Prevalence of exponential stability among nearly-integrable Hamiltonian systems

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Abstract : In the 70's, Nekhorochev proved that for an analytic nearly integrable Hamiltonian system, the action variables of the unperturbed Hamiltonian remain nearly constant over an exponentially long time with respect to the size of the perturbation, provided that the unperturbed Hamiltonian satisfies some generic transversality condition known as *steepness*. Recently, Guzzo has given examples of exponentially stable integrable Hamiltonians which are non steep but satisfy a weak condition of transversality which involves only the affine subspaces spanned by integer vectors.

We generalize this notion for an arbitrary integrable Hamiltonian and prove the Nekhorochev's estimates in this setting. The point in this refinement lies in the fact that we can exhibit a *generic* class of real analytic integrable Hamiltonians which are exponentially stable with *fixed* exponents.

Genericity is proved in the sense of measure since we exhibit a prevalent set of integrable Hamiltonian which satisfy the latter property. This is obtained by an application of a quantitative Sard theorem given by Yomdin.

Key words : Hamiltonian systems – Stability – Prevalence – Quantitative Morse-Sard's theory.

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I Introduction :

One of the main problem in Hamiltonian dynamic is the stability of motions in nearly-integrable systems (for example : the n-body planetary problem). The main tool of investigation is the construction of normal forms (see Giorgilli [4] for an introduction and a survey about these topics).

This yields two kinds of theorems :

i) Results of stability over infinite times provided by K.A.M. theory which are valid for solutions with initial conditions in a Cantor set of large measure but no information is given on the other trajectories. Russmann ([19], see also [2]) has given a minimal non degeneracy condition on the unperturbed Hamiltonian to ensure the persistence of invariant tori under perturbation. Namely, the image of the gradient map associated to the integrable Hamiltonian should not be included in an hyperplane and this condition is generic among real analytic numerical functions.

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ii) On the other hand, Nekhorochev ([13], [14]) have proved global results of stability over open sets of the following type:

Definition I.1. (exponential stability)

Consider an open set $\Omega \subset \mathbb{R}^n$, *an analytic integrable Hamiltonian* $h : \Omega \longrightarrow \mathbb{R}$ *and action-angle variables* $(I, \varphi) \in \Omega \times \mathbb{T}^n$ *with* $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

For an arbitrary $\rho > 0$, *let* $\mathcal{O}_\rho$ *be the space of analytic functions over a complex neighborhood* $\Omega_\rho \subset \mathbb{C}^{2n}$ *of size* ρ *around* $\Omega \times \mathbb{T}^n$ *equipped with the supremum norm* $||.||_\rho$ *over* Ω_ρ.

We say that the Hamiltonian h *is exponentially stable if there exists an open set* $\widetilde{\Omega} \subset \Omega$ *and positive constants* ρ, C_1, C_2, a, b *and* ε_0 *which depend only on* h *and* $\widetilde{\Omega}$ *such that:*

i) $h \in \mathcal{O}_\rho$.

ii) For any function $\mathcal{H} \in \mathcal{O}_\rho$ *such that* $||\mathcal{H} - h||_\rho = \varepsilon < \varepsilon_0$, *an arbitrary solution* $(I(t), \varphi(t))$ *of the Hamiltonian system associated to* $\mathcal{H}$ *with an initial action* $I(t_0) \in \widetilde{\Omega}$ *is defined over a time* $\exp(C_2/\varepsilon^a)$ *and satisfies:*

$$||I(t) - I(t_0)|| \leq C_1 \varepsilon^b \text{ for } |t - t_0| \leq \exp(C_2/\varepsilon^a)$$

a *and* b *are called stability exponents.*

Rk: Along the same lines, the previous definition can be extended to an integrable Hamiltonian in the Gevrey class (see [12]).

Definition I.2. ([13], [14], [16])

Consider an open set Ω *in* $\mathbb{R}^n$, *a real analytic function* $f : \Omega \longrightarrow \mathbb{R}$ *is said to be steep at a point* $I \in \Omega$ *along an affine subspace* Λ *which contains* I *if there are constants* $C > 0$, $\delta > 0$ *and* $p > 0$ *such that along any continuous curve* Γ *in* Λ *connecting* I *to a point at a distance* $d < \delta$, *the norm of the projection of the gradient* $\nabla f(x)$ *onto the direction of* Λ *is greater than* Cd^p *at some point;* (C, δ) *and* p *are respectively called the steepness coefficients and the steepness index.*

Under the previous assumptions, the function f *is said to be steep at the point* $I \in \Omega$ *if, for every* $m \in \{1, \ldots, n-1\}$, *there exist positive constants* C_m, δ_m *and* p_m *such that* f *is steep at* I *along any affine subspace of dimension* m *containing* I *uniformly with respect to the coefficients* (C_m, δ_m) *and the index* p_m.

Finally, a real analytic function f *is steep over a domain* $\mathcal{P} \subseteq \mathbb{R}^n$ *with the steepness coefficients* $(C_1, \ldots, C_{n-1}, \delta_1, \ldots, \delta_{n-1})$ *and the steepness indices* $(p_1, \ldots, p_{n-1})$ *if there are no critical points for* f *in* $\mathcal{P}$ *and* f *is steep at any point* $I \in \mathcal{P}$ *uniformly with respect to these coefficients and indices.*

For instance, convex functions are steep with all the steepness indices equal to one. On the other hand, $f(x, y) = x^2 - y^2$ is a typical non steep function but by adding a third order term (e.g. y^3) we recover steepness. Moreover, this definition is minimal since a function can be steep along all subspaces of dimension lower than or equal to $m < n-1$ and not steep for a subspace of dimension l greater than m (consider the function $f(x, y, z) = (x^2 - y)^2 + z$ at $(0, 0, 0)$ along all the lines and along the plane $z = 0$). Also, a quadratic form is steep if and only if it is sign definite.

In this setting, Nekhorochev proved the following:

Theorem I.3. ([13], [14])
If h is real analytic, non-degenerate ($|\nabla^2 h(I)| \neq 0$ for any $I \in \Omega$) and steep then h is exponentially stable.

In a previous paper ([16]), it has been proved a real analytic numerical function is steep if and only its restriction to any affine subspace admits only isolated critical points, hence the latter condition is sufficient to ensure exponential stability.

On the other hand, a partial reciprocal proposition was also proved: if a real analytic integrable Hamiltonian admits a restriction to an affine subspace spanned by integer vectors with an accumulation of critical points then there exists an arbitrary small perturbation which leads to a polynomial drift of the action variables. In the same direction, Guzzo ([6]) has given examples of exponentially stable integrable Hamiltonians which are non steep: consider $h(I_1, I_2) = I_1^2 - \delta I_2^2$ where δ is the square of a Diophantine number then its isotropic direction is the line directed by $(1, \sqrt{\delta})$ and this allows to prove that h is exponentially stable.

II Results of stability with a Diophantine steepness condition

With the previous observations, we introduce the following notion of *Diophantine steepness* on the unperturbed Hamiltonian.

For $m \in \{1, \dots, n\}$, we denote by $\mathrm{Graff}_R(n,m)$ the m-dimensional affine Grassmannian over $\overline{B}_R^{(n)} \subset \mathbb{R}^n$ (i.e.: the set of affine subspaces of dimension m in $\mathbb{R}^n$ which intersect the closed ball $\overline{B}_R^{(n)}$ of radius $R > 0$ around the origin) and $\mathrm{Graff}_R^K(n,m) \subset \mathrm{Graff}_R(n,m)$ is the set of affine subspaces of dimension m in $\mathbb{R}^n$ whose direction is generated by integer vectors of length $||\vec{k}||_1 = |k_1| + \dots + |k_n| \leq K$.

Definition II.1.
A differentiable function f defined on a neighborhood of $\overline{B}_R^{(n)} \subset \mathbb{R}^n$ is said to be $(\gamma,\tau)-$Diophantine steep for two positive constants γ and τ, if for any $m \in \{1, \dots, n\}$, there exists a fixed index $p_m \geq 1$ and a fixed coefficient $C_m > 0$ such that for any affine subspace $\Lambda_m \in \mathrm{Graff}_R^K(n,m)$ and any continuous curve Γ from $[0,1]$ to $\Lambda_m \cap B_R$ with $||\Gamma(0) - \Gamma(1)|| = r \leq \frac{\gamma}{K^\tau}$, we have:

$$\exists t_* \in [0,1] \text{ such that } ||\Gamma(0) - \Gamma(t_*)|| \leq r \text{ and } \left|\left|\mathrm{Proj}_{\overrightarrow{\Lambda}_m}(\nabla f(\gamma(t_*)))\right|\right| \geq C_m r^{p_m}$$

where $\overrightarrow{\Lambda}_m$ is the direction of Λ_m.

Rk: (i) Since $\mathbb{R}^n$ is an affine subspace generated by unitary integer vectors then along any arc in B_R of length $r \leq \gamma$ there exists a point where the norm of the gradient ∇f is greater or equal to $C_n r^{p_n}$ (the projection is reduced to the identity in this case).

(ii) With no loss of generality, we will assume that the coefficients $(C_1, \dots, C_n)$ are equal to one. Indeed, the problem can always be reduced to this case by using steepness indices slightly greater than the optimal ones.

Under the previous assumptions, we prove our main result of stability with reasonings already given in a previous paper ([15]) which rely on the construction of local resonant normal forms along each trajectory of the perturbed system together with the use of simultaneous Diophantine approximation as in Lochak's proof ([10]) of Nekhorochev's estimates. But this study ([15]) is generalized in three directions. First, we substitute the original Nekhorochev's condition of steepness by our weak assumption of Diophantine steepness given above. Moreover, thanks to a construction of the non-resonant sets directly in the frequency space, we can remove the non degeneracy condition on the frequency map ($\left|\nabla^2 h\right| \neq 0$) assumed in [15]. Finally, according to the remark given after the definition II.1, our integrable Hamiltonian h can admit critical points I (while $\nabla h(I) \neq 0$ was assumed in [15]) provided that h satisfies a global steepness condition on the full space $\mathbb{R}^n$. Following Nekhorochev's terminology ([13]), we will say that such a function is symmetrically steep (or S-steep).

Now, we specify the regularity of the perturbed Hamiltonian.

Consider a nearly integrable Hamiltonian $\mathcal{H}(I,\varphi) = h(I) + \varepsilon f(I,\varphi)$ where $(I,\varphi) \in \mathbb{R}^n \times \mathbb{T}^n$ with $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ are action-angle variables of the integrable Hamiltonian h.

We assume that $\mathcal{H}$ is *analytic* around a fixed complex neighborhood $V_{r,s}\mathcal{P} \subset \mathbb{C}^{2n}$ of a real domain $\mathcal{P} = B_R \times \mathbb{T}^n \subset \mathbb{R}^n \times \mathbb{T}^n$ where B_R is the ball of radius R centered at the origin and:

$$V_{r,s}\mathcal{P} = V_r\left(B_R\right) \times W_s\left(\mathbb{T}^n\right) = \left\{ \begin{array}{l} (I,\varphi) \in \mathbb{C}^{2n} \text{ such that } \mathrm{dist}(I, B_R) \leq r \text{ and} \\ \Re e(\varphi) \in \mathbb{T}^n \; ; \; \mathrm{Max}_{j\in\{1,\dots,n\}} \left|\Im m(\varphi_j)\right| \leq s \end{array} \right\}$$

with $1 > r > 0$, $s > 0$ and the distance to B_R given by the Euclidean norm in $\mathbb{C}^n$.

Let $||.||_{r,s}$ be the sup norm (L^∞) for numerical or vector-valued functions defined over $V_{r,s}\mathcal{P}$. We assume that $||f||_{r,s} \leq 1$ and that ε is a small parameter.

The Hessian matrix is assumed to be uniformly bounded with respect to the norm on the operators, also denoted by $||.||$, induced by the Euclidean norm: for some constant $M > 1$, we have $\left|\left|\partial_I^2 h(I)\right|\right|_{r,s} \leq M$ for all $I \in V_r(B_R)$.

With the previous assumptions, we can state the following result which will be proved in the appendix:

Theorem II.2.
Consider the positive constants

$$\beta = \frac{1}{1 + 2np_1 \dots p_{n-1}} \; ; \; a = \frac{\beta}{1+\tau} \; ; \; b = \frac{\beta}{p_n},$$

there exists $C_1 > 0$ and $C_2 > 0$ such that for a small enough perturbation $\varepsilon \leq C_1 \gamma^{1/a}$ and for any orbit of the perturbed system with initial conditions $(I(t_0), \varphi(t_0)) \in B_R \times \mathbb{T}^n$ far enough from the boundary of B_R, we have:

$$||I(t) - I(t_0)|| \leq R(\varepsilon) \textit{ for } |t| \leq \mathcal{T}(\varepsilon),$$

where $R(\varepsilon)$ and $\mathcal{T}(\varepsilon)$ are of order ε^b and $\exp(C_2/\varepsilon^a)$.

Rk. (i) In the regular case where there are no critical points for h and there exists a uniform lower bound on the norm of the gradient ∇h, the previous results are valid with the exponents:

$$a = \frac{1}{1 + (2n-1)(1+\tau)p_1 \dots p_{n-1}} \ ; \ b = (1+\tau)a \text{ and } \varepsilon \leq C_1 \gamma^{1/a}.$$

(ii) Moreover, under the classical assumption of steepness over any affine subspace, we have $\tau = 0$ and $a = b = \dfrac{1}{1 + (2n-1)p_1 \dots p_{n-1}}$.

Finally, in the quasi-convex case, the integrable Hamiltonian is strongly steep and all the indices are equal to one, hence we recover the same exponents $1/2n$ as Lochak and Poschel ([10], [11], [18]).

(iii) The fact that the exponents of stability are *independent* of γ is crucial for our subsequent reasonings. On the other hand, the threshold of application of this theorem depends of γ which is reminiscent of KAM theory where the latter quantities depend of $\sqrt{\gamma}$.

III Genericity of diophantine steepness among smooth functions

At first order, the set of Diophantine steep function is much wider than the initial class of steep functions defined in the introduction. Indeed, any linear Hamiltonian $h(I) = \omega.I$ with a (γ, τ)-Diophantine vector $\omega \in \mathbb{R}^n$ is Diophantine steep with indices equal to one (in order to get a coefficient $C_m = 1$).

At second order, one can prove that a quadratic integrable Hamiltonian is almost always Diophantine steep with indices equal to 2 (it can be shown that for any quadratic form $q(I) = {}^tIAI$ and for almost all $\lambda \in \mathbb{R}$, there exists $\gamma > 0$ such that the modified quadratic form $q_\lambda(I) = {}^tIAI + \lambda||I||^2$ is (γ, τ)-Diophantine steep for some $\gamma > 0$ provided that $\tau > n^2$).

In the spirit of these examples, we look for a full measure set of Diophantine steep function in the space of $\mathcal{C}^k$ numerical function defined on an open set in $\mathbb{R}^n$.

Actually, a set in an infinite dimensional space can be of zero measure only if it is a trivial set (see [7]). For this reason, Christensen [3], Hunt, Sauer and Yorke ([7]), Kaloshin ([8]) have introduced a weak notion of full measure set in an infinite dimensional space called prevalence which corresponds to the usual property in a finite dimensional space. In its simplest setting, a set $\mathcal{P}$ is said to be shy if there exists a *finite* dimensional subspace F called a probe space such that any affine subspace of direction F intersects $\mathcal{P}$ along a zero measure set for the usual Lebesgue measure on this subspace. A set is prevalent if its complement is shy. Stronger notions of prevalence can be defined (see [7], [8] and the section IV of this paper).

An example of prevalent set is given by the Morse functions in the Banach space $\left(\mathcal{C}^n\left(\overline{B}_R^{(n)}, \mathbb{R}\right), ||.||_{\mathcal{C}^n}\right)$ where $\overline{B}_R^{(n)}$ is the closed ball of radius R centered at the origin in $\mathbb{R}^n$. Indeed, for any function $f \in \mathcal{C}^n\left(\overline{B}_R^{(n)}, \mathbb{R}\right)$, by an application of Sard's theorem on the gradient map ∇f one can prove that for almost any linear form $\omega \in \mathcal{L}(\mathbb{R}^n, \mathbb{R})$, the function $f_\omega = f + \omega$ is Morse and the probe space is given by the linear forms. The modified function f_ω is called a morsification of f (see [1] and [5]).

Here we look for a set $\mathcal{P}$ of Diophantine steep functions with *fixed* indices which is prevalent in $\mathcal{C}^k\left(\overline{B}_R^{(n)}, \mathbb{R}\right)$ for a certain $k \in \mathbb{N}^*$. This will be obtained by introducing the class of Diophantine Morse function and proving its prevalence with reasonings similar to those used for the usual Morse functions but we have to substitute the Morse-Sard theory by a quantitative Morse-Sard theory developed by Yomdin ([20], [21]). Moreover, we show that the Diophantine Morse functions are Diophantine steep with indices equal to two.

III.1 Diophantine Morse functions

Definition III.1.1.

We denote by $\mathrm{Gr}(n,m)$ *the set of all vectorial subspaces of dimension* m *in* $\mathbb{R}^n$ *and, for* $K \in \mathbb{N}^*$, $\mathrm{Gr}_K(n,m) \subset \mathrm{Gr}(n,m)$ *is the set of vectorial subspaces in* $\mathbb{R}^n$ *generated by integer vectors of length* $||\vec{k}||_1 = |k_1| + \ldots + |k_n| \leq K$*, moreover* $\mathrm{Gr}(n) = \cup_{m=1}^n \mathrm{Gr}(n,m)$ *and* $\mathrm{Gr}_K(n) = \cup_{m=1}^n \mathrm{Gr}_K(n,m)$*.*

A twice differentiable function $f \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$ *defined on a neighborhood of the closed ball* $\overline{B}_{2R}^{(n)} \subset \mathbb{R}^n$ *of radius* $2R$ *centered at the origin is said to be* $(\gamma,\tau)-$*Diophantine Morse for some positive constants* γ *and* τ *if, for any* $K \in \mathbb{N}^*$*, any* $m \in \{1, \ldots, n\}$ *and any* $\Lambda \in \mathrm{Gr}_K(n,m)$*, there exists* $(e_1, \ldots, e_m)$ *(resp.* $(f_1, \ldots, f_{n-m})$*) an orthonormal basis of* Λ *(resp. of* $\Lambda^\perp$*) such that the function*

$$f_\Lambda(\alpha,\beta) := f\left(\alpha_1 e_1 + \ldots + \alpha_m e_m + \beta_1 f_1 + \ldots + \beta_{n-m} f_{n-m}\right),$$

which is twice differentiable on a neighborhood of $\overline{B}_R^{(m)} \times \overline{B}_R^{(n-m)}$*, satisfies:*

$$\left|\left|\frac{\partial f_\Lambda}{\partial \alpha}(\alpha,\beta)\right|\right| > \frac{\gamma}{K^\tau} \ \text{or} \ \left|\left|\frac{\partial^2 f_\Lambda}{\partial \alpha^2}_{|(\alpha,\beta)}(\eta)\right|\right| > \frac{\gamma}{K^\tau}||\eta|| \ (\forall \eta \in \mathbb{R}^m)$$

for all $(\alpha,\beta) \in \overline{B}_R^{(m)} \times \overline{B}_R^{(n-m)}$*.*

The link between the Diophantine Morse functions and the Diophantine steep functions is given in the following:

Proposition III.1.2.

With the previous notations, if a differentiable function $f \in \mathcal{C}^3(\mathbb{R}^n, \mathbb{R})$ *defined on a neighborhood of the closed ball* $\overline{B}_{2R}^{(n)} \subset \mathbb{R}^n$ *is* $(\gamma,\tau)-$*Diophantine Morse for some positive constants* γ *and* τ*, then* f *is* $(\gamma,\tau)-$*Diophantine steep over* $\overline{B}_R$ *with all the steepness indices equal to 2.*

Rk: Our definition of Diophantine Morse function relies on the choice of an orthonormal basis in any subspaces $\Lambda \in \mathrm{Gr}_K(n)$ and the eigenvalues of the Hessian matrix which are extrinsic. But the property of Diophantine steepness involves only the norm of the gradient ∇f since $\left|\left|\frac{\partial f_\Lambda}{\partial \alpha}(\alpha,\beta)\right|\right| = ||\mathrm{Proj}_\Lambda\left(\nabla f\left(\alpha_1 e_1 + \ldots + \alpha_m e_m + \beta_1 f_1 + \ldots + \beta_{n-m} f_{n-m}\right)\right)||$ which does not depend of the considered orthonormal basis.

Proof: Consider $f \in \mathcal{C}^3(\mathbb{R}^n, \mathbb{R})$ with $||f||_{\mathcal{C}^3} \leq M$ for some $M \geq 1$ over $\overline{B}_{2R}^{(n)}$ such that

$$\left|\left|\frac{\partial f_\Lambda}{\partial \alpha}(\alpha, \beta)\right|\right| > \frac{\gamma}{K^\tau} \text{ or } \left|\left|\frac{\partial^2 f_\Lambda}{\partial \alpha^2}_{|(\alpha,\beta)}(\eta)\right|\right| > \frac{\gamma}{K^\tau}||\eta|| \ (\forall \eta \in \mathbb{R}^m)$$

for all $(\alpha, \beta) \in \overline{B}_R^{(m)} \times \overline{B}_R^{(n-m)}$ with $\Lambda \in \mathrm{Gr}_K(n,m)$.

Then, for any continuous curve $\Gamma : [0,1] \longrightarrow \overline{B}_R^{(m)}$ of length $r \leq \mathrm{Inf}\left(\frac{\gamma}{2MK^\tau}, 1\right)$, we have either:

i) $\left|\left|\frac{\partial f_\Lambda}{\partial \alpha}(\Gamma(0), \beta)\right|\right| > \frac{\gamma}{K^\tau} > r \geq r^2$.

ii) otherwise, for $\alpha \in \mathbb{R}^m$ such that $||\alpha - \Gamma(0)|| < \frac{\gamma}{2MK^\tau}$ we have:

$$\left|\left|\frac{\partial^2 f_\Lambda}{\partial \alpha^2}_{|(\alpha,\beta)} - \frac{\partial^2 f_\Lambda}{\partial \alpha^2}_{|(\Gamma(0),\beta)}\right|\right| < \frac{\gamma}{2K^\tau}$$

and $\left|\left|\frac{\partial^2 f_\Lambda}{\partial \alpha^2}_{|(\Gamma(0),\beta)}(\eta)\right|\right| > \frac{\gamma}{K^\tau}||\eta|| \Longrightarrow \left|\left|\frac{\partial^2 f_\Lambda}{\partial \alpha^2}_{|(\alpha,\beta)}(\eta)\right|\right| > \frac{\gamma}{2K^\tau}||\eta||$ for all $\eta \in \mathbb{R}^m$.

The mean value theorem gives $\left|\left|\frac{\partial f_\Lambda}{\partial \alpha}(\alpha, \beta) - \frac{\partial f_\Lambda}{\partial \alpha}(\Gamma(0), \beta)\right|\right| = \left|\left|\frac{\partial^2 f_\Lambda}{\partial \alpha^2}_{|(\alpha_*,\beta)}(\alpha - \Gamma(0))\right|\right|$ for some α_* on the segment which connect $\Gamma(0)$ and α, it implies:

$$\left|\left|\frac{\partial f_\Lambda}{\partial \alpha}(\alpha, \beta) - \frac{\partial f_\Lambda}{\partial \alpha}(\Gamma(0), \beta)\right|\right| \geq \frac{\gamma}{2K^\tau}||\alpha - \Gamma(0)|| \geq ||\alpha - \Gamma(0)||^2$$

hence, we can ensure at least a variation of size r^2 on the norm $\left|\left|\frac{\partial f_\Lambda}{\partial \alpha}(\Gamma(t), \beta)\right|\right|$ along a path Γ of length $r \leq \mathrm{Inf}\left(\frac{\gamma}{2MK^\tau}, 1\right)$.

Moreover, the choice of the orthonormal basis $(e_1, \dots, e_m)$ gives

$$\frac{\partial f_\Lambda}{\partial \alpha}(\alpha, \beta) = \mathrm{Proj}_\Lambda\left(\nabla f\left(\alpha_1 e_1 + \ldots + \alpha_m e_m + \beta_1 f_1 + \ldots + \beta_{n-m} f_{n-m}\right)\right)$$

hence, for an arbitrary path $\widetilde{\Gamma}$ of length $r \leq \mathrm{Inf}\left(\frac{\gamma}{2MK^\tau}, 1\right)$ in the affine subspace $x + \Lambda$ with $\Lambda \in \mathrm{Gr}_K(n,m)$ and $x \in \Lambda^\perp$, there exists $t_* \in [0,1]$ such that $\left|\left|\mathrm{Proj}_\Lambda\left(\nabla f\left(\widetilde{\Gamma}(t_*)\right)\right)\right|\right| \geq r^2$.

Finally, any affine subspace spanned by integer vectors of lengths bounded by $K \in \mathbb{N}^*$ can be seen as the sum $x + \Lambda$ for some $x \in \Lambda^\perp \cap \overline{B}_R^{(n)}$ with the direction $\Lambda \in \mathrm{Gr}_K(n)$.

Hence, the definition of Diophantine steepness for f over B_R is satisfied.▶

III.2 Quantitative Morse-Sard theory and applications

Now we will show that, for a fixed $\tau > 0$ large enough, any sufficiently smooth function $f \in \mathcal{C}^p(\mathbb{R}^n, \mathbb{R})$ can be transformed into a (γ, τ)-Diophantine Morse function by adding almost any linear form. This latter results is proved thanks to an application of a quantitative version of Sard's theorem due to Yomdin ([20]). We recall the main results of this theory along the lines of a recent expository book of Yomdin and Comte ([21]).

For k, m and $n \in \mathbb{N}^*$ such that $m \leq n$, consider a mapping $g \in \mathcal{C}^{k+1}(\mathbb{R}^n, \mathbb{R}^m)$ defined on a neighborhood of the closed ball $\overline{B}_R^{(n)} \subset \mathbb{R}^n$ for some radius $R > 0$ with the bound $||g||_{\mathcal{C}^{k+1}} = M \geq 1$.

With the previous assumptions, the quantity $R_k(g) = \dfrac{M}{k!} R^{k+1}$ bounds the Taylor remainder term at order k over the closed ball $\overline{B}_R^{(n)}$.

For any matrix $A \in \mathcal{M}_{(m,n)}(\mathbb{R})$ with $1 \leq m \leq n$, the ordered singular values of A (i.e.: the eigenvalues of ${}^tAA$) are denoted $0 \leq \lambda_1(A) \leq \ldots \leq \lambda_m(A)$ and, for any $x \in \overline{B}_R^{(n)}$, the singular values of $dg(x)$ are denoted $\lambda_i(x)$ with $i \in \{1, \ldots, m\}$. In other word, $dg(x)$ maps the unit ball in $\mathbb{R}^n$ onto the ellipsoid of principal axes $0 \leq \lambda_1(x) \leq \ldots \leq \lambda_m(x)$ in $\mathbb{R}^m$.

For $\lambda = (\lambda_1, \ldots, \lambda_m)$ with $0 \leq \lambda_1 \leq \ldots \leq \lambda_m$, the set $\Sigma\left(g, \lambda, \overline{B}_R^{(n)}\right)$ of λ-critical points and the set $\Delta\left(g, \lambda, \overline{B}_R^{(n)}\right)$ of λ-critical values are defined as:

$$\Sigma\left(g, \lambda, \overline{B}_R^{(n)}\right) = \{x \in \overline{B}_R^{(n)} \text{ such that } \lambda_i(x) \leq \lambda_i, \text{ for } i = 1, \ldots, m\}$$

$$\text{and } \Delta\left(g, \lambda, \overline{B}_R^{(n)}\right) = g\left(\Sigma\left(g, \lambda, \overline{B}_R^{(n)}\right)\right).$$

Finally, for any relatively compact subset $\mathcal{A}$ in $\mathbb{R}^n$, we denote by $M(\varepsilon, \mathcal{A})$ the minimal number of closed balls of radius ε in $\mathbb{R}^n$ covering $\mathcal{A}$.

The cornerstone of the quantitative Sard theory is the following:

Theorem III.2.1. ([20], [21, theorem 9.2])

With the previous notations and assumptions, with $\lambda_0 = 1$ and $\lambda = (\lambda_1, \ldots, \lambda_m)$, we have:

$$M\left(\varepsilon, \Delta\left(g, \lambda, \overline{B}_R^{(n)}\right)\right) \leq c_e \sum_{j=0}^{m} \lambda_0\lambda_1 \ldots \lambda_j \left(\frac{R}{\varepsilon}\right)^j \quad \textit{for } \varepsilon \geq R_k(g)$$

$$M\left(\varepsilon, \Delta\left(f, \lambda, \overline{B}_R^{(n)}\right)\right) \leq c_i \sum_{j=0}^{m} \lambda_0\lambda_1 \ldots \lambda_j \left(\frac{R}{\varepsilon}\right)^j \left(\frac{R_k(f)}{\varepsilon}\right)^{\frac{n-j}{k+1}} \quad \textit{for } \varepsilon \leq R_k(g)$$

where $c_i > 0$ and $c_e > 0$ depend only on n, m and k.

Corollary III.2.2.

With the previous notations and assumptions, for any $\varepsilon \in]0, 1[$, we have:

$$M\left(\varepsilon, \Delta\left(g, \lambda, \overline{B}_R^{(n)}\right)\right) \leq C \sum_{j=0}^{m} \lambda_0\lambda_1 \ldots \lambda_j \left(\frac{1}{\varepsilon}\right)^{\frac{n+kj}{k+1}}$$

where $C > 0$ depend only on M, R, n, m and k.

If $\mathrm{Neigh}_\varepsilon(\mathcal{A}) = \cup_{x\in\mathcal{A}} B(x,\varepsilon)$ for a set $\mathcal{A} \subset \mathbb{R}^m$ then $\mathrm{Neigh}_\varepsilon\left(\Delta\left(g,\lambda,\overline{B}_R^{(n)}\right)\right)$ can be covered by $M\left(\varepsilon,\Delta\left(g,\lambda,\overline{B}_R^{(n)}\right)\right)$ balls of radius 2ε and, for the m-dimensional Lebesgue measure, we have:

$$\mathrm{Vol}\left(\mathrm{Neigh}_\varepsilon\left(\Delta\left(g,\lambda,\overline{B}_R^{(n)}\right)\right)\right) \leq V(m)(2\varepsilon)^m M\left(\varepsilon,\Delta\left(g,\lambda,\overline{B}_R^{(n)}\right)\right)$$

where $V(m)$ is the volume of the m-dimensional unit ball, finally:

$$\mathrm{Vol}\left(\mathrm{Neigh}_\varepsilon\left(\Delta\left(g,\lambda,\overline{B}_R^{(n)}\right)\right)\right) \leq \widetilde{C}\sum_{j=0}^{m} \lambda_0\lambda_1\dots\lambda_j\left(\frac{1}{\varepsilon}\right)^{\frac{n+kj}{k+1}-m}$$

for some constant $\widetilde{C}$ which depends only on M, R, n, m and k.

Corollary III.2.3.
For $\delta \in]0,1[$ and $\varepsilon = \delta^{\frac{k+1}{k}}$, we denote

$$\Delta_\delta = \Delta\left(g,(\delta,M,\dots,M),\overline{B}_R^{(n)}\right) \text{ and } \widetilde{\Delta}_\delta = \mathrm{Neigh}_\varepsilon\left(\Delta_\delta\right)$$

and we have the bounds:
i) $\mathrm{Vol}\left(\widetilde{\Delta}_\delta\right) \leq \overline{C}\delta^{\frac{k+1-n}{k}}$ *where $\overline{C} > 0$ depends only on M, R, n, m and k.*
ii) for $k = 2n$, we have $\mathrm{Vol}\left(\widetilde{\Delta}_\delta\right) \leq \overline{C}\delta^{\frac{n+1}{2n}}$.

Proof: Since $\varepsilon \in]0,1[$, we have:

$$\mathrm{Vol}\left(\widetilde{\Delta}_\delta\right) \leq \widetilde{C}\varepsilon^{m-\frac{n}{k+1}} + \widetilde{C}\sum_{j=1}^{m} M^{j-1}\delta\varepsilon^{m-\frac{n+kj}{k+1}} \leq \widetilde{C}\left(\varepsilon^{m-\frac{n}{k+1}} + \frac{M^m-1}{M-1}\delta\varepsilon^{m-\frac{n+km}{k+1}}\right)$$

and $m \geq 1$ implies $\mathrm{Vol}\left(\widetilde{\Delta}_\delta\right) \leq \overline{C}_1\varepsilon^{\frac{(k+1)m-n}{k+1}} + \overline{C}_2\delta\varepsilon^{\frac{m-n}{k+1}} \leq \overline{C}_1\varepsilon^{\frac{k+1-n}{k+1}} + \overline{C}_2\delta\varepsilon^{\frac{1-n}{k+1}}$.

Finally, the choice $\varepsilon = \delta^{\frac{k+1}{k}}$ yields $\mathrm{Vol}\left(\widetilde{\Delta}_\delta\right) \leq \overline{C}\delta^{\frac{k+1-n}{k}}$ and $k = 2n$ allows to obtain the second estimate.▶

Theorem III.2.4.
For $\kappa \in]0,1[$ and $g \in \mathcal{C}^{2n+1}(\mathbb{R}^n,\mathbb{R}^m)$ with $||g||_{\mathcal{C}^{k+1}} = M \geq 1$, there exists a subset $\mathcal{C}_\kappa \subset \mathbb{R}^m$ such that

$$\mathrm{Vol}\left(\mathcal{C}_\kappa\right) \leq \overline{C}\sqrt{\kappa}$$

and, for any $\omega \in \mathbb{R}^m \backslash \mathcal{C}_\kappa$, the function $g_\omega(x) = g(x) - \omega$ satisfies at any point $x \in \overline{B}_R^{(n)}$:

$$||g_\omega(x)|| > \kappa \text{ or } ||dg_\omega(x)\zeta|| > \kappa||\zeta|| \ (\forall\zeta \in \mathbb{R}^n).$$

Proof: We choose $\mathcal{C}_\kappa = \widetilde{\Delta}_\delta$ with $\delta = \kappa^{\frac{n}{n+1}}$, hence:

$$\mathrm{Vol}\,(\mathcal{C}_\kappa) \leq \overline{C}\delta^{\frac{n+1}{2n}} = \overline{C}\sqrt{\kappa}.$$

Now, with our bound on $||g||_{\mathcal{C}^{2n+1}}$, we have $\lambda_i(x) \leq M$ for any $i \in \{1, \dots, m\}$ and any $x \in \overline{B}^{(n)}_R$, hence:

$$\Delta_\delta = \{x \in \overline{B}^{(n)}_R \text{ such that } \lambda_1(x) \leq \delta\} = \{x \in \overline{B}^{(n)}_R \text{ such that } \exists \zeta \in \mathbb{R}^n \text{ with } ||dg(x)\zeta|| \leq \delta||\zeta||\}$$

Moreover $\varepsilon = \delta^{\frac{2n+1}{2n}} = \kappa^{\frac{2n+1}{2n+2}} > \kappa$ with $\kappa < 1$, then $||g_\omega(x)|| \leq \kappa$ implies $||g_\omega(x)|| < \varepsilon$ and $g(x) \notin \Delta_\delta$ since $\mathrm{Dist}(\omega, \Delta_\delta) \geq \varepsilon$, hence $||dg(x)\zeta|| > \delta||\zeta||$ for all $\zeta \in \mathbb{R}^n$.

Finally $\delta = \kappa^{\frac{n}{n+1}} > \kappa$ yields:

$$||dg_\omega(x)\zeta|| = ||dg(x)\zeta|| > \delta||\zeta|| > \kappa||\zeta|| \ (\forall \zeta \in \mathbb{R}^m)$$

and we obtain the second estimate.▶

Now, we consider the constants $\gamma > 0$, $\tau > 0$ and an arbitrary function $f \in \mathcal{C}^{2n+2}(\mathbb{R}^n, \mathbb{R})$ defined on a neighborhood of the closed ball $\overline{B}^{(n)}_R \subset \mathbb{R}^n$ with the bound $||f||_{\mathcal{C}^{2n+2}} = M \geq 1$.

Theorem III.2.5.

Consider $\nu \in \mathbb{N}^*$, $K \in \mathbb{N}^*$ and $\Lambda \in \mathrm{Gr}_K(n,m)$, there exists a subset $\mathcal{C}^{(\nu)}_\Lambda \subset \mathrm{B}^{(n)}_\nu$ where $\mathrm{B}^{(n)}_\nu$ is the open ball of radius ν centered at the origin in $\mathbb{R}^n$ with:

$$\mathrm{Vol}\left(\mathcal{C}^{(\nu)}_\Lambda\right) \leq \overline{C}^{(\nu)}_m \sqrt{\frac{\gamma}{K^\tau}}$$

where the constant $\overline{C}^{(\nu)}_m$ depends only of n, m, M, R and ν such that, for any $\Omega \in \mathcal{B}^{(n)}_\nu \backslash \mathcal{C}^{(\nu)}_\Lambda$ the modified function $f_\Omega(x) = f(x) - \Omega.x$ satisfies at any point $x \in \overline{B}^{(n)}_R$:

$$\left|\left|\partial_\alpha f_{(\Lambda,\Omega)}(\alpha,\beta)\right|\right| \geq \frac{\gamma}{K^\tau} \ \text{or} \ \left|\left|\partial^2_\alpha f_{(\Lambda,\Omega)}(\alpha,\beta)\eta\right|\right| \geq \frac{\gamma}{K^\tau}||\eta|| \ (\forall \eta \in \mathbb{R}^m)$$

(the function $f_{(\Lambda,\Omega)}$ is defined with respect to f_Ω along the lines of f_Λ with respect to f in the definition of a Diophantine Morse function).

Proof: We apply the latter theorem III.2.4. with the constant $\kappa = \gamma/K^\tau$ on the function $g(\alpha,\beta) = \partial_\alpha f_\Lambda(\alpha,\beta) \in \mathcal{C}^{2n+1}(\mathbb{R}^n, \mathbb{R}^m)$ in order to obtain a nearly critical set $\mathcal{C}_\kappa \subset \mathbb{R}^m$.

Then, for $\Omega \in \mathbb{R}^n$ such that $\mathrm{Proj}_\Lambda(\Omega) = \omega_1 e_1 + \ldots + \omega_m e_m$ with $\omega = (\omega_1, \dots, \omega_m) \notin \mathcal{C}_\kappa$, the function $f_\Omega(x) = h(x) - \Omega.x$ satisfies $\partial_\alpha f_{(\Lambda,\Omega)}(\alpha,\beta) = \partial_\alpha f_\Lambda(\alpha,\beta) - \omega = g_\omega(\alpha,\beta)$ and:

$$||g_\omega(\alpha,\beta)|| = \left|\left|\partial_\alpha f_{(\Lambda,\Omega)}(\alpha,\beta)\right|\right| \geq \frac{\gamma}{K^\tau} \text{ or } ||dg_\omega(\alpha,\beta)\zeta|| \geq \frac{\gamma}{K^\tau}||\zeta|| \ (\forall \zeta \in \mathbb{R}^n)$$

but the differential $\partial^2_\alpha f_{(\Lambda,\Omega)}(\alpha,\beta) = \partial^2_\alpha f_\Lambda(\alpha,\beta)$ is the restriction of dg to the subspace $\mathbb{R}^m \times \{0\} \subset \mathbb{R}^n$ and admits the same lower bound on its singular values as $dg = dg_\omega$.

Next, we consider the set :

$$\mathcal{C}_\Lambda^{(\nu)} = \left\{ \Omega \in \mathrm{B}_\nu^{(n)} \text{ such that } \mathrm{Proj}_\Lambda(\Omega) = \omega_1 e_1 + \ldots + \omega_m e_m \text{ with } \omega = (\omega_1, \ldots, \omega_m) \in \mathcal{C}_\kappa \right\}$$

then we have the estimate

$$\mathrm{Vol}\left(\mathcal{C}_\Lambda^{(\nu)}\right) = \mathrm{Vol}\left(\mathrm{Proj}_\Lambda^{-1}(\mathcal{C}_\kappa) \cap \mathrm{B}_\nu^{(n)}\right) \leq \mathrm{Vol}\left(\mathrm{B}_\nu^{(n)}\right) \mathrm{Vol}(\mathcal{C}_\kappa) \leq V(n)\nu^n \overline{C} \sqrt{\frac{\gamma}{K^\tau}}$$

where $V(n)$ is the volume of the unit ball in $\mathbb{R}^n$ and $\overline{C}$ is the constant in theorem III.2.4. computed for a function $g \in \mathcal{C}^{2n+1}(\mathbb{R}^n, \mathbb{R}^m)$ which depends only of M, R, n and m, finally :

$$\mathrm{Vol}\left(\mathcal{C}_\Lambda^{(\nu)}\right) = \overline{C}_m^{(\nu)} \sqrt{\frac{\gamma}{K^\tau}}$$

where the constant $\overline{C}_m^{(\nu)}$ depends only of n, m, M, R, ν.▶

Theorem III.2.6. (Prevalence of the Diophantine Morse functions)

Consider an arbitrary constant $\tau > 2(n^2+1)$ and a function $\mathcal{C}^{2n+2}\left(\overline{B}_R^{(n)}, \mathbb{R}\right)$ defined on a neighborhood of the closed ball $\overline{B}_R^{(n)} \subset \mathbb{R}^n$, then, for almost any $\Omega \in \mathbb{R}^n$ there exists $\gamma > 0$ such that the function $f_\Omega(x) = f(I) - \Omega.I$ is (γ, τ)–Diophantine Morse over $\overline{B}_R^{(n)}$.

Consequently, the set of $(\gamma, 2n^2+3)$-Diophantine Morse functions for some $\gamma > 0$ is prevalent in $\mathcal{C}^{2n+2}(\mathbb{R}^n, \mathbb{R})$.

Proof : For any $\nu \in \mathbb{N}^*$ and $K \in \mathbb{N}^*$, we consider the set $\mathcal{C}_K^{(\nu)} = \cup_{m=1}^n \cup_{\Lambda \in \mathrm{Gr}_K(n,m)} \mathcal{C}_\Lambda^{(\nu)}$ with an application of the latter theorem III.2.5, we obtain :

$$\mathrm{Vol}\left(\mathcal{C}_K^{(\nu)}\right) \leq \sum_{m=1}^n \mathrm{Card}\left(\mathrm{Gr}_K(n,m)\right) \overline{C}_m^{(\nu)} \sqrt{\frac{\gamma}{K^\tau}} \leq \left(\sum_{m=1}^n \overline{C}_m^{(\nu)}\right) K^{n^2} \sqrt{\frac{\gamma}{K^\tau}}.$$

Now, for a fixed $\gamma > 0$, the set $\mathcal{C}_\gamma^{(\nu)} = \cup_{K \in \mathbb{N}^*} \mathcal{C}_K^{(\nu)}$ satisfies

$$\mathrm{Vol}\left(\mathcal{C}_\gamma^{(\nu)}\right) \leq \left(\sum_{m=1}^n \overline{C}_m^{(\nu)}\right) \left(\sum_{K \in \mathbb{N}^*} K^{n^2 - \tau/2}\right) \sqrt{\gamma}$$

and this upper bound is convergent with our assumption on τ.

For $\mathcal{C}^{(\nu)} = \cap_{\gamma > 0} \mathcal{C}_\gamma^{(\nu)}$ we have $\mathrm{Vol}\left(\mathcal{C}^{(\nu)}\right) = 0$ and $\mathcal{C} = \cup_{\nu \in \mathbb{N}^*} \mathcal{C}^{(\nu)}$ satisfies $\mathrm{Vol}(\mathcal{C}) = 0$.

Finally, for any $\Omega \in \mathbb{R}^n \backslash \mathcal{C}$, the function $f_\Omega(x) = f(x) - \Omega.x$ is (γ, τ)–Diophantine Morse over $\overline{B}_R^{(n)}$ for some $\gamma > 0$ and we can choose $\tau = 2n^2 + 3 > 2(n^2+1)$.▶

IV Conclusion :

Going back to the dynamic, our result of exponential stability (theorem II.2) together with the prevalence of Diophantine Morse functions (theorem III.2.6) implies that for an arbitrary real analytic integrable Hamiltonian h and for almost all linear form $\omega \in \mathcal{L}(\mathbb{R}^n, \mathbb{R})$, the modified Hamiltonian $h_\omega(x) = h(x)+\omega(x)$ is exponentially stable with fixed exponents of stability (since the latter quantities depend only of the steepness indices), indeed:

Theorem IV.1. (Genericity of exponential stability)

Consider an arbitrary real analytic integrable Hamiltonian h defined on a neighborhood of the closed ball $\overline{B}_R^{(n)}$ of radius R centered at the origin in $\mathbb{R}^n$.

For almost any $\Omega \in \mathbb{R}^n$, the integrable Hamiltonian $h_\Omega(x) = h(I)-\Omega.I$ is exponentially stable with the exponents:

$$a = \frac{b}{2+n^2} \text{ and } b = \frac{1}{2(1+2^n n)}.$$

Proof:

If we choose $\tau = 3 + 2n^2$, the previous theorem III.2.6. implies that, for almost any $\Omega \in \mathbb{R}^n$, there exists $\gamma > 0$ such that the integrable Hamiltonian $h_\Omega(I) = h(I) + \Omega.I$ is $(\gamma, \tau)-$Diophantine Morse.

Hence, according to the theorem III.1.2. the integrable Hamiltonian h_Ω is $(\gamma, 3 + 2n^2)-$Diophantine steep with indices equal to two and finally the theorem II.2. ensures that h_Ω is exponentially stable with the desired exponents a and b.▶

We point out with the following example the difference between our result of stability with the generic theorems of stability which can be ensured with Nekhorochev's original work.

We have seen that the function $x^2 - y^2$ is not steep but $x^2 - y^2 + x^3$ is steep and, usually, a given function can be transformed in a steep function by adding higher order terms. Indeed, let $J_r(n)$ be the space of r-jets of the $\mathcal{C}^\infty$ numerical function of n variables, Nekhorochev ([13]) proved that the non-steep functions admit a r-jet in an algebraic set of $J_r(n)$ with a codimension which goes to infinity as r goes to infinity. With the geometric criterion for steepness given in [16], one also recover the previous algebraic genericity. The point is that we have to consider functions with arbitrary steepness indices and, consequently, we do not obtain *uniform* exponents of stability for a generic set of integrable Hamiltonian while here, fixed stability exponents are obtained on a prevalent set.

Different prevalent properties of dynamical systems have been proved in ([7], [8], [9]) but, up to the author knowledge, there is only one result of this kind for nearly integrable Hamiltonian system due to Perez-Marco ([17]) who proved that the Birkhoff's normal forms are convergent or divergent for a generic set of nearly-integrable Hamiltonian. Nevertheless, he uses a stronger notion of genericity than prevalence since the considered set intersects *any* finite-dimensional affine subspace along a full measure set with respect to the finite-dimensional Lebesgue measure.

Appendix: Exponential stability with a Diophantine steepness condition

A.I Description of our proof

First, we begin by an algebraic property: a vector $\omega \in \mathbb{R}^n$ is said to be rational if it is a multiple of a vector with integer components. In such a case, the scalar products $|k.\omega|$ for $k \in \mathbb{Z}^n$ such that $k.\omega \neq 0$ admit a lower bound $\ell > 0$. Hence, let $\omega \in \mathbb{R}^n$ be a rational vector and K a positive constant then there exist a small neighborhood V of ω and a lower bound $\ell > \ell' > 0$ which depends on K such that $|k.\omega'| \geq \ell'$ for any $\omega' \in V$ and all $k \in \mathbb{Z}^n \backslash < \omega >^{\perp}$ with $||k||_1 \leq K$. Hence, if we find a second rational vector $\widetilde{\omega} \in V$, then the scalar products $|k.\widetilde{\omega}|$ admit a uniform lower bound for all $k \in \mathbb{Z}^n \backslash \{< \omega >^{\perp} \cap < \widetilde{\omega} >^{\perp}\}$ and $||k||_1 \leq K$. Moreover, if ω and $\widetilde{\omega}$ are linearly independent, we have Dim $\left(< \omega >^{\perp} \cap < \widetilde{\omega} >^{\perp}\right) = n-2$. By a simple generalization, we can ensure that if we find a sequence $(\omega_1, \dots, \omega_n)$ of close enough rational vectors which are linearly independent (i.e.: $(\omega_1, \dots, \omega_n)$ form a basis of $\mathbb{R}^n$), then all the scalar products $|k.\omega_n|$ admit a uniform lower bound for $k \in \mathbb{Z}^n$ and $||k||_1 < K$ with $K > 0$.

Now, consider a trajectory of the perturbed system starting at a time t_0 which admits an increasing sequence of times $t_0 \leq t_1 \leq \dots \leq t_n \leq t_0 + \exp(cK)$ such that, for some constants $c > 0$ and $K > 0$, each frequency vector $\nabla h(I(t_k))$ is close to a rational vector ω_k for each $k \in \{1, \dots, n\}$ where $(\omega_1, \dots, \omega_n)$ is a basis of $\mathbb{R}^n$ composed of rational vectors which are close enough one to one another to satisfy the previous algebraic property with the constant K. Then, $I(t_n)$ is located in a resonance-free area up to some finite order and a local integrable normal form can be built up to an exponentially small remainder, this allows to confine the actions.

In section IV, we assume that there is a high enough density of rational vectors and prove that the mechanism of Nekhorochev together with the Diophantine steepness of the integrable Hamiltonian precisely ensure that any trajectory of the perturbed system satisfies one of the two following properties. Either, it admits a projection onto the action space over exponentially long time intervals included in a small area or the previous sequence of times $(t_1, \dots, t_n)$ and the basis $(\omega_1, \dots, \omega_n)$ can be built iteratively. The closeness of $\nabla h(I(t_k))$ to ω_k for $k \in \{1, \dots, n\}$ is given by an application of a theorem of Dirichlet about simultaneous Diophantine approximation, which yields a minimal rate of approximation of an arbitrary vector by rational vectors. This last argument gives an upper bound on the order K of normalization which can be carried out and imposes our value of the stability exponents a and b.

A.II Normal forms and non-resonant areas.

In order to avoid cumbersome expressions, we use the notations $u \lessdot\!\!* \, v$ (resp. $u *\!\!\lessdot \, v$, $u =\!\!* \, v$, $u *\!\!= v$) if there is $0 < \mathbf{C} \leq 1$ such that $u < \mathbf{C}v$ (resp. $u\mathbf{C} < v$, $u = \mathbf{C}v$ or $u\mathbf{C} = v$) and the constant $\mathbf{C}$ depends only on the dimension n, the bound M and the exponent τ but not on the small parameters ε and γ.

A.II.1 Normal forms

Let Λ be a sublattice of $\mathbb{Z}^n$. A subset $\mathcal{D} \subset B_R \subset \mathbb{R}^n$ is said to be (α, K)-non-resonant modulo Λ if at every point $I \in \mathcal{D}$, we have:

$$|k.\nabla h(I)| = |k.\omega(I)| \geq \alpha \text{ for all } I \in \mathcal{D} \text{ and } k \in \mathbb{Z}^n_K \backslash \Lambda$$

where $\mathbb{Z}_K^n = \{k \in \mathbb{Z}^n \text{ such that } ||k||_1 \leq K\}$ with a fixed $K > 0$.

In the neighborhood of such a set $\mathcal{D}$, the perturbed Hamiltonian $\mathcal{H}$ can be put in a Λ-resonant normal form $h + g + f_*$ where the Fourier expansion of g contains only harmonics in $\mathbb{Z}_K^n \cap \Lambda$ while the remainder f_* is a small general term.

Lemma A.II.1 (normal form, [18])

Suppose that $\mathcal{D} \subset B_R$ is (α, K)-non-resonant modulo Λ and that the following inequalities hold:

$$\varepsilon \lessdot \frac{\alpha r}{K} \;;\; r \lessdot \mathrm{Min}\left(\frac{\alpha}{K}, R\right) \;;\; 1 \ast\!\!< K.$$

Then we can define a real analytic, symplectic diffeomorphism $\Phi : V_{r_*,s_*}\mathcal{D} \longmapsto V_{r,s}\mathcal{D}$ where $r_* = \frac{r}{2}$, $s_* = \frac{s}{6}$ such that the pull-back of $\mathcal{H}$ by Φ is a Λ-resonant normal form $\mathcal{H} \circ \Phi = h + g + f_*$ up to a remainder f_* with

$$||g||_{r_*,s_*} \ast\!\!< \varepsilon \;\; \textit{and} \;\; ||f_*||_{r_*,s_*} \ast\!\!< \varepsilon \exp\left(-cK\right),$$

where c is some positive constant.

Moreover, $||\Pi_I \circ \Phi - Id_I||_{r_*,s_*} \lessdot r/6$ uniformly over $V_{r_*,s_*}\mathcal{D}$ where Π_I denotes the projection onto the action space and Id_I is the identity in the action space. Hence:

$$V_{r/3}\mathcal{D} \subset \Pi_I\left(\Phi\left(V_{r_*,s_*}\mathcal{D}\right)\right) \subset V_{2r/3}\mathcal{D}.$$

Corollary A.II.2.

With the notations of the previous lemma, consider a solution of the normalized system governed by $\mathcal{H} \circ \Phi$ and a time $t_k \in \mathbb{R}$. Let λ_k be the affine subspace which contains $I(t_k)$ and whose direction $\Lambda \otimes \mathbb{R}$ is the vector space spanned by Λ, then

$$\mathrm{dist}\left(I(t), \lambda_k\right) = ||I(t) - \lambda_k|| \ast\!\!< \varepsilon \textit{ for } |t - t_k| \lessdot \exp(cK) \textit{ and } |t| < \mathcal{T}_*$$

where $\mathcal{T}_*$ is the time of escape of $V_{r_*,s_*}\mathcal{D}$.

Proof: We denote by $\mathcal{Q}$ the orthogonal projection on $< \Lambda >^{\perp}$. Since $H \circ \Phi$ is in Λ-resonant normal form, we have $\frac{d}{dt}\mathcal{Q}(I(t)) = -\mathcal{Q}\left(\partial_\varphi f_*\right)$ and

$$||I(t) - \lambda_k|| \leq ||\mathcal{Q}\left(I(t) - I(t_k)\right)|| \leq ||\mathcal{Q}\left(\partial_\varphi f_*(I,\varphi)\right)||_{r_*,s_*} |t - t_k| \ast\!\!< \varepsilon$$

provided that $|t - t_k| \lessdot \exp(cK)$.▶

A.III Nearly periodic tori.

A vector $\omega \in \mathbb{R}^n$ is said to be *rational* if there is $t > 0$ such that $t\omega \in \mathbb{Z}^n$, in which case $T = \mathrm{Inf}\{t > 0 \;/\; t\omega \in \mathbb{Z}^n\}$ is called the *period* of ω.

Definition and theorem A.III.1

Consider $\delta > 0$ and $\omega \in \mathbb{R}^n \backslash \{0\}$ a rational vector of period T, the set

$$\mathcal{B}_\varrho(\omega) = \{I \in B_R \text{ such that } ||\nabla h(I) - \omega|| < \varrho\}$$

is called a nearly periodic torus.

For $K > 0$ and $\delta > 0$ such that $2MK\delta T < 1$, the set $\mathcal{B}_\delta(\omega)$ is $(\frac{1}{2T}, K)$-non-resonant modulo the $\mathbb{Z}$-module Λ spanned by $\mathbb{Z}^n_K \cap < \omega >^\perp$.

Proof:

Lemma A.III.2 *Let Ω be the hyperplane $< \omega >^\perp$, then for all $k \in \mathbb{Z}^n \backslash \Omega$ we have $|k.\omega| \geq 1/T$.*

Proof: We have $|k.\omega| = \frac{1}{T} |k.T\omega| = \frac{1}{T} |k.\alpha|$ for some $\alpha \in \mathbb{Z}^n$ and $|k.\alpha| \neq 0$ since $k \notin < \alpha >^\perp = < \omega >^\perp$.
Hence $|k.\alpha| \geq 1$ and $|k.\omega| \geq \frac{1}{T}$. ►

Then, since $\omega \neq 0$, we can ensure that $\dim(\Lambda) = n - 1$ and for all $I \in \mathcal{B}_\delta(\omega)$:

$$\forall k \in \mathbb{Z}^n_K \backslash \Lambda, \text{ we have } |k.\nabla h(I)| \geq |k.\omega| - |k| \, ||\nabla h(I) - \omega|| \geq \frac{1}{T} - MK\delta > \frac{1}{2T}.$$

with our threshold in the lemma. ►

Now, for an integer $m \in \{1, \dots, n\}$, consider a decreasing sequence of positive real numbers $\varrho_1 \geq \dots \geq \varrho_m$ and m rational vectors $(\omega_1, \dots, \omega_m)$ in $\mathbb{R}^n$ with respective periods *bounded* by an increasing sequence $T_1 \leq \dots \leq T_m$ such that $||\omega_{j+1} - \omega_j|| \leq \varrho_j$ for $j \in \{1, \dots, m-1\}$.

For $j \in \{1, \dots, m\}$, we denote by Ω_j the hyperplanes $\langle \omega_j \rangle^\perp$ and by $\mathcal{I}_j$ the sets $\Omega_1 \cap \dots \cap \Omega_j$.

Consider a positive constant K then the $\mathbb{Z}$-module (resp. the $\mathbb{R}$-vector space) spanned by $\mathbb{Z}^n_K \cap \mathcal{I}_j$ is denoted by Λ_j (resp. $\Lambda_j \otimes \mathbb{R}$).

Lemma A.III.3

With the previous notations, assume that the rational vectors $(\omega_1, \dots, \omega_m)$ are linearly independent, we can ensure that $\dim(\Lambda_j \otimes \mathbb{R}) \leq n - j$ for all $j \in \{1, \dots, m\}$ and if $2M(m - j + 1)K\varrho_j T_j < 1$ ($\forall j \in \{1, \dots, m\}$) then the nearly periodic tori:

$$\mathcal{B}_j = \{I \in B_R \text{ such that } ||\nabla h(I) - \omega_j|| < (m - j + 1)\varrho_j\}$$

are $\left(\frac{1}{2T_j}, K\right)$-non-resonant modulo Λ_j for $j \in \{1, \dots, m\}$.

Proof: Consider $j \in \{2, \dots, m\}$ and $I \in \mathcal{B}_j$, since the sequence $(\varrho_l)_{1 \leq l < j}$ is decreasing, we have $||\omega_l - \omega_j|| \leq \varrho_l + \dots + \varrho_{j-1} \leq (j - 1 - l)\varrho_l$ for all $l \in \{1, \dots, j-1\}$ and the assumption $||\nabla h(I) - \omega_j|| \leq (m - j + 1)\varrho_j$ yields:

$$||\nabla h(I) - \omega_l|| \leq (m - l)\varrho_l \leq (m - l + 1)\varrho_l \text{ for all } l \in \{1, \dots, j-1\}.$$

Then, the argument of the previous lemma (A.III.2) ensures that, for all $I \in \mathcal{B}_j$

$$\forall l \in \{1, \ldots, j\}\ ,\ \forall k \in \mathbb{Z}^n_K \backslash \Omega_l \text{ we have } |k.\nabla h(I)| \geq \frac{1}{T_l} - M(m-l+1)K\varrho_l \geq \frac{1}{2T_l}$$

with the thresholds of lemma A.III.3.

Hence, for all $k \in \cup_{l=0}^{j} \mathbb{Z}^n_K \backslash \Omega_l = \mathbb{Z}^n_K \backslash \cap_{l=0}^{j} \Omega_l = \mathbb{Z}^n_K \backslash \Lambda_j$, the scalar products $|k.\nabla h(I)|$ are lowered by $\frac{1}{2T_j}$ for any $I \in \mathcal{B}_j$.▶

A.IV Proof of the main theorem

Now, we build the sequence of rational vectors described in the previous section along any trajectory and ensure that a drift of the action variables leads to in an area which is free of resonances up to an exponentially small order. In this case, the main theorem of the paper is proved.

We will first build it formally with our assumption of steepness and, in the next section, the use of a simultaneous Diophantine approximation theorem of Dirichlet will prove that our formal scheme is convergent for an arbitrary trajectory.

Consider an arbitrary trajectory $(I(t), \varphi(t))$ of the perturbed system.

Let $(\omega_1, \ldots, \omega_m)$ be a sequence of rational vectors whose periods are bounded by $T_1 \leq \ldots \leq T_m$. As before, for $j \in \{1, \ldots, m\}$ we define $\Omega_j = \langle \omega_j \rangle^{\perp}$ and $\mathcal{I}_j = \Omega_1 \cap \ldots \cap \Omega_j$; let also Λ_j be the $\mathbb{Z}$-module spanned by $\mathbb{Z}^n_K \cap \mathcal{I}_j$, and finally $d_j = \dim(\Lambda_j \otimes \mathbb{R})$.

Definition A.IV.1 (fitted sequence)

The sequence of rational vectors $(\omega_1, \ldots, \omega_m)$ is a fitted sequence for a trajectory with an initial time t_0 if there are increasing real numbers $t_1 \leq \ldots \leq t_m$, decreasing positive real numbers $r_0 \geq \ldots \geq r_m$ and constants $K \ggg 1$ and $0 < \gamma < \frac{R}{3(m+1)}$ such that the following seven properties hold:

(i) $r_0 < \gamma$ *and* $r_1 < \mathrm{Inf}\left(\dfrac{\gamma}{K^\tau}, \dfrac{r_0^{p_n}}{2M}\right)$;

(ii) $t_j < t_0 + \exp(cK)$ *and* $||\nabla h(I(t_j)) - \omega_j|| \leq r_j$ *for* $j \in \{1, \ldots, m\}$;

(iii) $||\omega_{j+1} - \omega_j|| \leq \varrho_j = 3r_j$ *for* $j \in \{1, \ldots, m-1\}$;

(iv) *The dimensions* $(d_1, \ldots, d_m)$ *satisfies* : $d_1 > \ldots > d_j > \ldots > d_m = 0$;

(v) $r_{j+1} < \dfrac{1}{6}\left(\dfrac{r_j}{2M}\right)^{p_{d_k}}$ *for* $j \in \{1, \ldots, m-1\}$;

(vi) $2M(m-j+1)K\varrho_j T_j = 6M(m-j+1)Kr_jT_j < 1$ *for* $j \in \{1, \ldots, m\}$;

(vii) *The thresholds of lemma A.II.1 are satisfied with the parameters* ε, K, r_j *and* $\dfrac{1}{2T_j}$.

Rk : (i) The thresholds of lemma A.III.3 are satisfied according to (vi) and our last two assumptions make it possible to build a normalizing transformation Φ_j with respect to Λ_j on the complex $r_j/2$-neighborhoods of the nearly periodic tori $\mathcal{B}_j$:

$$V_{r_j/2}(\mathcal{B}_j) = V_{r_j/2}\left(\{I \in B_R \text{ such that } ||\nabla h(I) - \omega_j|| < (m-j+1)\varrho_j\}\right).$$

(ii) In the sequel, we will denote $\rho_0 = p_n$ and $\rho_k = p_{d_k}$.

Formal construction of a fitted sequence.
Theorem A.IV.2

Consider an arbitrary trajectory $(I(t), \varphi(t))$ of the perturbed system. If the perturbation is small enough, one of the following properties holds

(i) $I(t)$ is bounded over exponentially long time intervals

(ii) or, assuming that there are periodic vectors satisfying the assumptions of the previous definition A.IV.1, then the trajectory admits a fitted sequence.

Rk : (i) In any cases, the actions are bounded over exponentially long times.

(ii) We make our formal construction going forward in time but the same results are valid backward in time.

Proof : first step

Consider an arbitrary solution of the perturbed system with some initial condition $(I(t_0), \varphi(t_0))$ and a time $t_0^* \in [t_0, t_0 + \exp(cK)]$ such that $||I(t_0^*) - I(t_0)|| = r_0 \leq \gamma$, then our steepness assumption yields a time $t_1 \in [0, t_0^*]$ such that $||\nabla h(I(t_1))|| \geq r_0^{\rho_0}$ and $||I(t_1) - I(t_0)|| \leq r_0$; moreover

$$||\nabla h(I)|| \geq \frac{r_0^{\rho_0}}{2} \text{ for all } ||I - I(t_1)|| \leq \frac{r_0^{\rho_0}}{2M}.$$

In the regular case, we remove this first step since we can use a uniform lower bound on the gradient $\nabla h(I)$.

Assume that there exists a rational vector ω_1 with a period bounded by T_1 such that :

$$||\nabla h(I(t_1)) - \omega_1|| < r_1 \leq \text{Inf}\left(\frac{\gamma}{K^\tau}, \frac{r_0^{\rho_0}}{2M}\right)$$

and that $\left(\varepsilon, K, r_1, \frac{1}{2T_1}\right)$ satisfy the thresholds of lemma A.II.1 (we only need $2T_1 K\varepsilon \lessdot\!\!* \; r_1$ and $2T_1 r_1 K \lessdot\!\!* \; 1$ since $r_1 \leq r_0 < R$ by assumptions), then we have :

Lemma A.IV.3.

Denoting the projections $\text{Proj}_{\Omega_1} = \text{Proj}_{\overrightarrow{\Lambda_1}}$ *and* $\text{Proj}_{\langle\omega_1\rangle}$ *by* $\mathcal{Q}_1$ *and* $\widetilde{\mathcal{Q}}_1$*, if*

$$\varepsilon \lessdot\!\!* \; \frac{1}{2M}\left(\frac{r_1}{2M}\right)^{\rho_1} \quad \text{with } \rho_1 = p_{n-1} = p_{d_1},$$

then there exists a symplectic analytic transformation Φ_1 defined on a neighborhood of $I(t_1)$ such that in the normalized coordinates, we have :

$$(*) \text{ Either } ||I(t) - I(t_1)|| < \frac{r_1}{M} \text{ for all } t \in [t_0, t_0 + \exp(cK)]$$

$$(**) \text{ otherwise } \exists t_2 \in]t_0, t_0 + \exp(cK)[\text{ such that:}$$

$$||I(t_2)-I(t_1)|| \ < \frac{r_1}{M} \ \textit{and} \ ||\mathcal{Q}_1\left(\nabla h(I(t_2))\right)|| \geq \frac{1}{2}\left(\frac{r_1}{2M}\right)^{\rho_1}.$$

Proof :

Consider the domain $\mathcal{B}_1=\{I\in B_R \text{ such that } ||\nabla h(I)-\omega_1||<mr_1\}$. With our last two thresholds in the definition of a fitted sequence, there exists a normalization Φ_1 with respect to Λ_1 from $\mathcal{D}_1=\mathcal{V}_{r_1/2}\left(\mathcal{B}_1\right)$ to $\mathcal{V}_{r_1}(\mathcal{B}_1)$.

In the *normalized* coordinates, we use the canonical equations of motion and, according to the corollary A.II.2, we derive :

$$||I(t)-I(t_1)-\mathcal{Q}_1\left(I(t)-I(t_1)\right)||=\left|\left|\widetilde{\mathcal{Q}}_1\left(I(t)-I(t_1)\right)\right|\right| \ll \varepsilon$$

for $t\in[t_0,\mathrm{Inf}\,(t_0+\exp(cK);t_0^*)]$.

Then, our threshold on ε in lemma A.IV.3. and $\rho_1\geq 1$ yields :

$$||\nabla h\left(I(t)\right)-\nabla h\left(I(t_1)+\mathcal{Q}_1\left(I(t)-I(t_1)\right)\right)||\leq\frac{M}{2M}\left(\frac{r_1}{2M}\right)^{\rho_1}=\frac{1}{2}\left(\frac{r_1}{2M}\right)^{\rho_1} \qquad (A)$$

finally $I(t_1)+\mathcal{Q}_1\left(I(t)-I(t_1)\right)$ is a continuous path in $\lambda_1=I(t_1)+\Lambda_1\otimes\mathbb{R}$, hence we have :

$(*)$ either $||\mathcal{Q}_1\left(I(t)-I(t_1)\right)||\leq\dfrac{r_1}{2M}\Longrightarrow||I(t)-I(t_1)||\leq\dfrac{r_1}{M}$ for all $t\in[t_0,t_0+\exp(cK)]$

$(**)$ or there is a time of escape $t_1^*\in]t_0,t_0+\exp(cK)[$ with $||\mathcal{Q}_1(I(t_1^*)-I(t_1))||=\dfrac{r_1}{2M}$.

Then, the steepness of h yields $t_2\in[t_1,t_1^*]$ such that :

$$\left|\left|\mathrm{Proj}_{\overrightarrow{\Lambda_1}}\left(\nabla h\left(I(t_1)+\mathcal{Q}_1\left(I(t_2)-I(t_1)\right)\right)\right)\right|\right|\geq\left(\frac{r_1}{2M}\right)^{\rho_1} \text{ and } ||\mathcal{Q}_1\left(I(t_2)-I(t_1)\right)||\leq\frac{r_1}{2M}$$

and the inequality (A) implies

$$||\mathcal{Q}_1\left(\nabla h\left(I(t_2)\right)\right)||=\left|\left|\mathrm{Proj}_{\overrightarrow{\Lambda_1}}\left(\nabla h\left(I(t_2)\right)\right)\right|\right|\geq\frac{1}{2}\left(\frac{r_1}{2M}\right)^{\rho_1}$$

moreover $||I(t_2)-I(t_1)||\leq\dfrac{r_1}{M}$.▶

Now, we assume that there is a rational vector ω_2 with a period bounded by $T_2\geq T_1$ such that $||\nabla h(I(t_2))-\omega_2||\leq r_2$ with $r_2\leq\dfrac{1}{6}\left(\dfrac{r_1}{2M}\right)^{\rho_1}<r_1$.

Then, we can write :

$$||\mathcal{Q}_1\left(\omega_2\right)||\geq||\mathcal{Q}_1\left(\nabla h\left(I(t_2)\right)\right)||-||\mathcal{Q}_1\left(\nabla h(I(t_2))-\omega_2\right)||\geq\frac{1}{3}\left(\frac{r_1}{2M}\right)^{\rho_1}$$

hence (ω_1, ω_2) are linearly independent and we can ensure that $d_2 < d_1$, moreover

$$||\omega_1 - \omega_2|| \leq ||\omega_1 - \nabla h(I(t_1))|| + ||\nabla h(I(t_1)) - \nabla h(I(t_2))|| + ||\nabla h(I(t_2)) - \omega_2|| \leq 3r_1$$

In order to complete the proof of the first step of theorem A.IV.2. we assume that

$$2T_2 K\varepsilon <\!\!* \; r_2 \; ; \; 2T_2 r_2 K <\!\!* \; 1 \text{ and } 6M(m-1)Kr_2T_2 <\!\!* \; 1$$

which allows to apply the lemma A.II.1 and A.III.3.

Recurrence

Assume that an increasing sequence of times $t_1 \leq \ldots \leq t_j$ and a sequence of periodic vectors $(\omega_1, \ldots, \omega_j)$ with respective periods bounded by $T_1 \leq \ldots \leq T_j$ which satisfy the assumptions of a fitted sequence have been built up to order $j \in \{0, \ldots, m-1\}$, then a repeat of the arguments of lemma A.IV.3 shows:

Lemma A.IV.4.

If $\varepsilon <\!\!* \; \frac{1}{2M}\left(\frac{r_j}{2M}\right)^{\rho_j}$ *with* $\rho_j = p_{d_j}$*, then there exists a symplectic analytic transformation* Φ_j *defined on a neighborhood of the action* $I(t_j)$ *such that in the normalized coordinates, we have:*

$(*)$ *Either* $||I(t) - I(t_j)|| < \frac{r_j}{M}$ *for all* $t \in [t_j, t_0 + \exp(cK)]$

$(**)$ *or there is* $t_{j+1} \in \,]t_j, t_0 + \exp(cK)[$ *such that* $\left|\left|\mathrm{Proj}_{\Lambda_j \otimes \mathbb{R}}\left(\nabla h(I(t_{j+1}))\right)\right|\right| \geq \frac{1}{2}\left(\frac{r_j}{2M}\right)^{\rho_j}$

Now, we assume that there exists a rational vector ω_{j+1} with a period bounded by T_{j+1} such that

$$||\nabla h\left(I(t_{j+1})\right) - \omega_{j+1}|| \leq r_{j+1} \text{ with } r_{j+1} \leq \frac{1}{6}\left(\frac{r_j}{2M}\right)^{\rho_j},$$

which implies that $\left|\left|\mathrm{Proj}_{\Lambda_j \otimes \mathbb{R}}(\omega_{j+1})\right|\right| \geq \frac{1}{3}\left(\frac{r_j}{2M}\right)^{\rho_j}$ and $||\omega_{j+1} - \omega_j|| < 3r_j$.

Hence $< \omega_{j+1} >^{\perp} \not\subset \Lambda_j \otimes \mathbb{R}$ and we can ensure that $d_{j+1} < d_j$.

To complete the iterative proof of theorem A.IV.2. again we assume that

$$2T_{j+1}K\varepsilon <\!\!* \; r_{j+1} \; ; \; 2T_{j+1}r_{j+1}K <\!\!* \; 1 \text{ and } 6(m-j)KMr_{j+1}T_{j+1} <\!\!* \; 1.$$

Hence, the construction of a fitted sequence is completed for an arbitrary trajectory provided that the thresholds in definition $A.IV.1.$ and lemmas $A.IV.3.$ and $A.IV.4.$ are satisfied, this completes the proof of theorem $A.IV.2.$ ▶

Last step
Since $d_m = 0$, we have located a resonant-free area up to $\frac{1}{2T_m}$ given by

$$\mathcal{B}_m = \{I \in B_R \text{ such that } ||\nabla h(I) - \omega_m|| < \varrho_m = 3r_m\}$$

and, in order to confine the actions, we only need to build the integrable normal form over the domain $\mathcal{D}_m = V_{r_m}(\mathcal{B}_m)$. This is possible provided that the parameters $\left(\varepsilon, K, \dfrac{1}{2T_m}, r_m\right)$ satisfy only the thresholds of lemma A.II.1. and A.III.3 but we don't need to control the dynamics as in the lemma A.IV.4.

A.V Complete construction of a fitted sequence.

Now, we have to ensure the existence of a sequence of periodic vectors $(\omega_1, \dots, \omega_m)$ such that $||\nabla h(I(t_k)) - \omega_k|| < r_k$ for $k \in \{1, \dots, m\}$ with $r_k \leq \frac{1}{6}\left(\frac{r_{k-1}}{2M}\right)^{\rho_{k-1}}$.

We assume that $d_k = \mathrm{Dim}(\Lambda_k \otimes \mathbb{R}) = n - k$ hence $m = n$, with this assumption we have the lowest rate of decrease of the dimension of the resonant module and, consequently, the worst estimates on the radius and the time of stability.

In the sequel, we will denote $\pi_1 = 1$ and $\pi_k = \prod_{l=1}^{k-1} \rho_l = \prod_{l=1}^{k-1} p_{n-l}$ for $k \in \{2, \dots, n\}$.

Now, we introduce the Dirichlet's theorem on simultaneous Diophantine approximation (see Lochak, [10]), which gives the lowest rate of approximation of an arbitrary vector by an integer one.

Let $x \in \mathbb{R}^n$ and $Q \in \mathbb{N}^*$, we can renumber the indices in such a way that $x = \xi(\pm 1, x')$ for some $x' \in \mathbb{R}^{n-1}$ and $\xi = ||x||_\infty$ where $||\ ||_\infty$ is the maximum of the components.

The question is now reduced to approximation in $\mathbb{R}^{n-1}$. Dirichlet's theorem yields $q \in \mathbb{N}^*$ and $l' \in \mathbb{Z}^{n-1}$ such that $1 \leq q < Q$ and $||qx' - l'||_\infty \leq Q^{-\frac{1}{n-1}}$.

If $x^* = \xi\left(\pm 1, \dfrac{l'}{q}\right)$ we have:

$$||x^* - x||_\infty \leq \frac{\xi}{q} Q^{-\frac{1}{n-1}} \Longrightarrow ||x^* - x|| \leq \sqrt{n-1}\frac{\xi}{q} Q^{-\frac{1}{n-1}}$$

for the euclidean norm.

One checks easily that x^* is a rational vector of period $\mathcal{T} = \dfrac{q}{\xi}$ which satisfies

$$||x^* - x|| \leq \frac{\sqrt{n-1}}{\mathcal{T} Q^{\frac{1}{n-1}}} \text{ with } \frac{1}{\xi} \leq \mathcal{T} \leq \frac{Q}{\xi}.$$

If $x_k = \nabla h(I(t_k))$, with our assumptions on h, we have

$$\xi_k = ||x_k||_\infty \leq ||\nabla h(I(t_k))|| \leq M \text{ and } \xi_k = ||x_k||_\infty \geq \frac{1}{n}||\nabla h(I(t_k))|| \geq \frac{r_0^{\rho_0}}{2n}.$$

Hence, for all $k \in \{1, \ldots, n\}$, Dirichlet's theorem ensures the existence of a rational vector $x_k^* \in \mathbb{R}^n$ of period $\mathcal{T}_k$ such that

$$||x_k - x_k^*|| \leq \frac{\sqrt{n-1}}{\mathcal{T}_k Q_k^{\frac{1}{n-1}}} \text{ with } \frac{1}{M} \lessdot\!\!* \; \mathcal{T}_k \lessdot\!\!* \; \frac{2n}{r_0^{\rho_0}} Q_k$$

and in the regular case, we have the bounds $\frac{1}{M} \lessdot\!\!* \; \mathcal{T}_k \lessdot\!\!* \; Q_k$.

With this ingredient, we prove the following:

Theorem A.V.1.

There exists a positive constant C (resp. K_0) sufficiently large (resp. small) such that with

$$\beta = \frac{1}{1 + 2n\pi_n} \; ; \; a = \frac{\beta}{1+\tau} \; ; \; b = \frac{\beta}{\rho_0} \text{ and } \varepsilon \lessdot\!\!* \; \gamma^{1/a}$$

there is a fitted sequence for an arbitrary trajectory of the perturbed system with the parameters $K = K_0 \varepsilon^{-a}$ and $r_0 = \widetilde{C} \varepsilon^b$ with $\widetilde{C} = C^{1/\rho_0}$; $T_l \lessdot\!\!* \; \left(C\varepsilon^\beta\right)^{-n\pi_l}$ and $r_l \leq \left(C\varepsilon^\beta\right)^{\pi_l}$ for any $l \in \{1, \ldots, n\}$.

Rk: In the regular case where h does not admit critical points, one can find a fitted sequence with the parameters

$$a = \frac{1}{1 + (2n-1)(1+\tau)p_1 \ldots p_{n-1}} \; ; \; b = (1+\tau)a \text{ and } \varepsilon \leq C_1 \gamma^{1/a}.$$

Summary of the thresholds

We need upper bounds $T_1 \leq T_2 \leq \ldots \leq T_n$ on the periods $(\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_n)$ such that:

$$(i) \; 1 \lessdot\!\!* \; K \; ; \; (ii) \; r_0 < \gamma \; (iii) \; r_1 < \frac{\gamma}{K^\tau}$$

and for $k \in \{1, \ldots, n\}$:

$$(iv) \; \varepsilon \lessdot\!\!* \; r_{k-1}^{\rho_{k-1}} \; (v) \; \varepsilon T_k K \lessdot\!\!* \; r_k \; (vi) \; K T_k r_k \lessdot\!\!* \; 1 \; (vii) \; r_k \lessdot\!\!* \; r_{k-1}^{\rho_{k-1}}.$$

The iterative scheme

With the period $\mathcal{T}_1$ of the first rational vector, as in Lochak's paper we start with $r_1 = C\frac{\varepsilon^\beta}{\mathcal{T}_1}$ where β and C are positive constants.

Hence we need:

$$||\nabla h\left(I\left(t_1\right)\right) - x_1^*|| \leq \frac{\sqrt{n-1}}{\mathcal{T}_1 Q_1^{\frac{1}{n-1}}} < C\frac{\varepsilon^\beta}{\mathcal{T}_1} \text{ where } 1 \lessdot\!\!* \; \mathcal{T}_1 \leq \frac{2n}{r_0^{\rho_0}} Q_1$$

and we choose $Q_1 = \left(C\varepsilon^{\beta}\right)^{-n+1}$.

Here, thresholds (i) to (vii) are ensured if we choose $T_1 = \mathcal{T}_1$ and K, r_0, a, b and β as in the theorem A.V.1. with C (resp. K_0) large enough (resp. small enough) and $\varepsilon <\!\!* \gamma^{1/b}$.

For the regular case, we can choose $r_0 = r_1$, $a = \dfrac{\beta}{1+\tau}$, $b = \beta$ and $0 < a \leq \dfrac{1}{2n}$ always with the threshold $\varepsilon <\!\!* \gamma^{1/a}$.

Then, for $l \in \{2, \dots, n\}$, we assume the existence of a sequence of rational vectors $(\omega_1, \dots, \omega_{l-1})$ with periods bounded by $T_1 \leq \dots \leq T_{l-1}$ such that $||\nabla h\left(I(t_k)\right) - \omega_k|| \leq r_k$ with the parameters $(\varepsilon, K, T_k, r_k)$ which satisfy the thresholds (i) to (vii).

According to the previous paragraph we can find a time t_l with $I\left(t_l\right)$ in an area linked to resonances of lower multiplicity.

Now ω_l is build in the following way to ensure the last threshold $r_k <\!\!* r_{k-1}^{\rho_{k-1}}$:

— if there is a rational vector ω_l of period $\mathcal{T}_l$ such that $||\nabla h\left(I(t_l)\right) - \omega_l|| <\!\!* r_{l-1}^{\rho_{l-1}}$ and $\mathcal{T}_l < T_{l-1}^{\rho_{l-1}}$ then we *set* $r_l =\!\!* r_{l-1}^{\rho_{l-1}}$ and $T_l = T_{l-1}^{\rho_{l-1}}$,

— otherwise, we use Dirichlet's theorem to find a rational vector ω_l of period $\mathcal{T}_l$ such that $||\nabla h\left(I(t_l)\right) - \omega_l|| <\!\!* r_{l-1}^{\rho_{l-1}}$ with $T_{l-1}^{\rho_{l-1}} \leq \mathcal{T}_l \leq Q_l$ for some Q_l, then we *set* $r_l = \dfrac{\sqrt{n-1}}{\mathcal{T}_l Q_l^{\frac{1}{n-1}}}$ with $T_l = \mathcal{T}_l$ and $1 \;*\!\!< \mathcal{T}_l \leq \dfrac{2n}{r_0^{\rho_0}} Q_l$.

Let $(j_1, \dots, j_q)$ be the subsequence of $\{1, \dots, n\}$ where $\mathcal{T}_{j_k} = T_{j_k}$, hence $j_1 = 1$.

Then for $l \in \{j_k+1, \dots, j_{k+1}-1\}$, we have $\mathcal{T}_l < T_l$ and $r_l =\!\!* r_{l-1}^{\rho_{l-1}}$ which imply $r_{j_{k+1}-1} =\!\!* r_{j_k}^{\rho_{j_{k+1}-2} \cdots \rho_{j_k}}$ and we want $r_{j_{k+1}} <\!\!* r_{j_k}^{\rho_{j_{k+1}-1} \cdots \rho_{j_k}}$.

We use the Dirichlet's theorem at the j_k-th and j_{k+1}-th steps of the iterative scheme, with $\pi_l = \displaystyle\prod_{i=1}^{l-1} \rho_i$ for $l \in \{2, \dots, n\}$, hence we need:

$$\frac{1}{\mathcal{T}_{j_{k+1}} Q_{j_{k+1}}^{1/(n-1)}} <\!\!* \left(\frac{1}{\mathcal{T}_{j_k} Q_{j_k}^{1/(n-1)}}\right)^{\frac{\pi_{j_{k+1}}}{\pi_{j_k}}} \Longleftrightarrow \frac{1}{Q_{j_{k+1}}^{1/(n-1)}} <\!\!* \frac{\mathcal{T}_{j_{k+1}}}{\mathcal{T}_{j_k}^{\rho_{j_{k+1}-1} \cdots \rho_{j_k}}} \left(\frac{1}{Q_{j_k}^{1/(n-1)}}\right)^{\frac{\pi_{j_{k+1}}}{\pi_{j_k}}}$$

since $\mathcal{T}_{j_k}^{\rho_{j_{k+1}-1} \cdots \rho_{j_k}} = T_{j_k}^{\rho_{j_{k+1}-1} \cdots \rho_{j_k}} \leq T_{j_{k+1}} = \mathcal{T}_{j_{k+1}}$, we set $Q_{j_{k+1}} \;*\!\!= Q_{j_k}^{\pi_{j_{k+1}}/\pi_{j_k}}$ and

$$Q_{j_k} \;*\!\!= \left(*Q_{j_{k-1}}\right)^{\pi_{j_k}/\pi_{j_{k-1}}} = \left(*Q_{j_{k-2}}\right)^{\pi_{j_k}/\pi_{j_{k-2}}} = \dots = Q_{j_1}^{\pi_{j_k}} = Q_1^{\pi_{j_k}}$$

Finally, the bounds $Q_{j_k} \;*\!\!= \left(C\varepsilon^{\beta}\right)^{-(n-1)\pi_{j_k}}$ for $k \in \{1, \dots, q\}$ are sufficient to ensure our last threshold.

The previous inequality $r_l <\!\!* r_{l-1}^{\rho_{l-1}}$ is also satisfied by construction for any l in $\{1, \dots, n-1\} \backslash \{j_1, \dots, j_q\}$.

Then, we should check the other thresholds.

First, consider $l \in \{1, \dots, n\}$ with $\mathcal{T}_l = T_l$, hence $l \in \{j_1, \dots, j_q\}$ and

$$r_l = \frac{\sqrt{n-1}}{T_l Q_l^{\frac{1}{n-1}}} \text{ with } 1 \mathrel{\ast\!\!<} T_l \leq \frac{2n}{r_0^{\rho_0}} Q_l \text{ and } Q_l \mathrel{\ast\!\!=} \left(C\varepsilon^\beta\right)^{-(n-1)\pi_l}$$

which imply $T_l r_l \mathrel{\ast\!\!=} \left(C\varepsilon^\beta\right)^{\pi_l}$ and $\dfrac{1}{r_l} \mathrel{\ast\!\!=} \dfrac{T_l}{r_l} = \dfrac{T_l^2 Q_l^{\frac{1}{n-1}}}{\sqrt{n-1}} \mathrel{\ast\!\!<} \dfrac{\left(C\varepsilon^\beta\right)^{-(2n-1)\pi_l}}{r_0^{2\rho_0}}$.

For the second case, consider $l \in \{1, \dots, n\}$ with $\mathcal{T}_l < T_l$, hence $l \in \{j_k+1, \dots, j_{k+1}-1\}$ for some $k \in \{0, \dots, q-1\}$. With our construction, we have $r_l \mathrel{=\!\!>\!\!\ast} r_{j_k}^{\pi_l/\pi_{j_k}}$, $T_l = T_{j_k}^{\pi_l/\pi_{j_k}}$ and $Q_{j_k} \mathrel{\ast\!\!=} \left(C\varepsilon^\beta\right)^{-(n-1)\pi_{j_k}}$ imply $T_l r_l \mathrel{=\!\!>\!\!\ast} (r_{j_k} T_{j_k})^{\pi_l/\pi_{j_k}}$ and $T_l r_l \mathrel{\ast\!\!<} (C\varepsilon^\beta)^{\pi_l}$ together with $\dfrac{1}{r_l} \mathrel{\ast\!\!=} \dfrac{T_l}{r_l} \mathrel{\ast\!\!=} \left(\dfrac{T_{j_k}}{r_{j_k}}\right)^{\pi_l/\pi_{j_k}}$ and $\dfrac{1}{r_l} \mathrel{\ast\!\!=} \dfrac{T_l}{r_l} \mathrel{\ast\!\!<} \left(\dfrac{(C\varepsilon^\beta)^{-(2n-1)\pi_{j_k}}}{r_0^{2\rho_0}}\right)^{\pi_l/\pi_{j_k}}$

With $r_0 \leq 1$ and $\dfrac{\pi_l}{\pi_{j_k}} \leq \pi_l$, we see that the inequalities:

$$T_l r_l \mathrel{\ast\!\!<} \left(C\varepsilon^\beta\right)^{\pi_l} \text{ and } \frac{1}{r_l} \mathrel{\ast\!\!=} \frac{T_l}{r_l} \mathrel{\ast\!\!<} \left(\frac{\left(C\varepsilon^\beta\right)^{-2n+1}}{r_0^{2\rho_0}}\right)^{\pi_l}$$

are satisfied at any step of our iterative scheme.

In the regular case without critical points, we obtain the same estimates but with one instead of $r^{2\rho_0}$ at the denominator.

Finally, the previous choices for Q_{j_k} give the bounds:

$$T_l \mathrel{\ast\!\!<} \left(C\varepsilon^\beta\right)^{-n\pi_l} \text{ and } r_l \mathrel{\ast\!\!<} \left(C\varepsilon^\beta\right)^{\pi_l} \text{ for } l \in \{2, \dots, n\}$$

and in the regular case $T_l \mathrel{\ast\!\!<} \left(C\varepsilon^\beta\right)^{-(n-1)\pi_l}$.

Then all our thresholds are satisfied with our choices of parameters in the theorem A.V.1. ▶

Now, we can complete the proof of the main result (theorem II.2.). Indeed an action can drift at most of a distance r_l/M at each step of our iterative process (lemmas A.IV.3. and A.IV.4.) and we have a drift a length r_0 at the initial step.

Moreover, the size of each change of coordinates in our iterative scheme is bounded by $\frac{r_l}{6}$ for $l \in \{1, \dots, n\}$, hence in the original variables we have:

$$||I(t) - I(t_0)|| \leq r_0 + \frac{r_1}{M} + \ldots + \frac{r_n}{M} + \frac{2}{6}\left(r_1 + \ldots + r_n\right) \text{ if } |t - t_0| < \exp(cK)$$

with $K = K_0\varepsilon^{-b}$ and $r_l \ast\!\!< \varepsilon^{\beta\pi_l}$ and we get:

$$||I(t) - I(t_0)|| \leq (2n+1)r_0 \ast\!\!= \varepsilon^{\beta/\rho_0} \text{ if } |t - t_0| < \exp\left(cK_0\varepsilon^{-b}\right)$$

with the constant c in lemma A.II.1.

Now, our threshold on γ ensures that any trajectory remains in the domain of h all along this iterative process.

This complete the proof of theorem II.2. ►

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