we define the objects involved in the periodic problem and present the main result, Theorem 2, and its consequences. Section 3 contains the ideas of the proof of the main theorem, Theorem 2, while in Appendix A we give the full computations involved in this proof. Sections 4 and 5 deal with

the finite and infinite cases, respectively. Finally, Appendix B gives the necessary background on the theory of orthogonal polynomials on the unit circle.

## 2. The Main Results in the Periodic Case

We must first define our Hamiltonians $K_n$. Essentially, they are traces per volume of the powers of the extended CMV matrix $\mathcal{E}$.

Consider the periodic Ablowitz-Ladik problem with period $p$. If $p$ is even, then let $\mathcal{E}$ be the extended CMV matrix associated to these $\alpha$'s; if $p$ is odd, think of the sequence of Verblunsky coefficients as having period $2p$ and thus define the extended CMV matrix $\mathcal{E}$.

For each $n \geq 1$, we define the Hamiltonians we will be working with as:

$$(2.1) \qquad\qquad K_n = \frac{1}{n} \sum_{k=0}^{p-1} \mathcal{E}_{kk}^n.$$

For $n = 0$, we set

$$K_0 = \prod_{j=0}^{p-1} \rho_j^2.$$

Finally, for $\mathcal{A}$ a doubly-infinite matrix, we set $\mathcal{A}_+$ as the matrix with entries

$$(\mathcal{A}_+)_{jk} = \begin{cases} \mathcal{A}_{jk}, & \text{if } j < k; \\ \frac{1}{2}\mathcal{A}_{jj}, & \text{if } j = k; \\ 0, & \text{if } j > k. \end{cases}$$

The central theorem of our paper is:

**Theorem 2.** *The Lax pairs for the $n^{\mathrm{th}}$ Hamiltonian of the periodic defocusing Ablowitz-Ladik system are given by*

$$(2.2) \qquad\qquad \{\mathcal{E}, K_n\} = [\mathcal{E}, i\mathcal{E}_+^n]$$

*and*

$$(2.3) \qquad\qquad \{\mathcal{E}, \bar{K}_n\} = [\mathcal{E}, i(\mathcal{E}_+^n)^*]$$

*for all $n \geq 1$.*

Here we use $\{\mathcal{E}, f\}$ to denote the doubly-infinite matrix with $(j, k)$ entry $\{\mathcal{E}_{jk}, f\}$; also, $\mathcal{E}_+^n$ denotes $(\mathcal{E}^n)_+$.

**Remark 2.1.** The form of Theorem 2, and the main idea of the proof were inspired by the analogous result of van Moerbeke [18] for the periodic Toda lattice. But none of the two results implies the other.

Moreover, in the case of the Toda lattice, the necessary calculations are very simple due to the tri-diagonal, symmetric nature of the Jacobi matrices naturally associated to that problem. The analogue on the circle are CMV matrices, whose more complicated structure makes proving this result computationally much more involved.

But we are dealing with a finite dimensional problem, so we are interested in finding appropriate finite dimensional spaces to which we can restrict the operators in (2.2) and (2.3). Also, we want to express the Hamiltonians $K_n$ in terms of these restrictions.

The following lemma is an immediate consequence of the structure of $\mathcal{E}$; it can easily be proven by induction whenever $\mathcal{E}$ can be defined.

**Lemma 2.2.** *Let $n \geq 1$ be an integer. Then $\mathcal{E}_{j,k}^n$ is identically zero as a function of the Verblunsky coefficients if one of the following holds:*

$$|j - k| \geq 2n + 1$$
*or $j - k = 2n$ and $j$ and $k$ are even*
*or $j - k = -2n$ and $j$ and $k$ are odd.*

*In particular, the number of entries which are not identically zero (as functions of the $\alpha$'s) on any row of $\mathcal{E}^n$ is bounded by $4n$.*

Recall that the definition of $\mathcal{E}$ depends on the parity of the period $p$. This explains why we need to study the cases $p$ even and $p$ odd separately.

Let us first consider the case of the period $p$ being even. We denote by $X_{(d)}$ the subspace of $l^\infty(\mathbb{Z})$

$$X_{(d)} = \{u \in l^\infty(\mathbb{Z}) \,|\, u_{m+dp} = u_m\}$$

of sequences of period $dp$. As the Verblunsky coefficients are periodic with period $p$, we find that $\mathcal{E}_{j+p,k+p} = \mathcal{E}_{j,k}$ for any $j, k \in \mathbb{Z}$, and hence $\mathcal{E}^n$ restricts to $X_{(d)}$ for all $n \in \mathbb{Z}$ and $d \geq 1$. Moreover, if we denote by $\xi_k^{(d)}$, $k = 0, \ldots, dp - 1$, the $l^\infty(\mathbb{Z})$ vector given by

$$(\xi_k^{(d)})_j = 1 \text{ when } j \equiv k \,(\mathrm{mod}\, dp), \text{ and } 0 \text{ otherwise,}$$

we have that $\{\xi_0^{(d)}, \xi_1^{(d)}, \ldots, \xi_{dp-1}^{(d)}\}$ is a basis in $X_{(d)}$, and

$$(\mathcal{E}^n \xi_k^{(d)})_{j+p} = (\mathcal{E}^n \xi_k^{(d)})_j = \sum_{l \in \mathbb{Z}} \mathcal{E}_{j,k+ldp}^n.$$

Notice that this sum has only a finite number of nonzero terms for any choice of $n, j, k, p$, and $d$.

Let us denote by $\mathcal{Q}_{(d)}$ the matrix representation of the restriction $\mathcal{E} \upharpoonright X_{(d)}$ in the basis $\{\xi_0^{(d)}, \xi_1^{(d)}, \ldots, \xi_{dp-1}^{(d)}\}$. Then the matrix representing $\mathcal{E}^n \upharpoonright X_{(d)}$ in the same basis is $\mathcal{Q}_{(d)}^n$, whose entries are given by

$$(2.4) \qquad \mathcal{Q}_{(d),jk}^n = \sum_{l \in \mathbb{Z}} \mathcal{E}_{j,k+ldp}^n$$

for $0 \le j, k \le dp - 1$.

**Lemma 2.3.** *For $dp \ge 2n + 1$, we have that*

$$\frac{1}{d} \mathrm{Tr}(\mathcal{Q}_{(d)}^n)$$

*is independent of $d$ and equals $K_n$.*

*Proof.*

$$\frac{1}{d} \mathrm{Tr}(\mathcal{Q}_{(d)}^n) = \frac{1}{d} \sum_{k=0}^{dp-1} \mathcal{Q}_{(d),kk}^n \,.$$

From Lemma 2.2 we know that $\mathcal{E}_{jk}^n = 0$ for $|j - k| > 2n$. So for $dp \ge 2n + 1$ we get

$$\mathcal{Q}_{(d),kk}^n = \sum_{l \in \mathbb{Z}} \mathcal{E}_{k,k+ldp}^n = \mathcal{E}_{kk}^n \,.$$

From this and periodicity we can conclude that

$$\frac{1}{d} \mathrm{Tr}(\mathcal{Q}_{(d)}^n) = \frac{1}{d} \sum_{k=0}^{dp-1} \mathcal{E}_{kk}^n = \sum_{k=0}^{p-1} \mathcal{E}_{kk}^n = nK_n$$

is indeed independent of $d$. $\qquad \square$

If $p$ is odd, we consider the same objects as above, with the extra constraint that $dp$, and hence $d$, must always be even. Recall that in this case we define $\mathcal{E}$ by thinking of the Verblunsky coefficients as having period $2p$. For $d$ even, we can then define $X_{(d)}$ and $\mathcal{Q}_{(d)}$ as above, while always keeping in mind that we can use the results we just proved for $dp = \frac{d}{2} \cdot 2p$.

Therefore, if $d$ is even and large enough, we have that

$$\frac{2}{d} \mathrm{Tr}(\mathcal{Q}_{(d)}^n) = \frac{2}{d} \sum_{k=0}^{dp-1} \mathcal{E}_{kk}^n = \sum_{k=0}^{2p-1} \mathcal{E}_{kk}^n \,.$$

The last observation we need to make is that in this case the entries of $\mathcal{E}$ obey

$$\mathcal{E}_{jk} = \mathcal{E}_{k+p,j+p} \,.$$

This comes from the fact that

$$\mathcal{L}_{j+p,k+p} = \mathcal{M}_{jk},$$

and that $\mathcal{L}$ and $\mathcal{M}$ are symmetric. Hence

$$\mathcal{E}_{k+p,j+p} = \sum_{l\in\mathbb{Z}} \mathcal{L}_{k+p,l}\mathcal{M}_{l,j+p} = \sum_{l\in\mathbb{Z}} \mathcal{L}_{kl}\mathcal{M}_{lj} = \sum_{l\in\mathbb{Z}} \mathcal{L}_{lk}\mathcal{M}_{jl} = \mathcal{E}_{jk},$$

as claimed. A straightforward induction shows that

$$\mathcal{E}^n_{k+p,j+p} = \mathcal{E}^n_{jk}$$

for all $n$, and hence

$$\frac{1}{d}\mathrm{Tr}(\mathcal{Q}^n_{(d)}) = \frac{1}{2}\sum_{k=0}^{2p-1} \mathcal{E}^n_{kk} = \sum_{k=0}^{p-1}\mathcal{E}^n_{kk} = nK_n$$

also holds for $p$ odd, as long as $dp$ is even and $dp \geq 2n+1$.

So we proved that, with $K_n$ defined as in (2.1), we have

(2.5)
$$K_n = \frac{1}{dn}\mathrm{Tr}(\mathcal{Q}^n_{(d)})$$

for $dp$ even and greater than $2n+1$.

Let us note that relations (2.2) and (2.3) hold in the sense of bounded operators on $l^\infty(\mathbb{Z})$. Moreover, all the matrices in these relations obey the same periodicity conditions as $\mathcal{E}$, so it makes sense to restrict (2.2) and (2.3) to $X_{(d)}$ for $d \geq 1$. By doing this we get

**Corollary 2.4.** *For all $d \geq 1$, with $dp$ even, and $n \geq 1$, we have*

(2.6)
$$\{\mathcal{Q}_{(d)}, K_n\} = [\mathcal{Q}_{(d)}, i\mathcal{Q}^n_{(d),+}]$$

*and*

(2.7)
$$\{\mathcal{Q}_{(d)}, \bar{K}_n\} = [\mathcal{Q}_{(d)}, i(\mathcal{Q}^n_{(d),+})^*],$$

*where we denote by $\mathcal{Q}^n_{(d),+}$ the matrix representation of $(\mathcal{E}^n)_+ \upharpoonright X_{(d)}$ in the basis $\{\xi^{(d)}_0, \xi^{(d)}_1, \ldots, \xi^{(d)}_{dp-1}\}$.*

Note that $\mathcal{Q}^n_{(d),+}$ is not an upper triangular matrix, as it contains entries which are generically nonzero in its lower left corner.

Let us make an observation that will explain why we cannot simply use the traces of powers of $\mathcal{Q}_{(1)}$ even if $p$ is even, but also that we are not changing by much the Hamiltonians we are most interested in:

**Proposition 2.5.** *For $p$ even and $1 \leq n \leq \frac{p}{2} - 1$, we have that*

$$K_n = \frac{1}{n}\mathrm{Tr}(\mathcal{Q}_{(1)}^n),$$

*but*

$$\frac{2}{p}\mathrm{Tr}(\mathcal{Q}_{(1)}^{p/2}) = K_{p/2} + 2K_0^{1/2}.$$

*Proof.* From formula (2.4) and Lemma 2.2 we see that, for $n \leq \frac{p}{2} - 1$,

$$\mathcal{Q}_{(1),jj}^n = \sum_{l \in \mathbb{Z}} \mathcal{E}_{j,j+lp}^n = \mathcal{E}_{jj}^n$$

for all $j = 0, \ldots, p-1$. This follows since, for $|l| \geq 1$,

$$|j - (j + lp)| \geq p \geq 2n + 1.$$

Hence, using (2.1),

$$\frac{1}{n}\mathrm{Tr}(\mathcal{Q}_{(1)}^n) = \frac{1}{n}\sum_{j=0}^{p-1} \mathcal{E}_{jj}^n = K_n.$$

If $n = \frac{p}{2}$ and $j$ even, the formulae (2.4),(5.2), (5.3), and periodicity of the Verblunsky coefficients imply that

$$\begin{aligned}
\mathcal{Q}_{(1),jj}^n &= \sum_{l \in \mathbb{Z}} \mathcal{E}_{j,j+lp}^n \\
&= \mathcal{E}_{jj}^n + \mathcal{E}_{j,j+p}^n \\
&= \mathcal{E}_{jj}^n + \prod_{k=0}^{p-1} \rho_k
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{Q}_{(1),j+1,j+1}^n &= \sum_{l \in \mathbb{Z}} \mathcal{E}_{j+1,j+1+lp}^n \\
&= \mathcal{E}_{j+1,j+1}^n + \mathcal{E}_{j+1,j+1-p}^n \\
&= \mathcal{E}_{j+1,j+1}^n + \prod_{k=0}^{p-1} \rho_k.
\end{aligned}$$

Therefore,

$$\frac{2}{p}\mathrm{Tr}(\mathcal{Q}_{(1)}^{p/2}) = \frac{2}{p}\sum_{j=0}^{p-1} \mathcal{E}_{jj}^n + \frac{2p}{p}\prod_{k=0}^{p-1} \rho_k = K_{p/2} + 2K_0^{1/2},$$

as claimed. $\qquad\square$

**Remark 2.6.** An easy computation shows that

$$\{\alpha_j, 2\operatorname{Re}(K_1)\} = i\rho_j^2(\alpha_{j-1} + \alpha_{j+1})$$

and

$$\{\alpha_j, \log(K_0)\} = i\alpha_j$$

for all $0 \le j \le p-1$. Hence (1.1), the periodic defocusing Ablowitz-Ladik equation, is the evolution of the Verblunsky coefficients under the flow generated by the Hamiltonian $2\operatorname{Re}(K_1) - 2\log(K_0)$.

From Theorem 2 and Corollary 2.4 we can immediately conclude that

**Corollary 2.7.** *The Lax pairs for the Hamiltonians* $\operatorname{Re}(K_n)$ *and* $\operatorname{Im}(K_n)$, $n \ge 1$, *are given by*

(2.8) $$\{\mathcal{E}, 2\operatorname{Re}(K_n)\} = [\mathcal{E}, i\mathcal{E}_+^n + i(\mathcal{E}_+^n)^*]$$

*and*

(2.9) $$\{\mathcal{E}, 2\operatorname{Im}(K_n)\} = [\mathcal{E}, \mathcal{E}_+^n - (\mathcal{E}_+^n)^*],$$

*while the corresponding statements for* $\mathcal{Q}_{(d)}$, $d \ge 1$ *and* $dp$ *even, are given by*

(2.10) $$\{\mathcal{Q}_{(d)}, 2\operatorname{Re}(K_n)\} = [\mathcal{Q}_{(d)}, i\mathcal{Q}_{(d),+}^n + i(\mathcal{Q}_{(d),+}^n)^*]$$

*and*

(2.11) $$\{\mathcal{Q}_{(d)}, 2\operatorname{Im}(K_n)\} = [\mathcal{Q}_{(d)}, \mathcal{Q}_{(d),+}^n - (\mathcal{Q}_{(d),+}^n)^*].$$

In particular, relations (2.10) and (2.11), together with Remark 1.4 and (2.5), imply that

**Corollary 2.8.**

$$\{K_n, \operatorname{Re}(K_m)\} = \{K_n, \operatorname{Im}(K_m)\} = 0,$$

*and hence*

$$\{K_n, K_m\} = \{K_n, \bar{K}_m\} = 0.$$

Define the doubly-infinite matrix $\mathcal{P}$ by

$$\mathcal{P}_{lm} = (-1)^l \delta_{lm} \frac{i}{2} \Big( \prod_{k=0}^{p-1} \rho_k^2 \Big).$$

**Proposition 2.9.** *The Lax pair representation for the flow generated by* $K_0 = \prod_{j=0}^{p-1} \rho_j^2$ *is given by*

(2.12) $$\{\mathcal{E}, K_0\} = [\mathcal{E}, \mathcal{P}].$$

*In particular, we can conclude that*

(2.13) $$\{K_0, K_n\} = \{K_0, \bar{K}_n\} = 0,$$

*or, equivalently,*

(2.14) $$\{K_0, 2\operatorname{Re}(K_n)\} = \{K_0, 2\operatorname{Im}(K_n)\} = 0.$$

*Proof.* The Lax pair representation (2.12) is checked by a straightforward computation. It is based on the fact that the flow generated by $K_0$ rotates all the $\alpha$'s by the same angle

$$\{\alpha_j, K_0\} = iK_0\alpha_j,$$

while

$$[\mathcal{E}, \mathcal{P}]_{j,k} = \mathcal{E}_{j,k}(\mathcal{P}_{k,k} - \mathcal{P}_{j,j}).$$

The Poisson commutation relations (2.13) and (2.14) follow, as in the previous cases, by restricting the Lax pair to periodic subspaces and concluding that the flow preserves eigenvalues, and hence traces. $\square$

From (1.8), Corollary 2.8, and Proposition 2.9, we immediately get that $\prod_{j=0}^{p-1} \rho_j$ and the coefficients $c_k$ of $z^{p/2}\left(\prod_{j=0}^{p-1} \rho_j\right)[\Delta(z) - 2]$ Poisson commute. Note also that, by (1.8), we see that the connection between the $K$'s and the $c$'s cannot be explicitly written down. Hence one cannot write simple Lax pairs in terms of $\mathcal{E}$ for the flows generated by the $c$'s.

## 3. The Periodic Case: Proof of Theorem 2

The main technical ingredient in the proof of Theorem 2 is the following:

**Lemma 3.1.** *For all $n \geq 0$ and $j$ even, we have*

(3.1)
$$\begin{aligned}
\frac{\partial K_{n+1}}{\partial \alpha_j} &= -\frac{\bar{\alpha}_j\bar{\alpha}_{j+1}}{2\rho_j}\mathcal{E}_{j+1,j}^n - \frac{\bar{\alpha}_j\rho_{j+1}}{2\rho_j}\mathcal{E}_{j+2,j}^n - \frac{\bar{\alpha}_j\rho_{j-1}}{2\rho_j}\mathcal{E}_{j-1,j+1}^n \\
&\quad + \frac{\bar{\alpha}_j\alpha_{j-1}}{2\rho_j}\mathcal{E}_{j,j+1}^n - \bar{\alpha}_{j+1}\mathcal{E}_{j+1,j+1}^n - \rho_{j+1}\mathcal{E}_{j+2,j+1}^n
\end{aligned}$$

(3.2)
$$\begin{aligned}
\frac{\partial K_{n+1}}{\partial \bar{\alpha}_j} &= \rho_{j-1}\mathcal{E}_{j-1,j}^n - \alpha_{j-1}\mathcal{E}_{j,j}^n - \frac{\alpha_j\bar{\alpha}_{j+1}}{2\rho_j}\mathcal{E}_{j+1,j}^n \\
&\quad - \frac{\alpha_j\rho_{j+1}}{2\rho_j}\mathcal{E}_{j+2,j}^n - \frac{\alpha_j\rho_{j-1}}{2\rho_j}\mathcal{E}_{j-1,j+1}^n + \frac{\alpha_j\alpha_{j-1}}{2\rho_j}\mathcal{E}_{j,j+1}^n
\end{aligned}$$

(3.3)
$$\begin{aligned}
\frac{\partial K_{n+1}}{\partial \alpha_{j-1}} &= -\frac{\bar{\alpha}_{j-1}\rho_{j-2}}{2\rho_{j-1}}\mathcal{E}_{j,j-2}^n + \frac{\bar{\alpha}_{j-1}\alpha_{j-2}}{2\rho_{j-1}}\mathcal{E}_{j,j-1}^n - \frac{\bar{\alpha}_{j-1}\bar{\alpha}_j}{2\rho_{j-1}}\mathcal{E}_{j-1,j}^n \\
&\quad - \frac{\bar{\alpha}_{j-1}\rho_j}{2\rho_{j-1}}\mathcal{E}_{j-1,j+1}^n - \bar{\alpha}_j\mathcal{E}_{j,j}^n - \rho_j\mathcal{E}_{j,j+1}^n
\end{aligned}$$

$$(3.4) \quad \begin{aligned} \frac{\partial K_{n+1}}{\partial \bar{\alpha}_{j-1}} =& \rho_{j-2}\mathcal{E}^n_{j-1,j-2} - \alpha_{j-2}\mathcal{E}^n_{j-1,j-1} - \frac{\alpha_{j-1}\rho_{j-2}}{2\rho_{j-1}}\mathcal{E}^n_{j,j-2} \\ &- \frac{\alpha_{j-1}\bar{\alpha}_j}{2\rho_{j-1}}\mathcal{E}^n_{j-1,j} - \frac{\alpha_{j-1}\rho_j}{2\rho_{j-1}}\mathcal{E}^n_{j-1,j+1} + \frac{\alpha_{j-1}\alpha_{j-2}}{2\rho_{j-1}}\mathcal{E}^n_{j,j-1}. \end{aligned}$$

**Remark 3.2.** Note that, for any $n \geq 1$ and $0 \leq j \leq p-1$, we have

$$\frac{\partial \bar{K}_n}{\partial \beta_j} = \overline{\left(\frac{\partial K_n}{\partial \bar{\beta}_j}\right)}$$

and hence one can easily find the derivatives of $\bar{K}_n$ with respect to $\alpha_j$ and $\bar{\alpha}_j$ from Lemma 3.1

*Proof.* The proof reduces to direct computations once one notices that, by invariance of the trace under circular permutations,

$$(3.5)$$
$$\frac{\partial K_{n+1}}{\partial \beta_j} = \frac{1}{(n+1)d}\mathrm{Tr}\left(\frac{\partial \mathcal{Q}_{(d)}}{\partial \beta_j}\mathcal{Q}^n_{(d)} + \mathcal{Q}_{(d)}\frac{\partial \mathcal{Q}_{(d)}}{\partial \beta_j}\mathcal{Q}^{n-1}_{(d)} + \cdots + \mathcal{Q}^n_{(d)}\frac{\partial \mathcal{Q}_{(d)}}{\partial \beta_j}\right)$$
$$(3.6)$$
$$= \frac{1}{d}\mathrm{Tr}\left(\frac{\partial \mathcal{Q}_{(d)}}{\partial \beta_j}\mathcal{Q}^n_{(d)}\right).$$

We give here the complete proof of (3.1); (3.2) through (3.4) can be found in a similar way.

Notice that, for $j$ even, $\alpha_j$ appears in exactly $6d$ entries of $\mathcal{Q}_{(d)}$. So (3.1) follows by periodicity and by a straightforward computation from (3.5):

$$\begin{aligned} \frac{\partial K_{n+1}}{\partial \alpha_j} =& \frac{1}{d}\sum_{k,l}\frac{\partial \mathcal{Q}_{(d),kl}}{\partial \alpha_j}\mathcal{Q}^n_{(d),lk} \\ =& -\frac{\bar{\alpha}_j}{\rho_j}\bar{\alpha}_{j+1}\mathcal{E}^n_{j+1,j} - \frac{\bar{\alpha}_j}{\rho_j}\rho_{j+1}\mathcal{E}^n_{j+2,j} - \frac{\bar{\alpha}_j}{\rho_j}\rho_{j-1}\mathcal{E}^n_{j-1,j+1} \\ &+ \frac{\bar{\alpha}_j}{\rho_j}\alpha_{j-1}\mathcal{E}^n_{j,j+1} - \bar{\alpha}_{j+1}\mathcal{E}^n_{j+1,j+1} - \rho_{j+1}\mathcal{E}^n_{j+2,j+1}. \end{aligned}$$

$\square$

Before we embark on the proof of the main theorem, we provide another preliminary result; while the statement is almost certainly not new, we give a proof for the reader's convenience.

Consider an $N \times N$ matrix $A$ having the following *stair-shape*

$$
A = \begin{pmatrix}
\star & 0 & 0 & \cdots & 0 \\
\star & \star & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\star & \star & \star & \cdots & 0 \\
\star & \star & \star & \cdots & 0
\end{pmatrix},
$$

where the stars and 0's represent rectangular matrix blocks. Formally, that means that for any row number $i$ there exists a column number $j(i)$ so that $A_{ij} = 0$ for all $j > j(i)$, and the function $i \mapsto j(i)$ is non-decreasing. In particular, it is also true that for any column $j$ there exists a row $i(j)$ so that $A_{ij} = 0$ for $i < i(j)$. Note in passing that $j(i)$ and $i(j)$ are not equal.

We will say, somewhat informally, that another matrix $\tilde{A}$ *has the same shape as* $A$ if $\tilde{A}_{ij} = 0$ whenever $j > j(i)$ for all $i$.

**Lemma 3.3.** *Let $A$ be a matrix as above and $B$ an arbitrary $N \times N$ matrix. Then*

(3.7)
$$
[A, B_+]_{ij} = [A, B]_{ij}
$$

*for all $(i, j)$ with $j > j(i)$. This implies that, for the same indices $(i, j)$ with $j > j(i)$, we have*

(3.8)
$$
[A, B_-]_{ij} = 0.
$$

**Remark 3.4.** Note that:

- If $A$ and $B$ commute, then the commutators $[A, B_+]$ and $[A, B_-]$ have the same shape as $A$.
- Also, by transposing these equations, we obtain the same type of result for "lower triangle shapes."
- The same type of result holds for doubly-infinite matrices. In particular, if $\mathcal{A}$ and $\mathcal{B}$ are two doubly infinite, stair-shaped matrices such that the commutator $[\mathcal{A}, \mathcal{B}]$ makes sense and equals 0, then the commutators $[\mathcal{A}, \mathcal{B}_+]$ and $[\mathcal{A}, \mathcal{B}_-]$ are themselves stair-shaped.

*Proof.* We proceed by direct computation: Let $(i, j)$ be an index so that $j > j(i)$; equivalently, $i < i(j)$. Then

$$
\begin{aligned}
[A, B_+]_{ij} &= \sum_k A_{ik} B_{+,kj} - \sum_k B_{+,ik} A_{kj} \\
&= \sum_{k \leq j(i) < j} A_{ik} B_{+,kj} - \sum_{i < i(j) \leq k} B_{+,ik} A_{kj} \\
&= \sum_k A_{ik} B_{kj} - \sum_k B_{ik} A_{kj} \\
&= [A, B]_{ij}.
\end{aligned}
$$

Since $B_- = B - B_+$, we get that

$$
[A, B_-] = [A, B] - [A, B_+]
$$

and so the second relation is just a consequence of the first one.     □

We are now ready to prove Theorem 2.

*Proof.* We will first deal with relation (2.2) for $n + 1$, $n \geq 0$:

$$
\{\mathcal{E}, K_{n+1}\} = i[\mathcal{E}, \mathcal{E}_+^{n+1}]
$$

The left-hand side matrix has two types of entries: the ones outside the shape of a CMV matrix, which are identically zero, and the ones inside the shape.

The entries outside the shape are dealt with immediately by applying Lemma 3.3. Indeed, $\mathcal{E}$ and $\mathcal{E}^n$ are doubly-infinite matrices, and they commute; hence, by the third observation above, the commutator $[\mathcal{E}, \mathcal{E}_+^{n+1}]$ has the same shape as $\mathcal{E}$.

We are now left with the entries $(j, k)$ which are inside the shape. Before we start computing, we make a short observation. Consider the doubly-infinite matrix $\mathcal{U}$ given by

$$
\mathcal{U}_{jk} = \delta_{j,k+1}
$$

for all $j, k \in \mathbb{Z}$. In other words, $\mathcal{U}$ is the left-shift on $l^\infty(\mathbb{Z})$ in the usual basis. Note that for a doubly-infinite matrix $\mathcal{B}$ we have

$$
(\mathcal{U}^* \mathcal{B} \mathcal{U})_{jk} = B_{j-1,k-1} \quad \text{and} \quad (\mathcal{U} \mathcal{B} \mathcal{U}^*)_{jk} = B_{j+1,k+1}.
$$

Consider $\mathcal{E} = \mathcal{E}(\{\alpha_j\})$ to be a doubly-infinite CMV matrix. We know that $\mathcal{E} = \tilde{\mathcal{L}} \tilde{\mathcal{M}}$ with

$$
\tilde{\mathcal{L}} = \operatorname{diag}\left(\ldots, \Theta_0, \Theta_2, \Theta_4, \ldots\right)
$$

and

$$
\tilde{\mathcal{M}} = \operatorname{diag}\left(\ldots, \Theta_{-1}, \Theta_1, \Theta_3, \ldots\right).
$$

It is easily seen that

$$\mathcal{U}^*\tilde{\mathcal{L}}(\{\alpha_j\})^t\mathcal{U} = \tilde{\mathcal{M}}(\{\alpha_{j-1}\}) \quad \text{and} \quad \mathcal{U}^*\tilde{\mathcal{M}}(\{\alpha_j\})^t\mathcal{U} = \tilde{\mathcal{L}}(\{\alpha_{j-1}\}),$$

which implies that

$$(3.9) \qquad\qquad \mathcal{U}^*\mathcal{E}(\{\alpha_j\})^t\mathcal{U} = \mathcal{E}(\{\alpha_{j-1}\})$$

is also a doubly-infinite CMV matrix. The same is true for

$$\mathcal{U}\mathcal{E}(\{\alpha_j\})^t\mathcal{U}^* = \mathcal{E}(\{\alpha_{j+1}\}).$$

We use the notation $(2.2)_{kl}$ for the $(k,l)$ entry of relation $(2.2)$, and similarly for $(2.3)$. Assume we know $(2.2)_{kl}$ for a fixed pair of indices $(k,l)$. As for any $\{\alpha_j\}$ the matrix $\mathcal{U}^*\mathcal{E}^t\mathcal{U}$ is a doubly-infinite CMV matrix, we know that

$$(3.10) \qquad \{(\mathcal{U}^*\mathcal{E}^t\mathcal{U})_{kl}, K_{n+1}(\mathcal{U}^*\mathcal{E}^t\mathcal{U})\} = i[\mathcal{U}^*\mathcal{E}^t\mathcal{U}, (\mathcal{U}^*\mathcal{E}^t\mathcal{U})_+^{n+1}]_{kl}.$$

But

$$K_{n+1}(\mathcal{U}^*\mathcal{E}^t\mathcal{U}) = \frac{1}{(n+1)d}\text{Tr}\big((\mathcal{U}_{(d)}^*\mathcal{Q}_{(d)}^t\mathcal{U}_{(d)})^{n+1}\big)$$

$$= \frac{1}{(n+1)d}\text{Tr}\big(\mathcal{U}_{(d)}^*(\mathcal{Q}_{(d)}^t)^{n+1}\mathcal{U}_{(d)}\big)$$

$$= K_{n+1}(\mathcal{E})$$

and $\mathcal{U}$ is a constant matrix. Therefore,

$$\{(\mathcal{U}^*\mathcal{E}^t\mathcal{U})_{kl}, K_{n+1}(\mathcal{U}^*\mathcal{E}^t\mathcal{U})\} = \big(\mathcal{U}^*\{\mathcal{E}^t, K_{n+1}(\mathcal{E})\}\mathcal{U}\big)_{kl}$$

$$(3.11) \qquad\qquad = \{\mathcal{E}_{k-1,l-1}^t, K_{n+1}(\mathcal{E})\}$$

$$= \{\mathcal{E}_{l-1,k-1}, K_{n+1}(\mathcal{E})\}.$$

On the other hand,
$$(3.12)$$
$$i[\mathcal{U}^*\mathcal{E}^t\mathcal{U}, (\mathcal{U}^*\mathcal{E}^t\mathcal{U})_+^n]_{kl} = i\big(\mathcal{U}^*[\mathcal{E}^t, (\mathcal{E}^t)_+^{n+1}]\mathcal{U}\big)_{kl} = i[\mathcal{E}^t, (\mathcal{E}^t)_+^{n+1}]_{k-1,l-1}$$

$$= i[\mathcal{E}^t, (\mathcal{E}_-^{n+1})^t]_{k-1,l-1} = i[\mathcal{E}, \mathcal{E}_+^{n+1}]_{l-1,k-1}.$$

Plugging in $(3.11)$ and $(3.12)$ into $(3.10)$, one gets relation $(2.2)_{l-1,k-1}$:

$$\{\mathcal{E}_{l-1,k-1}, K_{n+1}\} = i[\mathcal{E}, \mathcal{E}_+^{n+1}]_{l-1,k-1}.$$

If instead of considering $\mathcal{U}^*\mathcal{E}^t\mathcal{U}$ we consider $\mathcal{U}\mathcal{E}^t\mathcal{U}^*$, we obtain that $(2.2)_{kl}$ implies $(2.2)_{l+1,k+1}$. In particular, this means that:

- $(2.2)_{kk} \quad \Leftrightarrow \quad (2.2)_{k+1,k+1}$
- $(2.2)_{k,k-1} \quad \Leftrightarrow \quad (2.2)_{k,k+1}$
- $(2.2)_{k+1,k-1} \quad \Leftrightarrow \quad (2.2)_{k,k+2}$
- $(2.2)_{k+1,k} \quad \Leftrightarrow \quad (2.2)_{k+1,k+2}.$

So the proof of relation (2.2) is complete once we prove it for the indices
$(k, k)$, $(k, k-1)$, $(k+1, k-1)$, and $(k+1, k)$ with $k$ even.

We note here that we can apply the same reasoning as above to $\mathcal{E}^*$
instead of $\mathcal{E}^t$, but we do not obtain anything new.

Finally, these relations are proved using Lemma 3.1. We give the
computational details in Appendix A.

The second part of the proof deals with relation (2.3) for $n+1$, $n \geq 0$:

$$\{\mathcal{E}, \bar{K}_{n+1}\} = [\mathcal{E}, i(\mathcal{E}_+^{n+1})^*].$$

We shall proceed in very much the same way as with (2.2), while in-
corporating the necessary computational adjustments.

Let us first note that

$$(\mathcal{E}_+^{n+1})^* = ((\mathcal{E}^*)^{n+1})_-.$$

So Lemma 3.3 and the subsequent remarks apply here too and we can
conclude that $[\mathcal{E}, i(\mathcal{E}_+^{n+1})^*]$ has the same shape as $\mathcal{E}$.

Turning our attention to the entries inside the shape of $\mathcal{E}$, let us note
that using exactly the same reasoning as for equation (2.2) shows that

- $(2.3)_{kk} \quad \Leftrightarrow \quad (2.3)_{k+1,k+1}$
- $(2.3)_{k,k-1} \quad \Leftrightarrow \quad (2.3)_{k,k+1}$
- $(2.3)_{k+1,k-1} \quad \Leftrightarrow \quad (2.3)_{k,k+2}$
- $(2.3)_{k+1,k} \quad \Leftrightarrow \quad (2.3)_{k+1,k+2}.$

So again we only have to check four relations; the only difference is that,
in this case, $(2.3)_{k+1,k+1}$, $(2.3)_{k,k+1}$, $(2.3)_{k,k+2}$, and $(2.3)_{k+1,k+2}$ turn out
to be computationally easier to verify. We do this in Appendix A.   $\square$

## 4. The Finite Case

In this section we prove Lax pair representations for the finite Ablowitz-
Ladik system.

We are interested in studying the system

$$-i\dot{\alpha}_j = \rho_j^2(\alpha_{j+1} + \alpha_{j-1})$$

for $0 \leq j \leq k-2$, with boundary conditions $\alpha_{-1} = \alpha_{k-1} = -1$. The
idea behind finding Lax pairs for this system is to take one of the $\alpha$'s
in the appropriate periodic problem to the boundary, and identify all
the objects obtained in this way. As it turns out, they are all naturally
related to both the Ablowitz-Ladik system and orthogonal polynomials
on the circle, and can be defined independently of the periodic setting.

Let us elaborate. As presented in Appendix B, if we start with a
finitely supported measure $\mu$ on $S^1$, the associated Verblunsky coeffi-
cients are $\alpha_0, \ldots, \alpha_{k-2} \in \mathbb{D}$, $\alpha_{k-1} \in S^1$. The CMV matrix is in this

case a unitary $k \times k$ matrix

$$\mathcal{C}_f = \mathcal{L}_f \mathcal{M}_f$$

with

$$\mathcal{L}_f = \begin{pmatrix} \bar{\alpha}_0 & \rho_0 & & & \\ \rho_0 & -\alpha_0 & & & \\ & & \ddots & & \\ & & & \bar{\alpha}_{k-2} & \rho_{k-2} \\ & & & \rho_{k-2} & -\alpha_{k-2} \end{pmatrix}$$

and

$$\mathcal{M}_f = \begin{pmatrix} 1 & & & & \\ & \bar{\alpha}_1 & \rho_1 & & \\ & \rho_1 & -\alpha_1 & & \\ & & & \ddots & \\ & & & & \bar{\alpha}_{k-1} \end{pmatrix}.$$

If, in addition, we restrict our attention to the case when $\alpha_{k-1} = -1$, then we obtain the following connection between the finite and the periodic cases:

**Lemma 4.1.** *Let $k$ be even and $\mathcal{C}_f$ as above with $\alpha_{k-1} = -1$. Define a doubly-infinite set of Verblunsky coefficients by periodicity: $\alpha_{nk+j} = \alpha_j$ for all $n \in \mathbb{Z}$ and $0 \le j \le k-1$. Then the extended CMV matrix $\mathcal{E}$ associated to these $\alpha$'s has the direct sum decomposition*

$$(4.1) \qquad \mathcal{E} = \bigoplus_{r \in \mathbb{Z}} S^r(\mathcal{C}_f),$$

*where $S : l^\infty(\mathbb{Z}) \to l^\infty(\mathbb{Z})$ is the right $k$-shift.*

*In particular, the following also hold:*

$$(4.2) \qquad \mathcal{Q}_{(d)} = \bigoplus_{r=0}^{d-1} S^r(\mathcal{C}_f)$$

*and*

$$(4.3) \qquad K_n(\mathcal{E}) = \frac{1}{n} \mathrm{Tr}(\mathcal{C}_f^n)$$

*for all $d \ge 1$ and $n \ge 1$.*

*Proof.* Relation (4.1) follows immediately if we observe that $\rho_{rk-1} = 0$ for all $r \in \mathbb{Z}$, and this implies that (see equation (B.10))

$$\tilde{\mathcal{M}} = \bigoplus_{r \in \mathbb{Z}} S^r(\mathcal{M}_f).$$

By periodicity, we always have

$$\tilde{\mathcal{L}} = \bigoplus_{r \in \mathbb{Z}} S^r(\mathcal{L}_f).$$

So (4.1) follows from the definition of $\mathcal{C}_f = \mathcal{L}_f \mathcal{M}_f$. Likewise, (4.2) is just the restriction of (4.1) to $X_{(d)}$. So then

$$\mathcal{Q}_{(d)}^n = \bigoplus_{r=0}^{d-1} S^r(\mathcal{C}_f^n)$$

and by taking the trace we get (4.3).                                   $\square$

Note also that the Poisson bracket (1.4) separates the $\alpha$'s, and hence it naturally restricts to the space of $(\alpha_0, \ldots, \alpha_{k-2}, \alpha_{k-1} = -1) \in \mathbb{D}^{k-1}$. If two functions $f$ and $g$ depend only on $\alpha_0, \ldots, \alpha_{k-2}$, then

$$\{f, g\} = \frac{1}{2} \sum_{j=0}^{k-2} \rho_j^2 \left[ \frac{\partial f}{\partial u_j} \frac{\partial g}{\partial v_j} - \frac{\partial f}{\partial v_j} \frac{\partial g}{\partial u_j} \right]$$

$$= i \sum_{j=0}^{k-2} \rho_j^2 \left[ \frac{\partial f}{\partial \bar{\alpha}_j} \frac{\partial g}{\partial \alpha_j} - \frac{\partial f}{\partial \alpha_j} \frac{\partial g}{\partial \bar{\alpha}_j} \right],$$

where, as before, $\alpha_j = u_j + i v_j$ for all $0 \le j \le k - 2$.

So the next theorem is an immediate consequence of Theorem 2:

**Theorem 3.** *Let*

$$K_n^f = K_n(\mathcal{C}_f) = \frac{1}{n} \mathrm{Tr}(\mathcal{C}_f^n)$$

*for all $n \ge 1$. Then the Lax pairs associated to these Hamiltonians are given by*

(4.4)                               $\{\mathcal{C}_f, K_n^f\} = [\mathcal{C}_f, i(\mathcal{C}_f^n)_+]$

*and*

(4.5)                               $\{\mathcal{C}_f, \bar{K}_n^f\} = [\mathcal{C}_f, i((\mathcal{C}_f^n)_+)^*]$

*for all $n \ge 1$.*

*Or, in terms of real-valued flows, we have*

(4.6)                   $\{\mathcal{C}_f, 2\,\mathrm{Re}(K_n^f)\} = [\mathcal{C}_f, i(\mathcal{C}_f^n)_+ + i((\mathcal{C}_f^n)_+)^*]$

*and*

(4.7)                   $\{\mathcal{C}_f, 2\,\mathrm{Im}(K_n^f)\} = [\mathcal{C}_f, (\mathcal{C}_f^n)_+ - ((\mathcal{C}_f^n)_+)^*]$

*for all $n \ge 1$.*

As in the periodic case, since $K_n^f$ is the trace of $\mathcal{C}_f^n$, we obtain Poisson commutativity of the Hamiltonians:

**Corollary 4.2.** *For all $m, n \geq 1$, we have that*

$$\{K_n^f, \operatorname{Re}(K_m^f)\} = \{K_n^f, \operatorname{Im}(K_m^f)\} = 0$$

*and*

$$\{K_n^f, K_m^f\} = \{K_n^f, \bar{K}_m^f\} = 0.$$

**Remark 4.3.** Note that, since $\alpha_{k-1} \equiv -1$, we get $\rho_{k-1} \equiv 0$, and so $K_0 = \prod_{j=0}^{k-1} \rho_j^2 \equiv 0$ on $\mathbb{D}^{k-1}$. But if we define

$$K_0^f = \prod_{j=0}^{k-2} \rho_j^2,$$

then

$$\{\alpha_m, K_0^f\} = iK_0^f \alpha_m,$$

or

$$\{\alpha_m, \log(K_0^f)\} = i\alpha_m.$$

But, even though $K_0^f$ acts on the $\alpha$'s in the finite case in the same way as $K_0$ does in the periodic case, there exists no Lax pair representation for $K_0^f$ in terms of $\mathcal{C}_f$. The reason is that

$$\operatorname{Tr}\{\mathcal{C}_f, K_0^f\} = -iK_0^f(\bar{\alpha}_0 - \alpha_{k-2}),$$

which is not identically zero on $\mathbb{D}^{k-1}$, while the trace of a commutator is always zero.

## 5. The Infinite Case

Finally, we deal with the infinite defocusing Ablowitz-Ladik system. By this, we mean that we consider the system whose first equation is

$$i\dot{\alpha}_j = \rho_j^2(\alpha_{j+1} + \alpha_{j-1})$$

for all $j \geq 0$, with the boundary condition $\alpha_{-1} = 0$. The idea behind constructing Lax pairs for this system is to use the finite AL result. Since each entry in a fixed power of the CMV matrix depends on only a bounded number of $\alpha$'s, extending the finite Lax pairs to the infinite case only requires an appropriate definition of the "infinite" Hamiltonians $K_n^i$ for all $n \geq 1$.

Let us explain these claims: Fix $n_0 \geq 1$ and $j_0, m_0 \geq 0$. Consider the finite problem with $k$ very large ($k \geq 20(j_0 + k_0 + n_0)$ is sufficient, though a much more precise bound can be found). In this case,

$$\mathcal{C}_{j,m}^n = (\mathcal{C}_f)_{j,m}^n$$

for all $0 \leq j, m \leq j_0 + 4, m_0 + 4$ respectively, and $1 \leq n \leq n_0$. Say we can define a $K_n^i$ such that its dependence on the first $k$ $\alpha$'s is the same

as that of $K_n^f$. Then, for $0 \le j, m \le j_0 + 4, m_0 + 4$ respectively, we can replace "finite" by "infinite" in $(4.4)_{j_0,m_0}$ and $(4.5)_{j_0,m_0}$.

So the last element we need is $K_n^i$, the $n^{\text{th}}$ Hamiltonian for the infinite problem. It is a function defined on sequences $\{\alpha_j\}_{j\ge 0}$ of numbers inside the unit disk, having a certain decay. The condition that it must satisfy is that

$$K_n^i(\{\alpha_0, \alpha_1 \ldots, \alpha_{k-1} = -1, 0, 0, \ldots\}) = K_n^f(\{\alpha_0, \alpha_1, \ldots, \alpha_{k-1} = -1\}).$$

Given that

$$K_n^f(\mathcal{C}_f) = \frac{1}{n} \text{Tr}(\mathcal{C}_f^n),$$

a natural guess for $K_n^i$ would be

$$K_n^i(\mathcal{C}) = \frac{1}{n} \text{``Tr''}(\mathcal{C}^n).$$

But recall that the CMV matrix is unitary, so it is not trace class. Nonetheless, given the special structure of $\mathcal{C}$, we can define our Hamiltonian $K_n^i$ following this intuition as the sum of the diagonal entries of $\mathcal{C}^n$. While this statement will be rigorously proved in the following lemma, the reason why one can sum the series of diagonal entries is that all of these entries have the same structure for shifted $\alpha$'s: They are the sum of a bounded number of "monomials." By "monomial" we mean a finite product of $\alpha$'s and $\rho$'s. All the monomials that appear as terms in the diagonal entries contain at least one $\alpha$ factor. Since all the $\alpha$'s and $\rho$'s have absolute values less than 1, and if we assume $l^1$-decay of the sequence of coefficients, the one $\alpha$ factor in each monomial will ensure convergence of the whole series.

The next Lemma and its proof explore in more detail the structure of the entries of powers of the CMV matrix and its consequences for the definition of Hamiltonians in the infinite case.

**Lemma 5.1.** *Let $\{\alpha_j\}_{j\ge 0} \in l^1(\mathbb{N})$ be a sequence of coefficients with $\alpha_j \in \mathbb{D}$ for all $j \ge 0$. Let $\mathcal{C}$ be the CMV matrix associated to these coefficients. Then the series*

$$(5.1) \qquad\qquad \sum_{k \ge 0} \mathcal{C}_{k,k}^n$$

*converges absolutely for any $n \ge 1$.*

*Moreover, for any $k \ge 0$, we have that $\mathcal{C}_{k,k}^n$ depends only on $\alpha_{k-(2n-1)}$, $\ldots, \alpha_{k+2n-1}$, where all the $\alpha$'s with negative indices are assumed to be identically zero.*

*Proof.* We prove these statements by making two important observations.

The first refers to the general, doubly-infinite case. Let $\{\alpha_j\}_{j\in\mathbb{Z}}$ be a sequence of complex numbers in $\mathbb{D}$, and $\mathcal{E}$ the associated extended CMV matrix. Notice that the structure of $\mathcal{E}$ is such that there exist functions $f^e_{1,d_1}$ and $f^o_{1,d_1}$ defined on $\mathbb{D}^3$ with

$$\mathcal{E}_{j,k} = f^e_{1,d_1}(\alpha_{j-1}, \alpha_j, \alpha_{j+1})$$

for all $j$ even and $j - k = d_1$, and

$$\mathcal{E}_{j+1,k} = f^o_{1,d_1}(\alpha_{j-1}, \alpha_j, \alpha_{j+1})$$

for all $j$ even and $(j + 1) - k = d_1$. Here $e$ and $o$ are used to denote "even" or "odd" respectively, and $-2 \leq d_1 \leq 1$.

Using this simple remark, one can prove by induction that, for all $n \geq 1$, there exist functions

$$f^e_{n,d_n}, f^o_{n,d_n} : \mathbb{D}^{4n-1} \to \mathbb{C}$$

with $-2n \leq d_n \leq 2n - 1$ such that

$$\mathcal{E}^n_{j,k} = f^e_{n,j-k}(\alpha_{j-(2n-1)}, \ldots, \alpha_{j+(2n-1)})$$

for $j$ even and $-2n \leq j - k \leq 2n - 1$,

$$\mathcal{E}^n_{j+1,k} = f^o_{n,j-k+1}(\alpha_{j-(2n-1)}, \ldots, \alpha_{j+(2n-1)})$$

for $j$ even and $-2n \leq j + 1 - k \leq 2n - 1$, and

$$\mathcal{E}^n_{l,m} = 0$$

for all the other indices $(l, m)$.

Moreover, for $|d_n| \leq 2n - 1$, each such function $f^{e/o}_{n,d_n}$ is a sum of at most $4^n$ monomials, that is, products of $\alpha$'s and $\rho$'s, and each monomial contains at least one $\alpha$ factor. The only entries containing only $\rho$'s are the extreme ones:

$$(5.2) \quad \begin{aligned} \mathcal{E}^n_{j,j+2n} &= f^e_{n,-2n}(\alpha_{j-(2n-1)}, \ldots, \alpha_{j+(2n-1)}) \\ &= \rho_j \rho_{j+1} \cdots \rho_{j+2n-1} \end{aligned}$$

and

$$(5.3) \quad \begin{aligned} \mathcal{E}^n_{j+1,j-(2n-1)} &= f^o_{n,-2n}(\alpha_{j-(2n-1)}, \cdots, \alpha_{j+(2n-1)}) \\ &= \rho_{j-(2n-1)} \rho_{j-(2n-2)} \cdots \rho_j \end{aligned}$$

for all $j$ even.

Fix $n \geq 1$. Each monomial in $f^{e/o}_{n,d_n}$ is bounded by the absolute value of one of the $\alpha$'s involved, and there are $4^n$ such monomial terms in each sum. Putting all of this together, we get that, for all $j$ even, we have

$$|\mathcal{E}^n_{j,j}|, \; |\mathcal{E}^n_{j+1,j+1}| \leq 4^n (|\alpha_{j-(2n-1)}| + \cdots + |\alpha_{j+2n-1}|).$$

The second observation we need to make in order to conclude the convergence of the series (5.1) concerns what changes in all of these formulae when we introduce a boundary condition $\alpha_{-1} = -1$.

From the discussion above, we see that actually

$$\mathcal{C}_{j,k}^n = \mathcal{E}_{j,k}^n$$

for $j, k \geq 4n$, as these entries only depend on $\alpha$'s with positive indices. (As we remarked earlier, these bounds are not optimal, but they are certainly sufficient for our purposes.) Hence we also get that

$$\left| \mathcal{C}_{j,j}^n \right|, \left| \mathcal{C}_{j+1,j+1}^n \right| \leq 4^n (|\alpha_{j-(2n-1)}| + \cdots + |\alpha_{j+2n-1}|)$$

for $j \geq 4n$ even. So, since the sequence of $\alpha$'s is in $l^1$, we get that, for any $n \geq 1$, the series (5.1) converges absolutely. $\qquad\square$

We can now define our Hamiltonians as

(5.4) $$K_n^i = K_n^i(\mathcal{C}) = \sum_{k=0}^{\infty} \mathcal{C}_{k,k}^n.$$

They are well-defined by the previous lemma, and, for any fixed $j \geq 0$, only a finite number of terms in the series depends on $\alpha_j$. We can therefore state our main theorem in the infinite case:

**Theorem 4.** *Let $\{\alpha_j\}_{j\geq 0}$ be an $l^1(\mathbb{N})$ sequence of complex numbers inside the unit disk, $\mathcal{C}$ the associated CMV matrix, and $K_n^i$ the function defined by (5.4). Then the Lax pairs associated to these Hamiltonians are given by*

(5.5) $$\{\mathcal{C}, K_n^i\} = [\mathcal{C}, i\mathcal{C}_+^n]$$

*and*

(5.6) $$\{\mathcal{C}, \bar{K}_n^i\} = [\mathcal{C}, i(\mathcal{C}_+^n)^*]$$

*for all $n \geq 1$.*

*Or, in terms of real-valued flows, we have*

(5.7) $$\{\mathcal{C}, 2\operatorname{Re}(K_n^i)\} = [\mathcal{C}, i\mathcal{C}_+^n + i(\mathcal{C}_+^n)^*]$$

*and*

(5.8) $$\{\mathcal{C}, 2\operatorname{Im}(K_n^i)\} = [\mathcal{C}, \mathcal{C}_+^n - (\mathcal{C}_+^n)^*]$$

*for all $n \geq 1$.*

*Proof.* For each fixed $n$ and entry $(j, l)$, there exists a $k$ large enough such that all the entries of $\mathcal{C}$ and $\mathcal{C}^n$ that appear in $(5.5)_{j,l}$ and $(5.6)_{j,l}$ are equal to the entries of $\mathcal{C}_f$ and $\mathcal{C}_f^n$, respectively, in the corresponding finite Lax pairs.

Moreover, since $\mathcal{C}_{j,l}$ depends on two $\alpha$'s, and these appear in only finitely many of the terms in $K_n^i$, the Poisson brackets on the left-hand side are well defined finite sums and equal the corresponding Poisson brackets in the finite case.

Therefore, the results of Theorem 4 follow directly from Theorem 3 and the observations in the proof of Lemma 5.1. $\qquad\square$

**Remark 5.2.** As in the finite case, we define

$$K_0^i = \prod_{j=0}^{\infty} \rho_j^2.$$

Recall that

$$\rho_j^2 = 1 - |\alpha_j|^2 \le 2(1 - |\alpha_j|),$$

and $\{\alpha_j\}_{j\ge 0} \in l^1(\mathbb{N})$. Therefore $K_0^i$ is well-defined and positive; also the following Poisson bracket makes sense

$$\{\alpha_j, \log(K_0^i)\} = -2i\alpha_j.$$

But, as in the finite case, we cannot hope to find a Lax pair representation for the flow generated by $K_0^i$ in terms of $\mathcal{C}$. The dependence of $\sum_{j\ge 0}\{\mathcal{C}_{jj}, K_0^i\}$ on $\bar{\alpha}_0$ is nontrivial, while $\sum_{j\ge 0}[\mathcal{C}, \mathcal{A}]_{jj}$ is identically zero for any infinite matrix $\mathcal{A}$ for which the commutator makes sense.

## Appendix A. Theorem 2: The Full Computations

We prove relation (2.2) for the necessary indices.

First let $k = l$ be even. Then

$$
\begin{aligned}
i\{\mathcal{E}_{kk}, K_{n+1}\} &= \sum_j \rho_j^2 \left[ \frac{\partial(-\alpha_{k-1}\bar{\alpha}_k)}{\partial \alpha_j} \frac{\partial K_{n+1}}{\partial \bar{\alpha}_j} - \frac{\partial(-\alpha_{k-1}\bar{\alpha}_k)}{\partial \bar{\alpha}_j} \frac{\partial K_{n+1}}{\partial \alpha_j} \right] \\
&= -\rho_{k-1}^2 \bar{\alpha}_k \frac{\partial K_{n+1}}{\partial \bar{\alpha}_{k-1}} + \rho_k^2 \alpha_{k-1} \frac{\partial K_{n+1}}{\partial \alpha_k} \\
&= -\rho_{k-1}^2 \bar{\alpha}_k \left[ \rho_{k-2}\mathcal{E}_{k-1,k-2}^n - \alpha_{k-2}\mathcal{E}_{k-1,k-1}^n - \frac{\alpha_{k-1}\rho_{-2}}{2\rho_{k-1}}\mathcal{E}_{k,k-2}^n \right. \\
&\quad \left. - \frac{\alpha_{k-1}\bar{\alpha}_k}{2\rho_{k-1}}\mathcal{E}_{k-1,k}^n - \frac{\alpha_{k-1}\rho_k}{2\rho_{k-1}}\mathcal{E}_{k-1,k+1}^n + \frac{\alpha_{k-1}\alpha_{k-2}}{2\rho_{k-1}}\mathcal{E}_{k,k-1}^n \right] \\
&\quad + \rho_k^2 \alpha_{k-1} \left[ -\frac{\bar{\alpha}_k\bar{\alpha}_{k+1}}{2\rho_k}\mathcal{E}_{k+1,k}^n - \frac{\bar{\alpha}_k\rho_{k+1}}{2\rho_k}\mathcal{E}_{k+2,k}^n - \frac{\bar{\alpha}_k\rho_{k-1}}{2\rho_k}\mathcal{E}_{k-1,k+1}^n \right. \\
&\quad \left. + \frac{\bar{\alpha}_k\alpha_{k-1}}{2\rho_k}\mathcal{E}_{k,k+1}^n - \bar{\alpha}_{k+1}\mathcal{E}_{k+1,k+1}^n - \rho_{k+1}\mathcal{E}_{k+2,k+1}^n \right].
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\left[\mathcal{E}, \mathcal{E}_+^{n+1}\right]_{k,k} &= \mathcal{E}_{k,k-1}\mathcal{E}_{k-1,k}^{n+1} - \mathcal{E}_{k,k+1}^{n+1}\mathcal{E}_{k+1,k} \\
&= \rho_{k-1}\bar{\alpha}_k\mathcal{E}_{k-1,k}^{n+1} + \alpha_{k-1}\rho_k\mathcal{E}_{k,k+1}^{n+1} \\
&= \rho_{k-1}\bar{\alpha}_k \left[ \rho_{k-2}\rho_{k-1}\mathcal{E}_{k-1,k-2}^n - \alpha_{k-2}\rho_{k-1}\mathcal{E}_{k-1,k-1}^n \right. \\
&\quad \left. - \alpha_{k-1}\bar{\alpha}_k\mathcal{E}_{k-1,k}^n - \alpha_{k-1}\rho_k\mathcal{E}_{k-1,k+1}^n \right] \\
&\quad + \alpha_{k-1}\rho_k \left[ \rho_{k-1}\bar{\alpha}_k\mathcal{E}_{k-1,k+1}^n - \alpha_{k-1}\bar{\alpha}_k\mathcal{E}_{k,k+1}^n \right. \\
&\quad \left. + \rho_k\bar{\alpha}_{k+1}\mathcal{E}_{k+1,k+1}^n + \rho_k\rho_{k+1}\mathcal{E}_{k+2,k+1}^n \right].
\end{aligned}
$$

After a few simple manipulations, we find that $i\{\mathcal{E}_{kk}, K_{n+1}\} + \left[\mathcal{E}, \mathcal{E}_+^{n+1}\right]_{kk}$ equals

$$
\begin{aligned}
&\frac{\alpha_{k-1}\bar{\alpha}_k}{2} \left[ \left( \mathcal{E}_{k,k-2}^n \rho_{k-2}\rho_{k-1} - \mathcal{E}_{k,k-1}^n \alpha_{k-2}\rho_{k-1} - \mathcal{E}_{k,k+1}^n \alpha_{k-1}\bar{\alpha}_k \right) \right. \\
&\quad \left. - \left( \rho_{k-1}\bar{\alpha}_k\mathcal{E}_{k-1,k}^n + \rho_k\bar{\alpha}_{k+1}\mathcal{E}_{k+1,k}^n + \rho_k\rho_{k+1}\mathcal{E}_{k+2,k}^n \right) \right] \\
&= \frac{\alpha_{k-1}\bar{\alpha}_k}{2} \left[ \left( \mathcal{E}^n\mathcal{E} \right)_{kk} - \left( \mathcal{E}\mathcal{E}^n \right)_{kk} \right] = 0,
\end{aligned}
$$

which concludes the proof of $(2.2)_{kk}$.

The second case we must consider is $(2.2)_{k,k-1}$ with $k$ even. Again, we look first at

$$
\begin{aligned}
i\{\mathcal{E}_{k,k-1}, K_{n+1}\} &= \sum_j \rho_j^2 \left[ \frac{\partial(\rho_{k-1}\bar{\alpha}_k)}{\partial \alpha_j} \frac{\partial K_{n+1}}{\partial \bar{\alpha}_j} - \frac{\partial(\rho_{k-1}\bar{\alpha}_k)}{\partial \bar{\alpha}_j} \frac{\partial K_{n+1}}{\partial \alpha_j} \right] \\
&= \rho_{k-1}^2 \left[ -\frac{\bar{\alpha}_{k-1}\bar{\alpha}_k}{2\rho_{k-1}} \frac{\partial K_{n+1}}{\partial \bar{\alpha}_{k-1}} + \frac{\alpha_{k-1}\bar{\alpha}_k}{2\rho_{k-1}} \frac{\partial K_{n+1}}{\partial \alpha_{k-1}} \right] - \rho_{k-1}\rho_k^2 \frac{\partial K_{n+1}}{\partial \alpha_k} \\
&= \frac{\rho_{k-1}\bar{\alpha}_k}{2} \left[ \rho_{k-2}\bar{\alpha}_{k-1}\mathcal{E}_{k-1,k-2}^n - \alpha_{k-2}\bar{\alpha}_{k-1}\mathcal{E}_{k-1,k-1}^n \right. \\
&\qquad\qquad \left. - \alpha_{k-1}\bar{\alpha}_k\mathcal{E}_{k,k}^n - \alpha_{k-1}\rho_k\mathcal{E}_{k,k+1}^n \right] \\
&\quad - \rho_{k-1}\rho_k \left[ \rho_k\rho_{k+1}\mathcal{E}_{k+2,k+1}^n + \rho_k\bar{\alpha}_{k+1}\mathcal{E}_{k+1,k+1}^n - \frac{\alpha_{k-1}\bar{\alpha}_k}{2}\mathcal{E}_{k,k+1}^n \right. \\
&\qquad\qquad \left. \frac{\rho_{k-1}\bar{\alpha}_k}{2}\mathcal{E}_{k-1,k+1}^n + \frac{\bar{\alpha}_k\rho_{k+1}}{2}\mathcal{E}_{k+2,k}^n + \frac{\bar{\alpha}_k\bar{\alpha}_{k+1}}{2}\mathcal{E}_{k+1,k}^n \right].
\end{aligned}
$$

On the other hand,

$$
[\mathcal{E}, \mathcal{E}_+^{n+1}]_{k,k-1} = -\rho_{k-1}\rho_k\mathcal{E}_{k,k+1}^{n+1} + \frac{\rho_{k-1}\bar{\alpha}_k}{2}\left( \mathcal{E}_{k-1,k-1}^{n+1} - \mathcal{E}_{kk}^{n+1} \right).
$$

If we write $\mathcal{E}^{n+1} = \mathcal{E}\mathcal{E}^n$ and plug in the appropriate entries in the expression above, we obtain

$$
\begin{aligned}
& i\{\mathcal{E}_{k,k-1}, K_{n+1}\} + [\mathcal{E}, \mathcal{E}_+^{n+1}]_{k,k-1} \\
&= -\frac{\rho_{k-1}\bar{\alpha}_k}{2} \left[ \alpha_{k-1}\rho_k\mathcal{E}_{k,k+1}^n + \rho_k\rho_{k+1}\mathcal{E}_{k+2,k}^n + \rho_k\bar{\alpha}_{k+1}\mathcal{E}_{k+1,k}^n \right. \\
&\qquad\qquad \left. + \rho_{k-1}\bar{\alpha}_k\mathcal{E}_{k-1,k}^n - \rho_{k-2}\rho_{k-1}\mathcal{E}_{k,k-2}^n - \rho_{k-1}\bar{\alpha}_k\mathcal{E}_{k,k-1}^n \right] \\
&= -\frac{\rho_{k-1}\bar{\alpha}_k}{2} \left[ (\mathcal{E}\mathcal{E}^n)_{kk} - (\mathcal{E}^n\mathcal{E})_{kk} \right] = 0,
\end{aligned}
$$

which proves $(2.2)_{k,k-1}$.

Having done these two cases in some detail, we will just present the main steps in the computations for $(2.2)_{k+1,k-1}$ and $(2.2)_{k+1,k}$. A useful observation is that, since both sides of our identities are polynomials in the $\alpha$'s and $\bar{\alpha}$'s, one can more easily identify terms by keeping track of the powers of $\frac{1}{2}$ that occur.

The right-hand side of $(2.2)_{k+1,k-1}$ gives us

$$
[\mathcal{E}, \mathcal{E}_+^{n+1}]_{k+1,k-1} = \frac{\rho_{k-1}\rho_k}{2}\left( \mathcal{E}_{k-1,k-1}^{n+1} - \mathcal{E}_{k+1,k+1}^{n+1} \right).
$$

So these are the terms we want to identify on the left-hand side:

$$i\{\mathcal{E}_{k+1,k-1}, K_{n+1}\} = \sum_j \rho_j^2 \left[ \frac{\partial(\rho_{k-1}\rho_k)}{\partial \alpha_j} \frac{\partial K_{n+1}}{\partial \bar{\alpha}_j} - \frac{\partial(\rho_{k-1}\rho_k)}{\partial \bar{\alpha}_j} \frac{\partial K_{n+1}}{\partial \alpha_j} \right]$$

$$= \frac{\rho_{k-1}\rho_k}{2} \left[ - \mathcal{E}^n_{k-1,k-2}\rho_{k-2}\bar{\alpha}_{k-1} - \mathcal{E}^n_{k-1,k-1}(-\alpha_{k-2}\bar{\alpha}_{k-1}) \right.$$
$$+ (-\alpha_{k-1}\rho_k)\mathcal{E}^n_{k,k+1} - \mathcal{E}^n_{k-1,k}\rho_{k-1}\bar{\alpha}_k$$
$$\left. + (-\alpha_k\bar{\alpha}_{k+1})\mathcal{E}^n_{k+1,k+1} + (-\alpha_k\rho_{k+1})\mathcal{E}^n_{k+2,k+1} \right]$$
$$+ \frac{|\alpha_{k-1}|^2}{4} \left[ \rho_{k-2}\rho_k\mathcal{E}^n_{k,k-2} + \bar{\alpha}_k\rho_k\mathcal{E}^n_{k-1,k} \right.$$
$$+ \rho_k^2\mathcal{E}^n_{k-1,k+1} - \alpha_{k-2}\rho_k\mathcal{E}^n_{k,k-1}$$
$$- \rho_{k-2}\rho_k\mathcal{E}^n_{k,k-2} + \alpha_{k-2}\rho_k\mathcal{E}^n_{k,k-1}$$
$$\left. - \bar{\alpha}_k\rho_k\mathcal{E}^n_{k-1,k} - \rho_k^2\mathcal{E}^n_{k-1,k+1} \right]$$
$$+ \frac{|\alpha_k|^2}{4} \left[ \rho_{k-1}\rho_{k+1}\mathcal{E}^n_{k+2,k} + \rho_{k-1}\bar{\alpha}_{k+1}\mathcal{E}^n_{k-1,k} \right.$$
$$+ \rho_{k-1}^2\mathcal{E}^n_{k-1,k+1} - \alpha_{k-1}\rho_{k-1}\mathcal{E}^n_{k,k+1}$$
$$- \rho_{k-1}\rho_{k+1}\mathcal{E}^n_{k+2,k} - \rho_{k-1}\bar{\alpha}_{k+1}\mathcal{E}^n_{k+1,k}$$
$$\left. - \alpha_{k-1}\rho_{k-1}\mathcal{E}^n_{k,k+1} - \rho_{k-1}^2\mathcal{E}^n_{k-1,k+1} \right]$$
$$= \frac{\rho_{k-1}\rho_k}{2} \left[ - \left(\mathcal{E}^n\mathcal{E}\right)_{k-1,k-1} + \mathcal{E}^n_{k-1,k+1}\mathcal{E}_{k+1,k-1} \right.$$
$$\left. + \left(\mathcal{E}\mathcal{E}^n\right)_{k+1,k+1} - \mathcal{E}_{k+1,k-1}\mathcal{E}^n_{k-1,k+1} \right]$$
$$= -[\mathcal{E}, \mathcal{E}^{n+1}_+]_{k+1,k-1} .$$

Finally, we deal with $(2.2)_{k+1,k}$. As before, we notice that

$$[\mathcal{E}, \mathcal{E}^{n+1}_+]_{k+1,k} = \rho_{k-1}\rho_k\mathcal{E}^{n+1}_{k-1,k} - \frac{\alpha_{k-1}\rho_k}{2}\left(\mathcal{E}^{n+1}_{kk} - \mathcal{E}^{n+1}_{k+1,k+1}\right).$$

The other side of the identity can be transformed as follows:

$$i\{\mathcal{E}_{k+1,k}, K_{n+1}\} = -\sum_j \rho_j^2 \Big[\frac{\partial(\rho_k\alpha_{k-1})}{\partial\alpha_j}\frac{\partial K_{n+1}}{\partial\bar{\alpha}_j} - \frac{\partial(\rho_k\alpha_{k-1})}{\partial\bar{\alpha}_j}\frac{\partial K_{n+1}}{\partial\alpha_j}\Big]$$

$$= -\rho_{k-1}\rho_k\Big[\mathcal{E}^n_{k-1,k-2}\rho_{k-2}\rho_{k-1} + \mathcal{E}^n_{k-1,k-1}(-\alpha_{k-2}\rho_{k-1})\Big]$$

$$+\frac{\alpha_{k-1}\rho_k}{2}\Big[\rho_{k-2}\rho_{k-1}\mathcal{E}^n_{k,k-2} + \rho_{k-1}\bar{\alpha}_k\mathcal{E}^n_{k-1,k}$$

$$+ \rho_{k-1}\rho_k\mathcal{E}^n_{k-1,k+1} - \alpha_{k-2}\rho_{k-1}\mathcal{E}^N_{k,k-1}$$

$$+ \rho_{k-1}\bar{\alpha}_k\mathcal{E}^n_{k-1,k} - \alpha_{k-1}\bar{\alpha}_k\mathcal{E}^n_{k,k}$$

$$+ \alpha_k\bar{\alpha}_{k+1}\mathcal{E}^n_{k+1,k+1} + \alpha_k\rho_{k+1}\mathcal{E}^n_{k+2,k+1}\Big]$$

$$+\frac{\alpha_{k-1}|\alpha_k|^2}{4}\Big[-\bar{\alpha}_{k+1}\mathcal{E}^n_{k+1,k} - \rho_{k+1}\mathcal{E}^n_{k+2,k} - \rho_{k-1}\mathcal{E}^n_{k-1,k+1}$$

$$+ \alpha_{k-1}\mathcal{E}^n_{k,k+1} + \bar{\alpha}_{k+1}\mathcal{E}^n_{k+1,k} + \rho_{k+1}\mathcal{E}^n_{k+2,k}$$

$$+ \rho_{k-1}\mathcal{E}^n_{k-1,k+1} - \alpha_{k-1}\mathcal{E}^n_{k,k+1}\Big]$$

$$= -\rho_{k-1}\rho_k\Big[\mathcal{E}^n_{k-1,k-2}\mathcal{E}_{k-2,k} + \mathcal{E}^n_{k-1,k-1}\mathcal{E}_{k-1,k}\Big]$$

$$+\frac{\alpha_{k-1}\rho_k}{2}\Big[\mathcal{E}^n_{k,k-2}\mathcal{E}_{k-2,k} + 2\rho_{k-1}\bar{\alpha}_k\mathcal{E}^n_{k-1,k}$$

$$- \mathcal{E}_{k+1,k-1}\mathcal{E}^n_{k-1,k+1} + 2\rho_{k-1}\rho_k\mathcal{E}^n_{k-1,k+1}$$

$$+ \mathcal{E}^n_{k,k-1}\mathcal{E}_{k-1,k} + \mathcal{E}^n_{k,k}\mathcal{E}_{k,k}$$

$$- \mathcal{E}_{k+1,k+1}\mathcal{E}^n_{k+1,k+1} - \mathcal{E}_{k+1,k+2}\mathcal{E}^n_{k+2,k+1}\Big]$$

$$= -\rho_{k-1}\rho_k(\mathcal{E}^n\mathcal{E})_{k-1,k} + \frac{\alpha_{k-1}\rho_k}{2}\Big[(\mathcal{E}^n\mathcal{E})_{k,k} - (\mathcal{E}\mathcal{E}^n)_{k+1,k+1}\Big]$$

$$= -[\mathcal{E}, \mathcal{E}^{n+1}_+]_{k+1,k}\,.$$

This concludes the proof of $(2.2)_{k+1,k}$, and hence of relation (2.2).

The second part of the proof deals with relation (2.3):

$$\{\mathcal{E}, \bar{K}_{n+1}\} = [\mathcal{E}, i(\mathcal{E}^{n+1}_+)^*].$$

We shall proceed in very much the same way as with (2.2), while incorporating the necessary computational adjustments.

Again, we only have to check four relations; the only difference is that in this case $(2.3)_{k+1,k+1}$, $(2.3)_{k,k+1}$, $(2.3)_{k,k+2}$, and $(2.3)_{k+1,k+2}$ turn out to be computationally easier to verify.

As before, we start with the diagonal entry, $(2.3)_{k+1,k+1}$, that we shall prove in some detail.

We start by analyzing the left-hand side and observing that

$$
i\{\mathcal{E}_{k+1,k+1}, \bar{K}_{n+1}\} = \sum_j \rho_j^2 \left[ \frac{\partial(-\alpha_k \bar{\alpha}_{k+1})}{\partial \alpha_j} \frac{\partial \bar{K}_{n+1}}{\partial \bar{\alpha}_j} - \frac{\partial(-\alpha_k \bar{\alpha}_{k+1})}{\partial \bar{\alpha}_j} \frac{\partial \bar{K}_{n+1}}{\partial \alpha_j} \right]
$$

$$
= -\rho_k^2 \bar{\alpha}_{k+1} \frac{\partial \bar{K}_{n+1}}{\partial \bar{\alpha}_k} + \rho_{k+1}^2 \alpha_k \frac{\partial \bar{K}_{n+1}}{\partial \alpha_{k+1}}.
$$

So by taking the complex conjugate in this relation, we get that

$$
\overline{i\{\mathcal{E}_{k+1,k+1}, \bar{K}_{n+1}\}} = -\rho_k^2 \alpha_{k+1} \frac{\partial K_{n+1}}{\partial \alpha_k} + \rho_{k+1}^2 \bar{\alpha}_k \frac{\partial K_{n+1}}{\partial \bar{\alpha}_{k+1}}
$$

$$
= \rho_k \alpha_{k+1} \Big[ \rho_k \bar{\alpha}_{k+1} \mathcal{E}_{k+1,k+1}^n + \rho_k \rho_{k+1} \mathcal{E}_{k+2,k+1}^n
$$

$$
+ \frac{\bar{\alpha}_k \bar{\alpha}_{k+1}}{2} \mathcal{E}_{k+1,k}^n + \frac{\bar{\alpha}_k \rho_{k+1}}{2} \mathcal{E}_{k+2,k}^n
$$

$$
+ \frac{\bar{\alpha}_k \rho_{k-1}}{2} \mathcal{E}_{k-1,k+1}^n - \frac{\alpha_{k-1} \bar{\alpha}_k}{2} \mathcal{E}_{k,k+1}^n \Big]
$$

$$
+ \bar{\alpha}_k \rho_{k+1} \Big[ \rho_k \rho_{k+1} \mathcal{E}_{k+1,k}^n - \alpha_k \rho_{k+1} \mathcal{E}_{k+1,k+1}^n
$$

$$
- \frac{\rho_k \alpha_{k+1}}{2} \mathcal{E}_{k+2,k}^n - \frac{\alpha_{k+1} \bar{\alpha}_{k+2}}{2} \mathcal{E}_{k+1,k+2}^n
$$

$$
- \frac{\alpha_{k+1} \rho_{k+2}}{2} \mathcal{E}_{k+1,k+3}^n + \frac{\alpha_k \alpha_{k+1}}{2} \mathcal{E}_{k+2,k+1}^n \Big].
$$

On the other hand, we have that

$$
\overline{[\mathcal{E}, (\mathcal{E}_+^{n+1})^*]_{k+1,k+1}} = \sum_{k-1 \le j \le k+2} \overline{\mathcal{E}_{k+1,j}} (\mathcal{E}_+^{n+1})_{k+1,j} - \sum_{k \le j \le k+3} (\mathcal{E}_+^{n+1})_{j,k+1} \overline{\mathcal{E}_{j,k+1}}
$$

$$
= \overline{\mathcal{E}_{k+1,k+2}} \mathcal{E}_{k+1,k+2}^{n+1} - \mathcal{E}_{k,k+1}^{n+1} \overline{\mathcal{E}_{k,k+1}}
$$

$$
= -\bar{\alpha}_k \rho_{k+1} \mathcal{E}_{k+1,k+2}^{n+1} - \rho_k \alpha_{k+1} \mathcal{E}_{k,k+1}^{n+1}.
$$

Notice that

$$\overline{i\{\mathcal{E}_{k+1,k+1}, \bar{K}_{n+1}\}} = -\rho_k^2 \alpha_{k+1} \frac{\partial K_{n+1}}{\partial \alpha_k} + \rho_{k+1}^2 \bar{\alpha}_k \frac{\partial K_{n+1}}{\partial \bar{\alpha}_{k+1}}$$

$$= \rho_k \alpha_{k+1} \Big[ (\mathcal{E} \cdot \mathcal{E}^n)_{k,k+1} - \rho_{k-1} \bar{\alpha}_k \mathcal{E}^n_{k-1,k+1} + \alpha_{k-1} \bar{\alpha}_k \mathcal{E}^n_{k,k+1}$$

$$+ \frac{\bar{\alpha}_k \bar{\alpha}_{k+1}}{2} \mathcal{E}^n_{k+1,k} + \frac{\bar{\alpha}_k \rho_{k+1}}{2} \mathcal{E}^n_{k+2,k}$$

$$+ \frac{\bar{\alpha}_k \rho_{k-1}}{2} \mathcal{E}^n_{k-1,k+1} - \frac{\alpha_{k-1} \bar{\alpha}_k}{2} \mathcal{E}^n_{k,k+1} \Big]$$

$$+ \bar{\alpha}_k \rho_{k+1} \Big[ (\mathcal{E}^n \cdot \mathcal{E})_{k+1,k+2} + \alpha_{k+1} \bar{\alpha}_{k+2} \mathcal{E}^n_{k+1,k+2} + \alpha_{k+1} \rho_{k+2} \mathcal{E}^n_{k+1,k+3}$$

$$- \frac{\rho_k \alpha_{k+1}}{2} \mathcal{E}^n_{k+2,k} - \frac{\alpha_{k+1} \bar{\alpha}_{k+2}}{2} \mathcal{E}^n_{k+1,k+2}$$

$$- \frac{\alpha_{k+1} \rho_{k+2}}{2} \mathcal{E}^n_{k+1,k+3} + \frac{\alpha_k \alpha_{k+1}}{2} \mathcal{E}^n_{k+2,k+1} \Big].$$

Therefore, we get that $\overline{i\{\mathcal{E}_{k+1,k+1}, \bar{K}_{n+1}\}} + [\mathcal{E}, (\mathcal{E}_+^{n+1})^*]_{k+1,k+1}$ equals

$$\frac{\bar{\alpha}_k \alpha_{k+1}}{2} \Big[ -\rho_{k-1} \rho_k \mathcal{E}^n_{k-1,k+1} + \alpha_{k-1} \rho_k \mathcal{E}^n_{k,k+1} + \alpha_k \rho_{k+1} \mathcal{E}^n_{k+2,k+1}$$

$$+ \rho_k \bar{\alpha}_{k+1} \mathcal{E}^n_{k+1,k} + \rho_{k+1} \bar{\alpha}_{k+2} \mathcal{E}^n_{k+1,k+2} + \rho_{k+1} \rho_{k+2} \mathcal{E}^n_{k+1,k+3} \Big]$$

$$= \frac{\bar{\alpha}_k \alpha_{k+1}}{2} \Big[ -(\mathcal{E} \cdot \mathcal{E}^n)_{k+1,k+1} + (\mathcal{E}^n \cdot \mathcal{E})_{k+1,k+1} \Big] = 0,$$

which ends the proof of $(2.3)_{k+1,k+1}$.

We now turn to $(2.3)_{k,k+1}$:

$$\overline{[\mathcal{E}, (\mathcal{E}_+^{n+1})^*]}_{k,k+1} = \rho_k \rho_{k+1} \mathcal{E}^{n+1}_{k+1,k+2} + \frac{\rho_k \alpha_{k+1}}{2} \Big( \mathcal{E}^{n+1}_{k+1,k+1} - \mathcal{E}^{n+1}_{k,k} \Big).$$

The left-hand side becomes

$$\overline{i\{\mathcal{E}_{k,k+1}, \bar{K}_{n+1}\}} = \overline{i\{\rho_k \bar{\alpha}_{k+1}, \bar{K}_{n+1}\}}$$

$$= -\rho_k \rho_{k+1} \Big( \mathcal{E}^n_{k+1,k} \mathcal{E}_{k,k+2} + \mathcal{E}^n_{k+1,k+1} \mathcal{E}_{k+1,k+2} \Big)$$

$$- \frac{\rho_k \alpha_{k+1}}{2} \Big( -\mathcal{E}_{k,k+2} \mathcal{E}^n_{k+2,k} + \mathcal{E}^n_{k+1,k+3} \mathcal{E}_{k+3,k+1}$$

$$- 2\mathcal{E}^n_{k+1,k+3} \rho_{k+1} \rho_{k+2} + \mathcal{E}^n_{k+1,k+2} \mathcal{E}_{k+2,k+1}$$

$$- 2\mathcal{E}^n_{k+1,k+2} \rho_{k+1} \bar{\alpha}_{k+2} - \mathcal{E}_{k,k-1} \mathcal{E}^n_{k-1,k}$$

$$- \mathcal{E}_{k,k} \mathcal{E}^n_{k,k} + \mathcal{E}^n_{k+1,k+1} \mathcal{E}_{k+1,k+1} \Big)$$

$$= -\rho_k \rho_{k+1} (\mathcal{E}^n \mathcal{E})_{k,k+1} - \frac{\rho_k \alpha_{k+1}}{2} \Big( (\mathcal{E} \mathcal{E}^n)_{k,k} + (\mathcal{E}^n \mathcal{E})_{k+1,k+1} \Big).$$

So we find what we wanted:

$$\{\mathcal{E}_{k,k+1}, \bar{K}_{n+1}\} = i[\mathcal{E}, (\mathcal{E}_+^{n+1})^*]_{k,k+1}.$$

The next entry that we analyze is $(2.3)_{k,k+2}$. Considering the right-hand side first, we get

$$\overline{[\mathcal{E}, (\mathcal{E}_+^{n+1})^*]_{k,k+2}} = \sum_j \overline{\mathcal{E}_{k,j}} \cdot (\mathcal{E}_+^{n+1})_{k+2,j} - \sum_j (\mathcal{E}_+^{n+1})_{j,k} \cdot \overline{\mathcal{E}_{j,k+2}}$$

$$= \frac{\rho_k \rho_{k+1}}{2} \left( \mathcal{E}_{k+2,k+2}^{n+1} - \mathcal{E}_{k,k}^{n+1} \right).$$

If we look at the left-hand side now, we get

$$\overline{i\{\mathcal{E}_{k,k+2}, \bar{K}_{n+1}\}} = \overline{i\{\rho_k \rho_{k+1}, \bar{K}_{n+1}\}}$$

$$= \rho_k^2 \left[ -\frac{\alpha_k \rho_{k+1}}{2\rho_k} \cdot \frac{\partial K_{n+1}}{\partial \alpha_k} + \frac{\bar{\alpha}_k \rho_{k+1}}{2\rho_k} \cdot \frac{\partial K_{n+1}}{\partial \bar{\alpha}_k} \right]$$

$$+ \rho_{k+1}^2 \left[ -\frac{\alpha_{k+1} \rho_k}{2\rho_{k+1}} \cdot \frac{\partial K_{n+1}}{\partial \alpha_{k+1}} + \frac{\bar{\alpha}_{k+1} \rho_k}{2\rho_{k+1}} \cdot \frac{\partial K_{n+1}}{\partial \bar{\alpha}_{k+1}} \right]$$

$$= \frac{\rho_k \rho_{k+1}}{2} \left[ \alpha_k \rho_{k+1} \mathcal{E}_{k+2,k+1}^n + \bar{\alpha}_k \rho_{k-1} \mathcal{E}_{k-1,k}^n \right.$$

$$- \alpha_{k-1} \bar{\alpha}_k \mathcal{E}_{k,k}^n + \alpha_{k+1} \bar{\alpha}_{k+2} \mathcal{E}_{k+2,k+2}^n$$

$$\left. + \alpha_{k+1} \rho_{k+2} \mathcal{E}_{k+2,k+3}^n + \bar{\alpha}_{k+1} \rho_k \mathcal{E}_{k+1,k}^n \right]$$

$$+ \frac{|\alpha_k|^2 \rho_{k+1}}{4} \left[ \bar{\alpha}_{k+1} \mathcal{E}_{k+1,k}^n + \rho_{k+1} \mathcal{E}_{k+2,k}^n \right.$$

$$+ \rho_{k-1} \mathcal{E}_{k-1,k+1}^n - \alpha_{k-1} \mathcal{E}_{k,k+1}^n$$

$$- \bar{\alpha}_{k+1} \mathcal{E}_{k+1,k}^n - \rho_{k+1} \mathcal{E}_{k+2,k}^n$$

$$\left. - \rho_{k-1} \mathcal{E}_{k-1,k+1}^n + \alpha_{k-1} \mathcal{E}_{k,k+1}^n \right]$$

$$+ \frac{|\alpha_{k+1}|^2 \rho_k}{4} \left[ \rho_k \mathcal{E}_{k+2,k}^n - \alpha_k \mathcal{E}_{k+2,k+1}^n \right.$$

$$+ \bar{\alpha}_{k+2} \mathcal{E}_{k+1,k+2}^n + \rho_{k+2} \mathcal{E}_{k+1,k+3}^n$$

$$- \rho_k \mathcal{E}_{k+2,k}^n + \alpha_k \mathcal{E}_{k+2,k+1}^n$$

$$\left. - \bar{\alpha}_{k+2} \mathcal{E}_{k+1,k+2}^n - \rho_{k+2} \mathcal{E}_{k+1,k+3}^n \right]$$

$$= \frac{\rho_k \rho_{k+1}}{2} \left[ (\mathcal{E}\mathcal{E}^n)_{k,k} - \mathcal{E}_{k,k+2} \mathcal{E}_{k+2,k}^n \right.$$

$$\left. - (\mathcal{E}^n \mathcal{E})_{k+2,k+2} + \mathcal{E}_{k+2,k}^n \mathcal{E}_{k,k+2} \right]$$

$$= -\overline{[\mathcal{E}, (\mathcal{E}_+^{n+1})^*]_{k,k+2}},$$

which immediately implies the equation $(2.3)_{k,k+2}$.

Finally, we turn to the last relation we have to prove, equation $(2.3)_{k+1,k+2}$. As above, we start with the right-hand side and observe

that

$$\overline{[\mathcal{E}, (\mathcal{E}_+^n)^*]}_{k+1,k+2} = \frac{\bar{\alpha}_k \rho_{k+1}}{2} \left( \mathcal{E}_{k+1,k+1}^{n+1} - \mathcal{E}_{k+2,k+2}^{n+1} \right) - \rho_k \rho_{k+1} \mathcal{E}_{k,k+1}^{n+1}.$$

Considering the left-hand side now, we have

$$\frac{}{i\{\mathcal{E}_{k+1,k+2}, \bar{K}_{n+1}\}} = - \rho_k^2 \rho_{k+1} \frac{\partial K_{n+1}}{\partial \alpha_k}$$

$$- \bar{\alpha}_k \rho_{k+1}^2 \left[ - \frac{\alpha_{k+1}}{2\rho_{k+1}} \frac{\partial K_{n+1}}{\partial \alpha_{k+1}} + \frac{\bar{\alpha}_{k+1}}{2\rho_{k+1}} \frac{\partial K_{n+1}}{\partial \bar{\alpha}_{k+1}} \right]$$

$$= \rho_k \rho_{k+1} \left[ \rho_k \bar{\alpha}_{k+1} \mathcal{E}_{k+1,k+1}^n + \rho_k \rho_{k+1} \mathcal{E}_{k+2,k+1}^n \right]$$

$$- \frac{\bar{\alpha}_k \rho_{k+1}}{2} \left[ - \rho_k \bar{\alpha}_{k+1} \mathcal{E}_{k+1,k}^n - \rho_k \rho_{k+1} \mathcal{E}_{k+2,k}^n \right.$$

$$- \rho_k \rho_{k-1} \mathcal{E}_{k-1,k+1}^n + \alpha_{k-1} \rho_k \mathcal{E}_{k,k+1}^n$$

$$+ \alpha_{k+1} \bar{\alpha}_{k+2} \mathcal{E}_{k+2,k+2}^n + \alpha_{k+1} \rho_{k+2} \mathcal{E}_{k+2,k+3}^n$$

$$\left. + \bar{\alpha}_{k+1} \rho_k \mathcal{E}_{k+1,k}^n - \alpha_k \bar{\alpha}_{k+1} \mathcal{E}_{k+1,k+1}^n \right]$$

$$- \frac{\bar{\alpha}_k |\alpha_{k+1}|^2}{4} \left[ \rho_k \mathcal{E}_{k+2,k}^n - \alpha_k \mathcal{E}_{k+2,k+1}^n \right.$$

$$+ \bar{\alpha}_{k+2} \mathcal{E}_{k+1,k+2}^n + \rho_{k+2} \mathcal{E}_{k+1,k+3}^n$$

$$- \rho_k \mathcal{E}_{k+2,k}^n - \bar{\alpha}_{k+2} \mathcal{E}_{k+1,k+2}^n$$

$$\left. - \rho_{k+2} \mathcal{E}_{k+1,k+3}^n + \alpha_k \mathcal{E}_{k+2,k+1}^n \right]$$

$$= \rho_k \rho_{k+1} \left[ \rho_k \bar{\alpha}_{k+1} \mathcal{E}_{k+1,k+1}^n + \rho_k \rho_{k+1} \mathcal{E}_{k+2,k+1}^n \right]$$

$$- \frac{\bar{\alpha}_k \rho_{k+1}}{2} \left[ - (\mathcal{E}^n \mathcal{E})_{k+2,k+2} + (\mathcal{E} \mathcal{E}^n)_{k+1,k+1} \right.$$

$$\left. - 2\rho_{k-1} \rho_k \mathcal{E}_{k-1,k+1}^n + 2\alpha_{k-1} \rho_k \mathcal{E}_{k,k+1}^n \right]$$

$$= \rho_k \rho_{k+1} \mathcal{E}_{k,k+1}^{n+1} - \frac{\bar{\alpha}_k \rho_{k+1}}{2} \left( \mathcal{E}_{k+1,k+1}^{n+1} - \mathcal{E}_{k+2,k+2}^{n+1} \right)$$

$$= -\overline{[\mathcal{E}, (\mathcal{E}_+^n)^*]}_{k+1,k+2} ,$$

which implies that

$$i\{\mathcal{E}_{k+1,k+2}, \bar{K}_{n+1}\} = -[\mathcal{E}, (\mathcal{E}_+^n)^*]_{k+1,k+2},$$

and hence $(2.3)_{k+1,k+2}$ holds.

## APPENDIX B. BACKGROUND: ORTHOGONAL POLYNOMIALS ON THE UNIT CIRCLE

In this Appendix we present some of the basic notions and results related to the theory of orthogonal polynomials on the unit circle. The

reader interested in more details can check Szegő's classical book [15]. In our presentation, we follow the upcoming two-volume treatise by Simon [13, 14].

Let us first recall the definition of the Verblunsky coefficients. Consider a probability measure $d\mu$ on $S^1$ which is supported at infinitely many points. By applying the Gram-Schmidt procedure to $1, z, z^2, \ldots$, one obtains the monic orthogonal polynomials $\{\Phi_n(z)\}_{n \geq 0}$ and the orthonormal polynomials

$$\phi_n(z) = \frac{\Phi_n(z)}{\|\Phi_n\|_{L^2(d\mu)}}.$$

These polynomials obey recurrence relations

(B.1)                $$\Phi_{k+1}(z) = z\Phi_k(z) - \bar{\alpha}_k \Phi_k^*(z),$$

(B.2)                $$\Phi_{k+1}^*(z) = \Phi_k^*(z) - \alpha_k z\Phi_k(z),$$

where the $\alpha_k$'s are the recurrence coefficients and $\Phi_k^*$ denotes the reversed polynomial:

(B.3)        $$\Phi_k(z) = \sum_{l=0}^{k} c_l z^l \quad \Rightarrow \quad \Phi_k^*(z) = \sum_{l=0}^{k} \bar{c}_{k-l} z^l.$$

Equivalently, $\Phi_k^*(z) = z^k \overline{\Phi_k(\bar{z}^{-1})}$. These recurrence equations imply

(B.4)        $$\|\Phi_k\|_{L^2(d\mu)} = \prod_{l=0}^{k-1} \rho_l \quad \text{where} \quad \rho_l = \sqrt{1 - |\alpha_l|^2},$$

from which the recurrence relations for the orthonormal polynomials are easily derived. The recurrence coefficients $\alpha_k$ are called Verblunsky coefficients and lie in the (open) unit disk $\mathbb{D}$.

The recursion relations for the orthonormal polynomials can be summarized as

$$\begin{bmatrix} \phi_n(z) \\ \phi_n^*(z) \end{bmatrix} = A(\alpha_{n-1}, z) \begin{bmatrix} \phi_{n-1}(z) \\ \phi_{n-1}^*(z) \end{bmatrix},$$

where

$$A(\alpha_k, z) = \frac{1}{\rho_k} \begin{bmatrix} z & -\bar{\alpha}_k \\ -\alpha_k z & 1 \end{bmatrix}.$$

We define the transfer matrix

(B.5)                $$T_n(z) = A(\alpha_{n-1}, z) \ldots A(\alpha_0, z)$$

for all $n \geq 1$; hence

$$\begin{bmatrix} \phi_n(z) \\ \phi_n^*(z) \end{bmatrix} = T_n(z) \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Consider the operator $f(z) \mapsto zf(z)$ in $L^2(d\mu)$. We want to represent this operator as a matrix. The most obvious choice of an orthonormal set of vectors in $L^2(d\mu)$ are the orthonormal polynomials, $\{\phi_n\}_{n\geq 0}$. This leads to a matrix whose entries can be expressed simply in terms of the $\alpha$'s. However, this matrix is typically not sparse: All entries above and including the sub-diagonal are non-zero; it is also unclear how to extend this matrix to a doubly-infinite matrix, which will turn out to be very important when we consider the case of periodic Verblunsky coefficients. Moreover, $\{\phi_n\}_{n\geq 0}$ is a basis in $L^2(d\mu)$ if and only if $\{\alpha_j\}_{j\geq 0}$ are not in $l^2(\mathbb{N})$.

An alternate approach, due to Cantero, Moral, and Velázquez [5], consists of defining two bases in $L^2(d\mu)$. Applying the Gram–Schmidt procedure to

$$1, z, z^{-1}, z^2, z^{-2}, \ldots$$

in $L^2(d\mu)$ produces the orthonormal basis

(B.6) $$\chi_k(z) = \begin{cases} z^{-k/2}\phi_k^*(z), & k \text{ even}; \\ z^{(1-k)/2}\phi_k(z), & k \text{ odd}, \end{cases}$$

where $k \geq 0$. As above, $\phi_k$ denotes the $k^{\text{th}}$ orthonormal polynomial and $\phi_k^*$, its reversal (cf. (B.3)). If we apply the procedure to $1, z^{-1}, z, z^{-2}, z^2, \ldots$, instead, then we obtain a second orthonormal basis:

(B.7) $$x_k(z) = \overline{\chi_k(1/\bar{z})} = \begin{cases} z^{-k/2}\phi_k(z), & k \text{ even}; \\ z^{(-1-k)/2}\phi_k^*(z), & k \text{ odd}. \end{cases}$$

It is natural to compute the matrix representation of $f(z) \mapsto zf(z)$ in $L^2(d\mu)$ with respect to these bases. The matrices with entries

$$\mathcal{L}_{i+1,j+1} = \langle \chi_i(z)|zx_j(z)\rangle \quad \text{and} \quad \mathcal{M}_{i+1,j+1} = \langle x_i(z)|\chi_j(z)\rangle$$

are block-diagonal; indeed,

(B.8) $$\mathcal{L} = \operatorname{diag}(\Theta_0, \Theta_2, \Theta_4, \ldots) \quad \text{and} \quad \mathcal{M} = \operatorname{diag}([1], \Theta_1, \Theta_3, \ldots),$$

where

$$\Theta_k = \begin{bmatrix} \bar{\alpha}_k & \rho_k \\ \rho_k & -\alpha_k \end{bmatrix}.$$

The representation of $f(z) \mapsto zf(z)$ in the $\{\chi_j\}$ basis is just

$$
\mathcal{C} = \mathcal{L}\mathcal{M} = \begin{pmatrix}
\bar{\alpha}_0 & \rho_0\bar{\alpha}_1 & \rho_0\rho_1 & 0 & 0 & \dots \\
\rho_0 & -\alpha_0\bar{\alpha}_1 & -\alpha_0\rho_1 & 0 & 0 & \dots \\
0 & \rho_1\bar{\alpha}_2 & -\alpha_1\bar{\alpha}_2 & \rho_2\bar{\alpha}_3 & \rho_2\rho_3 & \dots \\
0 & \rho_1\rho_2 & -\alpha_1\rho_2 & -\alpha_2\bar{\alpha}_3 & -\alpha_2\rho_3 & \dots \\
0 & 0 & 0 & \rho_3\bar{\alpha}_4 & -\alpha_3\bar{\alpha}_4 & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots
\end{pmatrix},
$$

which is called the CMV matrix, and in the $\{x_j\}$ basis, it is $\tilde{\mathcal{C}} = \mathcal{M}\mathcal{L}$. Let us note here that throughout the paper we index rows and columns of matrices starting with 0: for example $\mathcal{L}_{jj} = \bar{\alpha}_j$ for all $j \geq 0$. The (infinite) CMV matrix $\mathcal{C}$ is the matrix that we use in Section 4 to define Lax pairs for the flows generated by the Ablowitz-Ladik Hamiltonians on the coefficients $\alpha_j$, $j \geq 0$.

If the probability measure $\mu$ on the circle is supported at $k-1$ points, then we can define as above the orthogonal polynomials $\{\Phi_n(z)\}_{0 \leq n \leq k-2}$ and the corresponding orthonormal polynomials $\{\phi_n(z)\}_{0 \leq n \leq k-2}$. They still obey the same recurrence relations, which allow us to identify the Verblunsky coefficients $\alpha_0, \dots, \alpha_{k-2} \in \mathbb{D}$ and $\alpha_{k-1} \in S^1$. If, as in the infinite case, we represent the operator of multiplication by $z$ in the basis considered by Cantero, Moral, and Velázquez we obtain a finite CMV matrix

$$
\mathcal{C}_f = \mathcal{L}_f\mathcal{M}_f.
$$

Note that, since $|\alpha_{k-1}| = 1$,

$$
\Theta_{k-1} = \begin{bmatrix} \bar{\alpha}_{k-1} & 0 \\ 0 & -\alpha_{k-1} \end{bmatrix}
$$

decomposes as the direct sum of two $1 \times 1$ matrices. Hence, if we replace $\Theta_{k-1}$ by the $1 \times 1$ matrix that is its top left entry, $\bar{\alpha}_{k-1}$, and discard all $\Theta_m$ with $m \geq k$, we find that $\mathcal{L}_f$ and $\mathcal{M}_f$ are naturally $k \times k$ block-diagonal matrices. As in the infinite case, the finite CMV matrix $\mathcal{C}_f$ allows us to recast the Ablowitz-Ladik hierarchy of equations in Lax pair form.

Now we turn to the case of periodic Verblunsky coefficients. If the $\alpha$'s are periodic with period $p$ even, that is, they obey $\alpha_{j+p} = \alpha_j$ for all $j \geq 0$, then we can define a two-sided infinite sequence of coefficients by periodicity. The extended CMV matrix is

(B.9)                          $$\mathcal{E} = \tilde{\mathcal{L}}\tilde{\mathcal{M}},$$

where

(B.10)          $$\tilde{\mathcal{L}} = \bigoplus_{j \text{ even}} \Theta_j \qquad \text{and} \qquad \tilde{\mathcal{M}} = \bigoplus_{j \text{ odd}} \Theta_j,$$

with $\Theta_j$ defined on $l^2(\mathbb{Z})$ by

$$\Theta_j = \begin{bmatrix} \bar{\alpha}_j & \rho_j \\ \rho_j & -\alpha_j \end{bmatrix}$$

on the span of $\delta_j$ and $\delta_{j+1}$, and identically 0 otherwise. The extended CMV matrix $\mathcal{E}$ will play an important role in determining the Lax pairs associated to the Hamiltonian flows of the periodic Ablowitz-Ladik system.

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IRINA NENCIU, MATHEMATICS 253-37, CALTECH, PASADENA, CA 91125
*E-mail address*: nenciu@caltech.edu