Downstream asymptotics in exterior domains: from stationary wakes to time periodic flows

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Abstract

In this paper, we consider the time-dependent Navier-Stokes equations in the half-space $[x_0,\infty)\times\mathbf{R}\subset\mathbf{R}^2$, with boundary data on the line $x=x_0$ assumed to be time-periodic (or stationary) with a fixed asymptotic velocity $\mathbf{u}_\infty=(1,0)$ at infinity. We show that there exist (locally) unique solutions for all data satisfying a center-stable manifold compatibility condition in a certain class of fuctions. Furthermore, we prove that as $x\to\infty$, the vorticity decompose itself in a dominant stationary part on the parabolic scale $y\sim\sqrt{x}$ and corrections of order $x^{-\frac{3}{2}+\varepsilon}$, while the velocity field decompose itself in a dominant stationary part in form of an explicit multiscale expansion on the scales $y\sim\sqrt{x}$ and $y\sim x$ and corrections decaying at least like $x^{-\frac{9}{8}+\varepsilon}$. The asymptotic fields are made of linear combinations of universal functions with coefficients depending mildly on the boundary data. The asymptotic expansion for the component parallel to $\mathbf{u}_\infty$ contains 'non-trivial' terms in the parabolic scale with amplitude $\ln(x)x^{-1}$ and x^{-1}. To first order, our results also imply that time-periodic wakes behave like stationary ones as $x\to\infty$.

The class of functions used to prove these results is 'natural' in the sense that the well known 'Physically Reasonable' (in the sense of Finn & Smith) stationary solutions of the Navier-Stokes equations around an obstacle fall into that class if the half-space extends in the downstream direction and the boundary ($x=x_0$) is sufficiently far downstream. In that case, the coefficients appearing in the asymptotics can be linearly related to the net force acting on the obstacle. In particular, the asymptotic description holds for 'Physically Reasonable' stationary solutions in exterior domains, *without restrictions on the size of the drag acting on the obstacle*. To our knowledge, it is the first time that estimates uncovering the $\ln(x)x^{-1}$ correction are proved in this setting.

1 Introduction

In this paper, we consider the time-dependent Navier-Stokes equations[1]

$$\begin{aligned} \partial_t\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u} &= \frac{1}{\mathrm{Re}}\triangle\mathbf{u}-\nabla p \\ \nabla\cdot\mathbf{u} &= 0 \\ \mathbf{u}(\mathbf{x},t)|_{x=x_0} &= \mathbf{u}_b(y,t) \\ \lim_{|\mathbf{x}|\to\infty}\mathbf{u}(\mathbf{x},t) &= \mathbf{u}_\infty\equiv\begin{pmatrix}u_\infty\\0\end{pmatrix} \end{aligned} \tag{1.1}$$

in the half-space $\Omega_+=[x_0,\infty)\times\mathbf{R}$. We will restrict ourselves to the time-periodic setting, i.e. $\mathbf{u}_b(y,t)=\sum_{n\in\mathbf{Z}}e^{in\tau t}\mathbf{u}_{b,n}(y)$, with basic frequency (Strouhal number) τ and $\mathbf{u}_b\in l^1(\mathbf{Z},\mathcal{B})$ for some functional space $\mathcal{B}$ to be defined later on. Note that with appropriate scalings, we can set without loss of generality $|\mathbf{u}_\infty|=u_\infty=1$ and $\mathrm{Re}=1$. The scale of the Reynolds number then translates to the scale of $\mathbf{u}_b$.

[1] Vectors are denoted by boldface letters, generic positions in the physical space $\mathbf{R}^2$ are denoted either by $\mathbf{x}$ or by (x,y).

We consider this problem as a simplified version of the 'usual' exterior problem around an obstacle

$$\begin{aligned} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} &= \frac{1}{\mathrm{Re}}\triangle\mathbf{u} - \nabla p \\ \nabla\cdot\mathbf{u} &= 0 \\ \mathbf{u}(\mathbf{x},t)|_{\partial\Omega} &= 0 \\ \lim_{|\mathbf{x}|\to\infty}\mathbf{u}(\mathbf{x},t) &= \mathbf{u}_\infty \equiv \begin{pmatrix} u_\infty \\ 0 \end{pmatrix} \end{aligned} \tag{1.2}$$

in $\mathbf{R}^2\backslash\Omega$, where $\partial\Omega$, the obstacle, is compact and connected. Getting rid of the obstacle by considering the flow only in the downstream region is a brutal simplification. We hope that by capturing the main difficulty of (1.2), (the spatial asymetry introduced by (1.2.4), as seen in the slow decay of vorticity as $x \to \infty$ for instance), techniques used in this paper could shed a new light on the theory on the Navier-Stokes equations (1.2) which began with J. Leray's pioneering work in the 1930's (see also [7] and references therein).

The question we address here is to give a quantitative description of the flow in the so-called 'wake region' which extends downwards of the obstacle (i.e. as $x \to \infty$). In previous papers [9, 18] such a description has been obtained in the stationary case by assuming that the restriction of the solution of (1.2) on a given line $x = x_0 \gg 1$ was in a certain function class. Then it follows from [9, 18] that as $x \to \infty$, the velocity field $\mathbf{u}$ and the vorticity $\omega = \nabla\times\mathbf{u}$ satisfy

$$\mathbf{u}(x,y) = \mathbf{u}_\infty + \begin{pmatrix} \tilde{u}_{\mathbf{a}}(x,y) \\ \tilde{v}_{\mathbf{a}}(x,y) \end{pmatrix} + \mathcal{O}(x^{-1+\varphi_0})\,, \qquad \omega(x,y) = \omega_{\mathbf{a}}(x,y) + \mathcal{O}(x^{-\frac{3}{2}+\varphi_0})\,, \tag{1.3}$$

for some $\varphi_0 > 0$, where

$$\tilde{u}_{\mathbf{a}}(x,y) = \tfrac{a_1}{\sqrt{x}} f_0(\tfrac{y}{\sqrt{x}}) + \tfrac{1}{x}\Big(a_2 g_0(\tfrac{y}{x}) - a_3 g_1(\tfrac{y}{x})\Big) \tag{1.4}$$

$$\tilde{v}_{\mathbf{a}}(x,y) = \tfrac{a_1}{2x} f_1(\tfrac{y}{\sqrt{x}}) + \tfrac{1}{x}\Big(a_2 g_1(\tfrac{y}{x}) + a_3 g_0(\tfrac{y}{x})\Big) \tag{1.5}$$

$$\omega_{\mathbf{a}}(x,y) = \tfrac{a_1}{2x} f_1(\tfrac{y}{\sqrt{x}}) \tag{1.6}$$

for some $\mathbf{a} = (a_1, a_2, a_3)$ and

$$f_m(z) = \frac{z^m e^{-\frac{z^2}{4}}}{\sqrt{4\pi}}\,, \quad g_m(z) = \frac{1}{\pi}\frac{z^m}{1+z^2}\,.$$

This result was expected to hold for a long time, see e.g. [2]. It should be noted that the terms on the $y \sim x$ scale are of smaller order than the stated correction order. It is however argued in [2, 9, 18] that on the given scales ($y \sim x$ or $y \sim \sqrt{x}$) the velocity field indeed converge to its asymptotic form and furthermore that the upstream asymptotics ($x \to -\infty$ is given by (1.4) and (1.5) with $a_1 = 0$ *and the same coefficients* a_2 and a_3 as in the downstream direction. Integration of the equations (1.2) in the domain comprised between the lines $x = -x_0 \ll 0$ and $x = x_0 \gg 0$ then implies (see e.g. Appendix II in [21]) in the limit $x_0 \to \infty$ that $a_1 + 2a_2 = 0$ (mass conservation), and that the force $\mathbf{F}$ acting on the obstacle is given by

$$\mathbf{F} = \begin{pmatrix} 2a_2 \\ -2a_3 \end{pmatrix} + \int_{\mathbf{R}^2\backslash\Omega} \mathrm{d}\mathbf{x}\;\partial_t\mathbf{u}(\mathbf{x}) \equiv \begin{pmatrix} \text{drag} \\ \text{lift} \end{pmatrix}. \tag{1.7}$$

In other words, for stationary flows, this shows that a_2 resp. a_3 are linearly related to the drag resp. lift acting on the obstacle, see also [2, 9, 18] for more physical interpretations.

For completeness, we note that (1.4) and (1.5) can be easily derived heuristically in the two following steps. We first note that the field $(\tilde{u}_{\mathbf{a}}, \tilde{v}_{\mathbf{a}})$ with $a_1 = 0$ would be a solution of the Navier-Stokes equations (1.1) or (1.2) (for an appropriate pressure) but for the boundary conditions. And then, as we may expect $\partial_x^2\omega \ll \partial_x\omega$ as $x \to \infty$, the vorticity satisfies (to first order) $\partial_x\omega = \partial_y^2\omega$ whose solutions corresponding to decaying velocity fields behave asymptotically like (1.6). It is then easy to see using $\omega \approx -\partial_y u$ and $\partial_y v = -\partial_x u$ that the corresponding velocity fields are as stated in (1.4) and (1.5).

Unfortunately, the function class used in a first attempt to give a rigorous foundation to these heuristics in [9, 18] is rather unorthodox and the question wether the (restriction of) solutions of (1.2) were in this class was completely left open.

On the other hand, it is well known from experiences and numerical simulations that stationary solutions of (1.2) in exterior domains are only stable at low Reynolds numbers, and it is commonly believed (see e.g. [5, 14, 16, 17]) that at a (first) critical Reynolds number, the stationary flow loses its stability through a Hopf bifurcation and becomes time-periodic before eventually leading for larger Reynolds number to von Karman's vortex street and then to turbulence.

In this paper, we will give a more detailed asymptotic description than (1.3), as we will prove that in both the stationary and time-periodic case, the solutions of (1.1) satisfy

$$\mathbf{u}(x,y,t) = \mathbf{u}_\infty + \begin{pmatrix} u_{\mathbf{a}(t)}(x,y) \\ v_{\mathbf{a}(t)}(x,y) \end{pmatrix} + \mathcal{O}\begin{pmatrix} x^{-\frac{9}{8}+\varphi_0} \\ x^{-\frac{3}{2}+\varphi_0} \end{pmatrix} , \quad \omega(x,y) = \omega_{\mathbf{a}(t)}(x,y) + \mathcal{O}(x^{-\frac{3}{2}+\varphi_0}) , \tag{1.8}$$

uniformly in time, where $0 < \varphi_0 < \frac{1}{8}$, $\mathbf{a}(t) = (a_1, a_2(t), a_3(t), a_4, a_5, a_6)$ for some constants a_1, a_4, a_5 and a_6 and *time periodic functions* a_2 and a_3, $\omega_{\mathbf{a}}$ is as above and

$$\begin{aligned} u_{\mathbf{a}(t)}(x,y) &= \tfrac{a_1}{\sqrt{x}} f_0(\tfrac{y}{\sqrt{x}}) - \tfrac{1}{2x}\Big(a_5 h(\tfrac{y}{\sqrt{x}}) + (a_6 \ln(x) + a_4) f_1(\tfrac{y}{\sqrt{x}})\Big) + \tfrac{1}{x}\Big(a_2(t) g_0(\tfrac{y}{x}) - a_3(t) g_1(\tfrac{y}{x})\Big) \\ v_{\mathbf{a}(t)}(x,y) &= \tfrac{a_1}{2x} f_1(\tfrac{y}{\sqrt{x}}) + \tfrac{1}{x}\Big(a_2(t) g_1(\tfrac{y}{x}) + a_3(t) g_0(\tfrac{y}{x})\Big) \end{aligned} \tag{1.9}$$

where

$$f_m(z) = \frac{z^m e^{-\frac{z^2}{4}}}{\sqrt{4\pi}} , \quad g_m(z) = \frac{1}{\pi}\frac{z^m}{1+z^2} , \quad h(z) = f_0(z)^2 + \frac{1}{8\sqrt{\pi}}\, z\, \mathrm{erf}(\tfrac{z}{2}) e^{-\frac{z^2}{4}} .$$

By the use of functional spaces more adapted than in [9, 18], we will prove that existing results on (1.2) implies that (1.8) also holds for (1.2). This rigorous link between (1.1) and (1.2) will only be estabished for the stationary case, as this case has been widely studied in the litterature (see e.g. [7]). Though we believe it should also hold just after the Hopf bifurcation, we are not aware of any rigorous treatment of the exterior periodic problem in 2D (see [15] for some rigorous work on the 3D case).

In analogy with the stationary case, we may also expect that for the solution of (1.2), the asymptotic velocity field upstream ($x \to -\infty$) is given by (1.9) with $a_1 = a_4 = a_5 = a_6 = 0$ and the same coefficients $a_2(t)$ and $a_3(t)$ than in the downstream direction. If this holds, then integrating $\nabla \cdot \mathbf{u} = 0$ and $\omega = \nabla \times \mathbf{u}$ in the domain comprised between $x = -x_0 \ll 0$ and $x = x_0 \gg 0$, we get (letting $x_0 \to \infty$) $a_2(t) = -\frac{1}{2}a_1$ and $a_3(t) = \int_{\mathbf{R}^2\setminus\Omega} \omega(\mathbf{x})\, \mathrm{d}\mathbf{x}$. As this last quantity (the total vorticity) is preserved by (1.2), we see that $a_2(t)$ and $a_3(t)$ are in fact constant in time[2]. This implies that to the order given by (1.8), time-periodic

[2]Note that it would be wrong to conclude that the drag and lift are constant in time from the fact that a_2 and a_3 are constant, as the volume integral of $\partial_t\mathbf{u}$ in (1.7) will generically no longer be zero for time-periodic flows. This is in agreement with results of numerical simulations, see e.g. [10, 12].

wakes cannot be distinguished from stationary ones, though the actual value of the coefficients will differ from the to case. Without rigorous proof that the upstream asymptotics are as expected, we consider these physical interpretations as conjectural ones.

We end this section by noting that asymptotical results like (1.3) have been successfully used in numerical simulations of the stationary Navier-Stokes equations (1.2) in exterior domains, see [20, 21]. In particular, in fixed simulations domains, it allows to compute drag and lift coefficients with better accuracy than usual methods, while for fixed accuracy, smaller simulations domains can be used, thereby reducing significantly the CPU time needed for these computations. It is our hope that (1.8) could also be of such use in the time-periodic setting.

1.1 Reformulation of the problem

As in [9, 18], the starting point of the analysis is to write (1.1) as a dynamical system where x plays the rôle of time. To do so, we write $\mathbf{u} = \mathbf{u}_\infty + \mathbf{v}$ where $\mathbf{v} = (u, v)$ and introduce the vorticity $\omega = \partial_x v - \partial_y u$ and its derivative w.r.t. x as $\eta = \partial_x \omega$. Since the boundary data is assumed to be time-periodic, it is natural to assume that there is also a (discrete) Fourier decomposition of the various fields (this corresponds to the so-called *global mode* behavior, see also [13, 19, 22]) given by

$$\begin{aligned} u(x,y,t) &= \sum_{n\in\mathbf{Z}} e^{in\tau t}\, u_n(x,y) \;, \quad & v(x,y,t) &= \sum_{n\in\mathbf{Z}} e^{in\tau t}\, v_n(x,y) \;, \\ \omega(x,y,t) &= \sum_{n\in\mathbf{Z}} e^{in\tau t}\, \omega_n(x,y) \;, \quad & \eta(x,y,t) &= \sum_{n\in\mathbf{Z}} e^{in\tau t}\, \eta_n(x,y) \;. \end{aligned} \tag{1.10}$$

In terms of this decomposition, the n-th Fourier coefficient of (1.1) read (see also [9])

$$\begin{aligned} \partial_x \omega &= \eta \\ \partial_x \eta &= \eta - \partial_y^2 \omega + in\tau\omega + q \\ \partial_x u &= -\partial_y v \\ \partial_x v &= \partial_y u + \omega \;, \end{aligned} \tag{1.11}$$

with $q = u\partial_x\omega + v\partial_y\omega$, and where we dropped the n indices on the fields for concision. Namely, the third equation in (1.11) is the incompressibility relation $\nabla \cdot \mathbf{u} = 0$, the last one is the definition of the vorticity, while substituting the first one in the second, one recovers the 'dynamical' part of (1.1). We note here that using the incompressibility relation $\partial_x u = -\partial_y v$ and the definition of the vorticity, we may cast the nonlinearity q in the following equivalent form

$$q = \partial_x(u\omega) + \partial_y(v\omega) \equiv \partial_x(P) + \partial_y(Q) \;.$$

We also note that using $\partial_x u = -\partial_y v$ and defining $R = uv$, $S = \frac{1}{2}(v^2 - u^2)$, we have the decompositions

$$P = u\omega = \partial_x R + \partial_y S \;, \qquad Q = v\omega = -\partial_y R + \partial_x S \qquad \text{and} \qquad q = (\partial_x^2 + \partial_y^2) R + 2Q \;.$$

We interpret (1.11) as a new dynamical system where the space variable x plays the role of time (the x derivatives on the r.h.s. of (1.11) can be eliminated using $\eta - P$ instead of η as auxiliary field).

Using Duhamel's variation of constants formula, we can cast (1.11) in an integral form, whose structure (omitting also the time argument for concision) is given by

$$u(x) = K_1(x - x_0)\mathcal{L}_u w + K_0(x - x_0)\nu + \mathcal{F}_{1,u}(x) + \mathcal{F}_{2,u}(x) + \mathcal{L}_1 S(x) - \mathcal{L}_2 R(x) \;, \tag{1.12}$$

$$v(x) = K_1(x-x_0)\mathcal{L}_v w + K_0(x-x_0)\mu + \mathcal{F}_{1,v}(x) + \mathcal{F}_{2,v}(x) - \mathcal{L}_1 R(x) - \mathcal{L}_2 S(x) \ , \tag{1.13}$$
$$\omega(x) = K_1(x-x_0) w + \mathcal{F}_{1,\omega}(x) + \mathcal{F}_{2,\omega}(x) \ . \tag{1.14}$$

The derivation of this integral formulation for the solution of (1.1) is given in Section 2, as well as precise definitions of the different expressions. The terms involving the kernels K_0 and K_1 depend on ω and ν, which are functions defined on the boundary $x = x_0$, while μ is given by the Hilbert transform $\mu = \mathcal{H}\nu$ on the boundary (in particular, μ is *not* an independent quantity). The $\mathcal{F}$ terms depend on the values of P and Q on $[x_0, \infty) \times \mathbf{R}$ and *do not vanish* on the boundary $x = x_0$. Thus the integral formulation above does not satisfy the boundary condition $\mathbf{u}(x_0, y, t) = \mathbf{u}_b(y, t)$, unless for specific choices of ν and w, but then the boundary condition for the vorticity cannot be satisfied. This is related to the fact that the equation (1.1) with a boundary condition for the vorticity, the velocity and the pressure is ill-posed. The boundary values on $x = x_0$ thus have to be taken on the so-called central-stable manifold. As we will explain in section 7, the parametrization of that manifold by the functions w and ν is a convenient one.

We will give precise statements of our main results in subsection 1.3 below after the definition of some functional spaces and related norms. On an informal level, our results are twofold. We will use the integral formulation (1.12)-(1.14) to prove that if ω and ν are in a certain class $\mathcal{C}_i$, there exist a (locally) unique solution of (1.1) in the Banach space $\mathcal{W}$ defined in the next section. We will then show that the asymptotic structure of these solutions is indeed given by (1.8) with $\varphi_0 > 0$. On the other hand, time-periodic solutions of (1.2) must satisfy (1.1) for all x_0 sufficiently large. We will then show that for solutions of (1.2) in a certain class $\mathcal{C}_u$, the $\mathcal{F}$'s are well defined, and thus solutions of (1.2) in $\mathcal{C}_u$ must also satisfy (1.12)-(1.14) for certain functions w and ν. As $K_0(0) = K_1(0) = 1$, the functions w and ν can be determined by inverting any two of the three (linear and local) relations (1.12)-(1.14) at $x = x_0$, the third relation, which correspond to the central-stable manifold condition in the dynamical system formulation (1.11), is then automatically satisfied *since we know that the solution exist*. We will then show that the functions w and ν obtained in this way are in the class $\mathcal{C}_i$, which finally implies that solutions of (1.2) in $\mathcal{C}_u$ also satisfy (1.3) with $\varphi_0 > 0$.

1.2 Definitions

To state our main result, we need some definitions. We first give the topology we will use to control the decompositions (1.10). Let $1 \le p < \infty$, and $f(x,y,t) = \sum_{n\in\mathbf{Z}} e^{in\tau t} f_n(x,y)$ for $(x,y,t) \in [x_0,\infty) \times \mathbf{R} \times [0, \frac{2\pi}{\tau}]$. Let $\langle x \rangle = \sqrt{1+x^2}$, we define

$$\|f\|_{p,\sigma} = \sup_{x\ge x_0} \|f\|_{x,p,\sigma} \ , \quad \|f\|_{x,p,\sigma} = \langle x\rangle^{\sigma} \|f(x)\|_p = \langle x\rangle^{\sigma} \sum_{n\in\mathbf{Z}} \Big(\int_{\mathbf{R}} \mathrm{d}y \ |f_n(x,y)|^p \Big)^{\frac{1}{p}} \ ,$$
$$\|f\|_{\infty,\sigma} = \sup_{x\ge x_0} \|f\|_{x,\infty,\sigma} \ , \quad \|f\|_{x,\infty,\sigma} = \langle x\rangle^{\sigma} \|f(x)\|_{\infty} = \langle x\rangle^{\sigma} \sum_{n\in\mathbf{Z}} \operatorname*{ess\,sup}_{y\in\mathbf{R}} |f_n(x,y)| \ ,$$
$$\|f\|_{p,\{\sigma_1,\sigma_2\}} = \sup_{x\ge 0} \langle x\rangle^{-\sigma_1} x^{\sigma_2} \|f(x)\|_{\mathrm{L}^p} \ , \quad \|f(x)\|_{\mathrm{L}^p} = \sup_{n\in\mathbf{Z}} \Big(\int_{\mathbf{R}} \mathrm{d}y \ |f_n(x,y)|^p \Big)^{\frac{1}{p}} \ ,$$
$$\|f\|_{\infty,\{\sigma_1,\sigma_2\}} = \sup_{x\ge 0} \langle x\rangle^{-\sigma_1} x^{\sigma_2} \|f(x)\|_{\mathrm{L}^\infty} \ , \quad \|f(x)\|_{\mathrm{L}^\infty} = \sup_{n\in\mathbf{Z}} \operatorname*{ess\,sup}_{y\in\mathbf{R}} |f_n(x,y)| \ ,$$

where we use the notation $\|f(x)\|_p$ as a shorthand to the more rigorous $\|f(x,\cdot)\|_p$. We will refer to $\mathcal{P}$ and $\mathcal{P}_0$ as the projection operators on Fourier series defined by

$$\mathcal{P}\Big(\sum_{n\in\mathbf{Z}} e^{in\tau t} f_n \Big) = \sum_{n\in\mathbf{Z}, n\neq 0} e^{in\tau t} f_n \ , \quad \mathcal{P}_0\Big(\sum_{n\in\mathbf{Z}} e^{in\tau t} f_n \Big) = f_0 \ ,$$

as well as the operators $\mathcal{I}$ (the 'primitive'), $\mathcal{S}$ (the symmetrization) and $\mathcal{M}$ (the 'mean value') defined by

$$(\mathcal{I}f)(y) = \int_{-\infty}^{y} \mathrm{d}z\, \frac{f(z)}{2} - \int_{y}^{\infty} \mathrm{d}z\, \frac{f(z)}{2}\,, \quad (\mathcal{S}f)(y) = f(y) + f(-y)\,, \quad \mathcal{M}(f) = \int_{\mathbf{R}} f(y)\mathrm{d}y\,. \quad (1.15)$$

Note that $\mathcal{I}$ is the inverse of ∂_y (when it is defined). We can now specify our basic functional space

Definition 1.1 *Let $\mathcal{C}_0^\infty = \{\{(u_n, v_n, \omega_n)\}_{n\in\mathbf{Z}}$ s.t. $(u_n, v_n, \omega_n) \in \mathcal{C}_0^\infty([x_0,\infty)\times\mathbf{R},\mathbf{R}^3)\ \forall n \in \mathbf{Z}\}$. We denote by $\mathcal{W}$ the Banach space obtained by closure of $\mathcal{C}_0^\infty$ under the norm*

$$\begin{aligned}
\|(\mathbf{v},\omega)\| &= \sup_{x\geq x_0} \|(\mathbf{v},\omega)\|_x \\
\|(\mathbf{v},\omega)\|_x &= \|u\|_{x,\infty,\frac{1}{2}} + \|u\|_{x,q,\frac{1}{2}-\frac{1}{q}} + \|\partial_y u\|_{x,r,1-\frac{1}{2r}-\eta} \\
&\quad + \|v\|_{x,\infty,1-\varphi} + \|v\|_{x,p,1-\varphi-\frac{1}{p}} + \|\partial_y v\|_{x,r,\frac{3}{2}-\frac{1}{2r}-\xi} \\
&\quad + \|\omega\|_{x,2,\frac{3}{4}} + \||y|^\beta\omega\|_{x,2,\frac{3}{4}-\frac{\beta}{2}} + \|\partial_y\omega\|_{x,\infty,\frac{3}{2}} + \|\partial_y\omega\|_{x,1,1}\,.
\end{aligned}$$

This choice is discussed at the end of this section. Note at this point that the 'expected' asymptotic decomposition (1.3) is in $\mathcal{W}$ if $p > 1$. We now specify the class of solutions of (1.2) for which our results can be applied:

Definition 1.2 *A solution $(\mathbf{v},\omega)$ of (1.2) is in the class $\mathcal{C}_u$ if $\|(\mathbf{v},\omega)\| \leq \rho$ for some finite constant ρ and*

$$\begin{aligned}
&\tfrac{13}{7} \leq \beta \leq 3\,, \quad 1-\tfrac{1}{p} < \varphi < \tfrac{1}{2}\,, \quad 1 < p \leq q\,, \quad r > 2\,, \quad \tfrac{1}{2} \geq \xi \geq \eta \geq 0\,, \\
&\xi \geq \varphi\,, \quad \tfrac{1}{4} - \tfrac{\varphi}{2} - \eta > 0\,, \quad \tfrac{1}{2} - (1+\tfrac{1}{2r})\varphi > 0\,, \quad \tfrac{1}{2} + \xi - \eta - 2\varphi > 0\,, \\
&\tfrac{\langle\tau\rangle}{\tau} \leq \langle x_0\rangle^\varphi\,, \quad \tfrac{1}{2} + \eta - \xi - \tfrac{\varphi}{r} > 0\,.
\end{aligned} \qquad (1.16)$$

The condition $\frac{\langle\tau\rangle}{\tau} \leq \langle x_0\rangle^\varphi$ will be a convenient way to get bounds depending on x_0 only and not on the Strouhal number τ. It should be noted that this condition is only restrictive in the limit of vanishing Strouhal number. This is *not* expected to occur for time-periodic solutions of (1.2), if the Hopf bifurcation picture of [5, 14, 16, 17] is correct.

We now define the class $\mathcal{C}_i$, consisting essentially of those functions w, ν and μ for which the part of r.h.s. of (1.12)–(1.14) depending on w, ν and μ is in $\mathcal{W}$ (see Lemma 3.1, 3.3 and 3.6):

Definition 1.3 *We say that ν and w are in the class $\mathcal{C}_i$ if $\mathcal{M}(\mathcal{P}_0 w) = 0$ and $\|(\nu,\mu,w)\|_{x_0} \leq \rho$ for parameters satisfying (1.16).*

Note that $\mathcal{M}(\mathcal{P}_0 w)$ is always well defined if $\|(0,0,w)\|_{x_0} < \infty$ since $\|w\|_{\mathrm{L}^1} \leq \langle x_0\rangle^{-\frac{1}{2}}\|(0,0,w)\|_{x_0}$ (see Lemma A.1). In the case of symmetric flows (i.e. u even in y and v odd in y), $\mathcal{M}(\mathcal{P}_0 w) = 0$ is a trivial consequence of the fact that w is an odd function of y (it is also expected from (1.3) or (1.8)). Our results will in particulat imply that the vorticity decomposes itself in a first order part with zero mean value with second order corrections with generically non-zero mean value (see (1.12)).

We end this section by making some comments on Definition 1.1. First, for the v component, we will need $\varphi > 0$. Namely, as we will see, the optimal decay rate for v as $x \to \infty$ can only be obtained if $\mu \in \mathrm{L}^1(\mathbf{R},\mathrm{d}y)$, since the integral expression for v contains a convolution of μ with $K_0(x-x_0,y) = \frac{1}{\pi}\frac{x-x_0}{(x-x_0)^2+y^2}$. But (apart from symetric flows), $\mu(y) \sim 1/y$ as $y \to \infty$ (see (1.3)), so in general $\mu \notin \mathrm{L}^1(\mathbf{R},\mathrm{d}y)$.

The second comment is on the need of η and ξ. Basically, the problem is that $\partial_y u$ and $\partial_y v$ are naturally build of sum of functions on two length scales ($y \sim \sqrt{x}$ and $y \sim x$, see e.g. (1.9)). Dependence on r of the decay exponents as $x \to \infty$ of L^r norms of such functions either vary like $1/(2r)$ for functions on the shorter scale or like $1/r$ for functions on the longer one. Our choice of exponents are thus 'wrong' on the scale $y \sim x$ and is 'corrected' by introducing η and ξ. These additional parameters would not be needed if we choose $r = \infty$, but in that case we would lose the boundedness of the Dirichlet-Neumann operator (or exchange operator) $\mathbf{v} \to \mathcal{H}\mathbf{v}$ in $\mathcal{W}$, which is needed to compare solutions of (1.1) and (1.2) (see Section 7).

The last comment is that if (1.16) holds, we control the nonlinearities R, S, P and Q in terms of the $\|\cdot\|$-norm by

$$\begin{aligned} \|R\|_{m,\frac{3}{2}-\varphi-\frac{1}{m}} + \|S\|_{m,1-\frac{1}{m}} + \|\partial_y R\|_{r,\frac{3}{2}-\eta-\frac{1}{2r}} + \|\partial_y S\|_{r,\frac{3}{2}-\eta-\frac{1}{2r}} &\leq C\|(\mathbf{v},\omega)\|^2 , \\ \|P\|_{m,\frac{3}{2}-\frac{1}{2m}} + \|Q\|_{m,2-\varphi-\frac{1}{2m}} + \|\partial_y P\|_{n,2-\frac{1}{2n}-\eta} + \|\partial_y Q\|_{n,\frac{5}{2}-\frac{1}{2n}-\xi} &\leq C\|(\mathbf{v},\omega)\|^2 , \\ \||y|^\beta P\|_{2,\frac{5}{4}-\frac{\beta}{2}} + \||y|^\beta Q\|_{2,\frac{7}{4}-\varphi-\frac{\beta}{2}} &\leq C\|(\mathbf{v},\omega)\|^2 , \end{aligned} \tag{1.17}$$

for all $1 \leq m \leq \infty$ and $1 \leq n \leq r$. To establish these estimates, we used

$$\|\omega\|_{\infty,1} + \|\omega\|_{1,\frac{1}{2}} \leq \|(\mathbf{v},\omega)\| , \tag{1.18}$$

which follows (see Lemma A.1) from

$$\|\omega\|_{\mathrm{L}^1} \leq C_\beta \|\omega\|_{\mathrm{L}^2}^{1-\frac{1}{2\beta}} \||y|^\beta \omega\|_{\mathrm{L}^2}^{\frac{1}{2\beta}} \quad \text{and} \quad \|\omega\|_{\mathrm{L}^\infty} \leq \sqrt{\|\omega\|_{\mathrm{L}^2} \|\partial_y \omega\|_{\mathrm{L}^2}} .$$

1.3 Main results

We are now in position to state our results in a precise manner. The first one states that the topology of Definition 1.1 is well adapted to (1.1):

Theorem 1.4 *If x_0 is sufficiently large, then the two following statements are equivalent*

1. *There exist a unique solution to (1.1) in $\mathcal{C}_u$ with parameters satisfying (1.16).*
2. *ν and w are in the class $\mathcal{C}_i$ with parameters satisfying (1.16).*

Furthermore if 1. holds and additionally

$$\||y|^{\frac{1}{2}}\mathbf{v}(x_0)\|_4 + \||y|^{\frac{1}{2}-(1+\varepsilon)\varphi}\mathcal{S}\mathbf{v}(x_0)\|_1 \leq C(x_0, \|(\mathbf{v},\omega)\|) ,$$

then for all $\varepsilon > 0$, it holds

$$\||y|^{\frac{1}{2}-(1+\varepsilon)\varphi}\mathcal{S}\nu\|_1 + \||y|^{\frac{1}{2}-(1+\varepsilon)\varphi}\mathcal{S}\mu\|_1 \leq C_1(x_0, (\mathbf{v},\omega)\|) . \tag{1.19}$$

Our next result is that stationary solutions to the 'usual' exterior problem (1.2) are in the class $\mathcal{C}_u$:

Proposition 1.5 *For any stationary solution of (1.2) "Physically Reasonable" (PR) in the sense of Finn and Smith (see e.g. [6, 7, 8]), the fields u, v and ω satisfy $\|(\mathbf{v},\omega)\| \leq C$ with parameters satisfying (1.16) if x_0 is sufficiently large. Furthermore $\||y|^{\frac{1}{2}}\mathbf{v}(x_0)\|_4 + \||y|^{\frac{1}{2}-(1+\varepsilon)\varphi}\mathcal{S}\mathbf{v}(x_0)\|_1 < \infty$ for all $\varepsilon > 0$.*

From this, we conclude that (PR) solutions satisfy the integral formulation (1.12)-(1.14):

Corollary 1.6 *For any (PR) stationary solution of (1.2), ν and w are in the class $\mathcal{C}_i$ with parameters satisfying (1.16) and $\||y|^{\frac{1}{2}-(1+\varepsilon)\varphi}\mathcal{S}\nu\|_1 + \||y|^{\frac{1}{2}-(1+\varepsilon)\varphi}\mathcal{S}\mu\|_1 < \infty$ for all $\varepsilon > 0$.*

The proof of this Corollary follows directly from Theorem 1.4, Proposition 1.5 and uniqueness. Once these results are proved, we will use the integral formulation (1.12)-(1.14) to get the asymptotic structure of the solutions to (1.1) or (PR) solutions to (1.2):

Corollary 1.7 *Let $\mathbf{a}_1 = (-\mathcal{M}(\mathcal{I}\mathcal{P}_0 w) - \int_{\Omega_+} \mathcal{P}_0 Q(x,y)\,\mathrm{d}x\mathrm{d}y, 0, 0)$, then all solutions to (1.1) in $\mathcal{C}_u$ satisfy (1.3) with $\varphi_0 = (1+\varepsilon)\varphi > \varphi$.*

Note that since $\varphi > 0$, in (1.3), the terms containing a_2 and a_3 are of smaller order than the remainder, which explains why these parameters are not specifi ed in the Corollary. Once this Corollary is proved, we will be able to get the more precise asymptotic form as shows the

Corollary 1.8 *Assume that $\||y|^{\frac{1}{2}}\mathbf{v}(x_0)\|_4 < \infty$ and $\||y|^{\frac{1}{2}-(1+\varepsilon)\varphi}\mathcal{S}\nu\|_1 + \||y|^{\frac{1}{2}-(1+\varepsilon)\varphi}\mathcal{S}\mu\|_1 < \infty$, and let $a_1 = -\mathcal{M}(\mathcal{I}\mathcal{P}_0 w) - \int_{\Omega_+} \mathcal{P}_0 Q(x,y)\,\mathrm{d}x\mathrm{d}y$, $a_2 = \mathcal{M}(\mathcal{S}\nu) - \int_{\Omega_+} \mathcal{P}_0 Q(x,y)\,\mathrm{d}x\mathrm{d}y$, $a_3 = \mathcal{M}(\mathcal{S}\mu)$ and $\varphi < \varphi_0 = (1+\varepsilon)\varphi < \frac{1}{8}$. Then there exist a constant a_4 such that all solutions to (1.1) in $\mathcal{C}_u \subset \mathcal{W}$ satisfy (1.8) with $\mathbf{a} = (a_1, a_2, a_3, a_4, a_1^2, a_1\mathcal{P}_0 a_3)$, in $\|\cdot\|_\infty$ for $\mathbf{u}$, and in $\|\cdot\|_\infty + \|\cdot\|_1 + \||y|^{1-2(1+\varepsilon)\varphi}\cdot\|_2$ for ω.*

We conclude this section by noting that the constant a_1 can be expressed in the following equivalent forms:

Remark 1.9 *The constant a_1 is also given by the value of the following* constant *function*

$$
\begin{aligned}
\tilde{a}_1(x) &= \mathcal{M}\Big(\mathcal{I}\Big(\mathcal{P}_0\omega(x) + \int_x^\infty \mathrm{d}\tilde{x}\; e^{x-\tilde{x}}\mathcal{P}_0 P(\tilde{x})\Big) + \int_x^\infty \mathrm{d}\tilde{x}\; (e^{x-\tilde{x}} - 1)\mathcal{P}_0 Q(\tilde{x})\Big)\;, \\
&= \mathcal{M}\Big(\mathcal{I}\Big(\mathcal{P}_0\omega(x) + \int_x^\infty \mathrm{d}\tilde{x}\; e^{x-\tilde{x}}\mathcal{P}_0 P(\tilde{x})\Big) + \int_x^\infty \mathrm{d}\tilde{x}\; e^{x-\tilde{x}}\mathcal{P}_0 Q(\tilde{x}) + \mathcal{P}_0 S(x)\Big)\;,
\end{aligned}
$$

which is ‘almost local’ in x due to the exponential factors.

1.4 Structure of the paper

Our fi rst task in the remainder of this paper is to establish the integral formulation (1.12)-(1.14). This is done in the next section (Section 2). The proof of Theorem 1.4 is then split in two parts. The proof that (1.19) holds is delayed until Section 7 together with the proof that 1. implies 2. The converse is given in Section 4. The proof of Proposition 1.5 is also delayed until Section 8. Finally, the proof of Corollary 1.7 is given in Section 5, that of Corollary 1.8 in Section 6, while the proof that Remark 1.9 holds is left to the reader as it follows very easily from the integral formulation in Fourier space given in the next section.

2 Integral formulation

We now derive the integral formulation (1.12)-(1.14) of the solution of (1.11) and (1.1). All the material of this section is very similar to [9, 18] where the case $\tau = 0$ was treated. For completeness, we now reproduce some of the analysis here, encompassing the additional term proportional to the Strouhal number τ.

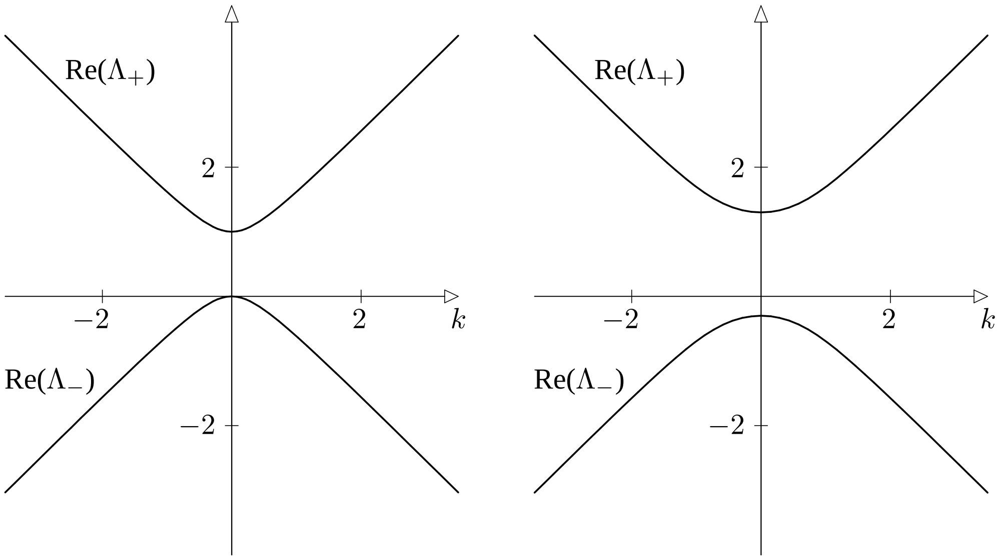

Figure 1: Dispersion relations $\Lambda_\pm$ as a function of wavelength k, with $n\tau = 0$ in left panel, and $n\tau = 1$ in right panel.

We first note that performing a (continuous) Fourier transform[3] $f(k) = \int_{\mathbf{R}} e^{iky} f(y)$ leads to a system of the form $\partial_x \mathbf{z} = L\mathbf{z} + \mathbf{q}$, with $\mathbf{z} = (\omega, \eta, u, v)$, $\mathbf{q} = (0, q, 0, 0)$ and

$$L(k) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ k^2 + in\tau & 1 & 0 & 0 \\ 0 & 0 & 0 & ik \\ 1 & 0 & -ik & 0 \end{pmatrix} .$$

As in [9], the matrix L can be diagonalized. Namely, define $\sigma(k) = \text{sign}(k)$, $\Lambda_0 = \sqrt{1 + 4(k^2 + in\tau)}$ and $\Lambda_\pm = \frac{1 \pm \Lambda_0}{2}$, and set $\mathbf{z} = S\mathbf{y}$, with $\mathbf{y} = (\omega_+, \omega_-, u_+, u_-)$ and

$$S(k) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ \Lambda_+ & \Lambda_- & 0 & 0 \\ \frac{ik}{\Lambda_+ + in\tau} & \frac{ik}{\Lambda_- + in\tau} & 1 & 1 \\ \frac{\Lambda_+}{\Lambda_+ + in\tau} & \frac{\Lambda_-}{\Lambda_- + in\tau} & -i\sigma & i\sigma \end{pmatrix} ,$$

then we get $S^{-1}LS = \text{diag}(\Lambda_+, \Lambda_-, |k|, -|k|)$ (see figure 2 for a graphical display of the dispersion relations $\Lambda_\pm$). The real part of Λ_+ being positive, the two equations corresponding to the '$+$' modes are linearly unstable. We thus integrate these modes backwards from $x = \infty$, where we set them to 0 (see also [18]). We then get

$$\omega_+(x) = -\int_x^\infty \mathrm{d}\tilde{x} \frac{e^{\Lambda_+(x-\tilde{x})}}{\Lambda_0} q(\tilde{x})$$

$$\omega_-(x) = e^{\Lambda_-(x-x_0)}\tilde{\omega}_0 - \int_{x_0}^x \mathrm{d}\tilde{x} \, \frac{e^{\Lambda_-(x-\tilde{x})}}{\Lambda_0} q(\tilde{x})$$

$$u_+(x) = -\frac{1}{2}\int_x^\infty \mathrm{d}\tilde{x} \, \frac{e^{|k|(x-\tilde{x})}}{ik - n\tau\sigma} q(\tilde{x})$$

[3] we distinguish functions and their Fourier transform only from their arguments 'k', resp. 'y' in Fourier resp. direct space.

$$u_-(x,k) = e^{-|k|(x-x_0)}\tilde{u}_0 + \frac{1}{2}\int_{x_0}^{x} \mathrm{d}\tilde{x}\ \frac{e^{-|k|(x-\tilde{x})}}{ik+n\tau\sigma} q(\tilde{x})\ ,$$

for some functions $\tilde{\omega}_0$ and $\tilde{u}_0$ to be specified. Integrating by parts the integrals involving $\partial_x P$ in $\omega_\pm$, replacing $q = (\partial_x^2 + \partial_y^2)R + 2\partial_y Q$ in $u_\pm$ and integrating twice by parts the term involving $\partial_x^2 R$, we find

$$\begin{aligned}
\omega_+(x) &= \frac{P(x)}{\Lambda_0} - \int_x^\infty \mathrm{d}\tilde{x}\ \frac{e^{\Lambda_+(x-\tilde{x})}}{\Lambda_0} q_+(\tilde{x}) \\
\omega_-(x) &= e^{\Lambda_-(x-x_0)} w - \frac{P(x)}{\Lambda_0} - \int_{x_0}^x \mathrm{d}\tilde{x}\ \frac{e^{\Lambda_-(x-\tilde{x})}}{\Lambda_0} q_-(\tilde{x}) \\
u_+(x) &= \frac{P(x) + ikS(x) + |k|R(x)}{2(ik - n\tau\sigma)} + \int_x^\infty \mathrm{d}\tilde{x}\ \frac{ike^{|k|(x-\tilde{x})}}{ik - n\tau\sigma} Q(\tilde{x}) \\
u_-(x) &= e^{-|k|(x-x_0)}\nu + \frac{P(x) + ikS(x) - |k|R(x)}{2(ik + n\tau\sigma)} - \int_{x_0}^x \mathrm{d}\tilde{x}\ \frac{ike^{-|k|(x-\tilde{x})}}{ik + n\tau\sigma} Q(\tilde{x})\ ,
\end{aligned}$$

where $q_\pm = \Lambda_\pm P - ikQ$, $\nu(k) = \tilde{u}_0(k) - \frac{P(x_0)+ikB(x_0)-|k|A(x_0)}{2(ik+n\tau\sigma)}$ and $w(k) = \tilde{\omega}_0(k) + \frac{P(x_0)}{\Lambda_0}$. For convenience, we also introduce $\mu(k) = i\sigma\nu(k)$. Then, a little algebra shows that when reconstructing ω, u and v, the terms involving $P(x)$ cancel out exactly, giving

$$\omega(x) = e^{\Lambda_-(x-x_0)} w - \int_{x_0}^x \mathrm{d}\tilde{x}\ \frac{e^{\Lambda_-(x-\tilde{x})}}{\Lambda_0} q_-(\tilde{x}) - \int_x^\infty \mathrm{d}\tilde{x}\ \frac{e^{\Lambda_+(x-\tilde{x})}}{\Lambda_0} q_+(\tilde{x}) \tag{2.1}$$

$$\begin{aligned}
u(x) = {} & \frac{ike^{\Lambda_-(x-x_0)}}{\Lambda_- + in\tau} w + e^{-|k|(x-x_0)}\nu + \frac{k^2 S(x) - |k|n\tau R(x)}{k^2 + (n\tau)^2} \\
& - \frac{ik}{\Lambda_0}\int_{x_0}^x \mathrm{d}\tilde{x}\ \frac{e^{\Lambda_-(x-\tilde{x})}}{\Lambda_- + in\tau} q_-(\tilde{x}) - \frac{ik}{\Lambda_0}\int_x^\infty \mathrm{d}\tilde{x}\ \frac{e^{\Lambda_+(x-\tilde{x})}}{\Lambda_+ + in\tau} q_+(\tilde{x}) \\
& - \int_{x_0}^x \mathrm{d}\tilde{x}\ \frac{ike^{-|k|(x-\tilde{x})}}{ik + n\tau\sigma} Q(\tilde{x}) + \int_x^\infty \mathrm{d}\tilde{x}\ \frac{ike^{|k|(x-\tilde{x})}}{ik - n\tau\sigma} Q(\tilde{x})
\end{aligned} \tag{2.2}$$

$$\begin{aligned}
v(x) = {} & \frac{\Lambda_- e^{\Lambda_-(x-x_0)}}{\Lambda_- + in\tau} w + e^{-|k|(x-x_0)}\mu - \frac{k^2 R(x) + |k|n\tau S(x)}{k^2 + (n\tau)^2} \\
& - \frac{\Lambda_-}{\Lambda_0}\int_{x_0}^x \mathrm{d}\tilde{x}\ \frac{e^{\Lambda_-(x-\tilde{x})}}{\Lambda_- + in\tau} q_-(\tilde{x}) - \frac{\Lambda_+}{\Lambda_0}\int_x^\infty \mathrm{d}\tilde{x}\ \frac{e^{\Lambda_+(x-\tilde{x})}}{\Lambda_+ + in\tau} q_+(\tilde{x}) \\
& + \int_{x_0}^x \mathrm{d}\tilde{x}\ \frac{|k|e^{-|k|(x-\tilde{x})}}{ik + n\tau\sigma} Q(\tilde{x}) + \int_x^\infty \mathrm{d}\tilde{x}\ \frac{|k|e^{|k|(x-\tilde{x})}}{ik - n\tau\sigma} Q(\tilde{x})\ .
\end{aligned} \tag{2.3}$$

Using inverse Fourier transform, we get (1.12)-(1.14) where the operator $K_1(x)$ is the convolution operator with the inverse Fourier transform of $K_1(x,k) = e^{\Lambda_- x}$, $K_0(x)$ is the convolution operator with $K_0(x,y) = \frac{1}{\pi}\frac{x}{x^2+y^2}$, and, in terms of their symbols, $\mathcal{L}_1 = \frac{k^2}{k^2+(n\tau)^2}$, $\mathcal{L}_2 = \frac{|k|n\tau}{k^2+(n\tau)^2}$, $\mathcal{L}_u = (\Lambda_- + in\tau)^{-1}ik$, $\mathcal{L}_v = (\Lambda_- + in\tau)^{-1}\Lambda_-$, and

$$\mathcal{F}_{1,\omega}(x) = \int_{x_0}^x \mathrm{d}\tilde{x}\ K_{1,1,\omega}(x-\tilde{x})P(\tilde{x}) + K_{1,2,\omega}(x-\tilde{x})Q(\tilde{x}) \tag{2.4}$$

$$\mathcal{F}_{2,\omega}(x) = \int_x^\infty \mathrm{d}\tilde{x}\ K_{2,1,\omega}(\tilde{x}-x)P(\tilde{x}) + K_{2,2,\omega}(\tilde{x}-x)Q(\tilde{x}) \tag{2.5}$$

with similar definitions for $\mathcal{F}_{1,u}$, $\mathcal{F}_{2,u}$, $\mathcal{F}_{1,v}$ and $\mathcal{F}_{2,v}$ and

$$\begin{array}{ll} K_{1,1,\omega} = -K_8 - K_{10} & K_{1,2,\omega} = -K_2 \\ K_{2,1,\omega} = -e^{-x}(K_1 + K_8 + K_{10}) & K_{2,2,\omega} = -e^{-x}K_2 \\ K_{1,1,u} = K_2 - K_{13} & K_{1,2,u} = -F - K_{12} \\ K_{2,1,u} = e^{-x}(K_2 - K_6) & K_{2,2,u} = F^* - e^{-x}K_5 \\ K_{1,1,v} = K_{1,1,\omega} + K_r + K_i & K_{1,2,v} = K_{1,2,\omega} + G + K_{13} \\ K_{2,1,v} = K_{2,1,\omega} - e^{-x}K_7 & K_{2,2,v} = K_{2,2,\omega} - G^* + e^{-x}K_6 \end{array}$$

with

$$\begin{array}{lll} K_1(x,k) = e^{\Lambda_- x} & K_2(x,k) = -\frac{ike^{\Lambda_- x}}{\Lambda_0} & K_5(x,k) = \frac{k^2 e^{\Lambda_- x}}{\Lambda_0(\Lambda_+ + in\tau)} \\ K_6(x,k) = \frac{kn\tau e^{\Lambda_- x}}{\Lambda_0(\Lambda_+ + in\tau)} & K_7(x,k) = -\frac{in\tau\Lambda_+ e^{\Lambda_- x}}{\Lambda_0(\Lambda_+ + in\tau)} & K_8(x,k) = \frac{\mathrm{Re}(\Lambda_-)}{\Lambda_0} e^{\Lambda_- x} \\ K_{10}(x,k) = \frac{i\mathrm{Im}(\Lambda_-)}{\Lambda_0} e^{\Lambda_- x} & K_{12}(x,k) = \frac{k^2 e^{\Lambda_- x}}{\Lambda_0(\Lambda_- + in\tau)} & K_{13}(x,k) = \frac{kn\tau e^{\Lambda_- x}}{\Lambda_0(\Lambda_- + in\tau)} \\ K_r(x,k) = \frac{in\tau\mathrm{Re}(\Lambda_-)e^{\Lambda_- x}}{\Lambda_0(\Lambda_- + in\tau)} & K_i(x,k) = \frac{-n\tau\mathrm{Im}(\Lambda_-)e^{\Lambda_- x}}{\Lambda_0(\Lambda_- + in\tau)} & K_0(x,y) = \frac{1}{\pi}\frac{x}{x^2+y^2}\ , \end{array}$$

and

$$F(x,k) = \frac{ike^{-|k|x}}{ik + n\tau\sigma}\ , \quad G(x,k) = \frac{|k|e^{-|k|x}}{ik + n\tau\sigma}\ .$$

Various estimates of these kernels are given in Appendix A. Intuitively, the two kernels F and G behave like Poisson's kernels $\frac{1}{\pi}\frac{x}{x^2+y^2}$ and $\frac{1}{\pi}\frac{y}{x^2+y^2}$, while all the other kernels behave like y derivatives or primitives of K_1 according to the expansion of their prefactor as $|k| \to 0$ or $|k| \to \infty$. We thus need to understand the basic properties of $e^{\Lambda_- x}$. To do so, we define

$$b(\alpha) = \tfrac{1}{4}\left(1 - \sqrt{\tfrac{1+\sqrt{1+16\alpha^2}}{2}}\right)\ ,\ \ c(\alpha) = \tfrac{1}{2}\sqrt{\tfrac{1+\sqrt{1+16\alpha^2}}{2+32\alpha^2}}\ ,$$

and note that (see also figure 2 on page 10)

$$\mathrm{Re}(\Lambda_-) \le \begin{cases} b(n\tau) - c(n\tau)k^2 & \forall |k| \le 1 \\ b(n\tau) - \frac{|k|}{2} & \forall |k| \ge 1 \end{cases} \quad \text{and} \quad \left|\frac{1}{\Lambda_0}\right| \le \begin{cases} (1 + (n\tau)^2)^{-1/4} \\ (1 + k^2)^{-1/2} \end{cases}\ .$$

For all practical purpose, the kernel K_1 corresponding to $e^{\Lambda_- x}$ thus behaves like a superposition of a kernel of Poisson's type with a heat kernel (see also Lemma A.10):

$$K_1(x,y) \approx e^{b(n\tau)x}\left(\frac{e^{-\frac{y^2}{4x}}}{\sqrt{4\pi\, x}} + \frac{1}{\pi}\frac{2x}{x^2 + 4y^2}\right)\ .$$

Actually, most results of Appendix A can be easily derived from this analogy. In particular since $b(0) = 0$ and $b(\tau) < 0$, it is easy to see that L^p estimates on K_1 will decay *at most algebraically* as $x \to \infty$, while the same estimates on $\mathcal{P}K_1$ will decay *exponentially faster*. We also easily see that $\partial_y^m K_1 \sim x^{-m}\langle x\rangle^{\frac{m}{2}} K_1$.

3 'Evolution' estimates

Our next task is to prove that for each boundary data in $\mathcal{C}_i$, there exist in $\mathcal{C}_u$ a unique solution to (1.12)-(1.14). This will be done in the next Section using a contraction mapping argument in $\mathcal{W}$. We thus have to show that the r.h.s. of (1.12)-(1.14) defines a Lipschitz map in (a ball of) $\mathcal{W}$. Subsection 3.2 is devoted to the terms involving ν, μ and w, Subsection 3.3 to those involving R and S, Subsection 3.4 to the terms $\mathcal{F}_{1,\cdot}$ and Subsection 3.5 to the terms $\mathcal{F}_{2,\cdot}$.

3.1 Preliminaries

In this whole section, we will encounter various convolution products like

$$K(x-z)f(z) \equiv \int_{-\infty}^{\infty} \mathrm{d}\tilde{y}\, K(x-z, y-\tilde{y}, n\tau) f_n(z,\tilde{y})$$

on which we will use repeatedly the following inequalities (see Subsection 1.2 for the definitions of the norms)

$$\| |y|^{\beta} K(x-z) f(z) \|_2 \leq \| |y|^{\beta} K(x-z) \|_{\mathrm{L}^2} \| f(z) \|_1 + \| K(x-z) \|_{\mathrm{L}^1} \| |y|^{\beta} f(z) \|_2 \ , \tag{3.1}$$

$$\| K(x-z) f(z) \|_s \leq \min \Big(\| K(x-z) \|_{\mathrm{L}^{p_1}} \| f(z) \|_{q_1} \ , \ \| K(x-z) \|_{\mathrm{L}^{p_2}} \| f(z) \|_{q_2} \Big) \tag{3.2}$$

$$\| K(x-z) f(z) \|_s \leq \min \Big(\| K(x-z) \|_{\mathrm{L}^{p_1}} \| f(z) \|_{q_1} \ , \ \| \partial_y K(x-z) \|_{\mathrm{L}^{p_2}} \| \mathcal{I} f(z) \|_{q_2} \Big) \ , \tag{3.3}$$

$$\| \partial_y (K(x-z) f(z)) \|_s \leq \min \Big(\| K(x-z) \|_{\mathrm{L}^{p_1}} \| \partial_y f(z) \|_{q_1} \ , \ \| \partial_y K(x-z) \|_{\mathrm{L}^{p_2}} \| f(z) \|_{q_2} \Big) \tag{3.4}$$

where $\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2} = 1 + \frac{1}{s}$. Note that (3.3) and (3.4) follow from $K(x-z)f(z) = \partial_y K(x-z) \mathcal{I} f(z)$ and $\partial_y(K(x-z)f(z)) = (\partial_y K(x-z))f(z) = K(x-z)(\partial_y f(z))$. In particular, (3.2)-(3.4) give a great freedom for the way we will actually do the estimates. Our main concern and difficulty will be to get optimal decay rates as $x \to \infty$. As a rule (particularly in Subsections 3.4 and 3.5), we will choose the p_1 as small as possible to cover regions where $|x-z|$ is small and p_2 as large as possible in regions where $|x-z|$ is large. For concision, we will often omit the arguments in the K's and f's when no confusion is possible. For the same reason, we will use (3.1)-(3.4) without reference or even sometimes without explicit statement of the choice made for the parameters.

We also note for further reference that using $\|f\|_\infty \leq (\|f\|_2 \|\partial_y f\|_2)^{\frac{1}{2}}$, the interpolation inequality, $0 < \varphi < \frac{1}{2}$ and $\frac{1}{2} + \eta - \xi \geq 0$, we have for some constant C that

$$\|(f,0,0)\| \leq C \|(0,f,0)\| \leq C \|(0,0,f)\| \ . \tag{3.5}$$

3.2 The 'linear' terms

In this subsection, we will prove the following inequalities,

$$\|(\mathcal{L}_u K_1(x-x_0) w(x_0), \mathcal{L}_v K_1(x-x_0) w(x_0), K_1(x-x_0) w(x_0))\| \leq C \|(0,0,w)\|_{x_0} \tag{3.6}$$

$$\|(K_0(x-x_0)\nu(x_0), K_0(x-x_0)\mu(x_0), 0)\| \leq C \|(\nu,\mu,0)\|_{x_0} \ , \tag{3.7}$$

which show that the $\|\cdot\|$ norm of the 'linear' terms in (1.12)–(1.14) is controlled by $\|(\nu,\mu,w)\|_{x_0}$, provided ν, μ and w are in the Class $\mathcal{C}_i$ of Definition 1.3. By (3.5), it will be enough to prove that

$$\|(0,0,K_1(x-x_0)w(x_0))\| \leq C \|(0,0,w)\|_{x_0} \tag{3.8}$$

$$\|(\mathcal{L}_u K_1(x-x_0) w(x_0), (\mathcal{L}_v - 1) K_1(x-x_0) w(x_0), 0)\| \leq C \|(0,0,w)\|_{x_0} \tag{3.9}$$

$$\|(K_0(x-x_0)\nu(x_0), K_0(x-x_0)\mu(x_0), 0)\| \leq C \|(\nu,\mu,0)\|_{x_0} \ . \tag{3.10}$$

For convenience, these three inequalities are proved in the three following Lemmas. The general idea of the proofs is to consider separately the regions $x_0 \leq x \leq 2x_0$ and $x \geq 2x_0$. In the first region, we will use the fact that $\|K_0(x-x_0)\|_{\mathrm{L}^1} + \|K_1(x-x_0)\|_{\mathrm{L}^1}$ is uniformly bounded (thus $K_0\cdot$ and $K_1\cdot$ are L^p-bounded operators for all $p \geq 1$), whereas in the region $x \geq 2x_0$, we will essentially use that $\|K_0(x-x_0)\|_{\mathrm{L}^p} + \|K_1(x-x_0)\|_{\mathrm{L}^p}$ decays as $x \to \infty$ as soon as $p > 1$.

Lemma 3.1 *Let $f = K_1(x - x_0)w(x_0)$. Assume that the parameters satisfy (1.16), then there exist a constant C such that*

$$\|(0,0,f)\| \le C\|(0,0,w)\|_{x_0} .$$

Proof. We first note that for $x_0 \le x \le 2x_0$, since $\|K_1\|_1 \le C$, we have

$$\|f\|_{x,2,\frac{3}{4}} + \|\partial_y f\|_{x,\infty,\frac{3}{2}} + \|\partial_y f\|_{x,1,1} \le C(\|w\|_{x_0,2,\frac{3}{4}} + \|\partial_y w\|_{x_0,\infty,\frac{3}{2}} + \|\partial_y w\|_{x_0,1,1}) ,$$

while for $x \ge 2x_0$,

$$\begin{aligned} \|f\|_{x,2,\frac{3}{4}} &\le (\|\mathcal{P}K_1\|_{x,2,\frac{3}{4}}\|w\|_1 + \|\partial_y K_1\|_{x,2,\frac{3}{4}}\|\mathcal{P}_0\mathcal{I}w\|_1) , \\ \|\partial_y f\|_{x,\infty,\frac{3}{2}} &\le (\|\mathcal{P}\partial_y K_1\|_{x,2,\frac{3}{4}}\|w\|_1 + \|\partial_y^2 K_1\|_{x,2,\frac{3}{4}}\|\mathcal{P}_0\mathcal{I}w\|_1) , \\ \|\partial_y f\|_{x,1,1} &\le (\|\mathcal{P}\partial_y K_1\|_{x,1,1}\|w\|_1 + \|\partial_y^2 K_1\|_{x,1,1}\|\mathcal{P}_0\mathcal{I}w\|_1) . \end{aligned}$$

Using Lemma A.5, that $x - x_0 \ge \frac{x}{2}$ if $x \ge 2x_0$, and that $\langle x\rangle^{\frac{1}{2}} e^{b(\tau)x} \le \frac{\langle\tau\rangle}{\tau} \le \langle x_0\rangle^{\frac{1}{2}}$, we get

$$\|f\|_{x,2,\frac{3}{4}} + \|\partial_y f\|_{x,\infty,\frac{3}{2}} + \|\partial_y f\|_{x,1,1} \le C(\langle x_0\rangle^{\frac{1}{2}}\|w\|_1 + \|\mathcal{P}_0\mathcal{I}w\|_1) .$$

Next, we note that for all $z \in \mathbf{R}$, we can write $|y|^\beta = |y - z|^\beta + L(y,z)$ with $|L(y,z)| \le C(|z|^\beta + |z||y - z|^{\beta-1})$, so that

$$\begin{aligned} \||y|^\beta\mathcal{P}f\|_{2,\frac{3}{4}-\frac{\beta}{2}} &\le C\langle x_0\rangle^{\frac{3}{4}-\frac{\beta}{2}}\||y|^\beta w\|_2 + C\sup_{x\ge x_0}\langle x\rangle^{\frac{3}{4}-\frac{\beta}{2}}\|\mathcal{P}|y|^\beta K_1\|_{\mathrm{L}^2}\|w\|_1 \\ &\le C\langle x_0\rangle^{\frac{3}{4}-\frac{\beta}{2}}\||y|^\beta w\|_2 + C\langle x_0\rangle^{\frac{1}{2}}\|w\|_1 , \\ \||y|^\beta\mathcal{P}_0 f\|_{2,\frac{3}{4}-\frac{\beta}{2}} &\le C\sup_{x\ge x_0}\langle x\rangle^{\frac{3}{4}-\frac{\beta}{2}}\Big(\|(|y|^\beta K_1)\mathcal{P}_0 w\|_2 + \||y|^\beta w\|_2 + \||y|^{\beta-1}K_1\|_{\mathrm{L}^2}\|yw\|_1\Big) \\ &\le C\Big(\sup_{x\ge x_0}\langle x\rangle^{\frac{3}{4}-\frac{\beta}{2}}\|\partial_y(|y|^\beta K_1)\|_{\mathrm{L}^2}\|\mathcal{I}\mathcal{P}_0 w\|_1\Big) + C\|(0,0,w)\|_{x_0} , \end{aligned}$$

where we used $\|(|y|^\beta K_1)\mathcal{P}_0 w\|_{\mathrm{L}^2} = \|(\partial_y|y|^\beta K_1)\mathcal{I}\mathcal{P}_0 w\|_{\mathrm{L}^2}$ and that since $\beta > \frac{3}{2}$, we have

$$\langle x\rangle^{\frac{3}{4}-\frac{\beta}{2}}\|yw\|_1 \le \langle x_0\rangle^{\frac{3}{4}-\frac{\beta}{2}}\|w\|_2 + \langle x_0\rangle^{\frac{3}{4}-\frac{\beta}{2}}\||y|^\beta w\|_2 \le \|(0,0,w)\|_{x_0} .$$

The proof is completed using $\langle x_0\rangle^{\frac{1}{2}}\|w\|_1 \le \|(0,0,w)\|_{x_0}$ (see (1.18)) and Lemma 3.2 below. ■

Lemma 3.2 *Let $\beta > \frac{3}{2}$ and $0 \le \gamma < \beta - \frac{3}{2}$. , $\mathcal{Z}_\beta = \{\|(1+|y|^\beta)f\|_{\mathrm{L}^2} < \infty$ and $\mathcal{M}(f) = \int_{\mathbf{R}} f(y)\mathrm{d}y = 0\}$. Then there exist constants $C_\beta, C_{\beta,\gamma}$ such that for all $f \in \mathcal{Z}_\beta$,*

$$\begin{aligned} \|\mathcal{I}f\|_{\mathrm{L}^\infty} &\le C_\beta\|f\|_{\mathrm{L}^2}^{1-\frac{1}{2\beta}}\||y|^\beta f\|_{\mathrm{L}^2}^{\frac{1}{2\beta}} , \\ \||y|^\gamma\mathcal{I}f\|_{\mathrm{L}^1} &\le C_{\beta,\gamma}\|f\|_{\mathrm{L}^2}^{1-\frac{3}{2\beta}-\frac{\gamma}{\beta}}\||y|^\beta f\|_{\mathrm{L}^2}^{\frac{3}{2\beta}+\frac{\gamma}{\beta}} . \end{aligned}$$

The first inequality is also valid if $\mathcal{M}(f) \ne 0$.

Proof. Let $\beta > \frac{3}{2}$ and $a > 0$. Since $\|\mathcal{I}f\|_{\mathrm{L}^\infty} \leq \|f\|_{\mathrm{L}^1}$, the first inequality follows from Lemma A.1. Then, since $\mathcal{M}(f) = 0$, we have

$$\mathcal{I}f(y) = -\int_y^\infty \mathrm{d}z\ f(z) = \int_{-\infty}^y \mathrm{d}z\ f(z)\ ,$$

from which we deduce

$$\begin{aligned} \||y|^\gamma \mathcal{I}f(y)\|_{\mathrm{L}^1} &\leq 2\Big(a\|f\|_{\mathrm{L}^2} + \||y|^\beta f\|_{\mathrm{L}^2}\Big) \int_0^\infty \mathrm{d}y\ |y|^\gamma \left(\int_y^\infty \mathrm{d}z\ (a+|z|^\beta)^{-2}\right)^{\frac{1}{2}} \\ &\leq \left(a^{\frac{3}{2\beta}+\frac{\gamma}{\beta}}\|f\|_{\mathrm{L}^2} + a^{\frac{3}{2\beta}+\frac{\gamma}{\beta}-1}\||y|^\beta f\|_{\mathrm{L}^2}\right) \int_0^\infty \mathrm{d}y\ |y|^\gamma \left(\int_y^\infty \frac{\mathrm{d}z}{(1+|z|^\beta)^2}\right)^{\frac{1}{2}}\ . \end{aligned}$$

Setting $a = \||y|^\beta f\|_{\mathrm{L}^2}/\|f\|_{\mathrm{L}^2}$ completes the proof, since the last integral is bounded if $\gamma < \beta - \frac{3}{2}$. ■

Lemma 3.3 *Let $f_u = K_1(x-x_0)\mathcal{L}_u w(x_0)$, $\tilde{\mathcal{L}}_v = \mathcal{L}_v - 1$ and $f_v = K_1(x-x_0)\tilde{\mathcal{L}}_v w(x_0)$. If (1.16) holds, then there exist a constant C such that*

$$\|(f_u, f_v, 0)\| \leq C\|(0,0,w)\|_{x_0}\ .$$

Proof. By Lemma A.5, for all $1 \leq s \leq \infty$, we have

$$\begin{aligned} \|\partial_y f_u\|_{r,1-\frac{1}{2r}-\eta} &\leq C \sup_{x_0 \leq x \leq 2x_0} \langle x\rangle^{1-\frac{1}{2r}-\eta}\|\partial_y \mathcal{L}_u w\|_r + \sup_{x\geq 2x_0} \langle x\rangle^{1-\frac{1}{2r}-\eta}\|\partial_y K_1\|_{\mathrm{L}^r}\|\mathcal{L}_u w\|_1 \\ &\leq C\|\mathcal{L}_u w\|_1 + \langle x_0\rangle^{1-\frac{1}{2r}-\eta}\|\partial_y \mathcal{L}_u w\|_r\ , \\ \|f_u\|_{s,\frac{1}{2}-\frac{1}{2s}} &\leq C \sup_{x_0\leq x\leq 2x_0} \langle x\rangle^{\frac{1}{2}-\frac{1}{2s}}\|\mathcal{L}_u w\|_s + C \sup_{x\geq 2x_0} \langle x\rangle^{\frac{1}{2}-\frac{1}{2s}}\|K_1\|_{\mathrm{L}^s}\|\mathcal{L}_u w\|_1 \\ &\leq C\|\mathcal{L}_u w\|_1 + \langle x_0\rangle^{\frac{1}{2}}\|\mathcal{L}_u w\|_\infty\ , \end{aligned}$$

since $\langle x_0\rangle^{\frac{1}{2}-\frac{1}{2p}}\|f\|_{\mathrm{L}^p} \leq \|f\|_{\mathrm{L}^1} + \langle x_0\rangle^{\frac{1}{2}}\|f\|_{\mathrm{L}^\infty}$ for all $1 \leq p \leq \infty$. We then note that $\tilde{\mathcal{L}}_v = \frac{-in\tau}{\Lambda_- + in\tau}$, in particular, $\mathcal{P}\tilde{\mathcal{L}}_v = \tilde{\mathcal{L}}_v$ and $\mathcal{P}e^{cb(n\tau)x} \leq e^{cb(\tau)x}$ for all $c > 0$. As in Lemma 3.1, since $x - x_0 \geq \frac{x}{2}$ for $x \geq 2x_0$ and $b(\tau) < 0$, we have

$$\begin{aligned} \|f_v\|_{s,1-\frac{1}{2s}-\varphi} &\leq C\langle x_0\rangle^{1-\frac{1}{2s}-\varphi}\|\tilde{\mathcal{L}}_v w\|_s + C \sup_{x\geq 2x_0} \langle x\rangle^{1-\frac{1}{2s}-\varphi}\|\mathcal{P}K_1\|_{\mathrm{L}^s}\|\tilde{\mathcal{L}}_v w\|_1\ , \\ &\leq C\langle x_0\rangle^{\frac{1}{2}-\varphi}\|\tilde{\mathcal{L}}_v w\|_1 + C\langle x_0\rangle^{1-\varphi}\|\tilde{\mathcal{L}}_v w\|_\infty + \langle x_0\rangle^{\frac{1}{2}-\varphi}\|\tilde{\mathcal{L}}_v w\|_1\ , \\ \|\partial_y f_v\|_{r,\frac{3}{2}-\frac{1}{2r}-\xi} &\leq C \sup_{x_0\leq x\leq 2x_0} \langle x\rangle^{\frac{3}{2}-\frac{1}{2r}-\xi}\|\partial_y \tilde{\mathcal{L}}_v w\|_r + C \sup_{x\geq 2x_0} \langle x\rangle^{\frac{3}{2}-\frac{1}{2r}-\xi}\|\mathcal{P}\partial_y K_1\|_{\mathrm{L}^r}\|\tilde{\mathcal{L}}_v w\|_1\ , \\ &\leq C\Big(\langle x_0\rangle^{\frac{3}{2}-\frac{1}{2r}-\xi}\|\partial_y \tilde{\mathcal{L}}_v \omega\|_r + \langle x_0\rangle^{\frac{1}{2}-\varphi}\|\tilde{\mathcal{L}}_v \omega\|_1\Big)\ . \end{aligned}$$

Since $\langle x\rangle e^{\frac{b(\tau)x}{8}} \leq \frac{\langle\tau\rangle}{\tau} \leq \langle x_0\rangle^{\frac{1}{2}}$ and $\xi \geq \varphi$. The proof is then completed using Lemma 3.4 and 3.5 below. ■

Lemma 3.4 (Mikhlin-Hörmander) *Let $m : \mathbf{R} \to \mathbf{C}$, and define $m_0 = \sup_{k\in\mathbf{R}} |m(k)| + |k\partial_k m(k)|$ and $m_1 = \sup_{k\in\mathbf{R}} |\partial_k m(k)|$. Let $\mathcal{F}$ denotes the (continuous) Fourier transform and $M : f \to \mathcal{F}^{-1}m(\cdot)\mathcal{F}f$. Then there exist constants C_p such that*

$$\|Mf\|_{\mathrm{L}^\infty} \leq C_\infty m_0\sqrt{\|f\|_{\mathrm{L}^2}\|\partial_y f\|_{\mathrm{L}^2}}\ , \qquad \|Mf\|_{\mathrm{L}^p} \leq C_p m_0\|f\|_{\mathrm{L}^p}$$

$$\|Mf\|_{L^1} \le C_1\Big(m_0\sqrt{\|f\|_{L^2}\|yf\|_{L^2}} + \sqrt{m_0 m_1}\|f\|_{L^2}\Big)$$

for all $1 < p < \infty$.

Proof. The L^p estimate for $1 < p < \infty$ is a consequence of the classical Mikhlin-Hörmander condition (see, e.g. [11]), the L^∞ and L^1 estimates are immediate consequences of the Sobolev and Plancherel inequalities. ■

Lemma 3.5 *Let* $\tilde{\mathcal{L}}_v = \mathcal{L}_v - 1$ *and* $\tilde{\mathcal{L}}_u = \mathcal{L}_u + \mathcal{I}\mathcal{P}_0$ *and assume that (1.16) holds, then*

$$\begin{aligned}
\|\mathcal{L}_u w\|_1 + \langle x_0\rangle^{\frac{1}{2}}\|\mathcal{L}_u w\|_\infty + \langle x_0\rangle^{1-\frac{1}{2r}-\eta}\|\partial_y\mathcal{L}_u w\|_r &\le C(\|\mathcal{I}\mathcal{P}_0 w\|_1 + \|(0,0,w)\|_{x_0})\,,\\
\|\tilde{\mathcal{L}}_u w\|_1 + \langle x_0\rangle^{\frac{1}{2}}\|\tilde{\mathcal{L}}_u w\|_\infty + \langle x_0\rangle^{1-\frac{1}{2r}-\eta}\|\partial_y\tilde{\mathcal{L}}_u w\|_r &\le C\|(0,0,w)\|_{x_0}\,,\\
\langle x_0\rangle^{\frac{1}{2}-\varphi}\|\tilde{\mathcal{L}}_v w\|_1 + \langle x_0\rangle^{1-\varphi}\|\tilde{\mathcal{L}}_v w\|_\infty + \langle x_0\rangle^{\frac{3}{2}-\frac{1}{2r}-\xi}\|\partial_y\tilde{\mathcal{L}}_v w\|_r &\le C\|(0,0,w)\|_{x_0}\,,\\
\langle x_0\rangle^{\frac{1}{2}-\varphi}\|\mathcal{L}_v w\|_1 + \langle x_0\rangle^{1-\varphi}\|\mathcal{L}_v w\|_\infty + \langle x_0\rangle^{\frac{3}{2}-\frac{1}{2r}-\xi}\|\partial_y\mathcal{L}_v w\|_r &\le C\|(0,0,w)\|_{x_0}\,.
\end{aligned}$$

Proof. We have $\mathcal{L}_u = -\mathcal{I}\mathcal{P}_0 + \tilde{\mathcal{L}}_u$. Then the symbol $T(k,n)$ of $\tilde{\mathcal{L}}_u$ is given by $T(k,n) = \frac{-ik}{\Lambda_- + in\tau}$ if $n \neq 0$ and $T(k,0) = \frac{-ik}{\Lambda_+}$, and it satisfies (uniformly in $n \in \mathbf{Z}$) the hypothesis of Lemma 3.4 with $m_0 = C\frac{\langle\tau\rangle}{\tau} \le C\langle x_0\rangle^{\frac{1}{2}}$ and $m_1 = C\frac{\langle\tau\rangle^2}{\tau^2} \le C\langle x_0\rangle$. Using $\partial_y\mathcal{I}f = f$ and Lemma 3.2 and 3.4, we get that

$$\begin{aligned}
\|\tilde{\mathcal{L}}_u w\|_1 &\le (\langle x_0\rangle^{\frac{3}{4}}\|w\|_2\langle x_0\rangle^{\frac{1}{4}}\|yw\|_2)^{\frac{1}{2}} + \langle x_0\rangle^{\frac{3}{4}}\|w\|_2\\
\langle x_0\rangle^{\frac{1}{2}}\|\tilde{\mathcal{L}}_u w\|_\infty &\le (\langle x_0\rangle^{\frac{3}{4}}\|w\|_2\langle x_0\rangle^{\frac{5}{4}}\|\partial_y w\|_2)^{\frac{1}{2}}\,,\\
\langle x_0\rangle^{1-\frac{1}{2r}-\eta}\|\partial_y\tilde{\mathcal{L}}_u w\|_r &\le \langle x_0\rangle^{\frac{3}{2}-\frac{1}{2r}-\eta}\|\partial_y w\|_r \le \langle x_0\rangle\|\partial_y w\|_1 + \langle x_0\rangle^{\frac{3}{2}}\|\partial_y w\|_\infty\,,\\
\langle x_0\rangle^{\frac{1}{2}}\|\mathcal{I}\mathcal{P}_0 w\|_\infty &\le \langle x_0\rangle^{\frac{1}{2}}\|\mathcal{P}_0 w\|_1 \le C(\langle x_0\rangle^{\frac{3}{4}}\|w\|_2\langle x_0\rangle^{\frac{1}{4}}\|yw\|_2)^{\frac{1}{2}}\,,\\
\langle x_0\rangle^{1-\frac{1}{2r}-\eta}\|\partial_y\mathcal{I}\mathcal{P}_0 w\|_r &\le \langle x_0\rangle^{1-\frac{1}{2r}-\eta}\|w\|_r \le \langle x_0\rangle^{\frac{1}{2}}\|w\|_1 + \langle x_0\rangle\|w\|_\infty\,.
\end{aligned}$$

Similarly, since $\mathcal{P}\tilde{\mathcal{L}}_v$ satisfies the hypothesis of Lemma 3.4 with $m_0 = 2\frac{\langle\tau\rangle}{\tau} \le 2\langle x_0\rangle^{\varphi} < 2\langle x_0\rangle^{\frac{1}{2}}$ and $m_1 = m_0^2 \le 4\langle x_0\rangle^{2\varphi}$, we get

$$\begin{aligned}
\langle x_0\rangle^{1-\varphi}\|\tilde{\mathcal{L}}_v w\|_\infty &\le \Big(\langle x_0\rangle^{\frac{3}{4}}\|w\|_2\langle x_0\rangle^{\frac{5}{4}}\|\partial_y w\|_2\Big)^{\frac{1}{2}}\\
\langle x_0\rangle^{\frac{1}{2}-\varphi}\|\tilde{\mathcal{L}}_v w\|_1 &\le C\Big(\langle x_0\rangle^{\frac{3}{4}}\|w\|_2\langle x_0\rangle^{\frac{1}{4}}\|yw\|_2\Big)^{\frac{1}{2}} + C\langle x_0\rangle^{\frac{3}{4}}\|w\|_2\\
\langle x_0\rangle^{\frac{3}{2}-\frac{1}{2r}-\xi}\|\partial_y\tilde{\mathcal{L}}_v w\|_r &\le \langle x_0\rangle^{\frac{3}{2}-\frac{1}{2r}}\|\partial_y\omega\|_r \le \langle x_0\rangle\|\partial_y\omega\|_1 + \langle x_0\rangle^{\frac{3}{2}}\|\partial_y\omega\|_\infty\,.
\end{aligned}$$

Since $\mathcal{L}_v = \tilde{\mathcal{L}}_v + 1$, the proof is completed using $\langle x_0\rangle^{\frac{1}{2}}\|w\|_1 + \langle x_0\rangle\|w\|_\infty \le C\|(0,0,w)\|_{x_0}$ (see also (1.18)) and $\langle x_0\rangle^{\frac{1}{4}}\|yw\|_2 \le \langle x_0\rangle^{\frac{3}{4}}\|w\|_2 + \langle x_0\rangle^{\frac{3}{4}-\frac{\beta}{2}}\||y|^{\beta}w\|_2 \le C\|(0,0,w)\|$. ■

Lemma 3.6 *Let* $g_u(x) = K_0(x-x_0)\nu(x_0)$ *and* $g_v(x) = K_0(x-x_0)\mu(x_0)$*, then if (1.16) holds,*

$$\begin{aligned}
\|(g_u, g_v, 0)\| &\le C\|(\nu,\mu,0)\|_{x_0}\,,\\
\|g_u(x)\|_\infty + \|g_v(x)\|_\infty &\le C\langle x\rangle^{-1+\varphi}\|(\nu,\mu,0)\|_{x_0}
\end{aligned}$$

for all $x \ge 2x_0$.

Proof. We first note that $\|K_0(x)\|_{\mathrm{L}^s} \le Cx^{\frac{1}{s}-1}$ and $\|\partial_y K_0(x)\|_{\mathrm{L}^s} \le Cx^{\frac{1}{s}-2}$. Then let $q \le p_0 \le \infty$ and $p \le p_1 \le \infty$, since $(x-x_0)^{\frac{1}{s}-1} \le C\langle x\rangle^{\frac{1}{s}-1}$ if $x \ge 2x_0$, we get

$$\begin{aligned}
\|g_u\|_{p_0,\frac{1}{2}-\frac{1}{p_0}} &\le C \sup_{x_0\le x\le 2x_0} \langle x\rangle^{\frac{1}{2}-\frac{1}{p_0}}\|\nu\|_{p_0} + C\sup_{x\ge 2x_0}\langle x\rangle^{\frac{1}{2}-\frac{1}{q}}\|\nu\|_q \ , \\
\|g_v\|_{p_1,1-\frac{1}{p_1}-\varphi} &\le C \sup_{x_0\le x\le 2x_0} \langle x\rangle^{1-\frac{1}{p_1}-\varphi}\|\mu\|_{p_1} + C\sup_{x\ge 2x_0}\langle x\rangle^{1-\frac{1}{p}-\varphi}\|\mu\|_p \ , \\
\|\partial_y g_u\|_{r,1-\frac{1}{2r}-\eta} &\le C \sup_{x_0\le x\le 2x_0} \langle x\rangle^{1-\frac{1}{2r}-\eta}\|\partial_y\nu\|_r + C\sup_{x\ge 2x_0}\langle x\rangle^{\frac{1}{2}-\frac{1}{q}}\|\nu\|_q \ , \\
\|\partial_y g_v\|_{r,\frac{3}{2}-\frac{1}{2r}-\xi} &\le C \sup_{x_0\le x\le 2x_0} \langle x\rangle^{\frac{3}{2}-\frac{1}{2r}-\xi}\|\partial_y\mu\|_r + C\sup_{x\ge 2x_0}\langle x\rangle^{1-\frac{1}{p}-\varphi}\|\mu\|_p \ ,
\end{aligned}$$

while for $x \ge 2x_0$, we have

$$\|g_u(x)\|_\infty + \|g_v(x)\|_\infty \le \langle x\rangle^{-1+\varphi}\Big(\langle x\rangle^{1-\frac{1}{p}-\varphi}(\|\mu\|_p + \|\mathcal{H}\mu\|_p)\Big) \le \langle x\rangle^{-1+\varphi}\Big(\langle x\rangle^{1-\frac{1}{p}-\varphi}\|\mu\|_p\Big)$$

The proof is completed since $\xi \ge \varphi$, $1 \le q < 2$ and $1-\frac{1}{p} \le \varphi < \frac{1}{2}$. ■

3.3 The ‘local’ terms

From now on, we begin the estimates of the contribution of the nonlinear terms in (1.12)-(1.14). We first consider the ‘local’ terms first.

Proposition 3.7 *Assume that (1.16) holds then for* $\kappa_0 = \min(\frac{\varphi}{2}, \frac{1}{2}-\eta+\xi-\varphi)$, *we have*

$$\begin{aligned}
\|(\mathcal{L}_1 S - \mathcal{L}_2 R, -\mathcal{L}_1 R - \mathcal{L}_2 S, 0)\| &\le C\langle x_0\rangle^{-\kappa_0}\|(\mathbf{v},\omega)\|^2 \ , \\
\|\mathcal{L}_1 S(x) - \mathcal{L}_2 R(x)\|_\infty + \|\mathcal{L}_1 R(x) + \mathcal{L}_2 S(x)\|_\infty &\le C\langle x\rangle^{-1}\|(\mathbf{v},\omega)\|^2 \ .
\end{aligned} \tag{3.11}$$

Proof. The proof follows at once from Lemma A.3 and (1.17). ■
We already see at this point (see (3.11)) that these terms are of smaller order as $x \to \infty$ than most terms of the preceding section.

3.4 The nonlinear terms I

In this section, we prove the

Theorem 3.8 *Assume that* P *and* Q *satisfy the bounds (1.17), and that the parameters satisfy (1.16), then there exist constants* C *and* $\kappa_1 > 0$ *such that*

$$\|(\mathcal{F}_{1,u}, \mathcal{F}_{1,v}, \mathcal{F}_{1,\omega})\| \le C\langle x_0\rangle^{-\kappa_1}\|(\mathbf{v},\omega)\|^2 \ . \tag{3.12}$$

This is incidentally the hardest part of the paper in that the parameters in (3.1)-(3.4) need to be chosen in the right way to get a bound that *decays* as $x_0 \to \infty$. The proof of (3.12) is split component-wise in the three Propositions ending this section. During the course of these proofs, we will encounter repeatedly the following functions

$$\begin{aligned}
A\Big[\begin{smallmatrix} p_2,q_2,s \\ p_1,q_1 \end{smallmatrix}\Big](x,x_0) &= \int_{x_0}^{x} \mathrm{d}\tilde{x}\ \min\Big(\frac{\langle\tilde{x}\rangle^{-q_1}}{(x-\tilde{x})^{p_1}}, \frac{\langle x\rangle^{s}\langle\tilde{x}\rangle^{-q_2}}{(x-\tilde{x})^{p_2}}\Big) \ , \\
B\Big[\begin{smallmatrix} p_2,q_2,s_2 \\ p_1,q_1,s_1 \end{smallmatrix}\Big](x,x_0) &= \int_{x_0}^{x} \mathrm{d}\tilde{x}\ e^{\frac{b(\tau)(x-\tilde{x})}{4}}\min\Big(\frac{\langle\tilde{x}\rangle^{-q_1}\langle x-\tilde{x}\rangle^{s_1}}{(x-\tilde{x})^{p_1}}, \frac{\langle\tilde{x}\rangle^{-q_2}\langle x-\tilde{x}\rangle^{s_2}}{(x-\tilde{x})^{p_2}}\Big) \ ,
\end{aligned}$$

which occur naturally from (3.1)-(3.4). For further reference, we note that these functions satisfy the

Lemma 3.9 *Let $p_1 < 1$, $s \geq 0$ and $p_2, q_1, q_2 \in \mathbf{R}$, there exist a constant C such that for all $x \geq x_0 \geq 1$, it holds*

$$A\left[\begin{smallmatrix} p_2,q_2,s \\ p_1,q_1 \end{smallmatrix}\right](x,x_0) \leq C\left(\langle x\rangle^{1-q_1-p_1} + \langle x\rangle^{s-p_2}\max(\langle x\rangle^{1-q_2}, \langle x_0\rangle^{1-q_2})\right) , \tag{3.13}$$

if $q_2 \neq 1$, while the same inequality holds with $\max(\langle x\rangle^{1-q_2}, \langle x_0\rangle^{1-q_2})$ replaced by $\ln(1+x)$ if $q_2 = 1$. If furthermore we have $s_1, s_2 \geq 0$ and $\frac{\langle\tau\rangle}{\tau} \leq \langle x_0\rangle^{\varphi}$, then for all $m \geq 0$, there exist a constant C such that for all $x \geq x_0 \geq 1$, it holds

$$B\left[\begin{smallmatrix} p_2,q_2,s_2 \\ p_1,q_1,s_1 \end{smallmatrix}\right](x,x_0) \leq C\Big(\langle x\rangle^{-q_1}\langle x_0\rangle^{2(1+s_1-p_1)\varphi} + \langle x\rangle^{-p_2-m}\langle x_0\rangle^{2(1+m+s_2)\varphi}\max(\langle x\rangle^{-q_2}, \langle x_0\rangle^{-q_2})\Big) .$$

Proof. We first note that for all $p > -1$, there exist a constant C such that

$$\int_{x_0}^{x} \mathrm{d}\tilde{x}\, e^{\frac{b(\tau)(x-\tilde{x})}{4}}(x-\tilde{x})^p \leq C\int_0^{x-x_0}\mathrm{d}z\, e^{\frac{-|b(\tau)|z}{4}} z^p \leq C|b(\tau)|^{-1-p} \leq C\tau^{-2(1+p)} \leq C\langle x_0\rangle^{2(1+p)\varphi} , \tag{3.14}$$

since $|b(\tau)| \leq C\tau^{-2} \leq C\langle x_0\rangle^{2\varphi}$. We then note that since $x \geq x_0 \geq 1$, we have $\frac{\langle x\rangle}{\sqrt{2}} \leq x \leq \langle x\rangle$. We first consider the case of finite x, that is precisely, $x_0 \leq x \leq 2x_0$, then

$$A\left[\begin{smallmatrix} p_2,q_2,s \\ p_1,q_1 \end{smallmatrix}\right](x,x_0) \leq C\langle x_0\rangle^{-q_1}(x-x_0)^{1-p_1} \leq C\langle x_0\rangle^{1-p_1-q_1} ,$$

$$B\left[\begin{smallmatrix} p_2,q_2,s_2 \\ p_1,q_1,s_1 \end{smallmatrix}\right](x,x_0) \leq \langle x_0\rangle^{-q_1}\int_{x_0}^{x}\mathrm{d}\tilde{x}\, e^{\frac{b(\tau)(x-\tilde{x})}{4}}(x-\tilde{x})^{s_1-p_1} \leq C\langle x_0\rangle^{-q_1+2(1+s_1-p_1)\varphi} .$$

However, in the applications of the result of this Lemma, we will generically have e.g. $1-q_1-p_1 < 0$, that is, the integrals we seek to bound *decay* as $x \to \infty$. To get the optimal decay rate, the idea is to consider $x \geq 2x_0$, and split the integration domain $x_0 \leq \tilde{x} \leq x$ in two equal parts. Since $x \geq 2x_0$ implies $\frac{x}{2} \leq (x-x_0) \leq x$ and $x_0 \leq \tilde{x} \leq \frac{x+x_0}{2}$ implies $\frac{x}{4} \leq \frac{x-x_0}{2} \leq x-\tilde{x} \leq x-x_0 \leq x$, we have

$$A\left[\begin{smallmatrix} p_2,q_2,s \\ p_1,q_1 \end{smallmatrix}\right](x,x_0) \leq C\langle x\rangle^{s-p_2}\int_{x_0}^{\frac{x+x_0}{2}}\mathrm{d}\tilde{x}\,\langle\tilde{x}\rangle^{-q_2} + C\langle x\rangle^{-q_1}\int_{\frac{x+x_0}{2}}^{x}\mathrm{d}\tilde{x}\,(x-\tilde{x})^{-p_1} .$$

The proof of (3.13) is completed using $\int_{x_0}^{\frac{x+x_0}{2}}\mathrm{d}\tilde{x}\,\langle\tilde{x}\rangle^{-q_2} \leq \int_{x_0}^{x}\mathrm{d}\tilde{x}\,\langle\tilde{x}\rangle^{-q_2}$ and considering separately $q_2 < 1$, $q_2 = 1$ and $q_2 > 1$. In the same way, we have

$$\begin{aligned} B\left[\begin{smallmatrix} p_2,q_2,s_2 \\ p_1,q_1,s_1 \end{smallmatrix}\right](x,x_0) \leq{}& \frac{C\max(\langle x\rangle^{-q_2}, \langle x_0\rangle^{-q_2})}{\langle x\rangle^{p_2+m}}\int_{x_0}^{\frac{x+x_0}{2}}\mathrm{d}\tilde{x}e^{\frac{b(\tau)(x-\tilde{x})}{4}}(x-\tilde{x})^{s_2+m} \\ &+ \frac{C}{\langle x\rangle^{q_1}}\int_{\frac{x+x_0}{2}}^{x}\mathrm{d}\tilde{x}e^{\frac{b(\tau)(x-\tilde{x})}{4}}(x-\tilde{x})^{s_1-p_1} , \end{aligned}$$

which completes the proof with the help of (3.14). ■

We now turn to the proof of the part of Theorem 3.8 that involves $\mathcal{F}_{1,\omega}$. To prepare the ground for the asymptotic results of Section 5, we also show that most terms in $\mathcal{F}_{1,\omega}$ have decay rates as $x \to \infty$ faster by (almost) $x^{-\frac{1}{2}+\varphi}$ than those of ω.

Proposition 3.10 *Assume that P and Q satisfy the bounds (1.17), and that the parameters satisfy (1.16), then there exist a constant C such that for $\kappa_{1,1} = \min(\frac{1}{4} - \frac{\varphi}{2} - \eta, \frac{1}{2} - \xi)$, we have*

$$\|(0,0,\mathcal{F}_{1,\omega})\| \leq C\langle x_0\rangle^{-\kappa_{1,1}}\|(\mathbf{v},\omega)\|^2 \ , \tag{3.15}$$

and defining $\mathcal{F}_{1,1,\omega}(x) = -\int_{x_0}^{x} \mathrm{d}\tilde{x}\ K_2(x-\tilde{x})Q(\tilde{x})$, we have

$$\begin{aligned} \|\mathcal{F}_{1,\omega}(x) - \mathcal{F}_{1,1,\omega}(x)\|_\infty &\leq C\langle x\rangle^{-\frac{3}{2}+\varphi}\|(\mathbf{v},\omega)\|^2 \\ \|\mathcal{F}_{1,\omega}(x) - \mathcal{F}_{1,1,\omega}(x)\|_1 &\leq C\langle x\rangle^{-1+\varphi}\|(\mathbf{v},\omega)\|^2 \\ \||y|^\beta(\mathcal{F}_{1,\omega}(x) - \mathcal{F}_{1,1,\omega}(x))\|_2 &\leq C\langle x\rangle^{-\frac{5}{4}+\frac{\beta}{2}+\varphi}\|(\mathbf{v},\omega)\|^2 \ . \end{aligned} \tag{3.16}$$

Proof. We write $\mathcal{F}_{1,\omega}(x) = \mathcal{F}_{1,1,\omega}(x) - \mathcal{F}_{1,2,\omega}(x) - \mathcal{F}_{1,3,\omega}(x)$, where $\mathcal{F}_{1,1,\omega}(x)$ is defined above and

$$\mathcal{F}_{1,2,\omega}(x) = \int_{x_0}^{x}\mathrm{d}\tilde{x}\ K_8(x-\tilde{x})P(\tilde{x}) \ , \qquad \mathcal{F}_{1,3,\omega}(x) = \int_{x_0}^{x}\mathrm{d}\tilde{x}\ K_{10}(x-\tilde{x})P(\tilde{x}) \ .$$

Then, from the results of Section A and (3.1)-(3.4), it follows easily that

$$\begin{aligned} \|(0,0,\mathcal{F}_{1,1,\omega})\|_x \leq{}& C\|(\mathbf{v},\omega)\|^2\Big(\langle x\rangle^{\frac{3}{4}}A\Big[\begin{smallmatrix}\frac{3}{4},\frac{3}{2}-\varphi,0\\ \frac{3}{4},\frac{3}{2}-\varphi\end{smallmatrix}\Big](x,x_0) + \langle x\rangle^{\frac{3}{4}-\frac{\beta}{2}}A\Big[\begin{smallmatrix}\frac{3}{4}-\frac{\beta}{2},\frac{3}{2}-\varphi,0\\ \frac{3}{4}-\frac{\beta}{2},\frac{3}{2}-\varphi\end{smallmatrix}\Big](x,x_0)\Big) \\ &+ C\|(\mathbf{v},\omega)\|^2\Big(\langle x\rangle^{\frac{3}{4}-\frac{\beta}{2}}A\Big[\begin{smallmatrix}\frac{1}{2},\frac{7}{4}-\frac{\beta}{2}-\varphi,0\\ \frac{1}{2},\frac{7}{4}-\frac{\beta}{2}-\varphi\end{smallmatrix}\Big](x,x_0)\Big) \\ &+ C\|(\mathbf{v},\omega)\|^2\Big(\langle x\rangle^{\frac{3}{2}}A\Big[\begin{smallmatrix}2,\frac{3}{2}-\varphi,\frac{1}{2}\\ \frac{3}{4},\frac{9}{4}-\xi\end{smallmatrix}\Big](x,x_0) + \langle x\rangle A\Big[\begin{smallmatrix}\frac{3}{2},\frac{3}{2}-\varphi,\frac{1}{2}\\ \frac{1}{2},2-\xi\end{smallmatrix}\Big](x,x_0)\Big) \ . \end{aligned}$$

Using Lemma 3.9 and $\beta \geq \frac{3}{2}$, we get

$$\|(0,0,\mathcal{F}_{1,1,\omega})\| \leq C\Big(\langle x_0\rangle^{-\frac{1}{2}+\varphi} + \langle x_0\rangle^{-\frac{1}{2}+\xi}\Big)\|(\mathbf{v},\omega)\|^2 \ . \tag{3.17}$$

Similarly, from the results of Lemma A.6, it follows easily, choosing $\xi_2 = 1-\varepsilon_1$ and $\xi_3 = 2-2\varepsilon_2$ with $\varepsilon_i > 0$, that

$$\begin{aligned} \|(0,0,\mathcal{F}_{1,2,\omega})\|_x \leq{}& C\|(\mathbf{v},\omega)\|^2\Big(\langle x\rangle^{\frac{3}{4}}A\Big[\begin{smallmatrix}1-\varepsilon_1,\frac{5}{4},0\\ 1-\varepsilon_1,\frac{5}{4}\end{smallmatrix}\Big](x,x_0) + \langle x\rangle^{\frac{3}{4}-\frac{\beta}{2}}A\Big[\begin{smallmatrix}\frac{5}{4}-\frac{\beta}{2},1,0\\ \frac{5}{4}-\frac{\beta}{2},1\end{smallmatrix}\Big](x,x_0)\Big) \\ &+ C\|(\mathbf{v},\omega)\|^2\Big(\langle x\rangle^{\frac{3}{4}-\frac{\beta}{2}}A\Big[\begin{smallmatrix}1-\varepsilon_1,\frac{5}{4}-\frac{\beta}{2},0\\ 1-\varepsilon_1,\frac{5}{4}-\frac{\beta}{2}\end{smallmatrix}\Big](x,x_0)\Big) \\ &+ C\|(\mathbf{v},\omega)\|^2\Big(\langle x\rangle^{\frac{3}{2}}A\Big[\begin{smallmatrix}2,\frac{3}{2},\frac{1}{2}\\ 1-\varepsilon_2,\frac{7}{4}-\eta\end{smallmatrix}\Big](x,x_0) + \langle x\rangle A\Big[\begin{smallmatrix}2,1,\frac{1}{2}\\ 1-\varepsilon_1,\frac{3}{2}-\eta\end{smallmatrix}\Big](x,x_0)\Big) \ , \\ \|\mathcal{F}_{1,2,\omega}(x)\|_\infty \leq{}& C\|(\mathbf{v},\omega)\|^2A\Big[\begin{smallmatrix}2,1,\frac{1}{2}\\ 1-\varepsilon_1,\frac{3}{2}\end{smallmatrix}\Big](x,x_0) \leq C\|(\mathbf{v},\omega)\|^2(\langle x\rangle^{-\frac{3}{2}+\varepsilon_1} + \langle x\rangle^{-\frac{3}{2}}\ln(x)) \ , \\ \|\mathcal{F}_{1,2,\omega}(x)\|_1 \leq{}& C\|(\mathbf{v},\omega)\|^2A\Big[\begin{smallmatrix}1,1,0\\ 1-\varepsilon_1,1\end{smallmatrix}\Big](x,x_0) \leq C\|(\mathbf{v},\omega)\|^2(\langle x\rangle^{-1+\varepsilon_1} + \langle x\rangle^{-1}\ln(x)) \ , \\ \||y|^\beta\mathcal{F}_{1,2,\omega}(x)\|_2 \leq{}& C\|(\mathbf{v},\omega)\|^2\Big(A\Big[\begin{smallmatrix}\frac{5}{4}-\frac{\beta}{2},1,0\\ \frac{5}{4}-\frac{\beta}{2},1\end{smallmatrix}\Big](x,x_0) + A\Big[\begin{smallmatrix}1-\varepsilon_1,\frac{5}{4}-\frac{\beta}{2},0\\ 1-\varepsilon_1,\frac{5}{4}-\frac{\beta}{2}\end{smallmatrix}\Big](x,x_0)\Big) \ . \end{aligned}$$

Using Lemma 3.9, $\ln(1+x) \le C\langle x\rangle^{\varphi}$ and $\varepsilon_i > 0$, we get

$$\|(0,0,\mathcal{F}_{1,2,\omega})\| \le C\Big(\langle x_0\rangle^{-\frac{1}{2}+\varphi} + \langle x_0\rangle^{-\frac{1}{2}+\varepsilon_1+\eta} + \langle x_0\rangle^{-\frac{1}{4}+\varepsilon_2+\eta}\Big)\|(\mathbf{v},\omega)\|^2\,, \tag{3.18}$$

$$\|\mathcal{F}_{1,2,\omega}(x)\|_\infty \le C\|(\mathbf{v},\omega)\|^2\langle x\rangle^{-\frac{3}{2}+\varepsilon_1}\,, \quad \|\mathcal{F}_{1,2,\omega}(x)\|_1 \le C\|(\mathbf{v},\omega)\|^2\langle x\rangle^{-1+\varepsilon_1}\,, \tag{3.19}$$

$$\||y|^\beta\mathcal{F}_{1,2,\omega}(x)\|_2 \le C\|(\mathbf{v},\omega)\|^2\langle x\rangle^{-\frac{5}{4}+\frac{\beta}{2}+\varepsilon_1}\,. \tag{3.20}$$

Finally, from the results of Lemma A.7, it follows easily that

$$\begin{aligned}
\|(0,0,\mathcal{F}_{1,3,\omega})\|_x \le{}& C\|(\mathbf{v},\omega)\|^2\Big(\langle x\rangle^{\frac{3}{4}}B\Big[\begin{smallmatrix}\frac{3}{4},1,0\\ \frac{3}{4},1,0\end{smallmatrix}\Big](x,x_0) + \langle x\rangle^{\frac{3}{4}-\frac{\beta}{2}}B\Big[\begin{smallmatrix}\frac{9}{8}-\frac{3\beta}{8},1,\frac{3}{8}+\frac{\beta}{8}\\ \frac{9}{8}-\frac{3\beta}{8},1,\frac{3}{8}+\frac{\beta}{8}\end{smallmatrix}\Big](x,x_0)\Big)\\
&+C\|(\mathbf{v},\omega)\|^2\Big(\langle x\rangle^{\frac{3}{4}-\frac{\beta}{2}}B\Big[\begin{smallmatrix}\frac{5}{8},\frac{5}{4}-\frac{\beta}{2},\frac{1}{8}\\ \frac{5}{8},\frac{5}{4}-\frac{\beta}{2},\frac{1}{8}\end{smallmatrix}\Big](x,x_0)\Big)\\
&+C\|(\mathbf{v},\omega)\|^2\Big(\langle x\rangle^{\frac{3}{2}}B\Big[\begin{smallmatrix}2,1,\frac{1}{2}\\ \frac{3}{4},\frac{7}{4}-\eta,0\end{smallmatrix}\Big](x,x_0) + \langle x\rangle B\Big[\begin{smallmatrix}\frac{13}{8},1,\frac{5}{8}\\ \frac{5}{8},\frac{3}{2}-\eta,\frac{1}{8}\end{smallmatrix}\Big](x,x_0)\Big)\,,\\
\|\mathcal{F}_{1,3,\omega}(x)\|_\infty \le{}& C\|(\mathbf{v},\omega)\|^2 B\Big[\begin{smallmatrix}1,1,0\\ \frac{5}{8},\frac{3}{2},\frac{1}{8}\end{smallmatrix}\Big](x,x_0) \le C\|(\mathbf{v},\omega)\|^2\langle x\rangle^{-\frac{3}{2}+\varphi}\,,\\
\|\mathcal{F}_{1,3,\omega}(x)\|_1 \le{}& C\|(\mathbf{v},\omega)\|^2 B\Big[\begin{smallmatrix}\frac{5}{8},1,\frac{1}{8}\\ \frac{5}{8},1,\frac{1}{8}\end{smallmatrix}\Big](x,x_0) \le C\|(\mathbf{v},\omega)\|^2\langle x\rangle^{-1+\varphi}\,,\\
\||y|^\beta\mathcal{F}_{1,3,\omega}(x)\|_2 \le{}& C\|(\mathbf{v},\omega)\|^2\Big(B\Big[\begin{smallmatrix}\frac{9}{8}-\frac{3\beta}{8},1,\frac{3}{8}+\frac{\beta}{8}\\ \frac{9}{8}-\frac{3\beta}{8},1,\frac{3}{8}+\frac{\beta}{8}\end{smallmatrix}\Big](x,x_0) + B\Big[\begin{smallmatrix}\frac{5}{8},\frac{5}{4}-\frac{\beta}{2},\frac{1}{8}\\ \frac{5}{8},\frac{5}{4}-\frac{\beta}{2},\frac{1}{8}\end{smallmatrix}\Big](x,x_0)\Big)\,,\\
\le{}& C\|(\mathbf{v},\omega)\|^2\langle x\rangle^{-\frac{5}{4}+\frac{\beta}{2}+\varphi}\,,
\end{aligned}$$

where in the last inequality, we used $\beta \ge \frac{1}{2}$. Using Lemma 3.9 and $\beta \ge 1$, we get

$$\|(0,0,\mathcal{F}_{1,3,\omega})\| \le C\langle x_0\rangle^{-\frac{1}{4}+\frac{\varphi}{2}+\eta}\|(\mathbf{v},\omega)\|^2\,. \tag{3.21}$$

The proof of (3.15) and (3.16) is completed choosing $\varepsilon_1 = \varphi$ and $\varepsilon_2 = \frac{\varphi}{2}$ in (3.18)-(3.20). ■

We now turn to $\mathcal{F}_{1,v}$. For further reference, we also show that substracting some terms to $\mathcal{F}_{1,v}$ gives improved decay rates compared to those of v.

Proposition 3.11 *Assume that P and Q satisfy the bounds (1.17), and that the parameters satisfy (1.16), then there exist a constant C such that for $\kappa_{1,2} = \min(\kappa_{1,1}, \frac{\varphi}{2}, \frac{1}{2} - \eta + \xi - 2\varphi)$, we have*

$$\|(0,\mathcal{F}_{1,v},0)\| \le C\langle x_0\rangle^{-\kappa_{1,2}}\|(\mathbf{v},\omega)\|^2\,, \tag{3.22}$$

$$\|\mathcal{F}_{1,v}(x) - \mathcal{F}_{1,\omega}(x) - \mathcal{F}_{1,3,v}(x)\|_\infty \le C\langle x\rangle^{-\frac{3}{2}+\varphi}\langle x_0\rangle^{\varphi}\|(\mathbf{v},\omega)\|^2\,,$$

where $\mathcal{F}_{1,3,v}(x) = \int_{x_0}^x \mathrm{d}\tilde{x}\; G(x-\tilde{x})Q(\tilde{x})$.

Proof. We fi rst note that we can write $\mathcal{F}_{1,v}(x) = \mathcal{F}_{1,\omega}(x) + \mathcal{F}_{1,1,v}(x) + \mathcal{F}_{1,2,v}(x) + \mathcal{F}_{1,3,v}(x)$ with $\mathcal{F}_{1,3}$ as above and

$$\mathcal{F}_{1,1,v}(x) = \int_{x_0}^x \mathrm{d}\tilde{x}\;(K_r(x-\tilde{x}) + K_i(x-\tilde{x}))P(\tilde{x})\,, \quad \mathcal{F}_{1,2,v}(x) = \int_{x_0}^x \mathrm{d}\tilde{x}\;K_{13}(x-\tilde{x})Q(\tilde{x})\,. \tag{3.23}$$

Using (3.5), we see that the contribution of $\mathcal{F}_{1,\omega}$ to (3.22) is already proved in Proposition 3.10. Then, from the results of Lemma A.8 and A.9, it follows easily that

$$\|K_r(x)+K_i(x)\|_{\mathrm{L}^1} \le Ce^{\frac{b(\tau)(x-\tilde{x})}{4}}\left(\frac{1}{x^{\frac{1}{2}}}+\frac{\langle x\rangle^{\frac{1}{8}}}{x^{\frac{1}{8}}}+\frac{\langle x\rangle^{\frac{1}{8}}\langle x_0\rangle^{\varphi}}{x^{\frac{1}{4}}}\right)\equiv CB_1(x)\ ,$$
$$\|K_{13}(x)\|_{\mathrm{L}^1} \le Ce^{\frac{b(\tau)(x-\tilde{x})}{4}}\left(1+\frac{\langle x_0\rangle^{\varphi}}{x^{\frac{1}{4}}}\right)\equiv CD_1(x)\ .$$

We then have

$$\begin{aligned}
\|(0,\mathcal{F}_{1,1,v},0)\|_x &\le C\|(\mathbf{v},\omega)\|^2\int_{x_0}^{x}\mathrm{d}\tilde{x}\langle x\rangle^{1-\varphi}B_1(x-\tilde{x})\langle\tilde{x}\rangle^{-\frac{3}{2}}\\
&\quad+C\|(\mathbf{v},\omega)\|^2\int_{x_0}^{x}\mathrm{d}\tilde{x}\langle x\rangle^{1-\varphi-\frac{1}{p}}B_1(x-\tilde{x})\langle\tilde{x}\rangle^{-\frac{3}{2}+\frac{1}{2p}}\\
&\quad+C\|(\mathbf{v},\omega)\|^2\int_{x_0}^{x}\mathrm{d}\tilde{x}\langle x\rangle^{\frac{3}{2}-\frac{1}{2r}-\xi}B_1(x-\tilde{x})\langle\tilde{x}\rangle^{-2+\frac{1}{2r}+\eta}\ ,\\
\|(0,\mathcal{F}_{1,2,v},0)\|_x &\le C\|(\mathbf{v},\omega)\|^2\int_{x_0}^{x}\mathrm{d}\tilde{x}\langle x\rangle^{1-\varphi}D_1(x-\tilde{x})\langle\tilde{x}\rangle^{-2+\varphi}\\
&\quad+C\|(\mathbf{v},\omega)\|^2\int_{x_0}^{x}\mathrm{d}\tilde{x}\langle x\rangle^{1-\varphi-\frac{1}{p}}D_1(x-\tilde{x})\langle\tilde{x}\rangle^{-2+\varphi+\frac{1}{2p}}\\
&\quad+C\|(\mathbf{v},\omega)\|^2\int_{x_0}^{x}\mathrm{d}\tilde{x}\langle x\rangle^{\frac{3}{2}-\frac{1}{2r}-\xi}D_1(x-\tilde{x})\langle\tilde{x}\rangle^{-\frac{5}{2}+\frac{1}{2r}+\xi}\ ,\\
\|\mathcal{F}_{1,1,v}(x)\|_\infty &\le C\|(\mathbf{v},\omega)\|^2\int_{x_0}^{x}\mathrm{d}\tilde{x}B_1(x-\tilde{x})\langle\tilde{x}\rangle^{-\frac{3}{2}}\ ,\\
\|\mathcal{F}_{1,2,v}(x)\|_\infty &\le C\|(\mathbf{v},\omega)\|^2\int_{x_0}^{x}\mathrm{d}\tilde{x}D_1(x-\tilde{x})\langle\tilde{x}\rangle^{-2+\varphi}\ .
\end{aligned}$$

Using Lemma 3.9, we get

$$\|(0,\mathcal{F}_{1,1,v},0)\|_x+\|(0,\mathcal{F}_{1,2,v},0)\|_x \le C\left(\langle x_0\rangle^{-\frac{1}{2}+\varphi}+\langle x_0\rangle^{-\frac{1}{2}+\eta-\xi+2\varphi}\right)\|(\mathbf{v},\omega)\|^2\ ,$$
$$\|\mathcal{F}_{1,1,v}(x)\|_\infty+\|\mathcal{F}_{1,2,v}(x)\|_\infty \le C\langle x\rangle^{-\frac{3}{2}+\varphi}\langle x_0\rangle^{\varphi}\|(\mathbf{v},\omega)\|^2\ .$$

In the same way, from the results of Lemma A.4, it follows easily that for all $q>1$ and $s\ge 1$, we have

$$\|\partial_y G(x)\|_{\mathrm{L}^2}\le Cx^{-\frac{3}{2}}\ ,\qquad \|G(x)\|_{\mathrm{L}^q}\le Cx^{-1+\frac{1}{q}}\left(1+\frac{\langle x_0\rangle^{\frac{1}{4q}}}{x^{\frac{1}{4q}}}\right)\equiv CE_q(x)\ .$$

We then have

$$\begin{aligned}
\|\mathcal{F}_{1,3,v}\|_{x,\infty,1-\varphi} &\le C\|(\mathbf{v},\omega)\|^2\langle x\rangle^{1-\varphi}\int_{x_0}^{x}\mathrm{d}\tilde{x}\ \langle\tilde{x}\rangle^{-\frac{3}{2}+\frac{\varphi}{2}}E_{\frac{1}{\varphi}}(x-\tilde{x})\\
\|\mathcal{F}_{1,3,v}\|_{x,p,1-\varphi-\frac{1}{p}} &\le C\|(\mathbf{v},\omega)\|^2\langle x\rangle^{1-\varphi-\frac{1}{p}}\int_{x_0}^{x}\mathrm{d}\tilde{x}\ \langle\tilde{x}\rangle^{-\frac{3}{2}+\varphi}E_p(x-\tilde{x})\\
\|\partial_y\mathcal{F}_{1,3,v}\|_{x,r,\frac{3}{2}-\frac{1}{2r}-\xi} &\le C\|(\mathbf{v},\omega)\|^2\langle x\rangle^{\frac{3}{2}-\frac{1}{2r}-\xi}\int_{x_0}^{x}\mathrm{d}\tilde{x}\ \min\left(\langle\tilde{x}\rangle^{-\frac{9}{4}+\frac{1}{2r}+\xi}E_2(x-\tilde{x}),\frac{\tilde{x}^{-\frac{7}{4}+\frac{1}{2r}}}{(x-\tilde{x})^{\frac{3}{2}}}\right)\ .
\end{aligned}$$

Using Lemma 3.9, and $r > 2$, we get

$$\|(0, \mathcal{F}_{1,3,v}, 0)\|_x \le C\langle x_0\rangle^{-\frac{\varphi}{2}} \|(\mathbf{v}, \omega)\|^2 .$$

The proof is completed. ■

We conclude this section by estimating $\mathcal{F}_{1,u}$. In the spirit of Proposition 3.10, we will also show that substracting the 'right' term to $\mathcal{F}_{1,u}$ improves its decay rate as $x \to \infty$.

Proposition 3.12 *Assume that P and Q satisfy the bounds (1.17), and that the parameters satisfy (1.16), then there exist a constant C such that for $\kappa_{1,3} = \min(\kappa_{1,2}, \frac{1}{2} - (1 + \frac{1}{2r})\varphi, \frac{1}{2} - \xi + \eta - \frac{\varphi}{r})$, we have*

$$\|(\mathcal{F}_{1,u}, 0, 0)\| \le \langle x_0\rangle^{-\kappa_{1,3}} \|(\mathbf{v}, \omega)\|^2 , \tag{3.24}$$

for all $x \ge x_0$. Furthermore, let

$$\mathcal{F}_{1,2,u}(x) = -\int_{x_0}^{x} \mathrm{d}\tilde{x}\; K_{12}(x - \tilde{x})Q(\tilde{x})$$

$$\mathcal{F}_{1,4,u}(x) = \int_{x_0}^{x} \mathrm{d}\tilde{x}\; \Big((\mathcal{P}K_2(x - \tilde{x}) - K_{13}(x - \tilde{x}))P(\tilde{x}) - \mathcal{P}K_{12}(x - \tilde{x})Q(\tilde{x})\Big) . \tag{3.25}$$

then for all $\varepsilon > 0$, there exists a constant C such that

$$\|\mathcal{F}_{1,u}(x) - \mathcal{F}_{1,2,u}(x)\|_\infty \le C\langle x\rangle^{-1+\varphi}\|(\mathbf{v}, \omega)\|^2 , \quad \|\mathcal{F}_{1,4,u}(x)\|_\infty \le C\langle x\rangle^{-\frac{3}{2}}\langle x_0\rangle^{\frac{5\varphi}{2}}\|(\mathbf{v}, \omega)\|^2 .$$

Proof. We first note that with $\mathcal{F}_{1,2,u}$ as above, we can write $\mathcal{F}_{1,u}(x) = \mathcal{F}_{1,1,u}(x) + \mathcal{F}_{1,2,u}(x) + \mathcal{F}_{1,3,u}(x)$ with

$$\mathcal{F}_{1,1,u}(x) = \int_{x_0}^{x} \mathrm{d}\tilde{x}\; (K_2(\tilde{x} - x) - K_{13}(\tilde{x} - x))P(\tilde{x}) , \quad \mathcal{F}_{1,3,u}(x) = -\int_{x_0}^{x} \mathrm{d}\tilde{x}\; F(\tilde{x} - x)Q(\tilde{x}) . \tag{3.26}$$

Then we note that $\|(\mathcal{F}_{1,3,u}, 0, 0)\| \le \|(0, \mathcal{F}_{1,3,u}, 0)\|$ (see (3.5)), and that $\mathcal{F}_{1,3,u}$ and $\mathcal{F}_{1,3,v}$ differ only by signs and the exchange of the Kernels F and G. The bound on $\mathcal{F}_{1,3,v}$ in the proof of Proposition 3.11 being insensitive to these details then apply mutatis mutandis, in particular, we have $\|\mathcal{F}_{1,3,u}(x)\|_\infty \le C\langle x\rangle^{-1+\varphi}\|(\mathbf{v}, \omega)\|^2$. Then, by Lemma A.8, we have

$$\|K_{12}(x)\|_{\mathrm{L}^p} \le C\frac{\langle x\rangle^{\frac{1}{2}-\frac{1}{2p}}}{x^{1-\frac{1}{p}}}\Big(1 + \frac{\langle x_0\rangle^{\frac{\varphi}{p}}}{x^{\frac{1}{4p}}}\Big) \equiv CE_p(x) ,$$

$$\|K_2(x)\|_{\mathrm{L}^p} + \|K_{13}(x)\|_{\mathrm{L}^p} \le C\left(\frac{1}{x^{1-\frac{1}{2p}}} + \frac{e^{\frac{b(\tau)x}{4}}\langle x\rangle^{\frac{1}{2}-\frac{1}{2p}}}{x^{1-\frac{1}{p}}}\Big(1 + \frac{\langle x_0\rangle^{\frac{\varphi}{p}}}{x^{\frac{1}{4p}}}\Big)\right) \equiv CH_p(x) ,$$

$$\|\partial_y K_2(x)\|_{\mathrm{L}^2} + \|\partial_y K_{13}(x)\|_{\mathrm{L}^2} \le C\left(\frac{\langle x\rangle^{\frac{1}{2}}}{x^{\frac{7}{4}}} + \frac{e^{\frac{b(\tau)x}{4}}\langle x\rangle^{\frac{3}{4}}}{x^{\frac{3}{2}}}\right) \equiv CJ(x) ,$$

so that for all $p_0 \in [q, \infty)$, we have

$$\|\mathcal{F}_{1,2,u}\|_{x,p_0,\frac{1}{2}-\frac{1}{p_0}} \le C\|(\mathbf{v}, \omega)\|^2\langle x\rangle^{\frac{1}{2}-\frac{1}{p_0}} \int_{x_0}^{x} \mathrm{d}\tilde{x}\; \min\Big(\langle\tilde{x}\rangle^{-2+\varphi+\frac{1}{2p_0}}E_1(x - \tilde{x}), \langle\tilde{x}\rangle^{-\frac{3}{2}+\varphi}E_{p_0}(x - \tilde{x})\Big)$$

$$\|\partial_y \mathcal{F}_{1,2,u}\|_{x,r,1-\frac{1}{2r}-\eta} \leq C\|(\mathbf{v},\omega)\|^2 \langle x\rangle^{1-\frac{1}{2r}-\eta} \int_{x_0}^{x} \mathrm{d}\tilde{x}\ \min\left(\langle\tilde{x}\rangle^{-2+\xi} E_r(x-\tilde{x}), \frac{\langle x\rangle^{1-\frac{1}{2r}}\langle\tilde{x}\rangle^{-\frac{3}{2}+\varphi}}{(x-\tilde{x})^{2-\frac{1}{r}}}\right),$$
$$\|\mathcal{F}_{1,1,u}\|_{p_0} \leq C\|(\mathbf{v},\omega)\|^2 \int_{x_0}^{x} \mathrm{d}\tilde{x}\ \min\left(\langle\tilde{x}\rangle^{-\frac{3}{2}+\frac{1}{2p_0}} H_1(x-\tilde{x}), \langle\tilde{x}\rangle^{-1} H_{p_0}(x-\tilde{x})\right)$$
$$\|\partial_y \mathcal{F}_{1,1,u}\|_{x,r,1-\frac{1}{2r}-\eta} \leq C\|(\mathbf{v},\omega)\|^2 \langle x\rangle^{1-\frac{1}{2r}-\eta} \int_{x_0}^{x} \mathrm{d}\tilde{x}\ \min\left(\langle\tilde{x}\rangle^{-\frac{3}{2}+\eta} H_r(x-\tilde{x}), \langle\tilde{x}\rangle^{-\frac{5}{4}+\frac{1}{2r}} J(x-\tilde{x})\right).$$

By Lemma 3.9, using these bounds with $p_0 = q$ and $p_0 = \infty$ and $\ln(1+x) \leq C\langle x\rangle^{\varphi}$, we get

$$\|(\mathcal{F}_{1,2,u},0,0)\|_x \leq C\left(\langle x_0\rangle^{-\frac{1}{2}+\varphi} + \langle x_0\rangle^{-\frac{1}{2}+\xi-\eta+\frac{\varphi}{r}-\frac{1}{4r}}\right)\|(\mathbf{v},\omega)\|^2,$$
$$\|(\mathcal{F}_{1,1,u},0,0)\|_x \leq C\left(\langle x_0\rangle^{-\frac{1}{2}+\varphi} + \langle x_0\rangle^{-\frac{1}{2}-\frac{1}{2r}+(1+\frac{3}{2r})\varphi}\right)\|(\mathbf{v},\omega)\|^2,$$
$$\|\mathcal{F}_{1,1,u}(x)\|_\infty \leq C\langle x\rangle^{-1+\varphi}\|(\mathbf{v},\omega)\|^2.$$

We finally note that

$$\|\mathcal{F}_{1,4,u}(x)\|_\infty \leq C\|(\mathbf{v},\omega)\|^2 \int_{x_0}^{x} \mathrm{d}\tilde{x}\ e^{\frac{b(\tau)(x-\tilde{x}}{4}}\left(1+\frac{1}{x^{\frac{1}{2}}}+\frac{\langle x_0\rangle^{\varphi}}{(x-\tilde{x})^{\frac{1}{4}}}\right)\langle\tilde{x}\rangle^{-\frac{3}{2}}$$
$$\leq C\langle x\rangle^{-\frac{3}{2}}\langle x_0\rangle^{\frac{5\varphi}{2}}\|(\mathbf{v},\omega)\|^2,$$

which completes the proof. ■

3.5 The nonlinear terms II

In this section, we prove the

Theorem 3.13 *Assume that P and Q satisfy the bounds (1.17), and that the parameters satisfy (1.16), then there exist constants C and $\kappa_2 > 0$ such that*

$$\|(\mathcal{F}_{2,u},\mathcal{F}_{2,v},\mathcal{F}_{2,\omega})\| \leq C\langle x_0\rangle^{-\kappa_2}\|(\mathbf{v},\omega)\|^2. \tag{3.27}$$

For convenience, the proof is split component-wise in the next three Propositions. For further reference, we will also point out that most decay rates on $\mathcal{F}_{2,\cdot}$ are in fact better than those of the related fields.

Proposition 3.14 *Assume that P and Q satisfy the bounds (1.17), and that the parameters satisfy (1.16), then there exist a constant C such that for $\kappa_{2,1} = \frac{1}{4} - \eta$, we have*

$$\|(0,0,\mathcal{F}_{2,\omega})\| \leq C\langle x_0\rangle^{-\kappa_{2,1}}\|(\mathbf{v},\omega)\|^2, \quad \|\mathcal{F}_{2,\omega}(x)\|_\infty \leq C\langle x\rangle^{-\frac{3}{2}}\|(\mathbf{v},\omega)\|^2$$
$$\|\mathcal{F}_{2,\omega}(x)\|_1 \leq C\langle x\rangle^{-1}\|(\mathbf{v},\omega)\|^2, \quad \||y|^{\beta}\mathcal{F}_{2,\omega}(x)\|_2 \leq C\langle x\rangle^{-\frac{5}{4}+\frac{\beta}{2}}\|(\mathbf{v},\omega)\|^2,$$

for all $x \geq x_0$.

Proof. From the results of section A, it follows easily that there are exponents $p \geq 0$ and $q < 1$ such that

$$\|e^x K_{2,1,\omega}\|_{1,\{p,q\}} + \|e^x K_{2,2,\omega}\|_{1,\{p,q\}} + \|e^x K_{2,1,\omega}\|_{2,\{p,q\}} + \|e^x K_{2,2,\omega}\|_{2,\{p,q\}} \leq C,$$
$$\|e^x|y|^{\beta} K_{2,1,\omega}\|_{1,\{p,q\}} + \|e^x|y|^{\beta} K_{2,2,\omega}\|_{1,\{p,q\}} \leq C,$$

while for all $x_0 \le x \le \tilde{x}$, we have

$$\begin{aligned}
\langle x\rangle^{\frac{3}{4}}(\|P(\tilde{x})\|_2 + \|Q(\tilde{x})\|_2) &\le \langle x\rangle^{-\frac{1}{2}}(\|P\|_{\tilde{x},2,\frac{5}{4}} + \|Q\|_{\tilde{x},2,\frac{7}{4}-\varphi})\ , \\
\langle x\rangle^{\frac{3}{2}}(\|\partial_y P(\tilde{x})\|_2 + \|\partial_y Q(\tilde{x})\|_2) &\le \langle x\rangle^{\eta-\frac{1}{4}}(\|\partial_y P\|_{\tilde{x},2,\frac{7}{4}-\eta} + \|\partial_y Q\|_{\tilde{x},2,\frac{9}{4}-\xi})\ , \\
\langle x\rangle(\|\partial_y P(\tilde{x})\|_1 + \|\partial_y Q(\tilde{x})\|_1) &\le \langle x\rangle^{\eta-\frac{1}{2}}(\|\partial_y P\|_{\tilde{x},1,\frac{3}{2}-\eta} + \|\partial_y Q\|_{\tilde{x},1,2-\xi})\ , \\
\langle x\rangle^{\frac{3}{4}-\frac{\beta}{2}}(\||y|^\beta P(\tilde{x})\|_2 + \||y|^\beta Q(\tilde{x})\|_2) &\le \langle x\rangle^{-\frac{1}{2}}(\||y|^\beta P\|_{\tilde{x},2,\frac{5}{4}-\frac{\beta}{2}} + \||y|^\beta Q\|_{\tilde{x},2,\frac{7}{4}-\varphi-\frac{\beta}{2}})\ , \\
\langle x\rangle^{\frac{3}{4}-\frac{\beta}{2}}(\|P(\tilde{x})\|_1 + \|Q(\tilde{x})\|_1) &\le \langle x\rangle^{-\frac{1}{2}}(\|P\|_{\tilde{x},1,1} + \|Q\|_{\tilde{x},1,\frac{3}{2}-\varphi})\ ,
\end{aligned}$$

since $\beta > \frac{3}{2}$, $\varphi \le \xi < \frac{1}{2}$ and $\frac{1}{2} - \xi + \eta \ge 0$. By (1.17), the above quantities are all bounded by $C\langle x\rangle^{\eta-\frac{1}{4}}\|(\mathbf{v},\omega)\|^2$, while $\langle x\rangle^{\frac{3}{2}}(\|P(\tilde{x})\|_\infty + \|Q(\tilde{x})\|_\infty) + \langle x\rangle(\|P(\tilde{x})\|_1 + \|Q(\tilde{x})\|_1) \le C\|(\mathbf{v},\omega)\|^2$. Easy estimates applied to (2.5) thus lead to

$$\begin{aligned}
\|(0,0,\mathcal{F}_{2,\omega})\|_x &\le C\langle x\rangle^{\eta-\frac{1}{4}}\|(\mathbf{v},\omega)\|^2 \int_x^\infty \mathrm{d}\tilde{x}\ \frac{e^{x-\tilde{x}}\langle \tilde{x}-x\rangle^p}{(\tilde{x}-x)^q}\ , \\
\|\mathcal{F}_{2,\omega}(x)\|_\infty &\le C\langle x\rangle^{-\frac{3}{2}}\|(\mathbf{v},\omega)\|^2 \int_x^\infty \mathrm{d}\tilde{x}\ \frac{e^{x-\tilde{x}}\langle \tilde{x}-x\rangle^p}{(\tilde{x}-x)^q}\ , \\
\|\mathcal{F}_{2,\omega}(x)\|_1 &\le C\langle x\rangle^{-1}\|(\mathbf{v},\omega)\|^2 \int_x^\infty \mathrm{d}\tilde{x}\ \frac{e^{x-\tilde{x}}\langle \tilde{x}-x\rangle^p}{(\tilde{x}-x)^q}\ , \\
\||y|^\beta\mathcal{F}_{2,\omega}(x)\|_2 &\le C\langle x\rangle^{-\frac{5}{4}+\frac{\beta}{2}}\|(\mathbf{v},\omega)\|^2 \int_x^\infty \mathrm{d}\tilde{x}\ \frac{e^{x-\tilde{x}}\langle \tilde{x}-x\rangle^p}{(\tilde{x}-x)^q}\ .
\end{aligned}$$

This completes the proof. ■

Proposition 3.15 *Assume that P and Q satisfy the bounds (1.17), and that the parameters satisfy (1.16), then there exist a constant C such that for $\kappa_{2,2} = \min(\kappa_{2,1}, \frac{\varphi}{2})$, we have*

$$\|(0,\mathcal{F}_{2,v},0)\| \le C\langle x_0\rangle^{-\kappa_{2,2}}\|(\mathbf{v},\omega)\|^2\ , \tag{3.28}$$

$$\|\mathcal{F}_{2,v}(x)\|_\infty \le C\langle x_0\rangle^{-\frac{3}{2}+(1+\varepsilon)\varphi}\|(\mathbf{v},\omega)\|^2\ . \tag{3.29}$$

for all $x \ge x_0$ and $0 < \varepsilon \le 1$.

Proof. We first note that we can write $\mathcal{F}_{2,v}(x) = \mathcal{F}_{2,\omega}(x) + \mathcal{F}_{2,1,v}(x) + \mathcal{F}_{2,2,v}(x)$ where

$$\mathcal{F}_{2,1,v}(x) = -\int_x^\infty \mathrm{d}\tilde{x}\ e^{-(\tilde{x}-x)}K_7(\tilde{x}-x)P(\tilde{x}) - e^{-(\tilde{x}-x)}K_6(\tilde{x}-x)Q(\tilde{x})\ , \tag{3.30}$$

$$\mathcal{F}_{2,2,v}(x) = -\int_x^\infty \mathrm{d}\tilde{x}\ G^*(\tilde{x}-x)Q(\tilde{x})\ . \tag{3.31}$$

Using (3.5), we see again that the contribution of $\mathcal{F}_{2,\omega}$ to (3.28) is already proved in Proposition 3.14. For the contribution of $\mathcal{F}_{2,1,v}$ to (3.28), we proceed as in Proposition 3.14. There are exponents $p \ge 0$ and $q < 1$ such that

$$\|K_6\|_{1,\{p,q\}} + \|K_7\|_{1,\{p,q\}} \le C\ ,$$

while for all $x_0 \leq x \leq \tilde{x}$, we have

$$\langle x\rangle^{1-\varphi}(\|P(\tilde{x})\|_\infty + \|Q(\tilde{x})\|_\infty) \leq \langle x\rangle^{-\frac{1}{2}-\varphi}(\|P\|_{\tilde{x},\infty,\frac{3}{2}} + \|Q\|_{\tilde{x},\infty,2-\varphi}) ,$$
$$\langle x\rangle^{1-\varphi-\frac{1}{p}}(\|P(\tilde{x})\|_p + \|Q(\tilde{x})\|_p \leq \langle x\rangle^{-\frac{1}{2}-\varphi-\frac{1}{2p}}(\|P\|_{\tilde{x},p,\frac{3}{2}-\frac{1}{2p}} + \|Q\|_{\tilde{x},p,2-\varphi-\frac{1}{2p}}) ,$$
$$\langle x\rangle^{\frac{3}{2}-\frac{1}{2r}-\xi}(\|\partial_y P(\tilde{x})\|_r + \|\partial_y Q(\tilde{x})\|_r) \leq \langle x\rangle^{-\frac{1}{2}}(\|\partial_y P\|_{\tilde{x},r,2-\frac{1}{2r}-\eta} + \|\partial_y Q\|_{\tilde{x},r,\frac{5}{2}-\frac{1}{2r}-\xi}) ,$$

since $\varphi \leq \xi < \frac{1}{2}$ and $\eta \leq \xi$. By (1.17), the above quantities are bounded by $C\langle x\rangle^{-\frac{1}{2}}\|(\mathbf{v},\omega)\|^2$, and we get

$$\|(0,\mathcal{F}_{2,1,v},0)\|_x \leq C\langle x\rangle^{-\frac{1}{2}}\|(\mathbf{v},\omega)\|^2 \int_x^\infty \mathrm{d}\tilde{x}\, \frac{e^{x-\tilde{x}}\langle \tilde{x}-x\rangle^p}{(\tilde{x}-x)^q} ,$$
$$\|\mathcal{F}_{2,1,v}(x)\|_\infty \leq C\langle x\rangle^{-\frac{3}{2}+\varphi}\|(\mathbf{v},\omega)\|^2 .$$

Next, we note that for $x \leq \tilde{x}$ and $q > 1$, we have

$$\|\mathcal{P}_0 G\|_{q,\{0,1-\frac{1}{q}\}} \leq C , \qquad \|\mathcal{P}G\|_{q,\{0,1-\frac{3}{4q}\}} \leq C\langle x_0\rangle^{\frac{1}{4q}} \leq C\langle \tilde{x}\rangle^{\frac{1}{4q}} ,$$

where in the last inequality, we used that $\tau^{-\frac{1}{4q}} \leq \langle\tau\rangle^{-\frac{1}{4q}}\langle x_0\rangle^{\frac{\varphi}{4q}} \leq \langle x_0\rangle^{\frac{1}{4q}}$. Then, for all $0 < \varepsilon \leq 1$, after the change of variables $\tilde{x} = xz$, we get

$$\|\mathcal{F}_{2,1,v}\|_{x,\infty,1-\varphi} \leq C\|(\mathbf{v},\omega)\|^2\langle x\rangle^{-\frac{1}{2}+\varepsilon\varphi}\int_1^\infty \mathrm{d}z\, \frac{z^{-\frac{3}{2}+(1-\varepsilon)\varphi}}{(z-1)^{1-2\varepsilon\varphi}}(1+(z-1)^{-\frac{\varphi}{4}}) ,$$
$$\|\mathcal{F}_{2,1,v}\|_{x,p,1-\varphi-\frac{1}{p}} \leq C\|(\mathbf{v},\omega)\|^2\langle x\rangle^{-\frac{1}{2}}\int_1^\infty \mathrm{d}z\, \frac{z^{-\frac{3}{2}+\varphi}}{(z-1)^{1-\frac{1}{p}}}(1+(z-1)^{-\frac{1}{4p}}) ,$$
$$\|\partial_y\mathcal{F}_{2,1,v}\|_{x,r,\frac{3}{2}-\frac{1}{2r}-\xi} \leq C\|(\mathbf{v},\omega)\|^2\langle x\rangle^{-\frac{1}{4}}\int_1^\infty \mathrm{d}z\, z^{-\frac{9}{4}+\frac{1}{2r}+\eta}((z-1)^{-\frac{1}{2}}+(z-1)^{-\frac{5}{8}}) .$$

This completes the proof. ■

Proposition 3.16 *Assume that P and Q satisfy the bounds (1.17), and that the parameters satisfy (1.16), then there exist a constant C such that for $\kappa_{2,3} = \kappa_{2,2}$*

$$\|(\mathcal{F}_{2,u},0,0)\| \leq C\langle x_0\rangle^{-\kappa_{2,3}}\|(\mathbf{v},\omega)\|^2 , \tag{3.32}$$
$$\|\mathcal{F}_{2,u}(x)\|_\infty \leq C\langle x\rangle^{-\frac{3}{2}+(1+\varepsilon)\varphi}\|(\mathbf{v},\omega)\|^2 , \tag{3.33}$$

for all $x \geq x_0$ and $0 < \varepsilon \leq 1$.

Proof. We first note that we can write

$$\mathcal{F}_{2,u}(x) = \mathcal{F}_{2,1,u}(x) + \mathcal{F}_{2,2,u}(x) ,$$
$$\mathcal{F}_{2,1,u}(x) = \int_x^\infty \mathrm{d}\tilde{x}\, e^{-(\tilde{x}-x)}(K_2(\tilde{x}-x) - K_6(\tilde{x}-x))P(\tilde{x}) + e^{-(\tilde{x}-x)}K_5(\tilde{x}-x)Q(\tilde{x}) , \tag{3.34}$$
$$\mathcal{F}_{2,2,u}(x) = \int_x^\infty \mathrm{d}\tilde{x}\, F^*(\tilde{x}-x)Q(\tilde{x}) . \tag{3.35}$$

We then note that

$$\|K_2\|_{1,\{0,\frac{1}{2}\}} + \|K_5\|_{1,\{0,\frac{1}{2}\}} + \|K_6\|_{1,\{0,\frac{1}{2}\}} \leq C ,$$

while for all $x_0 \leq x \leq \tilde{x}$, we have

$$\begin{aligned} \langle x\rangle^{\frac{1}{2}}(\|P(\tilde{x})\|_\infty + \|Q(\tilde{x})\|_\infty) &\leq \langle x\rangle^{-1}(\|P\|_{\tilde{x},\infty,\frac{3}{2}} + \|Q\|_{\tilde{x},\infty,2-\varphi}) , \\ \langle x\rangle^{\frac{1}{2}-\frac{1}{p}}(\|P(\tilde{x})\|_p + \|Q(\tilde{x})\|_p &\leq \langle x\rangle^{-1-\frac{1}{2p}}(\|P\|_{\tilde{x},p,\frac{3}{2}-\frac{1}{2p}} + \|Q\|_{\tilde{x},p,2-\varphi-\frac{1}{2p}}) , \\ \langle x\rangle^{1-\frac{1}{2r}-\eta}(\|\partial_y P(\tilde{x})\|_r + \|\partial_y Q(\tilde{x})\|_r) &\leq \langle x\rangle^{-1}(\|\partial_y P\|_{\tilde{x},r,2-\frac{1}{2r}-\eta} + \|\partial_y Q\|_{\tilde{x},r,\frac{5}{2}-\frac{1}{2r}-\xi}) , \end{aligned}$$

since $\varphi \leq \xi < \frac{1}{2}$ and $\frac{1}{2} - \xi + \eta \geq 0$. By (1.17), the above quantities are all bounded by $C\langle x\rangle^{-1}\|(\mathbf{v},\omega)\|^2$, thus we get

$$\|(\mathcal{F}_{2,1,u}, 0, 0)\|_x \leq C\langle x\rangle^{-1}\|(\mathbf{v},\omega)\|^2 \int_x^\infty \mathrm{d}\tilde{x}\, e^{x-\tilde{x}}(\tilde{x}-x)^{-\frac{1}{2}} . \tag{3.36}$$

Next, we use (3.5) and note that $\mathcal{F}_{2,2,u}$ and $\mathcal{F}_{2,2,v}$ differ only by signs and the exchange of the Kernels F and G (see (3.31) and (3.35)). The bounds on $\mathcal{F}_{2,2,v}$ in the proof of Proposition 3.15 being insensitive to these details then apply mutatis mutandis. Finally, the proof of (3.33) follows at once from (3.36). ■

4 Existence and uniqueness results

Our next task is now to prove existence and (local) uniqueness result in $\mathcal{C}_u$ for solutions of (1.1). This was stated as ”2. implies 1.” in Theorem 1.4, or, to rephrase it, that

Theorem 4.1 *If ν and w are in the class $\mathcal{C}_i$ with parameters satisfying (1.16) and x_0 is sufficiently large, then there exist a (locally) unique solution to (1.1) in $\mathcal{C}_u$ with parameters satisfying (1.16).*

Proof. The proof follows from the contraction mapping principle. For fixed ν and w in $\mathcal{C}_i$, we define the map $\mathcal{F} : \mathcal{W} \to \mathcal{W}$ by

$$\mathcal{F}(\mathbf{v},\omega) = \text{r.h.s. of } (1.12) - (1.14) .$$

By the results of Section 3, it follows that if the parameters satisfy (1.16), then for $\kappa = \min(\kappa_0,\kappa_1,\kappa_2)$, we have

$$\begin{aligned} \|\mathcal{F}(\mathbf{v},\omega)\| &\leq C_1\|(\nu,\mu,w)\|_{x_0} + C_2\langle x_0\rangle^{-\kappa}\|(\mathbf{v},\omega)\|^2 , \\ \|\mathcal{F}(\mathbf{v}_1,\omega_1) - \mathcal{F}(\mathbf{v}_2,\omega_2)\| &\leq C_2\langle x_0\rangle^{-\kappa}(\|(\mathbf{v}_1-\mathbf{v}_2,\omega_1-\omega_2)\|)(\|(\mathbf{v}_1+\mathbf{v}_2,\omega_1+\omega_2)\|) . \end{aligned}$$

Let $\rho > 0$ and $0 < \varepsilon < \frac{1}{2}$. We easily see that if $\|(\nu,\mu,w)\|_{x_0} \leq \rho$, the map $\mathcal{F}$ is a contraction in $\mathcal{B}_0((1+\varepsilon)C_1\rho) \subset \mathcal{W}$ if $\langle x_0\rangle > (C_1C_2\rho\varepsilon^{-1})^{\frac{1}{\kappa}}$. By classical arguments, the approximating sequence $(\mathbf{v}_{n+1},\omega_{n+1}) = \mathcal{F}(\mathbf{v}_n,\omega_n)$ for $n > 1$ and $(\mathbf{v}_1,\omega_1) = \mathcal{F}(0,0)$ converges to the unique solution of (1.1) in $\mathcal{B}_0((1+\varepsilon)C_1\rho) \subset \mathcal{W}$. This completes the proof. ■

5 Asymptotics

Now that we know that there exist (locally) unique solutions of (1.1) in $\mathcal{C}_u$, we can turn to their asymptotic description. As explained in Section 1.4, we will first prove the partial description of Corollary 1.7, and more precisely that

Theorem 5.1 *Let* $\mathbf{a}_1 = (-\mathcal{M}(\mathcal{I}\mathcal{P}_0 w) - \int_{\Omega_+} \mathcal{P}_0 Q(x,y)\,\mathrm{d}x\mathrm{d}y, 0, 0, 0, 0, 0)$ *and* $u_{\mathbf{a}_1}$, $\omega_{\mathbf{a}_1}$ *as in (1.9), then for all* $\varepsilon > 0$*, solutions to (1.1) in* $\mathcal{C}_u$ *satisfy*

$$\begin{aligned} \|u(x) - u_{\mathbf{a}_1}(x)\|_\infty &\leq C(x_0, \|(\mathbf{v},\omega)\|)\,\langle x\rangle^{-1+(1+\varepsilon)\varphi}\,, \\ \|\omega(x) - \omega_{\mathbf{a}_1}(x)\|_\infty &\leq C(x_0, \|(\mathbf{v},\omega)\|)\,\langle x\rangle^{-\frac{3}{2}+(1+\varepsilon)\varphi} \\ \|\omega(x) - \omega_{\mathbf{a}_1}(x)\|_1 &\leq C(x_0, \|(\mathbf{v},\omega)\|)\,\langle x\rangle^{-1+(1+\varepsilon)\varphi} \\ \||y|^{\beta_0}(\omega(x) - \omega_{\mathbf{a}_1}(x))\|_2 &\leq C(x_0, \|(\mathbf{v},\omega)\|)\,\langle x\rangle^{-\frac{5}{4}+\frac{\beta_0}{2}+(1+\varepsilon)\varphi} \end{aligned} \tag{5.1}$$

for all $\frac{1}{2} \leq \beta_0 \leq 1 - 2(1+\varepsilon)\varphi$ *and* $x \geq x_0$.

Note that here, in contrast with (1.3) or the statement of Corollary 1.7, we did not include the terms in the $y \sim x$ scale for u nor the v component, as they are of order x^{-1}, resp. $x^{-1+\varphi}$, which are smaller than the $\mathcal{O}(x^{-1+(1+\varepsilon)\varphi})$ correction. These terms will appear later in Section 6. Note that we need only prove (5.1) for $x \to \infty$ (we will in fact prove them for $x \geq 2x_0$), as they are trivially satisfied for finite x. Furthermore, for $x \geq 2x_0$, we can either compare, $u(x)$ to $u_{\mathbf{a}_1}(x)$ or $u_{\mathbf{a}_1}(x - x_0)$ and $\omega(x)$ to $\omega_{\mathbf{a}_1}(x)$ or $\omega_{\mathbf{a}_1}(x - x_0)$, as is proved in the

Lemma 5.2 *Let* $K_c(x) = \frac{e^{-\frac{y^2}{4x}}}{\sqrt{4\pi x}}$, $K_0(x) = \frac{1}{\pi}\frac{x}{x^2+y^2}$, $f(x) = K_c(x - x_0) - K_c(x)$ *and* $g(x) = K_0(x - x_0) - K_0(x)$*, then for all* $m \in \mathbf{N}$*, there exists a constant* C_m *such that*

$$\begin{aligned} &\|\partial_y^m g(x)\|_\infty \leq C_m\langle x\rangle^{-m-2}\langle x_0\rangle\,, \quad \|\partial_y^m \mathcal{H} g(x)\|_\infty \leq C_m\langle x\rangle^{-m-2}\langle x_0\rangle \\ &\|\partial_y^m f(x)\|_\infty \leq C_m\langle x\rangle^{-\frac{3+m}{2}}\langle x_0\rangle\,, \quad \|\partial_y^m f(x)\|_1 \leq C_m\langle x\rangle^{-\frac{2+m}{2}}\langle x_0\rangle\,, \\ &\|y\partial_y^m f(x)\|_2 \leq C_m\langle x\rangle^{-\frac{3+2m}{4}}\langle x_0\rangle\,, \end{aligned}$$

for all $x \geq 2x_0 \geq 2$.

Proof. Since $x - x_0 \geq \frac{x}{2}$ for $x \geq 2x_0$, we have

$$\begin{aligned} \|\partial_y^m g(x)\|_\infty + \|\partial_y^m \mathcal{H} g(x)\|_\infty &\leq \int_{-\infty}^{\infty} \mathrm{d}k|k|^m|e^{-|k|(x-x_0)} - e^{-|k|x}| \leq x_0\int_{-\infty}^{\infty} \mathrm{d}k|k|^{m+1}e^{-\frac{|k|x}{2}} \\ \|\partial_y^m f(x)\|_\infty &\leq \int_{-\infty}^{\infty} \mathrm{d}k|k|^m|e^{-k^2(x-x_0)} - e^{-k^2x}| \leq x_0\int_{-\infty}^{\infty} \mathrm{d}k|k|^{m+2}e^{-\frac{k^2x}{2}} \end{aligned}$$

and similarly

$$\begin{aligned} \|\partial_y^m f(x)\|_2^2 &\leq x_0^2\int_{-\infty}^{\infty} \mathrm{d}k|k|^{2(m+2)}e^{-k^2x} \leq C_m x_0^2 x^{-\frac{5}{2}-m}\,, \\ \|y\partial_y^m f(x)\|_2^2 &\leq x_0^2\int_{-\infty}^{\infty} \mathrm{d}k|k|^{2(m+1)}(1 + m^2 + m^2k^2x^2)e^{-k^2x} \leq C_m x_0^2 x^{-\frac{3}{2}-m}\,. \end{aligned}$$

The proof is completed with the use of Lemma A.1. ■

For convenience, the proof of Theorem 5.1 is split in the next two subsections. The terms coming from w and ν in (1.12)-(1.14) will be studied in the next subsection, the remainder in the second one. The basis of the proof of Theorem 5.1 is that the large time asymptotics of $K_1(x)f$ is captured by[4] $\mathcal{M}(f)K_1(x)$ if f decays sufficiently fast, which is the content of the next Lemma.

Lemma 5.3 *Let $0 \le \gamma \le 1$, $0 \le \gamma_2 \le 2$ and f satisfying $\|\langle y\rangle^\gamma f\|_1 < \infty$. Then for all $m \ge 0$, there exist a constant C_γ such that*

$$\begin{aligned}
\|\partial_y^m K_1(x)(f - \mathcal{M}(f))\|_\infty &\le C_\gamma \frac{\langle x\rangle^{\frac{1+m+\gamma}{2}}}{x^{1+m+\gamma}} \, \||y|^\gamma f\|_1 \,, \\
\|\partial_y^m K_1(x)(f - \mathcal{M}(f))\|_2 &\le C_\gamma \frac{\langle x\rangle^{\frac{1}{4}+\frac{m+\gamma}{2}}}{x^{\frac{1}{2}+m+\gamma}} \, \||y|^\gamma f\|_1 \,, \\
\|y\partial_y^m K_1(x)(f - \mathcal{M}(f))\|_2 &\le C_\gamma \frac{\langle x\rangle^{\frac{5}{4}+\frac{m}{2}}}{x^{\frac{3}{2}+m}} \, \||y| f\|_1 \,, \\
\|\partial_y^m K_1(x)(f - \mathcal{M}(f))\|_1 &\le C_\gamma \frac{\langle x\rangle^{\frac{3+\gamma}{4}+\frac{m}{2}}}{x^{1+m+\frac{\gamma}{2}}} \, \sqrt{\||y| f\|_1 \||y|^\gamma f\|_1} \,, \\
\|K_1(x)(f - \mathcal{M}(f)) - \partial_y K_1(x)\mathcal{M}(yf)\|_\infty &\le C_{\gamma_2} \frac{\langle x\rangle^{\frac{1+\gamma_2}{2}}}{x^{1+\gamma_2}} \, \||y|^{\gamma_2} f\|_1 \,, \\
\|K_{12}(x)(f - \mathcal{M}(f)) - \partial_y K_{12}(x)\mathcal{M}(yf)\|_\infty &\le C_{\gamma_2} \frac{\langle x\rangle^{\frac{1+\gamma_2}{2}}}{x^{1+\gamma_2}} \, \||y|^{\gamma_2} f\|_1 \,,
\end{aligned}$$

where $\mathcal{M}(f) = \int_{\mathbf{R}} f(y)\mathrm{d}y$.

Proof. Let

$$\begin{aligned}
R_1(x) &= K_{12}(x)(f - \mathcal{M}(f)) - \partial_y K_{12}(x)\mathcal{M}(yf) \\
R_2(x) &= K_1(x)(f - \mathcal{M}(f)) - \partial_y K_1(x)\mathcal{M}(yf)
\end{aligned}$$

and $R_3(x) = K_1(x)(f - \mathcal{M}(f))$. Using twice the Fourier Transform, we get

$$\begin{aligned}
\|R_1(x)\|_\infty + \|R_2(x)\|_\infty &\le \sum_{n\in\mathbf{Z}} \int_{-\infty}^{\infty} \mathrm{d}k \, \left||k|^{\gamma_2} e^{\Lambda_- x}\right| \int_{-\infty}^{\infty} \mathrm{d}y \, \left|\frac{e^{iky} - 1 - iky}{|ky|^{\gamma_2}}\right| \, |y|^{\gamma_2} |f_n(y)| \\
\|\partial_y^m R_3(x)\|_\infty &\le \sum_{n\in\mathbf{Z}} \int_{-\infty}^{\infty} \mathrm{d}k \, \left||k|^{m+\gamma} e^{\Lambda_- x}\right| \int_{-\infty}^{\infty} \mathrm{d}y \, \left|\frac{e^{iky} - 1}{|ky|^\gamma}\right| \, |y|^\gamma |f_n(y)| \\
\|\partial_y^m R_3(x)\|_2 &\le \sum_{n\in\mathbf{Z}} \left(\int_{-\infty}^{\infty} \mathrm{d}k \, \left||k|^{2m+2\gamma} e^{2\Lambda_- x}\right| \left| \int_{-\infty}^{\infty} \mathrm{d}y \, \left|\frac{e^{iky} - 1}{|ky|^\gamma}\right| \, |y|^\gamma |f_n(y)|\right|^2 \right)^{1/2} \,, \\
\|y\partial_y^m R_3(x)\|_2 &\le x \sum_{n\in\mathbf{Z}} \left(\int_{-\infty}^{\infty} \mathrm{d}k \, \left||k|^{2(m+2)} e^{2\Lambda_- x}\right| \, \|yf\|^2 \right)^{1/2} + \|\partial_y^m K_1(x)\|_{\mathrm{L}^2} \|\langle y\rangle f\|_1 \,.
\end{aligned}$$

The proof is completed using Lemma A.2 and $\|\partial_y^m R_3\|_1 \le (\|\partial_y^m R_3\|_2 \|y\partial_y^m R_3\|_2)^{\frac{1}{2}}$. ■

[4] by abuse of notation, K_1 is here considered as a function and not as a convolution operator.

5.1 The 'linear' terms

In this subsection, we consider the asymptotics of

$$U(x) = K_1(x-x_0)\mathcal{L}_u w + K_0(x-x_0)\nu \ , \quad W(x) = K_1(x-x_0)w \ , \tag{5.2}$$

as $x \to \infty$. We first note that by Lemma 3.5, 3.6 and A.5, we have

$$\begin{aligned} \|K_1(x-x_0)(\mathcal{L}_u + \mathcal{I}\mathcal{P}_0)w + K_0(x-x_0)\nu\|_\infty &\le C(x_0, \|(\nu,\mu,w)\|_{x_0})\ \langle x\rangle^{-1+\varphi} \ , \\ \|\mathcal{P}W(x)\|_\infty &\le C(x_0, \|(\nu,\mu,w)\|_{x_0})\ \langle x\rangle^{-\frac{3}{2}+\varphi} \ , \\ \|\mathcal{P}W(x)\|_1 &\le C(x_0, \|(\nu,\mu,w)\|_{x_0})\ \langle x\rangle^{-1+\varphi} \ , \\ \||y|^{\beta_0}\mathcal{P}W(x)\|_2 &\le C(x_0, \|(\nu,\mu,w)\|_{x_0})\ \langle x\rangle^{-\frac{5}{4}+\frac{\beta_0}{2}+\varphi} \ , \end{aligned}$$

for all $x \ge 2x_0$ and $0 \le \beta_0 \le \beta$. This means that the first order contribution of U and W to (5.1) is given by $U_1(x) = K_1(x-x_0)f$ and $W_1(x) = -\partial_y K_1(x-x_0)f$, where $f = -\mathcal{I}\mathcal{P}_0 w$. Using Lemma 5.3 and that by Lemma 3.2 we have $\|yf\|_1 \le C\|(0,0,w)\|_{x_0}$, we conclude that for $\mathbf{a}_{1,1} = (-\mathcal{M}(\mathcal{I}\mathcal{P}_0 w), 0, 0, 0, 0, 0)$, we have

$$\begin{aligned} \|U(x) - u_{\mathbf{a}_{1,1}}(x-x_0)\|_\infty &\le C(x_0, \|(\mathbf{v},\omega)\|)\ \langle x\rangle^{-1+(1+\varepsilon)\varphi} \ , \\ \|W(x) - \omega_{\mathbf{a}_{1,1}}(x-x_0)\|_\infty &\le C(x_0, \|(\mathbf{v},\omega)\|)\ \langle x\rangle^{-\frac{3}{2}+(1+\varepsilon)\varphi} \\ \|W(x) - \omega_{\mathbf{a}_{1,1}}(x-x_0)\|_1 &\le C(x_0, \|(\mathbf{v},\omega)\|)\ \langle x\rangle^{-1+(1+\varepsilon)\varphi} \\ \||y|^{\beta_0}(W(x) - \omega_{\mathbf{a}_{1,1}}(x-x_0))\|_2 &\le C(x_0, \|(\mathbf{v},\omega)\|)\ \langle x\rangle^{-\frac{5}{4}+\frac{\beta_0}{2}+(1+\varepsilon)\varphi} \end{aligned} \tag{5.3}$$

for all $\frac{1}{2} \le \beta_0 \le 1$ and $x \ge 2x_0$.

5.2 The nonlinear terms

We now turn to the asymptotics of

$$U_1(x) = \mathcal{F}_{1,u}(x) + \mathcal{F}_{2,u}(x) + \mathcal{L}_1 S(x) - \mathcal{L}_2 R(x) \ , \qquad W_1(x) = \mathcal{F}_{1,\omega}(x) + \mathcal{F}_{2,\omega}(x) \ .$$

It follows from Propositions 3.7 to 3.16 that for all $\frac{1}{2} \le \beta_0 \le \beta$ and $x \ge x_0$, we have

$$\begin{aligned} \|U_1(x) - \mathcal{F}_{1,2,u}(x)\|_\infty &\le C(x_0, \|(\mathbf{v},\omega)\|)\ \langle x\rangle^{-1+(1+\varepsilon)\varphi} \ , \\ \|W_1(x) - \mathcal{F}_{1,1,\omega}(x)\|_\infty &\le C(x_0, \|(\mathbf{v},\omega)\|)\ \langle x\rangle^{-\frac{3}{2}+(1+\varepsilon)\varphi} \\ \|W_1(x) - \mathcal{F}_{1,1,\omega}(x)\|_1 &\le C(x_0, \|(\mathbf{v},\omega)\|)\ \langle x\rangle^{-1+(1+\varepsilon)\varphi} \\ \||y|^{\beta_0}(W_1(x) - \mathcal{F}_{1,1,\omega}(x))\|_2 &\le C(x_0, \|(\mathbf{v},\omega)\|)\ \langle x\rangle^{-\frac{5}{4}+\frac{\beta_0}{2}+(1+\varepsilon)\varphi} \ , \end{aligned} \tag{5.4}$$

where

$$\mathcal{F}_{1,2,u}(x) = -\int_{x_0}^{x} \mathrm{d}\tilde{x}\ K_{12}(x-\tilde{x})Q(\tilde{x}) \quad \text{and} \quad \mathcal{F}_{1,1,\omega}(x) = -\int_{x_0}^{x} \mathrm{d}\tilde{x}\ K_2(x-\tilde{x})Q(\tilde{x}) \ .$$

The proof of Theorem 5.1 is then an immediate consequence of the preceding section, Lemma 5.2 and the

Proposition 5.4 *Assume that* Q *satisfies (1.17), and define* $\mathbf{a}_{1,2} = (-\int_{\Omega_+} \mathcal{P}_0 Q(x,y)\,\mathrm{d}x\mathrm{d}y, 0,0,0,0,0)$ *and*

$$D_1(x) = \mathcal{F}_{1,2,u}(x) - u_{\mathbf{a}_{1,2}}(x-x_0)\ , \quad D_2(x) = \mathcal{F}_{1,1,\omega}(x) - \omega_{\mathbf{a}_{1,2}}(x-x_0)\ ,$$

then for all $\varepsilon > 0$, *there exist a constant* C *such that*

$$\|D_1(x)\|_\infty \leq C\langle x\rangle^{-1+(1+\varepsilon)\varphi}\|(\mathbf{v},\omega)\|^2\ , \quad \|D_2(x)\|_\infty \leq C\langle x\rangle^{-\frac{3}{2}+(1+\varepsilon)\varphi}\|(\mathbf{v},\omega)\|^2$$
$$\|D_2(x)\|_1 \leq C\langle x\rangle^{-1+(1+\varepsilon)\varphi}\|(\mathbf{v},\omega)\|^2\ , \quad \||y|^{\beta_0}D_2(x)\|_2 \leq C\langle x\rangle^{-\frac{5}{4}+\frac{\beta_0}{2}+(1+\varepsilon)\varphi}\|(\mathbf{v},\omega)\|^2$$

for all $x \geq 2x_0$ *and* $\frac{1}{2} \leq \beta_0 \leq 1-2(1+\varepsilon)\varphi$.

Remark 5.5 *This result is expected in view of the corresponding classical theory on the nonlinear heat equation (see e.g. [3]). However in our case, we can prove that in fact* $\mathbf{a}_{1,2}$ does not *depend on* u, v *and* ω *on the whole domain* Ω_+, *but only on* u *and* v *on the boundary* $x = x_0$. *Namely, since* $Q = -\partial_y R + \partial_x S$, *we have*

$$\mathbf{a}_{1,2} = \mathcal{P}_0 \int_{\Omega_+} Q(x,y)\,\mathrm{d}x\mathrm{d}y = \mathcal{P}_0 \int_{\Omega_+} (\partial_x S(x,y) - \partial_y R(x,y))\,\mathrm{d}x\mathrm{d}y = -\mathcal{M}(\mathcal{P}_0 S(x_0))\ .$$

Proof. Let $D_{1,1}(x) = -\mathcal{P}\int_{x_0}^x \mathrm{d}\tilde{x}\, K_{12}(x-\tilde{x})Q(\tilde{x})$ and $D_{2,1}(x) = -\mathcal{P}\int_{x_0}^x \mathrm{d}\tilde{x}\, K_2(x-\tilde{x})Q(\tilde{x})$. Using Lemma 3.9, we have

$$\|D_{1,1}(x)\|_\infty \leq B\Big[\begin{smallmatrix} 1,\frac{3}{2}-\varphi,\frac{1}{2} \\ \frac{1}{2},\frac{7}{4}+\varphi,\frac{1}{4}\end{smallmatrix}\Big](x,x_0)\|(\mathbf{v},\omega)\|^2 \leq C\langle x\rangle^{-1+\varphi}\|(\mathbf{v},\omega)\|^2\ ,$$

$$\|D_{2,1}(x)\|_\infty \leq B\Big[\begin{smallmatrix} 1,\frac{3}{2}-\varphi,0 \\ \frac{1}{2},2-\varphi,0\end{smallmatrix}\Big](x,x_0)\|(\mathbf{v},\omega)\|^2 \leq C\langle x\rangle^{-\frac{3}{2}+\varphi}\|(\mathbf{v},\omega)\|^2\ ,$$

$$\|D_{2,1}(x)\|_1 \leq B\Big[\begin{smallmatrix} \frac{1}{2},1,0 \\ \frac{1}{2},1,0\end{smallmatrix}\Big](x,x_0)\|(\mathbf{v},\omega)\|^2 \leq C\langle x\rangle^{-1+\varphi}\|(\mathbf{v},\omega)\|^2\ ,$$

$$\begin{aligned}\big\||y|^\beta D_{2,1}(x)\big\|_2 &\leq \Big(B\Big[\begin{smallmatrix} \frac{3}{4}-\frac{\beta}{2},\frac{3}{2}-\varphi,0 \\ \frac{3}{4}-\frac{\beta}{2},\frac{3}{2}-\varphi,0\end{smallmatrix}\Big](x,x_0) + B\Big[\begin{smallmatrix} \frac{1}{2},\frac{7}{4}-\frac{\beta}{2}-\varphi,0 \\ \frac{1}{2},\frac{7}{4}-\frac{\beta}{2}-\varphi,0\end{smallmatrix}\Big](x,x_0)\Big)\|(\mathbf{v},\omega)\|^2 \\ &\leq C\langle x\rangle^{-\frac{5}{4}+\frac{\beta}{2}+\varphi}\|(\mathbf{v},\omega)\|^2\ .\end{aligned}$$

Now, let $D_{1,2}(x) = -\mathcal{P}_0\int_{x_0}^x \mathrm{d}\tilde{x}\, K_{12}(x-\tilde{x})Q(\tilde{x})$ and $D_{2,2}(x) = -\mathcal{P}_0\int_{x_0}^x \mathrm{d}\tilde{x}\, K_2(x-\tilde{x})Q(\tilde{x})$. Since $\mathcal{P}_0 K_{10} \equiv 0$, $\partial_x\mathcal{P}_0 K_{12} = -\partial_y K_2$, $\partial_x\mathcal{P}_0 K_2 = \partial_y K_8$ and $\mathcal{P}_0 K_2 = 2\partial_y\mathcal{P}_0 K_8 + \partial_y\mathcal{P}_0 K_1$ and $\mathcal{P}_0 K_{12} = -\mathcal{P}_0 K_1 - \mathcal{P}_0 K_8$, integrating by parts in $\tilde{x}$, we get

$$D_{1,2}(x) = \mathcal{P}_0\left(K_1(x-x_0) + K_8(x-x_0)\right)\int_{x_0}^x \mathrm{d}z\, Q(z) + \int_{x_0}^x \mathrm{d}\tilde{x}\,\partial_y K_2(x-\tilde{x})\int_{\tilde{x}}^x \mathrm{d}z\, Q(z)\ ,$$
$$D_{2,2}(x) = -\mathcal{P}_0\left(\partial_y K_1(x-x_0) + 2\partial_y K_8(x-x_0)\right)\int_{x_0}^x \mathrm{d}z\, Q(z) - \int_{x_0}^x \mathrm{d}\tilde{x}\,\partial_y K_8(x-\tilde{x})\int_{\tilde{x}}^x \mathrm{d}z\, Q(z)\ .$$

We then have

$$\left\|K_1(x-x_0)\int_x^\infty \mathrm{d}z\, Q(z)\right\|_\infty \leq C\|(\mathbf{v},\omega)\|^2\int_x^\infty \mathrm{d}z\,\langle z\rangle^{-2+\varphi} \leq C\|(\mathbf{v},\omega)\|^2\langle x\rangle^{-1+\varphi}\ ,$$
$$\left\|K_8(x-x_0)\int_{x_0}^x \mathrm{d}z\, Q(z)\right\|_\infty \leq C\|(\mathbf{v},\omega)\|^2(x-x_0)^{-1}\int_{x_0}^x \mathrm{d}z\,\langle z\rangle^{-\frac{3}{2}+\varphi} \leq C\|(\mathbf{v},\omega)\|^2\langle x\rangle^{-1}\ ,$$

$$
\begin{aligned}
\left\|\partial_y K_1(x-x_0)\int_x^\infty \mathrm{d}z\,Q(z)\right\|_\infty &\leq C\|(\mathbf{v},\omega)\|^2\frac{\langle x\rangle^{\frac{1}{2}}}{x-x_0}\int_x^\infty \mathrm{d}z\,\langle z\rangle^{-2+\varphi}\leq C\|(\mathbf{v},\omega)\|^2\langle x\rangle^{-\frac{3}{2}+\varphi}\ ,\\
\left\|\partial_y K_8(x-x_0)\int_{x_0}^x \mathrm{d}z\,Q(z)\right\|_\infty &\leq C\|(\mathbf{v},\omega)\|^2\frac{\langle x\rangle^{\frac{1}{4}}}{(x-x_0)^{\frac{7}{4}}}\int_{x_0}^x \mathrm{d}z\,\langle z\rangle^{-2+\varphi}\leq C\|(\mathbf{v},\omega)\|^2\langle x\rangle^{-\frac{3}{2}}\ ,\\
\left\|\partial_y K_1(x-x_0)\int_x^\infty \mathrm{d}z\,Q(z)\right\|_1 &\leq C\|(\mathbf{v},\omega)\|^2\frac{\langle x\rangle^{\frac{1}{2}}}{x-x_0}\int_x^\infty \mathrm{d}z\,\langle z\rangle^{-\frac{3}{2}+\varphi}\leq C\|(\mathbf{v},\omega)\|^2\langle x\rangle^{-1+\varphi}\ ,\\
\left\|\partial_y K_8(x-x_0)\int_{x_0}^x \mathrm{d}z\,Q(z)\right\|_1 &\leq C\|(\mathbf{v},\omega)\|^2\frac{\langle x\rangle^{\frac{1}{4}}}{(x-x_0)^{\frac{7}{4}}}\int_{x_0}^x \mathrm{d}z\,\langle z\rangle^{-\frac{3}{2}+\varphi}\leq C\|(\mathbf{v},\omega)\|^2\langle x\rangle^{-\frac{3}{2}}\ ,
\end{aligned}
$$

where in the last six inequalities we used $x \geq 2x_0$. Similarly, for $x \geq 2x_0$, we have

$$
\begin{aligned}
\left\||y|^{\beta_0}\partial_y K_1(x-x_0)\int_x^\infty \mathrm{d}z\,Q(z)\right\|_2 &\leq C\|(\mathbf{v},\omega)\|^2\langle x\rangle^{-\frac{5}{4}+\frac{\beta_0}{2}+\varphi}\ ,\\
\left\||y|^{\beta}\partial_y K_8(x-x_0)\int_{x_0}^x \mathrm{d}z\,Q(z)\right\|_2 &\leq C\|(\mathbf{v},\omega)\|^2\langle x\rangle^{-\frac{5}{4}+\frac{\beta}{2}+\varphi}\ .
\end{aligned}
$$

Note that the first of these two estimates is only valid if $\beta_0 < \frac{3}{2}-2\varphi$. Next, for $D_{1,3}(x)=\int_{x_0}^x \mathrm{d}\tilde{x}\,\partial_y K_2(x-\tilde{x})\int_{\tilde{x}}^x \mathrm{d}z\,Q(z)$ and $D_{2,3}(x)=\int_{x_0}^x \mathrm{d}\tilde{x}\,\partial_y K_8(x-\tilde{x})\int_{\tilde{x}}^x \mathrm{d}z\,Q(z)$, we have, (see Lemma 5.6 below for the definition of $D[\,\cdot\,](x,x_0)$ and related estimates), that

$$
\begin{aligned}
\|D_{1,3}(x)\|_\infty &\leq \langle x\rangle^{\frac{1}{2}}D\left[\begin{smallmatrix}2,\frac{3}{2}-\varphi\\ \frac{3}{2},2-\varphi\end{smallmatrix}\right](x,x_0)\|(\mathbf{v},\omega)\|^2\leq C\|(\mathbf{v},\omega)\|^2\langle x\rangle^{-1+\varphi}\ ,\\
\|D_{2,3}(x)\|_\infty &\leq \langle x\rangle^{\frac{1}{4}}D\left[\begin{smallmatrix}\frac{7}{4},2-\varphi\\ \frac{7}{4},2-\varphi\end{smallmatrix}\right](x,x_0)\|(\mathbf{v},\omega)\|^2\leq C\|(\mathbf{v},\omega)\|^2\langle x\rangle^{-\frac{3}{2}+\varphi}\ ,\\
\|D_{2,3}(x)\|_1 &\leq \langle x\rangle^{\frac{1}{4}}D\left[\begin{smallmatrix}\frac{7}{4},\frac{3}{2}-\varphi\\ \frac{7}{4},\frac{3}{2}-\varphi\end{smallmatrix}\right](x,x_0)\|(\mathbf{v},\omega)\|^2\leq C\|(\mathbf{v},\omega)\|^2\langle x\rangle^{-1+\varphi}\ .
\end{aligned}
$$

Along the same lines, we find $\left\||y|^{\beta}D_{2,3}(x)\right\|_2 \leq C\|(\mathbf{v},\omega)\|^2\langle x\rangle^{-\frac{5}{4}+\frac{\beta}{2}+\varphi}$. We finally define

$$
T(x,y)=u_{\mathbf{a}_{1,2}}(x)-\mathcal{P}_0K_1(x-x_0)\int_{x_0}^\infty \mathrm{d}z\,Q(z)\ .
$$

Since $\int_{x_0}^x \mathrm{d}z\,Q(z)=\int_{x_0}^\infty \mathrm{d}z\,Q(z)-\int_x^\infty \mathrm{d}z\,Q(z)$, we get, using Lemma 5.3 and $x\geq 2x_0$ that

$$
\begin{aligned}
\|D_1(x)\|_\infty &\leq \|T(x)\|_\infty + C\langle x\rangle^{-1+\varphi}\|(\mathbf{v},\omega)\|^2\ ,\\
&\leq C\langle x\rangle^{-\frac{1+\gamma}{2}}\int_{x_0}^\infty \mathrm{d}z\,\||y|^\gamma Q(z)\|_1 + C\langle x\rangle^{-1+\varphi}\|(\mathbf{v},\omega)\|^2\ ,\\
\|D_2(x)\|_\infty &\leq \|\partial_y T(x)\|_\infty + C\langle x\rangle^{-\frac{3}{2}+\varphi}\|(\mathbf{v},\omega)\|^2\ ,\\
&\leq C\langle x\rangle^{-1-\frac{\gamma}{2}}\int_{x_0}^\infty \mathrm{d}z\,\||y|^\gamma Q(z)\|_1 + C\langle x\rangle^{-\frac{3}{2}+\varphi}\|(\mathbf{v},\omega)\|^2\ ,\\
\|D_2(x)\|_1 &\leq \|\partial_y T(x)\|_1 + C\langle x\rangle^{-1+\varphi}\|(\mathbf{v},\omega)\|^2\ ,\\
&\leq C\langle x\rangle^{-\frac{3+\gamma}{4}}\int_{x_0}^\infty \mathrm{d}z\,\sqrt{\||y|Q(z)\|_1\||y|^\gamma Q(z)\|_1} + C\langle x\rangle^{-1+\varphi}\|(\mathbf{v},\omega)\|^2\ ,
\end{aligned}
$$

$$\||y|^{\beta_0} D_2(x)\|_2 \leq \left\||y|^{\beta_0}\partial_y T(x)\right\|_2 + C\langle x\rangle^{-\frac{5}{4}+\frac{\beta_0}{2}+\varphi}\|(\mathbf{v},\omega)\|^2 ,$$
$$\leq C\langle x\rangle^{-\frac{3}{4}+\frac{\gamma}{2}(1-\beta_0)}\int_{x_0}^{\infty}\mathrm{d}z\,\||y|^{\gamma}Q\|_1^{1-\beta_0}\||y|Q\|_1^{\beta_0} + C\langle x\rangle^{-\frac{5}{4}+\frac{\beta_0}{2}+\varphi}\|(\mathbf{v},\omega)\|^2 ,$$

for any $0 \leq \gamma \leq 1$ (we used $\||y|^{\beta_0} f\|_p \leq \|f\|_p^{1-\beta_0}\||y|f\|_p^{\beta_0}$ to establish the last estimate). Then, for any $\gamma_1 \leq 1$ and $\sigma > \frac{1}{2}$, we have

$$\int_{x_0}^{\infty}\mathrm{d}z\,\||y|^{\gamma_1}Q(z)\|_1 \leq C\int_{x_0}^{\infty}\mathrm{d}z\,\|(1+|y|)^{\sigma}|y|^{\gamma_1}Q(z)\|_2 \leq C\int_{x_0}^{\infty}\mathrm{d}z\,\langle z\rangle^{\varphi+\frac{\gamma_1+\sigma}{2}-\frac{7}{4}}\|(\mathbf{v},\omega)\|^2 ,$$

while with similar arguments we have for any $\gamma_2 \leq 1$ and $\gamma_3 \leq 1$,

$$\int_{x_0}^{\infty}\mathrm{d}z\,\sqrt{\||y|Q(z)\|_1\||y|^{\gamma_2}Q(z)\|_1} \leq C\int_{x_0}^{\infty}\mathrm{d}z\,\langle z\rangle^{\varphi+\frac{\gamma}{4}+\frac{\sigma}{2}-\frac{3}{2}}\|(\mathbf{v},\omega)\|^2 ,$$
$$\int_{x_0}^{\infty}\mathrm{d}z\,\||y|^{\gamma}Q\|_1^{1-\beta_0}\||y|Q\|_1^{\beta_0} \leq C\int_{x_0}^{\infty}\mathrm{d}z\,\langle z\rangle^{\varphi+\frac{\beta_0}{2}+\frac{\gamma(1-\beta_0)}{2}+\frac{\sigma}{2}-\frac{7}{4}}\|(\mathbf{v},\omega)\|^2 .$$

Choosing $\gamma_1 = 1-2(1+\varepsilon)\varphi$, $\gamma_2 = 1-4(1+\varepsilon)\varphi$, $\gamma_3 = 1-2(\frac{1+\varepsilon}{1-\beta_0})\varphi$ and $\sigma = \frac{1}{2}+\varepsilon\varphi$ with $\varepsilon > 0$ completes the proof. ■

Lemma 5.6 *Let* $0 \leq p_1, q_2 < 2$*, and* $p_2, q_1 \geq 0$*, then there exist a constant* C *such that*

$$D\left[\begin{smallmatrix} p_2,q_2 \\ p_1,q_1 \end{smallmatrix}\right](x,x_0) \equiv \int_{x_0}^{x}\mathrm{d}\tilde{x}\int_{\tilde{x}}^{x}\mathrm{d}z\,\min\left(\frac{\langle z\rangle^{-q_1}}{(x-\tilde{x})^{p_1}}, \frac{\langle z\rangle^{-q_2}}{(x-\tilde{x})^{p_2}}\right) \leq C(\langle x\rangle^{2-p_1-q_1} + \langle x\rangle^{2-p_2-q_2}) .$$

for all $x \geq 2x_0 \geq 2$.

Proof. The proof follows at once from

$$D\left[\begin{smallmatrix} p_2,q_2 \\ p_1,q_1 \end{smallmatrix}\right](x,x_0) \leq \frac{C}{(x-x_0)^{p_2}}\int_{x_0}^{\frac{x+x_0}{2}}\mathrm{d}\tilde{x}\int_{\tilde{x}}^{x}\mathrm{d}z\,\langle z\rangle^{-q_2} + C\langle x\rangle^{-q_1}\int_{\frac{x+x_0}{2}}^{x}\mathrm{d}\tilde{x}\,(x-\tilde{x})^{1-p_1}$$
$$\leq C(\langle x\rangle^{2-p_1-q_1} + \langle x\rangle^{2-p_2-q_2}) ,$$

see also the proof of Lemma 3.9 for related results. ■

6 Refined asymptotics

To complete the asymptotic description of solution of (1.1), we still have to prove the Corollary 1.8. Since the asymptotic description of ω is already proved in Theorem 5.1, it only remains to prove the

Theorem 6.1 *Let* $\varphi < \varphi_0 < \frac{1}{8}$*. Assume that* $\||y|^{\frac{1}{2}}\mathbf{v}(x_0)\|_4 < \infty$ *and* $\||y|^{\frac{1}{2}-\varphi_0}\mathcal{S}\nu\|_1 + \||y|^{\frac{1}{2}-\varphi_0}\mathcal{S}\mu\|_1 < \infty$*, and let* $a_1 = -\mathcal{M}(\mathcal{I}\mathcal{P}_0 w) - \int_{\Omega_+}\mathcal{P}_0 Q(x,y)\,\mathrm{d}x\mathrm{d}y$*,* $a_2 = \mathcal{M}(\mathcal{S}\nu) - \int_{\Omega_+}\mathcal{P}_0 Q(x,y)\,\mathrm{d}x\mathrm{d}y$ *and* $a_3 = \mathcal{M}(\mathcal{S}\mu)$*. Let* $\mathbf{a} = (a_1, a_2, a_3, a_4, a_1^2, a_1\mathcal{P}_0 a_3)$ *and* $u_{\mathbf{a}}, v_{\mathbf{a}}$ *as in (1.9), then there exist a constant* a_4 *such that for all* $\varepsilon > 0$*, solutions to (1.1) in* $\mathcal{C}_u$ *satisfy for all* $x \geq x_0$

$$\|u(x) - u_{\mathbf{a}}(x)\|_\infty \leq C(x_0, \|(\mathbf{v},\omega)\|)\,\langle x\rangle^{-\frac{9}{8}+\varphi_0} , \tag{6.1}$$
$$\|v(x) - v_{\mathbf{a}}(x)\|_\infty \leq C(x_0, \|(\mathbf{v},\omega)\|)\,\langle x\rangle^{-\frac{3}{2}+\varphi_0} \tag{6.2}$$

for $\varphi_0 = (1+\varepsilon)\varphi$ *and some constant* $C(x_0, \|(\mathbf{v},\omega)\|)$*.*

Here again, we note that we need only prove the estimates on u and v for $x \geq 2x_0$, and, using Lemma 5.2, we can choose to compare $u(x)$ and $v(x)$ either to $u_{\mathbf{a}}(x)$ and $v_{\mathbf{a}}(x)$ or to $u_{\mathbf{a}}(x-x_0)$ and $v_{\mathbf{a}}(x-x_0)$. The proof of Theorem 6.1 then stands on three pillars, the partial description of Theorem 5.1, Lemma 5.3 and its equivalent on F, G and K_0, which we now state:

Lemma 6.2 *Let $0 \leq \gamma \leq 1$ and f satisfying $\|\langle y\rangle^{\gamma} f\|_1 < \infty$. Then for all $m \geq 0$, there exist a constant C_{γ} such that*

$$\|\partial_y^m F(x)(f - \mathcal{M}(f))\|_\infty \leq C_\gamma x^{-1-m-\gamma} \, \||y|^\gamma f\|_1 \,,$$
$$\|\partial_y^m G(x)(f - \mathcal{M}(f))\|_\infty \leq C_\gamma x^{-1-m-\gamma} \, \||y|^\gamma f\|_1 \,,$$
$$\|\partial_y^m K_0(x)(f - \mathcal{M}(f))\|_\infty \leq C_\gamma x^{-1-m-\gamma} \, \||y|^\gamma f\|_1 \,,$$
$$\|\partial_y^m \mathcal{H} K_0(x)(f - \mathcal{M}(f))\|_\infty \leq C_\gamma x^{-1-m-\gamma} \, \||y|^\gamma f\|_1 \,,$$

where $\mathcal{M}(f) = \int_{\mathbf{R}} f(y)\mathrm{d}y$.

Proof. The proof follows along the same lines as that of Lemma 5.3, e.g.

$$\|\partial_y^m F(x)(f - \mathcal{M}(f))\|_\infty \leq \sum_{n\in\mathbf{Z}} \int_{-\infty}^{\infty} \mathrm{d}k \, \left||k|^{m+\gamma} e^{-|k|x}\right| \, \int_{-\infty}^{\infty} \mathrm{d}y \, \left|\frac{e^{iky}-1}{|ky|^\gamma}\right| \, |y|^\gamma |f_n(y)| \,.$$

The other estimates are similar. ■

For convenience, the proof of Theorem 6.1 will now be split in the following subsections. We will first come back to the terms proportional to w, ν and μ. Then, using the first order results on ω and u, we will prove (6.2) in a first round of estimates on the nonlinear terms. We will then use (6.2) to prove (6.1) in a second round of estimates on the nonlinear terms. In principle, this 'ping-pong' strategy could be systematically used to get higher order asymptotic developments.

6.1 Back to the 'linear' terms

In this subsection, we consider the asymptotics of

$$U(x) = K_1(x-x_0)\mathcal{L}_u w + K_0(x-x_0)\nu \,, \quad V(x) = K_1(x-x_0)\mathcal{L}_v w + K_0(x-x_0)\mu \,, \tag{6.3}$$

as $x \to \infty$. We first note that by Lemma A.5 and 3.5, for all $m \geq 0$ and $x \geq 2x_0$, we have

$$\|\mathcal{P}K_1(x-x_0)\mathcal{L}_u w\|_\infty + \|\mathcal{P}K_1(x-x_0)\mathcal{L}_v w\|_\infty \leq C(x_0, \|(\mathbf{v},\omega)\|, m)\langle x\rangle^{-m}$$

since $\|\mathcal{P}K_1(x)\|_1$ decays exponentially as $x \to \infty$. Then we note that for $x \geq 2x_0$, we have

$$\|\mathcal{P}_0 K_1(x-x_0)(\mathcal{L}_u + \mathcal{I})w\|_\infty \leq C\|\partial_y^2 K_1(x-x_0)\|_\infty \|\mathcal{I}(\mathcal{I}(\mathcal{L}_u+\mathcal{I})w)\|_1 \leq C\langle x\rangle^{-\frac{3}{2}} \|\mathcal{I}w\|_1 \,.$$

Thus, as in Section 5.1, the asymptotics of U and V are the same as those of $U_1(x) = K_1(x-x_0)f$ and $V_1(x) = -\partial_y K_1(x-x_0)f$, where $f = -\mathcal{I}\mathcal{P}_0 w$. Using Lemma 5.3 and that by Lemma 3.2 we have $\|y^\gamma f\|_1 \leq C(x_0, \|(0,0,w)\|_{x_0})$ for all $\gamma \leq \frac{3}{2} - 2(1+\varepsilon)\varphi$, we conclude that for $\mathbf{a}_1 = (-\mathcal{M}(\mathcal{I}\mathcal{P}_0 w)$, $a_{4,1} = -\mathcal{M}(y\mathcal{I}\mathcal{P}_0 w)$ and $\mathbf{a}_1 = (a_{1,1}, 0, 0, a_{4,1}, 0, 0)$, we have for $x \geq 2x_0$ that

$$\begin{aligned} \|U_1(x) - u_{\mathbf{a}_1}(x-x_0)\|_\infty &\leq C(x_0, \|(\mathbf{v},\omega)\|) \, \langle x\rangle^{-\frac{5}{4}+(1+\varepsilon)\varphi} \,, \\ \|V_1(x) - v_{\mathbf{a}_1}(x-x_0)\|_\infty &\leq C(x_0, \|(\mathbf{v},\omega)\|) \, \langle x\rangle^{-\frac{3}{2}+(1+\varepsilon)\varphi} \,. \end{aligned} \tag{6.4}$$

We then note that since $\mu = \mathcal{H}\nu$ and $\nu = -\mathcal{H}\mu$, we have

$$K_0(x-x_0)\nu = K_0(x-x_0)\mathcal{S}\nu - K_0(x-x_0)\mathcal{H}\mathcal{S}\mathcal{H}\nu = K_0(x-x_0)\mathcal{S}\nu - \mathcal{H}K_0(x-x_0)\mathcal{S}\mu\ ,$$
$$K_0(x-x_0)\mu = K_0(x-x_0)\mathcal{S}\mu - K_0(x-x_0)\mathcal{H}\mathcal{S}\mathcal{H}\mu = K_0(x-x_0)\mathcal{S}\mu + \mathcal{H}K_0(x-x_0)\mathcal{S}\nu\ .$$

Defining $a_{2,1} = \mathcal{M}(\mathcal{S}\nu)$, $a_{3,1} = \mathcal{M}(\mathcal{S}\mu)$ and $\mathbf{a}_2 = (0, a_{2,1}, a_{3,1}, 0, 0, 0)$, we get by Lemma 6.2 that for $x \geq 2x_0$,

$$\|K_0(x-x_0)\nu - u_{\mathbf{a}_2}(x-x_0)\|_\infty \leq Cc(\nu,\mu)\,\langle x\rangle^{-\frac{3}{2}+(1+\varepsilon)\varphi}$$
$$\|K_0(x-x_0)\mu - v_{\mathbf{a}_2}(x-x_0)\|_\infty \leq Cc(\nu,\mu)\,\langle x\rangle^{-\frac{3}{2}+(1+\varepsilon)\varphi}\ ,$$

where $c(\nu,\mu) = (\||y|^{\frac{1}{2}-(1+\varepsilon)\varphi}\mathcal{S}\nu\|_1 + \||y|^{\frac{1}{2}-(1+\varepsilon)\varphi}\mathcal{S}\mu\|_1)$. Using Lemma 5.2 and A.10, we get

$$\|U(x) - u_{\mathbf{a}_3}(x)\|_\infty \leq C(x_0, \|(\mathbf{v},\omega)\|, c(\nu,\mu))\,\langle x\rangle^{-\frac{5}{4}+(1+\varepsilon)\varphi}\ , \tag{6.5}$$
$$\|V(x) - v_{\mathbf{a}_3}(x)\|_\infty \leq C(x_0, \|(\mathbf{v},\omega)\|, c(\nu,\mu))\,\langle x\rangle^{-\frac{3}{2}+(1+\varepsilon)\varphi} \tag{6.6}$$

for $\mathbf{a}_3 = (a_{1,1}, a_{2,1}, a_{3,1}, a_{4,1}, 0, 0)$ and some constant $C(x_0, \|(\mathbf{v},\omega)\|, c(\nu,\mu))$.

6.2 Nonlinear terms, first round

To complete the proof of Theorem 6.1 we now have to give the asymptotic development of

$$u_1(x) = \mathcal{F}_{1,u}(x) + \mathcal{F}_{2,u}(x) + \mathcal{L}_1 S(x) - \mathcal{L}_2 R(x)\ ,$$
$$v_1(x) = \mathcal{F}_{1,v}(x) + \mathcal{F}_{2,v}(x) - \mathcal{L}_1 R(x) - \mathcal{L}_2 S(x)\ .$$

We first tackle the terms $u_2(x) = \mathcal{L}_1 S(x) - \mathcal{L}_2 R(x)$ and $v_2(x) = -\mathcal{L}_1 R(x) - \mathcal{L}_2 S(x)$. Let $P_{\mathbf{a}}(x) = u_{\mathbf{a}_1}(x)\omega_{\mathbf{a}_1}(x)$ where $\mathbf{a}_1 = (a_1, 0, 0, 0, 0, 0)$ and $\Delta S = S(x) - \mathcal{I}P_{\mathbf{a}}(x)$. We first note that by Theorem 5.1

$$\begin{aligned}\|\Delta S(x)\|_\infty &\leq \|v(x)\|_\infty^2 + \|u(x) - u_{\mathbf{a}_1}(x)\|_\infty \|u(x) + u_{\mathbf{a}_1}(x)\|_\infty \\ &\leq C(x_0, \|(\mathbf{v},\omega)\|)\,\langle x\rangle^{-\frac{3}{2}+(1+\varepsilon)\varphi}\ .\end{aligned}$$

Then, since $\mathcal{P}P_{\mathbf{a}}(x) = 0$ and $\mathcal{P}_0\mathcal{L}_2 = 0$ implies $\mathcal{L}_2 S = \mathcal{L}_2\Delta S$, we get

$$\|\mathcal{L}_1 R(x)\|_\infty + \|\mathcal{L}_2 R(x)\|_\infty \leq C(x_0, \|(\mathbf{v},\omega)\|)\,\langle x\rangle^{-\frac{3}{2}+\varphi}\ ,$$
$$\|\mathcal{L}_1 \Delta S(x)\|_\infty + \|\mathcal{L}_2 S(x)\|_\infty \leq C(x_0, \|(\mathbf{v},\omega)\|)\,\langle x\rangle^{-\frac{3}{2}+(1+\varepsilon)\varphi}\ ,$$

while $\mathcal{P}_0\mathcal{L}_1 = \mathbb{1}$ implies

$$\begin{aligned}\|u_2(x) - \mathcal{I}P_{\mathbf{a}}(x)\|_\infty &\leq C(x_0, \|(\mathbf{v},\omega)\|)\,\langle x\rangle^{-\frac{3}{2}+(1+\varepsilon)\varphi} \\ \|v_2(x)\|_\infty &\leq C(x_0, \|(\mathbf{v},\omega)\|)\,\langle x\rangle^{-\frac{3}{2}+(1+\varepsilon)\varphi}\ .\end{aligned} \tag{6.7}$$

It then follows from (6.7) and Propositions 3.10 to 3.16 that

$$\|u_1(x) - \mathcal{F}_{1,3,u}(x) - \mathcal{F}_{1,5,u}(x) - \mathcal{F}_{1,6,u}(x)\|_\infty \leq C(x_0, \|(\mathbf{v},\omega)\|)\,\langle x\rangle^{-\frac{3}{2}+(1+\varepsilon)\varphi}$$
$$\|v_1(x) - \mathcal{F}_{1,1,\omega}(x) - \mathcal{F}_{1,3,v}(x)\|_\infty \leq C(x_0, \|(\mathbf{v},\omega)\|)\,\langle x\rangle^{-\frac{3}{2}+(1+\varepsilon)\varphi}\ ,$$

where

$$\mathcal{F}_{1,5,u}(x) = \int_{x_0}^{x} \mathrm{d}\tilde{x}\; K_2(\tilde{x}-x)P(\tilde{x}) + 2\mathcal{I}P_{\mathbf{a}}(x)$$

$$\mathcal{F}_{1,6,u}(x) = -\int_{x_0}^{x} \mathrm{d}\tilde{x}\; K_{12}(\tilde{x}-x)Q(\tilde{x}) - \mathcal{I}P_{\mathbf{a}}(x)$$

The asymptotic development of $\mathcal{F}_{1,1,\omega}(x)$ is established in Proposition 5.4 above, that of $\mathcal{F}_{1,3,u}(x)$ and $\mathcal{F}_{1,3,v}(x)$ in Proposition 6.3 below, followed by that of $\mathcal{F}_{1,5,u}(x)$ in Proposition 6.4. The proof of Theorem 6.1 will be completed by the study of $\mathcal{F}_{1,6,u}(x)$ in Section 6.3.

Proposition 6.3 *Assume that Q satisfies (1.17), and define $\mathbf{a}_4 = (0, -\int_{\Omega_+} \mathcal{P}_0 Q(x,y)\,\mathrm{d}x\mathrm{d}y, 0,0,0,0)$, then for all $\varepsilon > 0$, there exist a constant C such that*

$$\|\mathcal{F}_{1,3,u}(x) - u_{\mathbf{a}_4}(x)\|_\infty + \|\mathcal{F}_{1,3,v}(x) - v_{\mathbf{a}_4}(x)\|_\infty \le C\langle x\rangle^{-\frac{3}{2}+(1+\varepsilon)\varphi}\|(\mathbf{v},\omega)\|^2$$

for all $x \ge 2x_0$.

Proof. The proof is very similar to the one of Proposition 5.4. We first note that $\|F\|_{\frac{1}{2\varepsilon\varphi},\{0,1-2\varepsilon\varphi\}} + \|G\|_{\frac{1}{2\varepsilon\varphi},\{0,1-2\varepsilon\varphi\}} \le C$, $\|Q\|_{\frac{1}{1-2\varepsilon\varphi},\frac{3}{2}-(1-\varepsilon)\varphi} \le C\|(\mathbf{v},\omega)\|^2$. Then, we define $T(x) = \int_{x_0}^{x} \mathrm{d}\tilde{x}\; F(x-\tilde{x})Q(\tilde{x})$. Since $\partial_x F(x) = \partial_y G(x)$, after integration by parts, we have

$$T(x) = F(x-x_0)\int_{x_0}^{\infty}\mathrm{d}z\; Q(z) - F(x-x_0)\int_{x}^{\infty}\mathrm{d}z\; Q(z) - \int_{x_0}^{x}\mathrm{d}\tilde{x}\; \partial_y G(x-\tilde{x})\int_{\tilde{x}}^{x}\mathrm{d}z\; Q(z)\,.$$

Let $T_1(x) = F(x-x_0)\int_x^\infty \mathrm{d}z\; Q(z)$ and $T_2(x) = \int_{x_0}^{x}\mathrm{d}\tilde{x}\;\partial_y G(x-\tilde{x})\int_{\tilde{x}}^{x}\mathrm{d}z\; Q(z)$. Since $x \ge 2x_0$, we have

$$\|T_1(x)\|_\infty \le C\|(\mathbf{v},\omega)\|^2(x-x_0)^{-1}\int_x^\infty \mathrm{d}z\; z^{-\frac{3}{2}+\varphi} \le C\|(\mathbf{v},\omega)\|^2\langle x\rangle^{-\frac{3}{2}+\varphi}$$

$$\|T_2(x)\|_\infty \le C\|(\mathbf{v},\omega)\|^2 D\left[\begin{smallmatrix} 2-2\varepsilon\varphi,\frac{3}{2}-\varphi(1-\varepsilon) \\ 2-2\varepsilon\varphi,\frac{3}{2}-\varphi(1-\varepsilon)\end{smallmatrix}\right](x,x_0) \le C\|(\mathbf{v},\omega)\|^2\langle x\rangle^{-\frac{3}{2}+(1+\varepsilon)\varphi}\,.$$

Then, we define $T_3(x) = F(x-x_0)\int_{x_0}^\infty \mathrm{d}z\; Q(z)$. Since $|\tau| \le \langle x_0\rangle^\varphi \le \langle x\rangle^\varphi$ and $x \ge 2x_0$, (using Lemma 5.6 in the second inequality)

$$\|\mathcal{P}T_3(x)\|_\infty \le C\langle x\rangle^{-2+\varphi}\|(\mathbf{v},\omega)\|^2\int_{x_0}^\infty \mathrm{d}z\;\langle z\rangle^{-\frac{3}{2}+\varphi} \le C\langle x\rangle^{-2+\varphi}\|(\mathbf{v},\omega)\|^2\,.$$

Finally, by Proposition 6.3 and using $\||y|^{\frac{1}{2}-(1+\varepsilon)\varphi}Q\|_1 \le C\||y|^{1-\varphi}Q\|_2 \le C\langle z\rangle^{-1-\frac{1}{4}(1-2\varphi)}\|(u,v,\omega)\|^2$, we have

$$\|\mathcal{P}_0 T_3(x) - u_{\mathbf{a}_4}(x-x_0)\|_\infty \le C\langle x\rangle^{-\frac{3}{2}+(1+\varepsilon)\varphi}\|(\mathbf{v},\omega)\|^2\int_{x_0}^\infty \mathrm{d}z\;\langle z\rangle^{-1-\frac{1}{4}(1-2\varphi)}\,,$$

where we used $\mathcal{P}_0 F = \mathcal{P}_0 K_0$. Since $\varphi < \frac{1}{2}$, the proof of the estimate on $\|\mathcal{F}_{1,3,u}(x) - u_{\mathbf{a}_4}(x)\|_\infty$ is completed using Lemma 5.2. The proof of the estimate on $\|\mathcal{F}_{1,3,v}(x) - v_{\mathbf{a}_4}(x)\|_\infty$ being very similar, we omit the details. ■

We now turn to the asymptotics of $\mathcal{F}_{1,5,u}$.

Proposition 6.4 *Assume that P satisfies (1.17) and let $P_{\mathbf{a}}(x) = u_{\mathbf{a}_1}(x)\omega_{\mathbf{a}_1}(x)$ where $\mathbf{a}_1 = (a_1, 0, 0, 0, 0, 0)$. Assume that*

$$\| |y|(u(x_0)^2 + v(x_0)^2) \|_2 \leq C \|(\mathbf{v}, \omega)\|^2 \,.$$

Let $\mathbf{a}_5 = (0, 0, 0, -\int_{\mathbf{R}} \mathcal{P}_0 u(x_0, y) v(x_0, y) \,\mathrm{d}y, a_1^2, 0)$, then for all $\varepsilon > 0$, there exist a constant $C = C(x_0, \|(\mathbf{v}, \omega)\|)$ such that

$$\|\mathcal{F}_{1,5,u}(x) - u_{\mathbf{a}_5}(x)\|_\infty \leq C(x_0, \|(\mathbf{v}, \omega)\|)\langle x\rangle^{-\frac{5}{4}+\varepsilon\varphi}$$

for all $x \geq 2x_0$.

Proof. We first note that Theorem 5.1 implies that $\|P - P_{\mathbf{a}}\|_{p,2-(1+\varepsilon)\varphi-\frac{1}{2p}} \leq C(x_0, \|(\mathbf{v}, \omega)\|)$ for all $1 \leq p \leq \infty$ and $\varepsilon > 0$, so that for $p = \frac{1}{1-\varepsilon\varphi}$, we have

$$\begin{aligned}\Big\| \int_{\frac{x+x_0}{2}}^{x} \mathrm{d}\tilde{x}\; K_2(x-\tilde{x})(P(\tilde{x}) - P_{\mathbf{a}}(\tilde{x})) \Big\|_\infty &\leq C\|(\mathbf{v}, \omega)\|^2 \int_{\frac{x+x_0}{2}}^{x} \frac{\mathrm{d}\tilde{x}\; \langle \tilde{x}\rangle^{-\frac{3}{2}+(1+\frac{3\varepsilon}{2})\varphi}}{(x-\tilde{x})^{1-\frac{\varepsilon\varphi}{2}}} \\ &\leq C\langle x\rangle^{-\frac{3}{2}+(1+2\varepsilon)\varphi}\|(\mathbf{v}, \omega)\|^2 \,.\end{aligned}$$

Then, we note that

$$P(x) - P_{\mathbf{a}}(x) = \partial_x R + \tfrac{1}{2}\partial_y(v(x)^2) - \tfrac{1}{2}\partial_y((u(x) - u_{\mathbf{a}_1}(x))(u(x) + u_{\mathbf{a}_1}(x)) \,.$$

Now, let $T_1(x) = \int_{x_0}^{\frac{x+x_0}{2}} \mathrm{d}\tilde{x}\; K_2(x-\tilde{x})\partial_{\tilde{x}} R(\tilde{x}) + K_2(x - x_0)R(x_0)$. By (1.17) and $\|\partial_x K_2\|_\infty \leq \|\partial_y K_8(x - \tilde{x})\|_\infty + \|\partial_y K_{10}(x - \tilde{x})\|_\infty$, integrating by parts, we find

$$\|T_1(x)\|_\infty \leq C\|(\mathbf{v}, \omega)\|^2 \langle x\rangle^{-\frac{3}{2}+\varphi}\Big(1 + \langle x_0\rangle^{\varphi}\Big) \,,$$

while Lemma 5.3 and A.10, together with $\| |y|^{\frac{1}{2}-\varepsilon\varphi} R(x_0)\|_1 \leq \| |y|(u(x_0)^2 + v(x_0)^2\|_2 \leq C\|(\mathbf{v}, \omega)\|^2$ show that

$$\|K_2(x - x_0)R(x_0) + u_{\mathbf{a}_5}(x - x_0)\|_\infty \leq C\|(\mathbf{v}, \omega)\|^2 \langle x\rangle^{-\frac{5}{4}+\varepsilon\varphi} \,.$$

Then, let $S_1(x) = v(x)^2$ and $S_2(x) = (u(x) - u_{\mathbf{a}_1}(x))(u(x) + u_{\mathbf{a}_1}(x))$. By Theorem 5.1, we have $\|S_2(x)\|_2 \leq C\langle x\rangle^{-1+\varepsilon\varphi}\|(\mathbf{v}, \omega)\|^2$ and $\|S_1(x)\|_1 \leq C\langle x\rangle^{-1+2\varphi}\|(\mathbf{v}, \omega)\|^2$. Therefore, for $x \geq 2x_0$, we have

$$\begin{aligned}\Big\| \int_{x_0}^{\frac{x+x_0}{2}} \mathrm{d}\tilde{x}\; \partial_y K_2(x-\tilde{x}) S_1(\tilde{x}) \Big\| &\leq C\langle x\rangle^{-\frac{3}{2}+2\varphi}\|(\mathbf{v}, \omega)\|^2 \\ \Big\| \int_{x_0}^{\frac{x+x_0}{2}} \mathrm{d}\tilde{x}\; \partial_y K_2(x-\tilde{x}) S_2(\tilde{x}) \Big\| &\leq C\langle x\rangle^{-\frac{5}{4}+\varepsilon\varphi}\|(\mathbf{v}, \omega)\|^2 \,.\end{aligned}$$

We then define $P_6(x) = \mathcal{P}_0 \int_{x_0}^{x} \mathrm{d}\tilde{x}\; (K_2(x-\tilde{x}) - \partial_y K_c(x-\tilde{x}))P_{\mathbf{a}}(\tilde{x})$ and we get

$$\|P_6(x)\|_\infty \leq C\|(\mathbf{v}, \omega)\|^2 \int_{x_0}^{x} \mathrm{d}\tilde{x} \int_{-\infty}^{\infty} \mathrm{d}k\; \mathcal{P}_0 e^{\Lambda_-(x-\tilde{x}) - \frac{k^2}{2}\tilde{x}}(|k|^5(x-\tilde{x}) + |k|^3)\tilde{x}^{-\frac{1}{2}}$$

$$\leq C\|(\mathbf{v},\omega)\|^2 \int_{x_0}^{x} \mathrm{d}\tilde{x} \min\Big(\frac{\langle x-\tilde{x}\rangle^3}{(x-\tilde{x})^5}, \frac{x-\tilde{x}}{\tilde{x}^3} + \frac{1}{\tilde{x}^2}\Big)\tilde{x}^{-\frac{1}{2}} \leq C\|(\mathbf{v},\omega)\|^2 \langle x\rangle^{-\frac{3}{2}} .$$

It remains to establish the asymptotic comportment of

$$T(x) = \int_{x_0}^{x} \mathrm{d}\tilde{x}\ \partial_y K_c(x-\tilde{x}) P_{\mathbf{a}}(\tilde{x}) + 2\mathcal{I} P_{\mathbf{a}}(x) .$$

We first note that $T(x)$ is conveniently computed in terms of its Fourier transform, which reads

$$\begin{aligned} \hat{T}(x,k) &= \tfrac{ika_1^2}{4}\mathrm{erf}(\tfrac{ik\sqrt{x_0}}{\sqrt{2}})e^{-k^2x} - \tfrac{ika_1^2}{4}\mathrm{erf}(\tfrac{ik\sqrt{x}}{\sqrt{2}})e^{-k^2x} - \tfrac{a_1^2 e^{-\frac{k^2x}{2}}}{2\sqrt{2\pi x}} , \\ &= \hat{T}_2(x,x_0,k) + x^{-\frac{1}{2}} H(k\sqrt{x}) . \end{aligned}$$

For $T_2(x,x_0,y)$, we note that for $x \geq x_0$, we have

$$\|T_2(x)\|_\infty \leq \int_{-\infty}^{\infty} \mathrm{d}k\ |\tfrac{ika_1^2}{4}\mathrm{erf}(\tfrac{ik\sqrt{x_0}}{\sqrt{2}})|e^{-k^2x} \leq a_1^2 \int_{-\infty}^{\infty}\mathrm{d}k\ k^2 e^{-\frac{k^2x}{2}} \leq C\|(\mathbf{v},\omega)\|^2\langle x\rangle^{-\frac{3}{2}} .$$

To complete the proof, we only have to prove that the inverse Fourier transform of $\hat{H}$ is $-\frac{a_1^2 h}{2}$. To do so, we note that $\hat{H}(k)$ satisfies

$$k\hat{H}'(k) + (2k^2-1)\hat{H}(k) - \tfrac{a_1^2 e^{-\frac{k^2}{2}}}{2\sqrt{2\pi}} = 0 , \quad \int_{-\infty}^{\infty}\mathrm{d}k\ \hat{H}(k) = -\tfrac{a_1^2}{4} , \quad \int_{-\infty}^{\infty}\mathrm{d}k\ k\hat{H}(k) = 0 ,$$

which after inverse Fourier transform leads to

$$H''(y) + \tfrac{y}{2}H'(y) + H(y) + \tfrac{a_1^2 e^{-\frac{y^2}{2}}}{8\pi} = 0 , \quad H(0) = -\tfrac{a_1^2}{8\pi} , \quad H'(0) = 0 ,$$

whose unique solution is $H(y) = -\frac{a_1^2 h(y)}{2}$. ■

6.3 Nonlinear terms, second round

In view of Remark 5.5 and the corresponding theory on nonlinear heat equations, (see e.g. [3]), we may guess that the decay rates of Proposition 5.4 on $\mathcal{F}_{1,2,u}$ would be improved using higher moments of Q, i.e. after substraction of $u_{\mathbf{a}_0}$ with $\mathbf{a}_0 = (\mathcal{P}_0 \int_{\Omega_+} Q(x,y)\,\mathrm{d}x\mathrm{d}y, 0, 0, \mathcal{P}_0 \int_{\Omega_+} yQ(x,y)\,\mathrm{d}x\mathrm{d}y, 0, 0)$. This is wrong since the first moment $\int_{\Omega_+} yQ(x,y)\,\mathrm{d}x\mathrm{d}y$ of Q is infinite in general[5]. However, with the estimates obtained so far on $v - v_{\mathbf{a}}$ and $\omega - \omega_{\mathbf{a}}$, we can show that higher moments are well defined for $Q - Q_{\mathbf{a}}$ as shows the

Lemma 6.5 *Let $Q_{\mathbf{a}} = v_{\mathbf{a}}\omega_{\mathbf{a}}$ where $v_{\mathbf{a}}$ and $\omega_{\mathbf{a}}$ are defined in (1.9) and $\mathbf{a} = (a_1,a_2,a_3,0,0,0)$. Then for all $\varepsilon > 0$, we have*

$$\begin{aligned} \|Q - Q_{\mathbf{a}}\|_{\infty,\frac{5}{2}-(1+\varepsilon)\varphi} + \|Q - Q_{\mathbf{a}}\|_{1,2-(1+\varepsilon)\varphi} &\leq C\|(\mathbf{v},\omega)\|^2 \\ \||y|^\gamma(Q - Q_{\mathbf{a}})\|_{1,\frac{9}{4}-\gamma-2\varphi(1+\frac{3\varepsilon}{4})} &\leq C\|(\mathbf{v},\omega)\|^2 \end{aligned} \quad (6.8)$$

for all $\frac{1}{2} \leq \gamma \leq \frac{5}{4} - 2\varphi(1+\varepsilon)$.

[5] except for symmetric flows where $\int_{\mathbf{R}} yQ(x,y)\,\mathrm{d}y = 0$

Proof. The estimate (6.8) follows at once from the boundedness of $\|(v-v_{\mathbf{a}})\|_{\infty,\frac{3}{2}-(1+\varepsilon)\varphi}$ and Theorem 5.1. Now let $\frac{1}{2}\leq\gamma\leq\frac{5}{4}-2\varphi(1+\varepsilon)$ and define $\varepsilon_1=1-\frac{1}{2\gamma}(1-(4+\varepsilon)\varphi)$ and $\beta_0=(1-\varepsilon_1)\gamma+\frac{1+\varepsilon}{2}\varphi$. By hypothesis on γ, we have $\gamma\varepsilon_1\leq 1$ and $0\leq\beta_0\leq 1-2(1+\varepsilon)\varphi$, so that

$$\begin{aligned}\||y|^{\gamma}(Q(x)-Q_{\mathbf{a}}(x))\|_1 &\leq \|v(x)-v_{\mathbf{a}}(x)\|_\infty\||y|^{\gamma}\omega(x)\|_1+\||y|^{\varepsilon_1\gamma}v_{\mathbf{a}}(x)\|_\infty\||y|^{\beta_0}(\omega(x)-\omega_{\mathbf{a}}(x))\|_2\\ &\leq C\|(\mathbf{v},\omega)\|^2\Big(\langle x\rangle^{-2+\frac{\gamma}{2}+2\varphi(1+\frac{3\varepsilon}{4})}+\langle x\rangle^{-\frac{9}{4}+\gamma+2\varphi(1+\frac{3\varepsilon}{4})}\Big)\,.\end{aligned}$$

This completes the proof since $\gamma\geq\frac{1}{2}$. ■

We can now conclude the proof of Theorem 6.1 by proving the following

Proposition 6.6 *Assume that Q satisfies (1.17), let $\mathbf{a}_6=(\int_{\Omega_+}\mathcal{P}_0Q(x,y)\,\mathrm{d}x\mathrm{d}y,0,0,a_{4,2},0,a_1\mathcal{P}_0a_3)$, $Q_{\mathbf{a}}$ as in Lemma 6.5. Then there exist $a_{4,2}\in\mathbf{R}$ such that*

$$\|\mathcal{F}_{1,6,u}(x)-u_{\mathbf{a}_6}(x)\|_\infty\leq C(x_0,\|(\mathbf{v},\omega)\|)\,\langle x\rangle^{-\frac{9}{8}+(1+\varepsilon)\varphi}\,.$$

for some constant $C=C(x_0,\|(\mathbf{v},\omega)\|)$.

Proof. We first note that we can write

$$\begin{aligned}\mathcal{P}_0Q_{\mathbf{a}}(x,y) &= \tfrac{a_1}{x}f_1(\tfrac{y}{\sqrt{x}})\Big(\tfrac{a_1}{x}f_1(\tfrac{y}{\sqrt{x}})+\tfrac{b}{x}g_0(\tfrac{y}{x})+\tfrac{c}{x}g_1(\tfrac{y}{x})\Big)\,,\\ &= \underbrace{\tfrac{a_1^2}{x^2}f_1(\tfrac{y}{\sqrt{x}})^2+\tfrac{a_1b}{x^2}f_1(\tfrac{y}{\sqrt{x}})}_{\equiv Q_{\mathbf{a},1}(x,y)}+\underbrace{\tfrac{a_1c}{x^{\frac{5}{2}}}f_2(\tfrac{y}{\sqrt{x}})g_0(\tfrac{y}{x})-\tfrac{a_1b}{x^3}f_3(\tfrac{y}{\sqrt{x}})g_0(\tfrac{y}{x})}_{\equiv Q_{\mathbf{a},2}(x,y)}\end{aligned}$$

where $f_m(z)=\frac{z^me^{-\frac{z^2}{4}}}{4\sqrt{\pi}}$, $g_m(z)=\frac{z^m}{1+z^2}$, $b=\mathcal{P}_0a_3$ and $c=\mathcal{P}_0a_2$. Now, since $|g_0(z)|\leq 1$, $Q_{\mathbf{a},2}$ satisfies the same estimates as $Q(x)-Q_{\mathbf{a}}(x)$ (with even better decay rates). To exploit this, we define $\Delta Q(x)=\mathcal{P}_0(Q(x)-Q_{\mathbf{a}}(x))+Q_{\mathbf{a},2}(x)$ and

$$\begin{aligned}T_{3,1}(x) &= -\int_{\frac{x+x_0}{2}}^{x}\mathrm{d}\tilde{x}\;K_{12}(x-\tilde{x})\Delta Q(\tilde{x})\,,\\ T_{3,2}(x) &= -\int_{x_0}^{\frac{x+x_0}{2}}\mathrm{d}\tilde{x}\;\Big(K_{12}(x-\tilde{x})(\Delta Q(\tilde{x})-\mathcal{M}(\Delta Q(\tilde{x})))-\partial_yK_{12}(x-\tilde{x})\mathcal{M}(y\Delta Q(\tilde{x}))\Big)\,.\end{aligned}$$

Using Lemma 5.3 and 6.5, as well as $x\geq 2x_0$, we get

$$\begin{aligned}\|T_{3,1}(x)\|_\infty &\leq C\,x\sup_{\xi\geq\frac{x+x_0}{2}}\|\Delta Q(\xi)\|_\infty\leq C\langle x\rangle^{-\frac{3}{2}+(1+\varepsilon)\varphi}\|(\mathbf{v},\omega)\|^2\,,\\ \|T_{3,2}(x)\|_\infty &\leq C\langle x\rangle^{-\frac{9}{8}+(1+\varepsilon)\varphi}\int_{x_0}^{\frac{x+x_0}{2}}\mathrm{d}\tilde{x}\;\||y|^{\frac{5}{4}-2\varphi(1+\varepsilon)}\Delta Q(\tilde{x})\|_1\\ &\leq C\langle x\rangle^{-\frac{9}{8}+(1+\varepsilon)\varphi}\|(\mathbf{v},\omega)\|^2\int_{x_0}^{\infty}\mathrm{d}\tilde{x}\;\langle\tilde{x}\rangle^{-1-\frac{\varepsilon\varphi}{2}}\,.\end{aligned}$$

We then define

$$T_{3,3}(x)=K_{12}(x-x_0)\int_{x_0}^{\frac{x+x_0}{2}}\mathrm{d}\tilde{x}\;\mathcal{M}(\Delta Q(\tilde{x}))-\int_{x_0}^{\frac{x+x_0}{2}}\mathrm{d}\tilde{x}\;K_{12}(x-\tilde{x})\mathcal{M}(\Delta Q(\tilde{x}))\,,$$

$$T_{3,4}(x) = \partial_y K_{12}(x-x_0)\int_{x_0}^{\frac{x+x_0}{2}} \mathrm{d}\tilde{x}\; \mathcal{M}(y\Delta Q(\tilde{x})) - \int_{x_0}^{\frac{x+x_0}{2}} \mathrm{d}\tilde{x}\; \partial_y K_{12}(x-\tilde{x})\mathcal{M}(y\Delta Q(\tilde{x}))\;,$$

and note that after integration by parts, using $\|\partial_x K_{12}(x)\|_\infty \le \|K_8(x)\|_\infty + \|K_{10}(x)\|_\infty \le C\langle x\rangle^{-\frac{3}{2}}\langle x_0\rangle^{\varphi}$ and $\|\partial_x\partial_y K_{12}(x)\|_\infty \le \|\partial_y K_8(x)\|_\infty + \|\partial_y K_{10}(x)\|_\infty \le C\langle x\rangle^{-2}\langle x_0\rangle^{\varphi}$ if $x>0$, we get

$$\|T_{3,3}(x)\|_\infty \le C\|(\mathbf{v},\omega)\|^2\langle x\rangle^{-\frac{3}{2}+\varphi}\int_{x_0}^{\frac{x+x_0}{2}}\mathrm{d}\tilde{x}\int_{\tilde{x}}^{\frac{x+x_0}{2}}\mathrm{d}z\;\langle z\rangle^{-2+(1+\varepsilon)\varphi} \le C\|(\mathbf{v},\omega)\|^2\langle x\rangle^{-\frac{3}{2}+(2+\varepsilon)\varphi}$$
$$\|T_{3,4}(x)\|_\infty \le C\|(\mathbf{v},\omega)\|^2\langle x\rangle^{-2+\varphi}\int_{x_0}^{\frac{x+x_0}{2}}\mathrm{d}\tilde{x}\int_{\tilde{x}}^{\frac{x+x_0}{2}}\mathrm{d}z\;\langle z\rangle^{-\frac{3}{2}+(1+\varepsilon)\varphi} \le C\|(\mathbf{v},\omega)\|^2\langle x\rangle^{-\frac{3}{2}+(2+\varepsilon)\varphi}\;,$$

while also for $x\ge 2x_0$, we have

$$\begin{aligned}\Big\|K_{12}(x-x_0)\int_{\frac{x+x_0}{2}}^{\infty}\mathrm{d}\tilde{x}\;\mathcal{M}(\Delta Q(\tilde{x}))\Big\|_\infty &\le C\|(\mathbf{v},\omega)\|^2\langle x\rangle^{-\frac{1}{2}}\int_{\frac{x+x_0}{2}}^{\infty}\mathrm{d}\tilde{x}\;\langle\tilde{x}\rangle^{-2+(1+\varepsilon)\varphi}\;,\\ &\le C\|(\mathbf{v},\omega)\|^2\langle x\rangle^{-\frac{3}{2}+(1+\varepsilon)\varphi}\\ \Big\|\partial_y K_{12}(x-x_0)\int_{\frac{x+x_0}{2}}^{\infty}\mathrm{d}\tilde{x}\;\mathcal{M}(y\Delta Q(\tilde{x}))\Big\|_\infty &\le C\|(\mathbf{v},\omega)\|^2\langle x\rangle^{-1}\int_{\frac{x+x_0}{2}}^{\infty}\mathrm{d}\tilde{x}\;\langle\tilde{x}\rangle^{-\frac{5}{4}+2(1+\frac{3\varepsilon}{2})\varphi}\;,\\ &\le C\|(\mathbf{v},\omega)\|^2\langle x\rangle^{-\frac{5}{4}+2(1+\frac{3\varepsilon}{2})\varphi}\;.\end{aligned}$$

Now, let $a_{1,3}=\int_{x_0}^{\infty}\mathrm{d}x\;\mathcal{M}(\Delta Q(x))$ and $a_{4,3}=\int_{x_0}^{\infty}\mathrm{d}x\;\mathcal{M}(y\Delta Q(x))$. As is easily shown using Lemma 6.5, $a_{1,3}$ and $a_{4,3}$ are bounded, and using Lemma A.10, we have for $x\ge 2x_0$

$$\|(K_{12}(x-x_0)-K_c(x-x_0))a_{1,3}\|_\infty \le C\langle x\rangle^{-\frac{3}{2}+\varphi}\|(\mathbf{v},\omega)\|^2\;,$$
$$\|\partial_y(K_{12}(x-x_0)-K_c(x-x_0))a_{4,3}\|_\infty \le C\langle x\rangle^{-\frac{3}{2}+\varphi}\|(\mathbf{v},\omega)\|^2\;.$$

for some constant C possibly depending on x_0. After collecting the results obtained so far, and using

$$\|(K_c(x-x_0)-K_c(x))a_{1,3}\|_\infty \le C\langle x\rangle^{-\frac{3}{2}+\varphi}\langle x_0\rangle\|(\mathbf{v},\omega)\|^2\;,$$
$$\|\partial_y(K_c(x-x_0)-K_c(x))a_{4,3}\|_\infty \le C\langle x\rangle^{-\frac{3}{2}+\varphi}\langle x_0\rangle\|(\mathbf{v},\omega)\|^2\;,$$

we get for $\mathbf{a}_7=(a_{1,3},0,0,a_{4,3},0,0)$ that

$$\Big\|\int_{x_0}^{x}\mathrm{d}\tilde{x}\;K_{12}(x-\tilde{x})\Delta Q(\tilde{x}) - u_{\mathbf{a}_7}(x)\Big\|_\infty \le C\|(\mathbf{v},\omega)\|^2\langle x\rangle^{-\frac{5}{4}+2(1+\varepsilon)\varphi}\;.$$

In other words, since $\mathcal{P}_0 Q = \Delta Q + \mathcal{P}_0 Q_{\mathbf{a},1}$, it only remains to establish the asymptotic comportment of

$$T_4(x) = -\int_{x_0}^{x}\mathrm{d}\tilde{x}\;\mathcal{P}_0 K_{12}(x-\tilde{x})Q_{\mathbf{a},1}(\tilde{x})\;.$$

To do so, we first define

$$T_5(x) = -\int_{x_0}^{x}\mathrm{d}\tilde{x}\;\mathcal{P}_0(K_{12}(x-\tilde{x})+K_c(x-\tilde{x}))Q_{\mathbf{a},1}(\tilde{x})\;,$$

on which we get

$$\|T_5(x)\|_\infty \leq C(x_0, \|(\mathbf{v},\omega)\|) \int_{x_0}^{x} \mathrm{d}\tilde{x} \int_{-\infty}^{\infty} \mathrm{d}k \, \mathcal{P}_0 e^{\Lambda - (x-\tilde{x}) - \frac{k^2\tilde{x}}{4}} \Big(k^2 + k^4(x-\tilde{x}) \Big) \tilde{x}^{-\frac{3}{2}}$$
$$\leq C(x_0, \|(\mathbf{v},\omega)\|) \int_{x_0}^{x} \mathrm{d}\tilde{x} \min \Big(\frac{\langle x-\tilde{x}\rangle^{\frac{5}{2}}}{(x-\tilde{x})^4} \, , \, \frac{1}{\tilde{x}^{\frac{3}{2}}} + \frac{x-\tilde{x}}{\tilde{x}^{\frac{5}{2}}} \Big) \tilde{x}^{-\frac{3}{2}} \leq C \|(\mathbf{v},\omega)\|^2 \langle x \rangle^{-\frac{3}{2}} \, .$$

We finally define

$$T_6(x) = \int_{x_0}^{x} \mathrm{d}\tilde{x} \; K_c(x-\tilde{x}) Q_{\mathbf{a},1}(\tilde{x}) \, .$$

As in Proposition 6.4, T_6 is conveniently computed in terms of its Fourier transform, which reads

$$\hat{D}_6(x,k) = a_1 b(\ln(x) - \ln(x_0)) ike^{-k^2 x} + \frac{a_1^2 e^{-k^2(x-\frac{x_0}{2})}}{4\sqrt{2\pi x_0}} - \frac{a_1^2 e^{-\frac{k^2 x}{2}}}{4\sqrt{2\pi x}} \, ,$$
$$= a_1 b(\ln(x) - \ln(x_0)) ike^{-k^2 x} + \frac{a_1^2 e^{-k^2 x}}{4\sqrt{2\pi x_0}} - \frac{a_1^2 e^{-\frac{k^2 x}{2}}}{4\sqrt{2\pi x}} + \frac{a_1^2 e^{-k^2(x-\frac{x_0}{2})}(1 - e^{-\frac{k^2 x_0}{2}})}{4\sqrt{2\pi x_0}} \, ,$$

from which we get finally

$$T_6(x,y) = a_1 b(\ln(x_0) - \ln(x)) \partial_y K_c(x,y) + \tfrac{a_1^2 K_c(x,y)}{4\sqrt{\pi x_0}} + \mathcal{I}P_{\mathbf{a}}(x,y) + R(x,y) \, ,$$

with

$$\|R(x)\|_\infty \leq C a^2 \sqrt{x_0} \int_{-\infty}^{\infty} \mathrm{d}k \; k^2 e^{-\frac{k^2 x}{2}} \leq C \|(\mathbf{v},\omega)\|^2 \langle x \rangle^{-\frac{3}{2}} \, .$$

We thus have proved that

$$\|\mathcal{F}_{1,6,u}(x) - u_{\mathbf{a}_8}(x)\|_\infty \leq \langle x \rangle^{-\frac{9}{8} + (1+\varepsilon)\varphi} \|(\mathbf{v},\omega)\|^2 \, ,$$

where $\mathbf{a}_8 = (a_{1,3} + \frac{a_1^2}{4\sqrt{\pi x_0}}, 0, 0, a_{4,3} + a_1 b \ln(x_0), 0, a_1 \mathcal{P}_0 a_3)$. It then follows by simple comparison with the result of Proposition 5.4 that $\mathbf{a}_8 = \mathbf{a}_6$ as claimed. ■

7 Estimates on the boundary data

In this section, we complete the proof of Theorem 1.4, which is

Theorem 7.1 *If x_0 is sufficiently large and there exist a unique solution to (1.1) in $\mathcal{C}_u$ with parameters satisfying (1.16), then ν and w are in the class $\mathcal{C}_i$ with parameters satisfying (1.16). If furthermore $\||y|^{\frac{1}{2}} \mathbf{v}(x_0)\|_4 + \||y|^{\frac{1}{2}-(1+\varepsilon)\varphi} \mathcal{S}\mathbf{v}(x_0)\|_1 \leq C\|(\mathbf{v},\omega)\|$, then for all $\varepsilon > 0$, it holds*

$$\||y|^{\frac{1}{2}-(1+\varepsilon)\varphi} \mathcal{S}\nu\|_1 + \||y|^{\frac{1}{2}-(1+\varepsilon)\varphi} \mathcal{S}\mu\|_1 \leq C_1(x_0, \|(\mathbf{v},\omega)\|) \, . \tag{7.1}$$

Proof. The functions ν and w are determined by the evaluation of (1.12)-(1.14) at $x = x_0$, which gives

$$\mathcal{L}_u w + \nu = u(x_0) - \mathcal{F}_{2,u}(x_0) + \mathcal{L}_1 S(x_0) - \mathcal{L}_2 R(x_0) \tag{7.2}$$
$$\mathcal{L}_v w + \mu = v(x_0) - \mathcal{F}_{2,v}(x_0) - \mathcal{L}_1 R(x_0) - \mathcal{L}_2 S(x_0) \tag{7.3}$$
$$w = \omega(x_0) - \mathcal{F}_{2,\omega}(x_0) . \tag{7.4}$$

Denote by (U, V, W) the r.h.s. of (7.2)-(7.4). By Propositions 3.7, 3.14, 3.15 and 3.16, (U, V, W) are well defined and $\|(U, V, W)\| \leq \|(\mathbf{v}, \omega)\|_{x_0} + C\langle x_0\rangle^{-\kappa}\|(\mathbf{v}, \omega)\|^2$ for $\kappa = \min(\kappa_0, \kappa_2)$. Note that unsurprisingly (the stationary Navier-Stokes system is elliptic), the system (7.2)-(7.4) is overdetermined. Nevertheless, since we know that the solution exists, the three relations have to be satisfied. We now use this as an extra freedom to derive properties on ν and w. We first note that using Propositions 3.7, 3.14, 3.15 and 3.16, we get

$$\|(\mathcal{L}_u w + \nu, \mathcal{L}_v w + \mu, w)\|_{x_0} \leq \|(\mathbf{v}, \omega)\|_{x_0} + C\langle x_0\rangle^{-\kappa}\|(\mathbf{v}, \omega)\|^2 \tag{7.5}$$

for some $\kappa > 0$, since (U, V, W) satisfies this estimate. In particular, it implies at once that

$$\|(0, 0, w)\|_{x_0} \leq \|(\mathbf{v}, \omega)\|_{x_0} + C\langle x_0\rangle^{-\kappa}\|(\mathbf{v}, \omega)\|^2 .$$

Then, by interpolation, we have

$$\langle x_0\rangle^{\frac{1}{2}-\frac{1}{2p}}\|\tilde{\mathcal{L}}_u w\|_{\mathrm{L}^p} \leq \|\tilde{\mathcal{L}}_u w\|_{\mathrm{L}^1} + \langle x_0\rangle^{\frac{1}{2}}\|\tilde{\mathcal{L}}_u w\|_{\mathrm{L}^\infty}$$
$$\langle x_0\rangle^{1-\frac{1}{2p}-\varphi}\|\mathcal{L}_v w\|_{\mathrm{L}^p} \leq \langle x_0\rangle^{\frac{1}{2}-\varphi}\|\mathcal{L}_v w\|_{\mathrm{L}^1} + \langle x_0\rangle^{1-\varphi}\|\mathcal{L}_v w\|_{\mathrm{L}^\infty} ,$$

where $\tilde{\mathcal{L}_u} = \mathcal{L}_u + \mathcal{I}\mathcal{P}_0$. Using these inequalities, $-\frac{1}{p} \leq -\frac{1}{2p}$ and Lemma 3.5, we get

$$\|(\tilde{\mathcal{L}}_u w, \mathcal{L}_v w, 0)\|_{x_0} \leq C_1\|(0, 0, w)\|_{x_0} ,$$

so that from (7.5), we get

$$\|(\nu - \mathcal{I}\mathcal{P}_0 w, \mu, w)\|_{x_0} \leq (1 + C_1)(\|(\mathbf{v}, \omega)\|_{x_0} + C\langle x_0\rangle^{-\kappa}\|(\mathbf{v}, \omega)\|^2) . \tag{7.6}$$

In particular, this implies that $\mu \in \mathrm{L}^p \cap \mathrm{L}^\infty$ and $\partial_y \mu \in \mathrm{L}^r$, which gives $\nu \in \mathrm{L}^p \cap \mathrm{L}^\infty$ using $\nu = -\mathcal{H}\mu$ (see Lemma 7.2 below). Since $q \geq p$, we get $\nu \in \mathrm{L}^q$, and then (7.6) also implies that $\mathcal{I}\mathcal{P}_0 w \in \mathrm{L}^q$ (because $\nu \in \mathrm{L}^q$ and $\nu - \mathcal{I}\mathcal{P}_0 w \in \mathrm{L}^q$). Thus $\mathcal{I}\mathcal{P}_0 w$ has to decay as $|y| \to \infty$, though maybe only in a weak sense. On the other hand, from the definition of $\mathcal{I}$ (see (1.15)), we have $\lim_{y\to\pm\infty} \mathcal{I}\mathcal{P}_0 w(y) = \pm\mathcal{M}(\mathcal{P}_0 w)$ (the limit exists since $(1 + |y|^\beta)\omega \in \mathrm{L}^2$ implies $w \in \mathrm{L}^1$). This is compatible with $\mathcal{I}\mathcal{P}_0 w \in \mathrm{L}^q$ only if $\mathcal{M}(\mathcal{P}_0 w)$ vanishes. We can thus use Lemma 3.2 and get that

$$\|\mathcal{I}\mathcal{P}_0 w\|_{\mathrm{L}^1} \leq C(\langle x_0\rangle^{\frac{3}{4}}\|w\|_{\mathrm{L}^2})^{1-\frac{3}{2\beta}}((\langle x_0\rangle^{\frac{3}{4}-\frac{\beta}{2}}\||y|^\beta w\|_{\mathrm{L}^2})^{\frac{3}{2\beta}} \leq C\|(0, 0, w)\|_{x_0} .$$

Using again Lemma 3.5, we thus get

$$\|(\mathcal{L}_u w, \mathcal{L}_v w, 0)\|_{x_0} \leq C_2\|(0, 0, w)\|_{x_0} ,$$

so that again from (7.5), we get

$$\|(\nu, \mu, w)\|_{x_0} \leq (1 + C_2)(\|(\mathbf{v}, \omega)\|_{x_0} + C\langle x_0\rangle^{-\kappa}\|(\mathbf{v}, \omega)\|^2) . \tag{7.7}$$

To complete the proof of the first part of Theorem 1.4, we still have to prove that (7.1) holds. This is done in Proposition 7.3 below. ■

Lemma 7.2 *Let $p, q > 1$. There exist a constant $C_{p,q}$ such that for all f satisfying $(f, \partial_y f) \in \mathrm{L}^p \cap \mathrm{L}^\infty \times \mathrm{L}^q$, we have $(\mathcal{H}f, \partial_y \mathcal{H}f) \in \mathrm{L}^p \cap \mathrm{L}^\infty \times \mathrm{L}^q$ and $\|\mathcal{H}f\|_{\mathrm{L}^\infty} \leq C_{p,q}(\|f\|_{\mathrm{L}^p} + \|\partial_y f\|_{\mathrm{L}^q})$.*

Proof. Note that $\mathcal{H}f \in \mathrm{L}^p$ and $\partial_y \mathcal{H}f \in \mathrm{L}^q$ for $1 < p, q < \infty$ is a classical result which follows from Lemma 3.4 (see page 15). Then, if $q' \equiv \frac{q}{q-1} \geq p$, the L^∞ estimate follows from $\|\mathcal{H}f\|_{\mathrm{L}^\infty} \leq (\|\mathcal{H}f\|_{q'}\|\partial_y \mathcal{H}f\|_q)^{\frac{1}{2}} \leq C(\|f\|_{q'}\|\partial_y f\|_q)^{\frac{1}{2}}$. However the $q' \geq p$ restriction is not essential: using the Cauchy-Schwartz inequality and integration by parts, we have

$$
\begin{aligned}
|\mathcal{H}f(y)| &= \left| \lim_{\varepsilon \to 0} \int_{|z| \geq \varepsilon} \frac{f(y-z)}{z} \mathrm{d}z \right| \leq C\|f\|_{\mathrm{L}^p} + \left| \lim_{\varepsilon \to 0} \int_{\varepsilon \leq |z| \leq 1} \frac{f(y-z)}{z} \mathrm{d}z \right| \\
&\leq C\|f\|_{\mathrm{L}^p} + \lim_{\varepsilon \to 0} \left| \ln(\varepsilon) \int_{y-\varepsilon}^{y+\varepsilon} \partial_z f(z) \mathrm{d}z \right| + \left| \int_{-1}^{1} \ln|z| \partial_y f(y-z) \mathrm{d}z \right| \\
&\leq C_{p,q}(\|f\|_{\mathrm{L}^p} + \|\partial_y f\|_{\mathrm{L}^q}) + \|\partial_y f\|_{\mathrm{L}^q} \lim_{\varepsilon \to 0} (2\varepsilon)^{1-\frac{1}{q}} |\ln(\varepsilon)| \ .
\end{aligned}
$$

This completes the proof. ■

Proposition 7.3 *Assume that $\||y|^{\frac{1}{2}}\mathbf{v}(x_0)\|_4 + \||y|^{\frac{1}{2}-(1+\varepsilon)\varphi}\mathcal{S}\mathbf{v}(x_0)\|_1 \leq C\|(\mathbf{v}, \omega)\|$, then for all $\varepsilon > 0$, it holds*

$$
\||y|^{\frac{1}{2}-(1+\varepsilon)\varphi}\mathcal{S}\nu\|_1 + \||y|^{\frac{1}{2}-(1+\varepsilon)\varphi}\mathcal{S}\mu\|_1 \leq C(x_0, \|(\mathbf{v}, \omega)\|) \ .
$$

Proof. In this proof, we will use repeatedly that $\||y|^a f\|_p \leq \|f\|_p^{1-a}\||y|f\|_p^a$ for all $p \geq 1$ and $0 \leq a \leq 1$, as well as $\||y|^{\frac{1}{2}-(1+\varepsilon)\varphi} f\|_1 \leq \||y|^{1-(1+\frac{\varepsilon}{2})\varphi} f\|_2$ or $\||y|^{\frac{1}{2}-(1+\varepsilon)\varphi} f\|_1 \leq \||y|f\|_2$. We first note that by Lemma 3.5 and 3.2, (using also that the symbols $\tilde{\mathcal{L}}_u$ and $\mathcal{L}_v$, together with their derivatives w.r.t. the Fourier variable 'k' are bounded), we have

$$
\begin{aligned}
\||y|^{\frac{1}{2}-(1+\varepsilon)\varphi}\mathcal{L}_u w\|_1 &\leq \||y|^{\frac{1}{2}-(1+\varepsilon)\varphi}\mathcal{I}\mathcal{P}_0 w\|_1 + \||y|\tilde{\mathcal{L}}_u w\|_2 \leq C\|(\mathbf{v}, \omega)\| \ , \\
\||y|^{\frac{1}{2}-(1+\varepsilon)\varphi}\mathcal{L}_v w\|_1 &\leq \||y|\mathcal{L}_v w\|_2 \leq C\|(\mathbf{v}, \omega)\| \ .
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\||y|^{\frac{1}{2}-\varepsilon}\mathcal{L}_1 S\|_1 &\leq \|y(\mathcal{L}_1 - \mathbb{1})\|_2\|S\|_1 + (1 + \|(\mathcal{L}_1 - \mathbb{1})\|_1)\|yS\|_2 \\
&\leq C\langle x_0\rangle^\varphi \|(\mathbf{v}, \omega)\|^2 + \||y|^{\frac{1}{2}}u(x_0)\|_4^2 + \||y|^{\frac{1}{2}}v(x_0)\|_4^2 \\
\||y|^{\frac{1}{2}-\varepsilon}\mathcal{L}_2 S\|_1 &\leq \|y\mathcal{L}_2\|_2\|S\|_1 + \|\mathcal{L}_2\|_1\|yS\|_2 \\
&\leq C\langle x_0\rangle^\varphi \|(\mathbf{v}, \omega)\|^2 + \||y|^{\frac{1}{2}}u(x_0)\|_4^2 + \||y|^{\frac{1}{2}}v(x_0)\|_4^2
\end{aligned}
$$

where we used $|\tau|^{-1} \leq \langle x_0\rangle^\varphi$. This shows that $\||y|^{\frac{1}{2}-\varepsilon}\mathcal{L}_1 S\|_1 + \||y|^{\frac{1}{2}-\varepsilon}\mathcal{L}_2 S\|_1 \leq C(x_0)\|(\mathbf{v}, \omega)\|^2$. The same holds for $\||y|^{\frac{1}{2}-\varepsilon}\mathcal{L}_1 R\|_1$ and $\||y|^{\frac{1}{2}-\varepsilon}\mathcal{L}_2 R\|_1$. We then note that

$$
\mathcal{F}_{2,v}(x) = \mathcal{F}_{2,\omega}(x) + \mathcal{F}_{2,1,v}(x) + \mathcal{F}_{2,2,v}(x) \ , \qquad \mathcal{F}_{2,u}(x) = \mathcal{F}_{2,1,u}(x) + \mathcal{F}_{2,2,u}(x) \ ,
$$

see Propositions 3.15 and 3.16, or (3.30), (3.31) and (3.34) and (3.35) for the definitions of the various terms appearing in this decomposition. By Proposition 3.14, the contribution of $\mathcal{F}_{2,\omega}$ is bounded by $C\|(\mathbf{v}, \omega)\|^2$. Then, there are exponents $p \geq 0$ and $q < 1$ such that

$$
\|K_2\|_{1,\{p,q\}} + \|K_5\|_{1,\{p,q\}} + \|K_6\|_{1,\{p,q\}} + \|K_7\|_{1,\{p,q\}} \leq C \ ,
$$

$$\||y|K_2\|_{2,\{p,q\}} + \||y|K_5\|_{2,\{p,q\}} + \||y|K_6\|_{2,\{p,q\}} + \||y|K_7\|_{2,\{p,q\}} \le C \ .$$

Using $\||y|^{\frac{1}{2}-(1+\varepsilon)\varphi}f\|_1 \le \||y|f\|_2$, this shows that the contributions of $\mathcal{F}_{2,1,v}$ and $\mathcal{F}_{2,1,u}$ is also bounded by $C\|(\mathbf{v},\omega)\|^2$. For the contribution of $\mathcal{F}_{2,2,v}$ and $\mathcal{F}_{2,2,u}$, we note that

$$\begin{aligned}\|\mathcal{S}G^*(\tilde{x}-x)Q(\tilde{x})\|_2 &\le C|\tilde{x}-x|^{-\frac{1}{2}}\langle\tilde{x}\rangle^{-\frac{3}{2}+\varphi}\|(\mathbf{v},\omega)\|^2 \ , \\ \|y(\mathcal{P}G^*(\tilde{x}-x)Q(\tilde{x}))\|_2 &\le C(|\tilde{x}-x|^{\frac{1}{2}}\langle\tilde{x}\rangle^{-\frac{3}{2}+\varphi} + \langle\tilde{x}\rangle^{-\frac{5}{4}+\varphi})\|(\mathbf{v},\omega)\|^2 \ ,\end{aligned}$$

while

$$\begin{aligned}\|y\mathcal{S}\mathcal{P}_0G^*(\tilde{x}-x)Q(\tilde{x})\|_2 \le{}& \left(\int_{-\infty}^{\infty}\mathrm{d}k\left(\partial_k e^{-|k||\tilde{x}-x|}\right)^2|Q(\tilde{x},k)|^2\right)^{\frac{1}{2}} \\ &+\left(\int_{-\infty}^{\infty}\mathrm{d}k\ e^{-|k||\tilde{x}-x|}|\partial_k(i\sigma(Q(\tilde{x},k)-Q(\tilde{x},-k))|^2\right)^{\frac{1}{2}} \\ \le{}& C(|\tilde{x}-x|^{\frac{1}{2}}\langle\tilde{x}\rangle^{-\frac{3}{2}+\varphi} + \langle\tilde{x}\rangle^{-\frac{5}{4}+\varphi})\|(\mathbf{v},\omega)\|^2 \ ,\end{aligned}$$

where we used that $|Q(\tilde{x},k)-Q(\tilde{x},-k)| \le |k|^{\frac{1}{2}-\varepsilon}\||y|^{\frac{1}{2}-\varepsilon}Q(\tilde{x})\|_1 \le |k|^{\frac{1}{2}-\varepsilon}\||y|Q(\tilde{x})\|_2$, so that the coefficient of the Dirac measure appearing when differentiating σ w.r.t. k in the above expression vanishes. This implies finally that

$$\begin{aligned}\||y|^{\frac{1}{2}-(1+\varepsilon)\varphi}\mathcal{S}G^*(\tilde{x}-x)Q(\tilde{x})\|_1 \le{}& \|\mathcal{S}G^*(\tilde{x}-x)Q(\tilde{x})\|_2^{(1+\frac{\varepsilon}{2})\varphi}\|y\mathcal{S}G^*(\tilde{x}-x)Q(\tilde{x})\|_2^{1-(1+\frac{\varepsilon}{2})\varphi} \\ \le{}& C|\tilde{x}-x|^{\frac{1}{2}-(1+\frac{\varepsilon}{2})\varphi}\langle\tilde{x}\rangle^{-\frac{3}{2}+\varphi}\|(\mathbf{v},\omega)\|^2 \\ &+ C|\tilde{x}-x|^{-\frac{1}{2}(1+\frac{\varepsilon}{2})\varphi}\langle\tilde{x}\rangle^{-\frac{5}{4}+\frac{6-\varepsilon}{8}\varphi}\|(\mathbf{v},\omega)\|^2 \ . \end{aligned} \tag{7.8}$$

The same estimate holds for $\||y|^{\frac{1}{2}-(1+\varepsilon)\varphi}\mathcal{S}F^*(\tilde{x}-x)Q(\tilde{x})\|_1$. Since $\varepsilon > 0$, integrating (7.8) from $\tilde{x} = x_0$ to $\tilde{x} = \infty$ completes the proof. ■

8 Checking the applicability to the usual exterior problem

In this section, we prove the Proposition 1.5. We will use the notation $r = \sqrt{x^2+y^2}$. From [1, 4, 7], we get that any "Physically Reasonable" (PR) solution solution satisfies the estimates

$$\begin{aligned}|u(x,y)| &\le C\begin{cases} r^{-\frac{1}{2}} & \text{if } r \ge C \\ r^{-\min(\frac{1+\sigma}{2},1-\varepsilon)} & \text{if } 1-\cos(\phi) \ge r^{-1+\sigma}\end{cases} \\ |v(x,y)| &\le Cr^{-1}\ln(r)\ , \qquad |\partial_y u(x,y)| \le Cr^{-1}\ln(r)^2\ , \qquad |\partial_y v(x,y)| \le Cr^{-\frac{3}{2}}\ln(r)^2 \\ \omega(x,y) &= c_1\partial_x(e^{\frac{x}{2}}\mathrm{K}_0(r)) + c_2\partial_y(e^{\frac{x}{2}}\mathrm{K}_0(r)) + \mathcal{O}\Big(e^{\frac{x-r}{4}}r^{-\frac{3}{2}}\ln(r)^2\Big)\ , \\ \partial_y\omega(x,y) &= c_1\partial_y\partial_x(e^{\frac{x}{2}}\mathrm{K}_0(r)) + c_2\partial_y^2(e^{\frac{x}{2}}\mathrm{K}_0(r)) + \mathcal{O}\Big(e^{\frac{x-r}{4}}r^{-2}\ln(r)^2\Big)\ ,\end{aligned}$$

where ε is arbitrarily small, $\sigma \in [0,1]$, $\tan(\phi) = \frac{y}{x}$, c_1 and c_2 are constants and K_0 is the modified Bessel function of the second type of order zero. From this, we get immediately $\|(\mathbf{v},\omega)\| \le C$ if x_0 is sufficiently large and $r > (2\min(\eta,\xi))^{-1}$ (using also $\ln(x) \le C\langle x\rangle^{\varphi}$). Namely, for the estimates of the velocity fields

u and v, the only difficulty is to prove that $\|u\|_{q,\frac{1}{2}-\frac{1}{q}} \le C$. This follows since for $\sigma = \frac{1}{q}$, $\varepsilon = \frac{1}{2} - \frac{1}{2q}$ and x_0 sufficiently large, we have

$$|u(x,y)| \le C \begin{cases} r^{-\frac{1}{2}} & \text{if } x \ge x_0 \text{ and } |y| < cx \\ r^{-\frac{1}{2}(1+\frac{1}{q})} & \text{if } x \ge x_0 \text{ and } |y| \ge cx \end{cases} ,$$

which gives

$$\|u\|_{q,\frac{1}{2}-\frac{1}{q}} \le C\Big(\frac{\langle x\rangle}{x}\Big)^{\frac{1}{2}-\frac{1}{q}} \left(\Big(\int_{-c}^{c} \frac{\mathrm{d}y}{(1+y^2)^{\frac{q}{4}}}\Big)^{\frac{1}{q}} + \frac{2}{x^{\frac{1}{2q}}}\Big(\int_{c}^{\infty} \frac{\mathrm{d}y}{(1+y^2)^{\frac{1+q}{4}}}\Big)^{\frac{1}{q}}\right) .$$

For the estimates on the vorticity, it follows, using that $|z|^p e^{-z} \le C_p$ for all $p \ge 0$ and the asymptotic development of K_0, that for $x \ge x_0$ sufficiently large we have

$$|\omega(x,y)| \le C e^{\frac{x}{4}-\frac{r}{4}} r^{-\frac{3}{2}} (|y| + \ln(x)^2) , \qquad |\partial_y \omega(x,y)| \le C\Big(e^{\frac{x}{4}-\frac{r}{4}} r^{-\frac{3}{2}}\Big) .$$

This shows at once that $\|\partial_y \omega\|_{\infty,\frac{3}{2}} \le C$. Then, for all $\alpha \ge 0$, after the change of variable $y = \sqrt{2xz + z^2}$ and using again that $|z|^p e^{-z} \le C_p$, we get that

$$\begin{aligned} \||y|^\alpha \omega\|_{\mathrm{L}^2} &\le C\Big(\int_0^\infty \mathrm{d}z\, \frac{e^{-\frac{z}{2}}}{\sqrt{z}} \frac{\big(\ln(x)^2 + \sqrt{z}\sqrt{2x+z}\big)^2 (z(2x+z))^\alpha}{(x+z)^2\sqrt{2x+z}}\Big)^{\frac{1}{2}} \\ &\le C x^{-\frac{3}{4}+\frac{\alpha}{2}} \Big(\int_0^\infty \mathrm{d}z\, \frac{e^{-\frac{z}{4}}}{\sqrt{z}}\Big)^{\frac{1}{2}} \le C\langle x\rangle^{-\frac{3}{4}+\frac{\alpha}{2}} , \qquad (8.1) \\ \|\partial_y \omega\|_{\mathrm{L}^1} &\le C \int_0^\infty \frac{\mathrm{d}z\, e^{-\frac{z}{4}}}{\sqrt{z}\sqrt{x+z}\sqrt{2x+z}} \le C x^{-1} \int_0^\infty \mathrm{d}z\, \frac{e^{-\frac{z}{4}}}{\sqrt{z}} \le C\langle x\rangle^{-1} . \end{aligned}$$

Using the estimate (8.1) with $\alpha = 0$ and $\alpha = \beta$ achieves the proof of $\|(0,0,\omega)\| \le C$. We then note that for $|y| \ge cx \ge cx_0$ with x_0, we have for all $q > 1$

$$|u(x,y)| + |v(x,y)| \le C r^{-\frac{1}{2}(1+\frac{1}{q})} , \qquad (8.2)$$

from which we deduce that $\||y|^{\frac{1}{2}} u(x)\|_4 + \||y|^{\frac{1}{2}} v(x)\|_4 \le C$. Finally, it follows from e.g. [7], section X.6, that there exist constants $\mathbf{m} = (m_1, m_2)$ such that for all $|y| \ge cx \ge cx_0$, we have

$$|u(x,y) - u_{\mathbf{m}}(x,y)| + |v(x,y) - v_{\mathbf{m}}(x,y)| \le C r^{-1} , \qquad (8.3)$$

where $u_{\mathbf{m}}$ and $v_{\mathbf{m}}$ are defined in terms of Oseen's tensor $\mathbf{E}$ by

$$\begin{pmatrix} u_{\mathbf{m}}(x,y) \\ v_{\mathbf{m}}(x,y) \end{pmatrix} = \mathbf{m} \cdot \mathbf{E}(x,y) . \qquad (8.4)$$

It then follows from (8.3), (8.4) and the explicit form of Oseen's tensor that

$$\||y|^{\frac{1}{2}-(1+\varepsilon)\varphi} \mathcal{S}u(x)\|_1 + \||y|^{\frac{1}{2}-(1+\varepsilon)\varphi} \mathcal{S}v(x)\|_1 \le C ,$$

where $(\mathcal{S}f)(y) \equiv f(y) + f(-y)$.

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Appendix A: Kernels estimates

A.1 Definitions and preliminaries

This section is devoted to estimates of all the kernels in various L^p and Sobolev spaces of the 'y' variable. Note that the Kernels are most conveniently expressed in terms of their Fourier transform, and though it is sometimes possible to calculate explicitly the inverse Fourier transform of the kernels, we will estimate the norms in Fourier space as often as possible. To do so, we will use the following Lemma which relates the L^1 norm in direct space to the H^1 in Fourier space, and the L^2 norm with weight $|y|^\beta$ for non-integer β to integer ones.

Lemma A.1 *Let $\beta > \frac{1}{2}$. There exist a constant C_β such that for all f with $\|(1+|y|^\beta)f\|_{\mathrm{L}^2} < \infty$, we have*

$$\|f\|_{\mathrm{L}^1} \leq \begin{cases} C_\beta \|f\|_{\mathrm{L}^2}^{1-\frac{1}{2\beta}} \||y|^\beta f\|_{\mathrm{L}^2}^{\frac{1}{2\beta}} , \\ C\sqrt{\|f\|_{\mathrm{L}^2}\|yf\|_{\mathrm{L}^2}} \leq C\sqrt{\|\hat{f}\|_{\mathrm{L}^2}\|\hat{f}'\|_{\mathrm{L}^2}} , \end{cases}$$

where $\hat{f}$ denote the (continuous) Fourier transform of f. Then, for all $s_1 \in [0,3]$ and $s_2 \in [0,2]$, we have

$$\||y|^{s_1} f\|_{\mathrm{L}^2} \leq \|f\|_{\mathrm{L}^2}^{1-\frac{s_1}{3}} \||y|^3 f\|_{\mathrm{L}^2}^{\frac{s_1}{3}} , \quad \||y|^{1+s_2} f\|_{\mathrm{L}^2} \leq \||y| f\|_{\mathrm{L}^2}^{1-\frac{s_2}{2}} \||y|^3 f\|_{\mathrm{L}^2}^{\frac{s_2}{2}} .$$

Proof. Let $a > 0$, then

$$\|f\|_{\mathrm{L}^1} \leq \|(a+|y|^\beta)f\|_{\mathrm{L}^2}\|(a+|y|^\beta)^{-1}\|_{\mathrm{L}^2} \leq C_\beta\Big(a^{\frac{1}{2\beta}}\|f\|_{\mathrm{L}^2} + a^{\frac{1}{2\beta}-1}\||y|^\beta f\|_{\mathrm{L}^2}\Big)$$

for some fi nite C_β. Setting $a = \||y|^\beta f\|_{\mathrm{L}^2}/\|f\|_{\mathrm{L}^2}$ completes the proof of the fi rst inequality. The second one follows from Plancherel's inequality, while the last two follow trivially from Young's inequality. ■

We then introduce the functions

$$B_{\mu,\varphi}(x,n\tau) = \int_{-\infty}^{\infty} \mathrm{d}k\, |k|^\varphi |\tfrac{k}{\Lambda_0}|^{2\mu}\, e^{2\,\mathrm{Re}(\Lambda_-)x} , \quad B_\varphi(x,n\tau) = \int_{-\infty}^{\infty} \mathrm{d}k\, |\tfrac{k}{\Lambda_0}|^{2\varphi} \tfrac{1}{|\Lambda_0|^2}\, e^{2\,\mathrm{Re}(\Lambda_-)x} .$$

through which most estimates on the kernels can be easily obtained, and which satisfy the

Lemma A.2 *Let $\mu \geq \frac{1}{2}$. Then for all $\varphi \geq 0$, there exist a constant C_φ such that for all $1 \leq \xi_1 \leq \mu + \frac{1}{2}$ we have*

$$B_{0,\varphi}(x,n\tau) \leq C_\varphi \frac{e^{b(n\tau)x}\langle x\rangle^{\frac{\varphi+1}{2}}}{x^{\varphi+1}} , \quad B_{\mu,\varphi}(x,n\tau) \leq C_\varphi \frac{e^{b(n\tau)x}\langle x\rangle^{\frac{\varphi}{2}}}{x^{\xi_1+\varphi}} , \quad B_\varphi(x,n\tau) \leq C_\varphi \frac{e^{b(n\tau)x}}{\langle x\rangle^{\frac{1}{2}+\varphi}}$$

*for all $x \geq 0$ and $n\tau \in$ **R**.*

Proof. We first have

$$B_{0,\varphi}(x,n\tau) \le Ce^{2b(n\tau)x}\left(\int_{|k|>1}\mathrm{d}k\,|k|^{\varphi}e^{-|k|x} + \int_{|k|\le 1}\mathrm{d}k\,|k|^{\varphi}e^{-2c(n\tau)xk^2}\right) ,$$

$$\le C\tfrac{e^{b(n\tau)x}}{x^{\varphi+1}}\left(1+(\varphi+1)^{\frac{\varphi+1}{2}}\left(\tfrac{c(n\tau)^{-1}xe^{\frac{b(n\tau)x}{\varphi+1}}}{\varphi+1}\right)^{\frac{\varphi+1}{2}}\right) \le C\frac{e^{b(n\tau)x}\langle x\rangle^{\frac{\varphi+1}{2}}}{x^{\varphi+1}} ,$$

because $c(n\tau)^{-1}\zeta e^{b(n\tau)\zeta} \le C(1+\zeta)$ for all $\zeta \ge 0$. Then, we note that since $|\frac{k}{\Lambda_0}|$ is uniformly bounded in k and $n\tau$, we trivially have $B_{\mu,\varphi}(x,n\tau) \le C_\mu B_{0,\varphi}(x,n\tau)$ for all $\mu \ge 0$. To get the more precise bound of the Lemma in the case $\mu \ge \frac{1}{2}$, we use that $|\frac{k}{\Lambda_0}| \le C$ and that by hypothesis on ξ_1, we have $0 \le \xi_1 - 1 \le 2\xi_1 - 1 \le 2\mu$, hence

$$B_{\mu,\varphi}(x,n\tau) \le Ce^{2b(n\tau)x}\left(\int_{|k|>1}\mathrm{d}k\,|k|^{\varphi+\xi_1-1}e^{-|k|x} + \int_{|k|\le 1}\mathrm{d}k\,\tfrac{|k|^{\varphi+2\xi_1-1}e^{-2c(n\tau)xk^2}}{(1+(n\tau)^2)^{\frac{\mu}{2}}}\right)$$

$$\le \frac{C}{x^{\xi_1+\varphi}}\left(e^{2b(n\tau)x} + \tfrac{(c(n\tau)^{-1}x)^{\frac{\varphi}{2}}e^{2b(n\tau)x}}{c(n\tau)^{\xi_1}(1+(n\tau)^2)^{\frac{\mu}{2}}}\right) .$$

Since $c(n\tau)^{-\mu-\frac{1}{2}}(1+(n\tau)^2)^{-\frac{\mu}{2}} \le C$ by hypothesis on μ and ξ_1, this completes the proof of the second inequality if $\varphi = 0$. If $\varphi > 0$, we use $c(n\tau)^{-1}\zeta e^{b(n\tau)\zeta} \le C(1+\zeta)$, so that

$$B_{\mu,\varphi}(x,n\tau) \le C\frac{e^{b(n\tau)x}}{x^{\xi_1+\varphi}}\left(1+(\tfrac{\varphi}{2})^{\frac{\varphi}{2}}(c(n\tau)^{-1}2x\varphi^{-1}e^{b(n\tau)2x\varphi^{-1}})^{\frac{\varphi}{2}}\right) \le C\frac{e^{b(n\tau)x}\langle x\rangle^{\frac{\varphi}{2}}}{x^{\xi_1+\varphi}} .$$

For the last inequality, we first note that $B_\varphi(x,n\tau) \le C_\varphi B_0(0,n\tau)$ (this follows again because $|\frac{k}{\Lambda_0}|$ is uniformly bounded). Then we have $B_0(0,n\tau) \le C$, so we only have to show that $B_\varphi(x,n\tau)$ decays at least like $e^{b(n\tau)x}x^{-\frac{1}{2}-\varphi}$ as $x \to \infty$, and this follows since

$$B_\varphi(x,n\tau) \le Ce^{2b(n\tau)x}\Big(\int_{|k|\le 1}\mathrm{d}k\,\tfrac{|k|^{2\varphi}e^{-2c(n\tau)xk^2}}{(1+(n\tau)^2)^{\frac{1+\varphi}{2}}} + \int_{|k|>1}\mathrm{d}k\,\tfrac{|k|^{2\varphi}e^{-|k|x}}{1+k^2}\Big)$$

$$\le \frac{Ce^{b(n\tau)x}}{x^{\frac{1}{2}+\varphi}}\left((c(n\tau)^{\frac{1}{2}+\varphi}(1+(n\tau)^2)^{\frac{1+\varphi}{2}})^{-1} + x^{-\frac{1}{2}-\varphi}\right) .$$

This completes the proof. ■

Note that in the bound on $B_{\mu,\varphi}(x)$ in Lemma A.2, the best decay rate as $x \to \infty$ improves as μ grows. The 'free' parameter ξ_1 gives a way to limit the growth of the divergence rate as $x \to 0$.

A.2 Actual estimates

We begin this section by an easy estimate on $\mathcal{L}_1$ and $\mathcal{L}_2$:

Lemma A.3 *Let* $\hat{\mathcal{L}}_1 = \frac{k^2}{k^2+(n\tau)^2}$ *and* $\hat{\mathcal{L}}_2 = \frac{|k|n\tau}{k^2+(n\tau)^2}$*, then*

$$\|\mathcal{L}_1 - \mathbb{1}\|_{1,\{0,0\}} + \|\mathcal{L}_2\|_{1,\{0,0\}} \le C .$$

In particular, $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *are* $\mathrm{L}^p \to \mathrm{L}^p$ *bounded operators for all* $p \in [1,\infty)$.

Proof. The proof follows immediately since using Fourier transform, we get that for fixed n, it holds $\|\hat{\mathcal{L}}_1 - 1\|_{\mathrm{L}^2} + \|\hat{\mathcal{L}}_2\|_{\mathrm{L}^2} \le C|n\tau|$ and $\|\partial_k(\hat{\mathcal{L}}_1 - 1)\|_{\mathrm{L}^2} + \|\partial_k\hat{\mathcal{L}}_2\|_{\mathrm{L}^2} \le C|n\tau|^{-1}$. ■

Lemma A.4 *For all $p>1$, $q\geq 2$ and $m\in\mathbf{N}$, there exists a constant $C>0$ such that*

$$\begin{aligned}\|\mathcal{P}_0 F\|_{p,\{0,1-\frac{1}{p}\}}+\|\mathcal{P}_0 G\|_{p,\{0,1-\frac{1}{p}\}}&\leq C\\ \|\partial_y^m F\|_{q,\{0,1+m-\frac{1}{q}\}}+\|\partial_y^m G\|_{q,\{0,1+m-\frac{1}{q}\}}&\leq C\\ \|\langle\tau x\rangle\mathcal{P}\partial_y^m F\|_{q,\{0,1+m-\frac{1}{q}\}}+\|\langle\tau x\rangle\mathcal{P}\partial_y^m G\|_{q,\{0,1+m-\frac{1}{q}\}}&\leq C\\ \|\mathcal{P}F\|_{1,\{0,\frac{1}{4}\}}+\|\mathcal{P}G\|_{1,\{0,\frac{1}{4}\}}&\leq C|\tau|^{-\frac{1}{4}}\,.\end{aligned}$$

Proof. After the change of variables $k=\xi/x$, we get

$$\begin{aligned}\|\langle x\tau\rangle\partial_y^m F\|_{q,\{0,1+m-\frac{1}{q}\}}&\leq\sup_{x\geq 0}\sup_{n\in\mathbf{Z}}\left(\int_{-\infty}^{\infty}\mathrm{d}\xi\left(\frac{\xi^2(1+(\tau x)^2)}{\xi^2+(n\tau x)^2}\right)^{\frac{q}{2(q-1)}}|\xi|^{\frac{qm}{q-1}}e^{-\frac{q|\xi|}{q-1}}\right)^{\frac{q-1}{q}}\\ &\leq\left[\int_{|\xi|\leq 1}\mathrm{d}\xi\,|\xi|^{\frac{qm}{q-1}}e^{-\frac{q|\xi|}{q-1}}+\int_{|\xi|\geq 1}\mathrm{d}\xi\,|\xi|^{\frac{q(1+m)}{q-1}}e^{-\frac{q|\xi|}{q-1}}\right]^{1/q}\leq C\,,\\ \|\partial_y^m F\|_{q,\{0,1+m-\frac{1}{q}\}}&\leq\sup_{x\geq 0}\sup_{n\in\mathbf{Z}}\left(\int_{-\infty}^{\infty}\mathrm{d}\xi\,|\xi|^{\frac{qm}{q-1}}e^{-\frac{q|\xi|}{q-1}}\right)^{\frac{q-1}{q}}\leq C\,,\end{aligned}$$

for any $m\in\mathbf{N}$ and $q\geq 2$. The same holds for G. We next note that $G=-i\sigma F$, so that

$$\begin{aligned}\partial_k G(x,k)&=-i\delta(k)F(x,k)-i\sigma\partial_k F(x,k)\\ &=\frac{-i\delta(k)}{1-\frac{in\tau}{|k|}}-i\sigma\partial_k F(x,k)=-i\delta_{n,0}-i\sigma\partial_k F(x,k)\,,\end{aligned}\qquad\text{(A.1)}$$

where $\delta_{n,0}=1$ if $n=0$ and $\delta_{n,0}=0$ if $n\neq 0$. We thus have $\partial_k\mathcal{P}_0 G(x,k)\notin\mathrm{L}^2$, so that we cannot use Lemma A.1 to bound $\|\mathcal{P}_0 G(x)\|_{\mathrm{L}^1}$. In fact, $\mathcal{P}_0 F$ and $\mathcal{P}_0 G$ can be explicitly computed, giving $\mathcal{P}_0 F(x,y)=\frac{1}{\pi}\frac{x}{x^2+y^2}$ and $\mathcal{P}_0 G(x,y)=\frac{1}{\pi}\frac{y}{x^2+y^2}$. This shows that $\mathcal{P}_0 G(x,y)\notin\mathrm{L}^1$, and gives an easy way to prove the estimate on $\|\mathcal{P}_0 F\|_{p,\{0,1-\frac{1}{p}\}}+\|\mathcal{P}_0 G\|_{p,\{0,1-\frac{1}{p}\}}$ for $p>1$ in direct space. On the other hand, (A.1) shows that $\|\mathcal{P}\partial_k G(x)\|_{\mathrm{L}^2}=\|\mathcal{P}\partial_k F(x)\|_{\mathrm{L}^2}$, and we have

$$\begin{aligned}\|\partial_k\mathcal{P}F\|_{2,\{0,0\}}+\|\partial_k\mathcal{P}G\|_{2,\{0,0\}}&=\sup_{x\geq 0}\sqrt{x}\sup_{n\in\mathbf{Z},n\neq 0}\left(\int_{-\infty}^{\infty}\mathrm{d}\xi\,\frac{e^{-2|\xi|}\left(\xi^4+(n\tau x)^2(1-|\xi|)^2\right)}{(\xi^2+(n\tau x)^2)^2}\right)^{1/2}\\ &\leq\sup_{x\geq 0}\frac{\sqrt{x}}{2}\sup_{n\in\mathbf{Z},n\neq 0}\left(\int_{|\xi|\leq 1}\mathrm{d}\xi\,\frac{1}{\xi^2+(n\tau x)^2}+\frac{C}{\langle n\tau x\rangle^2}\int_{|\xi|\geq 1}\mathrm{d}\xi\,e^{-|\xi|}\right)^{1/2}\\ &\leq\frac{C}{\sqrt{|\tau|}}\,.\end{aligned}$$

Using Lemma A.1, this proves the estimates on $\|\mathcal{P}F\|_{1,\{0,\frac{1}{4}\}}+\|\mathcal{P}G\|_{1,\{0,\frac{1}{4}\}}$ and completes the proof. ■

Lemma A.5 *There exist a constant $C>0$ such that for all $1\leq\beta\leq 3$, it holds*

$$\begin{aligned}\|K_1\|_{1,\{0,0\}}+\|K_1(x)\|_{\infty,\{\frac{1}{2},1\}}+\||y|^\beta K_1\|_{2,\{-\frac{1}{4}+\frac{\beta}{2},0\}}&\leq C\,,\\ \|\partial_y K_1\|_{1,\{\frac{1}{2},1\}}+\|\partial_y K_1\|_{\infty,\{1,2\}}+\||y|^\beta\partial_y K_1(x)\|_{2,\{-\frac{3}{4}+\frac{\beta}{2},0\}}&\leq C\,,\end{aligned}$$

$$\|\partial_y^2 K_1\|_{\infty,\{\frac{3}{2},3\}} + \|\partial_y^2 K_1\|_{1,\{1,2\}} \leq C \ ,$$
$$\|K_2\|_{1,\{0,\frac{1}{2}\}} + \|K_2\|_{\infty,\{0,1\}} + \||y|^{\beta} K_2\|_{2,\{-\frac{3}{4}+\frac{\beta}{2},0\}} \leq C \ ,$$
$$\|\partial_y K_2\|_{\infty,\{\frac{1}{2},2\}} + \|\partial_y K_2\|_{1,\{\frac{1}{2},\frac{3}{2}\}} \leq C \ ,$$
$$\|K_5\|_{1,\{0,\frac{1}{2}\}} + \|K_6\|_{1,\{0,\frac{1}{2}\}} + \|K_7\|_{1,\{\frac{1}{4},\frac{1}{4}\}} \leq C \ ,$$
$$\||y|K_5\|_{2,\{0,\frac{1}{4}\}} + \||y|K_6\|_{2,\{0,\frac{1}{4}\}} + \||y|K_7\|_{2,\{\frac{1}{2},\frac{1}{4}\}} \leq C \ .$$

The same estimates hold with K_n *replaced by* $e^{-\frac{b(\tau)x}{4}}\mathcal{P}K_n$ *for* $n = 1, 2, 5, 6, 7$.

Proof. We have $\left|\partial_k e^{\Lambda_- x}\right| \leq \frac{|k| x e^{\mathrm{Re}(\Lambda_-)x}}{|\Lambda_0|}$ and

$$\left|\partial_k^3 e^{\Lambda_- x}\right| \leq e^{\mathrm{Re}(\Lambda_-)x}\left(\frac{x^3|k|^3}{|\Lambda_0|^3} + \frac{x^2|k|}{|\Lambda_0|^2} + \frac{x|k|}{|\Lambda_0|^3}\right)$$
$$\left|\partial_k^3 (k e^{\Lambda_- x})\right| \leq e^{\mathrm{Re}(\Lambda_-)x}\left(\frac{x^3|k|^4}{|\Lambda_0|^3} + \frac{x^2|k|^2}{|\Lambda_0|^2} + \frac{x}{|\Lambda_0|}\right)$$
$$\left|\partial_k (\tfrac{k}{\Lambda_0} e^{\Lambda_- x})\right| \leq e^{\mathrm{Re}(\Lambda_-)x}\left(\frac{c_1}{|\Lambda_0|} + \frac{c_2 k^2 x}{|\Lambda_0|^2}\right) \ ,$$
$$\left|\partial_k^3 (\tfrac{k}{\Lambda_0} e^{\Lambda_- x})\right| \leq e^{\mathrm{Re}(\Lambda_-)x}\left(\frac{x^3|k|^4}{|\Lambda_0|^4} + \frac{x^2|k|^2}{|\Lambda_0|^3} + \frac{x}{|\Lambda_0|^2} + \frac{1}{|\Lambda_0|^3}\right) \ ,$$
$$\left|\partial_k (\tfrac{k^2}{\Lambda_0} e^{\Lambda_- x})\right| \leq |k| e^{\mathrm{Re}(\Lambda_-)x}\left(\frac{c_1}{|\Lambda_0|} + \frac{c_2 k^2 x}{|\Lambda_0|^2}\right) \ .$$

Similarly, we have

$$|K_5(x,k)| + |K_6(x,k)| \leq \frac{|k| e^{\mathrm{Re}(\Lambda_-)x}}{|\Lambda_0|} \ ,$$
$$|\partial_k K_5(x,k)| + |\partial_k K_6(x,k)| \leq e^{\mathrm{Re}(\Lambda_-)x}\left(\frac{c_3}{|\Lambda_0|} + \frac{c_4 k^2 x}{|\Lambda_0|^2}\right) \ ,$$
$$|K_7(x,k)| \leq C e^{\mathrm{Re}(\Lambda_-)x} \ ,$$
$$|\partial_k K_7(x,k)| \leq C e^{\mathrm{Re}(\Lambda_-)x}\left(\frac{1}{|\Lambda_0|} + \frac{|k|x}{|\Lambda_0|}\right) \ .$$

Finally, we note that for fixed x and n, we have

$$\begin{aligned}\|\partial_y K_1\|_{\mathrm{L}^1} &\leq C\left(\|kK_1\|_{\mathrm{L}^2}^2\left(\|K_1\|_{\mathrm{L}^2}^2 + \|k\partial_k K_1\|_{\mathrm{L}^2}^2\right)\right)^{\frac{1}{4}} \\ &\leq C\left(\|\partial_y K_1\|_{\mathrm{L}^2}^2\left(\|K_1\|_{\mathrm{L}^2}^2 + x^2\|\partial_y K_2\|_{\mathrm{L}^2}^2\right)\right)^{\frac{1}{4}} \ , \\ \|\partial_y^2 K_1\|_{\mathrm{L}^1} &\leq C\left(\|k^2 K_1\|_{\mathrm{L}^2}^2\left(2\|kK_1\|_{\mathrm{L}^2}^2 + \|k^2\partial_k K_1\|_{\mathrm{L}^2}^2\right)\right)^{\frac{1}{4}} \\ &\leq C\left(\|\partial_y^2 K_1\|_{\mathrm{L}^2}^2\left(2\|\partial_y K_1\|_{\mathrm{L}^2}^2 + x^2\|\partial_y^2 K_2\|_{\mathrm{L}^2}^2\right)\right)^{\frac{1}{4}} \ ,\end{aligned}$$

where here, $\mathrm{L}^1 \equiv \mathrm{L}^1(\mathbf{R}, \mathrm{d}y)$ and $\mathrm{L}^2 \equiv \mathrm{L}^2(\mathbf{R}, \mathrm{d}k)$. The proof is then easily completed using Lemma A.2 and $\mathcal{P}e^{\frac{b(n\tau)x}{4}} \leq e^{\frac{b(\tau)x}{4}}$, we omit the details. ∎

Lemma A.6 *For all* $1 \leq \beta \leq 3$, $\frac{1}{4} \leq \xi_2 \leq 1$ *and* $1 \leq \xi_3 \leq \frac{5}{2}$, *there exists a constant* $C > 0$ *such that*

$$\|K_8\|_{1,\{0,\xi_2\}} + \|K_8\|_{2,\{0,\frac{\xi_3}{2}\}} + \|K_8\|_{\infty,\{\frac{1}{2},2\}} + \|\partial_y K_8\|_{\infty,\{1,3\}} \leq C$$
$$\||y|^{\beta} K_8\|_{2,\{-\frac{5}{4}+\frac{\beta}{2},0\}} + \||y|^{\beta}\partial_y K_8\|_{2,\{-\frac{3}{4}+\frac{\beta}{2},1\}} + \|\partial_y K_8\|_{1,\{\frac{1}{4},\frac{1+\xi_3}{2}\}} \leq C \qquad \text{(A.2)}$$

The same estimate holds with K_8 *replaced by* $e^{-\frac{b(\tau)x}{4}}\mathcal{P}K_8$.

Proof. For any $0 \le \sigma \le 1$, we have

$$\begin{aligned}
|K_8(x,k)| &\le C \left|\frac{\mathrm{Re}(\Lambda_-)}{\Lambda_0}\right|^{1-\sigma} \frac{e^{\mathrm{Re}(\Lambda_-)x/2}}{(x|\Lambda_0|)^\sigma} \le C \frac{e^{\mathrm{Re}(\Lambda_-)\frac{x}{2}}}{(x|\Lambda_0|)^\sigma} ,\\
|\partial_k K_8(x,k)| &\le C\left(\frac{|k|}{|\Lambda_0|^2} + \frac{x|k\mathrm{Re}(\Lambda_-)|}{|\Lambda_0|^2}\right) e^{\mathrm{Re}(\Lambda_-)x} \le C \frac{|k|}{|\Lambda_0|^2} e^{\mathrm{Re}(\Lambda_-)\frac{x}{2}} ,\\
\left|\partial_k^3 K_8(x,k)\right| &\le C \left|\frac{\mathrm{Re}(\Lambda_-)}{\Lambda_0}\right| \left|\partial_k^3 e^{\Lambda_- x}\right| + \frac{|k\partial_k^2 e^{\Lambda_- x}|}{|\Lambda_0|^2} + \frac{|\partial_k e^{\Lambda_- x}|}{|\Lambda_0|^2} + \frac{|k e^{\Lambda_- x}|}{|\Lambda_0|^4} ,\\
&\le C\left(\frac{x^2|k|^3}{|\Lambda_0|^4} + \frac{x|k|}{|\Lambda_0|^3} + \frac{|k|}{|\Lambda_0|^4}\right) e^{\mathrm{Re}(\Lambda_-)\frac{x}{2}} .
\end{aligned}$$

Let $1 \le \xi_3 \le \frac{5}{2}$, $\sigma_3 = \frac{\xi_3}{2} - \frac{1}{4}$ and $\gamma_3 = \frac{\xi_3}{2} - \frac{1}{2}$. Since $0 \le \sigma_3, \gamma_3 \le 1$, for any fixed x, we have

$$\begin{aligned}
\|K_8(x)\|_{\mathrm{L}^2}^2 &\le C \sup_{n\in\mathbf{Z}} \left(x^{-2\sigma_3} \int_{|k|\le 1} \mathrm{d}k\, \frac{e^{b(n\tau)x - c(n\tau)xk^2}}{(1+(n\tau)^2)^{\frac{\sigma_3}{2}}} + x^{-2\gamma_3} \int_{|k|>1} \mathrm{d}k\, e^{b(n\tau)x - \frac{|k|x}{2}} \right)\\
&\le C \sup_{n\in\mathbf{Z}} e^{b(n\tau)x} \left(\frac{x^{-\frac{1}{2}-2\sigma_3}}{(1+(n\tau)^2)^{\frac{4\sigma_3-1}{8}}} + x^{-1-2\gamma_3} \right) \le C x^{-\xi_3} .
\end{aligned}$$

The bound on $\|K_8\|_{1,\{0,\xi_2\}} + \|K_8\|_{2,\{0,\frac{\xi_3}{2}\}} + \||y|^\beta K_8\|_{2,\{-\frac{5}{4}+\frac{\beta}{2},0\}}$ is completed using Lemma A.1, A.2 and $\mathcal{P}e^{\frac{b(n\tau)x}{4}} \le e^{\frac{b(\tau)x}{4}}$. To bound $\|\partial_y K_8\|_{1,\{\frac{1}{2},1+\xi_2\}}$, we note that for fixed x

$$\|\partial_y K_8(x)\|_{\mathrm{L}^1} \le C \sup_{n\in\mathbf{Z}} \left(\|kK_8(x)\|_{\mathrm{L}^2}^2 \left(\|K_8(x)\|_{\mathrm{L}^2}^2 + \|k\partial_k K_8(x)\|_{\mathrm{L}^2}^2\right)\right)^{\frac{1}{4}} \le C \left(\frac{1+x}{x^{2\xi_3+2}}\right)^{\frac{1}{4}} .$$

This completes the proof. ∎

Lemma A.7 *Let $x \ge 0$, then there exists a constant $C > 0$ such that for all $1 \le \beta \le 3$ we have*

$$\begin{aligned}
\|e^{-\frac{b(\tau)x}{4}} K_{10}\|_{\infty,\{0,1\}} + \|e^{-\frac{b(\tau)x}{4}} K_{10}\|_{2,\{0,\frac{3}{4}\}} + \|e^{-\frac{b(\tau)x}{4}} K_{10}\|_{1,\{\frac{1}{8},\frac{5}{8}\}} &\le C\\
\|e^{-\frac{b(\tau)x}{4}} |y|^\beta K_{10}\|_{2,\{\frac{3}{8}+\frac{\beta}{8},-\frac{9}{8}+\frac{3\beta}{8}\}} + \|e^{-\frac{b(\tau)x}{4}} \partial_y K_{10}\|_{\infty,\{\frac{1}{2},2\}} + \|e^{-\frac{b(\tau)x}{4}} \partial_y K_{10}\|_{1,\{\frac{5}{8},\frac{13}{8}\}} &\le C .
\end{aligned}$$

Proof. We have $|K_{10}(x,k)| \le C \frac{|n\tau|}{\langle n\tau\rangle} e^{\mathrm{Re}(\Lambda_-)x}$, and

$$\begin{aligned}
|\partial_k K_{10}(x,k)| &\le C\left(\frac{|n\tau k|}{|\Lambda_0|^4} + \frac{x|kn\tau|}{\langle n\tau\rangle|\Lambda_0|}\right) e^{\mathrm{Re}(\Lambda_-)x} \le \frac{\langle x\rangle |n\tau|}{\langle n\tau\rangle} \frac{|k| e^{\mathrm{Re}(\Lambda_- x)}}{|\Lambda_0|} ,\\
\left|\partial_k^3 K_{10}(x,k)\right| &\le C\left|\frac{\mathrm{Im}(\Lambda_-)}{\Lambda_0}\right| \left(\left|\partial_k^3 e^{\Lambda_- x}\right| + \left|\frac{k\partial_k^2 e^{\Lambda_- x}}{|\Lambda_0|^2}\right| + \left|\frac{\partial_k e^{\Lambda_- x}}{|\Lambda_0|^2}\right| + \left|\frac{k e^{\Lambda_- x}}{|\Lambda_0|^4}\right| \right) ,\\
&\le C \frac{|n\tau|}{\langle n\tau\rangle} e^{\mathrm{Re}(\Lambda_-)x} \left(\frac{x^3|k|^3}{|\Lambda_0|^3} + \frac{x^2|k|}{|\Lambda_0|} + \frac{1+x}{|\Lambda_0|} \right) .
\end{aligned}$$

In particular, we have $\mathcal{P}K_{10} = K_{10}$. The proof is then completed using Lemma A.2, that $\mathcal{P}e^{\frac{b(n\tau)x}{4}} \le e^{\frac{b(\tau)x}{4}}$ and $|n\tau|\langle n\tau\rangle^{-1}\langle x\rangle^{\frac{1}{2}} e^{\frac{b(n\tau)x}{4}} \le 2$, and that for fixed x, we have

$$\|\partial_y K_{10}(x)\|_{\mathrm{L}^1} \le C \left(\|kK_{10}(x)\|_{\mathrm{L}^2}^2 \left(\|K_{10}(x)\|_{\mathrm{L}^2}^2 + \|k\partial_k K_{10}(x)\|_{\mathrm{L}^2}^2\right)\right)^{\frac{1}{4}} ,$$

where $\mathrm{L}^1 \equiv \mathrm{L}^1(\mathbf{R}, \mathrm{d}y)$ and $\mathrm{L}^2 \equiv \mathrm{L}^2(\mathbf{R}, \mathrm{d}k)$. ∎

Lemma A.8 *Let $p \geq 2$. There exist a constant $C > 0$ such that*

$$\|K_{12}\|_{\infty,\{\frac{1}{2},1\}} + \|K_{12}\|_{2,\{\frac{1}{4},\frac{1}{2}\}} + \|\partial_y K_{12}\|_{p,\{1-\frac{1}{2p},2-\frac{1}{p}\}} \leq C \ ,$$

$$\|e^{-\frac{b(\tau)x}{2}} K_{13}\|_{\infty,\{\frac{1}{2},1\}} + \|e^{-\frac{b(\tau)x}{2}} K_{13}\|_{2,\{\frac{1}{4},\frac{1}{2}\}} + \|\partial_y e^{-\frac{b(\tau)x}{2}} K_{13}\|_{p,\{1-\frac{1}{2p},2-\frac{1}{p}\}} \leq C \ ,$$

while there exists a constant C such that for all $x \geq 0$, we have

$$\|K_{12}(x,n\tau)\|_{\mathrm{L}^1} + e^{-\frac{b(\tau)x}{2}} \|K_{13}(x,n\tau)\|_{\mathrm{L}^1} \leq C\Big(1 + \tfrac{\langle\tau\rangle}{|\tau| x^{\frac{1}{4}}}\Big) \ .$$

The estimates of this Lemma also hold with K_{12} replaced by $e^{-\frac{b(\tau)x}{4}}\mathcal{P}K_{12}$.

Proof. We first note that $\mathcal{P}_0 K_{13} = 0$ and $\mathcal{P}K_{13} = K_{13}$. We then have $|K_{12}(x,k)| + |K_{13}(x,k)| \leq Ce^{\mathrm{Re}(\Lambda_-)x}$ and

$$|\partial_k K_{12}(x,k)| + |\partial_k K_{13}(x,k)| \leq Ce^{\mathrm{Re}(\Lambda_-)x}\Big(\tfrac{m_n}{|\Lambda_0|} + \tfrac{|k|x}{|\Lambda_0|}\Big) \ ,$$

with $m_n = 1$ if $n = 0$ and $m_n = \frac{\langle n\tau\rangle}{|n\tau|}$ if $n \neq 0$. We then get e.g.

$$\begin{aligned} \|\partial_k K_{12}(x)\|^2_{\mathrm{L}^2} &\leq \sup_{n\in\mathbf{Z}} \Big(\int_{|k|\leq 1} \mathrm{d}k\, \tfrac{m_n^2 + k^2x^2}{\langle n\tau\rangle} e^{2b(n\tau)x - 2c(n\tau)xk^2} + \int_{|k|>1} \mathrm{d}k\, \tfrac{m_n^2+k^2x^2}{1+k^2} e^{2b(n\tau)x - |k|x}\Big) \\ &\leq C \sup_{n\in\mathbf{Z}} \left(e^{b(n\tau)x}\left(m_n^2 + \sqrt{x}\right)\right) \ . \end{aligned}$$

The proof is completed using Lemmas A.1 and A.2, that $|n\tau|\langle n\tau\rangle^{-1}\langle x\rangle^{\frac{1}{2}} e^{\frac{b(n\tau)x}{4}} \leq 2$ and $\mathcal{P}e^{\frac{b(n\tau)x}{4}} \leq e^{\frac{b(\tau)x}{4}}$ (see also the proof of Lemma A.7), we omit the details. ■

Lemma A.9 *Let $p \geq 2$. There exist a constant $C > 0$ such that*

$$\|e^{-\frac{b(\tau)x}{4}} K_r\|_{\infty,\{\frac{1}{2},1\}} + \|e^{-\frac{b(\tau)x}{4}} K_i\|_{\infty,\{\frac{1}{2},1\}} + \leq C \ ,$$

$$\|e^{-\frac{b(\tau)x}{4}} \partial_y K_r\|_{p,\{1-\frac{1}{2p},2-\frac{1}{p}\}} + \|e^{-\frac{b(\tau)x}{4}} \partial_y K_i\|_{p,\{1-\frac{1}{2p},2-\frac{1}{p}\}} \leq C \ ,$$

while there exists a constant C such that for all $x \geq 0$ and $p \geq 1$, we have

$$\|K_r(x)\|_{\mathrm{L}^1} + \|K_i(x)\|_{\mathrm{L}^1} \leq Ce^{\frac{b(\tau)x}{4}}\left(\tfrac{1}{x^{\frac{1}{2}}} + \tfrac{\langle x\rangle^{\frac{1}{8}}}{x^{\frac{1}{8}}}\left(1 + \tfrac{1}{|\tau|\sqrt{x}}\right)^{\frac{1}{4}}\right) \ ,$$

$$\|K_r(x)\|_{\mathrm{L}^p} + \|K_i(x)\|_{\mathrm{L}^p} \leq Ce^{\frac{b(\tau)x}{4}}\left(\tfrac{\langle x\rangle^{\frac{1}{2}-\frac{1}{2p}}}{x^{1-\frac{1}{2p}}} + \tfrac{\langle x\rangle^{\frac{1}{2}-\frac{3}{8p}}}{x^{1-\frac{7}{8p}}} + \tfrac{\langle x\rangle^{\frac{1}{2}-\frac{3}{8p}}}{|\tau|^{\frac{1}{4p}} x^{1-\frac{3}{4p}}}\right) \ .$$

Proof. We first note that $\mathcal{P}_0 K_i = 0$, $\mathcal{P}K_i = K_i$. We then have $|K_r(x,k)| \leq C\frac{|n\tau|}{\langle n\tau\rangle} e^{\mathrm{Re}(\Lambda_-)x}$, $|K_i(x,k)| \leq C\frac{(n\tau)^2 e^{\mathrm{Re}(\Lambda_-)x}}{k^2+(n\tau)^2} \leq C\min\left(\frac{(n\tau)^2 e^{b(n\tau)x}}{k^2+(n\tau)^2}\ ,\ e^{\mathrm{Re}(\Lambda_-)x}\right)$ and

$$|\partial_k K_r(x,k)| \leq Ce^{\mathrm{Re}(\Lambda_-)x}\tfrac{|n\tau|}{\langle n\tau\rangle}\left(\tfrac{1}{|\Lambda_0|} + \tfrac{|k|x}{|\Lambda_0|}\right) \ ,$$

$$|\partial_k K_i(x,k)| \leq Ce^{\mathrm{Re}(\Lambda_-)x}\left(\tfrac{1}{|n\tau|} + \tfrac{|k|x}{|\Lambda_0|}\right) \ .$$

This shows that

$$\|K_i(x)\|_{\mathrm{L}^\infty} \le C e^{\frac{b(\tau)x}{2}} \frac{\langle x\rangle^{\frac12}}{x} ,$$
$$\|K_i(x)\|_{\mathrm{L}^1} \le C \sup_{n\in\mathbf{Z}, n\neq 0} e^{\frac{b(n\tau)x}{2}} \min\left(|n\tau|, \frac{\langle x\rangle^{\frac12}}{x}\right)^{\frac14} \left(\frac{\langle x\rangle^{\frac12}}{(n\tau)^2 x} + \sqrt{x}\right)^{\frac14} .$$

The proof is completed using $|n\tau|^{-1} \le C|\tau|^{-1}$ if $|n|\ge 1$, $|n\tau|\langle n\tau\rangle^{-1}\langle x\rangle^{\frac12} e^{\frac{b(n\tau)x}{4}} \le 2$ and $\mathcal{P} e^{\frac{b(n\tau)x}{4}} \le e^{\frac{b(\tau)x}{4}}$ (see also the proof of Lemma A.7), we omit the details. ■

Lemma A.10 *Let* $K_c(x,y) = \mathcal{P}_0 \frac{e^{-\frac{y^2}{4x}}}{\sqrt{4\pi x}}$. *We have*

$$\|\partial_y^m (K_1 - K_c)\|_{\infty,\{\frac{m+5}{2}, m+4\}} + \|\partial_y (K_1 - K_c)\|_{1,\{3,\frac92\}} \le C\tau^{-2}\langle\tau\rangle^2$$
$$\|\partial_y^m (K_{12} - K_c)\|_{\infty,\{\frac{m+5}{2}, m+4\}} + \|K_2 - \partial_y K_c\|_{\infty,\{3,5\}} \le C\tau^{-2}\langle\tau\rangle^2$$

for all $m\in\mathbf{N}$.

Proof. We first note that $\mathcal{P}_0|\Lambda_- + k^2| \le Ck^4$, so that

$$\begin{aligned}
\|\partial_y^m(\mathcal{P}_0 K_1(x) - K_c(x))\|_{\mathrm{L}^\infty} &\le \sup_{n\in\mathbf{Z}} \int_{-\infty}^{\infty} \mathrm{d}k\, |k|^m \mathcal{P}_0 e^{\mathrm{Re}(\Lambda_-)x} |1 - e^{-(k^2+\Lambda_-)x}| \\
&\le C x B_{0,4+m}(x/2, 0) \le C\langle x\rangle^{\frac{m+5}{2}} x^{-m-4} , \\
\|\partial_y^m \mathcal{P} K_1(x)\|_{\mathrm{L}^\infty} &\le \sup_{n\in\mathbf{Z}} \int_{-\infty}^{\infty} \mathrm{d}k\, |k|^m \mathcal{P} e^{\mathrm{Re}(\Lambda_-)x} \le C B_{0,m}(x/2,\tau) \\
&\le C\langle x\rangle^{\frac{m+5}{2}} x^{-m-4} \sup_{x\ge 0} (x^3 \langle x\rangle^{-2} e^{\frac{b(\tau)x}{4}}) \le C\tau^{-2}\langle x\rangle^{\frac{m+5}{2}} x^{-m-4} , \\
\|\partial_y^m(\mathcal{P}_0 K_1(x) - K_c(x))\|_{\mathrm{L}^2}^2 &\le \sup_{n\in\mathbf{Z}} \int_{-\infty}^{\infty} \mathrm{d}k\, |k|^{2m} \mathcal{P}_0 e^{2\mathrm{Re}(\Lambda_-)x} |1 - e^{-(k^2+\Lambda_-)x}|^2 \\
&\le C x^2 B_{0,8+2m}(x, 0) \le C\langle x\rangle^{\frac{9+2m}{2}} x^{-7-2m} , \\
\|\partial_y^m \mathcal{P} K_1(x)\|_{\mathrm{L}^2}^2 &\le \sup_{n\in\mathbf{Z}} \int_{-\infty}^{\infty} \mathrm{d}k\, |k|^{2m} \mathcal{P} e^{2\mathrm{Re}(\Lambda_-)x} \le C B_{0,2m}(x,\tau) \\
&\le C\langle x\rangle^{\frac{9+2m}{2}} x^{-7-2m} \sup_{x\ge 0} (x^6 \langle x\rangle^{-4} e^{\frac{b(\tau)x}{4}}) \le C\tau^{-4}\langle x\rangle^{\frac{9+2m}{2}} x^{-7-2m} , \\
\|\partial_y(y(\mathcal{P}_0 K_1(x) - K_c(x)))\|_{\mathrm{L}^2}^2 &\le C x^2 \sup_{n\in\mathbf{Z}} \int_{-\infty}^{\infty} \mathrm{d}k\, |k|^4 \mathcal{P}_0 e^{2\mathrm{Re}(\Lambda_-)x} \left| \frac{2\Lambda_-}{\Lambda_0} + 1 - e^{-(k^2+\Lambda_-)x} \right|^2 \\
&\le C(x^2 B_{0,8}(x,0) + x^4 B_{0,12}(x,0)) \le C\langle x\rangle^{\frac{13}{2}} x^{-9} , \\
\|\partial_y(y\mathcal{P} K_1(x))\|_{\mathrm{L}^2}^2 &\le x^2 \sup_{n\in\mathbf{Z}} \int_{-\infty}^{\infty} \mathrm{d}k\, |k|^4 \mathcal{P} e^{2\mathrm{Re}(\Lambda_-)x} \le C x^2 B_{0,4}(x,\tau) \\
&\le C\langle x\rangle^{\frac{13}{2}} x^{-9} \sup_{x\ge 0} (x^6 \langle x\rangle^{-4} e^{\frac{b(\tau)x}{4}}) \le C\tau^{-4}\langle x\rangle^{\frac{13}{2}} x^{-9} .
\end{aligned}$$

The proof is completed using $\|\partial_y f\|_{\mathrm{L}^1} \le (\|\partial_y f\|_{\mathrm{L}^2}(\|f\|_{\mathrm{L}^2} + \|\partial_y(yf)\|_{\mathrm{L}^2}))^{\frac12}$, $K_2(x) = \partial_y(K_1(x) + K_8(x) + K_{10}(x))$ and $K_{12}(x) = K_1(x) + K_8(x) + K_{10}(x)$. ■

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