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# Integrated density of states and Wegner estimates for random Schrödinger Operators

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Abstract. We survey recent results on spectral properties of random Schrödinger operators. The focus is set on the integrated density of states (IDS). First we present a proof of the existence of a self-averaging IDS which is general enough to be applicable to random Schrödinger and Laplace-Beltrami operators on manifolds. Subsequently we study more specific models in Euclidean space, namely of alloy type, and concentrate on the regularity properties of the IDS. We discuss the role of the integrated density of states and its regularity properties for the spectral analysis of random Schrödinger operators, particularly in relation to localisation. Proofs of the central results are given in detail. Whenever there are alternative proofs, the different approaches are compared.

Resumen. Revisamos resultados recientes en propiedades espectrales de operadores de Schrödinger aleatorios. Nos enfocamos principialmente en la densidad integrada de estados (IDS). Primero presentamos una prueba de la existencia de la IDS y su propriedad auto-promediadora (self-averaging). El método es suficientemente general para ser aplicable a operadores de Schrödinger y de Laplace-Beltrami aleatorios en variedades de Riemann. Posteriormente estudiamos los modelos más específicos en el espacio Euclidiano, a saber de tipo aleación, y nos concentramos en las propiedades de la regularidad de la IDS. Discutimos el papel de la densidad integrada de estados y sus propiedades de regularidad para el análisis espectral de operadores de Schrödinger aleatorios, particularmente con relación a localización espectral. Las pruebas de los resultados centrales son descritas en detalle. Cuando hay pruebas alternativas, los enfoques diferentes se comparan.

## Contents



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## 1. Random operators

**1.1. Physical background.** Random Schrödinger operators are used as models of disordered solids within the framework of quantum mechanics.

A macroscopic solid consists of an order of magnitude of $10^{23}$ of nuclei and electrons. The resulting Hamiltonian taking into account all interactions is highly complicated. To arrive at a Schrödinger operator which can be studied in some detail one neglects the electron-electron interaction and treats the nuclei in the infinite mass approximation. Thus one arrives at an one-electron Schrödinger operator with an external potential due to the electric forces between the electron and the nuclei, which are assumed to be fixed in space.

In the case that the nuclei are arranged periodically on a lattice, the potential energy of the electron is a periodic function of the space variable.

On the other hand, the electron could be moving in an amorphous medium, in which case there is no large group of symmetries of the Hamiltonian. However, from the physical point of view it is reasonable to assume that the local structure of the medium will be translation invariant *on average*. This means that we consider the potential which the electron experiences as a particular realisation of a random process and assume stationarity with respect to some group of translations. Moreover, physical intuition suggests to assume that the local properties of the medium in two regions far apart (on the microscopic scale) are approximately independent from each other. Therefore the stochastic process describing the potential should have a correlation function which decays to zero, or — more precisely — should be ergodic.

There are interesting models which lie between the two extreme cases of lattice-periodic and amorphous media. They still have an underlying lattice structure which is, however, modified by disorder. Probably the best studied Hamiltonian with this properties is the alloy type model. We leave its precise definition for the next paragraph and introduce here a special case on the intuitive level. Consider first the potential

$$V_\omega(x) := \sum_{k\in\mathbb{Z}^d} u_k(\omega, x)$$

Each $k$ corresponds to a nucleus sitting on a lattice point. The function $u_k(\omega,\cdot)$ describes the atomic or nuclear potential at the site $k$ and depends on the random parameter $\omega$ which models the different realisations of the configuration of the nuclei. If there is only one type of atom present, which has a spherically symmetric potential, all the $u_k(\omega,\cdot)$ are the same, and $V_\omega$ is periodic. Now assume that there are two kinds $a$ and $b$ of atoms present, which have spherically symmetric atomic potentials of the same shape, but which differ in their nuclear charge numbers.

In this case the potential looks like

$$V_\omega(x) := \sum_{k \text{ occupied by } a} q_a\, u(x-k) + \sum_{k \text{ occupied by } b} q_b\, u(x-k)$$

If the two sorts of atoms are arranged on the lattice in a regular pattern, this again gives rise to a periodic potential.

However, there are physically interesting situations (e.g. binary alloys) where the type of atom sitting on site $k$ is random, for example obeying the law

$$\mathbb{P}\{k \text{ is occupied by atom } a\} = P, \qquad \mathbb{P}\{k \text{ is occupied by atom } b\} = 1-P \quad P>0$$

Here $\mathbb{P}\{\dots\}$ denotes the probability of an event. If we furthermore assume that the above probabilities are independent at each site and the parameter $P$ is the same for all $k$, we arrive at the continuum *Bernoulli-Anderson potential*

$$V_\omega(x) = \sum_k q_k(\omega)\, u(x-k)$$

Here $q_k(\omega) \in \{q_a, q_b\}, k \in \mathbb{Z}^d$ denotes a collection of independent, identically distributed Bernoulli random variables and $u$ is some atomic potential.

This model is a prototype which has motivated much research in the physics and mathematics literature, a part of which we will review in the present work.

Properties of disordered systems are discussed in the books [**35, 91, 208**] from the point of view of theoretical physics. The mathematical literature on random Schrödinger operators includes the books [**47, 241**], the introductory article [**158**],

a section on random Jacobi matrices in [**65**], the Lifshitz memorial issue [**207**], and a monograph specialised on localisation phenomena [**295**].

In the present work we focus on the spectral features of random Hamiltonians which are encoded in, or in close relation to the properties of the integrated density of states (IDS). We present the proofs of most of the results in detail. Indeed, for the two central theorems — the existence of an self-averaging integrated density of states and for the Wegner estimate — two independent proofs are given and compared.

**1.2. Model and Notation.** Let us start with some mathematical notation. The symbols $\mathbb{R}, \mathbb{Z}, \mathbb{N}, \mathbb{N}_0$ denote the set of reals, the set of integers, the set of natural numbers, and the set on non-negative integers, respectively. For a set $A$ we denote by $A^c$ its complement. An open subset of $\mathbb{R}^d$ will be denoted by $\Lambda$, and if there is a sequence of such sets its members will be denoted $\Lambda_1, \ldots, \Lambda_l, \ldots$. The symbol $|\Lambda|$ is used for the Lebesgue measure of $\Lambda$. We write $|x|$ for the norm of $x \in \mathbb{R}^d$, while the norm of a vector $f$ in a function space is denoted by $\|f\|$.

The Hilbert space of (equivalence classes of) measurable functions on $\Lambda$ which are square integrable with respect to Lebesgue measure is denoted by $L^2(\Lambda)$. Similarly, $L^p(\Lambda)$ with $p \geq 0$ stands for the space of measurable functions $f$ such that $|f|^p$ is integrable, while $L^\infty(\Lambda)$ is the set of measurable functions which are essentially bounded with respect to Lebesgue measure. The space of sequences $\{a_n\}_{n\in\mathbb{N}}$ such that $|a_n|^p$ is summable is denoted by $l^p(\mathbb{N})$. Note that the case $p \in ]0,1[$ is included in our notation. In our context we will often choose the exponent $p$ dependent on the dimension of the configuration space. In the following we denote by $p(d)$ any number in $[1, \infty[$ which satisfies

$$p(d) \begin{cases} \geq 2 & \text{if } d \leq 3, \\ > d/2 & \text{if } d \geq 4 \end{cases} \tag{1.1}$$

The symbols $C(\Lambda), C^\infty(\Lambda)$ stand for the continuous, respectively smooth, functions on $\Lambda$. The subscript $_c$ in $C_c(\Lambda), C_c^\infty(\Lambda), L_c^p(\Lambda)$ means that we consider only those functions which have compact support in $\Lambda$. In the sequel we will often consider potentials from the class of functions which are *uniformly locally* in $L^p$. More precisely, $f$ is in the set of uniformly locally $L^p$-functions, denoted by $L^p_{\mathrm{unif,loc}}(\mathbb{R}^d)$, if and only if there is a constant $C$ such that for each $y \in \mathbb{R}^d$

$$\int_{|x-y|<1} |f(x)|^p \, dx \leq C$$

The infimum over all such constants $C$ is $\|f\|^p_{p,\,\mathrm{unif,loc}}$.

Let $\Delta$ denote the Laplacian on $\mathbb{R}^d$. If we choose its *operator domain* $\mathcal{D}(\Delta)$ to be the Sobolev space $W_2^2(\mathbb{R}^d)$ of functions in $L^2(\mathbb{R}^d)$ whose second derivatives (in the sense of distributions) are square integrable, it becomes a selfadjoint operator. The restriction of $\Delta$ to a true open subset $\Lambda \subset \mathbb{R}^d$ becomes selfadjoint only if we specify appropriate boundary conditions (b.c.). Dirichlet b.c. are defined in Remark 2.2.3. For the definition of Neumann and periodic b.c. see for instance [**256**].

Let $A, B$ be two symmetric operators on a Hilbert space $\mathcal{H}$, whose norm we denote by $\|\cdot\|$. We say that $B$ is *(relatively) $A$-bounded* if the domains obey the inclusion $\mathcal{D}(A) \subset \mathcal{D}(B)$ and there are finite constants $a$ and $c_a$ such that for all

$f \in \mathcal{D}(A)$

$$\|Bf\| \leq a\|Af\| + c_a\|f\| \tag{1.2}$$

The infimum over all $a$ such that the estimate holds with some $c_a$ is called *relative bound* (of $B$ with respect to $A$). If $B$ is $A$-bounded with relative bound zero, we call it *infinitesimally $A$-bounded*. Let $A$ be selfadjoint, and $B$ symmetric and relatively $A$-bounded with relative bound smaller than one. Then the operator sum $A + B$ on the domain $\mathcal{D}(A)$ is selfadjoint by the Kato-Rellich Theorem. We will apply this result to the sum of the negative Laplacian and a potential. A multiplication operator by a function $V \in L^p_{\mathrm{unif,loc}}(\mathbb{R}^d)$ is infinitesimally $\Delta$-bounded if $p = p(d)$, cf. [**256**, Thm. XIII.96]. Moreover, the constant $c_a$ in (1.2) depends only on $\|V\|_{p,\,\mathrm{unif,loc}}$. Thus the sum $H := -\Delta + V$ is selfadjoint on $W^2_2(\mathbb{R}^d)$. If $B$ is relatively $A$-bounded in operator sense with relative bound $a$ it implies

$$\langle f, Bf\rangle \leq a\langle f, Af\rangle + C_a\langle f, f\rangle \text{ for some } C_a \in \mathbb{R}$$

which is called *relative form-boundedness* of the form $Q_B(f, g) := \langle f, Bg\rangle$ with respect to $Q_A(f, g) := \langle f, Ag\rangle$. See § VI.1.7 in [**152**] for more details.

The triple $(\Omega, \mathcal{B}_\Omega, \mathbb{P})$ stands for a probability space with associated $\sigma$-algebra and probability measure, while $\mathbb{E}\{\dots\}$ denotes the expectation value with respect to $\mathbb{P}$. A collection $T_j \colon \Omega \to \Omega, j \in \mathcal{J}$ of measure preserving transformations is called *ergodic* if all measurable sets in $\Omega$ which are invariant under the action of all $T_j, j \in \mathcal{J}$ have measure zero or one.

Definition 1.2.1. Let $p = p(d)$ be as in (1.1), $u \in L^p_c(\mathbb{R}^d)$ and $q_k \colon \Omega \to \mathbb{R}, k \in \mathbb{Z}^d$ be a sequence of bounded, independent, identically distributed random variables, called *coupling constants*. Then the family of multiplication operators given by the stochastic process

$$V_\omega(x) := \sum_{k\in\mathbb{Z}^d} q_k(\omega)\, u(x - k) \tag{1.3}$$

is called *alloy type potential*. The function $u$ is called *single site potential*. Let $H_0 := -\Delta + V_{\mathrm{per}}$ be a periodic Schrödinger operator with $V_{\mathrm{per}} \in L^p_{\mathrm{unif,loc}}(\mathbb{R}^d)$. The family of operators

$$H_\omega := H_0 + V_\omega, \quad \omega \in \Omega \tag{1.4}$$

is called *alloy type model*.

The distribution measure of the random variable $q_0$ will be called *single site distribution* and denoted by $\mu$. If not stated otherwise, in the sequel we assume that $\mu$ is absolutely continuous with respect to the Lebesgue measure and has a bounded density. The density function is denoted by $f$.

Due to our assumptions on the boundedness of the coupling constants, for each $a > 0$ there is a constant $c_a$ such that for all $\omega$ and all $\psi \in \mathcal{D}(\Delta)$

$$\|V_\omega\psi\| \leq a\|\Delta\psi\| + c_a\|\psi\|, \quad \|V_{\mathrm{per}}\psi\| \leq a\|\Delta\psi\| + c_a\|\psi\|$$

In particular $H_0$ and all $H_\omega$ are selfadjoint on the operator domain of $\Delta$. It will be of importance to us that the constant $c_a$ may be chosen independently of the random parameter $\omega$.

REMARK 1.2.2. (a) In several paragraphs we study Hamiltonians as in Definition 1.2.1, but where some of the hypotheses on the single site potential or the coupling constants are relaxed. More precisely, we will consider single site potentials with non-compact support and coupling constants which are unbounded, correlated, or do not have a bounded density.

(b) If the coupling constants are not bounded, one has to impose some moment condition to make sure that the alloy type model still makes sense. The main difference (to the bounded case) is that for $\omega$ in a set $\Omega' \subset \Omega$ of full measure the operator $H_\omega$ will be (essentially) selfadjoint, however this will fail to hold for $\omega$ in the complement $\Omega \setminus \Omega'$. See for example [**162, 163, 165**] for more details.

(c) There is a group of measure preserving transformations $T_k, k \in \mathbb{Z}^d$ on $(\Omega, \mathcal{B}_\Omega, \mathbb{P})$ such that (1.3) obeys

$$V_\omega(x-k) = V_{T_k\omega}(x)$$

In other words the stochastic process $V\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is *stationary* with respect to translations by vectors in $\mathbb{Z}^d$. Moreover, the group $T_k, k \in \mathbb{Z}^d$ acts ergodically on $\Omega$, therefore we call $V$ an $\mathbb{Z}^d$*-ergodic potential*.

To see that the above statements are true we pass over to the canonical probability space $\Omega = \times_{k\in\mathbb{Z}^d} \mathbb{R}$, equipped with the product measure $\mathbb{P} := \otimes_{k\in\mathbb{Z}^d} \mu$. Now the stochastic process $\{\pi_k\}_{k\in\mathbb{Z}^d}$, defined by $\pi_k(\omega) = \omega_k$ for all $k \in \mathbb{Z}^d$, has the same finite dimensional distributions as $\{q_k\}_k$. It is easily seen that the transformations $(T_k(\omega))_j := \omega_{j-k}$ are measure preserving and that the group $\mathbb{Z}^d$ acts ergodically on $\Omega$.

Using the stochastic process $\{\pi_k\}_k$ the alloy type potential can be written as

$$V_\omega(x) := \sum_{k\in\mathbb{Z}^d} \omega_k\, u(x-k) \tag{1.5}$$

which we will use without distinction in the sequel.

Abstracting the properties of stationarity and ergodicity we define general random potentials and operators with $\mathbb{Z}^d$-ergodic structure.

DEFINITION 1.2.3. Let $V\colon \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a stochastic process such that for almost all $\omega \in \Omega$ the realisation of the potential obeys $V_\omega \in L^p_{\mathrm{unif,loc}}(\mathbb{R}^d)$, $p = p(d)$ and additionally $\mathbb{E}\{\|V_\omega \chi_\Lambda\|_p^p\} < \infty$, where $\Lambda$ is an unit cube. Let $T_k, k \in \mathbb{Z}^d$ be a group of measure preserving transformations acting ergodically on $(\Omega, \mathcal{B}_\Omega, \mathbb{P})$ such that

$$V_\omega(x-k) = V_{T_k\omega}(x)$$

Then we call $\{V_\omega\}_\omega$ a ($\mathbb{Z}^d$*-ergodic*) *random potential* and $\{H_\omega\}_\omega$ with $H_\omega = -\Delta + V_\omega$ a ($\mathbb{Z}^d$*-ergodic*) *random operator*.

The restriction of $H_\omega$ to an open subset $\Lambda$ will be denoted by $H_\omega^\Lambda$ if we impose Dirichlet boundary conditions and by $H_\omega^{\Lambda,N}$ in the case of Neumann b.c. While we will be mainly concerned with $\mathbb{Z}^d$-ergodic operators we will give some comments as asides on their counterparts which are ergodic with respect to the group $\mathbb{R}^d$. The recent overview [**203**] is devoted to such models that model amorphous media. Insight in the research on almost periodic operators can be obtained for instance in [**298, 299, 20, 25, 65, 241**] and the references therein.

REMARK 1.2.4. All $\mathbb{Z}^d$-ergodic potentials can be represented in a form which resembles alloy type potentials. Namely, for such $V\colon \Omega \times \mathbb{R}^d \to \mathbb{R}$ there exists a

sequence $f_k, k \in \mathbb{Z}^d$ of random variables on $\Omega$ taking values in the separable Banach space $L^p(\mathbb{R}^d)$ such that $V$ can be written as

$$V_\omega(x) = \sum_{k \in \mathbb{Z}^d} f_k(\omega, x-k). \tag{1.6}$$

This representation is of interest because it ensures that after passing to an equivalent probability space and stochastic process one may assume that the sigma algebra on $\Omega$ is countably generated. See [**156**] and Remark 2.8 in [**199**] for more information.

**1.3. Transport properties and spectral types.** The main interest in the study of random operators is to understand the transport properties of the media they model. In the particular case of the quantum mechanical Hamiltonian of an electron in a disordered solid the electric conductance properties are the main object of interest.

The Hamiltonian governs the equation of motion, i.e. the time dependent Schrödinger equation

$$\frac{\partial \psi(t)}{\partial t} = -iH_\omega \psi(t)$$

The time evolution of the vector $\psi(t)$ in Hilbert space describes the movement of the electron. Since we chose the space representation in the Schrödinger picture, we can can think of $\psi(t)$ as a wave packet which evolves in time. The square of its absolute value $|\psi(t,\cdot)|^2 \in L^1(\mathbb{R}^d)$ is a probability density. More precisely, $\int_A |\psi(t,x)|^2 dx$ is the probability to find the electron in the set $A \subset \mathbb{R}^d$ at time $t$.

For a given initial state $\psi_0 := \psi(0)$ supported in a compact set $A$ one would like to know whether for large times the function $\psi(t)$ stays (essentially) supported near $A$, or moves away to infinity. In the first case one speaks of a *bound state*, since it remains localised near its original support for all times. The other extreme case would be that $\psi(t)$ leaves any compact region in $\mathbb{R}^d$ (and never comes back) as time goes to infinity. Such a state is called a *scattering* or *extended state*. By the RAGE theorem, cf. e.g. [**257, 65, 295**], it is possible to relate the dynamical properties of states just described to the spectral properties of the Hamiltonian. Roughly speaking, bound states correspond to pure point spectrum and scattering states to (absolutely) continuous spectrum. For a more precise statement consult for instance [**257, 65, 295**].

This motivates the systematic study of spectral properties of the introduced Schrödinger operators. If a random Schrödinger operator exhibits almost surely only pure point spectrum in an energy region one speaks of *Anderson* or *spectral localisation*. The name goes back to Anderson's seminal work [**13**]. This property has been established for a variety of random models. In most of those cases one can additionally prove that the corresponding eigenfunctions decay exponentially in configuration space, a phenomenon called *exponential (spectral) localisation*. The situation is different for random potentials with long range correlations, where sometimes only power-law decay of the eigenfunctions has been established [**168, 109, 329**].

If an energy interval contains almost surely only pure point spectrum, we call it *localisation interval*. An eigenfunction of $H_\omega$ which decays exponentially is called an *exponentially localised eigenstate*. The region or point in space where the localised

state has its highest amplitude will be called *localisation centre* (we will not need a mathematically precise definition of this notion).

However, it turns out that the spectrum captures only a rough view on the dynamical properties of the quantum mechanical system. A more detailed understanding can be obtained by studying the time evolution of the moments of the position operators. This led to a formulation of several criteria of *dynamical localisation*. The strongest characterisation of this phenomenon, namely *strong dynamical localisation in Hilbert-Schmidt* topology means that for all $q > 0$

$$\mathbb{E}\left\{\sup_{\|f\|_\infty \leq 1} \left\| |X|^{q/2} f(H_\omega) P^l_\omega(I) \chi_K \right\|^2_{HS}\right\} < \infty \tag{1.7}$$

Here $P^l_\omega(I)$ denotes the spectral projection onto the energy interval $I$ associated to the operator $H^l_\omega$, $\|\cdot\|_{HS}$ denotes the Hilbert-Schmidt norm, $K \subset \mathbb{R}^d$ is any compact set, and $|X|$ denotes the operator of multiplication with the function $|x|$. For the interpretation of (1.7) as non-spreading of wave-packets one chooses $f(y) = e^{-ity}$. Dynamical localisation (1.7) implies in particular that the random Hamiltonian $H_\omega$ exhibits spectral localisation in $I$. The reader can consult e.g. [**262, 118, 295, 120, 119, 6**] to get an insight how the notion of dynamical localisation evolved and for recent developments. In [**262, 70, 148**] examples are discussed where spectral localisation occurs, but certain dynamical criteria for localisation are not satisfied.

For the purpose of the present paper these distinctions are not crucial. In the case of the alloy type model, to which we devote most attention, spectral and dynamical localisation coincide, cf. [**67, 119**]. In the sequel we mean by localisation that the considered operator exhibits in a certain energy interval only pure point spectrum, and that the corresponding eigenfunctions decay sufficiently fast.

Since we are dealing not just with a single Hamiltonian, but with a whole family of them, we have to say something on how the spectral properties depend on the parameter $\omega$ describing the randomness: most properties of the spectrum of an operator pertaining to the family $\{H_\omega\}_\omega$ hold almost surely, i.e. for $\omega$ in a set such that its complement has measure zero in $\Omega$. This is at least true for the properties we discuss in the present paper.

We shortly describe what kinds of spectral types one expects from the physical point of view for random Schrödinger operators, say of alloy type. In case there are rigorous results which have confirmed this intuition we quote the reference.

In one space dimension the spectrum is pure point for all energies almost surely. Rigorous proofs of this statement can for instance be found in [**128, 193**].

In three or more dimensions it is expected that the spectrum is pure point near the boundaries of the spectrum while in the interior it is purely absolutely continuous. In the latter case one speaks also of an energy region with *delocalised* states. However, for alloy type Hamiltonians the proof of delocalisation is open. Some results on existence of absolutely continuous spectrum for random models of a different type can be found in [**176, 102, 160**]. The two regions with localised, respectively delocalised states are separated by a threshold, the so called *mobility edge*, for partial results see [**147, 160, 119, 103**].

The literature on the existence of pure point spectrum is extensive. We discuss it in more detail in § 3.2.

How large the intervals with point or continuous spectrum are, depends on the disorder present in the model. For instance, in (1.4) one could introduce a global

coupling constant $\lambda$ in front of the potential

$$H_\omega = H_0 + \lambda V_\omega$$

Now large $\lambda$ means large disorder, small $\lambda$ small disorder. The larger the disorder, the larger is the portion of the spectrum which contains localised states. For other types of random Schrödinger operators there are similar ways to introduce a disorder parameter.

The phenomenon that localised states emerge at the edges of the spectrum can be understood in terms of the so-called *fluctuation boundaries*. These are the regions of the spectrum which correspond to extremely rare configurations of the potential. Consequently, the *density of states* (or the spectral density function, see the next section for a precise definition) is very thin in this region. This has been first understood on physical grounds by Lifshitz. Today the tails of the density of states at the fluctuation boundaries bear the name of *Lifshitz-asymptotics* or *Lifshitz-tails*. We give a (non-exhaustive) list of works devoted the study of this asymptotics: [**238, 239, 235, 115, 164, 280, 167, 222, 223, 281, 305, 224, 181, 293, 184, 185, 183, 182**] and the references at the end of § 5.6.

The existence of localised states for random Schrödinger operators is in sharp contrast to the features of periodic operators. Indeed, for operators with periodic potential, satisfying some mild regularity assumptions, it is known that the spectrum is purely absolutely continuous, [**32, 288, 273, 274, 195**]. This difference might seem somewhat surprising, given the similarity of the structure of an alloy type and a periodic operator.

**1.4. Outline of the paper.** In the present section we discussed the physical motivation to study random operators and introduced the models we will analyse in the sequel. The next section is devoted to the proof of the existence of the IDS for these models. In the third section we discuss why regularity properties of the IDS are of interest. The following two sections give two independent proofs of the continuity of the IDS. Both approaches are suitable to extensions in various directions. We review some recently obtained results. For more details see the table of contents.

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## 2. Existence of the integrated density of states

Intuitively, the integrated density of states (IDS) measures how many electron energy levels can be found below a given energy per unit volume of a solid. It can be used to calculate the free energy and hence all basic thermodynamic quantities of the corresponding non-interacting many-particle system.

To define the IDS mathematically one uses an exhaustion procedure. More precisely, one takes an increasing sequence $\Lambda_l$ of open subsets of $\mathbb{R}^d$ such that each $\Lambda_l$ has finite volume and $\bigcup_l \Lambda_l = \mathbb{R}^d$. Then the operator $H_\omega^l$, which is the restriction of $H_\omega$ to $\Lambda_l$ with Dirichlet boundary conditions, is selfadjoint, bounded below and its spectrum consists of discrete eigenvalues $\lambda_1(H_\omega^l) \leq \lambda_2(H_\omega^l) \leq \cdots \leq \lambda_n(H_\omega^l) \to \infty$. Here $\lambda_n = \lambda_{n+1}$ means that the eigenvalue is degenerate and we take this into account in the enumeration.

The *normalised eigenvalue counting function* or *finite volume integrated density of states* $N_\omega^l$ is defined as

$$N_\omega^l(E) := \frac{\#\{n|\, \lambda_n(H_\omega^l) < E\}}{|\Lambda_l|} \tag{2.1}$$

The numerator can equally well be expressed using the trace of a spectral projection

$$\#\{n|\, \lambda_n(H_\omega^l) < E\} = \mathrm{Tr}\Big[P_\omega^l\big(]-\infty, E[\big)\Big]$$

Note that $N_\omega^l \colon \mathbb{R} \to [0, \infty[$ is a *distribution function* of a point measure for all $l \in \mathbb{N}$, i.e. $N_\omega^l(E) = \nu_\omega^l(]-\infty, E[)$. Here $\nu_\omega^l$ is the *finite volume density of states measure* defined by

$$\nu_\omega^l(I) := |\Lambda_l|^{-1}\, \#\{n|\, \lambda_n(H_\omega^l) \in I\}$$

By definition a distribution function is non-negative, left-continuous and monotone increasing. In particular, it has at most countably many points of discontinuity.

Under some additional conditions on the random operator and the exhaustion sequence $\Lambda_l, l \in \mathbb{N}$ one can prove that

(i) For almost all $\omega \in \Omega$ the sequence $N_\omega^l$ converges to a distribution function $N_\omega$ as $l$ goes to infinity. This means that we have $N_\omega^l(E) \to N_\omega(E)$ for all continuity points $E$ of the limit distribution $N_\omega$.

(ii) For almost all $\omega \in \Omega$ the distribution functions $N_\omega$ coincide, i.e. there is an $\omega$-independent distribution function $N$ such that $N = N_\omega$ for almost all $\omega$. This function $N$ is called the *integrated density of states*. Note that its independence of $\omega$ is not due to an explicit integration over the probability space $\Omega$, but only to the exhaustion procedure. This is the reason why the IDS is called *self-averaging*.

(iii) In most cases there is a formula for the IDS as an expectation value of a trace of a localised projection. For $\mathbb{Z}^d$-ergodic operators it reads

$$N(E) := \mathbb{E}\left\{\mathrm{Tr}\big[\chi_\Lambda P_\omega(]-\infty, E[)\big]\right\} \tag{2.2}$$

Here $\Lambda$ denotes the unit box $]0,1[^d$, which is the periodicity cell of the lattice $\mathbb{Z}^d$. Actually, one could choose certain other functions instead of $\chi_\Lambda$, yielding all the same result, cf. Formula (2.15). The equality (2.2) holds for $\mathbb{R}^d$-ergodic operators, too. It is sometimes called *Pastur-Šubin trace formula*.

In the following we prove the properties of the IDS just mentioned by two methods. In §§ 2.2 – 2.6 a complete proof is given using the Laplace transforms of the distribution functions $N_\omega^l$, while § 2.7 is devoted to a short sketch of an alternative method. It uses Dirichlet-Neumann bracketing estimates for Schrödinger operators, which carry over to the corresponding eigenvalue counting functions. These are thus super- or subadditive processes to which an ergodic theorem [**12**] can be applied.

Actually the proof using Laplace transforms will apply to more general situations than discussed so far, namely to more general geometries than Euclidean space. To be precise, we will consider random Schrödinger operators on Riemannian covering manifolds, where both the potential and the metric may depend on the randomness. This includes random Laplace-Beltrami operators.

We follow the presentation and proofs in [**243, 199**]. The general strategy we use was developed by Pastur and Šubin in [**237**] and [**298**] for random and almost-periodic operators in Euclidean space. A particular idea of this approach is to prove the convergence of the Laplace transforms $\mathcal{L}^l_\omega$ of the normalised finite volume eigenvalue counting functions $N^l_\omega$ instead of proving the convergence of $N^l_\omega$ directly. This is actually the main difference to the second approach we outline in § 2.7, which is taken from [**161**]. The Pastur-Šubin strategy seems to be better suited for geometries with underlying group structure which is non-abelian.

Indeed, one of the differences between random operators on manifolds and those on $\mathbb{R}^d$ is that the operator is equivariant with respect to a group which does not need to be commutative. This means that one has to use a non-abelian ergodic theorem to derive the convergence of the distribution functions $N^l_\omega$ or, alternatively, of their Laplace transforms $\mathcal{L}^l_\omega$.

This imposes some restriction on the strategy of the proof since the ergodic theorems which apply to non-abelian groups need more restrictive assumptions than their counterparts for commutative groups, cf. also Remark 2.6.2

For processes which are not additive, but only super- or subadditive, there is a non-abelian maximal ergodic theorem at disposal (cf. 6.4.1 Theorem in [**194**]) but so far no pointwise theorem. This is also the reason why the Dirichlet-Neumann bracketing approach of § 2.7 does not seem applicable to random operators living on a covering manifold with non-abelian deck-transformation group (covering transformation group).

**2.1. Schrödinger operators on manifolds: motivation.** In this section we study the IDS of random Schrödinger operators on manifolds. Let us first explain the physical motivation for this task.

Consider a particle or a system of particles which are constrained to a submanifold of the ambient (configuration) space. The classical and quantum Hamiltonians for such systems have been studied e.g. in [**226, 110**] (see also the references therein). To arrive at an effective Hamiltonian describing the constrained motion on the sub-manifold, a limiting procedure is used: a (sequence of) confining high-barrier potential(s) is added to the Hamiltonian defined on the ambient space to restrict the particle (system) to the sub-manifold. In [**226, 110**] one can find a discussion of the similarities and differences between the obtained effective quantum Hamiltonian and its classical analogue.

A important feature of the effective quantum Hamiltonian is the appearance of a so-called *extra-potential* depending on the extrinsic curvature of the sub-manifold and the curvature of the ambient space. This means that even if we disregard external electric forces the relevant quantum mechanical Hamiltonian of the constrained system is not the pure Laplacian but contains (in general) a potential energy term. This fact explains the existence of curvature-induced bound states in quantum waveguides and layers, see [**100, 89, 211, 90**] and the references therein.

As is mentioned in [**226**], the study of effective Hamiltonians of constrained systems is motivated by specific physical applications. They include stiff molecular

bonds in (clusters of) rigid molecules and molecular systems evolving along reaction paths. From the point of view of the present paper quantum wires, wave guides and layers are particularly interesting physical examples. Indeed, for these models (in contrast to quantum dots) at least one dimension of the constraint sub-manifold is of macroscopic size. Moreover, it is natural to assume that the resulting Hamiltonian exhibits some form of translation invariance in the macroscopic direction. E.g. it may be periodic, quasiperiodic or — in the case of a random model — ergodic.

For random quantum waveguides and layers the existence of dense point spectrum is expected, cf. the discussion of localisation in Paragraph 1.3. For a waveguide embedded in Euclidean space this has been rigorously proven in [**174, 175**]. The question of spectral localisation due to random geometries has been raised already in [**69**]. There the behaviour of Laplace-Beltrami operators under non-smooth perturbations of the metric is studied.

A second motivation for the analysis of operators on manifolds studied in this section comes from differential geometry. The spectral properties of periodic Schrödinger operators on manifolds have attracted the interest of various authors. A non-exhaustive list is [**19, 301, 186, 302, 303, 41, 40, 300, 150, 244, 245**]. By 'periodic' we mean that the operator acts on a covering manifold and is invariant under the unitary operators induced by the deck-transformations.

Particular attention was devoted to the analysis of the gap structure of the spectrum of periodic operators of Schrödinger type. More precisely, one is interested whether the spectrum in interrupted by *spectral gaps*, i.e. intervals on the real line which belong to the resolvent set. In case there are gaps: can one establish upper and lower bounds for the width and number of gaps and the spectral bands separating them? Although the gap structure of the spectrum is a mathematically intriguing question for its own sake, it is also important from the physical point of view. The features of gaps in the energy spectrum are relevant for the conductance properties of the physical system. Even for periodic Schrödinger operators in Euclidean space the gap structure is highly non-trivial. This is maybe best illustrated by works devoted to the Bethe-Sommerfeld conjecture, e.g. [**289, 285, 286, 287, 133**]. Another interesting feature of some periodic Laplace-Beltrami operators is the existence of $L^2$-eigenfunctions, cf. the discussion in Remark 3.1.3.

These periodic operators on manifolds are generalised by their random analogues studied in this section.

**2.2. Random Schrödinger operators on manifolds: definitions.** Let us explain the geometric setting in which we are working precisely: let $X$ be a complete $d$-dimensional Riemannian manifold with metric $g_0$. We denote the volume form of $g_0$ by $\mathrm{vol}_0$. Let $\Gamma$ be a discrete, finitely generated subgroup of the isometries of $(X, g_0)$ which acts freely and properly discontinuously on $X$ such that the quotient $M := X/\Gamma$ is a compact ($d$-dimensional) Riemannian manifold. Let $(\Omega, \mathcal{B}_\Omega, \mathbb{P})$ be a probability space on which $\Gamma$ acts by measure preserving transformations. Assume moreover that the action of $\Gamma$ on $\Omega$ is ergodic. Now we are in the position to define what we mean by a random metric and consequently a random Laplace-Beltrami operator.

Definition 2.2.1. Let $\{g_\omega\}_{\omega\in\Omega}$ be a family of Riemannian metrics on $X$. Denote the corresponding volume forms by $\mathrm{vol}_\omega$. We call the family $\{g_\omega\}_{\omega\in\Omega}$ a *random metric on* $(X, g_0)$ if the following five properties are satisfied:

(2.3) The map $\Omega \times TX \to \mathbb{R}$, $(\omega, v) \mapsto g_\omega(v,v)$ is jointly measurable.

(2.4) There is a $C_g \in ]0,\infty[$ such that

$$C_g^{-1}\, g_0(v,v) \le g_\omega(v,v) \le C_g\, g_0(v,v) \quad \text{for all} \quad v \in TX.$$

(2.5) There is a $C_\rho \in ]0,\infty[$ such that

$$|\nabla_0\, \rho_\omega(x)|_0 \le C_\rho \text{ for all } x \in X,$$

where $\nabla_0$ denotes the gradient with respect to $g_0$, $\rho_\omega$ is the unique smooth density of $\mathrm{vol}_0$ with respect to $\mathrm{vol}_\omega$, and $|v|_0^2 = g_0(v,v)$.

(2.6) There is a uniform lower bound $(d-1)K \in \mathbb{R}$ for the Ricci curvatures of all Riemannian manifolds $(X, g_\omega)$. Explicitly, $\mathrm{Ric}(g_\omega) \ge (d-1)K g_\omega$ for all $\omega \in \Omega$ and on the whole of $X$.

(2.7) The metrics are compatible in the sense that the deck transformations

$$\gamma\colon (X, g_\omega) \to (X, g_{\gamma\omega}), \quad \gamma\colon x \mapsto \gamma x$$

are isometries.

Property (2.7) implies in particular that the induced maps

$$U_{(\omega,\gamma)}\colon L^2(X, \mathrm{vol}_{\gamma^{-1}\omega}) \to L^2(X, \mathrm{vol}_\omega), \quad (U_{(\omega,\gamma)} f)(x) = f(\gamma^{-1}x)$$

are unitary operators.

The density $\rho_\omega$ appearing in (2.5) satisfies by definition

$$\int_X f(x)\, d\mathrm{vol}_0(x) = \int_X f(x)\rho_\omega(x)\, d\mathrm{vol}_\omega(x).$$

It is a smooth function and can be written as

$$\rho_\omega(x) = \big(\det g_0(e_\omega^i, e_\omega^j)\big)^{1/2} = \Big(\det g_\omega(e_0^i, e_0^j)\Big)^{-1/2}$$

Here $e_0^1, \dots, e_0^d$ denotes any basis of $T_xX$ which is orthonormal with respect to the scalar product $g_0(x)$, and $e_\omega^1, \dots, e_\omega^d \in T_xX$ is any basis orthonormal with respect to $g_\omega(x)$. It follows from (2.4) that

$$C_g^{-d/2} \le \rho_\omega(x) \le C_g^{d/2} \quad \text{for all } x \in X,\ \omega \in \Omega \tag{2.8}$$

which in turn, together with property (2.5) and the chain rule, implies

$$|\nabla_0\, \rho_\omega^{\pm\, 1/2}(x)|_0 \le C_g^{3d/4}\, |\nabla_0\, \rho_\omega(x)|_0 \quad \text{for all } x \in X,\ \omega \in \Omega \tag{2.9}$$

Moreover, for any measurable $\Lambda \subset X$ by (2.8) we have the volume estimate

$$C_g^{-d/2}\, \mathrm{vol}_0(\Lambda) \le \mathrm{vol}_\omega(\Lambda) \le C_g^{d/2}\, \mathrm{vol}_0(\Lambda) \tag{2.10}$$

We denote the Laplace-Beltrami operator with respect to the metric $g_\omega$ by $\Delta_\omega$.

Associated to the random metric just described we define a random family of operators.

Definition 2.2.2. Let $\{g_\omega\}$ be a random metric on $(X, g_0)$. Let $V\colon \Omega \times X \to \mathbb{R}$ be a jointly measurable mapping such that for all $\omega \in \Omega$ the *potential* $V_\omega := V(\omega, \cdot) \ge 0$ is in $L^1_{\mathrm{loc}}(X)$. For each $\omega \in \Omega$ let $H_\omega = -\Delta_\omega + V_\omega$ be a Schrödinger operator defined on a dense subspace $\mathcal{D}_\omega$ of the Hilbert space $L^2(X, \mathrm{vol}_\omega)$. The family $\{H_\omega\}_{\omega\in\Omega}$ is called a *random Schrödinger operator* if it satisfies for all $\gamma \in \Gamma$ and $\omega \in \Omega$ the following *equivariance* condition

$$H_\omega = U_{(\omega,\gamma)} H_{\gamma^{-1}\omega} U^*_{(\omega,\gamma)} \tag{2.11}$$

REMARK 2.2.3 (Restrictions, quadratic forms and selfadjointness). Some remarks are in order why the sum of the Laplace-Beltrami operator and the potential is selfadjoint. We consider the two cases of an operator on the whole manifold $X$ and on a proper open subset of $X$ simultaneously. The set of all smooth functions with compact support in an open set $\Lambda \subset X$ is denoted by $C_c^\infty(\Lambda)$. For each $\omega \in \Omega$ we define the quadratic form

$$\widetilde{Q}(H_\omega^\Lambda)\colon C_c^\infty(\Lambda) \times C_c^\infty(\Lambda) \to \mathbb{R}, \tag{2.12}$$

$$(f,h) \mapsto \int_\Lambda g_\omega(x)\Big(\nabla f(x), \nabla h(x)\Big)\, d\mathrm{vol}_\omega(x) + \int_\Lambda f(x) V_\omega(x) h(x)\, d\mathrm{vol}_\omega(x)$$

We infer from Theorem 1.8.1 in [**68**] that this quadratic from is closable and its closure $Q(H_\omega^\Lambda)$ gives rise to a densely defined, non-negative selfadjoint operator $H_\omega^\Lambda$. Actually, $Q(H_\omega^\Lambda)$ is the form sum of the quadratic forms of the negative Laplacian and the potential. The result in [**68**] is stated for the Euclidean case $X = \mathbb{R}^d$ but the proof works equally well for general Riemannian manifolds.

The unique selfadjoint operator associated to the above quadratic form is called *Schrödinger operator with Dirichlet boundary conditions*. It is the Friedrichs extension of the restriction $H_\omega^\Lambda|_{C_c^\infty(\Lambda)}$. If the potential term is absent we call it negative *Dirichlet Laplacian*.

There are special subsets of the manifold which will play a prominent role later:

DEFINITION 2.2.4. A subset $\mathcal{F} \subset X$ is called $\Gamma$-*fundamental domain* if it contains exactly one element of each orbit $O(x) := \{y \in X |\, \exists \gamma \in \Gamma : y = \gamma x\}$, $x \in X$.

In [**2**, Section 3] it is explained how to obtain a connected, polyhedral $\Gamma$-fundamental domain $\mathcal{F} \subset X$ by lifting simplices of a triangularisation of $M$ in a suitable manner. $\mathcal{F}$ consists of finitely many smooth images of simplices which can overlap only at their boundaries. In particular, it has piecewise smooth boundary.

To illustrate the above definitions we will look at some examples. Firstly, we consider covering manifolds with abelian deck-transformation group.

EXAMPLE 2.2.5 (Abelian covering manifolds). Consider a covering manifold $(X, g_0)$ with a finitely generated, abelian subgroup $\Gamma$ of the isometries of $X$. If the number of generators of the group $\Gamma$ equals $r$, it is isomorphic to $\mathbb{Z}^{r_0} \times \mathbb{Z}_{p_1}^{r_0} \times \ldots \mathbb{Z}_{p_n}^{r_n}$. Here $\sum r_i = r$ and $\mathbb{Z}_p$ is the cyclic group of order $p$. Assume as above that the quotient $X/\Gamma$ is compact. Periodic Laplace-Beltrami and Schrödinger operators on such spaces have been analised e.g. in [**302, 244, 245**].

In the following we will discus some examples studied by Post in [**244, 245**]. The aim of this papers was to construct covering manifolds, such that the corresponding Laplace operator has open spectral gaps. More precisely, for any given natural number $N$, manifolds are constructed with at least $N$ spectral gaps. For technical reasons the study is restricted to abelian coverings. In this case the Floquet decomposition of the periodic operator can be used effectively. Post studies two classes of examples with spectral gaps. In the first case a conformal perturbation of a given covering manifold is used to open up gaps in the energy spectrum of the Laplacian. The second type of examples in [**245**] is of more interest to us. There, one starts with infinitely many translated copies of a compact manifold and joins them by cylinders to form a periodic network of 'pipes'. By shrinking the radius of the connecting cylinders, more and more gaps emerge in the spectrum.

Such manifolds have in particular a non-trivial fundamental group and are thus topologically not equivalent to $\mathbb{R}^d$. On the other hand their deck-transformation group is rather easy to understand, since it is abelian. In particular, it is amenable (cf. Definition 2.3.4), which is a crucial condition in the study conducted later in this section. Some of the examples in [**244, 245**] are manifolds which can be embedded in $\mathbb{R}^3$ as surfaces. They can be thought of as periodic quantum waveguides and networks. See [**244**] for some very illustrative figures.

Furthermore, in [**245**] perturbations techniques for Laplace operators on covering manifolds have been developed, respectively carried over from earlier versions suited for compact manifolds, cf. [**51, 14, 114**]. They include conformal perturbations and local geometric deformations. Floquet decomposition is used to reduce the problem to an operator on a fundamental domain with quasi-periodic boundary conditions and discrete spectrum. Thereafter the min-max principle is applied to geometric perturbations of the Laplacian.

Related random perturbations of Laplacians are studied in [**200, 198**], cf. also Example 2.2.7. In particular a Wegner estimate for such operators is derived.

Now we give an instance of a covering manifold $X$ with non-abelian deck-transformation group $\Gamma$.

Example 2.2.6 (Heisenberg group). The Heisenberg group $H_3$ is the manifold of $3 \times 3$-matrices given by

$$H_3 = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\} \tag{2.13}$$

equipped with a left-invariant metric. The Lie-group $H_3$ is diffeomorphic to $\mathbb{R}^3$. Its group structure is not abelian, but nilpotent.

The subset $\Gamma = H_3 \cap M(3, \mathbb{Z})$ forms a discrete subgroup. It acts from the left on $H_3$ by isometries and the quotient manifold $H_3/\Gamma$ is compact.

Next we give examples of a random potential and a random metric which give rise to a random Schrödinger operator as in Definition 2.2.2. Both have an underlying structure which resembles alloy-type models (in Euclidean space).

Example 2.2.7. (a) Consider the case where the metric is fixed, i.e. $g_\omega = g_0$ for all $\omega \in \Omega$, and only the potential depends on the randomness in the following way:

$$V_\omega(x) := \sum_{\gamma \in \Gamma} q_\gamma(\omega)\, u(\gamma^{-1}x), \tag{2.14}$$

Here $u : X \to \mathbb{R}$ is a bounded, compactly supported measurable function and $q_\gamma \colon \Omega \to \mathbb{R}$ is a sequence of independent, identically distributed random variables. By considerations as in Remark 1.2.2 the random operator $H_\omega := -\Delta + V_\omega, \omega \in \Omega$ is seen to satisfy the equivariance condition.

(b) Now we consider the situation where the metric has an alloy like structure. Let $(g_0, X)$ be a Riemannian covering manifold and let a family of metrics $\{g_\omega\}_\omega$ be given by

$$g_\omega(x) = \left( \sum_{\gamma \in \Gamma} r_\gamma(\omega)\, u(\gamma^{-1}x) \right) g_0(x)$$

where $u \in C_c^\infty(X)$ and the $r_\gamma \colon \Omega \to ]0,\infty[$, $\gamma \in \Gamma$ are a collection of independent, identically distributed random variables. Similarly as in the previous example one sees that the operators $\Delta_\omega$ are equivariant.

**2.3. Non-randomness of spectra and existence of the IDS.** Here we state the main theorems on the non-randomness of the spectral components and the existence and the non-randomness of the IDS. They refer to random Schrödinger operators as defined in 2.2.2.

THEOREM 2.3.1. *There exists a subset $\Omega'$ of full measure in $(\Omega, \mathcal{B}_\Omega, \mathbb{P})$ and subsets of the real line $\Sigma$ and $\Sigma_\bullet$, where $\bullet \in \{disc, ess, ac, sc, pp\}$ such that for all $\omega \in \Omega'$*

$$\sigma(H_\omega) = \Sigma \quad \text{and} \quad \sigma_\bullet(H_\omega) = \Sigma_\bullet$$

*for any $\bullet = disc, ess, ac, sc, pp$. If $\Gamma$ is infinite, $\Sigma_{disc} = \emptyset$.*

The theorem is proven in [**201**], see Theorem 5.1. The arguments go to a large part along the lines of [**240, 196, 162**]. Compare also the literature on almost periodic Schrödinger operators, for instance [**298, 21**].

For the proof of the theorem one has to find random variables which encode the spectrum of $\{H_\omega\}_\omega$ and which are invariant under the action of $\Gamma$. By ergodicity they will be constant almost surely. The natural random variables to use are spectral projections, more precisely, their traces. However, since $\mathbb{R}$ is uncountable and one has to deal also with the different spectral components, some care is needed.

Random operators introduced in Definition 2.2.2 are naturally affiliated to a von Neumann algebra of operators which we specify in

DEFINITION 2.3.2. A family $\{B_\omega\}_{\omega\in\Omega}$ of bounded operators $B_\omega \colon L^2(X, \mathrm{vol}_\omega) \to L^2(X, \mathrm{vol}_\omega)$ is called a *bounded random operator* if it satisfies:

(i) $\omega \mapsto \langle g_\omega, B_\omega f_\omega\rangle$ is measurable for arbitrary $f, g \in L^2(\Omega \times X, \mathbb{P} \circ \mathrm{vol})$.
(ii) There exists a $\omega$-uniform bound on the norms $\|B_\omega\|$ for almost all $\omega \in \Omega$.
(iii) For all $\omega \in \Omega, \gamma \in \Gamma$ the equivariance condition

$$B_\omega = U_{(\omega,\gamma)} B_{\gamma^{-1}\omega} U^*_{(\omega,\gamma)}$$

holds.

By the results of the next paragraph § 2.4, $\{F(H_\omega)\}_\omega$ is a bounded random operator for any measurable, bounded function $F$.

It turns out that (equivalence classes of) bounded random operators form a von Neumann algebra. More precisely, consider two bounded random operators $\{A_\omega\}_\omega$ and $\{B_\omega\}_\omega$ as *equivalent* if they differ only on a subset of $\Omega$ of measure zero. Each equivalence class gives rise to a bounded operator on $L^2(\Omega \times X, \mathbb{P} \circ \mathrm{vol})$ by $(Bf)(\omega, x) := B_\omega f_\omega(x)$, see Appendix A in [**201**]. This set of operators is a von Neumann algebra $\mathcal{N}$ by Theorem 3.1 in [**201**]. On $\mathcal{N}$ a trace $\tau$ of type $\mathrm{II}_\infty$ is given by

$$\tau(B) := \mathbb{E}\left[\mathrm{Tr}(\chi_\mathcal{F} B_\bullet)\right]$$

Here $\mathrm{Tr} := \mathrm{Tr}_\omega$ denotes the trace on the Hilbert space $L^2(X, \mathrm{vol}_\omega)$. Actually for any choice of $u \colon \Omega \times X \to \mathbb{R}^+$ with $\sum_{\gamma\in\Gamma} u_{\gamma^{-1}\omega}(\gamma^{-1}x) \equiv 1$ for all $(\omega, x) \in \Omega \times X$ we have

$$\tau(B) = \mathbb{E}\left[\mathrm{Tr}(u_\bullet B_\bullet)\right] \tag{2.15}$$

In analogy with the case of operators which are Γ-invariant [**19**] we call $\tau$ the Γ-*trace*. The spectral projections $\{P_\omega\big(]-\infty,\lambda[\big)\}_\omega$ of $\{H_\omega\}_\omega$ onto the interval $]-\infty,\lambda[$ form a bounded random operator. Thus it corresponds to an element of $\mathcal{N}$ which we denote by $P(]-\infty,\lambda[)$. Consider the normalised Γ-trace of $P$

$$N_H(\lambda) := \frac{\tau(P\big(]-\infty,\lambda[\big)}{\mathbb{E}\left[\mathrm{vol}_\bullet(\mathcal{F})\right]} \tag{2.16}$$

The following is Theorem 3 in [**199**], see also [**201**].

THEOREM 2.3.3. *$P(]-\infty,\lambda[)$ is the spectral projection of the direct integral operator*

$$H := \int_\Omega^\oplus H_\omega \, d\mathbb{P}(\omega)$$

*and $N_H$ is the distribution function of its spectral measure. In particular, the almost sure spectrum $\Sigma$ of $\{H_\omega\}_\omega$ coincides with the points of increase*

$$\{\lambda \in \mathbb{R} |\, N_H(\lambda+\epsilon) > N_H(\lambda-\epsilon) \text{ for all } \epsilon > 0\}$$

*of $N_H$.*

That the IDS can be expressed in terms of a trace on a von Neumann Algebra was known long ago. In [**297**] and [**298**] Šubin establishes this relation for almost periodic elliptic diffential operators in Euclidean space. He attributes the idea of such an interpretation to Berezin, see the last sentence in Section 3 of [**298**].

We want to describe the self-averaging IDS by an exhaustion of the whole manifold $X$ along a sequence $\Lambda_l \to X$, $l \in \mathbb{N}$ of subsets of $X$. To ensure the existence of a sequence of subsets which is appropriate for the exhaustion procedure, we have to impose additional conditions on the group Γ.

DEFINITION 2.3.4. A group Γ is called *amenable* if it has an left invariant mean $m_L$.

Amenability enters as a key notion in Definition 2.3.6 and Theorem 2.3.8. For readers acquainted only with Euclidean geometry, its role is motivated in Remark 2.3.10.

Under some conditions on the group amenability can be expressed in other ways. A *locally compact* group Γ is amenable if for any $\epsilon > 0$ and compact $K \subset \Gamma$ there is a compact $G \subset \Gamma$ such that

$$m_L(G \Delta KG) < \epsilon \, m_L(G)$$

where $m_L$ denotes the left invariant Haar measure, cf. Theorem 4.13 in [**242**]. This is a geometric description of amenability of Γ. If Γ is a discrete, finitely generated group we chose $m_L$ to be the counting measure and write instead $|\cdot|$. In this case Γ is amenable if and only if a *Følner sequence* exists.

DEFINITION 2.3.5. Let Γ be a discrete, finitely generated group.

(i) A sequence $\{I_l\}_l$ of finite, non-empty subsets of Γ is called a *Følner sequence* if for any finite $K \subset \Gamma$ and $\epsilon > 0$

$$|I_l \Delta K I_l| \leq \epsilon \, |I_l|$$

for all $l$ large enough.

(ii) We say that a sequence $I_l \subset \Gamma, l \in \mathbb{N}$ of finite sets has the *Tempelman* or *doubling property* if it obeys

$$\sup_{l\in\mathbb{N}} \frac{|I_l I_l^{-1}|}{|I_l|} < \infty$$

(iii) We say that a sequence $I_l \subset \Gamma, l \in \mathbb{N}$ of finite sets has the *Shulman property* if it obeys

$$\sup_{l\in\mathbb{N}} \frac{|I_l I_{l-1}^{-1}|}{|I_l|} < \infty$$

(iv) A Følner sequence $\{I_l\}_l$ is called a *tempered Følner sequence* if it has the Shulman property.

In our setting $\Gamma$ is discrete and finitely generated. (Actually, $K := \{\gamma \in \Gamma \mid \gamma\mathcal{F} \cap \mathcal{F} \neq \emptyset\}$ is a finite generator set for $\Gamma$. This follows from the fact that the quotient manifold $X/\Gamma$ is compact, cf. § 3 in [**2**].) Under this circumstances a Følner sequence exists if and only if there is a sequence $J_l \subset \Gamma, l \in \mathbb{N}$ such that $\lim_{l\to\infty} \frac{|J_l \Delta \gamma J_l|}{|J_l|} = 0$ for all $\gamma \in \Gamma$. Moreover, for discrete, finitely generated, amenable groups there exists a Følner sequence which is increasing and exhausts $\Gamma$, cf. Theorem 4 in [**1**].

Both properties (ii) and (iii) control the growth of the group $\Gamma$. Lindenstrauss observed in [**210**] that each Følner sequence has a tempered subsequence. Note that this implies that every amenable group contains a tempered Følner sequence. One of the deep results of Lindenstrauss' paper is, that this condition is actually sufficient for a pointwise ergodic theorem, cf. Theorem 2.6.1. Earlier it was known that such theorems can be established under the more restrictive Tempelman property [**309, 194, 310**]. Shulman [**276**] first realised the usefulness of the relaxed condition (iii).

In the class of countably generated, discrete groups there are several properties which ensure amenability. Abelian groups are amenable. More generally, all solvable groups and groups of subexponential growth, in particular nilpotent groups, are amenable. This includes the (discrete) Heisenberg group considered in Example 2.2.6. Subgroups and quotient groups of amenable groups are amenable. On the other hand, the free group with two generators is not amenable.

For the discussion of combinatorial properties of Følner sequences in discrete amenable groups see [**1**].

Any finite subset $I \subset \Gamma$ defines a corresponding set

$$\phi(I) := \mathrm{int}\Big(\bigcup_{\gamma\in I} \gamma\overline{\mathcal{F}}\Big) \subset X$$

where $\mathrm{int}(\cdot)$ stands for the open interior of a set.

In the following we will need some notation for the thickened boundary. Denote by $d_0$ the distance function on $X$ associated to the Riemannian metric $g_0$. For $h > 0$, let $\partial_h\Lambda := \{x \in X \,|\, d_0(x, \partial\Lambda) \leq h\}$ be the *boundary tube of width* $h$ and $\Lambda_h$ be the interior of the set $\Lambda \setminus \partial_h\Lambda$.

Definition 2.3.6. (a) A sequence $\{\Lambda_l\}_l$ of subsets of $X$ is called *admissible exhaustion* if there exists an increasing, tempered Følner sequence $\{I_l\}_l$ with $\bigcup_l I_l = \Gamma$ such that $\Lambda_l = \phi(I_l^{-1})$, $l \in \mathbb{N}$.

(b) A sequence $\Lambda_l, l \in \mathbb{N}$ of subsets of $(X, g_0)$ is said to satisfy the *Van Hove property* [**313**] if

$$\lim_{l\to\infty} \frac{\mathrm{vol}_0(\partial_h \Lambda_l)}{\mathrm{vol}_0(\Lambda_l)} = 0 \text{ for all } h > 0 \tag{2.17}$$

REMARK 2.3.7. In our setting it is always possible to chose the sequences $\{I_l\}_l$ and $\{\Lambda_l\}_l$ in such a way that they exhausts the group, respectively the manifold. However, this is not really necessary for our results.

A simple instance where $\cup_l \Lambda_l \neq X$ can be given in one space dimension. Let $X = \mathbb{R}$, $\mathbb{Z}$, $I_l = \{0, \dots, l-1\}$, $\mathcal{F} = [0,1]$ and consequently $\Lambda_l = [0, l]$. One can use this sequence of sets to define the IDS of random Schrödinger operators although $\cup_l \Lambda_l = [0, \infty[$. A non-trivial example where the sets $\Lambda_l$ do not exhaust $X$ can be found in [**304, 306**]. There Sznitman considers random Schrödinger operators in hyperbolic spaces, in which setting the approach presented here does not work due to lack of amenability. However, it is interesting that Sznitman obtains the IDS by choosing a sequence $\Lambda_l$ which converges to a horosphere which is properly contained in the hyperbolic space.

Thus an admissible exhaustion always exists in our setting. By Lemma 2.4 in [**243**] every admissible exhaustion satisfies the van Hove property. Inequality (2.10) implies that for a sequence with the van Hove property

$$\lim_{l\to\infty} \frac{\mathrm{vol}_\omega(\partial_h \Lambda_l)}{\mathrm{vol}_\omega(\Lambda_l)} = 0 \text{ for all } h > 0$$

holds for all $\omega \in \Omega$. Let us remark that one could require for the sets $\Lambda_l$ in the exhaustion sequence to have smooth boundary, see Definition 2.1 in [**243**]. Such sequences also exist for any $X$ with amenable deck-transformation group $\Gamma$. This may be of interest, if one wants to study Laplacians with Neumann boundary conditions. For groups of polynomial growth it is possible to construct analoga of admissible exhaustions by taking metric open balls $B_{r_l}(o)$ around a fixed point $o \in X$ with increasing radii $r_1, \dots, r_n, \dots \to \infty$, cf. Theorem 1.5 in [**243**].

We denote by $H^l_\omega$ the Dirichlet restriction of $H_\omega$ to $\Lambda_l$, cf. Remark 2.2.3, and define the finite volume IDS by the formula

$$N^l_\omega(\lambda) := \mathrm{vol}_\omega(\Lambda_l)^{-1} \#\{n \mid \lambda_n(H^l_\omega) < \lambda\}$$

Now we are able to state the result on the existence of a self-averaging IDS.

THEOREM 2.3.8. *Let $\{H_\omega\}_\omega$ be a random Schrödinger operator and $\Gamma$ an amenable group. For any admissible exhaustion $\{\Lambda_l\}_l$ there exists a set $\Omega' \subset \Omega$ of full measure such that*

$$\lim_{l\to\infty} N^l_\omega(\lambda) = N_H(\lambda), \tag{2.18}$$

*for every $\omega \in \Omega'$ and every continuity point $\lambda \in \mathbb{R}$ of $N_H$.*

DEFINITION 2.3.9. The limit in (2.18) is called *integrated density of states*.

Thus all properties (i)–(iii) on page 106 can be established for the model under study. In particular, formula (2.18) is a variant of the Pastur-Šubin trace formula in the context of manifolds. The theorem is proven in §§ 2.4–2.6. It recovers in particular the result of Adachi and Sunada [**2**] on the existence of the IDS of periodic Schrödinger operators on manifolds.

REMARK 2.3.10. Let us motivate for readers acquainted only with Euclidean space why it is natural that the amenability requirement enters in the theorem. In the theory of random operators and in statistical mechanics one often considers a sequence of sets $\Lambda_l, l \in \mathbb{N}$ which tends to the whole space. Even in Euclidean geometry it is known that the exhaustion sequence $\Lambda_l, l \in \mathbb{N}$ needs to tend to $\mathbb{R}^d$ in an appropriate way, e.g. in the sense of Van Hove or Fisher [**265**]. Convergence in the sense of Van Hove [**313**] means that

$$\lim_{l\to\infty} \frac{|\partial_\epsilon \Lambda_l|}{|\Lambda_l|} = 0 \tag{2.19}$$

for all positive $\epsilon$.

If one chooses the sequence $\Lambda_l, l \in \mathbb{N}$, badly, one cannot expect the convergence of the finite volume IDS' $N^l_\omega, l \in \mathbb{N}$, to a limit. In a non-amenable geometry, any exhaustion sequence is bad, since (2.19) cannot be satisfied, cf. Proposition 1.1 in [**2**].

REMARK 2.3.11. We have assumed the potentials $V_\omega$ to be nonnegative and some of our proofs will rely on this fact.

However, the *statements* of Theorem 2.3.1 on the non-randomness of the spectrum and Theorem 2.3.8 on the existence of the IDS carry over to $V_\omega$ which are uniformly bounded below by a constant $C$ not depending on $\omega \in \Omega$. Indeed, in this case our results directly apply to the shifted operator family $\{H_\omega - C\}_{\omega\in\Omega}$. This implies immediately the same statements for the original operators, since the spectral properties we are considering are invariant under shifts of the spectrum.

**2.4. Measurability.** Since we want to study the operators $H^\Lambda_\omega$ as random variables we need a notion of measurability. To this aim, we extend the definition introduced by Kirsch and Martinelli [**162**] for random operators on a fixed Hilbert space to families of operators where the spaces and domains of definition vary with $\omega \in \Omega$.

To distinguish between the scalar products of the different $L^2$-spaces we denote by $\langle\cdot,\cdot\rangle_0$ the scalar product on $L^2(\Lambda, \mathrm{vol}_0)$ and by $\|\cdot\|_0$ the corresponding norm. Similarly, $\langle\cdot,\cdot\rangle_\omega$ and $\|\cdot\|_\omega$ are the scaler product and the norm, respectively, of $L^2(\Lambda, \mathrm{vol}_\omega)$.

DEFINITION 2.4.1. Consider a family of selfadjoint operators $\{H_\omega\}_\omega$, where the domain of $H_\omega$ is a dense subspace $\mathcal{D}_\omega$ of $L^2(\Lambda, \mathrm{vol}_\omega)$. The family $\{H_\omega\}_\omega$ is called a *measurable family of operators* if

$$\omega \mapsto \langle f_\omega, F(H_\omega) f_\omega\rangle_\omega \tag{2.20}$$

is measurable for all measurable and bounded $F\colon \mathbb{R} \to \mathbb{C}$ and all measurable functions $f\colon \Omega \times \Lambda \to \mathbb{R}$ with $f(\omega,\cdot) = f_\omega \in L^2(\Lambda, \mathrm{vol}_\omega)$ for every $\omega \in \Omega$.

THEOREM 2.4.2. *A random Schrödinger operator $\{H_\omega\}_{\omega\in\Omega}$ as in Definition 2.2.2 is a measurable family of operators. The same applies to the Dirichlet restrictions $\{H^\Lambda_\omega\}_{\omega\in\Omega}$ to any open subset $\Lambda$ of $X$.*

For the proof of this theorem we need some preliminary considerations.

Assumption (2.4) in our setting implies that it is sufficient to show the weak measurability (2.20) for functions $f$ which are constant in $\omega$. Note that $L^2(\Lambda, \mathrm{vol}_0)$ and $L^2(\Lambda, \mathrm{vol}_\omega)$ coincide as sets for all $\omega \in \Omega$, though not in their scalar products. Thus it makes sense to speak about $f_\omega \equiv f \in L^2(\Lambda, \mathrm{vol}_\omega)$ " = " $L^2(\Lambda, \mathrm{vol}_0)$.

LEMMA 2.4.3. *A random Schrödinger operator $\{H_\omega\}_\omega$ is measurable if and only if*

$$\omega \mapsto \langle f, F(H_\omega) f\rangle_\omega \text{ is measurable} \tag{2.21}$$

*for all measurable and bounded $F\colon \mathbb{R} \to \mathbb{C}$ and all $f \in L^2(\Lambda, \mathrm{vol}_0)$.*

PROOF. To see this, note that (2.21) implies the same statement if we replace $f(x)$ by $h(\omega, x) = g(\omega) f(x)$ where $g \in L^2(\Omega)$ and $f \in L^2(\Lambda, \mathrm{vol}_0)$. Such functions form a total set in $L^2(\Omega \times \Lambda, \mathbb{P} \circ \mathrm{vol})$.

Now, consider a measurable $h\colon \Omega \times \Lambda \to \mathbb{R}$ such that $h_\omega := h(\omega, \cdot) \in L^2(\Lambda, \mathrm{vol}_\omega)$ for every $\omega \in \Omega$. Then $h^n(\omega, x) := \chi_{h,n}(\omega)\, h(\omega, x)$ is in $L^2(\Omega \times \Lambda, \mathbb{P} \circ \mathrm{vol})$ where $\chi_{h,n}$ denotes the characteristic function of the set $\{\omega |\, \|h_\omega\|_{L^2(\Lambda, \mathrm{vol}_\omega)} \leq n\} \subset \Omega$. Since $\chi_{h,n} \to 1$ pointwise on $\Omega$ for $n \to \infty$ we obtain

$$\langle h^n_\omega, F(H_\omega) h^n_\omega\rangle_\omega \to \langle h_\omega, F(H_\omega) h_\omega\rangle_\omega$$

which shows that $\{H_\omega\}_\omega$ is a measurable family of operators. □

To prove Theorem 2.4.2 we will pull all operators $H^\Lambda_\omega$ onto the same Hilbert space using the unitary transformation $S_\omega$ induced by the density $\rho_\omega$

$$S_\omega\colon L^2(\Lambda, \mathrm{vol}_0) \to L^2(\Lambda, \mathrm{vol}_\omega), \quad (S_\omega f)(x) = \rho^{1/2}_\omega(x) f(x)$$

The transformed operators are

$$A_\omega := -S^{-1}_\omega\, \Delta^\Lambda_\omega\, S_\omega \tag{2.22}$$

$$A_\omega\colon S^{-1}_\omega\, \mathcal{D}(\Delta^\Lambda_\omega) \subset L^2(\Lambda, \mathrm{vol}_0) \longrightarrow L^2(\Lambda, \mathrm{vol}_0)$$

The domain of definition $S^{-1}_\omega\, \mathcal{D}(\Delta^\Lambda_\omega)$ is dense in $L^2(\Lambda, \mathrm{vol}_0)$ and contains all smooth functions of compact support in $\Lambda$.

The first fact we infer for the operators $A_\omega, \omega \in \Omega$ is that they are uniformly bounded with respect to each other, at least in the sense of quadratic forms. This is the content of Proposition 3.4 in [**199**] which we quote without proof.

PROPOSITION 2.4.4. *Let $Q_0, Q_\omega$ be the quadratic forms associated to the operators $-\Delta^\Lambda_0$ and $A_\omega$, and $\mathcal{D} \subset L^2(\Lambda, \mathrm{vol}_0)$ the closure of $C^\infty_c(\Lambda)$ with respect to the norm $\left(Q_0(f,f) + \|f\|^2_0\right)^{1/2}$. Then*

$$\mathcal{D} = \mathcal{D}(Q_0) = \mathcal{D}(Q_\omega)$$

*and there exists a constant $C_A$ such that*

$$C^{-1}_A \left(Q_0(f,f) + \|f\|^2_0\right) \leq Q_\omega(f,f) + \|f\|^2_0 \leq C_A \left(Q_0(f,f) + \|f\|^2_0\right). \tag{2.23}$$

*for all $f \in \mathcal{D}$ and $\omega \in \Omega$.*

Here $\mathcal{D}(Q)$ denotes the *domain of definition of the quadratic form* $Q$. In the proof of this proposition the bound (2.5) — more precisely (2.9) — on the gradient of the density $\rho_\omega$ is needed. It seems to be a technical assumption and in fact dispensable by using a trick from [**69**], at least if $\Lambda$ is precompact or of finite volume.

Since we are dealing now with a family of operators on a fixed Hilbert space, we are in the position to apply the theory developed in [**162**]. The following result is an extension of Proposition 3 there. It suits our purposes and shows that our notion of measurability is compatible with the one in [**162**].

Let $\mathcal{H}$ be a Hilbert space, $\mathcal{D} \subset \mathcal{H}$ a (fixed) dense subset and $B_\omega \colon \mathcal{D} \to \mathcal{H}, \omega \in \Omega$ nonnegative operators. Denote by $\tilde{\Sigma} = \overline{\bigcup_\omega \sigma(B_\omega)}$ the closure of all spectra, and by $\tilde{\Sigma}^c$ its complement. To establish the measurability of the family $\{B_\omega\}_\omega$ one can use one of the following classes of test functions:

- $\mathcal{F}_1 = \{\chi_{]-\infty,\lambda[} | \, \lambda \geq 0\}$,
- $\mathcal{F}_2 = \{x \mapsto e^{itx} | \, t \in \mathbb{R}\}$,
- $\mathcal{F}_3 = \{x \mapsto e^{-tx} | \, t \geq 0\}$,
- $\mathcal{F}_4 = \{x \mapsto (z-x)^{-1} | \, z \in \mathbb{C} \setminus \tilde{\Sigma}\}$,
- $\mathcal{F}_5 = \mathcal{F}_4(z_0) = \{x \mapsto (z_0 - x)^{-1}\}$ for a fixed $z_0 \in \mathbb{C} \setminus \tilde{\Sigma}$,
- $\mathcal{F}_6 = C_b = \{f \colon \mathbb{R} \to \mathbb{C} | \, f$ bounded, continuous$\}$,
- $\mathcal{F}_7 = \{f \colon \mathbb{R} \to \mathbb{C} | \, f$ bounded, measurable$\}$.

The following proposition says, that it does not matter which of the above sets of functions one chooses for testing the measurability of $\{B_\omega\}_\omega$.

PROPOSITION 2.4.5. *For $i = 1, \ldots, 7$ the following statements are equivalent:*

$(\mathbf{F}_i)$ $\qquad \omega \mapsto \langle f, F(B_\omega) h \rangle_{\mathcal{H}}$ *is measurable for all $f, h \in \mathcal{H}$ and $F \in \mathcal{F}_i$,*

PROOF. It is obvious that $(\mathbf{F}_4) \Rightarrow (\mathbf{F}_5)$, $(\mathbf{F}_7) \Rightarrow (\mathbf{F}_6)$, and $(\mathbf{F}_6) \Rightarrow (\mathbf{F}_3)$. The equivalence of $(\mathbf{F}_1)$, $(\mathbf{F}_2)$ and $(\mathbf{F}_4)$ can be found in [**162**].

To show $(\mathbf{F}_5) \Rightarrow (\mathbf{F}_4)$, consider the set

$$Z := \{z \in \tilde{\Sigma}^c | \, \omega \mapsto (z - H_\omega)^{-1} \text{ is weakly measurable } \}$$

in the topological space $\tilde{\Sigma}^c$. It is closed, since $z_n \to z$ implies the convergence of the resolvents, see e.g. [**258**, Theorem VI.5]. A similar argument using the resolvent equation and a Neumann series expansion shows that $z \in Z$ implies $B_\delta(z) \subset Z$ where $\delta := d(z, \tilde{\Sigma})$. Since $\tilde{\Sigma}^c$ is connected, $Z = \tilde{\Sigma}^c$ follows.

$(\mathbf{F}_3) \Rightarrow (\mathbf{F}_1)$: By the Stone-Weierstrass Theorem, see e.g. [**258**, Thm. IV.9], applied to $C([0,\infty])$ it follows that $\mathcal{F}_3$ is dense in the set of Functions $\{f \in C([0,\infty]) \, | \, f(\infty) = 0\} = C_\infty([0,\infty[)$. We may approximate any $\chi_{]-\infty,\lambda[}$ pointwise by a monotone increasing sequence $0 \leq f_n, n \in \mathbb{N}$ in $C_\infty(\mathbb{R})$. Polarisation, the spectral theorem, and the monotone convergence theorem for integrals imply that $\chi_{]-\infty,\lambda[}(H_\omega)$ is weakly measurable. An analogous argument shows $(\mathbf{F}_1) \Rightarrow (\mathbf{F}_7)$, since any non-negative $f \in \mathcal{F}_7$ can be approximated monotonously pointwise by non-negative step functions $f_n, n \in \mathbb{N}$. □

We use the following proposition taken from [**295**] (Prop. 1.2.6.) to show that $\{A_\omega\}_\omega$ is a measurable family of operators.

PROPOSITION 2.4.6. *Let $B_\omega, \omega \in \Omega$ and $B_0$ be nonnegative operators on a Hilbert space $\mathcal{H}$. Let $Q_\omega$, $\omega \in \Omega$ and $Q_0$ be the associated closed quadratic forms with the following properties:*

(2.24) *$Q_\omega$, $\omega \in \Omega$ and $Q_0$ are defined on the same dense subset $\mathcal{D} \subset \mathcal{H}$.*

(2.25) *There is a constant $C > 0$ such that*

$$C^{-1} \left( Q_0(f,f) + \|f\|_0^2 \right) \leq Q_\omega(f,f) + \|f\|_0^2 \leq C \left( Q_0(f,f) + \|f\|_0^2 \right)$$

*for all $\omega \in \Omega$ and $f \in \mathcal{D}$.*

(2.26) *For every $f \in \mathcal{D}$ the map $\omega \mapsto Q_\omega(f,f)$ is measurable.*

*Then the family $\{B_\omega\}_\omega$ of operators satisfies the equivalent properties of Proposition 2.4.5.*

By property ($\mathbf{F}_7$), this implies that $\{B_\omega\}_\omega$ is a measurable family of operators.

We apply the proposition to $B_\omega = A_\omega$, where $\{A_\omega\}_\omega$ is defined in Proposition 2.4.4. To do so we check that the properties (2.4.6)–(2.26) are satisfied: Since $C_c^\infty(\Lambda)$ is dense in $\mathcal{D}(Q_\omega)$ for all $\omega$, the closures of $C_c^\infty(\Lambda)$ with respect to any of the equivalent norms in (2.23) coincide, which shows assumption (2.4.6). Property (2.25) is just (2.23), property (2.26) is obvious for $f \in C_c^\infty(\Lambda)$ and follows by approximation for all $f \in \mathcal{D}$.

Proof of Theorem 2.4.2. We already know that the transformed 'kinetic' part $A_\omega$, $\omega \in \Omega$ of the Hamiltonian is measurable. To deal with the singular potential we introduce the cut off

$$V_\omega^n(x) := \min\{n, V_\omega(x)\} \text{ for } n \in \mathbb{N} \text{ and } \omega \in \Omega$$

The auxiliary potential $V_\omega^n$ is bounded and in particular its domain of definition is the whole Hilbert space $L^2(\Lambda, \mathrm{vol}_0)$. Thus the operator sum

$$A_\omega^n := A_\omega + V_\omega^n, \quad \omega \in \Omega$$

is well defined and [**162**, Prop. 4] implies that it forms a measurable family of operators. To recover the unbounded potential $V_\omega$, we consider the semigroups $\omega \mapsto \exp(-tA_\omega^n), t > 0$ which are weakly measurable.

The quadratic forms of $A_\omega^n$ converge monotonously to the form of $A_\omega^\infty := A_\omega + V_\omega$. Now Theorems VIII.3.13a and IX.2.16 in [**152**] imply that the semigroups of $A_\omega^n$ converge weakly towards the one of $A_\omega^\infty$ for $n \to \infty$. Thus $\exp(-tA_\omega^\infty)$ is weakly measurable, which implies the measurability of the family $A_\omega^\infty$.

Finally, since $S_\omega$ is multiplication with the measurable function $(x, \omega) \mapsto \rho_\omega(x)$, this implies the measurability of the family $H_\omega = S_\omega A_\omega^\infty S_\omega^{-1}$, $\omega \in \Omega$. □

For later use let us note that the trace of measurable operators is measurable. More precisely we will need the fact that the mappings

$$\omega \mapsto \mathrm{Tr}(\chi_\Lambda e^{-tH_\omega}) \quad \text{and} \quad \omega \mapsto \mathrm{Tr}(e^{-tH_\omega^\Lambda}) \tag{2.27}$$

are measurable. Note that one can chose an orthonormal basis for $L^2(\Lambda, \mathrm{vol}_\omega)$ which depends in a measurable way on $\omega$, cf. for instance Lemma II.2.1 in [**75**]. Thus (2.27) follows immediately from the Definition 2.4.1 of measurable operators.

**2.5. Bounds on the heat kernels uniform in $\omega$.** This paragraph is devoted to heat kernel estimates of the Schrödinger operators $H_\omega$. It consists of four parts. Firstly we discuss existence of $L^2$-kernels of $e^{-tH_\omega}, t > 0$ and derive rough upper bounds relying on results in [**68**]. Secondly, we infer Gaußian off-diagonal decay estimates of the kernels using estimates derived in [**204**]. We then present an idea of H. Weyl to derive the *principle of not feeling the boundary*, and finally we state a proposition which summarises the information on the heat kernel needed in the next section.

We have to control the dependence on the metric and potential of all these estimates since those quantities vary with the random parameter $\omega \in \Omega$.

As $H_\omega$ is non-negative the semigroup $e^{-tH_\omega}, t > 0$ consists of contractions. Moreover, the semigroup satisfies some nice properties formulated in the following definition which enable us to derive estimates on the corresponding heat kernel.

DEFINITION 2.5.1. Let $\Lambda \subset X$ be open and $\mu$ a $\sigma$-finite Borel measure on $\Lambda$. Let $A$ be a real, non-negative, selfadjoint operator on the Hilbert space $L^2(\Lambda,\mu)$. The semigroup $e^{-tA}, t > 0$ is called *positivity preserving* if $e^{-tA}f \geq 0$ for any $0 \leq f \in L^2(\Lambda,\mu)$ and $t > 0$. Furthermore, $e^{-tA}, t > 0$ is called a *Markov semigroup*, if it is well defined on $L^\infty(\Lambda,\mu)$ and the two following properties hold

$$e^{-tA}\colon L^2(\Lambda,\mu) \longrightarrow L^2(\Lambda,\mu) \qquad \text{is positivity preserving for every } t>0 \tag{2.28}$$

$$e^{-tA}\colon L^\infty(\Lambda,\mu) \to L^\infty(\Lambda,\mu) \qquad \text{is a contraction for every } t>0 \tag{2.29}$$

In this case $A$ is called a *Dirichlet form*.

A Markov semigroup $e^{-tA}$ is called *ultracontractive* if

$$e^{-tA}\colon L^2(\Lambda,\mu) \to L^\infty(\Lambda,\mu) \text{ is bounded for all } t>0 \tag{2.30}$$

The above (2.28) and (2.29) are called *Beurling-Deny* conditions [**28, 29**].

We infer from [**68**] the following facts: A Markov semigroup is a contraction on $L^p(\Lambda,\mu)$ for all $1 \leq p \leq \infty$ (and all $t > 0$). For all $\omega \in \Omega$ the Schrödinger operator $H_\omega^\Lambda$ on $L^2(\Lambda, \mathrm{vol}_\omega)$ is a Dirichlet form, [**68**, Thm. 1.3.5]. There the proof is given for $X = \mathbb{R}^d$, but it applies to manifolds, too. By Sobolev embedding estimates and the spectral theorem $e^{t\Delta_\omega^\Lambda}$ is ultracontractive. Thus by Lemma 2.1.2 in [**68**] each $e^{t\Delta_\omega^\Lambda}$ has a kernel, which we denote by $k_\omega^\Lambda$, such that for almost all $x, y \in \Lambda$

$$0 \leq k_\omega^\Lambda(t,x,y) \leq \|e^{t\Delta_\omega^\Lambda}\|_{1,\infty} =: C_\omega^\Lambda(t) \tag{2.31}$$

Here $\|B\|_{1,\infty}$ denotes the norm of $B\colon L^1 \to L^\infty$. For $\Lambda = X$ we use the abbreviation $k_\omega^X = k_\omega$.

To derive an analogous estimate to (2.31) for the full Schrödinger operator with potential we make use of the *Feynman-Kac formula*. Using the symbol $\mathbf{E}_x$ for the expectation with respect to the *Brownian motion* $b_t$ starting in $x \in X$ the formula reads

$$(e^{-tH_\omega}f)(x) = \mathbf{E}_x\left(e^{-\int_0^t V_\omega(b_s)\,ds}\, f(b_t)\right)$$

For a stochastically complete manifold $X$ and bounded, continuous $V_\omega$ the formula is proven, for instance, in Theorem IX.7A in [**93**]. It extends to general non-negative potentials which are in $L^1_{\mathrm{loc}}$ using semigroup and integral convergence theorems similarly as in the proof of Theorem X.68 in [**255**]. Since we consider (geodesically) complete manifolds whose Ricci curvature is bounded below, they are all stochastically complete, cf. for instance [**131**] or Theorem 4.2.4 in [**136**].

Since the potential is non-negative, the Feynman-Kac formula implies for non-negative $f \in L^1(\Lambda, \mathrm{vol}_\omega)$

$$0 \leq \left(e^{-tH_\omega^\Lambda}f\right)(x) \leq \left(e^{t\Delta_\omega^\Lambda}f\right)(x) \leq C_\omega^\Lambda(t)\,\|f\|_{L^1}$$

for almost every $x \in \Lambda$. Thus $e^{-tH_\omega^\Lambda}\colon L^1(\Lambda,\mathrm{vol}_\omega) \to L^\infty(\Lambda,\mathrm{vol}_\omega)$ has the same bound $C_\omega^\Lambda(t)$ as the semigroup where the potential is absent. This yields the pointwise estimate on the kernel $k_{H_\omega}^\Lambda$ of $e^{-tH_\omega^\Lambda}$:

$$0 \leq k_{H_\omega}^\Lambda(t,x,y) \leq C_\omega^\Lambda(t) \quad \text{for almost every } x, y \in X. \tag{2.32}$$

In the following we derive sharper upper bounds on the kernels which imply their decay in the distance between the two space arguments $x$ and $y$. Such estimates have been proven by Li and Yau [**204**] for *fundamental solutions* of the

heat equation. One would naturally expect that the fundamental solution and the $L^2$-heat kernel of the semigroup coincide under some regularity assumptions. This is actually the case as has been proven for instance in [**77**] for vanishing, and in [**199**] for smooth, non-negative potentials. The proof in the last cited source uses that $H_\omega$ is a Dirichlet form.

To formulate the results of Li and Yau [**204**] which we will be using, we denote by $d_\omega \colon X \times X \to [0, \infty[$ the Riemannian distance function on $X$ with respect to $g_\omega$. Note that the following proposition concerns the heat kernel of the pure Laplacian.

PROPOSITION 2.5.2. *For every $t > 0$ there exist constants $C(t) > 0$, $\alpha_t > 0$ such that*

$$k_\omega(t, x, y) \leq C(t) \exp\big( - \alpha_t \, d_0^2(x, y)\big) \tag{2.33}$$

*for all $\omega \in \Omega$ and $x, y \in X$.*

PROOF. For a fixed Schrödinger operator the estimate (with $d_0$ replaced by $d_\omega$) is contained in Corollary 3.1 in [**204**]. The properties (2.4), (2.8) and

$$C_g^{-1} d_0(x, y) \leq d_\omega(x, y) \leq C_g d_0(x, y)$$

ensure that the constants $C(t)$ and $\alpha_t$ in (2.33) may be chosen uniformly in $\omega$. Moreover, for measuring the distance between the points $x$ and $y$ we may always replace $d_\omega$ by $d_0$ by increasing $\alpha_t$. □

Let us collect various consequences of Proposition 2.5.2 which will be useful later on.

(i) The pointwise kernel bound on the left hand side of (2.32) can be chosen uniformly in $\omega \in \Omega$.

(ii) We stated Proposition 2.5.2 for the pure Laplacian, although Li and Yau treat the case of a Schrödinger operator with potential. The reason for this is that we want to avoid the regularity assumptions on the potential imposed in [**204**].

To recover from (2.33) the case where a (non-negative) potential is present we use again the Feynman-Kac formula. We need now a local version of the argument leading to (2.32), more precisely we consider $e^{-tH_\omega}$ as an operator form $L^1(B_\epsilon(y))$ to $L^\infty(B_\epsilon(x))$ for small $\epsilon > 0$. Thus we obtain

$$0 \leq k_{H_\omega}(t, x, y) \leq C(t) \exp\big( - \alpha_t \, d_0^2(x, y)\big)$$

(iii) The estimates derived so far immediately carry over to the case where the entire manifold is replaced by an open subset $\Lambda \subset X$.

$$0 \leq k^\Lambda_{H_\omega}(t, x, y) \leq k_{H_\omega}(t, x, y)$$

This is due to *domain monotonicity*, see for example [**68**, Thm. 2.1.6].

(iv) The *Bishop volume comparison* theorem controls the growth of the volume of balls with radius $r$, see for instance [**34**], [**50**, Thm. III.6] or [**42**]. It tells us that the lower bound (2.6) on the Ricci curvature is sufficient to bound the growth of the volume of balls as $r$ increases. The volume of the ball can be estimated by the volume of a ball with the same radius in a space with constant curvature $K$. The latter volume grows at most exponentially in the radius. For our purposes it is necessary to have an $\omega$-uniform version of the volume growth estimate. Using Properties (2.4), (2.6) and (2.8) we obtain the uniform bound

$$\mathrm{vol}_\omega\big(\{y |\, d_\omega(x, y) < r\}\big) \leq C_1 \, e^{C_2 r} \quad \text{for all } x \in X$$

where $C_1, C_2$ do not depend on $x$ and $\omega$. This implies that for all exponents $p > 0$, there exists a $M_p(t) < \infty$ such that the moment estimate

$$\int_\Lambda [k^\Lambda_{H_\omega}(t,x,y)]^p \, d\mathrm{vol}_\omega(y) \leq M_p(t)$$

holds uniformly in $\Lambda \subset X$ open, in $x \in \Lambda$ and $\omega \in \Omega$. We set $M(t) := M_1(t)$.

(v) The heat kernel estimates imply a uniform bound on the traces of the semigroup localised in space. Let $\Lambda \subset X$ be a (fixed) open set of finite volume. There exists a constant $C_{\mathrm{Tr}} = C_{\mathrm{Tr}}(\Lambda, t) > 0$ such that for all $\omega \in \Omega$

$$\mathrm{Tr}\big(\chi_\Lambda \, e^{-tH_\omega}\big) \leq C_{\mathrm{Tr}}$$

Intuitively this is the same as saying that $\int_\Lambda k_{H_\omega}(t,x,x) \, d\mathrm{vol}_\omega(x)$ is uniformly bounded. However, since the diagonal $\{(x,x) | \, x \in \Lambda\}$ is a set of measure zero, the integral does not make sense as long as we consider $k_{H_\omega}$ as an $L^2$-function. We do not want here to address the question of continuity of the kernel. Instead we use the semigroup property $e^{-2tH_\omega} = e^{-tH_\omega} e^{-tH_\omega}, t > 0$ and selfadjointness to express the trace as

(2.34)
$$\mathrm{Tr}\big(\chi_\Lambda \, e^{-tH_\omega}\big) = \int_\Lambda \int_\Lambda [k_{H_\omega}(t/2,x,y)]^2 \, d\mathrm{vol}_\omega(x) \, d\mathrm{vol}_\omega(y) \leq M_2(t/2) \, \mathrm{vol}_\omega(\Lambda)$$

By (2.10) this is bounded uniformly in $\omega \in \Omega$. Applying domain monotonicity once more, we obtain

$$\mathrm{Tr}\big(e^{-tH^\Lambda_\omega}\big) \leq M_2(t/2) \, \mathrm{vol}_\omega(\Lambda) \tag{2.35}$$

The following lemma is a maximum principle for Schrödinger operators with non-negative potentials. Combined with the off-diagonal decay estimates in Proposition 2.5.2 it will give us a proof of the principle of not feeling the boundary.

Lemma 2.5.3 (Maximum principle for heat equation with nonnegative potential). *Let $\Lambda \subset X$ be open with compact closure, $V$ be a non-negative function, and $u \in C([0,T[\times\overline{\Lambda}) \cap C^2(]0,T[\times\Lambda)$ be a solution of the heat equation $\frac{\partial}{\partial t}u + (-\Delta + V)u = 0$ on $]0,T[\times\Lambda$ with nonnegative supremum $s = \sup\{u(t,x) \mid (t,x) \in [0,T[\times\overline{\Lambda}\}$. Then,*

$$s = \max\left\{ \max_{x\in\overline{\Lambda}} u(0,x), \sup_{[0,T[\times\partial\Lambda} u(t,x) \right\}$$

Note that regularity of $V$ is not assumed explicitly, but implicitly by the requirements on $u$. They are e.g. satisfied if $V$ is smooth. Indeed, in that case the heat kernel is smooth, as can be seen following the proof of [**68**, Thm. 5.2.1].

Now we are in the position to state the second refined estimate on the heat kernels, the *principle of not feeling the boundary*. It is a formulation of the fact that the heat kernel of the Laplacian restricted to a large open set $\Lambda$ should not differ much from the heat kernel associated to the Laplacian on the whole manifold, as long as one stays away from the boundary of $\Lambda$. As before, we derive this estimate first for the pure Laplacian and then show that it carries over to Schrödinger operators with non-negative potential.

PROPOSITION 2.5.4. *For any fixed $t, \epsilon > 0$, there exists an $h = h(t, \epsilon) > 0$ such that for every open set $\Lambda \subset X$ and all $\omega \in \Omega$*

$$0 \leq k_\omega(t, x, y) - k_\omega^\Lambda(t, x, y) \leq \epsilon,$$

*for all $x \in \Lambda, y \in \Lambda_h$.*

PROOF. The first inequality is a consequence of domain monotonicity. So we just have to prove the second one.

Fix $\omega \in \Omega$ and $t, \epsilon > 0$. Choose $h > 0$ such that

$$C(t) \exp\Big( - \alpha_t \big(h/2\big)^2 \Big) \leq \epsilon$$

Note that the choice is independent of $\omega$. For any $y \in \Lambda_h$ and $0 < \delta < h/2$ denote by $B_\delta(y)$ the open $d_0$-ball around $y$ with radius $\delta$. Let $f_\delta \in C_0^\infty(B_\delta(y))$ be a non-negative approximation of the $\delta$-distribution at $y$.

We consider now the time evolution of the initial value $f$ under the two semigroups generated by $\Delta_\omega$ and $\Delta_\omega^\Lambda$, respectively.

$$u_1(t, x) := \int_X k_\omega(t, x, z) f_\delta(z) d\mathrm{vol}_\omega(z) = \int_\Lambda k_\omega(t, x, z) f_\delta(z) d\mathrm{vol}_\omega(z).$$
$$u_2(t, x) := \int_\Lambda k_\omega^\Lambda(t, x, z) f_\delta(z) d\mathrm{vol}_\omega(z).$$

The difference $u_1(t, x) - u_2(t, x)$ solves the heat equation $\frac{\partial}{\partial t} u = \Delta_\omega u$ and satisfies the initial condition $u_1(0, x) - u_2(0, x) = f_\delta(x) - f_\delta(x) = 0$ for all $x \in \Lambda$. Now, by domain monotonicity we know $k_\omega(t, x, z) - k_\omega^\Lambda(t, x, z) \geq 0$, thus

$$u_1(t, x) - u_2(t, x) = \int_\Lambda \big[k_\omega(t, x, z) - k_\omega^\Lambda(t, x, z)\big] f_\delta(z)\, d\mathrm{vol}_\omega(z) \geq 0$$

for all $t > 0$ and $x \in \Lambda$. The application of the maximum principle yields

$$u_1(t, x) - u_2(t, x) \leq \max_{]0,t] \times \partial\Lambda} \{u_1(s, w) - u_2(s, w)\}. \tag{2.36}$$

The right hand side can be further estimated by:

$$u_1(s, w) - u_2(s, w) \leq \int_\Lambda k_\omega(s, w, z) f_\delta(z)\, d\mathrm{vol}_\omega(z) = \int_{\Lambda_{h/2}} k_\omega(s, w, z) f_\delta(z)\, d\mathrm{vol}_\omega(z).$$

Since $w \in \partial\Lambda$ and $z \in \Lambda_{h/2}$, we conclude using Proposition 2.5.2:

$$\int_{\Lambda_{h/2}} k_\omega(s, w, z) f_\delta(z) d\mathrm{vol}_\omega(z) \leq C(t) \exp\big( - \alpha_t (h/2)^2 \big) \leq \epsilon$$

Since the bound is independent of $\delta$ we may take the limit $\delta \to 0$ which concludes the proof. □

One can prove the principle of not feeling the boundary by other means too, see for instance [**215, 79, 243**]. This alternative approach uses information on the behaviour of solutions of the wave equation. Unlike the solutions of the heat equation, they do not have the unphysical property that their support spreads instantaneously to infinity. Actually, the solutions of the wave equation have finite propagation speed [**307**]. Fourier transforms and the spectral theorem turn this information into estimates on the difference of the solutions of the free and restricted heat equation. Sobolev estimates lead then to the principle of not feeling the boundary.

See also Section 7 in [**254**].

REMARK 2.5.5. Similarly as in Lemma 2.5.3, one can prove the proposition if a potential is present. More precisely, Proposition 2.5.4 is valid for Schrödinger operators with potentials $V$ such that for continuous initial and boundary values the solution of the heat equation $\frac{\partial}{\partial t}u = -(-\Delta_\omega + V)u$ is in $C([0,T[\times\overline{\Lambda}) \cap C^2(]0,T[\times\Lambda)$. However, Proposition 2.5.4 implies an analogous estimate for the case where a non-negative potential is present, similarly as in (ii) on page 121.

Consider $e^{-tH_\omega} - e^{-tH_\omega^\Lambda}$ as an operator from $L^1(\Lambda_h)$ to $L^\infty(\Lambda)$, and denote by $\tau_x^\Lambda$ the *first exit time* from $\Lambda$ for a Brownian motion starting in $x$. By the Feynman-Kac formula, we have for $0 \leq f \in L^1(\Lambda_h)$

$$[(e^{-tH_\omega} - e^{-tH_\omega^\Lambda})f](x) = \mathbf{E}_x\Big(e^{-\int_0^t dsV(b_s)} f(b_t)\chi_{\{b|\,\tau_x^\Lambda \leq t\}}\Big)$$
$$\leq \mathbf{E}_x\Big(f(b_s)\chi_{\{b|\,\tau_x^\Lambda \leq t\}}\Big) = \int [k_\omega(t,x,y) - k_\omega^\Lambda(t,x,y)]f(y)\,d\mathrm{vol}_\omega \leq \epsilon \int f(y)\,d\mathrm{vol}_\omega$$

if we chose $h$ as in Proposition 2.5.4. Thus for almost all $x \in \Lambda, y \in \Lambda_h$

$$k_{H_\omega}(t,x,y) - k_{H_\omega}^\Lambda(t,x,y) \leq \|e^{-tH_\omega} - e^{-tH_\omega}\|_{L^1(\Lambda_h)\to L^\infty(\Lambda)} \leq \epsilon \tag{2.37}$$

The upper bounds on the heat kernel and the principle of not feeling the boundary enable us to prove a result on the traces of localised heat-semigroups.

In the macroscopic limit, as $\Lambda$ tends (in a nice way) to the whole of $X$, the two quantities

$$\mathrm{Tr}(\chi_\Lambda e^{-tH_\omega}) \quad \text{and} \quad \mathrm{Tr}(e^{-tH_\omega^\Lambda})$$

are approximately the same. The precise statement is contained in the following

PROPOSITION 2.5.6. *Let $\{\Lambda_l\}_{l\in\mathbb{N}}$, be an sequence of subsets of $X$ which satisfies the van Hove property 2.17 and let $\{H_\omega\}_\omega$ be a random Schrödinger operator. Then*

$$\lim_{l\to\infty} \sup_{\omega\in\Omega} \frac{1}{\mathrm{vol}_\omega(\Lambda_l)} \Big|\mathrm{Tr}(\chi_{\Lambda_l} e^{-tH_\omega}) - \mathrm{Tr}(e^{-tH_\omega^l})\Big| = 0$$

PROOF. We consider first a fixed $l \in \mathbb{N}$ and abbreviate $\Lambda = \Lambda_l$. For the operator $e^{-tH_\omega^\Lambda}$ we may write the trace in the same way as in (2.34) to obtain

$$\mathrm{Tr}(e^{-tH_\omega^\Lambda}) = \int_\Lambda \int_\Lambda [k_{H_\omega}^\Lambda(t/2,x,y)]^2 d\mathrm{vol}_\omega(x) d\mathrm{vol}_\omega(y) \tag{2.38}$$

We express the difference of (2.34) and (2.38) using

$$(k_{H_\omega})^2 - (k_{H_\omega}^\Lambda)^2 = (k_{H_\omega} - k_{H_\omega}^\Lambda)(k_{H_\omega} + k_{H_\omega}^\Lambda)$$

Next we chose $h = h(t/2,\epsilon) > 0$ as in Proposition 2.5.4 and decompose the integration domain according to

$$\Lambda \times \Lambda = (\Lambda \times \Lambda_h) \cup (\Lambda \times \partial_h \Lambda)$$

The difference of the traces can be now estimated as

$$\begin{aligned}(2.39)\quad &\mathrm{Tr}(\chi_\Lambda\, e^{-tH_\omega}) - \mathrm{Tr}(e^{-tH_\omega^\Lambda}) \\ &= \int_\Lambda \int_{\Lambda_h} \left[k_{H_\omega}(\frac{t}{2},x,y) - k^\Lambda_{H_\omega}(\frac{t}{2},x,y)\right]\left[k_{H_\omega}(\frac{t}{2},x,y) + k^\Lambda_{H_\omega}(\frac{t}{2},x,y)\right] d\mathrm{vol}_\omega(x,y) \\ &+ \int_\Lambda \int_{\partial_h\Lambda} \left[k_{H_\omega}(\frac{t}{2},x,y) - k^\Lambda_{H_\omega}(\frac{t}{2},x,y)\right]\left[k_{H_\omega}(\frac{t}{2},x,y) + k^\Lambda_{H_\omega}(\frac{t}{2},x,y)\right] d\mathrm{vol}_\omega(x,y)\end{aligned}$$

The first term is bounded by $2M(t/2)\,\epsilon\,\mathrm{vol}_\omega(\Lambda)$ and the second by

$$2M(t/2)\,C(t/2)\,\mathrm{vol}_\omega(\partial_h\Lambda)$$

It follows that

$$0 \leq \frac{1}{\mathrm{vol}_\omega(\Lambda)}\Big(\mathrm{Tr}(\chi_\Lambda\, e^{-tH_\omega}) - \mathrm{Tr}(e^{-tH_\omega^\Lambda})\Big) \leq 2M(t/2)\epsilon + 2M(t/2)C(t/2)\frac{\mathrm{vol}_\omega(\partial_h\Lambda)}{\mathrm{vol}_\omega(\Lambda)}$$

Now, we let $l$ go to infinity. Since the sequence $\Lambda_l$ satisfies the van Hove property (2.17) and since our bounds are uniform in $\omega$, the proposition is proven. □

**2.6. Laplace transforms and Ergodic Theorem.** This paragraph completes the proof of Theorem 2.3.8. It relies, apart from the results results established in the previous paragraphs 2.4–2.5, on a general ergodic theorem and a criterion for the convergence of distribution functions.

Lindenstrauss proved in [**210, 209**] an ergodic theorem which applies to locally compact, second countable amenable groups. It includes as a special case the following statement for discrete groups.

THEOREM 2.6.1. *Let $\Gamma$ be an amenable discrete group and $(\Omega, \mathcal{B}_\Omega, \mathbb{P})$ be a probability space. Assume that $\Gamma$ acts ergodically on $\Omega$ by measure preserving transformations. Let $\{I_l\}_l$ be a tempered Følner sequence in $\Gamma$. Then for every $f \in L^1(\Omega)$*

$$(2.40)\qquad \lim_{j\to\infty} \frac{1}{|I_l|} \sum_{\gamma\in I_l} f(\gamma\omega) = \mathbb{E}\,(f)$$

*for almost all $\omega \in \Omega$.*

In the application we have in mind $f \in L^\infty$, so the convergence holds in the $L^1$-topology, too.

REMARK 2.6.2. Some background on previous results can be found for instance in Section 6.6 of Krengel' s book [**194**], in Tempelman's works [**308, 309, 310**] or some other sources [**95, 15, 94, 236**]. The book [**310**] gives in § 5.6 a survey of Shulman's results [**276**]. Mean ergodic theorems hold in more general circumstances, see for instance [**194**, § 6.4] or [**310**, Ch. 6].

We will apply the ergodic theorem above not to the normalised eigenvalue counting functions $N^l_\omega$, but to their Laplace transforms $\mathcal{L}^l_\omega$. The reason is, that the $\mathcal{L}^l_\omega$ are bounded, while the original $N^l_\omega$ are not. The following criterion of Pastur and Šubin [**237, 298**] says that it is actually sufficient to test the convergence of the Laplace transforms.

LEMMA 2.6.3 (Pastur-Šubin convergence criterion). *Let $N_n$ be a sequence of distribution functions such that*

(i) *there exists a $\lambda_0 \in \mathbb{R}$ such that $N_l(\lambda) = 0$ for all $\lambda \leq \lambda_0$ and $l \in \mathbb{N}$,*

(ii) *there exists a function* $C : \mathbb{R}^+ \to \mathbb{R}$ *such that* $\mathcal{L}^l(t) := \int e^{-\lambda t} dN_l(\lambda) \leq C(t)$ *for all* $l \in \mathbb{N}$ *and* $t > 0$,
(iii) $\lim_{l\to\infty} \mathcal{L}^l(t) =: \mathcal{L}(t)$ *exists for all* $t > 0$.

*Then* $\mathcal{L}$ *is the Laplace transform of a distribution function* $N$ *and for all continuity points* $\lambda$ *of* $N$ *we have*

$$N(\lambda) := \lim_{l\to\infty} N_l(\lambda)$$

Finally, we present the proof of Theorem 2.3.8 on the existence of a self-averaging IDS:

Proof of Theorem 2.3.8. We have to check the conditions in the previous lemma for the normalised eigenvalue counting functions $N^l_\omega$. The first one is clearly satisfied for $\lambda_0 = 0$, since all operators we are dealing with are non-negative. To see (ii), express the Laplace transform by the trace of the heat semigroup

$$\mathcal{L}^l_\omega(t) = \frac{1}{\mathrm{vol}_\omega(\Lambda)} \sum_{n,\, \lambda_n\in\sigma} e^{-t\lambda_n} = \frac{1}{\mathrm{vol}_\omega(\Lambda)} \operatorname{Tr}(e^{-tH^l_\omega})$$

The sum extends over all eigenvalues $\lambda_n$ of $H^l_\omega$, counting multiplicities. Now, (2.35) implies condition (ii) of the Pastur-Šubin criterion.

To prove (iii) we will show for all $t > 0$ the convergence

$$\lim_{j\to\infty} \mathcal{L}^l_\omega(t) = \int_{\mathbb{R}} e^{-t\lambda} dN_H(\lambda)$$

in ($L^1$ and) $\mathbb{P}$ almost sure-sense.

Introduce for two sequences of random variables $a_l(\omega), b_l(\omega), l \in \mathbb{N}$ the equivalence relation $a_l \overset{j\to\infty}{\sim} b_l$ if they satisfy $a_l - b_l \to 0$, as $l \to \infty$, in $L^1$ and $\mathbb{P}$-almost surely.

For technical reasons we will deal separately with the convergence of the enumerator and denominator in

$$\mathcal{L}^l_\omega(t) = \mathrm{vol}_\omega(\Lambda_l)^{-1} \operatorname{Tr}(e^{-tH^l_\omega})$$

However, we need *some* normalisation, to avoid divergences. Consider first the enumerator with an auxiliary normalisation

$$|I_l|^{-1} \operatorname{Tr}(e^{-tH^l_\omega}) \tag{2.41}$$

By Proposition 2.5.6, the equivariance, and Lindenstrauss' ergodic theorem 2.6.1

$$\begin{aligned} |I_l|^{-1} \operatorname{Tr}(e^{-tH^l_\omega}) \quad &\overset{j\to\infty}{\sim} \quad |I_l|^{-1} \operatorname{Tr}(\chi_{\Lambda_l} e^{-tH_\omega}) = |I_l|^{-1} \sum_{\gamma\in I_l^{-1}} \operatorname{Tr}(\chi_{\gamma\mathcal{F}} e^{-tH_\omega}) \\ &= \quad |I_l|^{-1} \sum_{\gamma\in I_l} \operatorname{Tr}(\chi_{\mathcal{F}} e^{-tH_{\gamma\omega}}) \overset{j\to\infty}{\sim} \mathbb{E}\left\{\operatorname{Tr}(\chi_{\mathcal{F}} e^{-tH_\bullet})\right\} \end{aligned}$$

Similarly we infer for the normalised denominator

$$|I_l|^{-1}\mathrm{vol}_\omega(\Lambda_l) = |I_l|^{-1} \sum_{\gamma\in I_l^{-1}} \mathrm{vol}_\omega(\gamma\mathcal{F}) = |I_l|^{-1} \sum_{\gamma\in I_l} \mathrm{vol}_{\gamma\omega}(\mathcal{F}) \overset{j\to\infty}{\sim} \mathbb{E}\left\{\mathrm{vol}_\bullet(\mathcal{F})\right\}$$

Note that by (2.10) all terms in the above line are bounded form above and below uniformly in $\omega$. By taking quotients we obtain

$$\mathcal{L}^l_\omega(t) = \frac{|I_l|^{-1}\operatorname{Tr}(e^{-tH^l_\omega})}{|I_l|^{-1}\operatorname{vol}_\omega(\Lambda_l)} \overset{j\to\infty}{\sim} \frac{\mathbb{E}\left\{\operatorname{Tr}(\chi_{\mathcal{F}} e^{-tH_\bullet})\right\}}{\mathbb{E}\left\{\operatorname{vol}_\bullet(\mathcal{F})\right\}}$$

The right hand side is the Laplace transform of $N_H$, see the proof of Theorem 6.1 of [**201**] for a detailed calculation. □

**2.7. Approach using Dirichlet-Neumann bracketing.** We sketch briefly an alternative proof of the existence of the IDS due to Kirsch and Martinelli [**161**]. It applies to random Schrödinger operators on $\mathbb{R}^d$.

It relies on a ergodic theorem for superadditive processes by Akcoglu and Krengel [**12**] and estimates on the number of bound states essentially implied by the Weyl asymptotics.

Let us explain the notion of an superadditive process in our context. Denote by $Z$ the set of all multi-dimensional intervals or boxes $\Lambda$ in $\mathbb{R}^d$ such that $\Lambda = \{x |\, a_j < x_j < b_j, \text{ for } j = 1, \dots, d\}$ for some $a, b \in \mathbb{Z}^d$ with $a_j < b_j$ for all $j = 1, \dots, d$. The restriction of $H_\omega$ to an $\Lambda \in Z$ with Dirichlet boundary conditions is denoted by $H^\Lambda_\omega$ and with Neumann boundary conditions by $H^{\Lambda,N}_\omega$. Consider a group $\{T_k\}_{k\in\mathbb{Z}^d}$ (or semigroup $\{T_k\}_{k\in\mathbb{N}^d_0}$) of measure preserving transformations on the probability space $(\Omega, \mathcal{B}_\Omega, \mathbb{P})$.

Definition 2.7.1. A set function $F\colon Z \to L^1(\Omega)$ is called a (discrete) *superadditive process* (with respect to $\{T_k\}_k$) if the following conditions are satisfied

$$F_\Lambda \circ T_k = F_{\Lambda+k} \text{ for all } k \in \mathbb{Z}^d \text{ (or } \mathbb{N}^d_0) \text{ , } \Lambda \in Z \tag{2.42}$$

$$\text{if } \Lambda_1, \dots, \Lambda_n \in Z \text{ such that } \Lambda := \operatorname{int}\Big(\bigcup_{k=1}^n \overline{\Lambda}_k\Big) \in Z \text{ then } \; F_\Lambda \geq \sum_{k=1}^n F_{\Lambda_k} \tag{2.43}$$

$$\gamma := \gamma(F) := \sup_{\Lambda\in Z} |\Lambda|^{-1}\,\mathbb{E}\,\{F_\Lambda\} < \infty \tag{2.44}$$

$F$ is called *subadditive* if $-F$ is superadditive.

Similarly one can define continuous superadditive processes with respect to an action of $\mathbb{R}^d$ on $\Omega$.

We formulate the main result of [**12**] in the way it suits our needs.

Theorem 2.7.2. *Let $F$ be a discrete superadditive process and $\Lambda_l := [-l/2, l/2]^d$, $l \in \mathbb{N}$. Then the limit* $\lim_{l\to\infty} l^{-d} F_{\Lambda_l}$ *exists for almost all $\omega \in \Omega$.*

More generally, one can replace the $\Lambda_l, l \in \mathbb{N}$ by a so called *regular sequence*, cf. [**309, 12, 161**] or § 6.2 in [**194**].

To apply the superadditive ergodic theorem we consider for arbitrary, fixed $\lambda \in \mathbb{R}$ the processes given by the eigenvalue counting functions of the Dirichlet and Neumann Laplacian

$$F^D_\Lambda := F^D_\Lambda(\lambda, \omega) := \#\{n |\, \lambda_n(H^\Lambda_\omega) < \lambda\}, \quad \Lambda \in Z$$
$$F^N_\Lambda := F^N_\Lambda(\lambda, \omega) := \#\{n |\, \lambda_n(H^{\Lambda,N}_\omega) < \lambda\}, \quad \Lambda \in Z$$

where $H_\omega$ is a random operator as in Definition 1.2.3. Obviously for $\Lambda = \Lambda_l = [-l/2, l/2]^d$ we have $F^D_\Lambda(\lambda) = l^d N^l_\omega(\lambda)$. We will show that $F^D_\Lambda, \Lambda \in Z$ is a superadditive process, which is also true for $-F^N_\Lambda, \Lambda \in Z$. Property (2.42) follows from the equivariance of $\{H_\omega\}_\omega$, while (2.43) and (2.44) are implied by the following

LEMMA 2.7.3. *Let $H_\omega$ be a random operator as in Definition 1.2.3 and $\lambda$ a fixed energy value.*

(i) *For two cubes $\Lambda_1 \subset \Lambda_2$ we have $F^D_{\Lambda_2} \geq F^D_{\Lambda_1}$ and $F^N_{\Lambda_1} \geq F^D_{\Lambda_1}$.*

(ii) *If $\Lambda_1, \Lambda_2 \in Z$ are disjoint such that $\Lambda = \Lambda_1 \cup \Lambda_2 \cup M \in Z$ where $M \subset \mathbb{R}^d$ is a set of measure zero, then*

$$F^D_\Lambda \geq F^D_{\Lambda_1} + F^D_{\Lambda_2}$$
$$F^N_\Lambda \leq F^N_{\Lambda_1} + F^N_{\Lambda_2}$$

(iii) *There exists an $C_\lambda \in \mathbb{R}$ such that for all $\Lambda \in Z$ and $\omega \in \Omega$ we have $F^D_\Lambda(\omega) \leq C_\lambda |\Lambda|$.*

PROOF. The first two statements are known as Dirichlet-Neumann bracketing and are stated e.g. in Proposition XIII.15.4 in [**256**]. See also Section I.5 in [**50**] for analogous results on manifolds.

To prove (iii) we consider first the case where the potential is identically equal to zero and infer from the Weyl asymptotics that there is a constant $C(\lambda) \in \mathbb{R}$ such that

$$\#\{n | \lambda_n(-\Delta^\Lambda) < \lambda\} \leq C(\lambda) |\Lambda|$$

Here $\Delta^\Lambda$ denotes the Laplace operator on $\Lambda$ with Dirichlet boundary conditions. Since the potentials we consider are infinitesimally bounded with respect to the Laplacian, uniformly in $\omega$, the Min-Max principle for eigenvalues implies the same estimate for the full Schrödinger operator, with an suitably changed constant. See Sections XIII.3, 15 and 16 in [**256**] for more details and, for explicit estimates, the works [**264, 206, 205, 64**] deriving the Lieb-Thirring and Cwikel-Lieb-Rozenblum bounds on the number of bound states. □

Now we can state the main result of [**161**].

THEOREM 2.7.4. *There exists a set $\Omega' \subset \Omega$ of full measure such that*

$$N(\lambda) := \lim_{l \to \infty} N^l_\omega(\lambda) \tag{2.45}$$

*exists for every $\omega \in \Omega'$ and every continuity point $\lambda \in \mathbb{R}$ of $N$.*

PROOF. For a fixed $\lambda \in \mathbb{R}$ one applies the ergodic theorem of [**12**] to the superadditive process $F^D = F^D(\lambda, \omega)$. Since in our case the transformation group is ergodic, the limit $N(\lambda)$ equals $\gamma(F^D(\lambda))$, in particular it is independent of $\omega$. Almost sure convergence means that there is a set $\Omega_\lambda$ of measure one for which the convergence holds. Let $S \subset \mathbb{R}$ be dense countable set. Then $\Omega' = \cap_{\lambda \in S} \Omega_\lambda$ still has full measure and (2.45) holds for all $\lambda \in S$ and $\omega \in \Omega'$. Since $S$ was dense, this shows the convergence (2.45) at all continuity points of the limit function. Afterwards one modifies the limit function $N$ to be left continuous. See [**161, 157**] for details. □

For models which satisfy both the conditions of the previous theorem and of 2.3.8 the two IDS's coincide.

Under certain regularity conditions the theorem remains true if Neumann boundary conditions are used to define the IDS, and also for $\mathbb{R}^d$-ergodic potentials, cf. for instance [**161, 142**].

**2.8. Independence of the choice of boundary conditions.** Consider again the more general setting of Schrödinger operators on a Riemannian covering manifold $X$. If the open subset $\Lambda \subset X$ of finite volume is sufficiently regular, the Neumann Laplacian $H_\omega^{\Lambda,N}$ on $\Lambda$ has discrete spectrum. One condition which ensures this is the *extension property* of the domain $\Lambda$, see e.g. [**68**], which is in turn satisfied if the boundary $\partial\Lambda$ is piecewise smooth. Minimal conditions which ensure the extension property are discussed in § VI.3 of [**292**]. Thus it is possible to define the normalised eigenvalue counting function

$$N_\omega^{\Lambda,N}(\lambda) := \frac{1}{|\Lambda|}\,\#\{n \in \mathbb{N} |\, \lambda_n(H_\omega^{\Lambda,N}) < \lambda\}$$

Let $\Lambda_l$ be an admissible exhaustion $\Lambda_l \subset X, l \in \mathbb{N}$ of sets which all have the extension property. Consider the sequence of distribution functions $N_\omega^{l,N} := N_\omega^{\Lambda_l,N}$. It is natural to ask whether it converges almost surely, and, moreover, whether its limit coincides with $N$ as defined in (2.45). If this is true, the IDS is independent of the choice of Dirichlet or Neumann boundary conditions used for its construction. This indicates that boundary effects are negligible in the macroscopic limit.

However, this turns out not to be true for all geometric situations. Sznitman studied in [**304, 306**] the IDS of a random Schrödinger operator on hyperbolic space with potential generated by a Poissonian field. He showed that the IDS does depend on the choice of boundary condition used for its construction. Actually, he computes the Lifshitz asymptotics of the IDS at energies near the bottom of the spectrum and shows that it is different for Dirichlet and Neumann boundary conditions.

In contrast, in the case of Euclidean geometry $X = \mathbb{R}^d$, the question of boundary condition independence has been answered positively already some decades ago [**27, 161, 299, 88**] for a large class of $\mathbb{Z}^d$ or $\mathbb{R}^d$-ergodic random potentials. Recently, there has been interest in the same question if a magnetic field is included in the Hamiltonian, see also § 5.6. In this case the coincidence of the IDS defined by the use of Dirichlet and Neumann boundary conditions was established for bounded potentials in [**232**], for non-negative potentials in [**82**], and for certain potentials assuming both arbitrarily large positive and negative values in [**142**] and [**139**]. The last mentioned approach seems to be extensible to non-Euclidean geometries which is matter of current research.

## 3. Wegner estimate

In 1981 Wegner [**325**] proved on a physical level of rigour a lower and upper bound on the *density of states* (DOS) of the (discrete) Anderson model and similar lattice Hamiltonians. The density of states — if it exists — is the density function of the IDS. Wegner's argument did in particular not rely on any information about the type of the spectrum in the considered energy interval. This was important since before Wegner's result there where various conjectures in the physics community how the DOS should behave at the *mobility edge*, if it exists. This is the name for the critical energy which is supposed to form the boundary between an interval with pure point spectrum and another one with continuous spectrum. Note however that there is so far no rigorous proof of the existence of continuous spectrum for ergodic random Schrödinger operators.

There were arguments suggesting that the DOS should diverge to infinity at the mobility edge, others that it should vanish. Wegner's estimate discarded this misconceptions. In a sense it is a negative result: you cannot recognize the borderline of different spectral types by looking at the IDS.

In the sequel we concentrate on alloy type models as defined in 1.2.1 (and Remark 1.2.2). We will be concerned here with upper bounds on the DOS only. It is derived by considering its analoga on finite boxes. So what we are speaking about in this section is an estimate on

$$\mathbb{E}\left\{\mathrm{Tr}P^l_\omega(I)\right\} = |\Lambda_l|\,\mathbb{E}\left\{N^l_\omega(E_2) - N^l_\omega(E_1)\right\}$$

where for the moment for notational convenience we only consider half open energy intervals $I = [E_1, E_2[$. By the Čebyšev-inequality one sees

$$\mathbb{P}\left\{\sigma(H^l_\omega) \cap I \neq \emptyset\right\} = \mathbb{P}\left\{\mathrm{Tr}P^l_\omega(I) \neq 0\right\} \leq \mathbb{E}\left\{\mathrm{Tr}P^l_\omega(I)\right\} \tag{3.1}$$

This means that a bound on the averaged trace of the projection gives immediately a bound on the probability to find an eigenvalue in the considered energy interval. Actually, in the literature on Anderson localisation often the (weaker) bound on $\mathbb{P}\left\{\mathrm{Tr}P^l_\omega(I) \neq 0\right\}$ is called Wegner estimate, since it is sufficient for the purposes of multiscale analysis, see § 3.2.

In the following we will adopt the following notation: $\Lambda_l := ]-l/2, l/2[^d \subset \mathbb{R}^d$ denotes the cube of side length $l$ centred at zero. Occasionally we suppress the dependence on the size and just write $\Lambda$. A cube centred at $x \in \mathbb{R}^d$ is denoted by $\Lambda_l + x = \{y + x | \, y \in \Lambda_l\}$ or $\Lambda_l(x)$. The characteristic function of the unit cube $\Lambda_1 + j$ is abbreviated by $\chi_j$. For $l \in \mathbb{N}$ the symbol $\tilde{\Lambda}_l$ denotes the lattice points in $\mathbb{Z}^d$ such that

$$\mathrm{int}\Big(\bigcup_{j\in\tilde{\Lambda}_l} \overline{\Lambda_1(j)}\Big) = \Lambda_l$$

More explicitly: $\tilde{\Lambda}_l = \Lambda_l \cap \mathbb{Z}^d$.

**3.1. Continuity of the IDS.** The estimates on the expected number of energy levels in $I$, which most authors derive or use (for localisation proofs) are 'polynomial', more precisely

$$\mathbb{E}\left\{\mathrm{Tr}P^l_\omega(I)\right\} \leq C_W\,|I|^a\,|\Lambda_l|^b \tag{3.2}$$

Here $|I|$ and $|\Lambda_l|$ denote the (1-dimensional, respectively $d$-dimensional) Lebesgue measure of the energy interval $I$, or the set $\Lambda_l$, respectively. The *Wegner constant* $C_W$ depends on the various parameters of the model and for continuum Hamiltonians on the supremum of $I$. Actually, $C_W$ can be assumed to be a monotone non-decreasing function of $|\sup I|$. However, once $\sup I$ is fixed, $C_W$ is independent both of the energy interval length and the volume. It is obvious that the energy and volume exponents must satisfy $a \in ]0, 1], b \in [1, \infty[$. As far as the exponents are concerned, the Wegner estimate is optimal if the dependence on the volume and energy length is linear, i.e. $a = b = 1$.

For, if $b = 1$, the bound (3.2) carries over to the infinite volume IDS:

$$\lim_{l\to\infty} \mathbb{E}\left\{N^l_\omega(E_2) - N^l_\omega(E_2)\right\} = \lim_{l\to\infty} \frac{\mathbb{E}\left\{\mathrm{Tr}P^l_\omega([E_1, E_2[)\right\}}{|\Lambda_l|} \leq C_W\,|E_2 - E_1|^a \tag{3.3}$$

Since we know from Theorem 2.3.8 and dominated convergence that for $E_1, E_2$ in a dense set of energies

$$\lim_{l\to\infty} \mathbb{E}\left\{N^l_\omega(E_2) - N^l_\omega(E_1)\right\} = N(E_2) - N(E_1)$$

it follows

$$N(E_2) - N(E_1) \leq C_W \left|E_2 - E_1\right|^a$$

where $C_W = C_W(E_2)$. Now the monotonicity of the IDS implies its Hölder continuity. Moreover, if the estimate (3.2) is linear in the energy, i.e. $a = 1$, the IDS is even Lipschitz continuous. Thus its derivative, the *density of states*,

$$n(E) := \frac{dN(E)}{dE} \tag{3.4}$$

exists for almost every energy $E \in \mathbb{R}$ and is locally bounded

$$n(E) \leq C_W(E_2) \quad \text{for all } E \leq E_2$$

So the Wegner constant turns out to be a locally uniform bound on the DOS.

Remark 3.1.1. For certain models the bounds derived on $\mathbb{P}\left\{\sigma(H^l_\omega) \cap I \neq \emptyset\right\}$ are not polynomial in the volume. This is the case for one-dimensional Anderson or alloy type models where the coupling constants $\omega_j, j \in \mathbb{Z}$ are distributed according to the Bernoulli distribution: for some $p \in ]0,1[$ the random variable $\omega_0$ takes on the value 1 with probability equal to $p$ and the value 0 with probability $1-p$. Since this disorder regime is highly singular, the 'usual' proofs of the Wegner estimate, see Sections 4 and 5, fail. The ones that do work, yield somewhat weaker results. Namely, it is proven in [**46**] (cf. also [**275**]) for the discrete Anderson model and in [**66**] for the continuum alloy type model, that for a fixed compact energy interval $I$ and all $\beta \in ]0,1[, \gamma > 0$ there exist $l_0 \in \mathbb{N}$ and $\alpha > 0$ such that

$$\mathbb{P}\{d(\sigma(H^l_\omega), E) \leq e^{-\gamma l^\beta}\} \leq e^{-\alpha l^\beta} \tag{3.5}$$

for all $E \in I$ and all $l \geq l_0$. Here in the case of the continuum model it has to be assumed that $I$ is disjoint from a discrete set of exceptional energies.

This estimate obviously does not imply a continuity estimate for the infinite volume IDS. Interestingly, for these models, the Hölder continuity of the IDS is established using other techniques which are specifically tailored for the one dimensional case, see [**197**], [**46**, App. to § 5] and [**66**, Thm. 4.1]. Subsequently, this regularity result is used to derive the finite volume estimate (3.5). In higher dimensions, as we have discussed above, one proceeds in the other direction, carrying over finite volume estimates to the macroscopic limit.

The bound (3.5) is still sufficient as an input for the multiscale analysis which proves localisation, cf. e.g. [**320**] or our discussion in § 3.2. In the discussion of Spencer's example in § 3.2 we will obtain an insight why subexponentially small eigenvalue splittings are effective enough to prevent resonances.

Remark 3.1.2 (Continuity of the IDS on the lattice and in one dimension). In [**74**] Delyon and Souillard showed by a very simple argument that the IDS of the Anderson model on the lattice $\mathbb{Z}^d$ is continuous, regardless of the continuity of the distribution of the potential values. The potential may even be correlated over long distances, as long as it is an ergodic stochastic field. Delyon and Souillard use the unique continuation property of the discrete Schrödinger equation, to prove that no eigenvalue can be sufficiently degenerated to produce a jump of the IDS. At the end

of Remark 3.1.3 we contrast their theorem with the situation in graphs other than the lattice. In [**62**] a quantitative version of the continuity is proven: for random, ergodic lattice Hamiltonians the IDS is actually log-Hölder continuous. See also [**212**] and [**101, 80, 53, 220, 81**] for a related result for graph Hamiltonians.

Similarly, the IDS is continuous for one dimensional Hamiltonians, both on $\mathbb{Z}$ and on $\mathbb{R}$, [**240, 149, 21**]. Again, this result can be strengthened to log-Hölder continuity, cf. [**63**].

Remark 3.1.3 (Continuity of the IDS and geometry). So far we have only mentioned proven or expected assertions on the continuity of the IDS. One might ask whether there are interesting models which exhibit a discontinuous IDS. It turns out that this phenomenon may occur, if the configuration space has a more complicated geometry than $\mathbb{Z}^d$ or $\mathbb{R}^d$. Another example would be the IDS of the Landau Hamiltonian, cf. e.g. the references in § 5.6, in particular [**232**].

Maybe the simplest example to illustrate the difference between Euclidean and more general geometry is provided by periodic Schrödinger operators. Under mild assumptions on the $\mathbb{Z}^d$-periodic potential $V_{\mathrm{per}}$ the IDS of the Schrödinger operator $H_0 = -\Delta + V_{\mathrm{per}}$ on $\mathbb{R}^d$ is absolutely continuous, cf. e.g. [**256**, Problem 145],[**311**]. In particular the IDS cannot have jumps. However, precisely this can occur for Laplace-Beltrami operators (even without potential) on a Riemannian covering manifold $X$, as it was mentioned in [**301**, App. 2] and is a subject of current research [**198, 200**]. This phenomenon can be deduced from the fact that Laplacians on covering manifolds may have eigenvalues, as has been shown in [**186**]. Furthermore, the size of the jumps of the IDS is related to certain geometric invariants. Examples of such invariants are the order of the torsion subgroup of the deck transformation group $\Gamma$ of $X$ and the $L^2$-Betti numbers of $X$, which can be expressed in terms of the $\Gamma$-trace on a certain von Neumann algebra, establishing the connection to the representation of the IDS discussed before Theorem 2.3.3. [**76, 79, 80, 269, 81**]. Related material can be found in [**19, 302, 212, 300, 268, 130, 270, 220, 78, 213, 214**].

Some of the Wegner estimates we present in Sections 4 and 5 extend to alloy type models on manifolds. A particularly interesting phenomenon occurs if one considers a periodic Laplace-Beltrami operator with discontinuous IDS, and perturbs it randomly such that the IDS of the perturbed operator is continuous. This happens if either an appropriate alloy type potential is added to the Hamiltonian or if the metric is multiplied by an appropriate alloy type perturbation, see [**198, 200**].

A discontinuous IDS may also occur for models with a random geometry. This is the case for the tight-binding Hamiltonian defined on Delone sets studied in [**202, 173**]. Ideas related to the ones in [**173**] have been used in [**267**] and in § 2 of [**220**]. The paper [**267**] is devoted to the proof of the existence of spectral gaps for certain graph Hamiltonians.

We will discuss a different example, the *quantum percolation model*, in some more detail, since it fits readily in the class of models which we have described so far. This model has been studied amongst others in [**72, 71, 155, 272, 52, 314**]. We sketch the site percolation problem on $\mathbb{Z}^d$ with probability parameter $p$ above the percolation threshold: let $v_k \colon \Omega \to \{0, \infty\}, k \in \mathbb{Z}^d$ be a sequence of independent, identically distributed random variables which take on the value 0 with probability $p$ and the value $\infty$ with probability $1 - p$. Define $X_\omega$ to be the infinite component of the set of *active sites* $\{k \in \mathbb{Z}^d | \, v_k(\omega) = 0\}$. The graph $X_\omega$ is called the (active)

*infinite cluster*. For $p$ above a certain critical value $p_c$ it is known that almost surely an infinite cluster exists [**154, 132**] and is unique [**8, 116**].

One defines the Laplacian $h_\omega$ on $X_\omega$ as the restriction of the finite difference operator onto $l^2(X_\omega)$. For a cube $\Lambda_l$ one defines $X_\omega^l$ to be those active sites in $\Lambda_l \cap \mathbb{Z}^d$ which are connected to the boundary $\partial\Lambda_l$ by a chain of active sites. The finite volume Laplacian $h_\omega^l$ is the usual finite difference operator restricted to $l^2(X_\omega^l)$.

Although the finite active clusters, which would obviously give rise to bound states, are not taken into consideration, it turns out that $h_\omega$ has bound states. This was as first observed in [**155**]. Eigenstates with finite support in the infinite cluster are called *molecular states*. The existence of such states affects the properties of the IDS of $h_\omega$, which is defined in the following way. For each $l \in \mathbb{N}$ the normalised eigenvalue counting function of the Hamiltonian $h_\omega^l$ is defined as

$$N_\omega^l(E) := \frac{1}{\#(\Lambda_l \cap \mathbb{Z}^d)} \#\{n | \, \lambda_n(h_\omega^l) < E\}$$

which converges for $l \to \infty$ to an non-random limit almost surely [**52, 314**]. A simple construction shows that there are locally supported eigenfunctions, which depend only on the pattern of $X_\omega$ in a bounded region. Consequently the patterns and the associated localised eigenfunctions occur with a non-zero density along the infinite cluster and thus produce jumps of the IDS at the corresponding energy. In [**52**] it is shown by physical arguments that the discontinuities of $N$ constitute a dense set of energies.

Actually, uniqueness of the infinite cluster is not used in the arguments of [**52**] and a similar argument for constructing finitely supported eigenfunctions does work on the Bethe lattice as well, although there the infinite cluster is not unique. For the quantum percolation model on the Bethe lattice locally supported eigenfunctions have been constructed in [**46**, § 7].

A mathematically rigorous study of the quantum percolation model on amenable graphs is undertaken in [**314**]. There the discontinuities of the IDS are explained in terms of the breakdown of the unique continuation property of eigenfunctions of the adjacency operator, see also Remark 3.1.2. Moreover, the set of these energies is characterised in the case $X = \mathbb{Z}^d$. From a wider perspective, the properties of this set are related to the Atiyah conjecture, cf. [**78**].

Remark 3.1.4. While the continuity of the IDS has clearly to do with the distribution of eigenvalues of the random Hamiltonian, it only captures a part of the properties of this distribution. The theory of *level statistics* is concerned with the finer structure of the fluctuations of eigenvalues. It can be studied by an appropriate scaling procedure. This has been carried out for certain one-dimensional and discrete models in [**228, 227, 261, 260, 225, 123**].

**3.2. Application to Anderson localisation.** In the last paragraph the implications of Wegner estimates for the IDS were presented. Now we focus on the second main application of those bounds, namely *Anderson localisation*.

As we discussed earlier in § 1.3, this phenomenon tells us that a random family of Schrödinger operators exhibits in a certain energy interval dense pure point spectrum, almost surely. Moreover, the eigenfunctions of the eigenvalues lying in this interval decay exponentially. Even a stronger property, namely *dynamical localisation*, holds. See § 1.3 for more details and references.

For multi-dimensional configuration space there are two techniques at disposal to prove localisation: the *fractional moment method* and the *multiscale analysis* (MSA). The first one is also called *Aizenman-Molchanov* technique and was introduced in [**9, 4, 5, 129, 7**]. It was so far applicable only to lattice Hamiltonians, up to the recent extension to continuum configuration space [**6**]. For discrete models it has in fact been proven [**10, 11**] that in the energy regime where the MSA applies, the Aizenman-Molchanov method works, too.

However, we will discuss in a little more detail only the MSA, since it has found applications to a variety of models and since the Wegner estimate is a key ingredient in the MSA. We first sketch the basic ideas of the MSA, and then discuss shortly its history.

To carry through the MSA one needs two a priori estimates: the *initial scale estimate* and the *Wegner estimate*. These two conditions essentially determine for which single site potentials $u$, single site distribution measures $\mu$ and which energy intervals localisation can be derived. Note that $u$ and $\mu$ are parameters which determine our alloy type potential, see Definition 1.2.1.

In the literature one can find multiscale analyses which are adapted to operators describing the propagation of classical waves or to abstract families of differential operators, see among others [**104, 105, 61, 67, 295, 120, 177**]. In this context one has also to make sure that certain other conditions are satisfied, like the geometric resolvent inequality, the generalised eigenfunction expansion, a rough upper estimate on the trace of spectral projections of finite box operators (obtained e.g. by the Weyl asymptotics), etc. However, since we discuss here only (random) Schrödinger operators, these conditions are automatically satisfied, cf. [**295**, § 3.2] or [**119**, App. A].

The multiscale analysis is an induction argument over a sequence of increasing length scales $l_k, k \in \mathbb{N}$. Each scale $l_{k+1}$ is a power $l_k^\alpha$ of the preceding one, where $\alpha \in ]1, 2[$. Actually, for technical reasons one truncates the scales so that all $l_k$ lie in $6\mathbb{N}$.

One considers the restriction of the alloy type model $H_\omega$ to the open cube $\Lambda^{(k)} := \Lambda_{l_k}(0)$ of side length $l_k$. The corresponding restricted operator is denoted by $H_\omega^{(k)}$, where Dirichlet, Neumann or periodic boundary conditions ensure its selfadjointness. One wants to study decay properties of the Green's function of $H_\omega^{(k)}$, i.e. the integral kernel of the resolvent operator $R_\omega^{(k)}(z) = (H_\omega^{(k)} - z)^{-1}$, where $z$ is taken from the resolvent set $\mathbb{C} \setminus \sigma(H_\omega^{(k)})$. Since we are not interested in pointwise properties of the kernel of $R_\omega^{(k)}(z)$, and since they tend to be unpleasant near the diagonal, we may consider instead the sandwiched resolvent

$$\chi^{out}(H_\omega^l - E)^{-1}\chi^{in}$$

Here $\chi^{out}$ denotes the characteristic function of the boundary belt $\Lambda_{l-1} \setminus \Lambda_{l-3}$, and $\chi^{in}$ the characteristic function of the interior box $\Lambda_{l/3}$.

The initial scale estimate is stated in terms of the notion of *regular cubes*. A box $\Lambda_l = \Lambda_l(0)$ is called $(E, \gamma)$-*regular* if $l \in 6\mathbb{N}$, $E \notin \sigma(H_\omega^l)$, and

$$\|\chi^{out}(H_\omega^l - E)^{-1}\chi^{in}\| \leq e^{-\gamma l} \tag{3.6}$$

The exponent has to satisfy $\gamma \geq l^\beta$ for some $\beta > -1$. So regularity describes quantitatively how fast the Green's function on a finite box decays. The exponent $\beta$ must be greater than $-1$ otherwise the rhs of (3.6) could be just one.

The *initial scale estimate* is satisfied if there exist a scale $l_1 < \infty$ such that for some $\xi > 0$

$$\mathbb{P}\{\omega | \, \forall E \in I : \, \Lambda_l \text{ is } (E, \gamma)\text{-regular for } \omega\} \geq 1 - l^{-\xi} \tag{3.7}$$

for any $l \geq l_1$.

The *weak form* of the *Wegner estimate* as it is needed for the MSA is:

$$\mathbb{P}\{d(\sigma(H_\omega^l), E) \leq e^{-l^\theta}\} \leq l^{-q} \text{ for all } l \in 6\mathbb{N} \tag{3.8}$$

where $\theta < 1/2$ and $q > d$.

The initials scale estimate (3.7) serves as the induction anchor of the MSA. The induction step uses the Wegner estimate and proves that the exponential decay property holds on the subsequent scale $l_2$ with even higher probability, and that the decay exponent $\gamma$ essentially does not change. As one repeats the procedure on the scales $l_1, l_2, \ldots$ one proves that the decay of the Green's functions $\chi^{out}(H_\omega^{(k)} - E)^{-1}\chi^{in}$ holds with probability which tends to one, with error bounded polynomially in $l_k^{-1}$.

Thus one establishes the exponential decay of the sandwiched resolvent on arbitrary large cubes, where the decay exponent $\gamma$ is bounded away from zero uniformly in the scales. Now one uses polynomial bounds on the growth of eigenfunctions and subsolution estimates to prove spectral localisation, cf. for instance [**295**, § 3.3]. To prove dynamical localisation one has to do more work, see e.g. [**295**, § 3.4] or [**118, 67, 120**].

The assumptions for the MSA depend on several parameters, and so do the various versions of localisation which may be obtained by it. Recently Germinet and Klein showed in [**120**] how to optimize the dependence of the MSA on the various parameters, i.e. how to obtain with the weakest assumptions in the input the strongest conclusions.

The MSA was introduced by Fröhlich and Spencer in [**113**]. The method applied to the Anderson model on the lattice and experienced various improvements and applications [**218, 219, 112, 283, 73**].

Based on results from [**319**] and [**291**] von Dreifus and Klein presented in [**320**] a streamlined version of the MSA. Although results on localisation for continuum Hamiltonians existed earlier [**217, 193**], it was this simplification of the MSA, which made alloy type Schrödinger operators more accessible to systematic research.

There was a series of articles which proved various variants of the MSA for continuum models [**178, 54, 180, 170, 118, 109, 67, 295, 120**]. Other works concentrated onto identifying energy/disorder regimes where one can prove the assumptions needed for the MSA to work [**178, 179, 180, 23, 170, 169, 120, 294, 316, 317, 134, 329, 183**].

Remark 3.2.1. One dimensional models play a special role in the game of localisation. Namely, for $d = 1$ there are some specific techniques available which do not exist in higher dimensions, or are not as effective. Some examples are: the transfer-matrix method, study of the Ljapunov exponent, ODE techniques, Prüfer transformation.

Consequently, in one space dimension localisation has been proven for some models even before the MSA technique was available. See [**128, 228, 127, 44, 263, 193**] for various models on $\mathbb{Z}$ or $\mathbb{R}$. Furthermore there are certain models which even

now can be treated only in one dimension. This applies to the following random Schrödinger operators: random displacement model [**43, 284**], potentials generated by a Poissonian field [**43**], alloy type potentials with changing sign (at all energies) [**296**], discrete and continuous models with Bernoulli disorder [**46, 275, 66**]. This restriction to $d = 1$ is partially due to the fact, that there is no Wegner estimate at disposal in these cases.

The following paragraph gives an idea where the Wegner estimate is used in the MSA.

**3.3. Resonances of Hamiltonians on disjoint regions.** Rather than describing precisely how the Wegner estimate enters in the induction step of the MSA we will confine ourselves to present an illuminative example due to Spencer [**290**]. It was originally formulated for lattice Hamiltonians, but can also be considered in the continuum case, as we have learned from P. Müller.

As we mentioned earlier (3.2) implies for $0 \leq \delta < 1$

$$\mathbb{P}\{\omega |\, d(\sigma(H^l_\omega), E) \leq \delta\} \leq C_W(E)\, \delta^a\, |\Lambda_l|^b \tag{3.9}$$

This inequality implies that with respect to the parameter $\omega$ the eigenvalues of $H^l_\omega$ do not cluster on the energy axis. To give a more precise meaning to this statement, consider two box Hamiltonians $H^1 = H^{\Lambda_l(x)}_\omega$ and $H^2 = H^{\Lambda_l(y)}_\omega$. Assume that the boxes $\Lambda_l(x)$ and $\Lambda_l(x)$ are sufficiently far apart such that $H^1$ and $H^2$ are independent. Let $I$ be a bounded interval and consider the event

$$\Omega(\sigma_1, \sigma_2) := \{\omega |\, B_\delta(\sigma_1) \cap B_\delta(\sigma_2) \neq \emptyset\}$$

where $\sigma_i$ stands for $\sigma(H^i) \cap I$. Let $\lambda_1, \ldots, \lambda_N$ be the eigenvalues of $H^1$ in $I$. By Weyl's law we know that $N \leq C_I |\Lambda_l|$, where $C_I$ is independent of $\omega$. Since

$$\Omega(\sigma_1, \sigma_2) \subset \bigcup_{n=1}^{N} \Omega(n, \sigma_2) \quad \text{where } \Omega(n, \sigma_2) := \{\omega |\, B_{2\delta}(\lambda_n(\omega)) \cap \sigma_2 \neq \emptyset\}$$

we may use (3.9) to conclude

$$\mathbb{P}\{\Omega(\sigma_1, \sigma_1)\} \leq C \delta^a\, |\Lambda_l|^{b+1} \tag{3.10}$$

This means that resonances of $H^1$ and $H^2$, i.e. the occurrence of approximately the same eigenvalues for both operators are very unlikely.

The feature which is common to Spencers example and the MSA is the effect of resonances between Hamiltonians which are localised to disjoint cubes. As we mentioned earlier, in the induction step of the MSA one puts together boxes $\Lambda_l$ of side length $l$ to form a larger cube $\Lambda_L$ of side length $L$. Assume that one knows already that the Green's functions of the operators $H^l_\omega$ living on any one of the small cubes $\Lambda_l$ decays exponentially.

The Schrödinger operator $H^L_\omega$ on $\Lambda_L$ is obtained when we remove the boundary conditions which separate the smaller boxes $\Lambda_l$. The question is whether the Green's function on $\Lambda_L$ will still decay (with approximately the same rate). To answer this question affirmatively it is not enough to know the exponential decay of the individual Green's functions on the small boxes $\Lambda_l$, but it has to be ensured that they are not in resonance with each other.

Resonance means in this context that the spectra of two restriction $H^1, H^2$ of $H_\omega$ to disjoint cubes are very close to each other, and can be formulated quantitatively in terms of

$$d(\sigma_1, \sigma_2) := \inf\{d(\lambda_1, \lambda_2) | \, \lambda_1 \in \sigma_1, \lambda_2 \in \sigma_2\} \tag{3.11}$$

as we will see below.

The model situation we are about to consider is easier than the one occurring in the MSA because we do not introduce boundary conditions but confine ourselves to the analysis of the discrete spectrum below zero.

EXAMPLE 3.3.1 (Spencer's example [**290**, p. 903–904]). Consider two smooth potentials $V_1, V_2 \leq 0$ with compact support and set

$$\mathcal{V}_i := \mathrm{supp} V_i \subset B_r(a_i), r > 0, a_i \in \mathbb{R}^d, i = 1, 2 \tag{3.12}$$

It follows $d(\mathcal{V}_1, \mathcal{V}_2) \geq |a_1 - a_2| - 2r =: \varrho$. Consider furthermore the operators

$$H := H_0 + V_1 + V_2, \qquad H_0 := -\Delta \tag{3.13}$$

$$H_i := -\Delta + V_i, \quad i = 1, 2 \tag{3.14}$$

Denote by $\sigma_i := \sigma(H_i) \cap ]-\infty, 0[$, $i = 1, 2$ the negative spectra, which are purely discrete. We are interested in the localisation and decay properties of the corresponding eigenfunctions.

We look at two cases, where the first has a special symmetry and the second corresponds to the situation one expects to occur in a random medium.

**Case (A):**
Consider first the *exceptional* case in which $V_2$ is obtained from $V_1$ by a reflection along an axis of symmetry. Without loss of generality

$$V_2(x_1, x_2, \ldots, x_d) = V_1(-x_1, x_2, \ldots, x_d) \tag{3.15}$$

Thus $H$ commutes with the reflection operator

$$\Pi \colon L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2), \quad (\Pi f)(x_1, x_2, \ldots, x_d) = f(-x_1, x_2, \ldots, x_d) \tag{3.16}$$

In particular, for every eigenfunction $\psi$

$$H\psi = \lambda\psi, \quad \lambda < 0$$

the reflected function $\Pi\psi$ is an eigenvector of $H$ as well. If $\psi$ is localised around $a_1$ $\Pi\psi$ will be localised around $a_2$. Thus a typical vector form $\mathrm{span}\{\psi, \Pi\psi\}$ will have non negligible amplitudes both at $a_1$ and $a_2$, even for large distances $\varrho$. For short, eigenfunctions of $H$ do not need to have just one centre of localisation.

We are dealing with a resonance between the two disjoint regions $\mathcal{V}_1, \mathcal{V}_2$, or more precisely between the spectra of $H_1$ and $H_2$. Actually, we encountered the extreme case where the the spectra $\sigma_1$ and $\sigma_2$ are not only close to each other but identical.

The example we just considered exhibited a special symmetry, namely $[H, \Pi] = 0$. For random potentials we expect generically that such symmetries are absent and that the spectra $\sigma_1$ and $\sigma_2$ have positive distance. This situation is considered in

**Case (B):**
We give a condition on $d(\sigma_1, \sigma_2)$ which ensures that the eigenfunctions of $H$ (defined in (3.13)) are localised at *only one* of the potential wells $V_1, V_2$. Namely, assume that

$$d(\sigma_1, \sigma_2) \geq e^{-\sqrt{\varrho}} =: \epsilon \tag{3.17}$$

For an eigenvalue $\lambda \leq -\varrho^{-1}$, with corresponding equation $H\psi = \lambda\psi$, we have either

$$|\lambda_1^j - \lambda| \geq \epsilon/2, \quad \forall \lambda_1^j \in \sigma_1 \qquad \text{or} \qquad |\lambda_2^j - \lambda| \geq \epsilon/2, \quad \forall \lambda_2^j \in \sigma_2 \tag{3.18}$$

Assume without loss of generality the first case. The eigenfunction equation implies

$$-\psi = (H_1 - \lambda)^{-1} V_2 \psi$$

Applying twice the resolvent equation we obtain

$$-\psi = \left[R_0 - R_0 V_1 R_0 + R_0 V_1 R_1 V_1 R_0\right] V_2 \psi, \tag{3.19}$$

where $R_i := (H_i - \lambda)^{-1}$, $i = 0, 1$ denotes the resolvents. We show that the amplitude of $\psi$ on $\mathcal{V}_1$ is exponentially small in the parameter $\varrho$. Denote with $\chi^i$ the characteristic function of $\mathcal{V}_i$ for $i = 1, 2$ and multiply (3.19) with $\chi^1$

$$-\chi^1\psi = \chi^1 R_0 \chi^2 V_2 \psi - \chi^1 R_0 V_1 \chi^1 R_0 \chi^2 V_2 \psi + \chi^1 R_0 V_1 R_1 V_1 \chi^1 R_0 \chi^2 V_2 \psi. \tag{3.20}$$

The free resolvent decays exponentially, see e.g. [**3**] or [**255**, IX.30],

$$R^{1,2} := \|\chi^1 R_0 \chi^2\| \lesssim e^{-\varrho\sqrt{-\lambda}},$$

the terms $\|V_2\psi\|, \|\chi^1 R_0 V_1\|$ are bounded uniformly in $\varrho$, and (3.18) implies

$$\|R_1\| \leq \frac{2}{\epsilon} = 2e^{\sqrt{\varrho}}$$

Consequently

$$\|\chi^1\psi\| \leq R^{1,2}\ \|V_2\psi\| + \|\chi^1 R_0 V_1\|\ R^{1,2}\ \|V_2\psi\| + \|\chi^1 R_0 V_1\|\ \|R_1 V_1\|\ R^{1,2}\ \|V_2\psi\|$$

is bounded by a constant times $\exp(-\sqrt{\varrho})\,\|\psi\|$, since $\varrho^{-1} \leq -\lambda$.

Let us finish this section by discussing some aspects and contrasts of the two cases considered in the example.

(i) In general, the spectrum alone describes only general properties of the solution of the eigenvalue equation. In our example it is the additional information contained in (3.15) and (3.17), respectively, which allows us to analyse the eigenfunctions more precisely.

(ii) Obviously, in Case (A), the Green's function decays in space, too. However, this decay is not yet felt at the scale $\varrho$, since $|\psi(a_1)\psi(a_2)|$ converges to a positive constant for $\varrho \to \infty$. On the contrary, in Case (B), the amplitude is small either at $a_1$ or $a_2$, so

$$|\psi(a_1)\psi(a_2)| \lesssim e^{-\sqrt{\varrho}}.$$

(iii) A semiclassical analysis of double well potentials is carried out, for instance, in [**135**].

## 4. Wegner's original idea. Rigorous implementation

In this section we present a proof of Wegner's estimate following his original ideas in [**325**]. His proof was originally formulated for the discrete Anderson model. In the meantime, it has been cast into mathematically rigorous form and adapted for continuum Hamiltonians. We follow mostly the arguments of Kirsch [**159**]. There are proofs of Wegner's estimate by other authors, which make use of the ideas in [**325**]. Let us mention [**217, 216, 178, 179, 108, 57, 56**].

The theorem to be proven is

THEOREM 4.0.1. *Let $H_\omega$ be as in Definition 1.2.1 and assume additionally that there exists an $\kappa > 0$ such that*

$$u \geq \kappa \chi_{[-1/2,1/2]^d} \tag{4.1}$$

*Then for all $E_0 \in \mathbb{R}$ there exists a constant $C_W = C_W(E_0)$ such that for all $l \in \mathbb{N}$, $E \leq E_0$ and all $\epsilon \in [0,1]$*

$$\mathbb{E}\left\{ \mathrm{Tr}\big[P^l_\omega([E-\epsilon, E+\epsilon])\big] \right\} \leq C_W \ \epsilon \ l^{2d} \tag{4.2}$$

This theorem is proven in the next section. Its bound with respect to the volume term $l^d$ is quadratic and does not yield a continuity statement for the IDS. Subsequently we show how this estimate was improved in [**60**]. Denote by $\omega_+$ and $\omega_-$ the largest, respectively the smallest value a coupling constant may take.

**4.1. Spectral averaging of the trace of the spectral projection.** We show that the expectation over the randomness smears out the eigenvalues of $H^l_\omega$ and thus regularises the trace $P^l_\omega(I)$.

By definition the spectral projection $P^l_\omega(I) = \chi_I(H^l_\omega)$ is the characteristic function of $H^l_\omega$. For certain purposes it will be necessary to differentiate this function with respect to the energy parameter, which motivates the introduction of the following smooth 'switch function'.

Let $\rho$ be a smooth, non-decreasing function such that on $]-\infty, -\epsilon]$ it is identically equal to $-1$, on $[\epsilon, \infty[$ it is identically equal to zero and $\|\rho'\|_\infty \leq 1/\epsilon$. Then

$$\chi_{]E-\epsilon, E+\epsilon[}(x) \leq \rho(x-E+2\epsilon) - \rho(x-E-2\epsilon) = \int_{-2\epsilon}^{2\epsilon} dt\, \rho'(x-E+t)$$

Thus by the spectral theorem

$$P^l_\omega(]E-\epsilon, E+\epsilon[) \leq \int_{-2\epsilon}^{2\epsilon} dt\, \rho'(H^l_\omega - E + t)$$

in the sense of quadratic forms. Since $B_\epsilon(E) = ]E-\epsilon, E+\epsilon[$ is bounded and $\sigma(H^l_\omega)$ discrete, the above operators are trace class and we may estimate:

$$\mathrm{Tr}\Big[P^l_\omega(B_\epsilon(E))\Big] \leq \mathrm{Tr}\Big[\int_{-2\epsilon}^{2\epsilon} dt\, \rho'(H^l_\omega - E + t)\Big] = \sum_{n\in\mathbb{N}} \int_{-2\epsilon}^{2\epsilon} dt\, \rho'(\lambda^l_n(\omega) - E + t)$$

where $\lambda^l_n(\omega)$ denotes the eigenvalues of $H^l_\omega$ enumerated in non-decreasing order and counting multiplicities. Only a finite number of terms in the sum are non-zero. More generally the above arguments prove the following

LEMMA 4.1.1. *Let $H$ be an operator with purely discrete spectrum. Denote by $\lambda_1 \leq \lambda_2 \leq \dots$ the eigenvalues of $H$. Then for $E \in \mathbb{R}$ and $\epsilon > 0$*

$$\mathrm{Tr}\Big[\chi_{B_\epsilon(E)}(H)\Big] \leq \sum_{n\in\mathbb{N}} \int_{-2\epsilon}^{2\epsilon} dt\, \rho'(\lambda_n - E + t)$$

In the following we analyse the behaviour of the spectrum of the Schrödinger operator under the perturbation $\omega_j\, u(\cdot - j)$. Fix a box-size $l \in \mathbb{N}$, a lattice site $j \in \tilde{\Lambda}$ and a configuration of coupling constants $\omega \in \Omega$ and consider the one-parameter family of operators

$$t \mapsto H_t := H + tU, \text{ where } H = H^l_\omega \text{ and } U = u(\cdot - j)$$

By the arguments in §1.2 the single site potential is infinitesimally bounded with respect to $H$, thus $H_t$ forms a holomorphic family of type (A) in the sense of Kato [**152**] for $t$ in a neighbourhood of the real line, cf. e.g. XII.§ 2 in [**256**]. Moreover, $H_t$ has compact resolvent by XII.§ 14 in [**256**]. Hence one may apply a theorem of Rellich [**259**], see also Theorem VII.§3.9 in [**152**]. It says that the eigenvalues and eigenvectors of $H_t$ can be chosen to be real analytic on $\mathbb{R}$. Actually, each eigenvalue is holomorphic on a neighbourhood of $\mathbb{R}$ in the complex plane, but their intersection may contain only $\mathbb{R}$.

If $\lambda_n(t)$ is a non-degenerate eigenvalue of $H_t$, first order perturbation theory tells us that there exists a normalised eigenfunction $\psi_n(t)$ such that

$$\frac{d\lambda_n}{dt}(t_0) = \langle \psi_n(t_0), U\psi_n(t_0)\rangle \tag{4.3}$$

REMARK 4.1.2. This is sometimes called *Hellmann-Feynman formula*, and it holds true also if the eigenvalue $\lambda_n$ happens to be degenerate at $t = t_0$, cf. for instance [**146**]. One has however to chose the enumeration of the eigenvalues $\lambda_n$ and eigenvectors $\psi_n$ in such a way that the pair $\lambda_n(t), \psi_n(t), t < t_0$ continues holomorphically into $\lambda_n(t), \psi_n(t), t > t_0$. Note that this is actually not the case with the enumeration we chose earlier, where $\lambda_n(t)$ denotes the $n$-th eigenvalue of $H_t$. There are two possibilities to solve the problem: either one chooses a somewhat unintuitive enumeration of eigenvalues which makes them — together with the eigenvectors — holomorphic functions of $t$. Or one sums over the eigenvalues. Namely, formula (4.3) remains true if we sum over all eigenvalues which correspond to a degeneracy. More precisely, for a degenerate eigenvalue $\lambda_n(t_0)$ denote by $l, k \in \mathbb{N}$ the largest numbers such that $\lambda_{n-l}(t_0) = \dots = \lambda_n(t_0) = \dots = \lambda_{n+k}(t_0)$ and set $S(t) = \sum_{m=n-l}^{n+k} \lambda_m(t)$. Then we have

$$\frac{dS}{dt}(t_0) = \sum_{m=n-l}^{n+k} \langle \psi_m(t_0), U\psi_m(t_0)\rangle$$

In the application of (4.3) in the next proposition we will be considering all eigenvalues below a certain energy. Thus if we consider one eigenvalue participating in a degeneracy we will actually take into account all participating eigenvalues.

The results for one parameter families of operators carry over to the multi-parameter family $\omega \mapsto H^l_\omega$. Thus we have

$$\sum_{j\in\tilde{\Lambda}} \frac{\partial \lambda_n(H^l_\omega)}{\partial \omega_j} = \sum_{j\in\tilde{\Lambda}} \langle \psi_n, u(\cdot - j)\psi_n\rangle$$

where $\psi_n$ are normalised eigenvectors corresponding to $\lambda_n(H^l_\omega)$. By assumption (4.1) we have

$$\sum_{j\in\tilde\Lambda}\langle\psi_n, u(\cdot-j)\psi_n\rangle \geq \kappa > 0 \tag{4.4}$$

Now the chain rule

$$\sum_{j\in\tilde\Lambda}\frac{\partial\rho(\lambda_n(H^l_\omega)-E+t)}{\partial\omega_k} = \rho'(\lambda_n(H^l_\omega)-E+t)\sum_{j\in\tilde\Lambda}\frac{\partial\lambda_n(H^l_\omega)}{\partial\omega_k}$$

implies

$$\rho'(\lambda_n(H^l_\omega)-E+t) \leq \kappa^{-1}\sum_{j\in\tilde\Lambda}\frac{\partial\rho(\lambda_n(H^l_\omega)-E+t)}{\partial\omega_j}. \tag{4.5}$$

Due to monotonicity, integrating over one coupling constant we obtain

$$\begin{aligned}\int d\omega_j\, f(\omega_j)\,\frac{\partial\rho(\lambda_n(H^l_\omega)-E+t)}{\partial\omega_j} &\leq \|f\|_\infty\int d\omega_j\,\frac{\partial\rho(\lambda_n(H^l_\omega)-E+t)}{\partial\omega_j}\\ &= \|f\|_\infty\big[\rho(\lambda_n(\omega, j=\max)-E+t)-\rho(\lambda_n(\omega,j=\min)-E+t)\big]\end{aligned}$$

where $\lambda_n(\omega, j=\max)$ denotes the $n$-th eigenvalue of the operator

$$H^l_\omega(j=\max) := H^l_\omega + (\omega_+-\omega_j)\,u(x-j)$$

where $\omega_j$ takes its maximal value. Analogously we use the notation $\lambda_n(\omega, j=\min)$. This proves

Proposition 4.1.3.

$$\begin{aligned}&\mathbb{E}\left\{\mathrm{Tr}\big[P^l_\omega([E-\epsilon,E])\big]\right\}\\ &\leq\frac{\|f\|_\infty}{\kappa}\sum_{n\in\mathbb{N}}\int_{-2\epsilon}^{2\epsilon}dt\sum_{j\in\tilde\Lambda}\mathbb{E}\left\{\rho[\lambda_n(\omega,j=\max)-E+t]-\rho[\lambda_n(\omega,j=\min)-E+t]\right\}\end{aligned}$$

The upper bound can be also written as

$$\frac{\|f\|_\infty}{\kappa}\int_{-2\epsilon}^{2\epsilon}dt\sum_{j\in\tilde\Lambda}\mathbb{E}\left\{\mathrm{Tr}\big[\rho[H^l_\omega(j=\max)-E+t]-\rho[H^l_\omega(j=\min)-E+t]\big]\right\}$$

Since $\rho\leq 0$

$$\begin{aligned}&\sum_{n\in\mathbb{N}}\rho[\lambda_n(\omega,j=\max)-E+t]-\rho[\lambda_n(\omega,j=\min)-E+t]\\ &\leq -\sum_{n\in\mathbb{N}}\rho[\lambda_n(\omega,j=\min)-E+t] \leq C_{E+3\epsilon}\;l^d \leq C_{E_0+3}\;l^d\end{aligned} \tag{4.6}$$

by bound (iii) in Lemma 2.7.3. This proves Theorem 4.0.1.

Remark 4.1.4. The suboptimality of the volume bound in Theorem 4.0.1 is due to the rough estimate (4.6). The right hand side of the inequality is the net increase of the number of eigenvalues in the energy interval $]E-t-\epsilon, E-t+\epsilon[$ due to the decrease of the $j$-th coupling constant from its maximal to its minimal value. This quantity is expected to be independent of $\Lambda$. However, in (4.6) we estimated it by the total number of eigenvalues below the energy $E+3\epsilon$, which is by Weyl's

law proportional to the volume of $\Lambda$. Thus we get an extra volume factor in the upper bound of the Wegner estimate.

**4.2. Improved volume estimate.** In [**60**] Combes, Hislop and Nakamura obtained a Wegner estimate analogous to the one in the last section, but with upper bound linear in the volume. The Wegner estimate they proof is somewhat weaker in the energy parameter.

More precisely the main result in [**60**] may be formulated as follows. Let $I_G := ]E_-, E_+[, I_G \cap \sigma(H_0) = \emptyset$ be a spectral gap of $H_0$. Denote by $\tilde{H}^l_\omega$ the operator $H_0 + \sum_{k\in\tilde{\Lambda}_l} \omega_k u(\cdot - k)$ and by $\tilde{P}^l_\omega$ the corresponding spectral projector.

THEOREM 4.2.1. *Let $H_\omega$ be as in Definition 1.2.1 and assume additionally that the periodic potential $V_{\rm per}$ is bounded below, $0 \le u \in C_c(\mathbb{R}^d)$ and $u$ is not identically equal to zero. Let $E \in I_G$, $0 < \epsilon_0 := \frac{1}{2} d(E, I_G^c)$ and $\alpha < 1$ then there exists a finite constant $C$ such that*

$$\mathbb{E}\left\{\mathrm{Tr}\big[\tilde{P}^l_\omega(B_\epsilon(E))\big]\right\} \le C\,\epsilon^\alpha\; l^d \quad \text{for all } 0 \le \epsilon < \epsilon_0 \text{ and } l \in \mathbb{N} \tag{4.7}$$

Note that $\tilde{H}^l_\omega$ is *not* an operator restricted to a bounded domain. This is the reason, why the theorem does not apply to energies inside the unperturbed spectrum $\sigma(H_0)$. Estimate (4.7) implies the Hölder continuity of the IDS outside the spectrum of the unperturbed operator $H_0$. A sharper bound, linear in $\epsilon$, was obtained in [**23**]. However, its proof is based on a different technique, cf. Section 5. We will prove here a slightly simpler fact than the above, namely

THEOREM 4.2.2. *Let $H_\omega$ be as in Definition 1.2.1 and assume additionally that the periodic potential $V_{\rm per}$ is bounded below, $u \in L^p_c(\mathbb{R}^d), p = p(d)$ and for some $\kappa > 0$*

$$u \ge \kappa \chi_{[0,1]^d}$$

*Then, there exists for any $E_0 \in \mathbb{R}$ and $\alpha < 1$ a finite constant $C$ such that*

$$\mathbb{E}\left\{\mathrm{Tr}\big[P^l_\omega(B_\epsilon(E)\big]\right\} \le C\,\epsilon^\alpha\; l^d$$

*for all $\epsilon \in [0,1]$, $E \le E_0$ and $l \in \mathbb{N}$.*

This result is proven in Section 3 of [**189**] using partially different methods. Note that the earlier papers [**193, 54**] contain sharper estimates, cf. Section 5.

The fundamental contribution of [**60**] was that it replaced the Weyl-type volume estimate (4.6), whose drawback we explained in Remark 4.1.4. We present now the improved volume estimate of [**60**]. Together with the argument from the previous § 4.1 it proves Theorem 4.2.2.

The trace

$$\mathrm{Tr}\big[\rho[H^l_\omega(\omega, j = \max) - E + t] - \rho[H^l_\omega(\omega, j = \min) - E + t]\big] \tag{4.8}$$

can be expressed using the *spectral shift function* $\xi = \xi(\cdot, H + U, H)$, abbreviated SSF, of the pair of operators

$$H := H^l_\omega(\omega, j = \min) \text{ and } H + U, \text{ where } U = (\omega_+ - \omega_-)\, u(x - j)$$

The necessary estimates on the SSF are collected in Appendix A. Since the difference of our operators is not trace class, we have to use the indirect definition of the SSF by the *invariance principle*, cf. (A.9). Let $C_0 \in \mathbb{R}$ be such that $H_\omega, H_0 \ge C_0$ for all $\omega$ and $g(x) = (x - C_0 + 1)^{-k}$ for some $k \in \mathbb{N}$.

For $k > \frac{d+4}{2}, k \in \mathbb{N}$ the operator $g(H+U) - g(H)$ is trace class and the invariance principle implies

$$\mathrm{Tr}\big(\rho(H+U-E-t) - \rho(H-E-t)\big) = -\int \rho'(\lambda)\xi\big(g(\lambda), g(H+U), g(H)\big)\, d\lambda$$

$$\leq -\big(\textstyle\int \rho'(\lambda)^q d\lambda\big)^{1/q} \left(\displaystyle\int_{\mathrm{supp}\rho'} \xi\big(g(\lambda), g(H+U), g(H)\big)^p\, d\lambda\right)^{1/p} \tag{4.9}$$

In the last line we used the Hölder inequality and $p, q \geq 1$ are conjugate exponents $\frac{1}{p} + \frac{1}{q} = 1$. Remember that we choose $\rho$ depending on $\epsilon$. Thus, its derivative is bounded by $\epsilon^{-1}$ times a constant and

$$\big(\textstyle\int \rho'(\lambda)^q d\lambda\big)^{1/q} \leq C\big(1/\epsilon\big)^{\frac{q-1}{q}} \big(\textstyle\int \rho'(\lambda) d\lambda\big)^{1/q} = C\,\epsilon^{\frac{1}{q}-1} = C\,\epsilon^{\frac{1}{p}}$$

Since on the support of $\rho'$ the function $g$ is uniformly bounded away from zero a transformation of variables gives

$$-\left(\int_{\mathrm{supp}\rho'} \xi(g(\lambda), g(H+U), g(H))^p\, d\lambda\right)^{1/p} \leq C\left(\int_{\mathbb{R}} \xi(\lambda, g(H+U), g(H))^p\, d\lambda\right)^{1/p}$$
$$= C\,\|g(H+U) - g(H)\|_{J_{1/p}}^{1/p} \leq \tilde{C}$$

where we used in the last line Theorems A.1 and A.6 from the Appendix. There one can find the definition of the *super-trace class ideal* $J_{1/p}$ and its norm. The constant $\tilde{C}$ is independent of the box $\Lambda_l$, the lattice site $j$, the configuration of the coupling constants $\omega_k, k \neq j$, and of $\epsilon$.

Hence we have a volume independent bound for (4.8), in contrast to the estimate discussed in Remark 4.1.4. The bound is Hölder continuous in the energy parameter. In view of Proposition 4.1.3 the proof of Theorem 4.2.1 is finished.

REMARK 4.2.3. Recently Hundertmark, Killip and the author [**137**] found a different, shorter way to prove the super-trace class estimates and to apply them to bound the SSF. The basic observation is, that one can control the singular values of the difference of two Schrödinger semigroups. In fact the singular values decay almost exponentially, and the semigroup difference is therefore in any super-trace class ideal.

We state without proofs their result on the decay of the singular values and the estimate on the SSF it implies. For simplicity we consider here the case where the magnetic vector potential is absent.

THEOREM 4.2.4. *Let $H_1 = -\Delta + V$ and $H_2 = H_1 + u$, with $V, u \in L^1_{\mathrm{loc}}(\mathbb{R}^d), V, u \geq -\frac{1}{2}C_0$. Denote by $H_1^l, H_2^l$ the corresponding Dirichlet restrictions to the cube $\Lambda_l = [-l/2, l/2]^d$. Set $V_{\mathrm{eff}}^l := e^{-H_1^l} - e^{-H_2^l}$. There are finite positive constants $c_1, c_2$ such that the singular values $\mu_n$ of the operator $V_{\mathrm{eff}}^l$ obey*

$$\mu_n \leq c_1\, e^{-c_2\, n^{1/d}} \tag{4.10}$$

*The constants depend only on $d$, $C_0$ and the diameter of the support of $u$.*

Let $\rho$ be a switch function as above.

THEOREM 4.2.5. *There is a constant depending only on $d$, $C_0$,* $\operatorname{diam}\operatorname{supp} u$ *and* $E+\epsilon$ *such that*

$$\operatorname{Tr}\left[\rho(H_2^l - E) - \rho(H_1^l - E)\right] \leq const\, \log(1+1/\epsilon)^d$$

*for all* $\epsilon > 0$ *and* $l \in \mathbb{N}$.

**4.3. Sparse potentials.** Form the physical point of view there are some interesting models which have a potential

$$V_\omega(x) = \sum_{k\in\Gamma} \omega_k u(x-k) \tag{4.11}$$

resembling the alloy type model. However, the set $\Gamma$ may be much more general than the lattice $\mathbb{Z}^d$. A class of particular interest are surface models where $\Gamma = \{0\}\times\mathbb{Z}^\nu$ and $\nu < d$ is the dimension of a hyperplane in whose neighbourhood the potential is concentrated. The literature on such models includes [**48, 188, 189, 36**], see also [**96, 97**]. The results in this paragraph are taken from [**172**].

Here we will consider arbitrary sets $\Gamma$, which are *uniformly discrete* in the following sense

$$\sup_{x\in\mathbb{R}^d} \#\{\Gamma \cap B_1(x)\} < \infty$$

For uniformly discrete $\Gamma$ the number of points of $\Gamma$ contained in the cube $\Lambda_l(x)$ can be bounded linearly in the volume of the cube and independently of its centre $x$.

Consider a background Schrödinger operator $H_0 = -\Delta + V_{\rm per}$ with a periodic potential $V_{\rm per} \in L^p_{\rm unif,loc}(\mathbb{R}^d)$ where $p = p(d)$ is as in (1.1). By adding a constant we may assume that $\inf\sigma(H_0) = 0$. Let $H_\omega = H_0 + V_\omega$ be an random operator with an *generalised alloy type potential* $V_\omega$ as in (4.11).

As before $H^l_\omega$ stands for the restriction of $H_\omega$ to the cube $\Lambda_l$ with Dirichlet boundary conditions (we may as well use Neumann or periodic ones), and $P^l_\omega$ denotes the corresponding spectral projection.

THEOREM 4.3.1. *Assume that the single site potential $u \in L^\infty_c(\mathbb{R}^d)$ is non positive, and that the single site distribution $\mu$ has a density $f \in L^\infty_c([0,\infty[)$.*

*Then, for any $\alpha < 1$ and $-E' < 0$ there exists a finite $C$ such that for any $E\in\mathbb{R}, \epsilon \geq 0$ satisfying $E + 3\epsilon \leq -E'$:*

$$\mathbb{E}\left[\operatorname{Tr} P^l_\omega(B_\epsilon(E))\right] \leq C\,\epsilon^\alpha\, l^d$$

The proof of the theorem follows from the arguments of § 4.1 and § 4.2, once a replacement for the estimate (4.5) has been established. This is provided by the following lemma. Denote by $\Lambda^+_l = \{k\in\Gamma | \operatorname{supp} u(\cdot - x)\cap\Lambda_l \neq \emptyset\}$ the set of indices whose coupling constants influence the value of the potential in the cube $\Lambda_l$. Recall that the supremum of the support of $f$ is denoted by $\omega_+$.

LEMMA 4.3.2. *Assume that the $n$-th eigenvalue of the operator $H^l_\omega$ satisfies $\lambda^l_n(\omega) \leq -E' < 0$. Then*

$$\rho'(\lambda^l_n(\omega) - E + t) \leq \frac{\omega_+}{E'}\left[-\sum_{k\in\Lambda^+} \frac{\partial\rho(\lambda^l_n(\omega) - E + t)}{\partial\omega_k}\right]$$

PROOF. Let $\psi_n$ be the normalised eigenfunction corresponding to $\lambda^l_n(\omega)$. Then $\psi_n$ satisfies by definition $\langle\psi_n, (H^l_0 - \lambda^l_n(\omega))\psi_n\rangle = -\langle\psi_n, V_\omega\psi_n\rangle$. We have

$$\sum_{k\in\Lambda^+} \omega_k \langle\psi_n, -u_k(\cdot - k)\psi_n\rangle = -\langle\psi_n, V_\omega\psi_n\rangle = \langle\psi_n, (H^l_0 - \lambda^l_n(\omega))\psi_n\rangle \geq E'$$

Now we have by the Hellmann-Feynman theorem

$$-\sum_{k\in\Lambda^+}\frac{\partial\lambda_n^l(\omega)}{\partial\omega_k}=\sum_{k\in\Lambda^+}\langle\psi_n,-u_k(\cdot-k)\psi_n\rangle\geq\omega_+^{-1}\sum_{k\in\Lambda^+}\omega_k\,\langle\psi_n,-u_k(\cdot-k)\psi_n\rangle\geq\frac{E'}{\omega_+}$$

This gives

$$\begin{aligned}\rho'(\lambda_n^l(\omega)-E+t)&=-\left[-\sum_{k\in\Lambda^+}\frac{\partial\lambda_n^l(\omega)}{\partial\omega_k}\right]^{-1}\sum_{k\in\Lambda^+}\frac{\partial\rho(\lambda_n^l(\omega)-E+t)}{\partial\omega_k}\\&\leq\frac{\omega_+}{E'}\left[-\sum_{k\in\Lambda^+}\frac{\partial\rho(\lambda_n^l(\omega)-E+t)}{\partial\omega_k}\right]\end{aligned}\tag{4.12}$$

□

Note that since $\rho$ is monotone increasing and $u$ is non-positive, (4.12) is a non-negative real.

Related models and results as presented in this paragraph are discussed in Section 3.1 of [**57**]. In [**172**] two more classes of generalised alloy type models are analysed. Firstly, the case where the number of points in $\Gamma\cap B_1(x)$ is not uniformly bounded, but grows at a controlled rate as $x$ goes to infinity. Secondly, the case where $\Gamma$ is itself a random point process, for example of Poissonian type, cf. also [**54**].

**4.4. Locally continuous coupling constants.** In this paragraph we present a Wegner estimate which requires the coupling constants $\omega_k$ to have a continuous distribution merely in a neighbourhood of their extremal value $\omega_+=\sup\operatorname{supp}f$. Both the result and its proof are taken from [**172**].

Consider a background Schrödinger operator $H_0=-\Delta+V_{\rm per}$ with a periodic potential $V_{\rm per}\in L^p_{\rm unif,loc}(\mathbb{R}^d)$. Let $H_\omega=H_0+V_\omega$ be an random operator with an alloy type potential $V_\omega$. Assume that the coupling constants $\omega_k,k\in\mathbb{Z}^d$ take values in the bounded interval $[\omega_-,\omega_+]$. By modifying the periodic background potential we may consider only the case that the coupling constants are non-negative. For a value $\omega_c\in[0,\omega_+]$ introduce the auxiliary periodic potential $V_c=\omega_c\sum_{k\in\mathbb{Z}^d}u(x-k)$ and the threshold energy $E_c=\inf\sigma(H_0+V_c)$.

THEOREM 4.4.1. *Assume that the single site potential $u\in L^\infty_c(\mathbb{R}^d)$ is non positive, and that the restriction of the single site distribution $\mu_c:=\mu|_{]\omega_c,\omega_+]}$ has a density $f\in L^\infty$.*

*Then, for any $\alpha<1$ and $E'<E_c$ there exists a $C$ such that for any $E\in\mathbb{R},\epsilon\geq 0$ satisfying $E+3\epsilon\leq E'$:*

$$\mathbb{E}\left[\mathrm{Tr}P^l_\omega(B_\epsilon(E))\right]\leq C\,\epsilon^\alpha\,l^d$$

PROOF. The value $\omega_c$ is a critical one for the random variable $\omega_k$ in the sense that for $\omega_k>\omega_c$ we know that it is continuously distributed, while for smaller values we do not know anything. We introduce a corresponding decomposition of the 'probability' space $\Omega_l:=\times_{k\in\Lambda^+}\mathbb{R}\cong\mathbb{R}^L$. This is the part of the randomness on which the restricted Hamiltonian $H^l_\omega$ depends. For a given configuration of coupling constants $\{\omega_k\}_{k\in\Lambda^+}$ set

$$\Lambda^{ac}(\omega)=\{k\in\Lambda^+\,|\,\omega_k>\omega_c\}$$

This defines an equivalence relation on $\Omega_l$ by setting for any $A \subset \Lambda^+$

$$\Omega(A) := \{\omega |\, \Lambda^{ac}(\omega) = A\}$$

Consequently

$$\sum_{A\subset\Lambda^+} \int_{\mathbb{R}^L} \prod_{k\in\Lambda^+} d\mu_k(\omega_k)\, \chi_{\Omega(A)}(\omega) = 1. \tag{4.13}$$

Split the potential now into two parts, a singular and an absolutely continuous one. The singular one

$$V^s_\omega(x) := \sum_{k\in\Lambda^+,\omega_k\le\omega_c} \omega_k u_k(x-k) + \sum_{k\in\Lambda^+,\omega_k>\omega_c} \omega_c u_k(x-k) \ge V_c(x)$$

will be considered as part of the background operator, while the absolutely continuous one

$$V^{ac}_\omega(x) := \sum_{k\in\Lambda^+,\omega_k>\omega_c} r_k u_k(x-k) = \sum_{k\in\Lambda^{ac}} r_k u_k(x-k), \quad \text{with } r_k = \omega_k - \omega_c > 0$$

will be used for spectral averaging.

Consider an eigenvalue $\lambda^l_n \le E' < E_c$ and an eigenfunction $H^l_\omega\psi_n = \lambda^l_n\psi_n$ and set $\delta = E_c - E'$. We have

$$-\langle\psi_n, V^{ac}_\omega\psi_n\rangle = \langle\psi_n, (H^l_0 + V^s_\omega - \lambda^l_n)\psi_n\rangle \ge \langle\psi_n, (H^l_0 + V_c - \lambda^l_n)\psi_n\rangle \ge \delta$$

which implies similarly as in Lemma 4.3.2

$$-\sum_{j\in\Lambda^{ac}} \frac{\partial\lambda_n(\omega)}{\partial\omega_j} \ge \frac{1}{\omega_+ - \omega_c} \sum_{j\in\Lambda^{ac}} r_j\, \langle\psi_n, -u_j(\cdot - j)\psi_n\rangle = -\frac{\langle\psi_n, V^{ac}_\omega\psi_n\rangle}{\omega_+ - \omega_c} \ge \frac{\delta}{\omega_+ - \omega_c}$$

Consider first the case $\emptyset \ne A \subset \Lambda^+$ and estimate

$$\int_{\mathbb{R}^L} \prod_{k\in\Lambda^+} d\mu(\omega_k)\chi_{\Omega(A)}(\omega) \sum_{n\in\mathbb{N}} \int_{-2t}^{2t} \rho'(\lambda_n(\omega) - E + t)$$

$$\le \frac{\omega_+ - \omega_c}{\delta} \int_{\mathbb{R}^L} \prod_{k\in\Lambda^+} d\mu(\omega_k)\chi_{\Omega(A)}(\omega) \sum_{n\in\mathbb{N}} \int_{-2t}^{2t} dt \left[ -\sum_{j\in\Lambda^{ac}} \frac{\partial\rho(\lambda_n(\omega) - E + t)}{\partial\omega_j} \right]$$

As we know that all sites $j \in \Lambda^{ac}$ correspond to coupling constants $\omega_j$ with values in the absolutely continuous region of the conditional density $f$ we may estimate as in § 4.1:

$$\begin{aligned} &-\sum_{n\in\mathbb{N}} \int_{\mathbb{R}} d\mu(\omega_j)\, \chi_{\Omega(A)}(\omega)\, \frac{\partial\rho(\lambda_n(\omega) - E + t)}{\partial\omega_j} \\ &= \quad -\sum_{n\in\mathbb{N}} \int_{\omega_c}^{\omega_+} f(\omega_j) d\omega_j\, \frac{\partial\rho(\lambda_n(\omega) - E + t)}{\partial\omega_j} \\ &\le \quad \|f\|_\infty \sum_{n\in\mathbb{N}} [\rho(\lambda_n(\omega, \omega_j = \omega_c) - E + t) - \rho((\lambda_n(\omega, \omega_j = \omega_+) - E + t)] \end{aligned}$$

which can be estimated as in § 4.2. We have to say something how we deal with the special case $A = \emptyset$. In this situation $V^{ac}_\omega \equiv 0$ and $H_\omega = H_0 + V^s_\omega \ge H_0 + V_c \ge E_c$. Thus there are no eigenvalues in the considered energy interval for this potential configuration.

Finally we use the decomposition (4.13) to finish the proof:

$$\begin{aligned}
&\mathbb{E}\left(\mathrm{Tr}P^l_\omega([E-\epsilon,E+\epsilon])\right)\\
&\leq \sum_{A\subset\Lambda^+}\int_{\mathbb{R}^L}\prod_{k\in\Lambda^+}d\mu(\omega_k)\,\chi_{\Omega(A)}(\omega)\sum_{j\in A}\frac{(\omega_+-\omega_c)}{\delta}\,4\epsilon\|f\|_\infty C(\alpha)\epsilon^{\alpha-1}\\
&\leq 4C(\alpha)\frac{(\omega_+-\omega_c)}{\delta}\|f\|_\infty\epsilon^\alpha l^d
\end{aligned}$$

□

**4.5. Potentials with small support.** In § 4.1 we used in a crucial step in the derivation of the Wegner estimate that the single site potentials were lower bounded by a partition of unity

$$\sum_{k\in\mathbb{Z}^d}u(x-k)\geq\kappa\quad\text{ on }\mathbb{R}^d$$

It is of natural interest, whether a Wegner estimate holds if this condition is relaxed. In this paragraph we consider the case that $u$ is of fixed sign, but has small support. More precisely, we assume throughout this paragraph merely that there is an open set $\mathcal{O}\subset\mathbb{R}^d$ and a positive $\kappa$ such that

$$u(x)\geq\kappa\chi_{\mathcal{O}}\tag{4.14}$$

The first Wegner estimates under this relaxed condition on the single site potential were derived for spectral boundaries, i.e. for energies either near the bottom of the spectrum, or near an internal spectral boundary. The case of the infimum of the spectrum was treated e.g. in [**180, 159**], and internal spectral boundaries in [**170**]. These works derived a Wegner estimate where the volume dependence of the bound was growing faster than linearly. Thus they were not sufficient to derive a result on the regularity of the IDS, cf. our discussion in § 3.1. A linear bound for the same energy regimes was found in [**23, 60**].

By now there are Wegner estimates which under the relaxed condition (4.14) derive Wegner estimates valid for any bounded interval on the energy axis. We consider first the one-dimensional case where the result is particularly clear and the proof simple. We follow [**171**] in the presentation, see [**122, 56**] for other proofs.

Assume that the single site potential $u$ and the periodic potential $V_{\mathrm{per}}$ are bounded.

THEOREM 4.5.1. *Assume that $u$ is compactly supported and obeys* (4.14). *For any $E_0\in\mathbb{R}$ there exist a constant $C$ such that*

$$\mathbb{E}\left[\mathrm{Tr}P^l_\omega(B_\epsilon(E))\right]\leq C\,\epsilon\,l,\quad\forall\,\epsilon\in[0,1],E\leq E_0,l\in\mathbb{N}\tag{4.15}$$

Thus the IDS is Lipschitz-continuous.

PROOF. First we show how to replace (4.4) in the case of small support. By shifting the origin of $\mathbb{R}^d$ we may assume without loss of generality that there is a $s>0$ such that $\Lambda_s(0)\subset\mathcal{O}$. Likewise, we may assume $\kappa=1$ by rescaling the single site potential and the coupling constants.

We set $S=\bigcup_{k\in\tilde\Lambda}\Lambda_s(k)$. The Hellmann-Feynman theorem gives us

$$\sum_{k\in\tilde\Lambda}\frac{\partial\lambda^l_n(\omega)}{\partial\omega_k}=\sum_{k\in\tilde\Lambda}\langle\psi_n,u(\cdot-k)\psi_n\rangle\geq\int_S|\psi_n|^2.$$

where $\psi_n$ is a normalised eigenfunction corresponding to $\lambda_n^l(\omega)$.

If the integral on the right hand side would extend over the whole of $\Lambda_l$ it would be equal to 1 due to the normalisation of $\psi_n$. A priori the integral over $S$ could be arbitrary close to zero, but the following Lemma shows that this is not the case.

LEMMA 4.5.2. *Let $I$ be a bounded interval and $s > 0$. There exists a constant $c > 0$ such that*

$$\int_{\Lambda_s(k)} |\psi|^2 \geq c \int_{\Lambda_1(k)} |\psi|^2$$

*for all $l \in \mathbb{N}$, all $k \in \tilde{\Lambda}_l$ and for any eigenfunction $\psi$ corresponding to an eigenvalue $E \in I$ of $H_\omega^l$.*

PROOF OF THE LEMMA. For

$$\phi(y) := \int_{\Lambda_s(k+y)} dx\, |\psi(x)|^2 = \int_{\Lambda_s(k)} dx\, |\psi(x-y)|^2$$

one has

$$\begin{aligned} \left| \frac{\partial}{\partial y} \phi(y) \right| &= \left| \int_{\Lambda_s(k)} dx \left[ \frac{\partial}{\partial y} \psi(x-y) \right] \overline{\psi(x-y)} + \int_{\Lambda_s(k)} dx\, \psi(x-y)\, \frac{\partial}{\partial y} \overline{\psi(x-y)} \right| \\ &\leq 2 \, \|\psi\|_{L^2(\Lambda_s(k+y))} \, \|\psi'\|_{L^2(\Lambda_s(k+y))} \, . \end{aligned}$$

Sobolev norm estimates (e.g. Theorems 7.25 and 7.27 in [**124**]) imply

$$\|\psi'\|_{L^2(\Lambda_s(k+y))} \leq C_5 \, \|\psi\|_{L^2(\Lambda_s(k+y))} + \|\psi''\|_{L^2(\Lambda_s(k+y))}$$

By the eigenvalue equation we have

$$\left| \frac{\partial}{\partial y} \phi(y) \right| \leq C_6 \, \|\psi\|^2_{L^2(\Lambda_s(k+y))} = C_6 \, \phi(y), \qquad C_6 = C_6(\|V_{\rm per} + V_\omega - E\|_\infty) \tag{4.16}$$

Gronwall's Lemma implies $\phi(y) \leq \exp(C_6|y|)\, \phi(0)$ and thus

$$\int_{\Lambda_1(k)} |\psi|^2 \leq e^{C_6} \; s^{-1} \int_{\Lambda_s(k)} |\psi|^2$$

□

Thus $\int_S |\psi|^2 \geq c \int_{\Lambda_l} |\psi|^2$ with the same constant as in Lemma 4.5.2.

It remains to estimate the spectral shift

$$\sum_{n \in \mathbb{N}} \left[ \rho(\lambda_n^l(\omega, j = \max) - E + t) - \rho(\lambda_n^l(\omega, j = \min) - E + t) \right] \tag{4.17}$$

We may assume without loss of generality that the single site potential $u$ is supported in the interval $[-R, R]$. Introduce now the operator $H_\omega^{l,D}(j = \max)$ which coincides with $H_\omega^l(j = \max)$ up to additional Dirichlet boundary conditions at the points $j - R$ and $j + R$. Likewise, $H_\omega^{l,N}(j = \min)$ coincides with $H_\omega^l(j = \min)$ up to additional Neumann boundary conditions at the same points. Their eigenvalues are $\lambda_n^{l,D}(\omega, j = \max)$ and $\lambda_n^{l,N}(\omega, j = \min)$, respectively. By Dirichlet-Neumann bracketing, the square brackets in (4.17) are bounded by

$$\rho(\lambda_n^{l,D}(\omega, j = \max) - E + t) - \rho(\lambda_n^{l,N}(\omega, j = \min) - E + t) \tag{4.18}$$

Since for both $*=D,N$ the Hamiltonian $H_\omega^{l,*}$ is a direct sum of an operator $H_\omega^{j,*}$ acting on $L^2(j-R,j+R)$ and another one $H_\omega^{c,*}$ acting on $L^2(\Lambda_l \setminus [j-R,j+R])$ the sum over the terms in (4.18) can be separated:

$$\sum_n \rho(\lambda_n^{c,D}(\omega) - E + t) - \rho(\lambda_n^{c,N}(\omega) - E + t) \tag{4.19}$$

$$+ \quad \sum_n \rho(\lambda_n^{j,D}(\omega, j = \max) - E + t) - \rho(\lambda_n^{j,N}(\omega, j = \min) - E + t) \tag{4.20}$$

Note that the eigenvalues in (4.19) are independent of $\omega_j u(\cdot - j)$. Since the difference in the boundary conditions is a rank two perturbation in resolvent sense (see e.g. [**282**]), the interlacing theorem says that

$$\rho(\lambda_n^{c,D}(\omega, j = \max) - E + t) \leq \rho(E_{n+2}^{c,N}(\omega, j = \max) - E + t)$$

A telescoping argument bounds the whole sum in (4.19) by twice the total variation of $\rho$, which is equal to one. The sum in (4.20) we estimate by

$$\begin{aligned} &\operatorname{Tr}\left[\chi_{[E-3\epsilon,\infty[}(H_\omega^{j,D}(j = \max)) - \chi_{]E+3\epsilon,\infty[}(H_\omega^{j,N}(j = \min))\right] \\ &\quad \leq 2 + \operatorname{Tr}\left[\chi_{[E-3\epsilon,\infty[}(H_\omega^{j,D}(j = \min) + \|u_j\|_\infty) - \chi_{]E+3\epsilon,\infty[}(H_\omega^{j,D}(j = \min))\right] \end{aligned}$$

which is bounded by a constant, that is independent of $\Lambda_l$, $j \in \tilde{\Lambda}_l$ and $\epsilon > 0$ (on which $\rho$ depends). □

In the remainder of this section we give an overview of various Wegner estimates which are based or related to techniques presented at the beginning of Section 4. However, we refrain from giving the proofs of this results but refer to the original articles.

In [**56**] Combes, Hislop and Klopp study multi-dimensional alloy type models with single site potentials of small support, and establish the Hölder continuity of the IDS at all energies. They consider the case where the single site potential $u \in L_c^\infty(\mathbb{R}^d)$ is non-negative and not identically equal to zero, and treat three different situations. In all of them the unperturbed background operator $H_0 = (-i\nabla - A)^2 + V_0$ may include a magnetic vector potential $A$ and a (scalar) electric potential $V_0$. They have to satisfy some regularity conditions such that $H_0$ is selfadjoint and has $C_0^\infty(\mathbb{R}^d)$ as an operator core. The coupling constants are distributed according to a bounded, compactly supported density.

(i) The background operator $H_0$ has an IDS $N_0$, which is Hölder continuous

$$|N_0(E_2) - N_0(E_1)| \leq C_0 |E_2 - E_1|^{\tilde{\alpha}}$$

with Hölder exponent $\tilde{\alpha} \in ]0,1]$. The constant $C_0 = C_0(I)$ can be chosen uniformly for $E_2, E_1$ in a given compact interval $I$.

(ii) The background operator $H_0$ is periodic with respect to the lattice $\mathbb{Z}^d$ and has the unique continuation property, cf. for instance [**327**]. The set $\{x | \, u(x) > 0\}$ contains an open subset of $\mathbb{R}^d$.

(iii) Let the space dimension be $d = 2$. Let $H_0 = (-i\nabla - A)^2 + V_{\text{per}}$ be a Landau Hamiltonian with vector potential $A(x_1, x_2) = \frac{B}{2}(-x_2, x_1)$ where $B > 0$ is the (constant) magnetic field strength. The magnetic flux trough a unit cell satisfies the rationality condition

$$B \in 2\pi\,\mathbb{Q} \tag{4.21}$$

The scalar potential $V_{\text{per}}$ is a $\mathbb{Z}^d$-periodic function in $L^2_{\text{loc}}(\mathbb{R}^d)$.

In case (i) set $\alpha_c = \frac{\tilde{\alpha}}{\tilde{\alpha}+2}$, otherwise $\alpha_c = 1$.

In a follow up work [**58**] on the Landau Hamiltonian in collaboration with Raikov condition (4.21) has been removed.

THEOREM 4.5.3. *Let $H_\omega$ be an alloy type model satisfying either one of the above conditions* (i)–(iii). *Then, for each $\alpha \in ]0, \alpha_c[$, the IDS of $H_\omega$ is Hölder continuous at all energies, with Hölder exponent $\alpha$.*

**4.6. Hölder continuous coupling constants.** There is special interest to extend the known Wegner estimates to coupling constants with singular distribution. The reason is the intuitive interpretation of the coupling consonants as nuclear charge numbers modulating the strength of atomic potentials. In this case their distribution would correspond to a pure point measure.

So far the best result in this direction for multi-dimensional alloy type models was obtained by Stollmann in [**294**]. In Remarks 3.1.1 and 3.2.1 we mentioned already results for one-dimensional models with singular randomness.

Stollmann's result applies to a single site measure $\mu$ which has compact support $[\omega_-, \omega_+]$ and which is merely Hölder continuous. For $\epsilon \geq 0$ denote

$$s(\epsilon) := s(\mu, \epsilon) := \sup\{\mu([a,b]) |\, b - a \leq \epsilon\}$$

THEOREM 4.6.1. *Let $H_\omega$ be an alloy type model as in Definition 1.2.1, but let the single site measure be merely Hölder continuous. Assume additionally that the single site potential obeys $u \geq \chi_{[0,1]^d}$. Then for any $E \in \mathbb{R}$ there exists a constant $C$ such that for any open interval $I \subset ]-\infty, E[$ and any $l \in 2\mathbb{N}$*

$$\mathbb{P}\{\omega |\, \sigma(H^l_\omega) \cap I \neq \emptyset\} \leq C\, s(|I|)\, l^{2d}$$

*holds.*

**4.7. Single site potentials with changing sign.** First Wegner estimates for indefinite alloy type potentials were derived in [**180**]. In [**134**] Hislop and Klopp combine the techniques from [**180**] and [**60**] to prove a Wegner estimate valid for general indefinite single site potentials and for energy intervals at edges of $\sigma(H_\omega)$. They assume the single site potential $u \in C_c(\mathbb{R}^d)$ satisfies $u(0) \neq 0$. The density $f \in L^\infty_c$ of the random variable $\omega_0$ (which may be in fact the conditional density with respect to $\omega^{\perp 0} := (\omega_k)_{k \neq 0}$) is assumed to be piecewise absolutely continuous. For any $\alpha < 1$ and any compact energy interval $I$ strictly below the spectrum of the unperturbed operator $H_0$ they prove

$$\mathbb{P}\{\sigma(H^l_\omega) \cap I \neq \emptyset\} \leq C\, |I|^\alpha\, l^d$$

where the constant $C$ depends only on $\alpha, d$ and the distance between the interval $I$ and $\sigma(H_0)$. With a sufficiently small global coupling constant $\lambda$ the same result holds for the operator $H_0 + \lambda V_\omega$ for $I$ in an internal spectral gap of $H_0$. The results of [**134**] extend to more general models including certain operators with random magnetic field.

In § 5.5 we discuss in more detail an alternative technique to obtain a Wegner estimate valid for single site potentials which change sign. It applies to a more restricted class of potentials but yields stronger results. In particular, it proves the Lipschitz continuity of the IDS at all energies.

**4.8. Uniform Wegner estimates for long range potentials.** Kirsch, Stollmann and Stolz proved in [**169**] a Wegner estimate for single site potentials which do not need to have compact support, but merely need to decay sufficiently fast. They consider $u$ of polynomial decay

$$|u(x)| \leq C(1+|x|^2)^{-m/2} \tag{4.22}$$

where $m > 0$ is required to be larger than $3d$. For certain applications they can also deal with the case where $m$ is only larger than $2d$, cf. [**168, 329**].

For such single site potentials the restrictions of the alloy type potential to two finite cubes may be correlated, even if the cubes are far apart. This makes it necessary to use a enhanced version of the multiscale analysis for the proof of localisation. Among others, this requires a *uniform Wegner estimate*. By this we mean a Wegner estimate for the Hamiltonian $H_\omega^l$ restricted to the cube $\Lambda_l$ which is uniform in the coupling constants $\omega_k$ with index $\|k\|_\infty > r$ where $r$ is a function of $l$.

To formulate the Wegner estimate from [**169**] let us first introduce some notation. For any cube $\Lambda \subset \mathbb{R}^d$ and $\tilde{\Lambda} = \Lambda \cap \mathbb{Z}^d$ we denote by $\Pi_\Lambda$ the projection

$$\Pi_\Lambda \colon \Omega \mapsto \bigtimes_{\tilde{\Lambda}} \operatorname{supp} \mu \qquad \Pi_\Lambda(\omega) := \{\omega_k\}_{k\in\tilde{\Lambda}}$$

For a measurable set $A \subset \Omega$ we denote by $A_\Lambda^*$ the cylinder set

$$A_\Lambda^* := \Pi_\Lambda^{-1}(\Pi_\Lambda A) = \{\omega \in \Omega |\, \exists \omega' \in A \text{ such that } \Pi_\Lambda(\omega') = \Pi_\Lambda(\omega)\}$$

The following observation plays a crucial role in the enhanced multiscale analysis.

LEMMA 4.8.1. *For two disjoint cubes $\Lambda, \Lambda'$ and two events $A, B \in \Omega$, the induced events $A_\Lambda^*$ and $B_{\Lambda'}^*$ are independent.*

The following lemma allows one to turn a 'usual' Wegner estimate, as we have it discussed before, into a uniform Wegner estimate. It relies on the polynomial decay of the single site potential (4.22). Let $I$ be a compact interval, $E \in I$ and $\epsilon \in ]0,1]$. We denote by $A(E,\epsilon,l)$ the event $\{\omega |\, d(E, \sigma(H_\omega^l)) < \epsilon\}$ and use the abbreviations $\Pi_l := \Pi_{\Lambda_l}, A_l^* := A_{\Lambda_l}^*$.

LEMMA 4.8.2. *Under the above assumptions there exists a finite constant $c$, independent of $\omega \in \Omega$, $l, r \in \mathbb{N}$ and $\epsilon \leq 1$ such that*

$$\mathbb{P}\{A(E,\epsilon,l)_{l+r}^*\} \leq \mathbb{P}\{A(E,\epsilon + cr^{-(m-d)}, l)\}$$

PROOF. By definition, for an $\omega \in A_{l+r}^*$ there exists an $\omega' \in A$ such that

$$\Pi_{l+r}\omega' = \Pi_{l+r}\omega$$

Thus, the coupling constants of $\omega$ and $\omega'$ with index $k$ within the cube of size $l+r$ coincide and we have for $x \in \Lambda_l$

$$|V_\omega(x) - V_{\omega'}(x)| \leq \sum_{|k|_\infty > l+r} |\omega_k - \omega_k'|\, u(x-k) \leq c' \sum_{|k|_\infty > l+r} |x-k|^{-m} \leq cr^{-(m-d)}$$

Therefore $d(E, \sigma(H_{\omega'}^l)) < \epsilon$ implies $d(E, \sigma(H_\omega^l)) < \epsilon + cr^{-(m-d)}$, which proves the lemma. □

Let us have a look on the implications of the preceding lemma for a concrete example. Assume that the single site potential is bounded below on the unit cube around zero by $\kappa > 0$. Then we have by Theorem 5.0.1

$$\mathbb{P}\big\{\omega|\sigma(H^l_\omega) \cap [E-\epsilon, E+\epsilon] \neq \emptyset\big\} \leq C_W(I)\ \epsilon\ l^d$$

for all $E, \epsilon$ such that $[E-\epsilon, E+\epsilon]$ is contained in the open interval $I$. This Wegner estimate implies its uniform analog

$$\mathbb{P}\Big(\big\{\omega|\sigma(H^l_\omega) \cap [E-\epsilon, E] \neq \emptyset\big\}^*_{l+r}\Big) \leq C_W(I)\ (\epsilon + cr^{-(m-d)})\ l^d \tag{4.23}$$

for sufficiently large $r > 0$. In the application in the multiscale analysis, both $\epsilon$ and $r$ are chosen as functions of $l$. From the estimate in (4.23) it might seem to be sufficient to choose $m > d$. This is also the minimal requirement to make the alloy type model with long range single site potentials well defined as a densely defined operator. However, for technical reasons, for the multiscale analysis to work one has to assume at least $m > 2d$. Under this assumption one can prove that the spectrum of $H_\omega$ is almost surely pure point near its bottom, and the corresponding eigenfunctions decay faster than any polynomial, [**168, 329**]. To obtain exponential decay of the eigenfunctions, one has to require $m > 3d$ [**169**].

In the paper [**329**] by Zenk the above results have been extended to a model which incorporates random displacements of the single site potentials.

## 5. Lipschitz continuity of the IDS

In [**193**] Kotani and Simon extended to continuum alloy type models certain arguments previously used for the derivation of Wegner's estimate for the discrete Anderson model. They treated only the case where the single site potential is the characteristic function of the unit cube, but Combes and Hislop showed in [**54**] that the same argument extends to non-negative single site potentials with uniform lower bound on the unit cube. There also some steps of the proof have been streamlined.

One of the ideas in [**193**] is that in the same way as rank one perturbations are used for discrete Laplacians, positive perturbations may be used in the continuum case. This is related to the Aronszajn-Donoghue Theory [**16, 17, 18, 83**]. See [**45, 192, 283, 282**] for more background and references. This was essential, since a finite rank potential in the continuum may be a Dirac distribution, but not a function.

THEOREM 5.0.1. *Let $H_\omega$ as in Definition 1.2.1 and assume additionally that there exists an $\kappa > 0$ such that*

$$u \geq \kappa \chi_{[-1/2,1/2]^d}$$

*Then for all $E \in \mathbb{R}$ there exists a constant $C_W = C_W(E)$ such that for all $l \in \mathbb{N}$ and all intervals $I \subset ]-\infty, E]$*

$$\mathbb{E}\left\{\mathrm{Tr}\big[P^l_\omega(I)\big]\right\} \leq C_W\ |I|\ l^d \tag{5.1}$$

REMARK 5.0.2. (a) It is sufficient to prove the theorem for the case $\kappa = 1$. Since $\omega_0 u = \kappa\omega_0\, \kappa^{-1} u$, the general case follows by rescaling the coupling constants and single site potentials.

(b) The statement of the theorem remains true if one uses Neumann or periodic boundary conditions for $H^l_\omega$.

(c) An explicit formula for the Wegner constant $C_W$ is given in (5.17). Since (5.1) is linear in the volume it follows $|N(E_2) - N(E_1)| \leq C_W\,|E_2 - E_1|$. Thus, as we discussed already in § 3.1, the density of states $n(E) := dN(E)/dE$ exists almost everywhere and is bounded by $n(E) \leq C_W(E_2)$ for all $E \leq E_2$.

The next four paragraphs are devoted to the proof of Theorem 5.0.1. Up to some modifications we follow the line of argument in Section 4 of [**54**].

**5.1. Partition of the trace into local contributions.** In the present paragraph we derive preparatory estimates on

$$\mathbb{E}\left\{\mathrm{Tr} P^l_\omega(I)\right\} \tag{5.2}$$

where we do not yet use the specific alloy-type structure of the potential. They have two aims. Firstly, to decompose the trace to contributions of unit cubes in $\Lambda_l$. This will later facilitate the averaging procedure with respect to random parameters, whose effect on the potential is felt only locally. Secondly, it allows us to reduce the averaging of the trace of the spectral projection to the averaging of the quadratic form of the resolvent. The latter is technically easier to perform.

Denote by $\Delta^l$ and $\Delta^l_N$ the Laplace operator on $\Lambda_l$ with Dirichlet, respectively Neumann boundary conditions. In § 1.2 we saw that the potential $V = V_{\mathrm{per}} + V_\omega$ is infinitesimally bounded with respect to $-\Delta$ and that the constants in the bound can be chosen uniformly in $\omega \in \Omega$. This implies that $V$ is infinitesimally form bounded with respect to any of the operators $-\Delta$, $-\Delta^l$ and $-\Delta^l_N$ with bounds uniform in $\omega \in \Omega, l \in \mathbb{N}$ and the choice of Dirichlet or Neumann boundary conditions. Consequently, there is a $C_0 < \infty$ such that for all $\omega \in \Omega$ and $l \in \mathbb{N}$

$$|\langle \phi, V\phi\rangle| \leq \frac{1}{2}\langle \phi, -\Delta^l_N \phi\rangle + C_0\|\phi\|^2$$

which implies

$$\langle \phi, H^l_\omega \phi\rangle \geq \langle \phi, -\frac{1}{2}\Delta^l_N\phi\rangle - C_0\|\phi\|^2 \geq -C_0\|\phi\|^2 \tag{5.3}$$

Thus $H^l_\omega + C_0$ is a non-negative operator.

Definition 5.1.1. A monotone decreasing, convex function $r\colon [0,\infty[ \to ]0,\infty[$ such that

$$C_{\mathrm{Tr}} := C_{\mathrm{Tr}}(r) := \sum_{n\in\mathbb{Z}^d, n_j\geq 0} r\left(\frac{\pi^2}{2}\sum_{j=1}^d n_j^2\right) < \infty \tag{5.4}$$

will be called *trace regularising*.

Throughout the rest of this section we denote by $\Lambda$ the unit cube centred at zero.

Remark 5.1.2. The bound (5.4) means that the operator $r(-\frac{1}{2}\Delta^N_\Lambda)$ has finite trace. Namely, the eigenvalues of the Neumann Laplacian on the unit cube are given by

$$\pi^2\sum_{j=1}^d n_j^2 \quad \text{for all } n \in \mathbb{Z}^d \text{ such that } n_j \geq 0, j = 1,\dots,d$$

cf. for instance [**256**], page 266. By the spectral mapping theorem the eigenvalues of $r(-\frac{1}{2}\Delta^N_\Lambda)$ are just $r\big(\frac{1}{2}\pi^2\sum_{j=1}^d n_j^2\big)$.

Examples of functions $r$ which are trace-regularising are the exponential functions $r\colon x \mapsto e^{-tx}$ for $t > 0$. They have been used in [**54**] to implement the procedure outlined in this section. Another choice for $r$ is a sufficiently high power of the resolvent $x \mapsto (x+1)^{-k}$ for $k > d/2$, which was used in [**193**]. That the operator $x \mapsto (-\frac{1}{2}\Delta_\Lambda^N + 1)^{-k}$ is actually trace class can be inferred from [**279**].

The possibility to choose $r$ from a large class of functions is of interest if one wants to give explicit upper bounds on the density of states. For instance, Section 3.2 of [**141**] is devoted to deriving such explicit upper estimates. However, there, following [**54**], the function $r(x) = e^{-tx}$ is used. Due to this choice, the upper bound on the density of states is exponentially growing in the energy. This can be improved to a merely polynomial growing bound. Furthermore, if one studies coupling constants which may take on arbitrarily negative values, the choice of $r$ will determine which moment conditions one has to impose on the negative part of $\omega_0$, see also § 5.6.

PROPOSITION 5.1.3. *With $C_0$ as in* (5.3)

$$\mathbb{E}\left\{\mathrm{Tr} P_\omega^l(I)\right\} \leq r(E_2 + C_0)^{-1}\, C_{\mathrm{Tr}}(r) \sum_{j\in\tilde{\Lambda}_l} \left\| \mathbb{E}\left\{\chi_j P_\omega^l(I)\chi_j\right\}\right\|$$

PROOF. Since $\frac{1}{r}$ is well-defined and bounded on the compact interval $I := [E_1, E_2]$, we have

$$\mathrm{Tr}\left[P_\omega^l(I)\right] = \mathrm{Tr}\left[r(H_\omega^l + C_0)^{-1}\, P_\omega^l(I)\, r(H_\omega^l + C_0)\right]$$

Furthermore, by spectral calculus and since for positive operators $A, B$ we have $\mathrm{Tr}(AB) \leq \|A\|\ \mathrm{Tr}(B)$, the above line is bounded by

$$r(E_2 + C_0)^{-1}\mathrm{Tr}\left[P_\omega^l(I)\, r(H_\omega^l + C_0)\right]$$

According to the direct sum decomposition

$$L^2(\Lambda_l) = \bigoplus_{j\in\tilde{\Lambda}_l} L^2(\Lambda + j)$$

we consider the Laplace operators $-\Delta^{j,N}$ on $L^2(\Lambda + j)$ with Neumann boundary conditions. Dirichlet-Neumann bracketing implies

$$H_\omega^l + C_0 \geq -\frac{1}{2}\Delta_N^l \geq -\frac{1}{2}\bigoplus_{j\in\tilde{\Lambda}_l} \Delta^{j,N} =: \oplus H \tag{5.5}$$

For a normalised eigenfunction $\phi$ of $H_\omega^l$ corresponding to the eigenvalue $\lambda$ we have by the spectral mapping theorem

$$\langle\phi, r(H_\omega^l + C_0)\phi\rangle = r(\lambda + C_0) = r(\langle\phi, (H_\omega^l + C_0)\phi\rangle) \leq r(\langle\phi, \oplus H\phi\rangle) \tag{5.6}$$

Applying Jensen's inequality to the spectral measure of $\oplus H$ we estimate (5.6) from above by $\langle\phi, r(\oplus H)\phi\rangle$. Let $\phi_n, n \in \mathbb{N}$ be an orthonormal basis of eigenvectors of $H_\omega^l$ with corresponding eigenvalues $\lambda_n, n \in \mathbb{N}$. We apply the above estimates to the trace

$$\begin{aligned}\mathrm{Tr}\left[P_\omega^l(I) r(H_\omega^l + C_0)\right] &\leq \sum_{n\in\mathbb{N},\lambda_n\in I} \langle\phi_n, r(H_\omega^l + C_0)\phi_n\rangle \\ &\leq \sum_{n\in\mathbb{N},\lambda_n\in I} \langle\phi_n, r(\oplus H)\phi_n\rangle \leq \mathrm{Tr}\left[P_\omega^l(I) r(\oplus H)\right]\end{aligned}$$

For the next step we write down the trace with respect to different basis. For each $j \in \tilde{\Lambda}$ let $\{\psi_n^j | \, n \in \mathbb{N}\}$ be an orthonormal basis of $L^2(\Lambda + j)$, then $\{\psi_n^j | \, n \in \mathbb{N}, j \in \tilde{\Lambda}\}$ is an orthonormal basis of $L^2(\Lambda_l)$. Since $r(\oplus H)\psi_n^j = \chi_j r(-\frac{1}{2}\Delta^{j,N})\chi_j \psi_n^j$ it follows for the trace

$$\begin{aligned}
\operatorname{Tr}\left[P_\omega^l(I) r(\oplus H)\right] &= \sum_{j \in \tilde{\Lambda}_l} \sum_{n \in \mathbb{N}} \langle \psi_{j,n}, P_\omega^l(I) r(\oplus H) \psi_{j,n} \rangle \\
&= \sum_{j \in \tilde{\Lambda}_l} \sum_{n \in \mathbb{N}} \langle \psi_{j,n}, \chi_j P_\omega^l(I) \chi_j r(-\tfrac{1}{2}\Delta^{j,N}) \chi_j \psi_{j,n} \rangle \\
&= \sum_{j \in \tilde{\Lambda}_l} \operatorname{Tr}\left[\chi_j P_\omega^l(I) \chi_j r(-\tfrac{1}{2}\Delta^{j,N}) \chi_j\right].
\end{aligned}$$

Thus we have decomposed the trace to contributions from each unit cube. We summarize the estimates so far:

$$\operatorname{Tr} P_\omega^l(I) \leq r(E_2 + C_0)^{-1} \sum_{j \in \tilde{\Lambda}_l} \operatorname{Tr}\left[\chi_j P_\omega^l(I) \chi_j r(-\tfrac{1}{2}\Delta^{j,N}) \chi_j\right]$$

Since $I$ is a bounded interval and $V_{\mathrm{per}} + V_\omega$ is an infinitesimally small perturbation of $-\Delta^l$ independently of $\omega$, it follows that the dimension of $P_\omega^l(I) L^2(\Lambda_l)$ is bounded by a constant $C_3$ uniformly in $\omega$. Thus

$$\operatorname{Tr}\left[\chi_j P_\omega^l(I) \chi_j r(-\tfrac{1}{2}\Delta^{j,N}) \chi_j\right] \leq C_3 \, r(0) \text{ for all } \omega \in \Omega$$

is an upper bound by an integrable majorant and we are able to interchange the trace and the expectation by Lebesgue's theorem on dominated convergence

$$\begin{aligned}
&\mathbb{E}\left\{\operatorname{Tr}\left[\chi_j P_\omega^l(I) \chi_j r(-\tfrac{1}{2}\Delta^{j,N}) \chi_j\right]\right\} = \operatorname{Tr}\left[\mathbb{E}\left\{\chi_j P_\omega^l(I) \chi_j r(-\tfrac{1}{2}\Delta^{j,N}) \chi_j\right\}\right] \\
&= \operatorname{Tr}\left[\mathbb{E}\left\{\chi_j P_\omega^l(I) \chi_j\right\} \chi_j r(-\tfrac{1}{2}\Delta^{j,N}) \chi_j\right] \leq \left\|\mathbb{E}\left\{\chi_j P_\omega^l(I) \chi_j\right\}\right\| \operatorname{Tr}\left[r(-\tfrac{1}{2}\Delta^{0,N})\right]
\end{aligned}$$

By assumption, $r$ is trace regularising, so the trace in the last line is finite. □

**5.2. Spectral averaging of resolvents.** Now we consider how resolvents are averaged when integrated over a random parameter. Together with the partition result in the previous paragraph § 5.1 this will enable us to complete in § 5.4 the proof of Theorem 5.0.1.

Apart from this, the spectral averaging result bears in itself a meaning. Consider a nonnegative operator $H$ with discrete spectrum. Its resolvent $R(E) = (H - E)^{-1}$ has singularities at the eigenvalues of $H$ which are of the form $(\lambda_n - E)^{-1}, \lambda_n \in \sigma(H)$ and thus are not integrable over the energy axis. In other words, for a general vector $\phi$ the function $E \mapsto \langle \phi, R(E)\phi \rangle$ will not have a convergent integral. Now, if $H = H_\lambda$ depends on a random parameter $\lambda$, we might hope that the averaged resolvent $E \mapsto \int d\mathbb{P}(\lambda) \langle \phi, R_\lambda(E)\phi \rangle$ will be integrable. This would mean that the singularities of the resolvent have been smeared out sufficiently by the integral over $\lambda$. The lemma in this paragraph shows that this is actually the case for operators which depend in a specific way on the random parameter.

Consider the following operators on a Hilbert space $\mathcal{H}$. Let $H$ be a selfadjoint operator, $W$ symmetric and infinitesimally bounded with respect to $H$, and $J$ nonnegative with $J^2 \leq W$. Choose two parameters

$$\begin{aligned}
z \in \mathbb{C}_- &:= \{z \in \mathbb{C} | \, \operatorname{Im} z < 0\} \\
\zeta \in \overline{\mathbb{C}_+} &:= \{\zeta \in \mathbb{C} | \, \operatorname{Im} \zeta \geq 0\}
\end{aligned}$$

and set

$$H(\zeta) := H + \zeta W,\ K(\zeta, z) := J(H(\zeta) - z)^{-1} J \tag{5.7}$$

The following lemma is a slight generalisation of Lemma 4.1 in [**54**].

LEMMA 5.2.1. *For all $z \in \mathbb{C}_-$, all $t > 0$ and any normalised $\phi \in \mathcal{H}$ we have*

$$\left| \int_{\mathbb{R}} \langle \phi, K(\zeta, z)\phi \rangle \, \frac{d\zeta}{1 + t\zeta^2} \right| \leq \pi \tag{5.8}$$

PROOF. By Pythagoras we have $|\langle \phi, (A + iB)\phi \rangle|^2 = |\langle \phi, A\phi \rangle|^2 + |\langle \phi, B\phi \rangle|^2$ for any two selfadjoint operators $A, B$. Thus the norm of $K(\zeta, z)$ is bounded by $|\operatorname{Im} z|^{-1} \|J\|^2$. On the other hand, the equation

$$-\operatorname{Im} K(\zeta, z) = B[(H(\bar{\zeta}) - \bar{z})^{-1}[(\operatorname{Im} \zeta)W - \operatorname{Im} z](H(\zeta) - z)^{-1}]\, B$$

implies

$$\|K(\zeta, z)\| \leq |\operatorname{Im} \zeta|^{-1} \tag{5.9}$$

Here we used that $W(H(\zeta) - z)^{-1}$ is a bounded operator. Now observe that for all $z \in \mathbb{C}_-$ the function $\zeta \mapsto K(\zeta, z)$ is holomorphic and bounded on $\overline{\mathbb{C}_+}$. The residue theorem, integration over a closed curve in $\mathbb{C}$ and the bounds on $K$ imply

$$\left| \int_{\mathbb{R}} \langle \phi, K(\zeta, z)\phi \rangle \, \frac{d\zeta}{1 + t\zeta^2} \right| = \frac{\pi}{\sqrt{t}} \, \|K\big(i/\sqrt{t}, z\big)\| \tag{5.10}$$

Together with (5.9), this completes the proof. □

REMARK 5.2.2. The lemma shows that for the particular family of operators $H(\zeta)$ in (5.7), where $\zeta$ is a random variable with measure $\frac{d\zeta}{1+t\zeta^2}$, the $\zeta$-averaged resolvents are indeed integrable with respect to the energy. Thus the singularities of the resolvent have been smeared out.

**5.3. Stone's formula and spectral averaging of projections.** Stone's formula allows one to express the spectral projection in terms of the resolvent. This is handy because the resolvent has some nice analytic properties. In our case we use Stone's formula to derive the analog of (5.8) for spectral projections.

A sequence of bounded operators $A_n, n \in \mathbb{N}$ on the Hilbert space $\mathcal{H}$ converges *strongly* (or in *strong topology*) to $A$ if for every $\phi \in \mathcal{H}$

$$\lim_{n \to \infty} \|A\phi - A_n\phi\| = 0$$

LEMMA 5.3.1 (Stone's formula). *Let $H$ be a selfadjoint operator with spectral family denoted by $P(\cdot)$. Then the following limit holds in the strong topology*

$$\lim_{\delta \searrow 0} \frac{1}{2\pi i} \int_{E_1}^{E_2} \left[ (H - E - i\delta)^{-1} - (H - E + i\delta)^{-1} \right] dE = \frac{1}{2} \Big[ P\big([E_1, E_2]\big) + P\big(]E_1, E_2[\big) \Big]$$

PROOF. The function

$$f_\delta(x) := \frac{1}{\pi}\left(\arctan\frac{x-E_1}{\delta} - \arctan\frac{x-E_2}{\delta}\right) \tag{5.11}$$
$$= \frac{1}{2\pi i}\int_{E_1}^{E_2}\left[(x-E-i\delta)^{-1} - (x-E+i\delta)^{-1}\right]dE$$
$$= -\frac{1}{\pi}\,\mathrm{Im}\int_{E_1}^{E_2}(x-E+i\delta)^{-1}dE$$

converges for $\delta \searrow 0$ to

$$\frac{1}{2}\big(\chi_{[E_1,E_2]} + \chi_{]E_1,E_2[}\big).$$

Now one applies the spectral theorem to $f_\delta(H)$. □

More details on Stone's formula can be found in [**258**], or [**326**] where the spectral calculus is actually introduced in this way in Section 7.3.

Now let $H(\zeta)$ be as in the last paragraph and $P(\zeta, I)$ the corresponding spectral projection onto an interval $I$. For a normalised vector $\psi$ in $\mathcal{H}$ denote $\mathcal{P}(\zeta) := \langle\psi, JP(\zeta, I)J\psi\rangle$. The next lemma contains a spectral averaging estimate for $\mathcal{P}$.

LEMMA 5.3.2. *Let $\rho \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$. Then*

$$\int_{\mathbb{R}} \rho(\zeta)\,\mathcal{P}(\zeta)\,d\zeta \le \|\rho\|_\infty |I| \tag{5.12}$$

While Combes and Hislop [**54**] considered only compactly supported $\rho$, it was first observed in [**107**] that densities with non-compact support can be treated. There this extension was necessary to derive estimates for Gaussian random potentials.

PROOF. We first consider the special density $\frac{d\zeta}{1+t\zeta^2}$ and an open interval $I$. By Stone's formula

$$\int_{\mathbb{R}} \frac{d\zeta}{1+t\zeta^2}\,\mathcal{P}(\zeta) \le -\int_{\mathbb{R}} \frac{d\zeta}{1+t\zeta^2}\,\lim_{\delta\to 0}\frac{1}{\pi}\,\mathrm{Im}\int_I dE\langle\psi, K(\zeta, E-i\delta)\psi\rangle \tag{5.13}$$

Note that $\frac{d\zeta}{1+t\zeta^2}$ is a finite Borel measure on $\mathbb{R}$ and that (5.11) implies that $|f_\delta(\cdot)|$, and hence $\|f_\delta(H(\zeta))\|$, is bounded by one. Thus we may apply the dominated convergence theorem to interchange the limit and the integration, and bound (5.13) by

$$\frac{1}{\pi}\lim_{\delta\to 0}\left|\int_I dE\int_{\mathbb{R}}\frac{d\zeta}{1+t\zeta^2}\,\langle\psi, K(\zeta, E-i\delta)\psi\rangle\right| \le |I| \tag{5.14}$$

The last inequality follows from Lemma 5.2.1. This implies for all $\rho \in L^\infty$ with compact support:

$$\begin{aligned}\int_{\mathbb{R}} \rho(\zeta)\,\mathcal{P}(\zeta)\,d\zeta &\le \sup_{\mathrm{supp}\rho}\left[\rho(\zeta)(1+t\zeta^2)\right]\int_{\mathbb{R}}\frac{\mathcal{P}(\zeta)}{1+t\zeta^2}\,d\zeta\\ &\le \sup_{\mathrm{supp}\rho}\left[\rho(\zeta)(1+t\zeta^2)\right]|I|\\ &\to \|\rho\|_\infty |I| \text{ for } t\to 0\end{aligned}$$

Finally, assume only that $\rho \in L^\infty \cap L^1$. Set $\rho^y = \rho\,\chi_{\{x\,|\,|x|<y\}}$ and decompose $\rho = \rho^y + \rho_y$. For $y \to \infty$, $\rho_y$ tends to zero pointwise. Since $\mathcal{P}$ is bounded by one,

$\rho \in L^1(\mathbb{R}, d\zeta)$ is a $y$-uniform majorant for $\rho_y \mathcal{P}$ and we may apply the dominated convergence theorem to conclude

$$\int_{\mathbb{R}} \rho(\zeta)\, \mathcal{P}\,(\zeta)\, d\zeta = \lim_{y\to\infty} \int_{\mathbb{R}} \rho^y(\zeta)\, \mathcal{P}\,(\zeta)\, d\zeta \leq \|\rho\|_\infty\, |I| \tag{5.15}$$

If $I$ is not open, we write it as an intersection of open, decreasing sets and use monotone convergence to conclude (5.12). □

**5.4. Completion of the proof of Theorem 5.0.1.** The results on the localisation of the trace to unit cubes and spectral averaging of projections allow us to assemble the proof of Theorem 5.0.1.

To estimate the operator norm appearing in Proposition 5.1.3 we may as well bound the corresponding quadratic form since

$$\left\| \mathbb{E}\left\{\chi_j P^l_\omega(I)\chi_j\right\}\right\| = \sup_{\|\phi\|=1} \langle \phi, \mathbb{E}\left\{\chi_j P^l_\omega(I)\chi_j\right\}\phi\rangle \tag{5.16}$$

Now one can apply Fubini's Theorem and Lemma 5.3.2 with the choice $\rho = f$, $\mathcal{H} = L^2(\Lambda_l)$, $J = \chi_j$, $H = H_0 + \sum_{k\in\tilde{\Lambda}\setminus j} \omega_k\, u(\cdot - k)$, $\zeta = \omega_j$ and $W = u(x-j)$:

$$\langle \phi, \mathbb{E}\left\{\chi_j P^l_\omega(I)\chi_j\right\}\phi\rangle \leq \|f\|_\infty\, |I|$$

This bound is $j$-independent and thus yields

$$\mathbb{E}\left\{\mathrm{Tr} P^l_\omega(I)\right\} \leq r(E_2 + C_0)^{-1}\, C_{\mathrm{Tr}}(r)\, \|f\|_\infty\, |I|\, |\tilde{\Lambda}_l| \tag{5.17}$$

□

Now it becomes clear why we introduced the operator $r(H^l_\omega + C_0)^{-1} r(H^l_\omega + C_0) = \mathrm{Id}$ in Proposition 5.1.3: without this regularisation of the trace we could have estimated

$$\mathbb{E}\left\{\mathrm{Tr}\chi_j P^l_\omega(I)\chi_j\right\} \leq const.\, |I|\, |\tilde{\Lambda}_l|$$

However, this would lead to a Wegner estimate with quadratic volume bound.

The role played by $r$ resembles the one of the function $g$ in paragraph 4.2 and Appendix A.

**5.5. Single site potentials with changing sign.** In § 4.7 we saw an extension of the Wegner-Kirsch approach to single site potentials of changing sign. The Kotani-Simon-Combes-Hislop proof of Wegner's estimate also allows such a generalisation [**315, 317**], which we present in this section. Its main shortcoming in comparison to the results in § 4.7 is that it is restricted to single site potentials which have a (generalised) step function form. On the other hand, it is valid not only at spectral boundaries, but on the whole energy axis. Furthermore, it yields the existence of the density of states as a function and upper bounds on it.

THEOREM 5.5.1. *Let $L^p_c(\mathbb{R}^d) \ni w \geq \kappa\chi_{[0,1]^d}$ with $\kappa > 0$ and $p(d)$ be as in* (1.1). *Let $\Gamma \subset \mathbb{Z}^d$ be finite, the* convolution vector $\alpha = (\alpha_k)_{k\in\Gamma} \in \mathbb{R}^\Gamma$ *satisfy $\alpha^* := \sum_{k\neq 0} |\alpha_k| < |\alpha_0|$, and the single site potential be of* generalised step function *form:*

$$u(x) = \sum_{k\in\Gamma} \alpha_k\; w(x-k). \tag{5.18}$$

*Assume that the density satisfies $f \in W^{1,1}_c(\mathbb{R})$. Then for all $E \in \mathbb{R}$ there exists a constant $C_W = C_W(E)$ such that*

$$\mathbb{E}\left\{\mathrm{Tr} P^l_\omega(I)\right\} \leq C_W\; |I|\, l^d, \quad \text{for all } l \in \mathbb{N} \text{ and } I \subset ]-\infty, E] \tag{5.19}$$

The theorem implies that the DOS, the derivative of the IDS, exists for a.e. $E$ and is locally uniformly bounded: $dN(E)/dE \leq C(E_1)$ for all $E \leq E_1$.

PROOF. For simplicity we assume $w = \chi_{[0,1]^d}$. To estimate $\mathbb{E}\{\langle \phi, \chi_j P^l_\omega(I)\chi_j\phi\rangle\}$ for any normalised $\phi \in L^2(\Lambda_l)$ we introduce a transformation of coordinates on the probability space $\Omega$.

For each cube $\Lambda = \Lambda_l$ denote $\Lambda^+ := \{\lambda - \gamma |\ \lambda \in \tilde{\Lambda}, \gamma \in \Gamma\}$ and $L = \#\Lambda^+$. The operator $H^l_\omega$ depends only on the truncated random vector $(\omega_k)_{k\in\Lambda^+} \in \mathbb{R}^L$. On such vectors acts a multi-level Toeplitz matrix $A_\Lambda := \{\alpha_{j-k}\}_{j,k\in\Lambda^+}$ induced by the convolution vector $\alpha$. The transformation has an inverse $B_\Lambda = \{b_{k,j}\}_{k,j\in\Lambda^+} = A_\Lambda^{-1}$ which is bounded in the row-sum norm $\|B_\Lambda\| \leq \frac{1}{1-\alpha^*}$. Note that the bound is uniform in $\Lambda_l$. We drop now the subscript $\Lambda$ and denote with $\eta := A\omega$ the vector of the transformed random coordinates. They have the common density

$$k(\eta) = |\det B|\, F(A^{-1}\eta) \tag{5.20}$$

where $F(\omega) = \prod_{k\in\Lambda^+} f(\omega_k)$ is the original density of the $\omega_k$. We calculate the potential $V_\omega$ written as a function of $\eta$ (and $x \in \Lambda$):

$$V_\omega(x) = V_{B\eta}(x) = \sum_{j\in\tilde{\Lambda}} \eta_j \chi_j(x)$$

In the new representation of the potential the single site potentials are non-negative, so we can make use of the spectral averaging formula in Lemma 5.3.2

$$\int_{\mathbb{R}} d\eta_j\, k(\eta)\, s(\eta) \leq |I| \sup_{\eta_j} |k(\eta)|, \qquad \text{where } s(\eta) := \langle \phi, \chi_j P^l_{B\eta}(I)\chi_j\phi\rangle \tag{5.21}$$

Fubini, (5.21), and the fundamental theorem of calculus give

$$\int_{\mathbb{R}^L} d\eta\, k(\eta)\, s(\eta) \leq |I| \int_{\mathbb{R}^{L-1}} d\eta^{\perp j} \sup_{\eta_j} |k(\eta)| \leq |I| \int_{\mathbb{R}^L} d\eta\, |(\partial_j k)(\eta)| \tag{5.22}$$

Here $\eta^{\perp j}$ is an abbreviation for $\{\eta_k |\ k \in \Lambda^+ \setminus j\}$. The last integral equals

$$|\det A| \int_{\mathbb{R}^L} d\omega\, |(\partial_j k)(A\omega)|$$

which is bounded by $\|f'\|_{L^1} \sum_{k\in\Lambda^+} |b_{k,j}|$. The proof of the theorem is finished by the estimate

$$\mathbb{E}\left\{\langle \phi, \chi_j P^l_\omega(I)\chi_j\phi\rangle\right\} \leq |I|\ \|f'\|_{L^1}\|B\| \tag{5.23}$$

and Proposition 5.1.3. □

One drawback of Theorem 5.5.1 is the requirement of the weak differentiability of $f$. This excludes in particular the uniform distribution on an interval. However, in a joint work [**191**] with Kostrykin we have proven:

PROPOSITION 5.5.2. *Let the assumptions of Theorem 5.5.1 be satisfied with the only difference that $f$ is the uniform density on an interval. Let $\Gamma \subset \{k \in \mathbb{Z}^d |\ k_i \geq 0\ \forall\ i = 1, \ldots, d\}$. Then* (5.19) *holds true.*

In fact, it turns out that the proof of Theorem 5.5.1 can be extended to density functions of finite total variation. This covers in particular linear combinations of functions in $W^{1,1}_c$ and (finite) step functions. More precisely:

PROPOSITION 5.5.3. *Let the assumptions of Theorem 5.5.1 be satisfied, but require $f$ merely to have finite total variation $\|f\|_{\mathrm{Var}} < \infty$. Then the Wegner estimate* (5.19) *holds.*

The difference to Theorem 5.5.1 is that the constant $C_W$ now depends on $\|f\|_{\mathrm{Var}}$ instead of $\|f'\|_{L^1}$, cf. (5.23). The result in the Proposition 5.5.3 is proven in [**190**]. Moreover, there we discuss how the condition $\sum_{k\neq 0} |\alpha_k| < |\alpha_0|$ can be relaxed using the theory of Toeplitz matrices. Let us give an example in the one dimensional case.

The *symbol* of the Toeplitz matrix $A$ is the function

$$s_A \colon \mathbb{T} \to \mathbb{C}, \quad s_A(e^{i\theta}) = \sum_{j\in\mathbb{Z}} \alpha_j \, e^{i\, j\theta}, \quad \theta \in ]-\pi, \pi]$$

Since we assume that only finitely many components of $\alpha$ are different from zero, $s_A$ is actually a trigonometric polynomial, and thus uniformly continuous and bounded. Invertibility criteria for $A_\Lambda$ as well as bounds on $B_\Lambda = A_\Lambda^{-1}$ and $B = A^{-1}$ may be established by studying the symbol $s_A$. Consider the case that the symbol $s_A$ has no zeros and the winding number of $s_A$ with respect to $0 \in \mathbb{C}$ vanishes. A theorem of Baxter [**24**], see also [**125**, Thm. III.2.1], states that

$$\sup_{l\in\mathbb{N}} (\|B\|, \|B_{\Lambda_l}\|) \leq const. < \infty \tag{5.24}$$

In this case we have again:

PROPOSITION 5.5.4. *Let $d = 1$ and the assumptions of Theorem 5.5.1 be satisfied, but require for the convolution vector $\alpha = (\alpha_k)_{k\in\Gamma}$ merely that the symbol $s_A$ of the associated Toeplitz matrix has no zeros. Then the Wegner estimate* (5.19) *holds.*

REMARK 5.5.5 (Anderson model). For the discrete Anderson model $h_\omega = h_0 + V_\omega$ there is a result analogous to Theorem 5.5.1. Here $h_0$ is the finite difference Laplacian on $l^2(\mathbb{Z}^d)$ and $(V_\omega\psi)(n) = V_\omega(n)\psi(n)$, $\forall\, n \in \mathbb{Z}^d$, a multiplication operator as in the continuum case. This is not surprising, since the arguments in § 5.2 and §5.3 rely only on abstract functional analysis. If fact, as we mentioned earlier, Kotani and Simon were motivated in their treatment [**193**] of the alloy type model by its discrete counterpart. Moreover, since on $l^2(\mathbb{Z}^d)$ the trace can be expressed using the canonical basis as

$$\mathrm{Tr}[P_\omega^l(I)] = \sum_{j\in\tilde{\Lambda}_l} \langle \delta_j, P_\omega^l(I)\delta_j\rangle$$

the use of a trace regularising function is not necessary. Here $P_\omega^l$ denotes the spectral projection of the truncation $h_\omega^l$ of the Anderson model $h_\omega$. More precisely, $h_\omega^l$ is the finite matrix $\{\langle \delta_j, h_\omega \delta_k\rangle\}_{j,k\in\tilde{\Lambda}_l}$.

Note that in the discrete case $\chi_j$ is just $\delta_j$. Under the assumptions of Theorem 5.5.1 on the coupling constants $\{\omega_j\}_j$ and the single site potential $u$ we have the following Wegner estimate for the Anderson model:

$$\mathbb{E}\left\{\mathrm{Tr}[P_\omega^l(I)\right\} \leq \frac{\|f'\|_{L^1}}{1-\alpha^*} \; |I| \; |\tilde{\Lambda}_l| \tag{5.25}$$

REMARK 5.5.6. Theorem 5.5.1 can also be understood as a Wegner estimate for the alloy type potential

$$V_\eta(x) = \sum_{k\in\mathbb{Z}^d} \eta_k \, \chi_k$$

where the coupling constants $\{\eta_j\}_j$ are not any more independent, but correlated satisfying certain conditions. See § 4.2 in [**318**] for a precise formulation. Wegner estimates for correlated coupling constants can also be found in [**59**] (cf. [**141**], too).

The use of the common density $F$, respectively $k$, in the proof of Theorem 5.5.1 is conceptually new. One could try to use conditional densities instead by considering the indefinite potential $V_\omega$ in its representation $V_{B\eta}$ as an alloy type potential with dependent coupling constants. However, this would require to have uniform upper bounds on the conditional densities, cf. [**59, 141**]. They do not seem to be easy to establish for the model considered in this paragraph, and in fact sometimes fail to hold as can be seen in the following example.

EXAMPLE 5.5.7. It is sufficient to consider only one space dimension $d = 1$. Let the density function be $f = \chi_{[0,1]}$ and the single site potential $u = \chi_{[0,1]} - \alpha\chi_{[1,2]}$ with $-\alpha \in ]-1, 0[$. To this model the results of Propositions 5.5.2 and 5.5.3 apply.

The restriction of $H_\omega$ to the interval $]-1/2, l-1/2[$ of length $l$ depends only on the coupling constants $\omega_j$ with indices $j \in \{-1, \ldots, l-1\} =: \Lambda^+$. They are transformed by the Toeplitz matrix $A$ into new random variables $\eta_j, j \in \{-1, \ldots, l-1\}$, as in the proof of Theorem 5.5.1. Here the convolution vector is given by $\alpha_0 = 1, \alpha_1 = -\alpha$.

The conditional density $\rho_j(\eta) = \rho_j^l(\eta)$ of the variable $\eta_j$ with respect to the remaining coupling constants $\eta^{\perp j} = (\eta_k)_{k\in\Lambda^+\setminus j}$ in $\Lambda^+$ is given by $\rho_j(\eta) = \frac{k(\eta)}{g_j(\eta)}$. Here $g_j(\eta) = \int k(\eta) d\eta_j$ denotes the marginal density. The question is whether $\sup_j \rho(\eta)$ is finite.

One calculates the common density to be $k(\eta) = \prod_{k=-1}^{l-1} \chi_{[0,1]}(\sum_{\nu=-1}^{k} \alpha^{k-\nu}\eta_\nu)$. For $\eta_{j+1} \in [0,1], \eta_k = 0, \forall k \neq j+1$ we have

$$k(\eta) = \prod_{k=j+1}^{l-1} \chi_{[0,1]}(\alpha^{k-j-1}\eta_\nu) = 1.$$

The marginal density

$$\begin{aligned} g_j(\eta) &= \prod_{k=-1}^{j-1} \chi_{[0,1]}\left(\sum_{\nu=-1}^{k} \alpha^{k-\nu}\eta_\nu\right) \int \prod_{k=j}^{l-1} \chi_{[0,1]}\left(\sum_{\nu=-1}^{k} \alpha^{k-\nu}\eta_\nu\right) d\eta_j \\ &\leq \int d\eta_j \prod_{k=j}^{j+1} \chi_{[0,1]}\left(\sum_{\nu=-1}^{k} \alpha^{k-\nu}\eta_\nu\right) \end{aligned}$$

has for $\eta_{j+1} \in [0,1], \eta_k = 0, \forall k \notin \{j, j+1\}$ the upper bound

$$\int_0^1 \chi_{[0,1]}(\alpha\eta_j + \eta_{j+1}) d\eta_j \leq \alpha^{-1}(1 - \eta_{j+1})$$

Particularly, $g_j(\eta) \searrow 0$ for $\eta_{j+1} \nearrow 1$ and thus

$$\sup_\eta \rho_j(\eta) = \infty$$

Therefore, proofs of a Wegner estimate which require the conditional density to be bounded cannot be applied to this alloy type potential. See also § 4.3 of [**318**] for another example.

**5.6. Unbounded coupling constants and magnetic fields.** Motivated by certain physical applications, e.g. the study of the quantum hall effect (see for instance [**26, 234, 111, 271, 153, 92**]), it is desirable to extend the results on the continuity of the IDS to include Hamiltonians with magnetic fields. This is, for instance, done in the papers [**60, 134, 141**].

We discuss here the results on alloy type potentials obtained in [**141**] by Hupfer, Leschke, Müller, and Warzel, since they are build on the method presented in the preceding §§ 5.1–5.4. Moreover, their result allows the coupling constants to be unbounded, as long as very negative fluctuations are exponentially rare. Actually, the primary interest of their research are Hamiltonians with Gaussian random potentials, so they need to cope with unbounded fluctuations of the potential. The proof is based on earlier techniques from [**107**] — which in turn use [**54**] — and Dirichlet-Neumann bracketing for magnetic Schrödinger operators, as discussed in Appendix A of [**141**]. The results concerning alloy type potentials are summarised in § 4.1 of their paper, which we review shortly.

Let $A\colon \mathbb{R}^d \to \mathbb{R}^d$ be a measurable vector potential with the property $|A|^2 \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Denote with $H_0$ the selfadjoint closure of $\sum_{i=j}^d (i\partial_j + A_j)^2$ defined on smooth functions with compact support. The alloy type Schrödinger operator $H_\omega = H_0 + V_\omega$ now incorporates a magnetic field. In [**141**] it is proven that Theorem 5.0.1 essentially remains true if the magnetic field is included. (Their conditions on the single site potential are slightly different.)

Moreover, two cases are discussed, where the coupling constants are unbounded random variables, and the theorem still remains true. In the first one, it is assumed that $\omega_0$ is non-negative and a certain moment condition is satisfied, roughly $\mathbb{E}\{\omega_0^{2d+2}\} < \infty$.

The second one concerns the case where $\omega_0$ is distributed according to the Laplace distribution: $\mathbb{P}\{\omega_0 \in I\} = \frac{1}{a}\int_I dx\, e^{|x|/a}$. Note that the probability that $\omega_0$ assumes very negative values is exponentially small. In fact, in [**141**] it is noted, that this is a necessary requirement for their techniques to work. The reason for this is that they use $r(x) = e^{-tx}$ as the trace regularising function, cf. Definition 5.1.1. A different choice of $r$ would allow for more general distributions unbounded from below.

We conclude this section by listing further literature on random Hamiltonians with magnetic fields. Works treating the regularity of the IDS of random Schrödinger operators with magnetic field include [**321, 22, 323, 145, 134**], while the question of the (in)dependence of the IDS on boundary conditions for these models has been treated in [**232, 82, 141, 139**]. A related problem is the analysis of the semigroup kernels of magnetic operators [**38, 39**]. In [**166, 253**] the behaviour of the IDS in a strong magnetic field is identified.

The asymptotic behaviour of the IDS near the boundaries of the spectrum in the presence of random magnetic fields was the object of study of the articles [**230, 231, 233**] which prove high energy and Lifshitz asymptotics for certain models. The high energy asymptotics has been analysed already in [**221, 312**].

For Schrödinger operators with constant magnetic field and random potential generated by a Poissonian process the different possible behaviours of the IDS at the bottom of the spectrum are analysed in [**37, 98, 143, 144, 99, 324, 138**].

The analysis of Landau Hamiltonians in the single band approximation is done in [**84, 85, 246, 247, 145**]. Examples of localisation proofs which allow for magnetic fields can be found in [**84, 55, 85, 86, 322, 22, 118, 87, 109, 247, 6**].

## Appendix A. Properties of the spectral shift function

For a exposition of the theory of the spectral shift function (SSF) see [**33, 282**] or the last chapter of [**328**]. We review here the relevant facts in our context.

For two trace class operators $A, B$ the SSF $\xi(\cdot, A, B)$ may be defined by the formula

$$\mathrm{Tr}(f(A) - f(B)) = \int f'(\lambda)\, \xi(\lambda, A, B)\, d\lambda \tag{A.1}$$

for functions $f \in C_c^\infty$. Actually, it holds for more general functions, too. By Theorem 8.3.3 in [**328**], it is sufficient to assume $f \in C^1(\mathbb{R})$ and that $f'$ is the Fourier transform of a finite complex measure. One can define the SSF also via the perturbation determinant from scattering theory

$$\xi(\lambda, A, B) := \frac{1}{\pi} \lim_{\epsilon \searrow 0} \arg\det[1 + (A - B)(B - \lambda - i\epsilon)^{-1})]$$

In this case the equality (A.1) is called *Krein trace formula*. The SSF can be bounded in terms of the properties of $A - B$, namely

$$\|\xi(\lambda, A, B)\|_1 \leq \|A - B\|_{J_1} \tag{A.2}$$

Here $J_1$ denotes the ideal of trace class operators and $\|\cdot\|_{J_1}$ the *trace norm*. On the other hand, if $A - B$ is finite rank

$$\|\xi(\lambda, A, B)\|_\infty \leq \mathrm{rank}(A - B) \tag{A.3}$$

Since we have an estimate on $\xi$ in the $L^1$ and $L^\infty$-norms, it is natural to ask whether an estimate for the $L^p$-norm, $p \in ]1, \infty[$, may be derived. This indeed turns out to be true and can be understood as an interpolation result, cf. the proof of Theorem 2.1 in [**60**].

To formulate this bound we have to introduce ideals of 'better than trace class' operators. For a compact operator $C$ denote by $\mu_n(C), n \in \mathbb{N}$ its singular values, in non-increasing order. If $C$ is trace class, the sum of the singular values is finite and equals $\|C\|_{J_1}$. We denote by $J_\beta$ the class of compact operators such that

$$\|C\|_{J_\beta} := \Big( \sum_{n \in \mathbb{N}} \mu_n(C)^\beta \Big)^{1/\beta} < \infty \tag{A.4}$$

The theory of such operators is classical for $\beta \geq 1$. However, since we want to interpolate between (A.2) and (A.3), we need to consider operators whose singular values converge not slower, but *faster* than a $l^1$-sequence to zero. This leads us to consider operators such that $\|C\|_{J_\beta}$ is finite, for $\beta$ *smaller than one*. In particular, all such operators are trace class, which explains why they are sometimes called *super-trace class*. It follows that the SSF may be defined for such operators. They have been studied in [**126, 30, 31**], while their relevance in the present context was recognised in [**60**].

Form these sources we infer the following properties of $J_\beta$. Since for any compact operator $A$ and bounded $B$ the singular values of the products obey

$$\mu_n(AB) \leq \|B\|\,\mu_n(A) \quad \text{and} \quad \mu_n(BA) \leq \|B\|\,\mu_n(A) \tag{A.5}$$

the set $J_\beta$ is an two-sided ideal in the algebra of bounded operators for all $\beta > 0$. For $\beta \geq 1$ the functional $A \mapsto \|A\|_{J_\beta}$ is a norm, which is not true for $\beta < 1$. More precisely, in this case we have only

$$\|A+B\|_{J_\beta}^\beta \leq \|A\|_{J_\beta}^\beta + \|B\|_{J_\beta}^\beta$$

This property implies that $\|\cdot\|_{J_\beta}$ is a *quasi-norm* and that

$$\mathrm{dist}_\beta(A,B) = \|A-B\|_{J_\beta}^\beta$$

is a well defined metric on $J_\beta$. The pair $(J_\beta, \mathrm{dist}_\beta)$ forms a complete, separable linear metric space, in which the finite rank operators form a dense subset.

In [**60**] the following $L^p$-bound on the SSF was proven.

THEOREM A.1. *Let $p \geq 1$ and $A, B$ be selfadjoint operators whose difference is in $J_\beta$ where $\beta = 1/p$. Then the spectral shift function $\xi(\cdot, A, B)$ is in $L^p(\mathbb{R})$ and*

$$\|\xi(\cdot, A, B)\|_{L^p} \leq \big\|A-B\big\|_{J_\beta}^\beta \tag{A.6}$$

This estimate is sufficient for our purposes. There exists a sharp version proven by Hundertmark and Simon in [**140**]. It is used in the result described in Remark 4.2.3.

THEOREM A.2. *Let $F\colon [0,\infty[ \to [0,\infty[$ be a convex function such that $F(0) = 0$. Let $A, B$ be bounded and $C$ a non-negative compact operator such that for all $N \in \mathbb{N}$*

$$\sum_{n=N}^\infty \mu_n(|A-B|) \leq \sum_{n=N}^\infty \mu_n(C) \tag{A.7}$$

*Then*

$$\int F\big(|\xi(\lambda, A, B)|\big)d\lambda \;\leq\; \int F\big(|\xi(\lambda, C, 0)|\big)d\lambda \;=\; \sum_{n\in\mathbb{N}} \big[F(n) - F(n-1)\big]\,\mu_n(C)$$

Condition (A.7) is in particular satisfied if $|A-B| \leq C$. Of course, to apply Theorem A.1, we need a criterion for the operators which arise in our situation to be in $J_\beta$ for $\beta = 1/p \leq 1$. So, let's have a closer look at the application of the theory of the SSF to Schrödinger operators.

Since we are studying Schrödinger operators, we cannot expect to deal with trace class perturbations. However, the theory extends to operator pairs such that the difference of a sufficiently high power of their resolvents is trace class. More precisely, assume that $H+u, H$ is a pair of lower bounded operators such that

$$H + u \geq C_0, H \geq C_0 \text{ and } g(H+u) - g(H) \in J_1 \tag{A.8}$$

where $g(x) = (x - C_0 + 1)^{-k}$ for some sufficiently large $k > 0$. Following [**60**], we denote $g(H+u) - g(H)$ by $V_{\text{eff}}$. This is the 'effective' perturbation, although it is obviously not a multiplication operator. One defines the SSF of the pair $H+u, H$ as

$$\xi(\lambda, H+u, H) := -\xi\big(g(\lambda), g(H+u), g(H)\big) \quad \text{for } \lambda \geq 0 \tag{A.9}$$

and $\xi = 0$ otherwise. This definition of $\xi$ is independent of the choice of the exponent $k > 0$ in $g$. By Theorem 8.9.1 in [**328**] the trace formula (A.1) holds if $f \in C^2(\mathbb{R})$ and $f'$ has compact support. This conditions are clearly satisfied by the switch function $\rho$ we use in § 4.1 and § 4.2.

The purpose of the theorem we are heading to now is twofold: firstly, to establish that $V_{\text{eff}}$ is in $J_\beta$ for suitable $\beta < 1$. Thus, we will be able to apply Theorem A.1. Secondly, to control the upper bound $\|V_{\text{eff}}\|_{J_\beta}^\beta$ appearing in (A.6).

It is well known that operators which may be formally written as $f(x)g(-i\nabla)$ are in the Hilbert-Schmidt class if $f, g \in L^2(\mathbb{R}^d)$. The product of such two operators is trace class. Extending this idea, we want to show for certain operators that they are in some (super-trace) ideal $J_\beta$, $\beta < 1$ by writing them as a product of sufficiently many operators of the type $f(x)g(-i\nabla)$. For this purpose it is useful to note that the Hölder inequality extends also to the case of exponents smaller than one: let $a_i \colon \mathbb{N} \to \mathbb{C}$, $i = 1, \dots, N$ be such that $|a_i(n)|^{p_i}$ is summable, where $p_i > 0$ for all $i = 1, \dots, N$, and set $\frac{1}{r} := \sum_{i=1}^N \frac{1}{p_i}$. Then the pointwise product $\prod_{i=1}^N a_i$ is in $l^r(\mathbb{N})$ and

$$\Big\| \prod_{i=1}^N a_i \Big\|_r \leq \prod_{i=1}^N \|a_i\|_{p_i}$$

By applying this to the sequence of singular values of compact operators, we obtain the following

Lemma A.3. *Let $A_i \in J_{p_i}$ for $i = 1, \dots, N$, then $\prod_{i=1}^N A_i$ is in $J_r$ where $\frac{1}{r} := \sum_{i=1}^N \frac{1}{p_i}$ and*

$$\Big\| \prod_{i=1}^N A_i \Big\|_{J_r} \leq \prod_{i=1}^N \|A_i\|_{J_{p_i}} \tag{A.10}$$

See also [**31**] Corollary 11.11. The following result is taken from [**232**], cf. Lemma 11 and its proof. For $l \in \mathbb{N}$ we abbreviate $\Lambda = \Lambda_l$.

Lemma A.4. *Let $q > d/2, q \in 2\mathbb{N}$, $f \in L_c^\infty(\mathbb{R}^d)$ and $C_0 \in \mathbb{R}$ be such that $V_{\text{per}} + V_\omega \geq C_0$ for all $\omega$. Then the operator product $f\,(H_\omega^l - C_0 + 1)^{-1}$ is in the ideal $J_q$ and*

$$\|f\chi_\Lambda\,(H_\omega^l - C_0 + 1)^{-1}\|_{J_q} \leq \|f\,(-\Delta + 1)^{-1}\|_{J_q} \leq C(q)\ \|f\|_q$$

Proof. There exist a bounded extension operator $\mathcal{E} \colon W^{2,2}(\Lambda_l) \to W^{2,2}(\mathbb{R}^d)$ and its norm is independent of $l \in \mathbb{N}$, cf. Section IV.3.2 in [**292**]. Thus we have

$$f\chi_\Lambda (H_\omega^l - C_0 + 1)^{-1} = f\chi_\Lambda (H_\omega - C_0 + 1)^{-1}(H_\omega - C_0 + 1)\mathcal{E}(H_\omega^l - C_0 + 1)^{-1}$$

Since $\mathcal{E}$ is an extension operator, $\chi_\Lambda\,\mathcal{E}$ is the identity on $W^{2,2}(\Lambda_l)$. By the ideal property of $J_q$ and the boundedness of $\mathcal{E}$ we have

$$\begin{aligned}\|f\chi_\Lambda\,(H_\omega^l - C_0 + 1)^{-1}\|_{J_q} &\\ \leq \|f(H_\omega - C_0 + 1)^{-1}\|_{J_q}\, &\|(H_\omega - C_0 + 1)\mathcal{E}(H_\omega^l - C_0 + 1)^{-1}\| \\ &\leq \mathit{const.}\, \|f(H_\omega - C_0 + 1)^{-1}\|_{J_q}\end{aligned}$$

By the *Kato-Simon* [**151, 277**] or *diamagnetic inequality* we have

$$|f(H_\omega - C_0 + 1)^{-1}\psi| \leq f(-\Delta + 1)^{-1}|\psi| \tag{A.11}$$

for $\psi \in L^2(\mathbb{R}^d)$, cf. proof of Theorem 3.3 in [**139**]. For the 'free' case we know by Theorem 4.1 in [**278**] $f\,(-\Delta+1)^{-1} \in J_q$ and

$$\|f\,(-\Delta+1)^{-1}\|_{J_q} \leq \|f\|_q \|h\|_q$$

where $h(x) = (x^2+1)^{-1}$. The pointwise inequality (A.11) implies by Theorem 2.13 in [**278**] $f\,(H_\omega - C_0 + 1)^{-1} \in J_q$ and

$$\|f\,(H_\omega - C_0 + 1)^{-1}\|_{J_q} \leq \|f\,(-\Delta+1)^{-1}\|_{J_q}$$

□

Remark A.5. The result remains true if we consider Neumann boundary conditions instead of Dirichlet ones, and if we include a bounded magnetic vector potential in the background operator $H_0$, see Lemma 10 in [**232**]. This fact relies on the existence of an appropriate extension operator which takes (magnetic) Sobolev functions on $\Lambda$ to (magnetic) Sobolev functions on $\mathbb{R}^d$.

The following result establishes that $V_{\text{eff}} = g(H^l_\omega + \omega_+ u) - g(H^l_\omega + \omega_- u)$ is indeed super-trace class and that its quasi-norm $\|V_{\text{eff}}\|_{J_p}$ can be bounded independently of the cube $\Lambda$.

Theorem A.6. *Let $H_1 = -\Delta + V$ and $H_2 = H_1 + u$, with $\frac{1}{2}C_0 \leq V, u \in L^p_{\text{loc,unif}}(\mathbb{R}^d)$, where $p$ is as in* (1.1)*. Denote by $H^l_1, H^l_2$ the corresponding Dirichlet restrictions to the cube $\Lambda_l$. Assume*

$$\|V\|_{p,\,\text{unif,loc}} \leq C_1, \qquad \text{and} \qquad \|u\|_{p,\,\text{unif,loc}} \leq C_2$$

*and that the support of $u$ is contained in the ball $B_R(x)$. For any $\beta > 0$ choose $k \in \mathbb{N}$ with $k > \frac{d+4}{2\beta}$. For $g(x) = (x - C_0 + 1)^{-k}$ set $V_{\text{eff}} := g(H_2) - g(H_1)$.*

*Then $V_{\text{eff}} \in J_\beta$ and $\|V_{\text{eff}}\|_{J_\beta}$ is bounded by a constant which is independent of $\Lambda_l$ and $x$ and depends on $V$ and $u$ only trough $C_0, C_1, C_2$ and $R$.*

Proof. Choose a function $f \in C^\infty_c(B_{2R}(x))$ such that $f \equiv 1$ on $B_R(x)$. An iteration of the resolvent formula yields

$$\begin{aligned} V_{\text{eff}} &= -\sum_{m=0}^{k-1} (H_2 - C_0 + 1)^{-(k-m)}\, u\, (H_1 - C_0 + 1)^{-(m+1)} \\ &= -\sum_{m=0}^{k-1} \left[f^{k-m}\,(H_2 - C_0 + 1)^{-(k-m)}\right]^*\, u\, \left[f^{m+1}\,(H_1 - C_0 + 1)^{-(m+1)}\right] \end{aligned} \tag{A.12}$$

By Lemma 9 and Appendix A in [**232**] we have the following representation

$$f^\nu\,(H_1 - C_0 + 1)^{-\nu} = \sum_{i=1}^{N} \prod_{j=1}^{\nu} f_{ij}\,(H_1 - C_0 + 1)^{-1} B_{ij}$$

and analogously for $H_2$. Here $N = N(\nu)$ is an integer which depends only on $\nu$, the functions $f_{ij} \in C^\infty_c(B_{2R}(x))$ are linear combinations of derivatives of $f$, and $B_{ij}$ are bounded operators with norms independent of $l$. Set

$$C_{q,f,B} = C(q)\, \max\{\|B_{ij}\|\, \|f_{ij}\|_{L^q} \,|\, i = 1, \ldots, N, j = 1, \ldots, \nu\}$$

where $q \in ]\frac{d}{2}, \frac{d}{2}+2]$ and $C(q)$ are as in Lemma A.4. By the same lemma, the ideal property of $J_\beta$ and the Hölder inequality (A.10) we have

(A.13)

$$\|f^\nu\,(H_1 - C_0 + 1)^{-\nu}\|_{J_t}^t \leq \sum_{i=1}^{N} \prod_{j=1}^{\nu} \|f_{ij}\,(H_1 - C_0 + 1)^{-1}\|_{J_q}^t \,\|B_{ij}\|^t \leq N\, C_{q,f,B}^q$$

if $t = q/\nu < 1$. For $t \geq 1$, $\|\cdot\|_{J_t}$ is even a norm and a similar estimate holds. Since $u$ is infinitesimally bounded with respect to the Laplacian

$$C_u := \max_{i,j} \left\{ \|u f_{ij}\,(H_1 - C_0 + 1)^{-1} B_{ij}\| \right\}$$

is finite. Thus, in analogy to (A.13)

$$\|u f^{\nu+1}\,(H_1 - C_0 + 1)^{-\nu+1}\|_{J_t}^t \leq N\, C_u^t\, C_{q,f,B}^q$$

From the preceding we see that (A.12) factorises $V_{\text{eff}}$ as a product of bounded operators and $k$ operators which are in $J_q$. All the involved operator and super-trace class norms can be bounded independently of $\Lambda_l$ and $x$. Using the ideal property and Hölder's inequality we see that $V_{\text{eff}}$ is in $J_r$ for all $r \geq q/k$ and $\|V_{\text{eff}}\|_{J_r}$ is bounded by a constant which is independent of $\Lambda_l$ and $x$. The way we choose $k$ and $q$ makes it possible to take $r = \beta$. □

Remark A.7 (Properties and relevance of the SSF). An exposition of the role played by the SSF in scattering theory can be found in [**328**]. The SSF has proven useful in the study of random operators, particularly in problems related to surface models, e.g. the definition of the density of surface states [**48, 49, 188, 189**].

Various of its properties are discussed in the literature: monotonicity and concavity [**117, 121, 187**], the asymptotic behaviour in the large coupling constant [**251, 266, 252**] and semiclassical limit [**229**], and some other bounds [**249, 250, 248**].

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