Scattering of TM waves by an impedance cylinder immersed halfway between two half spaces

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Scattering of TM waves by an impedance cylinder halfway immersed between two half spaces of different electromagnetic properties has been studied. Solutions are obtained from an application of a discrete index of Hankel function transform. Expressions for the fields in both half spaces are given.

Introduction: To investigate the features of various media by means of electromagnetic radiation it is necessary to know the field scattered by inhomogeneties of these media. This problem can be tackled by using as a basis a rigorous solution of a basic structure. One of the basic structures is the one considered in this letter. The problem under consideration has also acquired practical relevance in fields of optical engineering such as the study of contaminated surfaces and the detection of defects. Additionally the solution of canonical problems such as the one under consideration is important in the sense of scattering and diffraction theories. The aim of this letter is to present solutions, in terms of a discrete index of Hankel function transform, to the problem of the scattering of TM waves by a circular impedance cylinder immersed halfway between two half spaces of different electromagnetic properties. Other configurations of cylinder and two half spaces have been dealt with before in the literature (references [1,2] using Fourier series expansions on the angular variable and reference [3] using integral equations methods) but the one discussed in this letter, to the best of the author's knowledge, has not been addressed before.

Formulation: We look into the problem of scattering by an impedance cylinder of surface impedance Z_s , radius a with half of which in medium one and the other half in medium two. The interface between the two media is the $y = 0$ plane. The axis of the cylinder is the z axis of the coordinate system. The cylinder is infinite in the z direction. k_1, ϵ_1, and μ_1 are respectively the wave number, permittivity, and permeability in medium one. k_2, ϵ_2 and μ_2 are corresponding quantities in medium two.

Continuity of tangential field components apply at the interface between the two media and impedance boundary conditions apply on the cylinder surface.

A time harmonic infinite electric line current source (in the z direction located at (ρ', ϕ') in medium one provides the illumination. Time dependence $e^{-i\omega t}$ is assumed and suppressed from the analysis.

It should be pointed out that other forms of excitation are admitted. To pass from a result derived from a *unit* strength line source located at (ρ', ϕ') to the result for a *unit* amplitude plane wave incident along the direction ϕ', one first lets $\rho' \to \infty$ in the fields' expressions then sets $[\frac{1}{4}\sqrt{\frac{2}{\pi k_1 \rho'}}\ e^{i(k_1\rho' + \frac{\pi}{4})} = 1]$. The fields due to beam excitation are derived from those of plane wave excitation by attaching a profile to the incident plane wave and making use of the superposition principle.

By using the symmetry of the problem structure with respect to the plane $\varphi = \pm\pi/2$, we split the problem into two independent subproblems. The boundary conditions on the symmetry plane correspond to either a perfect electric conductor (PEC) or a perfect magnetic conductor (PMC). So without loss of generality, we confine our attention to the case of PEC wall only.

Mathematical model: We propose to solve the problem by means of a discrete index of Hankel function transform.

The transform pair is given by [4]

$f(\rho) = \sum_p A_p \Phi_p(k\rho)$

$A_p = \int_a^\infty \frac{1}{\rho} f(\rho)\Phi_p(k\rho)d\rho$

$\Phi_p(k\rho) = \{-\pi i \frac{\nu_p b(\nu_p)}{[\frac{\partial}{\partial\nu} d(\nu)]_{\nu_p}}\}^{1/2} H^{(1)}_{\nu_p}(k\rho)$

$b(\nu) = J'_\nu(ka) + i\overline{C} J_\nu(ka)$

$d(\nu) = H'^{(1)}_\nu(ka) + i\overline{C} H^{(1)}_\nu(ka)$

$J_\nu(z)$ is Bessel function; $J'_\nu(z) = \frac{d}{dz}J_\nu(z)$; $H^{(1)}_\nu(z)$ is Hankel function of type one; $H'^{(1)}_\nu(z) = \frac{d}{dz}H^{(1)}_\nu(z)$; $\overline{C} = Z/Z_s$; $d(\nu_p) = 0$ for $\{\nu_p\}$ located in the first quadrant of the complex ν plane; and $Z = \sqrt{\mu/\epsilon}$ the medium impedance. Passivity requirement is met if Re $\overline{C} \succeq 0$.

Such an index transform has been used before [4] to analyze diffraction by an impedance cylinder in free space.

We represent the electric field in the z direction in medium one, $E^{(1)}_z$, as the sum over $E^{(d)}_z$ which accounts for the source discontinuity (in the ϕ direction) and an additional field $E^{(r)}_z$ which accounts for the rest of the field. In medium two we represent the electric field in the z direction as $E^{(2)}_z$.

$$E^{(d)}_z = \sum_p [\frac{\sin\nu_{1p}(\frac{\pi}{2} - \phi_>)\cos\nu_{1p}\phi_<}{2\nu_{1p}\cos\nu_{1p}\frac{\pi}{2}}\Phi_p(k_1\rho')]\Phi_p(k_1\rho) \tag{1}$$

$$E^{(r)}_z = \sum_p A_1(\nu_{1p})\sin\nu_{1p}(\phi - \frac{\pi}{2})\Phi_p(k_1\rho) \tag{2}$$

$$E_z^{(2)} = \sum_p A_2(\nu_{2p}) \sin \nu_{2p}(\phi + \frac{\pi}{2}) \Phi_p(k_2\rho) \tag{3}$$

From E_z the rest of the field components are derived

$H_\rho = \frac{-i\omega\epsilon}{k^2} \frac{1}{\rho} \frac{\partial E_z}{\partial \phi}$

$H_\phi = \frac{i\omega\epsilon}{k^2} \frac{\partial E_z}{\partial \rho}$

The continuity of the tangential fields on the interface between the two media leads to

$E_z^{(d)} + E_z^{(r)} = E_z^{(2)}$ at $\phi = 0$

$\epsilon_1 (\frac{\partial E_z^{(d)}}{\partial \phi} + \frac{\partial E_z^{(r)}}{\partial \phi}) = \epsilon_2 \frac{\partial E_z^{(2)}}{\partial \phi}$ at $\phi = 0$

$\sum_p \{ \frac{\sin \nu_{1p}(\frac{\pi}{2} - \phi')}{2\nu_{1p} \cos \nu_{1p} \frac{\pi}{2}} \Phi_p(k_1\rho') - A_1(\nu_{1p}) \sin \nu_{1p} \frac{\pi}{2} \} \ \Phi_p(k_1\rho) = \sum_p A_2(\nu_{2p}) \sin \nu_{2p} \frac{\pi}{2}$ $\Phi_p(k_2\rho)$

$\sum_p \nu_{1p} A_1(\nu_{1p}) \cos \nu_{1p} \frac{\pi}{2} \ \Phi_p(k_1\rho) = r \sum_p \nu_{2p} A_2(\nu_{2p}) \cos \nu_{2p} \frac{\pi}{2} \ \Phi_p(k_2\rho)$

with $r = \frac{\epsilon_2}{\epsilon_1}$

The impedance boundary condition $E_z = Z_s H_\phi$ on the surface of the cylinder is built in the eigen functions $\Phi_p(k_{1,2}\rho)$. This is one advantage of using the discrete index of Hankel transform for problems with boundaries along $\rho =$ constant.

We utilize the orthogonality relation of the ρ eigen functions to reach

$\frac{\sin \nu_{1q}(\frac{\pi}{2} - \phi')}{2\nu_{1q} \cos \nu_{1q} \frac{\pi}{2}} \Phi_q(k_1\rho') - A_1(\nu_{1q}) \sin \nu_{1q} \frac{\pi}{2} = \sum_p A_2(\nu_{2p}) \sin \nu_{2p} \frac{\pi}{2} C_{pq} \ \forall q$

$\nu_{1q} A_1(\nu_{1q}) \cos \nu_{1q} \frac{\pi}{2} = r \sum_p \nu_{2p} A_2(\nu_{2p}) \cos \nu_{2p} \frac{\pi}{2} \ C_{pq} \quad \forall q$

$C_{pq} = \int_a^\infty \frac{1}{\rho} \Phi_q(k_1\rho) \Phi_p(k_2\rho) d\rho$

We cast the linear system as

$\mathbf{S} - \mathbf{D}[\sin \nu_1 \frac{\pi}{2}] \mathbf{A}_1 = \mathbf{C} \ \mathbf{D}[\sin \nu_2 \frac{\pi}{2}] \mathbf{A}_2$

$\mathbf{D}[\nu_1 \cos \nu_1 \frac{\pi}{2}] \mathbf{A}_1 = \mathbf{C} \ \mathbf{D}[r\nu_2 \cos \nu_2 \frac{\pi}{2}] \mathbf{A}_2$

where S is the vector $S = \{ \frac{\sin \nu_{1q}(\frac{\pi}{2} - \phi')}{2\nu_{1q} \cos \nu_{1q} \frac{\pi}{2}} \Phi_q(k_1\rho') \}$;

$\mathbf{D}[.]$ are diagonal matrices with diagonal elements [.] and $\mathbf{A}_{1,2}$ are the vectors of spectral amplitudes.

From the above system we derive

$\mathbf{A}_1 = \mathbf{M}^{-1} \ \mathbf{S}$,

$\mathbf{A}_2 = \mathbf{D}^{-1}[r\nu_2 \cos \nu_2 \frac{\pi}{2}] \mathbf{C}^{-1} \ \mathbf{D}[\nu_1 \cos \nu_1 \frac{\pi}{2}] \mathbf{A}_1$

where

$\mathbf{M} = \mathbf{D}[\sin \nu_1 \frac{\pi}{2}] + \mathbf{C}\mathbf{D}[\sin \nu_2 \frac{\pi}{2}] \mathbf{D}^{-1}[r\nu_2 \cos \nu_2 \frac{\pi}{2}] \mathbf{C}^{-1} \mathbf{D}[\nu_1 \cos \nu_1 \frac{\pi}{2}]$

It is not difficult to show, from the asymptotic expansion of Bessel and Hankel functions as $q \to \infty$ [4], that the series representations in equations (1-3) converge exponentially with order of the p^{th}term $O(e^{i\nu_p|\phi - \phi'|})$ for equation (1) and $O(e^{i\nu_p(|\phi| + \phi')})$ for equations (2,3). The only exceptions are when $\phi = \phi'$ in equations (1) and when $\phi = \phi' = 0$ in equations (2,3) where the corresponding series diverges. A remedy is given in the next section.

Had we used Fourier expansion on the angular variable instead of the discrete Hankel index transform, the lack of an orthogonality relation for the Hankel functions of integer order would have resulted in the appearance of additional dense matrices of the form $\int_a^\infty \frac{1}{\rho}H_q^{(1)}(k_1\rho)H_p^{(1)}(k_1\rho)d\rho$, $\int_a^{\rho'} \frac{1}{\rho}H_q^{(1)}(k_1\rho)J_p(k_1\rho)d\rho$ and $\int_{\rho'}^\infty \frac{1}{\rho}H_q^{(1)}(k_1\rho)H_p^{(1)}(k_1\rho)d\rho$ in the linear system; and the convergence of the series is algebraic.

These are other advantages of using the discrete Hankel index transform for problems with boundaries along $\phi =$ constant.

The isovelocity case: It is always instructive to look into the isovelocity case ($k_1 = k_2$) wherein, for $Z_s = 0$ or ∞, $\nu_{1p} = \nu_{2p}$, C_{pq} turns diagonal and the linear system simplifies to

$$\frac{\sin\nu_{1q}(\frac{\pi}{2}-\phi')}{2\nu_{1q}\cos\nu_{1q}\frac{\pi}{2}}\Phi_q(k_1\rho') - A_1(\nu_{1q})\sin\nu_{1q}\frac{\pi}{2} = A_2(\nu_{1q})\sin\nu_{1q}\frac{\pi}{2}$$

$$A_1(\nu_{1q}) = rA_2(\nu_{1q})$$

$$A_2(\nu_{1q}) = \frac{1}{1+r}\frac{\sin\nu_{1q}(\frac{\pi}{2}-\phi')}{2\nu_{1q}\cos\nu_{1q}\frac{\pi}{2}\sin\nu_{1q}\frac{\pi}{2}}\Phi_q(k_1\rho')$$

leading to the above closed form analytic expressions for the coefficients $A_{1,2}(\nu_{1q})$.

Therefore

$$E_z^{(2)} = \frac{1}{1+r}\sum_q \frac{\sin\nu_{1q}(\frac{\pi}{2}-\phi')}{2\nu_{1q}\cos\nu_{1q}\frac{\pi}{2}\sin\nu_{1q}\frac{\pi}{2}}\sin\nu_{1q}(\phi+\frac{\pi}{2})\Phi_q(k_1\rho')\Phi_q(k_1\rho)$$

$$E_z^{(r)} = \frac{r}{1+r}\sum_q \frac{\sin\nu_{1q}(\frac{\pi}{2}-\phi')}{2\nu_{1q}\cos\nu_{1q}\frac{\pi}{2}\sin\nu_{1q}\frac{\pi}{2}}\sin\nu_{1q}(\phi-\frac{\pi}{2})\Phi_q(k_1\rho')\Phi_q(k_1\rho)$$

For $E_z^{(d)}$, if one must compute the fields along $\phi = \phi'$ where the series representation diverges and is not, relying on field continuity, satisfied by interpolating from neighboring points then a convergent representation for $\phi = \phi'$ is derived by

a) manipulating the summand part $[\frac{\sin\nu_{1p}(\frac{\pi}{2}-\phi_>)\cos\nu_{1p}\phi_<}{2\nu_{1p}\cos\nu_{1p}\frac{\pi}{2}}]$ into $[\frac{1}{4}\{ie^{i\nu_{1p}|\phi-\phi'|} - \frac{2ie^{i\pi\nu_{1p}}}{1+e^{i\pi\nu_{1p}}}\cos\nu_{1p}(\phi-\phi') + \frac{\sin\nu_{1p}(\frac{\pi}{2}-\phi-\phi')}{\cos\nu_{1p}\frac{\pi}{2}}\}]$. The second and third terms result in convergent series for all observation angles;

b) applying a Watson transform [5] on the partial series of the first term reduces it to $\frac{i}{16}[\int_\gamma e^{i\nu|\phi-\phi'|}\{H_\nu^{(2)}(k_1\rho_<)H_\nu^{(1)}(k_1\rho_>) - H_\nu^{(1)}(k_1\rho)H_\nu^{(1)}(k_1\rho')\frac{H_\nu^{(2)}(k_1a)}{H_\nu^{(1)}(k_1a)}\}d\nu]$

with γ the contour around the zeros of $d(\nu) = 0$ in the first quadrant of the complex ν plane.

After asymptotic evaluation, the integrals are recognized as the geometrical optical incident and reflected fields respectively.

The same treatment is applicable to $E_z^{(r),(2)}$ when $\phi = \phi' = 0$.

The presented solution method is extendable to other geometries: a cylinder in the vicinity of two half spaces and a cylinder less|more than halfway buried. These and extension to source excitations leading to coupled TE|TM polarizations will be presented elsewhere.

References

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