Counting Eigenvalues of Biharmonic Operators with Magnetic Fields

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An analysis is given of the spectral properties of perturbations of the magnetic bi-harmonic operator $\Delta_{\mathbf{A}}^2$ in $L^2(\mathbf{R}^n)$, n=2,3,4, where $\mathbf{A}$ is a magnetic vector potential of Aharonov-Bohm type, and bounds for the number of negative eigenvalues are established. Key elements of the proofs are newly derived Rellich inequalities for $\Delta_{\mathbf{A}}^2$ which are shown to have a bearing on the limiting cases of embedding theorems for Sobolev spaces $H^2(\mathbf{R}^n)$.

1 Introduction

Let

$$D := -\frac{\partial^2}{\partial r^2} - \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Lambda_\omega \tag{1.1}$$

in $L^2(\mathbf{R}^n), n \geq 2$, where (r, ω) are polar co-ordinates in $\mathbf{R}^n$ and Λ_ω is a non-negative self-adjoint operator with domain $\mathcal{D}(\Lambda_\omega)$ in $L^2(\mathbf{S}^{n-1})$ with a discrete spectrum. In [7] it was proved that for all f in the set

$$\mathcal{D}_0 := \{f : f \in C_0^\infty(\mathbf{R}^n \setminus \{0\}), \;\; f(r, \cdot) \in \mathcal{D}(\Lambda_\omega) \text{ for } \; 0 < r < \infty, \;\; Df \in L^2(\mathbf{R}^n)\}, \tag{1.2}$$

we have

$$\int_{\mathbf{R}^n} |Df|^2 d\mathbf{x} \geq C(n) \int_{\mathbf{R}^n} \frac{|f|^2}{|\mathbf{x}|^4} d\mathbf{x} \tag{1.3}$$

where

$$C(n) = \inf_{m \in \mathcal{I}} \left\{ \lambda_m + \frac{n(n-4)}{4} \right\}^2 \tag{1.4}$$

and $\{\lambda_m\}_{m \in \mathcal{I}}$ is the set of eigenvalues of Λ_ω. The celebrated inequality of Rellich (see [15, 16]) is the special case $D = -\Delta$ and Λ_ω is then the Laplace-Beltrami operator. The main motivation behind [7] was to investigate the case of $n = 4$ when the Rellich inequality fails and the case $n = 2$ when the function class has to be restricted. Our approach was reminiscent of that of Laptev and Weidl in [10] for the Hardy inequality which is invalid in $\mathbf{R}^2$. We took $D = -\Delta_{\mathbf{A}}$, the *magnetic Laplacian* associated with a magnetic potential $\mathbf{A}$ of Aharonov-Bohm type. The magnetic field $\mathrm{curl}\mathbf{A}$ is supported on a co-ordinate hyperplane $\mathcal{L}_n$ of co-dimension 2 in $\mathbf{R}^n$, so that $\mathbf{R}^n \setminus \mathcal{L}_n$ is *not* simply connected. Problems for Schrödinger operators involving such Aharonov-Bohm type magnetic fields in $\mathbf{R}^3$ with support on the $x_3 - \mathrm{axis}$ are considered in [11].

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Intimately connected with the Rellich inequality for $D = -\Delta$ are analogues of the Cwikel-Lieb-Rosenblum inequalities, namely, for $0 \le V \in L^{n/4}(\mathbf{R}^n)$ and $n > 4$, the number $N(\Delta^2 - V)$ of negative eigenvalues of $\Delta^2 - V$ satisfies

$$N(\Delta^2 - V) \le const. \int_{\mathbf{R}^n} V(\mathbf{x})^{n/4} d\mathbf{x}. \tag{1.5}$$

When $n \le 4$ and $V \in L^{n/4}(\mathbf{R}^n)$, there is no such bound; indeed $\Delta^2 - V$ may not even be bounded below. Estimates of different types were derived in [5] for $n = 3$ and [4] for $n = 2$. There are some results for the case $n = 4$ in [3], [4], [19], [20], but the article of greatest relevance to us here is [9] where an upper bound is obtained for $N(\Delta^2 + \frac{c}{|\mathbf{x}|^2} - V)$ (c a positive constant) which coincides with (1.5) when V is radial.

In this paper we analyse the spectral properties of perturbations of the *magnetic bi-harmonic operator* $\Delta^2_{\mathbf{A}}$, mainly in the cases $n = 2, 3, 4$. The perturbations are of the form $B_+ - B_-$, where the $B_\pm$ are non-negative symmetric operators which are small in the form sense relative to $\Delta^2_{\mathbf{A}}$ and are such that the essential spectrum of $\Delta^2_{\mathbf{A}} + B_+ - B_-$ coincides with $[0, \infty)$. Upper bounds of Cwikel-Lieb-Rosenblum type are derived for $N(\Delta^2_{\mathbf{A}} + B_+ - B_-)$ when the "magnetic flux" $\tilde{\Psi}$ is not an integer. Similar results for the magnetic Laplacian in $\mathbf{R}^2$ were obtained in [1].

To establish our main results, various inequalities are proved which have an interesting bearing on the limiting cases of embedding theorems for the Sobolev spaces $H^2(\mathbf{R}^n)$. Denoting the completion of $C_0^\infty(\mathbf{R}^n \setminus \mathcal{L}_n)$ by $H_{\mathbf{A}}(\mathbf{R}^n)$, with norm given by

$$\|f\|^2_{\mathbf{A}} := \|\Delta_{\mathbf{A}} f\|^2 + \|f\|^2,$$

where $\|\cdot\|$ denotes the $L^2(\mathbf{R}^n)$ norm, it is proved, in particular, that $H_{\mathbf{A}}(\mathbf{R}^4) \hookrightarrow L^\infty(\mathbf{R}_+; L^2(\mathbf{S}^{n-1}), dr)$ and $H_{\mathbf{A}}(\mathbf{R}^2) \hookrightarrow \{f : \int_{\mathbf{S}^1} f(\cdot, \omega) d\omega \in C^{0,1}(\mathbf{R}^2)\}$. These embeddings are not valid when $\tilde{\Psi} \in \mathbf{Z}$.

We shall write $a \lesssim b$ to mean that a is bounded above by a constant multiple of b, the multiple being independent of any variables in a and b.

2 Some inequalities

We first establish some integral inequalities which play a pivotal role in subsequent analysis.

Theorem 1 *For D and $\mathcal{D}_0$ defined in (1.1) and (1.2),*

$$\begin{aligned} &\|Df\|^2 + \max_m \{\lambda_m(2 - \lambda_m)\} \int_{\mathbf{R}^n} \frac{|f(\mathbf{x})|^2}{|\mathbf{x}|^4} d\mathbf{x} \\ &\quad \ge \sup_{r \in (0,\infty)} \left\{ r^{n-2} \int_{\mathbf{S}^{n-1}} |\tfrac{\partial f}{\partial r}|^2 d\omega + 2 \min_m \{\lambda_m\} r^{n-4} \int_{\mathbf{S}^{n-1}} |f|^2 d\omega \right\} \end{aligned} \tag{2.1}$$

for $f \in \mathcal{D}_0$.

Proof. Let $L_r := -\frac{\partial^2}{\partial r^2} - \frac{n-1}{r} \frac{\partial}{\partial r}$. For all $f \in \mathcal{D}_0$ set

$$F_m(r) := \int_{\mathbf{S}^{n-1}} f(r, \omega) \overline{u_m(\omega)} d\omega, \tag{2.2}$$

where u_m, $m \in \mathcal{I}$, are the normalised eigenvectors of Λ_m; since Λ_m is assumed to have a discrete spectrum, $\{u_m\}_{m \in \mathcal{I}}$ is an orthonormal basis of $L^2(\mathbf{S}^{n-1})$. We have on using Parseval's identity that

$$\begin{aligned} \int_{\mathbf{R}^n} |Df|^2 d\mathbf{x} = &\int_{\mathbf{R}^n} |L_r f|^2 d\mathbf{x} + 2\Re e[\int_{\mathbf{R}^n} L_r f \overline{\Lambda_\omega f} \tfrac{d\mathbf{x}}{|\mathbf{x}|^2}] + \int_{\mathbf{R}^n} |\Lambda_\omega f|^2 \tfrac{d\mathbf{x}}{|\mathbf{x}|^4} \\ = &\sum_m \{ \int_0^\infty |L_r F_m|^2 r^{n-1} dr + 2\Re e[\lambda_m \int_0^\infty \overline{F_m} L_r F_m r^{n-3} dr] \\ &+ \lambda_m^2 \int_0^\infty |F_m(r)|^2 r^{n-5} dr \} \\ =: &\sum_m \{ I_1 + 2\lambda_m I_2 + \lambda_m^2 I_3 \}. \end{aligned} \tag{2.3}$$

It follows on integration by parts that

$$\begin{aligned} I_1 &= \int_0^\infty \left[|F_m''|^2 + 2\frac{n-1}{r}\Re e\{F_m''\overline{F_m'}\} + \frac{(n-1)^2}{r^2}|F_m'|^2\right] r^{n-1}dr \\ &= \int_0^\infty \left[|F_m''|^2 + \frac{n-1}{r^2}|F_m'|^2\right] r^{n-1}dr; \qquad (2.4) \\ I_2 &= \int_0^\infty [|F_m'|^2 r^{-2} + (n-4)|F_m|^2 r^{-4}] r^{n-1}dr; \qquad (2.5) \end{aligned}$$

and

$$I_3 = \int_0^\infty \frac{|F_m|^2}{r^4} r^{n-1}dr.$$

Thus,

$$\|Df\|^2 = \sum_m \big\{ \int_0^\infty \big(|F_m''|^2 + \tfrac{n-1+2\lambda_m}{r^2}|F_m'|^2 + \tfrac{2(n-4)\lambda_m + \lambda_m^2}{r^4}|F_m|^2\big) r^{n-1}dr \big\}. \qquad (2.6)$$

Since $F_m \in C_0^\infty(0,\infty)$,

$$2\Re e \int_0^r t^{n-4}\overline{F_m(t)}F_m'(t)dt = r^{n-4}|F_m(r)|^2 - (n-4)\int_0^r t^{n-5}|F_m(t)|^2 dt$$

and

$$2\Re e \int_0^r t^{n-2}\overline{F_m'(t)}F_m''(t)dt = r^{n-2}|F_m'(r)|^2 - (n-2)\int_0^r t^{n-3}|F_m'(t)|^2 dt,$$

which imply that

$$r^{n-4}|F_m(r)|^2 \le \int_0^r |F_m'(t)|^2 t^{n-3}dt + (n-3)\int_0^r t^{n-5}|F_m(t)|^2 dt$$

and

$$r^{n-2}|F_m'(r)|^2 \le \int_0^r |F_m''(t)|^2 t^{n-1}dt + (n-1)\int_0^r t^{n-3}|F_m'(t)|^2 dt.$$

By substituting these inequalities into (2.6) and using Parseval's identity, we may conclude that, for $0 < r < \infty$,

$$\begin{aligned} \|Df\|^2 \ge\ & \sum_m \big\{ r^{n-2}|F_m'(r)|^2 + 2\lambda_m r^{n-4}|F_m(r)|^2 \\ & + \int_0^\infty \tfrac{\lambda_m(\lambda_m - 2)}{r^4}|F_m(r)|^2 r^{n-1}dr \big\} \\ \ge\ & r^{n-2}\int_{\mathbf{S}^{n-1}} |\tfrac{\partial f}{\partial r}|^2 d\omega + 2\min_m\{\lambda_m\} r^{n-4}\int_{\mathbf{S}^{n-1}} |f|^2 d\omega \\ & -\max_m\{\lambda_m(2-\lambda_m)\}\int_{\mathbf{R}^n} \tfrac{|f(\mathbf{x})|^2}{|\mathbf{x}|^4} d\mathbf{x} \end{aligned}$$

whence (2.1). □

Corollary 1 *For all* $f \in \mathcal{D}_0$

$$\begin{aligned} \Big\| r^{n-2}\|\tfrac{\partial f}{\partial r}\|^2_{L^2(\boldsymbol{S}^{n-1})} + 2 & \min_m\{\lambda_m\} r^{n-4}\|f\|^2_{L^2(\boldsymbol{S}^{n-1})} \Big\|_{L^\infty(0,\infty)} \\ & \le \|Df\|^2 + \max_m\{\lambda_m(2-\lambda_m)\}\||\mathbf{x}|^{-2}f\|^2 \qquad (2.7) \\ & \le \left(1 + \tfrac{\max_m\{\lambda_m(2-\lambda_m)\}}{C(n)}\right)\|Df\|^2 \end{aligned}$$

if, for the last inequality, the constant $C(n)$ in (1.4) is not zero.

P r o o f. The proof follows from (1.3) and Theorem 1 above. □

Note that $\max\{\lambda_m(2-\lambda_m)\} \leq 1$, with equality attained only if some $\lambda_m = 1$. In particular, when $n = 4$ and $\min_m\{\lambda_m\} > 0$, then

$$\| \|f\|_{L^2(\mathbf{S}^3)} \|_{L^\infty(0,\infty)} \lesssim \|Df\|^2$$

Hence, for radial $f \in \mathcal{D}_0$, it follows that $f \in L^\infty(0,\infty)$.

We shall be concerned with the case when $D = -\Delta_{\mathbf{A}} := (\nabla_{\mathbf{A}})^2$, where $\nabla_{\mathbf{A}} := \nabla - i\mathbf{A}$. We shall assume, without loss of generality (see [21], Section 8.4.2) that $\mathbf{A}\cdot\mathbf{x} = 0$ (Poincaré gauge) and $\mathbf{A}$ is of Aharonov-Bohm type. The associated magnetic field $\operatorname{curl} \mathbf{A} = \mathbf{0}$ outside a co-ordinate hyperplane $\mathcal{L}_n$ and specifically, in the cases $n = 2, 3, 4$, which are our main concern, we have the following from [7], §3:

n=2: Let $|\mathbf{x}| = r, \omega = \mathbf{x}/|\mathbf{x}| = (\cos\theta, \sin\theta)$ and for $\mathbf{x} \notin \mathcal{L}_2 = \{0\}$,

$$\mathbf{A}(r,\theta) = \frac{1}{r}\Psi(\theta)(-\sin\theta, \cos\theta), \quad \Psi \in L^\infty(\mathbf{S}^1), \quad \Psi(0) = \Psi(2\pi).$$

Then,

$$-\Delta_{\mathbf{A}} = -\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Lambda_\omega, \qquad \Lambda_\omega = \left(i\frac{\partial}{\partial\theta} + \Psi(\theta)\right)^2.$$

The eigenvalues of Λ_ω are $\lambda_m = (m+\tilde{\Psi})^2, m \in \mathbf{Z}$, where $\tilde{\Psi} = \frac{1}{2\pi}\int_0^{2\pi}\Psi(\theta)d\theta$ is the magnetic flux. By gauge invariance, we may assume that $\tilde{\Psi} \in [0,1)$. It follows that the constant $C(2)$ in (1.4) is

$$\begin{aligned} C(2) &= \inf_{m\in\mathbf{Z}}\{(m+\tilde{\Psi})^2 - 1\}^2 \\ &= \begin{cases} (\tilde{\Psi}^2 - 1)^2 & \text{if } \tilde{\Psi} \in [\frac{1}{2}, 1) \\ \tilde{\Psi}^2(\tilde{\Psi} - 2)^2 & \text{if } \tilde{\Psi} \in [0, \frac{1}{2}). \end{cases} \end{aligned}$$

n=3: For $\omega = (\cos\theta_1, \sin\theta_1\cos\theta_2, \sin\theta_1\sin\theta_2)$, $\theta_1 \in (0,\pi)$, $\theta_2 \in (0,2\pi)$ and for $\mathbf{x} \notin \mathcal{L}_3 = \{\mathbf{x} : r\sin\theta_1 = 0\}$

$$\mathbf{A}(r,\omega) = \frac{1}{r\sin\theta_1}\Psi(\theta_2)(0, -\sin\theta_2, \cos\theta_2),$$

with $\Psi \in L^\infty(\mathbf{S}^1)$, and $\Psi(0) = \Psi(2\pi)$. In this case, we have

$$-\Delta_{\mathbf{A}} = -\frac{\partial^2}{\partial r^2} - \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Lambda_\omega$$

and

$$\Lambda_\omega = -\frac{\partial^2}{\partial\theta_1^2} - \cot\theta_1\frac{\partial}{\partial\theta_1} + \frac{1}{\sin\theta_1^2}\left(i\frac{\partial}{\partial\theta_2} + \Psi(\theta_2)\right)^2.$$

The eigenvalues of Λ_ω can be enumerated as

$$\lambda_m = (m-\tilde{\Psi})(m-\tilde{\Psi}+1), m \in \mathbf{Z}',$$

where $\mathbf{Z}' = \{m \in \mathbf{Z} : (m-\tilde{\Psi})(m-\tilde{\Psi}+1) \geq 0\}$. It follows that

$$C(3) = \inf_{m\in\mathbf{Z}'}\left\{(m-\tilde{\Psi})(m-\tilde{\Psi}+1) - \frac{3}{4}\right\}^2.$$

Note that $C(3) = 0$ if $\tilde{\Psi} = 1/2$.

n=4: In this case $\omega = (\cos\theta_1, \sin\theta_1\cos\theta_2, \sin\theta_1\sin\theta_2\cos\theta_3, \sin\theta_1\sin\theta_2\sin\theta_3)$, where $\theta_1,\theta_2 \in (0,\pi), \theta_3 \in (0,2\pi)$. For $\mathbf{x} \notin \mathcal{L}_4 = \{\mathbf{x} : r\sin\theta_1\sin\theta_2 = 0\}$,

$$\mathbf{A}(r,\omega) = \frac{1}{r\sin\theta_1\sin\theta_2}\Psi(\theta_3)(0,0,-\sin\theta_3,\cos\theta_3),$$

with $\Psi \in L^\infty(\mathbf{S}^1)$, $\Psi(0) = \Psi(2\pi)$. Now,

$$-\Delta_{\mathbf{A}} = -\frac{\partial^2}{\partial r^2} - \frac{3}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Lambda_\omega$$

and

$$\Lambda_\omega = -\frac{\partial^2}{\partial\theta_1^2} - 2\cot\theta_1\frac{\partial}{\partial\theta_1} + \frac{1}{\sin\theta_1^2}\left[-\frac{\partial^2}{\partial\theta_2^2} - \cot\theta_2\frac{\partial}{\partial\theta_2} + \frac{1}{\sin\theta_2^2}\left(i\frac{\partial}{\partial\theta_3} + \Psi(\theta_3)\right)^2\right].$$

The eigenvalues of Λ_ω can be enumerated as

$$\lambda_m = (m+\tilde{\Psi})^2 - 1, \;\; m \in \mathbf{Z}^{''},$$

where $\mathbf{Z}^{''} = \{m \in \mathbf{Z} : (m+\tilde{\Psi})^2 \geq 1\}$. It follows that

$$C(4) = \min\{(1+\tilde{\Psi})^2 - 1, (-2+\tilde{\Psi})^2 - 1\}\}.$$

From above we see that for $n = 2,4$, $C(n) > 0$ and $\min\{\lambda_{\rm m}\} > 0$ if $\tilde{\Psi} \in (0,1)$. For $n = 3$, $\min\{\lambda_{\rm m}\} > 0$ if $\tilde{\Psi} \in (0,1)$ and $C(3) > 0$ if $\tilde{\Psi} \in [0,\frac{1}{2}) \cup (\frac{1}{2},1)$. A consequence of Corollary 1 is therefore

Corollary 2 *If $\tilde{\Psi} \in (0,1)$ when $n = 2,4$ and $\tilde{\Psi} \in (0,\frac{1}{2}) \cup (\frac{1}{2},1)$ when $n = 3$, we have*

$$\left\| r^{n-2}\|\partial f/\partial r\|_{L^2(\mathbf{S}^{n-1})}\right\|_{L^\infty(0,\infty)}, \; \left\| r^{n-4}\|f\|_{L^2(\mathbf{S}^{n-1})}\right\|_{L^\infty(0,\infty)} \lesssim \|\Delta_{\mathbf{A}} f\|^2 \tag{2.8}$$

for all $f \in \mathcal{D}_0$.

3 Forms and operators

We shall assume hereafter that $n = 2,3$, or 4, adopt the notation of Section 2, and make the assumptions necessary for Corollary 2 to hold.

Let $\mathcal{D}_0^{'} = C_0^\infty(\mathbf{R}^n \setminus \mathcal{L}_n)$ and let $S_{\mathbf{A}}$ denote the Friedrichs extension of the restriction of $-\Delta_{\mathbf{A}}$ to $\mathcal{D}_0^{'}$. Clearly $\mathcal{D}_0^{'} \subseteq \mathcal{D}_0$ and so Corollary 2 holds on $\mathcal{D}_0^{'}$. The form domain of $S_{\mathbf{A}}$, $\mathcal{Q}(S_{\mathbf{A}})$, is the completion of $\mathcal{D}_0'$ with respect to $[\|\nabla_{\mathbf{A}} f\|^2 + \|f\|^2]^{\frac{1}{2}}$. Let $\mathcal{H}(S_{\mathbf{A}})$ be the Hilbert space defined by the inner product

$$\begin{aligned}(\varphi,\psi)_{S_{\mathbf{A}}} &= ((S_{\mathbf{A}}+i)\varphi, (S_{\mathbf{A}}+i)\psi)_{L^2(\mathbf{R}^n)} \\ &= (S_{\mathbf{A}}\varphi, S_{\mathbf{A}}\psi)_{L^2(\mathbf{R}^n)} + (\psi,\phi)_{L^2(\mathbf{R}^n)}, \qquad \varphi,\psi \in \mathcal{D}(S_{\mathbf{A}}),\end{aligned}$$

which induces the graph norm associated with $S_{\mathbf{A}} : \mathcal{D}(S_{\mathbf{A}}) \to L^2(\mathbf{R}^n)$.

Lemma 1 *Suppose that the hypothesis of Corollary 2 is satisfied and let B_+ be the operator of multiplication by the function b_+, where*

$$0 \leq b_+ \in L^1(\mathbf{R}_+; L^\infty(\mathbf{S}^{n-1}); r^3 dr) \equiv L^1(\mathbf{R}_+; r^3 dr) \otimes L^\infty(\mathbf{S}^{n-1}).$$

Then, $B_+^{\frac{1}{2}} : \mathcal{H}(S_{\mathbf{A}}) \to L^2(\mathbf{R}^n)$ is bounded and $B_+^{\frac{1}{2}}(S_{\mathbf{A}}+i)^{-1}$ is compact on $L^2(\mathbf{R}^n)$.

Proof. For $\varphi \in \mathcal{D}_0' = C_0^\infty(\mathbf{R}^n \setminus \mathcal{L}_n)$

$$\begin{aligned} |(B_+\varphi,\varphi)| &= \int_{\mathbf{S}^{n-1}}\int_0^\infty b_+(r,\omega)|\varphi(r,\omega)|^2 r^{n-1}drd\omega \\ &\leq \int_0^\infty \|b_+\|_{L^\infty(\mathbf{S}^{n-1})} r^3 dr \sup_{0<r<\infty}\left(r^{n-4}\int_{\mathbf{S}^{n-1}}|\varphi|^2 d\omega\right) \\ &\lesssim \|b_+\|_{L^1(\mathbf{R}_+;L^\infty(\mathbf{S}^{n-1});r^3dr)}\|S_{\mathbf{A}}\varphi\|^2 \end{aligned} \tag{3.1}$$

by Corollary 1. Thus, $\mathcal{D}(S_{\mathbf{A}})$ lies in the form domain of B_+ and $B_+^{\frac{1}{2}} : \mathcal{H}(S_{\mathbf{A}}) \to L^2(\mathbf{R}^n)$ is bounded.

Let $\varphi_\ell \rightharpoonup 0$ in $L^2(\mathbf{R}^n)$ and set $\psi_\ell = (S_{\mathbf{A}}+i)^{-1}\varphi_\ell$. Then, $\psi_\ell \in \mathcal{D}(S_{\mathbf{A}})$ and $\psi_\ell \rightharpoonup 0$ in $\mathcal{H}(S_{\mathbf{A}})$. Given $\epsilon > 0$, choose $\tilde{b}_+$ such that

$$\tilde{b}_+ \in C_0^\infty(\mathbf{R}_+; L^\infty(\mathbf{S}^{n-1})), \quad \text{supp}\, \tilde{b}_+ \subset \Omega_\epsilon = B(0;k_\epsilon)\setminus B(0;1/k_\epsilon),$$

$$\|\tilde{b}_+\|_{L^\infty(\mathbf{R}^n)} < k_\epsilon, \text{ and } \left\|\|b_+ - \tilde{b}_+\|_{L^\infty(\mathbf{S}^{n-1})}\right\|_{L^1(\mathbf{R}_+;r^3dr)} < \epsilon$$

for some $k_\epsilon > 1$.

For some constant $C > 0$

$$\begin{aligned} \|B_+^{\frac{1}{2}}(S_{\mathbf{A}}+i)^{-1}\varphi_\ell\|^2 &= \|B_+^{\frac{1}{2}}\psi_\ell\|^2 = (B_+\psi_\ell,\psi_\ell) \\ &= \int_{\mathbf{R}^n}\tilde{b}_+|\psi_\ell|^2 d\mathbf{x} + \int_{\mathbf{R}^n}(b_+ - \tilde{b}_+)|\psi_\ell|^2 d\mathbf{x} \\ &\leq k_\epsilon \int_{\Omega_\epsilon}|\psi_\ell|^2 d\mathbf{x} \\ &\quad + \|\|b_+ - \tilde{b}_+\|_{L^\infty(\mathbf{S}^{n-1})}\|_{L^1(\mathbf{R}_+;r^3dr)}\sup_{0<r<\infty}\left\{r^{n-4}\int_{\mathbf{S}^{n-1}}|\psi_\ell|^2 d\omega\right\} \\ &\leq k_\epsilon\int_{\Omega_\epsilon}|\psi_\ell|^2 d\mathbf{x} + \epsilon C\|S_{\mathbf{A}}\psi_\ell\|^2 \end{aligned} \tag{3.2}$$

by (2.8).

For $u \in \mathcal{D}_0' = C_0^\infty(\mathbf{R}^n \setminus \mathcal{L}_n)$

$$\left(\frac{\partial}{\partial x_j}|u|\right)(\mathbf{x}) = \begin{cases} \Re e\left(\frac{\overline{u}}{|u|}\frac{\partial}{\partial x_j}u\right)(\mathbf{x}), & u(\mathbf{x}) \neq 0 \\ 0, & u(\mathbf{x}) = 0. \end{cases}$$

Since $\Re e[\overline{u}\frac{\partial}{\partial x_j}u] = \Re e[\overline{u}(\frac{\partial}{\partial x_j} + iA_j)u]$, then we have the diamagnetic inequality

$$\left|\nabla|u(\mathbf{x})|\right| \leq \left|\nabla_{\mathbf{A}}u(\mathbf{x})\right| \tag{3.3}$$

as in [12], p. 193. Since

$$\begin{aligned} \|\nabla_{\mathbf{A}}\psi_\ell\|^2 &= (S_{\mathbf{A}}\psi_\ell,\psi_\ell) \leq \|(S_{\mathbf{A}}+i)\psi_\ell\|^2/2 \\ &= \|\phi_\ell\|^2/2 \end{aligned}$$

it follows from (3.3) that the sequence $\{|\psi_\ell|\}$ must be bounded in $H^1(\mathbf{R}^n)$. Since $H^1(\Omega_\epsilon)$ is compactly embedded in $L^2(\Omega_\epsilon)$, it follows that $\psi_\ell \to 0$ in $L^2(\Omega_\epsilon)$. The result now follows from (3.2) and the fact that ϵ can be chosen arbitrarily small.

□

Remark 1 *The compactness of $B_+^{\frac{1}{2}}(S_{\mathbf{A}}+i)^{-1} : L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$ established in Lemma 1 implies that $B_+^{\frac{1}{2}}$ is $S_{\mathbf{A}}$-compact, and consequently, by* [6] *(Corollary III.7.7), $B_+^{\frac{1}{2}}$ has $S_{\mathbf{A}}$-bound zero. This implies that the form (B_+u,u) is relatively bounded with respect to the form $(S_{\mathbf{A}}u, S_{\mathbf{A}}u)$ with relative bound zero. Therefore, $\Delta_{\mathbf{A}}^2 + B_+$ is defined in the form sense and has form domain $\mathcal{D}(S_{\mathbf{A}})$, cf. Kato* [8] *Theorem VI.1.33.*

Lemma 2 *Let $n = 4$ and suppose that the hypothesis of Corollary 2 is satisfied. For*

$$0 \le V \in L^1(\mathbf{R}_+; L^\infty(\mathbf{S}^3), r^3 dr),$$

let B_- be a nonnegative self-adjoint operator with form domain $\mathcal{D}(S_{\mathbf{A}})$ which is such that, given $\epsilon > 0$,

$$(B_-\varphi, \varphi) \le \epsilon \int_0^\infty \int_{\mathbf{S}^3} r \Big| \frac{\partial}{\partial r}\varphi(r,\omega)\Big|^2 d\omega dr + k(\epsilon) \int_0^\infty \int_{\mathbf{S}^3} V(r,\omega)|\varphi(r,\omega)|^2 r^3 d\omega dr. \tag{3.4}$$

for all $\varphi \in \mathcal{D}(S_{\mathbf{A}})$ and some constant $k(\epsilon)$. Then $B_-^{\frac{1}{2}}(S_{\mathbf{A}} + i)^{-1}$ is compact on $L^2(\mathbf{R}^4)$.

P r o o f. As in the proof of Lemma 1, given $\delta > 0$, choose $\tilde{V}$ such that for some $k_\delta > 1$

$$\tilde{V} \in C_0^\infty(\mathbf{R}_+; L^\infty(\mathbf{S}^3)), \quad \text{supp}\, \tilde{V} \subset \Omega_\delta = B(0; k_\delta) \setminus B(0; 1/k_\delta),$$

$$\|\tilde{V}\|_{L^\infty(\mathbf{R}^4)} < k_\delta, \text{ and } \big\| \|V - \tilde{V}\|_{L^\infty(\mathbf{S}^3)} \big\|_{L^1((0,\infty); r^3 dr)} < \delta.$$

Let $\varphi_\ell \rightharpoonup 0$ in $L^2(\mathbf{R}^4)$ with $\|\varphi_\ell\| \le 1$ and set $\psi_\ell = (S_{\mathbf{A}} + i)^{-1}\varphi_\ell$. Then, $\psi_\ell \rightharpoonup 0$ in $\mathcal{H}(S_{\mathbf{A}})$ and, on using (3.4)

$$\begin{aligned} \|B_-^{\frac{1}{2}}(S_{\mathbf{A}} + i)^{-1}\varphi_\ell\| \le \quad & \epsilon \textstyle\int_0^\infty \int_{\mathbf{S}^3} r \big| \frac{\partial}{\partial r}\psi_\ell(r,\omega)\big|^2 d\omega dr \\ & + k(\epsilon)\big\{k_\delta \textstyle\int_{\Omega_\delta} |\psi_\ell(\mathbf{x})|^2 d\mathbf{x} + \delta C \sup\limits_{0<r<\infty} \int_{\mathbf{S}^3} |\psi_\ell(r,\omega)|^2 d\omega\big\} \\ \le \quad & \epsilon \textstyle\int_0^\infty \int_{\mathbf{S}^3} r \big| \frac{\partial}{\partial r}\psi_\ell(r,\omega)\big|^2 d\omega dr \\ & + k(\epsilon)\big\{k_\delta \textstyle\int_{\Omega_\delta} |\psi_\ell(\mathbf{x})|^2 d\mathbf{x} + \delta C \|S_{\mathbf{A}}\psi_\ell\|^2\big\} \end{aligned}$$

by (2.8). Now note that from (2.4) for the case $n = 4$,

$$3 \int_0^\infty \int_{\mathbf{S}^3} r \Big| \frac{\partial}{\partial r}\psi_\ell(r,\omega)\Big|^2 d\omega dr \le \sum_m I_1 \le \|S_{\mathbf{A}}\psi_\ell\|^2$$

by (2.3). Consequently,

$$\|B_-^{\frac{1}{2}}(S_{\mathbf{A}} + i)^{-1}\varphi_\ell\| \le \quad \tfrac{\epsilon}{3}\|\varphi_\ell\|^2 + k(\epsilon)\big\{k_\delta \textstyle\int_{\Omega_\delta} |\psi_\ell(\mathbf{x})|^2 d\mathbf{x} + \delta C \|\varphi_\ell\|^2\big\}.$$

On allowing $\ell \to \infty$ we may conclude as in the proof of Lemma 1 that the last line is bounded by

$$\epsilon + Ck(\epsilon)\delta.$$

Since δ and ϵ are arbitrary, the lemma follows.

□

Examples of multiplication operators B_- which satisfy the hypothesis of Lemma 2 are given by

Lemma 3 *Let $b(r) \ge 0$ on $(0, \infty)$ and*

$$\int_0^\infty \int_r^\infty b(s)s^2 ds dr < \infty, \qquad \int_0^\infty r\Big(\int_r^\infty b(s)s^2 ds\Big)^2 dr < \infty. \tag{3.5}$$

Then there is a function $W \in L^1((0,\infty); r^3 dr)$ such that for any $\varepsilon > 0$,

$$\int_0^\infty b(r)|\varphi(r)|^2 r^3 dr \le \epsilon \int_0^\infty r|\varphi'(r)|^2 dr + k(\epsilon) \int_0^\infty W(r)|\varphi(r)|^2 r^3 dr \tag{3.6}$$

for all $\varphi \in C_0^\infty(0,\infty)$ and some constant $k(\varepsilon)$. We can take

$$r^3 W(r) = r\Big(\int_r^\infty b(s)s^2 ds\Big)^2 + \int_r^\infty b(s)s^2 ds. \tag{3.7}$$

P r o o f. Let

$$r^{\frac{3}{2}}\sqrt{\omega(r)} = \int_r^\infty b(s)s^2 ds. \tag{3.8}$$

According to Opic and Kufner [13], Theorem 5.9, p.63, the inequality

$$\int_0^\infty b(r)|\varphi(r)|^2 r^3 dr \le c\int_0^\infty \frac{d}{dr}(r|\varphi(r)|^2) r^{\frac{3}{2}}\sqrt{\omega(r)}dr \tag{3.9}$$

is satisfied for some $c > 0$ if and only if

$$C := \sup_{0<r<\infty}\Big[\int_r^\infty t^2 b(t)dt \cdot \sup_{0<t<r}\big\{[t^{\frac{3}{2}}\sqrt{\omega(t)}]^{-1}\big\}\Big] < \infty$$

with $c = C$ the best possible constant for (3.9). On choosing (3.8) it follows that $C \le 1$. From (3.9) with $c \le 1$

$$\begin{aligned}\int_0^\infty b(r)|\varphi(r)|^2 r^3 dr \le\ & 2\int_0^\infty r|\varphi(r)\varphi'(r)|r^{\frac{3}{2}}\sqrt{\omega(r)}dr + \int_0^\infty |\varphi(r)|^2 r^{\frac{3}{2}}\sqrt{\omega(r)}dr \\ \le\ & \epsilon\int_0^\infty r|\varphi'(r)|^2 dr + \tfrac{1}{\epsilon}\int_0^\infty |\varphi(r)|^2\omega(r)r^4 dr + \int_0^\infty |\varphi(r)|^2 r^{\frac{3}{2}}\sqrt{\omega(r)}dr.\end{aligned}$$

The choice (3.7) yields (3.6) with $k(\varepsilon) = \varepsilon^{-1} + 1$ and $W \in L^1((0,\infty); r^3 dr)$ in view of (3.5). □

Theorem 2 *Assume the hypothesis of Lemma 1, and when $n = 4$ assume the hypothesis of Lemma 2. Then we have the following.*

(i) *The form $(S_{\mathbf{A}}u, S_{\mathbf{A}}v)$ is closed with core $\mathcal{D}_0'$ and $S_{\mathbf{A}}^2$ is the associated self-adjoint operator.*

(ii) *The symmetric form $\mathbf{t}_{\mathbf{A}}[u,v] = (S_{\mathbf{A}}u, S_{\mathbf{A}}v) + (B_+u, v)$ is closed and bounded below with core $\mathcal{D}_0'$. Let $T_{\mathbf{A}}^2 = S_{\mathbf{A}}^2 + B_+$ denote the operator associated with $\mathbf{t}_{\mathbf{A}}$. It has form domain $\mathcal{Q}(T_{\mathbf{A}}^2) = \mathcal{Q}(S_{\mathbf{A}}^2) = \mathcal{D}(S_{\mathbf{A}})$ and $\sigma_{ess}(T_{\mathbf{A}}^2) = \sigma_{ess}(S_{\mathbf{A}}^2) = [0,\infty)$.*

(iii) *For $T_{\mathbf{A}}$ defined as the positive square root of $T_{\mathbf{A}}^2$ and $n = 4$, $B_-^{\frac{1}{2}}(T_{\mathbf{A}} + i)^{-1}$ is compact on $L^2(\mathbf{R}^4)$ and $T_{\mathbf{A}}^2 - B_-$ is defined in the form sense with form domain $\mathcal{D}(S_{\mathbf{A}})$. Moreover,*

$$\sigma_{ess}(S_{\mathbf{A}}^2 + B_+ - B_-) = \sigma_{ess}(S_{\mathbf{A}}^2) = [0,\infty).$$

P r o o f. (i) The proof of (i) follows as in [8], Examples VI.2.13 & VI.1.23.

(ii) The first part follows from Remark 1. The fact that $\mathcal{Q}(T_{\mathbf{A}}^2) = \mathcal{Q}(S_{\mathbf{A}}^2) = \mathcal{D}(S_{\mathbf{A}})$ is a consequence of the second representation theorem, [8], p.331.

Since $B_+^{\frac{1}{2}}(S_{\mathbf{A}} + i)^{-1}$ is compact in $L^2(\mathbf{R}^n)$ by Lemma 1, then Theorem IV.4.4 of [6] applies (with $p_2 = 0$) showing that (vi) of Theorem IV.4.2 of [6] holds. (Equivalently, we have that the form $(B_+\cdot,\cdot)$ is relatively form compact with respect to the form $(S_{\mathbf{A}}\cdot, S_{\mathbf{A}}\cdot)$ - see Reed and Simon [14], p. 369.) This fact implies that $\sigma_{ess}(T_{\mathbf{A}}^2) = \sigma_{ess}(S_{\mathbf{A}}^2)$.

(iii) For $f \in \mathcal{D}(S_{\mathbf{A}})$

$$\|S_{\mathbf{A}}f\|^2 \le \|T_{\mathbf{A}}f\|^2 = \|S_{\mathbf{A}}f\|^2 + (B_+f, f)$$

implying that we have for some $C > 0$

$$\|(S_{\mathbf{A}} + i)f\|^2 \le \|(T_{\mathbf{A}} + i)f\|^2 = C\|(S_{\mathbf{A}} + i)f\|^2$$

by (3.1). Then for $f = (T_{\mathbf{A}} + i)^{-1}g$, we have that

$$\|(S_{\mathbf{A}} + i)(T_{\mathbf{A}} + i)^{-1}g\| \le \|g\|,$$

so that from Lemma 2 we have that $B_-^{\frac{1}{2}}(T_{\mathbf{A}} + i)^{-1}$ is compact on $L^2(\mathbf{R}^4)$. The remainder of the proof for part (iii) follows as in the proof for part (ii) given above. □

4 Estimating the number of eigenvalues

Theorem 3 *Let the hypothesis of Lemma 2 be satisfied. Then*

(i) $L_{\mathbf{A}} := S_{\mathbf{A}}^2 + B_+ - B_-$ *is a self-adjoint operator defined in the form sense;*

(ii) $B_-^{\frac{1}{2}}(T_{\mathbf{A}} + i)^{-1}$ *is compact in* $L^2(\mathbf{R}^4)$*, where* $T_{\mathbf{A}}^2 = S_{\mathbf{A}}^2 + B_+$*;*

(iii) $\sigma_{ess}(L_{\mathbf{A}}) = [0, \infty)$*;*

(iv) *if* $\tilde{\Psi} \in (0,1)$*, there exists a positive constant* $C = C(\tilde{\Psi})$ *such that the number* $N(L_{\mathbf{A}})$ *of negative eigenvalues of* $L_{\mathbf{A}}$ *satisfies*

$$N(L_{\mathbf{A}}) \leq C(\tilde{\Psi}) \big\| \|V\|_{L^\infty(\mathbf{S}^3)} \big\|_{L^1((0,\infty); r^3 dr)} \tag{4.1}$$

where V *is given in (3.4) and* $C(\tilde{\Psi})$ *depends on the distance of* $\tilde{\Psi}$ *from* $\{0, 1\}$*.*

Proof. Parts (i)-(iii) are included here for completeness. We refer the reader to Theorem 2 for proofs. For part (iv), we see from (2.6) that for $n = 2, 3, 4$,

$$\|\Delta_{\mathbf{A}} f\|^2 = \sum_m \int_0^\infty \overline{F_m} D_m F_m r^{n-1} dr$$

where F_m is given by (2.2) and

$$D_m = \frac{1}{r^{n-1}} \frac{d^2}{dr^2} \big(r^{n-1} \frac{d^2}{dr^2} \big) - \frac{(n-1) + 2\lambda_m}{r^{n-1}} \frac{d}{dr} \big(r^{n-3} \frac{d}{dr} \big) + \frac{2(n-4)\lambda_m + \lambda_m^2}{r^4}. \tag{4.2}$$

Define

$$W(r) := \|V(r, \cdot)\|_{L^\infty(\mathbf{S}^3)}.$$

Thus, when $n = 4$, since

$$B_- \leq -\frac{\varepsilon}{r^3} \frac{d}{dr} (r \frac{d}{dr}) + k(\epsilon) W(r)$$

from (3.4), we have

$$\begin{aligned} \Delta_{\mathbf{A}}^2 + B_+ - B_- \geq \;& \Delta_{\mathbf{A}}^2 - B_- \\ \geq \;& \bigoplus_{m \in \mathbf{Z}''} \Big\{ \big[D_m + \tfrac{\epsilon}{r^3} \tfrac{d}{dr} (r \tfrac{d}{dr}) - k(\epsilon) W(r) \big] \otimes \mathbf{I}_m \Big\} \end{aligned} \tag{4.3}$$

where

$$\mathbf{Z}'' := \{ m \in \mathbf{Z} : (m + \tilde{\Psi})^2 \geq 1 \},$$

$\mathbf{I}_m$ is the identity on the orthonormal basis $\{u_m\}_{m \in \mathbf{Z}''}$, of $L^2(\mathbf{S}^3)$, and $\lambda_m = (m + \tilde{\Psi})^2 - 1$; see Section 2 above. In (4.3)

$$D_m + \frac{\epsilon}{r^3} \frac{d}{dr} \big(r \frac{d}{dr} \big) = \frac{1}{r^3} \frac{d^2}{dr^2} \big(r^3 \frac{d^2}{dr^2} \big) - \frac{3 + 2\lambda_m - \epsilon}{r^3} \frac{d}{dr} \big(r \frac{d}{dr} \big) + \frac{\lambda_m^2}{r^4}.$$

We also have that

$$\Delta^2 + \frac{c}{r^4} = \bigoplus_{|m| \geq 1} \big\{ [D_m^0 + \frac{c}{r^4}] \otimes \mathbf{I}_m \big\}$$

in which

$$D_m^0 + \frac{c}{r^4} = \frac{1}{r^3} \frac{d^2}{dr^2} \big(r^3 \frac{d^2}{dr^2} \big) - \frac{3 + 2\lambda_m^0}{r^3} \frac{d}{dr} \big(r \frac{d}{dr} \big) + \frac{(\lambda_m^0)^2 + c}{r^4}$$

with $\lambda_m^0 = m^2 - 1$. If $m \in \mathbf{Z}''$, then either $m \geq 1$, in which case

$$\lambda_m \geq \lambda_m^0 + \tilde{\Psi}^2, \qquad \lambda_m^2 \geq (\lambda_m^0)^2 + \tilde{\Psi}^4,$$

or $m \le -2$ implying that

$$\begin{aligned} \lambda_m \ge & \quad (m+1)^2 - 1 + (1-\tilde{\Psi})^2 = \lambda^0_{m+1} + (1-\tilde{\Psi})^2, \\ \lambda_m^2 \ge & \quad (\lambda^0_{m+1})^2 + (1-\tilde{\Psi})^4. \end{aligned}$$

As a consequence, for $m \ge 1$

$$D_m + \frac{\epsilon}{r^3}\frac{d}{dr}\Big(r\frac{d}{dr}\Big) \ge D^0_m + \frac{c}{r^4}$$

if $\epsilon < 2\tilde{\Psi}^2$ and $c < \tilde{\Psi}^4$. For $m \le -2$

$$D_m + \frac{\epsilon}{r^3}\frac{d}{dr}\Big(r\frac{d}{dr}\Big) \ge D^0_{m+1} + \frac{c}{r^4}$$

if $\epsilon < 2(1-\tilde{\Psi})^2$ and $c < (1-\tilde{\Psi})^4$. Hence, if $\epsilon < 2\min\{\tilde{\Psi}^2, (1-\tilde{\Psi})^2\}$ and $c < \min\{\tilde{\Psi}^4, (1-\tilde{\Psi})^4\}$, then

$$\begin{aligned} & N\Big(\bigoplus_{m\ge 1}\big[D_m + \tfrac{\epsilon}{r^3}\tfrac{d}{dr}\big(r\tfrac{d}{dr}\big) - k(\epsilon)W(r)\big]\otimes \mathbf{I}_m\Big) \\ & \qquad \le N\Big(\bigoplus_{m\ge 1}\big[D^0_m + \tfrac{c}{r^4} - k(\epsilon)W(r)\big]\otimes \mathbf{I}_m\Big) \end{aligned}$$

and

$$\begin{aligned} & N\Big(\bigoplus_{m\le -2}\big[D_m + \tfrac{\epsilon}{r^3}\tfrac{d}{dr}\big(r\tfrac{d}{dr}\big) - k(\epsilon)W(r)\big]\otimes \mathbf{I}_m\Big) \\ & \qquad \le N\Big(\bigoplus_{m\le -1}\big[D^0_m + \tfrac{c}{r^4} - k(\epsilon)W(r)\big]\otimes \mathbf{I}_m\Big). \end{aligned}$$

Now, Theorem 1.2 of Laptev and Netrusov [9] and the last two inequalities imply (4.1). □

Theorem 4 *Let $\tilde{\Psi} \in (0,1)$ for $n = 2, 4$ and $\tilde{\Psi} \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ for $n = 3$. Let $V(\mathbf{x}) \ge 0$ and*

$$V \in L^1(\mathbf{R}_+; L^\infty(\mathbf{S}^{n-1}), r^3 dr).$$

Then, the operator $S^2_{\mathbf{A}} - V$ is defined in the form sense and has essential spectrum $[0,\infty)$. Moreover, for λ_m given in § 2 and $n = 2, 3, 4$,

$$N(S^2_{\mathbf{A}} - V) \le {\sum}' \frac{4}{|4\lambda_m + n(n-4)|\sqrt{n^2 + 8\lambda_m}} \int_0^\infty r^3 \|V(r,\cdot)\|_{L^\infty(\mathbf{S}^{n-1})} dr$$

where ${\sum}'$ indicates that all summands less than 1 are omitted.

P r o o f. The fact that $S^2_{\mathbf{A}} - V$ is defined in the form sense and has essential spectrum $[0,\infty)$ follows from Lemma 1 and Theorem 2.

For all $f \in \mathcal{D}'_0 = C_0^\infty(\mathbf{R}^n \setminus \mathcal{L}_n)$ and

$$F_m(r) := \int_{\mathbf{S}^{n-1}} f(r,\omega)\overline{u_m(\omega)}d\omega,$$

we have from (2.6) with $n = 2, 3, 4$,

$$\begin{aligned} \|\Delta_{\mathbf{A}} f\|^2 = & \quad \sum_m \Big\{ \int_0^\infty \big(|F''_m|^2 + \tfrac{n-1+2\lambda_m}{r^2}|F'_m|^2 + \tfrac{2(n-4)\lambda_m + \lambda_m^2}{r^4}|F_m|^2\big) r^{n-1} dr\Big\} \\ \ge & \quad \sum_m \Big\{ \int_0^\infty \big(\tfrac{\frac{1}{4}(n-2)^2 + n - 1 + 2\lambda_m}{r^2}|F'_m|^2 + \tfrac{2(n-4)\lambda_m + \lambda_m^2}{r^4}|F_m|^2\big) r^{n-1} dr\Big\} \end{aligned}$$

by Hardy's inequality. Making the substitutions

$$c(n,\lambda_m) := n^2 + 8\lambda_m \quad \text{and} \quad \varphi_m(r) := \frac{\sqrt{c(n,\lambda_m)}}{2} r^{(n-3)/2} F_m(r)$$

we have that

$$\|\Delta_{\mathbf{A}} f\|^2 \geq \sum_m \int_0^\infty \left[|\varphi_m'|^2 + \tfrac{(n-3)(n-5)+16\lambda_m(\lambda_m+2(n-4))c(n,\lambda_m)^{-1}}{4r^2}|\varphi_m|^2\right]dr.$$

Therefore, for $f \in \mathcal{D}_0'$ and

$$K(n,\lambda_m) := (n-3)(n-5) + 16\lambda_m(\lambda_m + 2(n-4))(n^2+8\lambda_m)^{-1}$$

it follows that

$$((\Delta_{\mathbf{A}}^2 - V)f,f) \geq \sum_m \int_0^\infty \left[|\varphi_m'|^2 + \tfrac{K(n,\lambda_m)}{4r^2}|\varphi_m|^2 - \tfrac{4r^2}{n^2+8\lambda_m}W(r)|\varphi_m|^2\right]dr \tag{4.4}$$

with $W(r) := \|V(r,\cdot)\|_{L^\infty(\mathbf{S}^{n-1})}$. Bargmann's bound for the number of negative eigenvalues (see [2] and [18]) applies to the Sturm-Liouville operator associated with the integral on the right-hand side of (4.4), i.e.,

$$\tau(n,m) := -\frac{d^2}{dr^2} + \frac{K(n,\lambda_m)}{4r^2} - \frac{4r^2}{n^2+8\lambda_m}W(r), \quad n = 2,3,4,$$

if

$$K(n,\lambda_m) > -1. \tag{4.5}$$

In that case,

$$N(\tau(n,m)) < \frac{4}{(n^2+8\lambda_m)\sqrt{K(n,\lambda_m)+1}} \int_0^\infty r^3 W(r)dr.$$

We first note that

$$K(n,\lambda_m) + 1 = [4\lambda_m + n(n-4)]^2/(n^2+8\lambda_m) \geq 0$$

since $\min\{\lambda_m\} > 0$. In fact, it is easy to show that the strict inequality (4.5) holds under the hypothesis of the theorem on substituting the values of λ_m given in §2, namely

$$\begin{array}{ll} \lambda_m = (m+\tilde{\Psi})^2, \ \ m \in \mathbf{Z}, & \text{for } \ n=2; \\ \lambda_m = (m-\tilde{\Psi})(m-\tilde{\Psi}+1), \ \ m \in \mathbf{Z}', & \text{for } \ n=3; \\ \lambda_m = (m+\tilde{\Psi})^2 - 1, \ \ m \in \mathbf{Z}'', & \text{for } \ n=4. \end{array} \tag{4.6}$$

In view of (4.4), the proof is complete. □

Corollary 3 *Assume the hypothesis of Theorem 4. Then, $S_{\mathbf{A}}^2 - V$ has no eigenvalues if for $n = 2$*

$$\int_0^\infty r^3 \|V(r,\cdot)\|_{L^\infty(\mathbf{S}^{n-1})}dr < \left\{ \begin{array}{ll} 2\tilde{\Psi}(2-\tilde{\Psi})\sqrt{3-4\tilde{\Psi}+2\tilde{\Psi}^2} & \textit{for } \ \tilde{\Psi} \in (0,\frac{1}{2}], \\ 2(1-\tilde{\Psi}^2)\sqrt{1+2\tilde{\Psi}^2} & \textit{for } \ \tilde{\Psi} \in (\frac{1}{2},1); \end{array} \right. \tag{4.7}$$

for $n = 3$

$$\int_0^\infty r^3 \|V(r,\cdot)\|_{L^\infty(\mathbf{S}^{n-1})}dr < \left\{ \begin{array}{ll} |\tilde{\Psi}(\tilde{\Psi}+1) - \frac{3}{4}|\sqrt{9+8\tilde{\Psi}(\tilde{\Psi}+1)} & \textit{for } \ \tilde{\Psi} \in [0,\frac{1}{2}), \\ |\tilde{\Psi}^2 - 3\tilde{\Psi} + \frac{5}{4}|\sqrt{25-24\tilde{\Psi}+8\tilde{\Psi}^2} & \textit{for } \ \tilde{\Psi} \in (\frac{1}{2},1); \end{array} \right. \tag{4.8}$$

for $n = 4$

$$\int_0^\infty r^3 \|V(r,\cdot)\|_{L^\infty(\mathbf{S}^{n-1})}dr < \left\{ \begin{array}{ll} 2^{\frac{3}{2}}\tilde{\Psi}(2+\tilde{\Psi})\sqrt{2+2\tilde{\Psi}+\tilde{\Psi}^2} & \textit{for } \ \tilde{\Psi} \in (0,\frac{1}{2}], \\ 2^{\frac{3}{2}}((2-\tilde{\Psi})^2-1)\sqrt{1+(2-\tilde{\Psi})^2} & \textit{for } \ \tilde{\Psi} \in (\frac{1}{2},1). \end{array} \right. \tag{4.9}$$

P r o o f. Define
$$B(\lambda_m, n) := \frac{1}{4}|4\lambda_m + n(n-4)|\sqrt{n^2+8\lambda_m}.$$
Then by Theorem 4 there will be no eigenvalues if
$$\int_0^\infty r^3 \|V(r,\cdot)\|_{L^\infty(\mathbf{S}^{n-1})} dr < \min_m \{B(\lambda_m, n)\}$$
for $m \in \mathbf{Z}$ further restricted according to (4.6).

It is easy to see that the functions $B(x, n)$, $n = 2, 3, 4$, are minimized on $[0, \infty)$ for some $x \in (0, 2)$ and accordingly, in order to minimize $B(\lambda_m, n)$ we may restrict our attention to those λ_m given in (4.6) that lie in the interval $(0, 2)$. Noting that $\lambda_m = \lambda_m(\tilde{\Psi})$, the estimate (4.7) follows from the fact that
$$\min_{m\in\mathbf{Z}} B(\lambda_m, 2) = \min_{\tilde{\Psi}\in(0,1)} \{B(\lambda_0, 2), B(\lambda_{-1}, 2)\};$$
estimate (4.8) follows from the fact that
$$\min_{m\in\mathbf{Z}} B(\lambda_m, 3) = \min_{\tilde{\Psi}\in[0,1)} \{B(\lambda_{-1}, 3), B(\lambda_1, 3)\};$$
and estimate (4.9) follows from the fact that
$$\min_{m\in\mathbf{Z}} B(\lambda_m, 4) = \min_{\tilde{\Psi}\in(0,1)} \{B(\lambda_1, 4), B(\lambda_{-2}, 4)\}.$$
□

5 Additional remarks on embedding results

The following optimal embedding results for the Sobolev space $H^2(\mathbf{R}^n) \equiv W^{2,2}(\mathbf{R}^n)$ are known (see [12], p.213 and [6], p.263):
$$H^2(\mathbf{R}^n) \hookrightarrow \begin{cases} L^q(\mathbf{R}^n), & \forall q \in [2, 2n/(n-4)] \quad \text{for } n > 4, \\ L^q(\mathbf{R}^n), & \forall q \in [2, \infty) \quad \text{if } n = 4, \\ C^{0,\gamma}(\mathbf{R}^n), & 0 < \gamma < 1 \quad \text{if } n = 2, \\ C^{0,\gamma}(\mathbf{R}^n), & 0 < \gamma < \frac{1}{2} \quad \text{if } n = 3, \end{cases} \tag{5.1}$$
where $C^{0,\gamma}(\Omega)$ is the subspace of the space of continuous functions $C(\Omega)$ consisting of functions satisfying a local Hölder condition on Ω.

If we denote by $H_{\mathbf{A}}(\mathbf{R}^n)$ the completion of $\mathcal{D}_0 = \mathcal{D}(\Delta_{\mathbf{A}})$ with the norm
$$\|f\|_{\mathbf{A}}^2 := \|\Delta_{\mathbf{A}} f\|^2 + \|f\|^2$$
we then obtain from Theorem 1 the following results which are valid in the limiting cases of (5.1).

Theorem 5

(i) *For all* $f \in H_{\mathbf{A}}(\mathbf{R}^4)$, $f \in L^\infty(\mathbf{R}_+; L^2(\mathbf{S}^3), dr)$ *and*
$$\sup_{0<r<\infty} \int_{\mathbf{S}^3} |f(r,\omega)|^2 d\omega \lesssim \|\Delta_{\mathbf{A}} f\|^2.$$

(ii) *For all* $f \in H_{\mathbf{A}}(\mathbf{R}^2)$, $\int_{\mathbf{S}^1} f(\cdot,\omega)d\omega \in C^{0,1}(\mathbf{R}^2)$, *(i.e., Lipschitz) and for* $0 < s < r < \infty$

$$\Big| \int_{\mathbf{S}^1} \frac{f(r,\omega) - f(s,\omega)}{r-s} d\omega \Big| \lesssim \|\Delta_{\mathbf{A}} f\|^2.$$

(iii) *For all* $f \in H_{\mathbf{A}}(\mathbf{R}^3)$, $\int_{\mathbf{S}^2} |f(\cdot,\omega)|^2 d\omega \in C^{0,1}(\mathbf{R}^3)$ *and*

$$\Big| \int_{\mathbf{S}^2} \frac{|f(r,\omega)|^2 - |f(s,\omega)|^2}{r-s} d\omega \Big| \lesssim \|\Delta_{\mathbf{A}} f\|^2.$$

P r o o f. Part (i) is immediate from (2.8).

In part (ii) we have for $0 < s < r < \infty$,

$$\begin{aligned} \Big| \int_{\mathbf{S}^1} \tfrac{f(r,\omega)-f(s,\omega)}{r-s} d\omega \Big| &= \Big| (r-s)^{-1} \int_{\mathbf{S}^1} \big(\int_s^r \tfrac{\partial}{\partial t} f(t,\omega) dt \big) d\omega \Big| \\ &\le |r-s|^{-1} \int_s^r \big\{ \int_{\mathbf{S}^1} \big| \tfrac{\partial}{\partial t} f(t,\omega) \big|^2 d\omega |\mathbf{S}^1| \big\}^{\frac{1}{2}} dt \\ &\lesssim \|\Delta_{\mathbf{A}} f\|^2 \end{aligned}$$

by (2.8).

In part (iii) we have for $0 < s < r < \infty$ and any $\epsilon > 0$

$$\begin{aligned} \tfrac{|f(r,\omega)|^2 - |f(s,\omega)|^2}{r-s} &= \tfrac{1}{r-s} \int_s^r 2\Re e \big[\overline{f(t,\omega)} \tfrac{\partial}{\partial t} f(t,\omega) \big] dt \\ &\le \tfrac{1}{r-s} \Big\{ \epsilon \int_s^r t \big| \tfrac{\partial f}{\partial t} \big|^2 dt + \tfrac{1}{\epsilon} \int_s^r \tfrac{1}{t} |f(t)|^2 dt \Big\} \end{aligned}$$

and, with $F(r) := \int_{\mathbf{S}^2} |f(r,\omega)|^2 d\omega$

$$\begin{aligned} \big| \tfrac{F(r)-F(s)}{r-s} \big| &\le \tfrac{1}{r-s} \int_{\mathbf{S}^2} \Big\{ \epsilon \int_s^r t \big| \tfrac{\partial f}{\partial t} \big|^2 dt + \tfrac{1}{\epsilon} \int_s^r \tfrac{1}{t} |f(t)|^2 dt \Big\} d\omega \\ &= \tfrac{1}{r-s} \Big\{ \epsilon \int_s^r t \big(\int_{\mathbf{S}^2} \big| \tfrac{\partial f}{\partial t} \big|^2 \big) dt + \tfrac{1}{\epsilon} \int_s^r \tfrac{1}{t} \int_{\mathbf{S}^2} |f(t)|^2 d\omega dt \Big\} \\ &\lesssim \|\Delta_{\mathbf{A}} f\| \end{aligned}$$

by (2.8).

□

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