The translation invariant massive Nelson model: I. The bottom of the spectrum

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August, 2004

Abstract

In this paper we analyze the bottom of the energy-momentum spectrum of the translation invariant Nelson model, describing one electron linearly coupled to a second quantized massive scalar field. Our results are non-perturbative and include an HVZ theorem, non-degeneracy of ground states, existence of isolated groundstates in dimensions 1 and 2, non-existence of ground states embedded in the bottom of the essential spectrum in dimensions 3 and 4, (i.e., at total momenta where no isolated groundstate eigenvalue exists), and we study regularity and monotonicity properties of the bottom of the essential spectrum, as a function of total momentum.

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1 Introduction and results

In this section we introduce the Nelson model and formulate our main results. The notation we use is standard, but for the sake of completeness we give the basic constructions in Subsect. 2.1.

1.1 Non-relativistic QED: An overview

In the last decade there has been a surge of interest in non-relativistic QED, sparked by a string of papers by Hübner and Spohn, and by Bach, Fröhlich, and Sigal. See e.g. [5, 4, 39, 38]. The purpose of this subsection is to give an overview over different aspects of the problem and place the model we study, as well as the results derived, into context.

The fundamental Hamiltonian in non-relativistic QED, describing one charged particle, with mass $M > 0$ and charge e, coupled to a radiation field, is the *minimally coupled* one

$$H_{\min} \;:=\; \mathbb{1}\otimes d\Gamma(|k|) \;+\; \frac{1}{2M}\big(p\otimes\mathbb{1} - e\,A(\mathrm{x})\big)^2\,, \text{ on } L^2(\mathbb{R}^3_{\mathrm{x}})\otimes\Gamma(L^2(\mathbb{R}^3_k))\,. \quad (1.1)$$

Here $d\Gamma(|k|)$ is the kinetic energy of the radiation field, $p = \mathrm{i}\nabla_{\mathrm{x}}$ is the particle momentum operator, and A is the second quantized (massless) Maxwell field in the Coulomb gauge, i.e. $\nabla_{\mathrm{x}}\cdot A = 0$. The Hilbert space $\Gamma(L^2(\mathbb{R}^3_k))$ is the bosonic Fock-space. See [34] and [5, 41]. In order to make sense of this operator (a priori as a form) one must introduce an ultraviolet cutoff into A. We recall that the model is translation invariant, in the sense that it commutes with the operator of total momentum $P := p\otimes\mathbb{1} + \mathbb{1}\otimes d\Gamma(k)$. We remark that often the second quantized Pauli operator is taken as a starting point instead of (1.1). It is defined by replacing $(p - eA)^2$ by $(\sigma\cdot(p - eA))^2$, where σ is the vector of Pauli matrices. This operator differs from (1.1) by a magnetic term $\sigma\cdot(\nabla_{\mathrm{x}}\times A)$ (and with $L^2(\mathbb{R}^3_{\mathrm{x}})$ replaced by $L^2(\mathbb{R}^3_{\mathrm{x}})\otimes\mathbb{C}^2$, thus taking into account the spin of the particle).

The study of $H_{\min}$ is a natural starting point in non-relativistic QED. In particular in the context of scattering theory, where the dynamics of $H_{\min}$ is a natural choice for ”free” dynamics. Unfortunately there are not many non-perturbative (where e is here viewed as a coupling constant) rigorous results established for the minimally coupled model, as it is formulated in (1.1). We refer the reader to [35, 40]. Most results obtained in the literature are for $H_{\min}$ perturbed by an electric potential, and results then pertain to existence and properties of ground states for the perturbed model, or localization in $L^2(\mathbb{R}^3_{\mathrm{x}})$ of states below an ionization threshold. See [29, 30, 41, 42].

There are a number of different ways to obtain simpler problems. Some involve passing to phenomenological Hamiltonians, which are simpler to analyze than (1.1). We list some choices typically considered in the litterature:

S1) Consider the problem perturbatively, i.e., in the limit of e small.

S2) Replace the massless photons by massive photons, which amounts to replacing the massless dispersion relation $k \to |k|$ by a massive one $k \to \sqrt{k^2+m^2}$, $m > 0$. This removes the infrared problem.

S2’) Set the interaction between soft photons (photons with small momenta) equal to zero.

S3) Replace the minimal coupling with a linear coupling to a scalar field, i.e. replace $H_{\min}$ by $H = \mathbb{1} \otimes d\Gamma(|k|) + \frac{1}{2M}p^2 \otimes \mathbb{1} + \Phi(v)$, where $\Phi(v)$ is a field operator.
S4) Place the system in a confining external electric potential V, that is $\lim_{|\mathrm{x}|\to\infty} V(\mathrm{x}) = \infty$. This breaks the translation invariance of the problem. An extreme version of this are the spin-boson and Wigner-Weisskopf models.
S4') Place the system in an external potential V such that $p^2 + V$ has isolated eigenvalues below the essential spectrum. Then consider $H_{\min} + V \otimes \mathbb{1}$ in a low energy regime where states are isolated bound states of $p^2 + V$ dressed with photons.
S5) A combination of the above.

In this paper we consider the massive translation invariant linearly coupled model in any dimension, which can be viewed as a simplification of the minimally coupled model, by applying **S2)** and **S3)** as mentioned above. This model was considered by Nelson in [46], and it is distinguished by being renormalizable in a Hamiltonian setting, cf. also [10, 32, 52]. This model is often referred to as the *Nelson model*, a convention also adopted here. The models discussed in this introduction is part of a body of models sometimes referred to as *Pauli-Fierz models*. In this paper we do not consider renormalized operators. In addition we note that we work with more general dispersion relations ω and Ω than $\sqrt{k^2 + m^2}$ and $p^2/2M$ respectively. We emphasize that we are interested in *non-perturbative* results. See Subsect. 1.2 below for a more detailed description of the model.

We remark that one can formulate the model and the simplifications discussed above for multiple particles coupled to a radiation field. For confined versions of the model, cf. **S4)** and **S4')** above, this makes no difference. However, for translation invariant models, not much is known.

We pause to remind the reader that translation invariance, the fact that $[H, P] = 0$, gives a direct integral representation $\oint H(\xi)d\xi$ of the Hamiltonian. What we study in this paper is the bottom of the spectrum and essential spectrum of $H(\xi)$ as functions of total momentum ξ. The former function is also called the *ground state mass shell*, or simply the mass shell. We note that in the massive case isolated excited states could exist and would give rise to excited mass shells.

We are mainly inspired by papers of Fröhlich [19, 20], Spohn [54], and one of Dereziński and Gérard [14]. Fröhlich considered non-perturbative properties of the ground state mass shell for the massless translation invariant Nelson model. Most of his results hold (suitably translated) also for massive photons. Dereziński and Gérard were concerned with confined, in the sense of **S3)** above, massive linearly coupled models. Using non-perturbative methods they give a geometric proof of a HVZ theorem, thus locating the essential spectrum. (They furthermore apply Mourre theory and time-dependent scattering theory to the model.) Spohn proved a HVZ theorem for the translation invariant model, using in part ideas of Glimm and Jaffe (via a reference to [20]). He furthermore showed, in dimension 1 and 2, that the Hamiltonian at fixed total momentum admits an isolated groundstate. The results of Spohn are for a class of massive and subadditive dispersion relations ω. The result on existence of groundstates requires an additional assumption which excludes the dispersion relation $\sqrt{k^2 + m^2}$, $m > 0$.

In this paper we prove the following results for the structure of the bottom of the spectrum of the massive translation invariant Nelson model: An HVZ theorem, Theorem 1.2 (valid for ω which are not necessarily subadditive). The ground state mass shell is non-degenerate, Theorem 1.3, using a Perron-Frobenius argument of [19]. Existence of an isolated groundstate for all total momenta, Theorem 1.6 i) ($\nu = 1, 2$), thus extending the result of Spohn to the case $\omega(k) = \sqrt{k^2 + m^2}$. Non-existence of a ground state embedded in the essential spectrum, Theorem 1.6 ii) ($\nu = 3, 4$). Analyticity of the bottom of the essential spectrum, away from a closed countable set, Theorem 1.11. Maximality of the spectral gap and analyticity at local minima for the bottom of the essential spectrum, Theorem 1.12. See Subsect. 1.3 for a precise formulation of the main results. In Subsect. 4.2 we discuss how to extend the results to the model with a cutoff in the photon number operator.

The models considered in this paper only fails to include the socalled (optical mode) polaron model of an electron in a crystal by the requirement that $\omega(k) \to \infty$, $|k| \to \infty$. This requirement is a consequence of our use of geometric methods to prove the HVZ theorem, and an adoption of the Glimm-Jaffe approach, as used in [20], might remedy this. However, the geometric approach is important for future work on Mourre and scattering theory. For mathematical work on the polaron model see [32, 43, 52, 53, 54], and for a textbook discussion see [18].

We remark that there are not many non-perturbative results on the translation invariant Nelson model, other than what we have already mentioned above. See however [31], Lemma 4.1 in this paper. In [53] upper and lower bounds on the effective mass are obtained (the effective mass is the inverse of the Hessian of the ground state mass shell at zero total momentum). There are more complete results available if one imposes a cutoff at small photon number, cf. [23] (the massless case with at most one photon).

In the perturbative case there are more results, cf. [11, 21, 47]. See also [33, 36, 37]. (We remark however, that although the photon dispersion relation in [21] is massless, the interaction is of the type mentioned in **S2')** above, and the model thus retains massive features.)

Finally we recall that for confined massive models, cf. **S4)** and **S4')** above, quite strong non-perturbative results are available. See, apart from [14] mentioned above, the papers [2, 3, 22]. As for the massless confined model we refer the reader to [7, 9, 24, 26, 38] for non-perturbative results.

1.2 The translation invariant Nelson model

We consider a particle moving in $\mathbb{R}^\nu$ and interacting with a scalar radiation field. We write x and $p = -\mathrm{i}\nabla_{\mathrm{x}}$ for the particle position and momentum respectively. The particle Hilbert space is

$$\mathcal{K} \;:=\; L^2(\mathbb{R}^\nu_{\mathrm{x}})\,,$$

and the Hamiltonian for a free particle is taken to be $\Omega(p)$, where $\Omega : \mathbb{R}^\nu \to \mathbb{R}$ is a smooth dispersion relation. We are primarily interested in the standard non-relativistic and relativistic choices, i.e. $\Omega(p) = \frac{p^2}{2M}$ and $\Omega(p) = \sqrt{p^2 + M^2}$. Here $M > 0$ is the mass of the particle.

The photon coordinates will be denoted by $x = \mathrm{i}\nabla_k$ and k respectively and the one-photon space is

$$\mathfrak{h}_{\mathrm{ph}} \;:=\; L^2(\mathbb{R}^\nu_k)\,.$$

The Hilbert space for the radiation field is the bosonic Fock-space

$$\mathcal{F} \;\equiv\; \Gamma(\mathfrak{h}_{\mathrm{ph}}) \;:=\; \bigoplus_{n=0}^{\infty} \mathcal{F}^{(n)}\,, \;\text{ where }\; \mathcal{F}^{(n)} \equiv \Gamma^{(n)}(\mathfrak{h}_{\mathrm{ph}}) \;:=\; \mathfrak{h}_{\mathrm{ph}}^{\otimes_s n}\,. \tag{1.2}$$

We write $\Omega = (1,0,0,\dots)$ for the vacuum. The creation and annihilation operators, $\mathbf{a}^*(k)$ and $\mathbf{a}(k)$ satisfy the canonical commutation relations (CCR for short)

$$[\mathbf{a}^*(k),\mathbf{a}^*(k')] \;=\; [\mathbf{a}(k),\mathbf{a}(k')] \;=\; 0\,, \;\; [\mathbf{a}(k),\mathbf{a}^*(k')] \;=\; \delta(k-k')\,, \tag{1.3}$$

and $\mathbf{a}(k)\Omega = 0$. The free photon energy is the second quantization of the one-photon dispersion relation ω

$$d\Gamma(\omega) \;:=\; \int_{\mathbb{R}^\nu} \omega(k)\,\mathbf{a}^*(k)\,\mathbf{a}(k)\,dk\,, \;\text{ where }\; \omega(k) \;:=\; \sqrt{k^2 + m^2}\,. \tag{1.4}$$

Here $m > 0$ is the mass of the scalar photon. Our methods do not extend to the case of massless photons, $m = 0$. The full Hilbert space of the combined system is

$$\mathcal{H} \;:=\; \mathcal{K} \otimes \mathcal{F}\,.$$

We will make the following identification

$$\mathcal{H} \;\equiv\; L^2(\mathbb{R}^\nu_{\mathrm{x}};\mathcal{F}).$$

The interaction considered here is linear in the field operator and is given by

$$V \;:=\; \int_{\mathbb{R}^\nu} \Big\{ e^{\mathrm{i}k\cdot\mathrm{x}} v(k)\,\mathbb{1}_{\mathcal{K}} \otimes \mathbf{a}^*(k) \;+\; e^{-\mathrm{i}k\cdot\mathrm{x}}\,\overline{v(k)}\,\mathbb{1}_{\mathcal{K}} \otimes \mathbf{a}(k) \Big\}\,dk\,,$$

where the physical form of the interaction is $v(k) = \chi(k)/\sqrt{\omega(k)}$ and χ is an ultraviolet cutoff, which ensures that $v \in \mathfrak{h}_{\mathrm{ph}}$. The free and coupled Hamiltonians for the combined system are

$$H \;:=\; H_0 \;+\; V\,, \;\text{ where }\; H_0 \;:=\; \Omega(p) \otimes \mathbb{1}_{\mathcal{F}} \;+\; \mathbb{1}_{\mathcal{K}} \otimes d\Gamma(\omega)\,. \tag{1.5}$$

The total momentum for the combined system is given by

$$P \;:=\; p \otimes \mathbb{1}_{\mathcal{F}} \;+\; \mathbb{1}_{\mathcal{K}} \otimes d\Gamma(k)\,.$$

The property of translation invariance is contained in the statement that the Hamiltonian commutes with the total momentum. That is, the energy momentum vector (P,H)

has mutually commuting coordinates. Translation invariance implies that H_0 and H are fibered operators. We introduce a unitary transformation

$$I_{\rm fib} := F\,\Gamma(e^{-\mathrm{i}k\cdot \mathrm{x}}) \ : \ \mathcal{H} \ \to \ L^2(\mathbb{R}^\nu_\xi\,;\,\mathcal{F})\ ,$$

where F is the Fourier transform $F : L^2(\mathbb{R}^\nu_{\mathrm{x}};\mathcal{F}) \to L^2(\mathbb{R}^\nu_\xi;\mathcal{F})$ and $\Gamma(e^{-\mathrm{i}k\cdot\mathrm{x}})$ restricted to $\mathcal{K}\otimes\mathcal{F}^{(n)}$ is multiplication by $e^{-\mathrm{i}(k_1+\cdots+k_n)\cdot\mathrm{x}}$. We have

$$I_{\rm fib}\, H_0\, I_{\rm fib}^* \ = \ \oint_{\mathbb{R}^\nu} H_0(\xi)\, d\xi \ \text{ and } \ I_{\rm fib}\, H\, I_{\rm fib}^* \ = \ \oint_{\mathbb{R}^\nu} H(\xi)\, d\xi\ .$$

The fiber operators $H_0(\xi)$ and $H(\xi)$, $\xi\in\mathbb{R}^\nu$, are operators on $\mathcal{F}$ given by

$$H(\xi) \ = \ H_0(\xi) \ + \ \Phi(v) \ \text{ where } \ H_0(\xi) \ = \ d\Gamma(\omega) \ + \ \Omega(\xi - d\Gamma(k)) \tag{1.6}$$

and the interaction is

$$\Phi(v) \ = \ \int_{\mathbb{R}^\nu} \big\{ v(k)\,\mathbf{a}^*(k) \ + \ \overline{v(k)}\,\mathbf{a}(k)\big\}\, dk\ . \tag{1.7}$$

We will in general use the notation $v\in\mathfrak{h}_{\rm ph}$ to denote a form-factor. In this paper we study the properties of the bottom of the joint spectrum of the vector (P,H).

1.3 Main results

In this subsection we will formulate precise conditions and state our main results. Proofs will be given in Section 3. The first condition is on the particle dispersion relation. We use the standard notation $\langle t\rangle := (1+t^2)^{1/2}$.

Condition 1.1. (The particle dispersion relation) *Let $\Omega\in C^\infty(\mathbb{R}^\nu)$. There exists $s_\Omega\in\{0,1,2\}$ such that*

i) There exists C such that $\Omega(\eta)\geq C^{-1}\langle\eta\rangle^{s_\Omega} - C$.

ii) For any multi-index α there exists C_α such that $|\partial^\alpha\Omega(\eta)|\leq C_\alpha\langle\eta\rangle^{s_\Omega-|\alpha|}$.

We note that the standard choices $\Omega(p) = \frac{p^2}{2M}$ and $\Omega(p) = \sqrt{p^2+M^2}$ satisfy this condition with $s_\Omega = 2$ and $s_\Omega = 1$ respectively.

Condition 1.2. (The photon dispersion relation) *Let $\omega\in C^\infty(\mathbb{R}^\nu)$ satisfy*

i) There exists $m>0$, the photon mass, such that $\inf_{k\in\mathbb{R}^\nu}\omega(k) = \omega(0) = m$.

ii) $\omega(k)\to\infty$, in the limit $|k|\to\infty$.

iii) There exists $s_\omega\geq 0$ and C_ω such that for any multi-index α, with $|\alpha|\geq 1$,

$$\omega(k)\geq C_\omega^{-1}\langle k\rangle^{s_\omega} - C_\omega \ \ \textit{and} \ \ |\partial_k^\alpha\omega(k)|\leq C_\alpha\langle k\rangle^{s_\omega-|\alpha|}\ .$$

The condition iii) is used in connection with pseudo differential calculus. The physical choice of ω used in (1.4) satisfies this condition (with $s_\omega = 1$), and so does $\omega(k) = k^2 + m$ (with $s_\omega = 2$).

We introduce a space of test functions

$$\mathcal{C}_0^\infty \;:=\; \Gamma_{\rm fin}(C_0^\infty(\mathbb{R}^\nu))\,. \tag{1.8}$$

Note that since $H_0(\xi)$ is a bounded from below multiplication operator on each n-particle sector, we find that it is essentially self-adjoint on $\mathcal{C}_0^\infty$. We recall the following result, cf. [46], [19], and [20]. For completeness we give a proof in the beginning of Section 3

Proposition 1.1. *Let $v \in L^2(\mathbb{R}^\nu)$. Assume Ω and ω, satisfy Conditions 1.1 and 1.2 i) respectively. Then*

i) $\mathcal{D}(H_0(\xi))$ is independent of ξ and we denote it by $\mathcal{D}$.

ii) $\Phi(v)$ is $H_0(\xi)$-bounded with relative bound 0. In particular $H(\xi)$ is bounded from below, self-adjoint on $\mathcal{D}(H(\xi)) = \mathcal{D}(H_0(\xi))$, and essentially self-adjoint on $\mathcal{C}_0^\infty$.

iii) The bottom of the spectrum of the fiber Hamiltonians, $\xi \to \Sigma_0(\xi) := \inf\sigma(H(\xi))$, is Lipschitz continuous.

We introduce some notation. First the bottom of the spectrum of the full operator:

$$\Sigma_0 \;:=\; \inf_{\xi\in\mathbb{R}^\nu}\, \Sigma_0(\xi) \;>\; -\infty\,.$$

For $n \geq 1$ and $\underline{k} = (k_1,\dots,k_n) \in \mathbb{R}^{n\nu}$ we often write $k^{(n)} = k_1 + \cdots + k_n$. We now introduce the bottom of the spectrum for a composite system at total momentum ξ, consisting of an interacting system at total momentum $\xi - k^{(n)}$ and n non-interacting photons with momenta $\underline{k}$:

$$\Sigma_0^{(n)}(\xi;\underline{k}) \;:=\; \Sigma_0^{(n)}\big(\xi - k^{(n)}\big) \;+\; \sum_{j=1}^n \omega(k_j)\,. \tag{1.9}$$

The following functions are thresholds due to ground states dressed by n photons, at critical momenta:

$$\Sigma_0^{(n)}(\xi) \;:=\; \inf_{\underline{k}\in\mathbb{R}^{n\nu}} \Sigma_0^{(n)}(\xi;\underline{k})\,. \tag{1.10}$$

The bottom of the essential spectrum (see Theorem 1.2 below)

$$\Sigma_{\rm ess}(\xi) \;:=\; \inf_{n\geq 1}\, \Sigma_0^{(n)}(\xi)\,. \tag{1.11}$$

We have the following elementary properties of the functions introduced above. Namely

$$0 \;\leq\; \Sigma_{\rm ess}(\xi) \;-\; \Sigma_0(\xi) \;\leq\; m \tag{1.12}$$

$$\Sigma_0(\xi) \;=\; \Sigma_0 \;\Rightarrow\; \Sigma_{\rm ess}(\xi) \;=\; \Sigma_0(\xi) \;+\; m \tag{1.13}$$

$$\lim_{|\xi|\to\infty} \Sigma_0(\xi) \;=\; \lim_{|\xi|\to\infty} \Sigma_{\rm ess}(\xi) \;=\; \lim_{|\xi|\to\infty} \Sigma_0^{(n)}(\xi) \;=\; \infty \tag{1.14}$$

$$\lim_{n\to\infty} \Sigma_0^{(n)}(\xi) \;=\; \infty\,. \tag{1.15}$$

Our first result is

Theorem 1.2. (HVZ) *Let $v \in L^2(\mathbb{R}^\nu)$. Assume Conditions 1.1, and 1.2. Then*

i) Eigenvalues of $H(\xi)$ below $\Sigma_{\mathrm{ess}}(\xi)$ have finite multiplicity and can only accumulate at $\Sigma_{\mathrm{ess}}(\xi)$.

ii) $\sigma_{\mathrm{ess}}(H(\xi)) = [\Sigma_{\mathrm{ess}}(\xi), \infty)$.

The method of proof for the HVZ theorem is geometric and follows ideas of [14], cf. Subsect. 3.2. See also [2, 1, 9, 15, 24]. The name "HVZ" (Hunziker–van Winter–Zhislin) is used because the geometric idea of the proof is quite similar to that employed in the proof of the standard HVZ theorem for N-body Schrödinger operators, cf. [13, Theorem 6.2.2]. We recall that there is another method, due to Glimm and Jaffe [28], one can employ to obtain an HVZ theorem. See [54, Section 4], for the case of subadditive dispersion relations ω, and in addition [8, 20].

We have the following result on non-degeneracy of groundstates. This type of result is not new, cf. [31, Section 6] and [19, Section 3.2].

Theorem 1.3. (Non-degeneracy of ground states) *Let $v \in L^2(\mathbb{R}^\nu)$. Assume Conditions 1.1 and 1.2. Suppose furthermore that $v(k) > 0$ a.e. Then, if $\Sigma_0(\xi)$ is an eigenvalue for $H(\xi)$, it is non-degenerate.*

We note that the result of Gross [31] is for zero total momentum only, and assumed that $p \to \exp(-t\Omega(p))$ is a positive definite function for all $t > 0$. However, Gross does not assume v to have a sign. This is because one can pass to the Schrödinger representation of the Fock-space, where $H_0(\xi)$ is positivity improving if and only if $\xi = 0$.

In the following we will impose

Condition 1.3. *$\omega \in C^\infty(\mathbb{R}^\nu)$ satisfies*

i) Subbadditivity: For $k_1, k_2 \in \mathbb{R}^\nu$ we have $\omega(k_1 + k_2) \le \omega(k_1) + \omega(k_2)$.

i') Strict subadditivity: For $k_1, k_2 \in \mathbb{R}^\nu$ we have $\omega(k_1 + k_2) < \omega(k_1) + \omega(k_2)$.

The standard dispersion relation $\omega(k) = \sqrt{k^2 + m^2}$ satisfies Condition 1.3 i'), but $\omega(k) = k^2 + m$ does not. Below we will discuss consequences of imposing Condition 1.3.

If ω is (strictly) subadditive we find, for all $\xi \in \mathbb{R}^\nu$,

$$\Sigma_0^{(n)}(\xi) \overset{(<)}{\le} \Sigma_0^{(n')}(\xi)\,, \quad \text{for} \;\; n \;<\; n'\,. \tag{1.16}$$

We thus get the following supplement to the HVZ Theorem, cf. also [54, Section 4],

Corollary 1.4. *Let $v \in L^2(\mathbb{R}^\nu)$. Assume Conditions 1.1, 1.2, and 1.3 i). Then $\Sigma_{\mathrm{ess}}(\xi) = \Sigma^{(1)}(\xi)$.*

The following simple lemma can be used to check for subadditivity.

Lemma 1.5. *Let $\omega \in C^\infty(\mathbb{R}^\nu)$ be convex and satisfy: For any $k \in \mathbb{R}^\nu$, we have $\omega(k) - k \cdot \nabla\omega(k) \overset{(>)}{\ge} 0$. Then ω is (strictly) subadditive.*

We introduce the notation

$$\mathcal{I}_0 \ := \ \{ \, \eta \, \in \, \mathbb{R}^\nu \ : \ \Sigma_0(\eta) \ < \ \Sigma_{\rm ess}(\eta) \, \} \, . \tag{1.17}$$

We prove the following result on the nature of the bottom of the spectrum, at a total momentum with no isolated ground state eigenvalue.

Theorem 1.6. (Existence/Non-existence of ground states) *Let $v \in L^2(\mathbb{R}^\nu)$. Assume Conditions 1.1, 1.2, and 1.3 i'). Suppose furthermore that $v(k) > 0$ a.e., locally uniformly in $k \in \mathbb{R}^\nu$. We have:*

- *i) If $1 \leq \nu \leq 2$, then $\mathcal{I}_0 = \mathbb{R}^\nu$, that is; $\Sigma_0(\xi)$ is an isolated eigenvalue of $H(\xi)$ for any $\xi \in \mathbb{R}^\nu$.*
- *ii) If $3 \leq \nu \leq 4$ and $\xi \notin \mathcal{I}_0$, then $H(\xi)$ has no ground state; i.e. $\Sigma_0(\xi)$ is not an eigenvalue.*

The statement i) above is an extension to the Nelson model of a result of Spohn, [54, Section 5]. We give a new proof replacing Spohn's functional integral approach by the pull-through formula.

The remaining results are derived under the following condition

Condition 1.4. *The functions $\Omega, \omega \in C^\infty(\mathbb{R}^\nu)$ and $v \in L^2(\mathbb{R}^\nu)$ and*

- *i) Invariance under rotations: For any $\xi \in \mathbb{R}^\nu$ and $O \in O(\nu)$ (the orthogonal group), we have: $\Omega(O\xi) = \Omega(\xi)$, $\omega(O\xi) = \omega(\xi)$, and $v(Ok) = v(k)$ a.e.*
- *ii) ω is convex.*
- *iii) Ω and ω are analytic.*

The rotation invariance of Ω, ω, and v, implies that the ground state mass shell $\Sigma_0(\xi)$ is invariant under rotations, and so are $\Sigma_0^{(n)}(\xi)$.

For $\xi \in \mathbb{R}^\nu$ and $n \in \mathbb{N}$ we define

$$\mathcal{I}_0^{(n)}(\xi) \ := \{ \, \underline{k} \, \in \, \mathbb{R}^{n\nu} \ : \ \xi \ - \ k^{(n)} \ \in \ \mathcal{I}_0 \, \} \, . \tag{1.18}$$

Our last theorem is concerned with the regularity of the functions $\xi \to \Sigma_0^{(n)}(\xi)$. Our strategy is to study local minima of $\underline{k} \to \Sigma_0^{(n)}(\xi; \underline{k})$. The following lemma, in conjunction with (1.16), ensures that under Condition 1.3, the relevant local minima, i.e. global minima, are located in $\mathcal{I}_0^{(n)}(\xi)$, where the bottom of the spectrum is smooth.

Lemma 1.7. *Let $v \in L^2(\mathbb{R}^\nu)$. Assume Conditions 1.1 and 1.2 i). Let $\xi \in \mathbb{R}^\nu$, $n \geq 1$ and $\underline{k} \in \mathbb{R}^{n\nu}$. If $\Sigma_0^{(n)}(\xi; \underline{k}) < \inf_{n'>n} \Sigma_0^{(n')}(\xi)$, then $\underline{k} \in \mathcal{I}_0^{(n)}(\xi)$.*

The following lemma allows us to restrict the analysis to one dimension.

Lemma 1.8. *Let $v \in L^2(\mathbb{R}^\nu)$. Assume Conditions 1.1, 1.2 i), and 1.4 i),ii). Let $\xi \in \mathbb{R}^\nu$ and $n \in \mathbb{N}$. Any local minimum $\underline{k} \in \mathcal{I}_0^{(n)}(\xi)$ of $\underline{k} \to \Sigma_0^{(n)}(\xi; \underline{k})$ is of the form $k_1 = \cdots = k_n = \theta\xi$, for some $\theta \in \mathbb{R}$.*

Let $\vec{u}$ be a unit vector in $\mathbb{R}^\nu$. We write $\sigma(t) = \Sigma_0(t\vec{u})$, for $t \in \mathbb{R}$. By rotation invariance, σ is independent of $\vec{u}$. Similarly we write $\sigma^{(n)}(t) := \Sigma_0^{(n)}(t\vec{u})$ and $\sigma_{\mathrm{ess}}(t) := \Sigma_{\mathrm{ess}}(t\vec{u})$. With a slight abuse of notation we write $\omega(t) = \omega(t\vec{u})$ and $\mathcal{I}_0$ to denote the set of t's such that $t\vec{u} \in \mathcal{I}_0$. We furthermore use the symbol $\mathcal{I}_0^{(n)}(t)$, $n > 0$ (not necessarily integer), to denote the set $\{s \in \mathbb{R} : t - ns \in \mathcal{I}_0\}$.

In light of the previous lemma, we introduce now, for $n > 0$ and not necessarily integer, the following functions

$$\sigma^{(n)}(t;s) \;=\; \sigma(t\,-\,ns) \;+\; n\,\omega(t) \;\text{ and }\; \sigma^{(n)}(t) \;=\; \inf_{s\in\mathbb{R}} \sigma^{(n)}(t;s)\,.$$

Note that by Lemma 1.8 we have, for integer n, $\Sigma_0^{(n)}(\xi) = \sigma^{(n)}(|\xi|)$, and in particular $\Sigma_{\mathrm{ess}}(\xi) = \sigma^{(1)}(|\xi|)$. In this connection we mention that a local minimum for $\Sigma_0^{(n)}(t;\cdot)$ induces a local minimum for $\sigma^{(n)}(t;\cdot)$. Conversely however, a local minimum for $\sigma^{(n)}(t;\cdot)$, which is not a global minimum, could be associated with a saddle point for $\Sigma_0^{(n)}(t;\cdot)$.

We have, cf. also [19, Lemma 1.6],

Proposition 1.9. *Assume Conditions 1.1, 1.2, and 1.4. Let $\lambda < \Sigma_0$. The family of self adjoint operators $t \to (H(t\vec{u}) - \lambda)^{-1}$ is analytic of type A. Furthermore, the map $\mathcal{I}_0 \ni t \to \sigma(t)$ is analytic.*

We introduce an index for a local minimum of $s \to \sigma^{(n)}(t;s)$.

Definition 1.10. *Let $n > 0$, $t \in \mathbb{R}$ and $s \in \mathcal{I}_0^{(n)}(t)$. Assume s is a local minimum. We define the index to be $\mathrm{Ind}^{(n)}(t;s) = \min\{\ell \in \mathbb{N} : \partial_s^{2\ell}\sigma^{(n)}(t;s) > 0\}$, with the convention that the index is ∞ if $\partial_s^{2\ell}\sigma^{(n)}(t;s) = 0$ for all ℓ. For simplicity we define $\mathrm{Ind}^{(n)}(t;s) = 0$ if $s \in \mathcal{I}_0^{(n)}(t)$ is not a local minimum for $s' \to \partial_s\sigma^{(n)}(t;s')$.*

We have the following regularity result

Theorem 1.11. *Assume Conditions 1.1, 1.2, 1.3, and 1.4. Let $n > 0$. There exists a closed countable set $\mathcal{T}^{(n)} \subset \mathbb{R}$, and an analytic map $\mathbb{R}\backslash\mathcal{T}^{(n)} \ni t \to \Theta^{(n)}(t) \in \mathcal{I}_0^{(n)}(t)$ with the property that the maps $s \to \sigma^{(n)}(t;s)$, $t \in \mathbb{R}\backslash\mathcal{T}^{(n)}$, has a unique global minimum at $s = \Theta^{(n)}(t)$, with $\mathrm{Ind}^{(n)}(t;\Theta^{(n)}(t)) = 1$. In particular $\mathbb{R}\backslash\mathcal{T}^{(n)} \ni t \to \sigma^{(n)}(t)$ is analytic and*

$$\frac{d}{dt}\,\sigma^{(n)}(t) \;=\; \partial\omega\big(\Theta^{(n)}(t)\big)\,, \;\; \textit{for}\;\; t \,\in\, \mathbb{R}\setminus\mathcal{T}^{(n)}\,. \tag{1.19}$$

Our final main result is concerned with the structure of the spectrum near local minima of the essential spectrum

Theorem 1.12. *Assume Conditions 1.1, 1.2, 1.3, and 1.4. Let t_0 be a local minimum of $t \to \sigma_{\mathrm{ess}}(t)$. Then the spectral gap at t_0 is maximal, i.e. $\sigma_{\mathrm{ess}}(t_0) - \sigma(t_0) = m$, the map $t \to \sigma(t)$ has a local minimum at t_0, the map $t \to \sigma_{\mathrm{ess}}(t)$ is analytic near t_0, and*

$$\partial^2\sigma_{\mathrm{ess}}(t_0) \;=\; \frac{\partial^2\omega(0)\,\partial^2\sigma(t_0)}{\partial^2\omega(0)\;+\;\partial^2\sigma(t_0)}\,.$$

2 Notation and preliminaries

In this section we recall known facts. The reader is urged to consult in particular [14], where most of the results pertaining to second quantization can be found.

2.1 The second quantization functor Γ

Let $\mathfrak{h}$ be a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$, which is conjugate linear in the first variable and linear in the second. We use the standard notation $\Gamma(\mathfrak{h})$ for the associated bosonic Fock-space, see (1.2). For a (not necessarily dense) subspace $\mathcal{C}\subset\mathfrak{h}$, we write $\Gamma_{\mathrm{fin}}(\mathcal{C})$ for the subspace of $\Gamma(\mathfrak{h})$ consisting of finite linear combinations of elements of the algebraic tensor products $\mathcal{C}^{\otimes_s n}$, $n\geq 0$. If $\mathcal{C}$ is dense in $\mathfrak{h}$, then $\Gamma_{\mathrm{fin}}(\mathcal{C})$ is dense in $\Gamma(\mathfrak{h})$.

We write $\mathbf{a}^*(f)$ and $\mathbf{a}(f)$, $f\in\mathfrak{h}$, for the creation and annihilation operators. Recall that for $u\in\Gamma^{(n)}(\mathfrak{h}):=\mathfrak{h}^{\otimes_s n}$, the n-particle sector; $\mathbf{a}^*(f)u=\sqrt{n+1}S_{n+1}f\otimes u\in\Gamma^{(n+1)}(\mathfrak{h})$. Here S_k is the symmetrization operator on $\mathfrak{h}^{\otimes k}$. We furthermore recall that $\mathbf{a}^*(f)$ and $\mathbf{a}(f)$ are closed and densely defined, and that $\mathcal{D}(\mathbf{a}(f))=\mathcal{D}(\mathbf{a}^*(f))$. They satisfy the CCR:

$$[\mathbf{a}^*(f),\mathbf{a}^*(g)]\;=\;[\mathbf{a}(f),\mathbf{a}(g)]\;=\;0\,,\quad[\mathbf{a}(f),\mathbf{a}^*(g)]\;=\;\langle f,g\rangle \tag{2.1}$$

and $\mathbf{a}(f)\Omega=0$, for $f\in\mathfrak{h}$. The field operator

$$\Phi(f)\;:=\;\mathbf{a}^*(f)\;+\;\mathbf{a}(f) \tag{2.2}$$

is self-adjoint on $\mathcal{D}(\mathbf{a}^*(f))=\mathcal{D}(\mathbf{a}(f))$ and essentially self-adjoint on $\Gamma_{\mathrm{fin}}(\mathfrak{h})$. In the case $\mathfrak{h}=\mathfrak{h}_{\mathrm{ph}}$ we have the relation with (1.3): $\mathbf{a}^*(f)=\int_{\mathbb{R}^\nu}f(k)\mathbf{a}^*(k)dk$ and $\mathbf{a}(f)=\int_{\mathbb{R}^\nu}\overline{f(k)}\mathbf{a}(k)dk$. In particular (2.2) and (1.7) coincide. We frequently write $\mathbf{a}^\#(k)$ to denote either $\mathbf{a}(k)$ or $\mathbf{a}^*(k)$. Similarly for $\mathbf{a}^\#(f)$. Recall that $\mathbf{a}(k)$ is well-defined on $\mathcal{C}_0^\infty=\Gamma_{\mathrm{fin}}(C_0^\infty(\mathbb{R}^\nu))$, but it is not closable. The domain of its adjoint $(\mathbf{a}(k))^*$ equals $\{0\}$. The "operator" $\mathbf{a}^*(k)$ should be understood as a form. See the monograph by Berezin [6].

Let b be a bounded operator between Hilbert spaces $\mathfrak{h}_1$ and $\mathfrak{h}_2$. We define $\Gamma(b):\Gamma(\mathfrak{h}_1)\to\Gamma(\mathfrak{h}_2)$ by its restriction to $\Gamma^{(n)}(\mathfrak{h}_1)$

$$\Gamma(b)_{|\Gamma^{(n)}(\mathfrak{h}_1)}\;:=\;\overbrace{b\otimes\cdots\otimes b}^{n\ \mathrm{times}}\,.$$

In particular we have $\Gamma(b)\Omega=\Omega$. Recall that $\Gamma(b)$ is bounded if and only if $\|b\|_{\mathcal{B}(\mathfrak{h}_1;\mathfrak{h}_2)}\leq 1$.

We introduce $d\Gamma(a)$ for operators $a:\mathfrak{h}\to\mathfrak{h}$ with domain $\mathcal{D}(a)$ by

$$d\Gamma(a)_{|\Gamma^{(n)}(\mathfrak{h})}\;:=\;a\otimes\mathbb{1}_{\mathfrak{h}}\otimes\cdots\otimes\mathbb{1}_{\mathfrak{h}}\;+\;\cdots\;+\;\mathbb{1}_{\mathfrak{h}}\otimes\cdots\otimes\mathbb{1}_{\mathfrak{h}}\otimes a\,, \tag{2.3}$$

a priori on the domain $\Gamma_{\mathrm{fin}}(\mathcal{D}(a))$. In particular; $d\Gamma(a)\Omega=0$. The operators $\Gamma(b)$ and $d\Gamma(a)$ are related through the formula $\Gamma(e^a)=e^{d\Gamma(a)}$ (suitably interpreted). It is easy to see that if a is closed (or closable) on $\mathcal{D}(a)$ then $d\Gamma(a)$ is closable on $\Gamma_{\rm fin}(\mathcal{D}(a))$. See [24, Section 3.2] for a simple proof, which applies also to similar situations below. In addition, if a is self-adjoint, then $d\Gamma(a)$ is essentially self-adjoint on $\Gamma_{\rm fin}(\mathcal{D}(a))$, cf. [50, Subsect. VIII.10, Theorem VIII.33 and Example 2]. For closed a we will by $d\Gamma(a)$ understand the closure of (2.3). Otherwise $d\Gamma(a)$ denotes the operator in (2.3) with the a priori domain $\Gamma_{\rm fin}(\mathcal{D}(a))$.

For a quadratic form a with form-domain $\mathcal{Q}(a)$ we also write $d\Gamma(a)$ for the quadratic form defined on $\Gamma_{\rm fin}(\mathcal{Q}(a))$ by (2.3).

An important operator is the number operator

$$N \ := \ d\Gamma(\mathbb{1}_{\mathfrak{h}})\,, \tag{2.4}$$

which in the case $\mathfrak{h} = \mathfrak{h}_{\rm ph}$ can be written as $N = \int_{\mathbb{R}^\nu} \mathbf{a}^*(k)\mathbf{a}(k)dk$. See also (1.4).

Let a and b be densely defined operators on $\mathfrak{h}$ and $v \in \mathcal{D}(a)$. We have the following commutation properties, which should be interpreted as forms on $\Gamma_{\rm fin}(\mathcal{D}(a^*)\cap\mathcal{D}(b^*)) \times \Gamma_{\rm fin}(\mathcal{D}(a)\cap\mathcal{D}(b))$ and $\Gamma_{\rm fin}(\mathcal{D}(a^*)) \times \Gamma_{\rm fin}(\mathcal{D}(a))$ respectively.

$$\begin{gathered} \mathrm{i}[d\Gamma(a), d\Gamma(b)] \ = \ d\Gamma(\mathrm{i}[a,b])\,, \\ [\mathbf{a}^*(v), d\Gamma(a)] \ = \ -\mathbf{a}^*(av)\,, \ \ [\mathbf{a}(v), d\Gamma(a)] \ = \ \mathbf{a}(av)\,, \\ \text{and} \ \ \mathrm{i}[\Phi(v), d\Gamma(a)] \ = \ -\,\Phi(\mathrm{i}av)\,. \end{gathered} \tag{2.5}$$

Let $b : \mathfrak{h}_1 \to \mathfrak{h}_2$ be a contraction and $a : \mathfrak{h}_1 \to \mathfrak{h}_2$ with domain $\mathcal{D}(a)$. We define $d\Gamma(b,a) : \Gamma(\mathfrak{h}_1) \to \Gamma(\mathfrak{h}_2)$ on $\Gamma_{\rm fin}(\mathcal{D}(a))$ by

$$d\Gamma(b,a)_{|\Gamma^{(n)}(\mathfrak{h}_1)} \ := \ a \otimes b \otimes \cdots \otimes b \ + \ \cdots \ + \ b \otimes \cdots \otimes b \otimes a\,. \tag{2.6}$$

In particular (in the case $\mathfrak{h}_1 = \mathfrak{h}_2 = \mathfrak{h}$) $d\Gamma(\mathbb{1}_{\mathfrak{h}}, a) = d\Gamma(a)$; cf. (2.3). If a is closed (or closable) we find, as above, that $d\Gamma(b,a)$ is closable on $\Gamma_{\rm fin}(\mathcal{D}(a))$. As for $d\Gamma(a)$ we use the notation $d\Gamma(b,a)$ also in the case where a is a form on $\mathfrak{h}_2 \times \mathfrak{h}_1$.

Let $b : \mathfrak{h}_1 \to \mathfrak{h}_2$ be a contraction, $a_1 : \mathfrak{h}_1 \to \mathfrak{h}_1$ and $a_2 : \mathfrak{h}_2 \to \mathfrak{h}_2$ be densely defined. As a form on $\Gamma_{\rm fin}(\mathcal{D}(a_2^*)) \times \Gamma_{\rm fin}(\mathcal{D}(a_1))$ we have

$$(\Gamma(b)\, d\Gamma(a_1) \ - \ d\Gamma(a_2)\,\Gamma(b)) \ = \ d\Gamma(b, (ba_1 - a_2 b))\,. \tag{2.7}$$

2.2 Basic estimates involving Γ

We have the following lemma

Lemma 2.1. *For $f \in \mathfrak{h}$ and $s \geq 0$, we have $\mathbf{a}^\#(f) : \mathcal{D}(N^{s+1/2}) \to \mathcal{D}(N^s)$ and the following holds true*

i) Let $f_1, \ldots, f_n \in \mathfrak{h}$. Then

$$\big\|(N+1)^k\, \mathbf{a}^\#(f_1)\cdots \mathbf{a}^\#(f_n)\,(N+1)^{-\frac{n}{2}-k}\big\| \ \leq \ C_{k,n}\|f_1\|\cdots\|f_n\|\,.$$

ii) The map

$$\mathfrak{h}^n \ni (f_1, \ldots, f_n) \to (N+1)^k \, \mathbf{a}^\#(f_1) \cdots \mathbf{a}^\#(f_n) \, (N+1)^{-\frac{n}{2}-k} \in \mathcal{B}(\Gamma(\mathfrak{h}))$$

is norm-continuous.

iii) Let $\{f_{1,l}\}_{l\in\mathbb{N}}, \ldots, \{f_{n,l}\}_{l\in\mathbb{N}}$ be uniformly bounded sequences, converging weakly to zero in $\mathfrak{h}$. Then

$$\mathrm{s}-\lim_{l\to\infty} (N+1)^k \, \mathbf{a}(f_{1,l}) \cdots \mathbf{a}(f_{n,l}) \, (N+1)^{-\frac{n}{2}-k} = 0 \, .$$

Suppose $b \in \mathcal{B}(\mathfrak{h}_1; \mathfrak{h}_2)$ is a contraction, $a_1 : \mathfrak{h}_1 \to \tilde{\mathfrak{h}}$ and $a_2 : \mathfrak{h}_2 \to \tilde{\mathfrak{h}}$. Define a as a form on $\mathcal{D}(a_2) \times \mathcal{D}(a_1)$ by $(f, ag) := (a_2 f, a_1 g)$. Then, for $v \in \Gamma_{\mathrm{fin}}(\mathcal{D}(a_1))$ and $u \in \Gamma_{\mathrm{fin}}(\mathcal{D}(a_2))$,

$$|\langle u, d\Gamma(b,a)v\rangle| \leq \langle u, d\Gamma(a_2^* a_2)u\rangle^{\frac{1}{2}} \, \langle v, d\Gamma(a_1^* a_1)v\rangle^{\frac{1}{2}} \, . \tag{2.8}$$

Here $a_\#^* a_\#$ denote the obvious forms on $\mathfrak{h}_\#$. Taking in particular $\tilde{\mathfrak{h}} = \mathfrak{h}_2$, $a_2 = 1\!\!1_{\mathfrak{h}_2}$, and $a_1 = a$ we get, for $v \in \Gamma_{\mathrm{fin}}(\mathcal{D}(a))$,

$$\|(N+1)^{-\frac{1}{2}} \, d\Gamma(b,a)v\| \leq \langle v, d\Gamma(a^* a)v\rangle^{\frac{1}{2}} \, . \tag{2.9}$$

In connection with this bound we also use the easy property

$$a \leq b \implies d\Gamma(a) \leq d\Gamma(b) \, , \tag{2.10}$$

where a and b are self-adjoint operators (or symmetric forms) on $\mathfrak{h}$. We also make use of the following estimate, cf. [27, Lemma A.2]. Let $k \in \mathbb{N}$ and let a and b be self-adjoint operators on $\mathfrak{h}$. If $0 \leq a^\ell \leq b^\ell$ for all $1 \leq \ell \leq k$, with $\ell \in \mathbb{N}$. Then

$$(d\Gamma(a))^k \leq (d\Gamma(b))^k \, . \tag{2.11}$$

We note that there are several bounds involving powers of second quantized operators, cf. e.g. [15, Lemma 3.2] and [24, Section 3.2] for a selection.

2.3 The extended space and $\check{\Gamma}$

Let $\mathfrak{h}_0$ and $\mathfrak{h}_\infty$ be two Hilbert spaces. We will use the standard unitary identification $U : \Gamma(\mathfrak{h}_0 \oplus \mathfrak{h}_\infty) \to \Gamma(\mathfrak{h}_0) \otimes \Gamma(\mathfrak{h}_\infty)$, which is determined uniquely by linearity and the two properties

$$U\Omega = \Omega \otimes \Omega \tag{2.12}$$

$$U \, \mathbf{a}^*((f,g)) = \big(\mathbf{a}^*(f) \otimes 1\!\!1_{\Gamma(\mathfrak{h}_\infty)} + 1\!\!1_{\Gamma(\mathfrak{h}_0)} \otimes \mathbf{a}^*(g)\big) \, U \, . \tag{2.13}$$

Let $a_0 : \mathfrak{h}_0 \to \mathfrak{h}_0$ and $a_\infty : \mathfrak{h}_\infty \to \mathfrak{h}_\infty$. We have the intertwining property

$$U \, d\Gamma(a_0 \oplus a_\infty) = \big(d\Gamma(a_0) \otimes 1\!\!1_{\Gamma(\mathfrak{h}_\infty)} + 1\!\!1_{\Gamma(\mathfrak{h}_0)} \otimes d\Gamma(a_\infty)\big) \, U \, , \tag{2.14}$$

as an identity on $\Gamma_{\rm fin}(\mathcal{D}(a_0)\oplus\mathcal{D}(a_\infty))$.

Let $\mathfrak{h}$, $\mathfrak{h}_0$ and $\mathfrak{h}_\infty$ be Hilbert spaces and let $b=(b_0,b_\infty)$, where $b_0\in\mathcal{B}(\mathfrak{h};\mathfrak{h}_0)$ and $b_\infty\in\mathcal{B}(\mathfrak{h};\mathfrak{h}_\infty)$. We view b as an element of $\mathcal{B}(\mathfrak{h};\mathfrak{h}_0\oplus\mathfrak{h}_\infty)$ and define the associated operator $\check\Gamma(b)$ by

$$\check\Gamma(b) \ := \ U\,\Gamma(b) \ : \ \Gamma(\mathfrak{h}) \ \to \ \Gamma(\mathfrak{h}_0)\otimes\Gamma(\mathfrak{h}_\infty)\,. \tag{2.15}$$

In this paper we always require $b_0^*b_0+b_\infty^*b_\infty=\mathbb{1}_{\mathfrak{h}}$, which implies $\|b\|_{\mathcal{B}(\mathfrak{h};\mathfrak{h}_0\oplus\mathfrak{h}_\infty)}=1$ and $\check\Gamma(b)$ is an isometry:

$$\check\Gamma(b)^*\,\check\Gamma(b) \ = \ \mathbb{1}_{\Gamma(\mathfrak{h})}\,. \tag{2.16}$$

We interpret $\check\Gamma(b)$ as a partition of unity.

Let $b=(b_0,b_\infty)$ be as above, and let $a=(a_0,a_\infty)$ be an operator from $\mathfrak{h}$ to $\mathfrak{h}_1\oplus\mathfrak{h}_2$, with domain $\mathcal{D}(a)=\mathcal{D}(a_0)\cap\mathcal{D}(a_\infty)$. We introduce the operator $d\check\Gamma(b,a):\Gamma_{\rm fin}(\mathcal{D}(a))\to\Gamma(\mathfrak{h}_0)\otimes\Gamma(\mathfrak{h}_\infty)$ by

$$d\check\Gamma(b,a) \ := \ U\,d\Gamma(b,a)\,. \tag{2.17}$$

We use the same notation for forms $a=(a_0,a_\infty)$, where $a_\#$ are forms on $\mathfrak{h}_\#\times\mathfrak{h}$.

Let $r:\mathfrak{h}\to\mathfrak{h}$, $q_0:\mathfrak{h}_0\to\mathfrak{h}_0$ and $q_\infty:\mathfrak{h}_\infty\to\mathfrak{h}_\infty$, be densely defined operators. We have the following intertwining relation, viewed as an identity between forms on $\{\Gamma_{\rm fin}(\mathcal{D}(q_0^*))\otimes\Gamma_{\rm fin}(\mathcal{D}(q_\infty^*))\}\times\Gamma_{\rm fin}(\mathcal{D}(r))$:

$$\check\Gamma(b)d\Gamma(r) \ - \ \big(d\Gamma(q_0)\otimes\mathbb{1}_{\Gamma(\mathfrak{h}_\infty)} \ + \ \mathbb{1}_{\Gamma(\mathfrak{h}_0)}\otimes d\Gamma(q_\infty)\big)\,\check\Gamma(b) \ = \ d\check\Gamma(b,a)\,, \tag{2.18}$$

where $a=(b_0r-q_0b_0,b_\infty r-q_\infty b_\infty)$ has form-domain $\{\mathcal{D}(q_0^*)\oplus\mathcal{D}(q_\infty^*)\}\times\mathcal{D}(r)$.

2.4 Basic estimates involving $\check\Gamma$

Let $b=(b_0,b_\infty)$ be as in (2.17). Let $a_{\#,1}:\mathfrak{h}\to\tilde{\mathfrak{h}}_\#$ and $a_{\#,2}:\mathfrak{h}_\#\to\tilde{\mathfrak{h}}_\#$, where $\tilde{\mathfrak{h}}_\#$ are auxiliary Hilbert spaces. Here $\#$ denotes 0 and ∞. We define a form $a=(a_0,a_\infty)$ on $\{\mathcal{D}(a_{0,2})\oplus\mathcal{D}(a_{\infty,2})\}\times\{\mathcal{D}(a_{0,1})\cap\mathcal{D}(a_{\infty,1})\}$ by prescribing the forms a_0 and a_∞ as follows: $(f,a_\# g):=(a_{\#,2}f,a_{\#,1}g)$ on $\mathcal{D}(a_{\#,2})\times\mathcal{D}(a_{\#,1})$.

Let $u_0\in\Gamma_{\rm fin}(\mathcal{D}(a_{0,2}))$, $u_\infty\in\Gamma_{\rm fin}(\mathcal{D}(a_{\infty,2}))$, $v\in\Gamma_{\rm fin}(\mathcal{D}(a_{0,1})\cap\mathcal{D}(a_{\infty,1}))$. The following key estimate follows from (2.14) and (2.8)

$$\begin{aligned}
&|\langle u_0\otimes u_\infty, d\check\Gamma(b,a)v\rangle|\\
&\quad\le\quad \big\{\langle u_0,d\Gamma(a_{0,2}a_{0,2}^*)u_0\rangle^{\frac12}\,\|u_\infty\| \ + \ \|u_0\|\,\langle u_\infty,d\Gamma(a_{\infty,2}a_{\infty,2}^*)u_\infty\rangle^{\frac12}\big\}\\
&\qquad\quad\times\langle v,d\Gamma(a_{0,1}^*a_{0,1}+a_{\infty,1}^*a_{\infty,1})v\rangle^{\frac12}\,.
\end{aligned}\tag{2.19}$$

Again $a_{\#,2}^*a_{\#,2}$ denote the obvious forms on $\mathcal{D}(a_{\#,2})$, and $a_{0,1}^*a_{0,1}+a_{\infty,1}^*a_{\infty,1}$ is a form on $\mathcal{D}(a_{0,1})\cap\mathcal{D}(a_{\infty,1})$.

As for (2.9) this implies (here $\tilde{\mathfrak{h}}_\#=\mathfrak{h}_\#$, $a_{\#,2}=\mathbb{1}_{\mathfrak{h}_\#}$, and $a_{\#,1}=a_\#$)

$$\|(N_0+N_\infty)^{-\frac12}d\check\Gamma(b,a)v\| \ \le \ \langle v,d\Gamma(a_{0,1}^*a_{0,1}+a_{\infty,1}^*a_{\infty,1})v\rangle^{\frac12}\,. \tag{2.20}$$

Here and in the following we use the notation (cf. (2.4))

$$N_0 \ = \ d\Gamma(\mathbb{1}_{\mathfrak{h}_0})\otimes\mathbb{1}_{\Gamma(\mathfrak{h}_\infty)} \quad\text{and}\quad N_\infty \ = \ \mathbb{1}_{\Gamma(\mathfrak{h}_0)}\otimes d\Gamma(\mathbb{1}_{\mathfrak{h}_\infty})\,. \tag{2.21}$$

2.5 Auxiliary spaces and operators

In this subsection we introduce some notation which will be used in the proof of the HVZ theorem in Subsect. 3.2.

We introduce auxiliary Hilbert spaces for an interacting system accompanied by a fixed number $\ell \geq 1$ of auxiliary photons

$$\mathcal{H}^{(\ell)} := \mathcal{F} \otimes \mathcal{F}^{(\ell)} \equiv L^2_{\mathrm{sym}}(\mathbb{R}^{\ell\nu}; \mathcal{F}) .$$

Here the subscript *sym* indicates that functions are symmetric under permutation, i.e. $f(k_{\tau(1)}, \dots, k_{\tau(\ell)}) = f(k_1, \dots, k_\ell)$ a.e., for any $\tau \in S(\ell)$ the group of permutations of the set $\{1, \dots, \ell\}$.

For $\ell \in \mathbb{N}$ we extend the notation for second quantization as follows

$$d\Gamma^{(\ell)}(a) = d\Gamma(a) \otimes \mathbb{1}_{\mathcal{F}^{(\ell)}} + \mathbb{1}_{\mathcal{F}} \otimes d\Gamma(a)_{|\mathcal{F}^{(\ell)}} ,$$

for operators a on $\mathfrak{h}_{\mathrm{ph}}$. Again $d\Gamma(a)$ defined on $\Gamma_{\mathrm{fin}}(\mathcal{D}(a)) \otimes \mathcal{D}(a)^{\otimes_s \ell}$ is closable (essentially self-adjoint) if a is closable (essentially self-adjoint). For the Hamiltonian we write

$$H^{(\ell)}(\xi) := H_0^{(\ell)}(\xi) + \Phi(v) \otimes \mathbb{1}_{\mathcal{F}^{(\ell)}} , \tag{2.22}$$

where

$$H_0^{(\ell)}(\xi) := d\Gamma^{(\ell)}(\omega) + \Omega\big(\xi - d\Gamma^{(\ell)}(k)\big) . \tag{2.23}$$

We note that $H_0^{(\ell)}(\xi)$ is essentially self-adjoint on

$$\mathcal{C}_0^{\infty(\ell)} := \mathcal{C}_0^\infty \otimes \Gamma^{(\ell)}(C_0^\infty(\mathbb{R}^\nu)) . \tag{2.24}$$

and write $\mathcal{D}^{(\ell)} = \mathcal{D}(H_0^{(\ell)}(\xi))$, which is independent of ξ. Observe that there is no interaction between the ℓ auxiliary photons, nor are they coupled with the interacting system (apart from the coupling coming from the dispersive structure). Note that as for Proposition 1.1, $\Phi(v) \otimes \mathbb{1}_{\mathcal{F}^{(\ell)}}$ is $H_0^{(\ell)}(\xi)$-bounded with relative bound 0, so $H^{(\ell)}(\xi)$ is essentially self-adjoint on $\mathcal{C}_0^{\infty(\ell)}$ and self-adjoint on $\mathcal{D}^{(\ell)}$.

Using a direct integral representation we can write the auxiliary Hamiltonian for each total momentum ξ as

$$H^{(\ell)}(\xi) = \int^{\oplus}_{\mathbb{R}^{\ell\nu}} H^{(\ell)}(\xi; k)\, d^{\ell\nu}k , \tag{2.25}$$

where

$$H^{(\ell)}(\xi; k) := H(\xi - k) + \big(\sum_{j=1}^{\ell} \omega(k_j)\big)\, \mathbb{1}_{\mathcal{F}} . \tag{2.26}$$

Here $d^{\ell\nu}k = \Pi_{j=1}^{\ell} d^\nu k_j$. We have a similar fibration of $H_0^{(\ell)}(\xi)$. The fiber operators, being spectral translates of a Hamiltonian at a different total momentum, are clearly self-adjoint on $\mathcal{D}$ and essentially self-adjoint on $\mathcal{C}_0^\infty$.

We note the following important observations

$$\Sigma_0^{(\ell)}(\xi;k) \;=\; \inf\left\{\sigma\big(H^{(\ell)}(\xi;k)\big)\right\}, \tag{2.27}$$

$$\Sigma_0^{(\ell)}(\xi) \;=\; \inf\left\{\sigma\big(H^{(\ell)}(\xi)\big)\right\}. \tag{2.28}$$

2.6 Geometric partition of unity and extended operators

In the analysis of the many-body problem, a central tool is a geometric partition of unity in the configuration space; cf. [13]. Here we will need a similar notion, made complicated by the fact that we have to partition an infinite number of particles. The type of partition of unity used here was introduced in [14] and subsequently used by many authors, cf. [2, 1, 15, 22, 24, 27].

Here $\mathfrak{h} = \mathfrak{h}_0 = \mathfrak{h}_\infty = \mathfrak{h}_{\mathrm{ph}}$. Let $j_0, j_\infty \in C^\infty(\mathbb{R}^\nu)$ be non-negative functions satisfying: $j_0 = 1$ on $\{k : |k| \le 1\}$, $j_0 = 0$ on $\{k : |k| > 2\}$, and finally $j_0^2 + j_\infty^2 = 1$. By j^R, $R > 1$, we understand the operator $j^R = (j_0(x/R), j_\infty(x/R))$. Recall that $x = \mathrm{i}\nabla_k$ is a differential operator. We view j^R as a map from $\mathfrak{h}_{\mathrm{ph}}$ into $\mathfrak{h}_{\mathrm{ph}} \oplus \mathfrak{h}_{\mathrm{ph}}$ and the operator $\check{\Gamma}(j^R)$ is an isometry, see (2.16),

$$\check{\Gamma}(j^R) : \mathcal{F} \;\to\; \mathcal{F}^{\mathrm{ext}} \;:=\; \mathcal{F} \otimes \mathcal{F} \quad\text{and}\quad \check{\Gamma}(j^R)^* \,\check{\Gamma}(j^R) \;=\; \mathbb{1}_{\mathcal{F}}\,. \tag{2.29}$$

The partition of unity is used to decouple photons at infinity from photons near the electron. In fact the reader should think of the first component as the Fock-space for interacting photons and the second component as the Fock-space for non-interacting photons at infinity.

As in the previous section we extend the notation for second quantization to these extended spaces. We will in general call operators constructed this way, extended operators. The simplest extended operator is the extended number operator, already encountered in Subsect. 2.4

$$N^{\mathrm{ext}} \;:=\; N_0 \;+\; N_\infty\,.$$

This is a particular case of the following notation, which will be used for operators a on $\mathfrak{h}_{\mathrm{ph}}$,

$$d\Gamma^{\mathrm{ext}}(a) \;=\; d\Gamma(a) \otimes \mathbb{1}_{\mathcal{F}} \;+\; \mathbb{1}_{\mathcal{F}} \otimes d\Gamma(a)\,. \tag{2.30}$$

As in the previous section $d\Gamma^{\mathrm{ext}}(a)$ is closable (essentially self-adjoint) if a is closable (essentially self-adjoint). Using this notation we introduce the extended Hamiltonian as

$$H^{\mathrm{ext}}(\xi) \;:=\; H_0^{\mathrm{ext}}(\xi) \;+\; \Phi(v) \otimes \mathbb{1}_{\mathcal{F}}\,, \tag{2.31}$$

where

$$H_0^{\mathrm{ext}}(\xi) \;:=\; d\Gamma^{\mathrm{ext}}(\omega) \;+\; \Omega\big(\xi \;-\; d\Gamma^{\mathrm{ext}}(k)\big)\,. \tag{2.32}$$

The free extended Hamiltonian (2.32) is essentially self-adjoint on $\mathcal{C}_0^\infty \otimes \mathcal{C}_0^\infty$ and we write $\mathcal{D}^{\mathrm{ext}} = \mathcal{D}(H_0^{\mathrm{ext}}(\xi))$, which is independent of ξ. Note that as for Proposition 1.1, $\Phi(v) \otimes \mathbb{1}_{\mathcal{F}}$ is $H_0^{\mathrm{ext}}(\xi)$-bounded with relative bound 0, so $H^{\mathrm{ext}}(\xi)$ is essentially self-adjoint on $\mathcal{C}_0^\infty \otimes \mathcal{C}_0^\infty$ and self-adjoint on $\mathcal{D}^{\mathrm{ext}}$.

Using the notation introduced in the previous subsection we have

$$\mathcal{F}^{\mathrm{ext}} \;=\; \mathcal{F} \,\oplus\, \Big\{ \bigoplus_{\ell=1}^{\infty} \mathcal{H}^{(\ell)} \Big\}\,, \tag{2.33}$$

and

$$H^{\mathrm{ext}}(\xi) \;=\; H(\xi) \oplus \Big\{ \bigoplus_{\ell=1}^{\infty} H^{(\ell)}(\xi) \Big\}\,. \tag{2.34}$$

2.7 The pull-through formula

In the following we use that $\mathbf{a}(k)$ makes sense as an operator on $\mathcal{C}^0 = \Gamma_{\mathrm{fin}}(\mathfrak{h}_{\mathrm{ph}} \cap C^0(\mathbb{R}^\nu))$. Here $C^0(\mathbb{R}^\nu)$ denotes the space of continuous functions on $\mathbb{R}^\nu$. Note that $\mathbf{a}(k) : \mathcal{C}^0 \to \mathcal{C}^0$, $\mathbf{a}(k) : \mathcal{C}_0^\infty \to \mathcal{C}_0^\infty$, and under the assumption $v \in L^2(\mathbb{R}^\nu) \cap C^0(\mathbb{R}^\nu)$, we have $H(\xi) : \mathcal{C}_0^\infty \to \mathcal{C}^0$. For the definition of $\mathcal{C}_0^\infty$, see (1.8). The type of formula presented here has been used previously in the study of ground states of translation invariant models, cf. [19], and confined models, see e.g. [5, 24, 26].

Proposition 2.2. *Suppose $v \in L^2(\mathbb{R}^\nu) \cap C^0(\mathbb{R}^\nu)$. Let $\xi \in \mathbb{R}^\nu$, $n \geq 1$, $\underline{k} \in \mathbb{R}^{n\nu}$, and $z \in \mathbb{C}$. For $\psi \in \mathcal{C}_0^\infty$ we have the identity*

$$\begin{aligned} &\mathbf{a}(k_1) \cdots \mathbf{a}(k_n)\, (H(\xi) \,-\, z)\, \psi \\ &\quad= \;\; \Big(H(\xi \,-\, k^{(n)}) \,+\, \sum_{i=1}^{n} \omega(k_i) \,-\, z \Big)\, \mathbf{a}(k_1) \cdots \mathbf{a}(k_n)\, \psi \\ &\qquad\quad + \sum_{i=1}^{n} v(k_i)\, \mathbf{a}(k_1) \cdots \widehat{\mathbf{a}(k_i)} \cdots \mathbf{a}(k_n)\, \psi\,, \end{aligned}$$

where $k^{(n)} = k_1 + \cdots + k_n$.

The notation $\widehat{\mathbf{a}(k_i)}$ indicates that the term $\mathbf{a}(k_i)$ is omitted from the product.

For $n = 1$ we formulate another pull through formula. Note that for $\psi \in \mathcal{D}(N^{\frac{1}{2}})$, the map $k \to \mathbf{a}(k)\psi$ is in $L^2(\mathbb{R}^\nu; \mathcal{F})$. In general, for $\psi \in \mathcal{F}$ we have $k \to \mathbf{a}(k)\psi$ in $L^2(\mathbb{R}^\nu; \mathcal{D}(N^{\frac{1}{2}})^*)$. The following proposition can be proved directly as in [26, Proposition 3.4], or by using Proposition 2.2 and an approximation argument.

Proposition 2.3. *Suppose $v \in L^2(\mathbb{R}^\nu)$. Let $\xi \in \mathbb{R}^\nu$ and $z \in \mathbb{C}$, $\mathrm{Im} z \neq 0$. For $\psi \in \mathcal{D}$, we have the $L^2(\mathbb{R}^\nu; \mathcal{F})$-identity*

$$\begin{aligned} &\big(H(\xi \,-\, k) \,+\, \omega(k) \,-\, z \big)^{-1}\, \mathbf{a}(k)\, (H(\xi) \,-\, z)\, \psi \\ &\quad = \;\; \mathbf{a}(k)\, \psi \;+\; v(k)\, \big(H(\xi \,-\, k) \,+\, \omega(k) \,-\, z \big)^{-1}\, \psi\,. \end{aligned}$$

3 Spectral theory

We start this section by giving a proof of Proposition 1.1. First some simple observations.

Since $1^\ell \leq m^{-\ell}\omega(k)^\ell$ for any $\ell \geq 0$, we obtain from (2.11) that $N^k \leq m^{-k} d\Gamma(\omega)^k$, for $k \in \mathbb{N}$. Since $0 \leq d\Gamma(\omega) \leq H_0(\xi)$ and they commute, we find that $d\Gamma(\omega)^k \leq H_0(\xi)^k$ for any $k \in \mathbb{N}$. We thus get $N^k \leq m^{-k} H_0(\xi)^k$, for $k \in \mathbb{N}$. This estimate in particular shows that for $k \in \mathbb{N}$

$$N^{\frac{k}{2}} \text{ is } H_0(\xi)^{\frac{k}{2}} - \text{bounded and } N^{\mathrm{ext}\,\frac{k}{2}} \text{ is } H_0^{\mathrm{ext}}(\xi)^{\frac{k}{2}} - \text{bounded}\,. \tag{3.1}$$

Proof of Proposition 1.1: We begin by showing that $D(H_0(\xi))$ is independent of ξ. We compute on $\mathcal{C}_0^\infty$ as an operator identity $H(\xi) - H(0) = \xi \cdot \int_0^1 \nabla\Omega(t\xi - d\Gamma(k))dt$. By Condition 1.1 i)–ii) and the estimate $ab \leq a^q + b^p$, $q^{-1} + p^{-1} = 1$ we obtain $\|(H(\xi) - H(0))\psi\| \leq \epsilon\|\Omega(d\Gamma(k))\psi\| + C(\epsilon,\xi)\|\psi\|$, for any $\epsilon > 0$ and $\psi \in \mathcal{C}_0^\infty$. That the domain is independent of ξ now follows from the Kato-Rellich theorem [48, Theorem X.12].

As for ii), the observation (3.1) (applied with $k = 1$), together with the $N^{1/2}$-boundedness of $\Phi(v)$, cf. Lemma 2.1 i), implies the result.

The last part follows from the variational principle and an argument similar to the one given for i). We leave it to the reader. □

Clearly Proposition 1.1 also holds with $\{H_0(\xi), H(\xi)\}$ replaced by either of the pairs $\{H_0^{\mathrm{ext}}(\xi), H^{\mathrm{ext}}(\xi)\}$ or $\{H_0^{(\ell)}(\xi), H^{(\ell)}(\xi)\}$.

We note the following consequence, for $k \in \{1,2\}$,

$$N^{\frac{k}{2}} \text{ is } H(\xi)^{\frac{k}{2}} - \text{bounded}\,,\ N^{\mathrm{ext}\,\frac{k}{2}} \text{ is } H^{\mathrm{ext}}(\xi)^{\frac{k}{2}} - \text{bounded}\,, \tag{3.2}$$

$$N^{(\ell)\,\frac{k}{2}} \text{ is } H^{(\ell)}(\xi)^{\frac{k}{2}} - \text{bounded}\,. \tag{3.3}$$

Here $N^{(\ell)} := d\Gamma^{(\ell)}(\mathbb{1}_{\mathfrak{h}_{\mathrm{ph}}})$.

3.1 Localization errors

In this subsection we show that localization errors arising when we apply $\check{\Gamma}(j^R)$ are small for large R.

Lemma 3.1. *Let $s \in \mathbb{N}_0 \cap [0, s_\Omega]$ and $f \in C^\infty(\mathbb{R}^\nu)$ satisfy the bound $|(\partial^\alpha f)(\eta)| \leq C_\alpha\langle\eta\rangle^{s-|\alpha|}$, for any multi-index α. Let $t = 1$, if $s = 0$, and $t = (1 + s_\Omega - s)/2$ if $s \geq 1$. We have as a form on $\mathcal{F}^{\mathrm{ext}} \times \mathcal{F}$,*

$$\begin{aligned}(H_0^{\mathrm{ext}}(\xi) - \mathrm{i})^{-1}\big(\check{\Gamma}(j^R)\, f(\xi - d\Gamma(k)) \;-\; f(\xi - d\Gamma^{\mathrm{ext}}(k))\,\check{\Gamma}(j^R)\big)(H_0(\xi) - \mathrm{i})^{-1}\\ = \; (H_0^{\mathrm{ext}}(\xi) - \mathrm{i})^{-t} B_1(R) = B_2(R)(H_0(\xi) - \mathrm{i})^{-t}\,,\end{aligned}$$

where B_1 and B_2 are families of bounded operators which satisfy $\|B_1(R)\| + \|B_2(R)\| = O(R^{-1/2})$, as $R \to \infty$, locally uniformly in ξ.

Proof. As a first step we compute as a form on $(\mathcal{C}_0^\infty \otimes \mathcal{C}_0^\infty) \times \mathcal{C}_0^\infty$, for $1 \le p \le \nu$,

$$\check{\Gamma}(j_R)\, d\Gamma(k_{;p}) \;-\; d\Gamma^{\mathrm{ext}}(k_{;p})\, \check{\Gamma}(j_R) \;=\; d\check{\Gamma}(j^R, s_p^R)\,, \tag{3.4}$$

$s_p^R = ([j_0^R, k_{;p}], [j_\infty^R, k_{;p}])$. Clearly s_p^R are bounded operators and

$$[j_\#^R, k_{;p}] \;=\; O(R^{-1})\,, \;\; \text{as } R \to \infty\,. \tag{3.5}$$

Here we used the notation $k_{;p}$ to denote the p'th coordinate of a vector $k \in \mathbb{R}^\nu$. (This notation should not be confused with the labeling k_j of a family of vectors $k_j \in \mathbb{R}^\nu$.)

We consider first the case $s = 0$. Let $\tilde{f} \in C^\infty(\mathbb{C}^\nu)$ denote an almost analytic extension of f. Let $\chi \in C_0^\infty(\mathbb{R}^\nu)$ be equal to 1 near 0. Write $\chi_n(\eta) = \chi(\eta/n)$. Then $f_n = \chi_n f$ has almost analytic extensions $\tilde{f}_n$ satisfying that, for all $z \in \mathbb{C}^\nu$: $\bar{\partial}\tilde{f}_n(z) \to \bar{\partial}\tilde{f}(z)$, and the estimates

$$|\bar{\partial}\tilde{f}_n(z)| \le C_\ell \langle z\rangle^{-1-\ell} |\mathrm{Im} z|^\ell \tag{3.6}$$

hold uniformly in n, cf. (A.3). If we take for example the Borel construction (A.2), for $\tilde{f}$ and the $\tilde{f}_n$'s, then this property is easy to verify. This well-known approximation technique has been used by many authors (in the case $\nu = 1$), see e.g. [51, Section 5] and [45, Section 4].

We use (3.4) to compute as a form on $(\mathcal{C}_0^\infty \otimes \mathcal{C}_0^\infty) \times \mathcal{C}_0^\infty$, for $\mathrm{Im} z \neq 0$,

$$\begin{aligned} & T(z;R) \;:=\; \check{\Gamma}(j^R)\, |\xi - d\Gamma(k) - z|^2 \;-\; |\xi - d\Gamma^{\mathrm{ext}}(k) - z|^2\, \check{\Gamma}(j^R) \\ = \;\; & \sum_{p=1}^{\nu} \Big\{ d\check{\Gamma}(j^R, s_p^R)\, (\xi_{;p} - d\Gamma(k_{;p}) - z_{;p}) + (\xi_{;p} - d\Gamma^{\mathrm{ext}}(k_{;p}) + z_{;p})\, d\check{\Gamma}(j^R, s_p^R) \Big\}\,. \end{aligned}$$

Using (2.10), (2.20) (with $\mathfrak{h} = \tilde{\mathfrak{h}}_\# = \mathfrak{h}_\# = \mathfrak{h}_{\mathrm{ph}}$ and $a_{\#,1} = [j_\#^R, k_{;p}]$), and (3.5), we conclude the following estimate

$$\begin{aligned} & (N^{\mathrm{ext}} + 1)^{-\frac{1}{2}} |\xi - d\Gamma^{\mathrm{ext}}(k) - z|^{-1} T(z;R)\, |\xi - d\Gamma(k) - z|^{-1} (N+1)^{-\frac{1}{2}} \\ & \;=\; O\big(|Imz|^{-1} R^{-1}\big)\,. \end{aligned} \tag{3.7}$$

The estimate is valid uniformly in ξ and $\mathrm{Re} z = \{\mathrm{Re} z_1, \dots, \mathrm{Re} z_\nu\}$.

We proceed to compute

$$\begin{aligned} & \check{\Gamma}(j^R)\, |\xi - d\Gamma(k) - z|^{-2\nu} \;-\; |\xi - d\Gamma^{\mathrm{ext}}(k) - z|^{-2\nu}\, \check{\Gamma}(j^R) \\ = \;\; & -|\xi - d\Gamma(k) - z|^{-2\nu} \Big\{ \check{\Gamma}(j^R)\, |\xi - d\Gamma(k) - z|^{2\nu} \\ & \quad - |\xi - d\Gamma^{\mathrm{ext}}(k) - z|^{2\nu}\, \check{\Gamma}(j^R) \Big\}\, |\xi - d\Gamma(k) - z|^{-2\nu} \\ = \;\; & -\sum_{j=0}^{\nu-1} |\xi - d\Gamma(k) - z|^{-2(\nu-j)} T(z;R)\, |\xi - d\Gamma(k) - z|^{-2(j+1)}\,. \end{aligned} \tag{3.8}$$

Combining this identity with (3.7), we obtain the estimate

$$\begin{aligned}
&(N^{\mathrm{ext}}+1)^{-\frac{1}{2}}\Big\{\check{\Gamma}(j^R)\,|\xi-d\Gamma(k)-z|^{-2\nu} \\
&\qquad -|\xi-d\Gamma^{\mathrm{ext}}(k)-z|^{-2\nu}\,\check{\Gamma}(j^R)\Big\}(N+1)^{-\frac{1}{2}} \\
&= \ |\xi-d\Gamma^{\mathrm{ext}}(k)-z|^{-1}\,O\big(|Imz|^{-2\nu}R^{-1}\big) \\
&= \ O\big(|Imz|^{-2\nu}R^{-1}\big)\,|\xi-d\Gamma(k)-z|^{-1}\,. \qquad (3.9)
\end{aligned}$$

A small calculation using (3.4) (and again the estimates (2.10), (2.20), and (3.5)) in conjunction with (3.6) and (3.9) gives the following estimate for all $1\leq p\leq\nu$ and $\ell\geq 0$

$$\begin{aligned}
&\bar{\partial}_p\tilde{f}_n(z)(N^{\mathrm{ext}}+1)^{-\frac{1}{2}}\,\Big\{\check{\Gamma}(j^R)\,(\xi_{;p}-d\Gamma(k_{;p})+z_{;p})\,|\xi-d\Gamma(k)-z|^{-2\nu} \\
&\quad-(\xi_{;p}-d\Gamma^{\mathrm{ext}}(k_{;p})+z_{;p})\,|\xi-d\Gamma^{\mathrm{ext}}(k)-z|^{-2\nu}\,\check{\Gamma}(j^R)\Big\}(N+1)^{-\frac{1}{2}} \\
&= \ O\big(\langle z\rangle^{-\ell-1}|\mathrm{Im}z|^{\ell-2\nu}R^{-1}\big)\,. \qquad (3.10)
\end{aligned}$$

By choosing $\ell=2\nu$, in order to dampen the singularity at the real axis, we get an integrable weight factor $\langle z\rangle^{-2\nu-1}$, uniformly in n. We can now invoke the Lebesgue theorem on dominated convergence, and remove the cutoff by taking $n\to\infty$ in the representation formula (A.4). This gives finally

$$\begin{aligned}
&(N^{\mathrm{ext}}+1)^{-\frac{1}{2}}\Big\{\check{\Gamma}(j^R)\,f(\xi-d\Gamma(k)) \ - \ f(\xi-d\Gamma^{\mathrm{ext}}(k))\,\check{\Gamma}(j^R)\Big\}(N+1)^{-\frac{1}{2}} \\
&= \ O(R^{-1})\,.
\end{aligned}$$

Note that the term in the brackets above is a bounded operator with norm bounded uniformly in R and ξ. We thus get by interpolation (and since powers of N can be moved around as we please) for $0\leq\rho\leq 1/2$.

$$\begin{aligned}
&(N^{\mathrm{ext}}+1)^{\rho-\frac{1}{2}}\Big\{\check{\Gamma}(j^R)\,f(\xi-d\Gamma(k)) \ - \ f(\xi-d\Gamma^{\mathrm{ext}}(k))\,\check{\Gamma}(j^R)\Big\}(N+1)^{-\rho} \\
&= \ O(R^{-\frac{1}{2}})\,. \qquad (3.11)
\end{aligned}$$

By (3.1), this concludes the proof for the case $s=0$.

Next we consider the case $s=1$ (and hence $s_\Omega\in\{1,2\}$). Use Taylor's formula to write $f(\eta)=f(0)+\eta\cdot F_0(\eta)$, where $F_0(\eta)=\int_0^1(\nabla f)(t\eta)dt$. It is easy to verify that F_0's coordinate functions satisfy the assumption of the lemma with $s=0$. From (3.4) (again combined with (2.10), (2.20), and (3.5)) and (3.11) we get, as a form estimate on $(\mathcal{C}_0^\infty\otimes\mathcal{C}_0^\infty)\times\mathcal{C}_0^\infty$,

$$\begin{aligned}
&(N^{\mathrm{ext}}+1)^{\rho-\frac{1}{2}}\Big\{\check{\Gamma}(j^R)\,f(\xi-d\Gamma(k)) \ - \ f(\xi-d\Gamma^{\mathrm{ext}}(k))\,\check{\Gamma}(j^R)\Big\}(N+1)^{-\rho} \\
&= \ O(R^{-\frac{1}{2}}) \ + \ \sum_{p=1}^{\nu}(\xi_{;p}-d\Gamma^{\mathrm{ext}}(k_{;p}))\,O(R^{-\frac{1}{2}}) \\
&= \ O(R^{-\frac{1}{2}}) \ + \ \sum_{p=1}^{\nu}O(R^{-\frac{1}{2}})\,(\xi_{;p}-d\Gamma(k_{;p}))\,. \qquad (3.12)
\end{aligned}$$

Note that if $s_\Omega = 1$ then $d\Gamma(k)$ is $H_0(\xi)$-bounded, and if $s_\Omega = 2$ then $d\Gamma(k)$ is $H_0(\xi)^{1/2}$-bounded. Corresponding relative bounds for the extended operators hold as well. This implies the lemma for $s = 1$.

In the remaining case $s = 2$ (and hence $s_\Omega = 2$). We proceed in a similar fashion, writing $f(\eta) = f(0) + \eta \cdot F_1(\eta)$, where F_1's coordinate functions satisfy the assumptions of the lemma with $s = 1$. Since in this case $d\Gamma(k)$ and $F_1(\xi - d\Gamma(k))$ are $H_0(\xi)^{1/2}$-bounded, the result follows (by a similar argument) from the $s = 1$ case. $\square$

Lemma 3.2. *We have as a form on $\mathcal{F}^{\mathrm{ext}} \times \mathcal{F}$,*

$$\begin{aligned}
&(H_0^{\mathrm{ext}}(\xi) - \mathrm{i})^{-1}\big\{\check{\Gamma}(j^R)\, H(\xi) \; - \; H^{\mathrm{ext}}(\xi)\, \check{\Gamma}(j^R)\big\}(H_0(\xi) - \mathrm{i})^{-1} \\
&\quad = \; (H_0^{\mathrm{ext}}(\xi) - \mathrm{i})^{-\frac{1}{2}} B_1(R) = B_2(R)(H_0(\xi) - \mathrm{i})^{-\frac{1}{2}} \; ,
\end{aligned}$$

where B_1 and B_2 are families of bounded operators satisfying $\|B_1(R)\| + \|B_2(R)\| = o(1)$, as $R \to \infty$, locally uniformly in ξ.

Proof. By Lemma 3.1, applied with $f = \Omega$ and $s = s_\Omega$, we only need to prove the lemma with $H(\xi)$ replaced by $d\Gamma(\omega)$ and $\Phi(v)$, and $H^{\mathrm{ext}}(\xi)$ replaced by $d\Gamma^{\mathrm{ext}}(\omega)$ and $\Phi(v) \otimes \mathbb{1}_{\mathcal{F}}$ respectively.

We begin by computing as a form on $\mathcal{D}^{\mathrm{ext}} \times \mathcal{D}$

$$\check{\Gamma}(j_R)\, d\Gamma(\omega) \; - \; d\Gamma^{\mathrm{ext}}(\omega)\, \check{\Gamma}(j_R) \; = \; d\check{\Gamma}(j^R, r^R) \; ,$$

where $r^R = ([j_0^R, \omega], [j_\infty^R, \omega])$. By Condition 1.2 iii) and pseudo differential calculus, the components of r^R satisfies, as operators on $\mathcal{D}(\omega^{\frac{1}{2}})^*$,

$$\omega^{-\frac{1}{2}}\, [j_\#^R, \omega]\, \omega^{-\frac{1}{2}} \; = \; O(R^{-1}) \; , \quad \text{for} \;\; R \to \infty \; .$$

(Alternatively one could also use here the calculus of almost analytic extensions.) The contribution to B_1 and B_2 coming from $d\Gamma(\omega)$ thus satisfies the required bounds by (2.10), (2.19), and (3.1). Here we choose $\mathfrak{h} = \mathfrak{h}_\# = \mathfrak{h}_{\mathrm{ph}}$, $\tilde{\mathfrak{h}}_\# = \mathcal{D}(\omega^{\frac{1}{2}})^*$, $a_{\#,2} = \omega^{\frac{1}{2}}$, and $a_{\#,1} = \{\omega^{-\frac{1}{2}}[j_\#^R, \omega]\omega^{-\frac{1}{2}}\}\omega^{\frac{1}{2}}$, when applying (2.19).

It remains to treat the contribution from the perturbation. We compute as a form on $\mathcal{D}^{\mathrm{ext}} \times \mathcal{D}$, using [14, Lemma 2.14 (iii)]

$$\begin{aligned}
&\check{\Gamma}(j^R)\, \Phi(v) \; - \; \Phi(v) \, \otimes \, \mathbb{1}_{\mathcal{F}}\, \check{\Gamma}(j^R) \\
&\quad = \; -\frac{1}{\sqrt{2}}\Big\{ \big(\mathbf{a}^*((1 - j_0^R)v) \, \otimes \, \mathbb{1}_{\mathcal{F}} \; + \; \mathbb{1}_{\mathcal{F}} \, \otimes \, \mathbf{a}^*(j_\infty^R v)\big)\, \check{\Gamma}(j^R) \\
&\qquad + \check{\Gamma}(j^R)\, \mathbf{a}((1 - j_0^R)v) \Big\} \; .
\end{aligned}$$

Eq. (3.1) and Lemma 2.1 ii) now yield the result, since $\mathrm{s}{-}\lim_{R\to\infty} j_\infty^R = \mathrm{s}{-}\lim_{R\to\infty}(1 - j_0^R) = 0$ and $v \in L^2(\mathbb{R}^\nu)$. $\square$

We immediately get the following two corollaries.

Corollary 3.3. *We have for any $R > 1$*

$$\check{\Gamma}(j^R) \;:\; \mathcal{D} \;\to\; \mathcal{D}^{\mathrm{ext}}_{1/2} \;\; \textit{and} \;\; \check{\Gamma}(j^R)^* \;:\; \mathcal{D}^{\mathrm{ext}} \;\to\; \mathcal{D}_{1/2}\,,$$

where $\mathcal{D}_{1/2} = \mathcal{D}(H_0(\xi)^{1/2})$ and $\mathcal{D}^{\mathrm{ext}}_{1/2} = \mathcal{D}(H_0^{\mathrm{ext}}(\xi)^{1/2})$ are independent of ξ.

The first part of the following corollary follows from Lemma 3.2 while the second part follows from the first part and the calculus of almost analytic extensions (with $\nu = 1$), as presented in Subsect. A.1.

Corollary 3.4. *We have, in the limit $R \to \infty$,*

i) The following estimate holds true locally uniformly in ξ and $z \in \mathbb{C}$ with $\mathrm{Im} z \neq 0$

$$\check{\Gamma}(j^R)\,(H(\xi)-z)^{-1} \;-\; (H^{\mathrm{ext}}(\xi)-z)^{-1}\,\check{\Gamma}(j^R) \;=\; |\mathrm{Im} z|^{-2}\,o(1)\,.$$

ii) For $f \in C_0^\infty(\mathbb{R})$, we have uniformly in ξ

$$\check{\Gamma}(j^R)\,f(H(\xi)) \;-\; f(H^{\mathrm{ext}}(\xi))\,\check{\Gamma}(j^R) \;=\; o(1)\,.$$

3.2 The HVZ-Theorem

In this Subsect. we prove Theorem 1.2.

Recall the abbreviations $\underline{k} = (k_1,\dots,k_n) \in \mathbb{R}^{n\nu}$ and $k^{(n)} = k_1 + \cdots + k_n$. We start by establishing three lemmas

Proof of Lemma 1.7: Suppose to the contrary that $\underline{k} \notin \mathcal{I}_0^{(n)}(\xi)$, that is $\Sigma_0(\xi - k^{(n)}) \geq \Sigma_{\mathrm{ess}}(\xi - k^{(n)})$, cf. (1.18). Then there exist $\ell \geq 1$ and $k_{n+1},\dots,k_{n+\ell}$, cf. (1.11), such that (writing $k^{(n+\ell)} = \sum_{i=1}^{n+\ell} k_i$)

$$\begin{aligned} \Sigma_0^{(n)}(\xi;\underline{k}) \;&=\; \Sigma_0\big(\xi \,-\, k^{(n)}\big) \;+\; \sum_{i=1}^{n}\omega(k_i) \\ &\geq\; \Sigma_0\big(\xi \,-\, k^{(n+\ell)}\big) \;+\; \sum_{i=1}^{n+\ell}\omega(k_i) \\ &\geq\; \Sigma_0^{(n+\ell)}(\xi) \;>\; \Sigma_0^{(n)}(\xi;\underline{k})\,, \end{aligned}$$

which is a contradiction. This proves the lemma. □

Lemma 3.5. *Let $n \geq 1$, and $B \in L^2_{\mathrm{sym}}(\mathbb{R}^{n\nu};\mathcal{B}(\mathcal{F}))$. Suppose $B(\underline{k})$ commute with N for almost all $\underline{k} \in \mathbb{R}^{n\nu}$. Define for $\psi \in \mathcal{C}_0^\infty$ the map*

$$\mathbf{a}(B)\,\psi \;:=\; \int_{\mathbb{R}^{n\nu}} B(\underline{k})\,\mathbf{a}(k_1)\cdots\mathbf{a}(k_n)\,\psi\,d^{n\nu}\underline{k}\,.$$

Then $(N+1)^{-n/2}\mathbf{a}(B)$ extends from $\mathcal{C}_0^\infty$ to a bounded operator on $\mathcal{F}$ and there exists $C = C(n)$ such that

$$C^{-1}\,\big\|(N+1)^{-n/2}\,\mathbf{a}(B)\big\|_{\mathcal{B}(\mathcal{F})} \;\leq\; \|B\| \;:=\; \Big(\int_{\mathbb{R}^{n\nu}} \|B(\underline{k})\|^2_{\mathcal{B}(\mathcal{F})}\,d^{n\nu}\underline{k}\Big)^{\frac{1}{2}}\,. \qquad (3.13)$$

Proof. Let $\psi \in \mathcal{C}_0^\infty$ and $\varphi \in \mathcal{F}$, with $\|\varphi\| = 1$. We estimate

$$\begin{aligned}
&\big|\big\langle \varphi, (N+n+1)^{-n/2}\, \mathbf{a}(B)\, \psi \big\rangle\big| \\
&\le \int_{\mathbb{R}^{n\nu}} \big|\big\langle \varphi, (N+n+1)^{-\frac{n}{2}} B(\underline{k})\, \mathbf{a}(k_1)\cdots \mathbf{a}(k_n)\, \psi \big\rangle\big|\, d^{n\nu}\underline{k} \\
&= \int_{\mathbb{R}^{n\nu}} \big|\big\langle \varphi, B(\underline{k})\, \mathbf{a}(k_1)\cdots \mathbf{a}(k_n)\, (N+1)^{-\frac{n}{2}} \psi \big\rangle\big|\, d^{n\nu}\underline{k} \\
&\le \int_{\mathbb{R}^{n\nu}} \|B(\underline{k})\|_{\mathcal{B}(\mathcal{F})}\, \big\|\mathbf{a}(k_1)\cdots \mathbf{a}(k_n)\, (N+1)^{-\frac{n}{2}} \psi\big\|\, d^{n\nu}\underline{k} \\
&\le \|B\| \Big(\int_{\mathbb{R}^{n\nu}} \big\|\mathbf{a}(k_1)\cdots \mathbf{a}(k_n)\, (N+1)^{-\frac{n}{2}} \psi\big\|^2\, d^{n\nu}\underline{k} \Big)^{\frac{1}{2}} \\
&\le \|B\|\, \|\psi\| \,.
\end{aligned}$$

Here we used the representation $N = \int_{\mathbb{R}^\nu} \mathbf{a}^*(k)\mathbf{a}(k) d^\nu k$ repeatedly in the last step. This estimate yields the lemma (with $C = ((n+1)/2)^{n/2}$). □

Lemma 3.6. *Let $\chi \in C_0^\infty(\mathbb{R})$ and $\xi \in \mathbb{R}^\nu$. Then, for all $k, \ell \ge 0$, the form $N^k \chi(H(\xi)) N^\ell$ extends from $\mathcal{C}_0^\infty$ to a bounded form on $\mathcal{D}^*$.*

Remark. We employ the standard triple: $\mathcal{D} \subset \mathcal{F} \subset \mathcal{D}^*$ continuously and densely.
Proof: Recall from [14, Lemma 3.2] that $N^k \chi(H(\xi)) N^\ell$ extends to a bounded form on $\mathcal{F}$. It remains to prove that it extends further by continuity to $\mathcal{D}^*$. It is sufficient to verify that $H(\xi) N^k \chi(H(\xi))$, viewed as a form on $\mathcal{C}_0^\infty \times \mathcal{F}$, extends to a bounded form on $\mathcal{F} \otimes \mathcal{F}$.

Let $\psi \in \mathcal{C}_0^\infty$ and $\varphi \in \mathcal{F}$. We compute for $k \ge 1$,

$$\begin{aligned}
&\big\langle H(\xi)\, \psi,\, (N+1)^k \chi(H(\xi))\, \varphi \big\rangle \\
&\quad = \big\langle \Phi(v)\, \psi,\, (N+1)^k \chi(H(\xi))\, \varphi \big\rangle \;+\; \big\langle (N+1)^k \psi,\, H_0(\xi)\, \chi(H(\xi))\, \varphi \big\rangle \\
&\quad = \big\langle (N+1)^{-\frac{1}{2}} \Phi(v)\, \psi,\, (N+1)^{k+\frac{1}{2}} \chi(H(\xi))\, \varphi \big\rangle \\
&\qquad + \big\langle (N+1)^k \psi,\, H(\xi)\, \chi(H(\xi))\, \varphi \big\rangle \\
&\qquad - \big\langle (N+1)^{-k-\frac{1}{2}} \Phi(v)\, (N+1)^k \psi,\, (N+1)^{k+\frac{1}{2}}\, \chi(H(\xi)) \varphi \big\rangle \,.
\end{aligned}$$

An application of Lemma 2.1 i) now yields the result. □

Proof of Theorem 1.2: We begin with i). Let $\xi \in \mathbb{R}^\nu$ and let $f \in C_0^\infty(\mathbb{R})$ be such that $\operatorname{supp} f \subset (-\infty, \Sigma_{\text{ess}}(\xi))$. By definition of $\Sigma_{\text{ess}}(\xi)$ (see (1.9–1.11)), (2.21), (2.33), (2.34), and (2.28), we observe that

$$\begin{aligned}
H^{\text{ext}}(\xi)\, \mathbb{1}(N_\infty \ge 1) \;&=\; \bigoplus_{\ell=1}^{\infty} H^{(\ell)}(\xi) \\
\ge\;& \bigoplus_{\ell=1}^{\infty} \Sigma_0^{(\ell)}(\xi)\, \mathbb{1}_{\mathcal{H}^{(\ell)}} \;\ge\; \Sigma_{\text{ess}}(\xi)\, \mathbb{1}(N_\infty \ge 1) \,.
\end{aligned}$$

Here we used the identification $\mathbb{1}_{\mathcal{H}^{(\ell)}} = \mathbb{1}(N_\infty = \ell)$. The lower bound above, together with (2.29) and Corollary 3.4 ii), yields

$$\begin{aligned} f(H(\xi)) &= \check{\Gamma}(j^R)^* f(H^{\mathrm{ext}}(\xi))\, \check{\Gamma}(j^R) \;+\; o(1) \\ &= \Gamma(j_0^R)\, f(H(\xi))\, \Gamma(j_0^R) \\ &\quad + \check{\Gamma}(j^R)^* f(H^{\mathrm{ext}}(\xi))\, \mathbb{1}(N_\infty \geq 1)\, \check{\Gamma}(j^R) \;+\; o(1) \\ &= \Gamma(j_0^R)\, f(H(\xi))\, \Gamma(j_0^R) \;+\; o(1)\,, \quad \text{for} \;\; R \to \infty\,. \end{aligned}$$

The first term on the right-hand side is compact, by a standard argument using Condition 1.2 ii). This implies that $f(H(\xi))$ is a compact operator, and hence; that the spectrum of $H(\xi)$ below $\Sigma_{\mathrm{ess}}(\xi)$ is locally finite.

As for ii), fix $\xi \in \mathbb{R}^\nu$ and $\lambda \geq \Sigma_{\mathrm{ess}}(\xi)$. We wish to show that there exists $n_0 \geq 1$ and $\underline{\eta} = (\eta_1, \dots, \eta_{n_0}) \in \mathbb{R}^{n_0\nu}$ such that

$$\lambda \;=\; \Sigma_0(\xi - \eta^{(n_0)}) \;+\; \sum_{i=1}^{n_0} \omega(\eta_i) \;\; \text{and} \;\; \underline{\eta} \in \mathcal{I}_0^{(n_0)}(\xi)\,, \tag{3.14}$$

where $\eta^{(n_0)} = \sum_{i=1}^{n_0} \eta_i$.

Let n_0 be given by $n_0 + 1 = \min\{n : \lambda < \min_{n' \geq n} \Sigma_0^{(n')}(\xi)\}$. The minima exist, and $n_0 \geq 1$, due to (1.11) and (1.15). There exists $\underline{k} = (k_1, \dots, k_{n_0})$ such that $\Sigma_0^{(n_0)}(\xi) = \Sigma_0(\xi - k^{(n_0)}) + \sum_{i=1}^{n_0} \omega(k_i) \leq \lambda$, where $k^{(n_0)} = k_1 + \cdots + k_{n_0}$. By Condition 1.2 ii), (1.14), and continuity of $\Sigma_0(\xi)$, cf. Proposition 1.1, we can find $\underline{\eta}$ such that the first part of (3.14) is fulfilled. The choice of n_0 and Lemma 1.7 implies the last part.

By i); $\Sigma_0(\xi - \eta^{(n_0)})$, given by (3.14), is an eigenvalue for $H(\xi - \eta^{(n_0)})$. We write φ_0 for a corresponding ground state; $H(\xi - \eta^{(n_0)})\varphi_0 = \Sigma_0(\xi - \eta^{(n_0)})\varphi_0$. Let $f \in C_0^\infty(\mathbb{R}^\nu)$ with $f \geq 0$ and $f(0) = 1$. Write $f_{i,\ell}(k) = \ell^{\nu/2} f(\ell(k - \eta_i))$. Then $\{f_{1,\ell}\}_{\ell\in\mathbb{N}}, \dots, \{f_{n_0,\ell}\}_{\ell\in\mathbb{N}}$ is a family of uniformly bounded sequences in $\mathfrak{h}_{\mathrm{ph}}$, which all converge weakly to 0.

Let $\psi_\ell = \mathbf{a}^*(f_{n_0,\ell}) \cdots \mathbf{a}^*(f_{1,\ell})\varphi_0$. The rest of the proof is concerned with showing that ψ_ℓ is a Weyl sequence for the energy λ. Note that by Lemma 3.6 and Lemma 2.1 i), we have $\varphi_0 \in \mathcal{D}(\mathbf{a}^*(f_{n_0,\ell}) \cdots \mathbf{a}^*(f_{1,\ell}))$. Lemma 2.1 iii) furthermore implies that $\{\psi_\ell\}_{\ell\in\mathbb{N}}$ converges weakly to zero in $\mathcal{F}$.

For ψ_ℓ to be a Weyl sequence it must satisfy $\|\psi_\ell\| > 0$ uniformly in ℓ. Let $S(n)$ denote the group of permutations of n elements, and write $(\sigma\underline{k})_j = k_{\sigma(j)}$, for $\sigma \in S(n)$ and $\underline{k} \in \mathbb{R}^{n\nu}$.

Let n be such that $\varphi_0^{(n)} \neq 0$. Pick a compact (and non-empty) set $\mathcal{K} \subset \mathbb{R}^{n\nu}$ with the following properties: ($\mathcal{K}$1) If $\underline{k} \in \mathcal{K}$ then $\sigma\underline{k} \in \mathcal{K}$, $\sigma \in S(n)$. ($\mathcal{K}$2) For $\underline{k} \in \mathcal{K}$ we have $k_i \neq \eta_j$, $1 \leq i \leq n$ and $1 \leq j \leq n_0$. ($\mathcal{K}$3) $\mathbb{1}(\underline{k} \in \mathcal{K})\varphi_0^{(n)} \neq 0$.

Let $\psi_{\mathcal{K}}$ be defined by $\psi_{\mathcal{K}}^{(n')} := 0$, for $n' \neq n$, and $\psi_{\mathcal{K}}^{(n)} := \mathbb{1}(\underline{k} \in \mathcal{K})\varphi_0^{(n)}$. By property ($\mathcal{K}$2), there exists ℓ_0 such that $a(f_{j,\ell})\psi_{\mathcal{K}} = 0$, for any $1 \leq j \leq n_0$, and $\ell \geq \ell_0$.

By the CCR (2.1) we thus get, for $\ell \geq \ell_0$,

$$\begin{aligned}\left\langle \mathbf{a}^*(f_{n_0,\ell}) \cdots \mathbf{a}^*(f_{1,\ell})\, \psi_{\mathcal{K}}, \psi_\ell \right\rangle &= \sum_{\sigma \in S(n_0)} \left(\Pi_{j=1}^{n_0} \langle f_{j,\ell}, f_{\sigma(j),\ell} \rangle \right) \langle \psi_{\mathcal{K}}, \varphi_0 \rangle \\ &= \sum_{\sigma \in S(n_0)} \left(\Pi_{j=1}^{n_0} \langle f_j, f_{\sigma(j)} \rangle \right) \left\langle \varphi_0^{(n)}, \mathbb{1}(\underline{k} \in \mathcal{K})\, \varphi_0^{(n)} \right\rangle \\ &\geq \|f\|^{2n_0} \|\mathbb{1}(\underline{k} \in \mathcal{K})\, \varphi_0^{(n)}\|^2 .\end{aligned}$$

This estimate and property ($\mathcal{K}3$), implies $\|\psi_\ell\| > 0$ uniformly in $\ell \geq \ell_0$.

It remains to prove that $\|(H(\xi) - \lambda)\psi_\ell\| \to 0$ as $\ell \to \infty$.

Let $\tilde{v} \in L^2(\mathbb{R}^\nu) \cap C(\mathbb{R}^\nu)$. Write $\widetilde{H}(\xi)$ for the fiber Hamiltonian with the interaction $\Phi(v)$ replaced by $\Phi(\tilde{v})$. Compute, as an identity on $\mathcal{D}$,

$$\begin{aligned}&\widetilde{H}(\xi - k^{(n_0)}) \ - \ H(\xi - \eta^{(n_0)}) \\ &\quad = \ (k^{(n_0)} - \eta^{(n_0)}) \cdot (\nabla \Omega)(\xi - \eta^{(n_0)} - d\Gamma(k)) \\ &\qquad + \left\langle (k^{(n_0)} - \eta^{(n_0)}), T(k^{(n_0)}, \eta^{(n_0)})\, (k^{(n_0)} - \eta^{(n_0)}) \right\rangle \ + \ \Phi(\tilde{v} - v) ,\end{aligned} \tag{3.15}$$

where $T(\zeta_1, \zeta_2) = \int_0^1 (1-t)(\nabla^2 \Omega)(\xi - t\zeta_1 - (1-t)\zeta_2 - d\Gamma(k))dt$. Note that this operator is continuous and bounded uniformly in ζ_1 (and ζ_2) and commutes with the number operator.

Abbreviate

$$\omega^\Sigma(\underline{k}, \underline{\eta}) \ := \ \sum_{j=1}^{n_0} \left(\omega(k_j) \ - \ \omega(\eta_j) \right) .$$

By (3.14), (3.15), and the pull-through formula, Proposition 2.2, we get for $\psi \in \mathcal{C}_0^\infty$

$$\begin{aligned}&\langle \varphi_0, \mathbf{a}(k_1) \cdots \mathbf{a}(k_{n_0})\, (\widetilde{H}(\xi) - \lambda)\, \psi \rangle \\ &= \ \left\langle \left\{ \widetilde{H}(\xi - k^{(n_0)}) \ - \ H(\xi - \eta^{(n_0)}) \ + \ \omega^\Sigma(\underline{k}, \underline{\eta}) \right\} \varphi_0, \mathbf{a}(k_1) \cdots \mathbf{a}(k_{n_0})\, \psi \right\rangle \\ &\quad + \sum_{i=1}^{n_0} \tilde{v}(k_i) \left\langle \varphi_0, \mathbf{a}(k_1) \cdots \widehat{\mathbf{a}(k_i)} \cdots \mathbf{a}(k_{n_0})\, \psi \right\rangle \\ &= \ \langle \Phi(\tilde{v} - v)\varphi_0, \mathbf{a}(k_1) \cdots \mathbf{a}(k_n)\, \psi \rangle \ + \ \omega^\Sigma(\underline{k}, \underline{\eta})\, \langle \varphi_0, \mathbf{a}(k_1) \cdots \mathbf{a}(k_n)\, \psi \rangle \\ &\quad - (k^{(n_0)} - \eta^{(n_0)}) \cdot \left\langle (\nabla \Omega)(\xi - \eta^{(n_0)} - d\Gamma(k))\, \varphi_0, \mathbf{a}(k_1) \cdots \mathbf{a}(k_{n_0})\, \psi \right\rangle \\ &\quad + \left\langle \left\langle (k^{(n_0)} - \eta^{(n_0)}), T(k^{(n_0)}, \eta^{(n_0)})(k^{(n_0)} - \eta^{(n_0)}) \right\rangle \varphi_0, \mathbf{a}(k_1) \cdots \mathbf{a}(k_n)\, \psi \right\rangle \\ &\quad + \sum_{i=1}^{n_0} \tilde{v}(k_i) \left\langle \varphi_0, \mathbf{a}(k_1) \cdots \widehat{\mathbf{a}(k_i)} \cdots \mathbf{a}(k_{n_0})\, \psi \right\rangle .\end{aligned}$$

Abbreviate

$$\begin{aligned}B_\ell^1(\underline{k}) &:= \omega^\Sigma(\underline{k}, \underline{\eta})\, \Pi_{j=1}^{n_0} f_{j,\ell}(k_j)\, \mathbb{1}_{\mathcal{F}} , \\ B_{p,\ell}^2(\underline{k}) &:= (k_{;p}^{(n_0)} \ - \ \eta_{;p}^{(n_0)})\, \Pi_{j=1}^{n_0} f_{j,\ell}(k_j)\, \mathbb{1}_{\mathcal{F}} , \\ B_\ell^3(\underline{k}) &:= \left\langle (k^{(n_0)} - \eta^{(n_0)}), T(k^{(n_0)}, \eta^{(n_0)})(k^{(n_0)} \ - \ \eta^{(n_0)}) \right\rangle \Pi_{j=1}^{n_0} f_{j,\ell}(k_j) .\end{aligned}$$

By construction of the $f_{j,\ell}$'s we find (see (3.13) for the definition of the norm)

$$\|B^1_\ell\| \;+\; \sum_{p=1}^{\nu} \|B^2_{p,\ell}\| \;+\; \|B^3_\ell\| \;\to\; 0\,, \quad \text{for}\ \ \ell \to \infty\,. \tag{3.16}$$

Using the notation introduced in Lemma 3.5, we can now compute

$$\begin{aligned}
\langle \psi_\ell, (\widetilde{H}(\xi) - \lambda)\psi \rangle \;&=\; \langle \varphi_0, \Phi(\tilde{v} - v)\mathbf{a}(f_{1,\ell}) \cdots \mathbf{a}(f_{n_0,\ell})\psi \rangle \\
&+ \langle \varphi_0, \mathbf{a}(B^1_\ell)\,\psi \rangle \;+\; \big\langle \varphi_0, \mathbf{a}(B^3_\ell)\,\psi \big\rangle \\
&+ \sum_{p=1}^{\nu} \big\langle \partial_p \Omega(\xi - \eta^{(n_0)} - d\Gamma(k))\,\varphi_0, \mathbf{a}(B^2_{\ell,p})\,\psi \big\rangle \\
&+ \sum_{i=1}^{n_0} \big\langle f_{i,\ell}, \tilde{v} \big\rangle \langle \mathbf{a}^*(f_{n_0,\ell}) \cdots \widehat{\mathbf{a}^*(f_{i,\ell})} \cdots \mathbf{a}^*(f_{1,\ell})\varphi_0, \psi \big\rangle
\end{aligned}$$

By Lemma 2.1 ii) we can take the limit $\tilde{v} \to v$ in $L^2(\mathbb{R}^\nu)$. This amounts to replacing $\tilde{v}$ by v and $\widetilde{H}(\xi)$ by $H(\xi)$ in the equation above. The resulting identity together with Condition 1.1, Lemma 2.1 i), and Lemma 3.6 implies that $\psi_\ell \in \mathcal{D}$ and

$$\begin{aligned}
\big\|(H(\xi) - \lambda)\,\psi_\ell\big\| \;&\le\; C\,\big\|(N+1)^{\frac{n_0}{2}}\varphi_0\big\|\,\big(\|B^1_\ell\| \;+\; \|B^3_\ell\|\big) \\
&+ C \sum_{p=1}^{\nu} \big\|\partial_p \Omega(\xi - \eta^{(n_0)} - d\Gamma(k))\,(N+1)^{\frac{n_0}{2}}\varphi_0\big\|\,\|B^2_{\ell,p}\| \\
&+ C_{0,n_0-1}\,\big(\max_{1\le j\le\nu} |\langle f_{j,\ell}, v\rangle|\big)\,\big\|(N+1)^{\frac{n_0-1}{2}}\;\varphi_0\big\|\,\sum_{i=1}^{n_0} \Pi_{k\neq i}\|f_{k,\ell}\|\,.
\end{aligned}$$

By (3.16) and the fact that $\mathrm{w}-\lim_{\ell\to\infty} f_{j,\ell} = 0$, we thus find $\|(H(\xi) - \lambda)\psi_\ell\| \to 0$ as $\ell \to \infty$, and hence; ψ_ℓ is a Weyl-sequence. This concludes the proof. □

3.3 Uniqueness, existence, and non-existence of ground states

We begin by applying the Perron-Frobenius theorem of Farris, which is presented in Appendix A.2. See also Fröhlich [19].

We write $\mathfrak{h}_{\mathrm{ph}} = \mathfrak{h}_{\mathrm{ph}_\mathbb{R}} \oplus \mathrm{i}\mathfrak{h}_{\mathrm{ph}_\mathbb{R}}$, where $\mathfrak{h}_{\mathrm{ph}_\mathbb{R}}$ is the real Hilbert space consisting of the real valued functions in $\mathfrak{h}_{\mathrm{ph}}$. We define $\mathcal{H}_\mathbb{R} := \oplus_{n=0}^\infty {\mathfrak{h}_{\mathrm{ph}}}_\mathbb{R}^{\otimes_s n}$, which is also a real Hilbert space. We take as a Hilbert cone, cf. Definition A.1,

$$\begin{aligned}
\mathbf{C} \;&:=\; \times_{n=0}^{\infty}\,\mathbf{C}^{(n)}\,, \\
\mathbf{C}^{(n)} \;&:=\; \{\, f \,\in\, {\mathfrak{h}_{\mathrm{ph}}}_\mathbb{R}^{\otimes_s n} \;:\; (-1)^n f \;\ge\; 0 \,\}\,.
\end{aligned} \tag{3.17}$$

In this section we assume that the coupling function $v \in L^2(\mathbb{R}^\nu)$ is strictly positive almost everywhere.

Clearly $f(H_0(\xi))$ is positivity preserving in the sense of Definition A.2 ii), for any bounded non-negative Borel function f.

For $\mu > 0$ sufficiently large, the Neumann series

$$(H(\xi)+\mu)^{-1} \;=\; \sum_{k=0}^{\infty}(H_0(\xi)+\mu)^{-1}\left\{(-\Phi(v))\,(H_0(\xi)+\mu)^{-1}\right\}^k \tag{3.18}$$

converge. Note that $\|\Phi(v)(H_0(\xi)+\mu)^{-1}\| \leq C\mu^{-\frac{1}{2}}$; cf. Lemma 2.1 i) and (3.1). Since $v \geq 0$ a.e., by assumption, we find from this formula that $(H(\xi)+\mu)^{-1}$ is positivity preserving. In fact, we find from (3.18) that, the resolvent $(H(\xi)+\mu)^{-1}$ is a sum of terms of the form

$$(H_0(\xi)+\mu)^{-1}\left\{\mathbf{a}^{\#}(v)\,(H_0(\xi)+\mu)^{-1}\right\}^k,$$

where all powers k and combinations of $\mathbf{a}^*(v)$ and $\mathbf{a}(v)$ occur. Furthermore each of these terms are positivity preserving.

Let $u \in \mathbf{C}\backslash\{0\}$. There exists $n \geq 0$ such that $u_n \in {\mathfrak{h}_{\mathrm{ph}}}_c^{\otimes_s n}$, the projection onto the n-particle sector, is non-vanishing; $u_n \neq 0$. We wish to prove that $(H(\xi)+\mu)^{-1}u$ is strictly positive in the sense of Definition A.2 i). Let $v \in \mathbf{C}\backslash\{0\}$. There exists $n' \geq 0$ such that $v_{n'} \in {\mathfrak{h}_{\mathrm{ph}}}_c^{\otimes_s n'}$ is non-zero; $v_{n'} \neq 0$. We estimate

$$\begin{aligned}
&\big\langle (H(\xi)+\mu)^{-1}u, v\big\rangle \;\geq\; \big\langle (H(\xi)+\mu)^{-1}u_n, v_{n'}\big\rangle \\
&\quad\geq\quad \Big\langle \big\{\mathbf{a}(v)\,(H_0(\xi)+\mu)^{-1}\big\}^n u_n, \big\{(H_0(\xi)+\mu)^{-1}\,\mathbf{a}(v)\big\}^{n'}(H_0(\xi)+\mu)^{-1}v_{n'}\Big\rangle \\
&\quad\geq\quad \mu^{-n-n'-1}\int_{\mathbb{R}^{\nu n}} v(k_1)\cdots v(k_n)\,(-1)^n u_n(k_1,\ldots,k_n)\,dk_1\cdots dk_n \\
&\qquad\qquad \times\int_{\mathbb{R}^{\nu n'}} v(k_1)\cdots v(k_{n'})\,(-1)^{n'}v_{n'}(k_1,\ldots,k_n)\,dk_1\cdots dk_n'\,.
\end{aligned}$$

The right-hand side is strictly positive and hence; $(H(\xi)+\mu)^{-1}u$ is strictly positive. Since $u \in \mathbf{C}\backslash\{0\}$ was arbitrary we conclude that $(H(\xi)+\mu)^{-1}$ is positivity improving in the sense of Definition A.2 iii). The abstract result of Faris, Theorem A.3 now implies that a ground state, if it exists, is unique and strictly positive in the sense of Definition A.2 i). This proves Theorem 1.3.

Before continuing with Theorem 1.6 we give the following:

Proof of Lemma 1.5: We Taylor expand $\omega(k_1)$ and $\omega(k_2)$ around k_1+k_2 and estimate the result using Condition 1.3

$$\begin{aligned}
\omega(k_1) \;+\; \omega(k_2) \;&=\; 2\,\omega(k_1\,+\,k_2) \;-\; \nabla\omega(k_1\,+\,k_2)\cdot(k_1\,+\,k_2) \\
&\quad+\frac{1}{2}\int_0^1(1-t)^2\left\{\big\langle k_2,\nabla^2\omega(k_{1,t})k_2\big\rangle \;+\; \big\langle k_1,\nabla^2\omega(k_{2,t})k_1\big\rangle\right\}dt \\
&\overset{(>)}{\geq}\quad \omega(k_1\,+\,k_2)\,.
\end{aligned}$$

Here $k_{1,t} := k_1+(1-t)k_2$ and $k_{2,t} := k_2+(1-t)k_1$. $\square$

Proof of Theorem 1.6 ii): Let ξ be such that $\Sigma_0(\xi) = \Sigma_{\rm ess}(\xi)$. Assume $\Sigma_0(\xi)$ is an eigenvalue. By Theorem 1.3, the eigenvalue is non-degenerate and we can choose an eigenfunction $\psi_\xi \in \mathbf{C}$ which is strictly positive.

Recall from Corollary 1.4 that $\Sigma_{\rm ess}(\xi) = \Sigma_0^{(1)}(\xi)$, under Condition 1.3. Let $\mathcal{M} := \{k \in \mathbb{R}^\nu : \Sigma_0^{(1)}(\xi;k) = \Sigma_0^{(1)}(\xi)\}$ be the set of minimizers. By (1.16) and Lemma 1.7, $\mathcal{M}$ is a compact subset of the open set $\mathcal{I}_0^{(1)}(\xi)$. There exists $k_0 \in \partial\mathcal{M}$, a unit vector $\vec{u} \in \mathbb{R}^\nu$, and a number $r > 0$, with the following property: For any $\delta > 0$ we have

$$\Omega_\delta^r \ := \ \{k \ \in \ \mathbb{R}^\nu \ : \ \|k \ - \ k_0\| \ \leq r \ \text{ and } \ (k \ - \ k_0)\cdot\vec{u} \geq \delta\,\} \ \subset \ \mathcal{I}_0^{(1)}(\xi)\setminus\mathcal{M}\,.$$

We also use this notation with $\delta = 0$.

For any $\delta > 0$ there exists $C(\delta)$ such that

$$\inf_{k\in\Omega_\delta^r} \Sigma_0(\xi-k) \ + \ \omega(k) \ - \ \Sigma_0(\xi-k_0) \ \geq \ C(\delta)^{-1}\,. \tag{3.19}$$

Recall that $\Sigma_0(\xi-k)$, $k \in \Omega_0^r$, are isolated eigenvalues and, again by Theorem 1.3, they are non-degenerate and we can choose eigenfunctions $\psi_{\xi-k} \in \mathbf{C}$ which are strictly positive. Since $\mathcal{I}_0^{(1)}(\xi) \ni k \to \psi_{\xi-k}$ is continuous, we find

$$\inf_{k\in\Omega_0^r} \langle\psi_{\xi-k},\psi_\xi\rangle \ > \ 0\,. \tag{3.20}$$

Let $N_\delta := d\Gamma(\mathbb{1}(k \in \Omega_\delta^r)) = \int_{\Omega_\delta^r} \mathbf{a}^*(k)\mathbf{a}(k)d^\nu k$. Note that $0 \leq N_\delta \leq N$, and hence $\psi_\xi \in \mathcal{D}(N_\delta)$ with $\|N_\delta\psi_\xi\| \leq \|N\psi_\xi\| < \infty$ uniformly in $\delta > 0$. Using Proposition 2.3, (3.19), and the Lebesgue theorem on dominated convergence (to replace z, ${\rm Im} z \neq 0$, by $z = \Sigma_0(\xi)$), we get

$$\begin{aligned}
&\langle\psi_\xi, N_\delta\psi_\xi\rangle \\
&\quad\geq \int_{\Omega_\delta^r} v(k)^2 \left\|\big(H(\xi-k) \ + \ \omega(k) \ - \ \Sigma_0(\xi)\big)^{-1}\psi_\xi\right\|^2 dk \\
&\quad\geq \int_{\Omega_\delta^r} v(k)^2 \big(\Sigma_0(\xi-k) \ + \ \omega(k) \ - \ \Sigma_0(\xi)\big)^{-2} |\langle\psi_{\xi-k},\psi_\xi\rangle|^2 dk \\
&\quad\geq \inf_{k\in\Omega_0^r}\{|\langle\psi_{\xi-k},\psi_\xi\rangle|^2 v(k)^2\} \int_{\Omega_\delta^r} \big(\Sigma_0(\xi-k) \ + \ \omega(k) \ - \ \Sigma_0(\xi)\big)^{-2} dk\,.
\end{aligned} \tag{3.21}$$

Since $\Sigma_0(\xi-k)$ is a smooth function of k in $\mathcal{I}_0^{(1)}(\xi)$ and k_0 is a global minimum of the function $k \to \Sigma_0(\xi-k)+\omega(k)$, we find that there exists $C > 0$ such that

$$0 \ \leq \ \Sigma_0(\xi \ - \ k) \ + \ \omega(k) \ - \ \Sigma_0(\xi) \ \leq \ C\,|k \ - \ k_0|^2, \ \text{ for } \ k \in \Omega_0^r\,.$$

This estimate together with (3.20), (3.21), and the assumption $3 \leq \nu \leq 4$ implies that $|\langle\psi_\xi, N_\delta\psi_\xi\rangle| \to \infty$, as $\delta \to 0$. This contradicts $\psi_\xi \in \mathcal{D}(N)$, and hence; $\Sigma_0(\xi)$ is not an eigenvalue. $\square$

The first step in the proof of Theorem 1.6 i) is the following Lemma.

Lemma 3.7. *Let $\xi \in \mathbb{R}^\nu$ and $z < \Sigma_0(\xi)$. Then*

$$\Omega(\xi) \;-\; z \;-\; \int_{\mathbb{R}^\nu} v(k)^2 \big\langle \Omega, (H(\xi-k)+\omega(k)-z)^{-1}\Omega \big\rangle \, dk \;>\; 0\,.$$

Proof. Let $P_\Omega := |\Omega\rangle\langle\Omega|$, and $\overline{P}_\Omega := \mathbb{1}_{\mathcal{F}} - P_\Omega$. Using the Feshbach projection method, cf. e.g. [4, Section II], we find

$$\big\langle \Omega, (H(\xi)-z)^{-1}\Omega \big\rangle \;=\; \Big(\Omega(\xi) \;-\; z \;-\; \big\langle v, (\overline{H}(\xi)-z)^{-1} v\big\rangle_{\mathrm{Ran}\,\overline{P}_\Omega}\Big)^{-1}\,. \tag{3.22}$$

Here $\overline{H}(\xi) = \overline{P}_\Omega H(\xi)\overline{P}_\Omega$ as an operator on $\mathrm{Ran}\,\overline{P}_\Omega$, and v is viewed as an element of the one-particle space which is contained in $\mathrm{Ran}\,\overline{P}_\Omega$. By the spectral theorem the left-hand side of (3.22) is strictly positive and hence

$$\Omega(\xi) \;-\; z \;-\; \big\langle v, (\overline{H}(\xi)-z)^{-1} v\big\rangle_{\mathrm{Ran}\,\overline{P}_\Omega} \;>\; 0\,. \tag{3.23}$$

Viewing $(\overline{H}(\xi)-z)^{-1}v$ as an element of $\mathcal{F}$ we write

$$\big\langle v, (\overline{H}(\xi)-z)^{-1} v\big\rangle_{\mathrm{Ran}\,\overline{P}_\Omega} \;=\; \int_{\mathbb{R}^\nu} v(k)\, \big\langle \Omega, a(k)(\overline{H}(\xi)-z)^{-1} v\big\rangle \, dk\,. \tag{3.24}$$

Applying the pull-through formula, Theorem 2.3, with $\psi = (\overline{H}(\xi)-z)^{-1}v \in \mathcal{D}$, yields as an $L^2(\mathbb{R}^\nu;\mathcal{F})$ identity

$$\begin{aligned} &a(k)\,(\overline{H}(\xi)-z)^{-1}v \\ &\quad= \;\; (H(\xi-k)+\omega(k)-z)^{-1}\, a(k)\,(H(\xi)-z)\,(\overline{H}(\xi)-z)^{-1}v \\ &\qquad\; - v(k)\,(H(\xi-k)+\omega(k)-z)^{-1}(\overline{H}(\xi)-z)^{-1}v\,. \end{aligned} \tag{3.25}$$

We now make two observations. The first is the identity

$$a(k)\,(H(\xi)-z)\,(\overline{H}(\xi)-z)^{-1}v \;=\; a(k)\,v \;=\; v(k)\,\Omega\,. \tag{3.26}$$

The second observation is that $(\overline{H}(\xi)-z)^{-1}$ is positivity preserving, with respect to the cone $\mathbf{C}$ introduced in (3.3) (after extending it by zero to the vacuum sector). This follows by a Neumann expansion, as for $(H(\xi)+\mu)^{-1}$ in (3.18), and Lemma A.4. Since $(H(\xi-k)+\omega(k)-z)^{-1}$ is also positivity preserving we find that, for a.e. $k \in \mathbb{R}^\nu$,

$$\big\langle \Omega, (H(\xi-k)+\omega(k)-z)^{-1}(\overline{H}(\xi)-z)^{-1}v\big\rangle \;\le\; 0\,. \tag{3.27}$$

Combining (3.25)–(3.27) we get the following estimate a.e.

$$v(k)\,\big\langle \Omega, a(k)\,(\overline{H}(\xi)-z)^{-1}v\big\rangle \;\ge\; v(k)^2\,\big\langle \Omega, (H(\xi-k)+\omega(k)-z)^{-1}\Omega\big\rangle\,.$$

This estimate in conjunction with (3.23) and (3.24) concludes the proof. □

Proof of Theorem 1.6 i): Assume that the statement is false at ξ, i.e. $\Sigma_0(\xi) = \Sigma_{\mathrm{ess}}(\xi)$.

The aim is to show that the equation

$$\Omega(\xi) - z \;=\; \int_{\mathbb{R}^\nu} v(k)^2 \big\langle \Omega, (H(\xi-k) + \omega(k) - z)^{-1}\Omega \big\rangle \, dk$$

has a solution $z < \Sigma_{\mathrm{ess}}(\xi)$, which would by Lemma 3.7 provide a contradiction.

In the limit $z \to -\infty$ the left-hand side dominates the right-hand side. A solution exists (and is necessarily unique by monotonicity) if we can show that the righthand side diverges as z approaches $\Sigma_{\mathrm{ess}}(\xi)$ from below.

As in the proof of Theorem 1.6 ii) we choose a minimizer $k_0 \in \mathbb{R}^\nu$ satisfying $\Sigma_0^{(1)}(\xi; k_0) = \Sigma_0^{(1)}(\xi) = \Sigma_{\mathrm{ess}}(\xi)$. Then, by (1.16) and Lemma 1.7, $k_0 \in \mathcal{I}_0^{(1)}(\xi)$ and there exists a neighbourhood $\mathcal{O} \subset \mathcal{I}_0^{(1)}(\xi)$ of k_0 satisfying $\inf_{k\in\mathcal{O}} \langle \psi_{\xi-k}, \Omega\rangle > 0$. Here $\psi_{\xi-k} \in \mathbf{C}$, $k \in \mathcal{O}$, are the strictly positive ground state eigenfunctions of $H(\xi - k)$. We thus get

$$\begin{aligned} &\int_{\mathbb{R}^\nu} |v(k)|^2 \big\langle \Omega, (H(\xi-k) + \omega(k) - z)^{-1}\Omega \big\rangle \, dk \\ &\geq \inf_{k\in\mathcal{O}} \{\langle \psi_{\xi-k}, \Omega\rangle^2 v(k)^2\} \int_{\mathcal{O}} (\Sigma_0(\xi-k) + \omega(k) - z)^{-1} dk \,. \end{aligned}$$

Since the righthand side diverges in dimension 1 and 2, as $z \to \Sigma_{\mathrm{ess}}(\xi)$ from below, we conclude the result. □

3.4 Regularity of $t \to \sigma_{\mathrm{ess}}(t)$

We begin with

Proof of Lemma 1.8: Let $\underline{k}$ be a local minimum of $\mathcal{I}_0^{(n)}(\xi) \ni \underline{k} \to \Sigma_0^{(n)}(\xi; \underline{k})$. That the k_j's must be equal follows from strict convexity of ω: Assume $n \geq 2$. Let $k_{j,s} = (1-s)k_j + s\frac{1}{2}(k_1 + k_2)$, $j = 1, 2$ and $0 \leq s \leq 1$. Note that $k_{1,s} + k_{2,s} = k_1 + k_2$, so that substituting $k_{1,s}, k_{2,s}$ for k_1, k_2 only changes the contribution to $\Sigma_0^{(n)}(\xi; \underline{k})$ coming from ω. We compute

$$\frac{d}{ds}\big\{\omega(k_{1,s}) \;+\; \omega(k_{2,s})\big\} \;=\; \frac{1}{2}(k_2 \,-\, k_1)\,\big\{\nabla\omega(k_{1,s}) \;-\; \nabla\omega(k_{2,s})\big\} \,.$$

Since $\nabla\omega(k_1) - \nabla\omega(k_2) = (\int_0^1 \nabla^2\omega(tk_1 + (1-t)k_2)dt(k_1 - k_2)$, we find that the derivative is strictly negative at $s = 0$, unless $k_1 = k_2$.

Write $k_1 = \cdots = k_n = \Theta$. We proceed to argue that Θ is a multiple of ξ. A local minimum is in particular a critical point, i.e. it satisfies $\nabla_j \Sigma^{(n)}(\xi; \underline{k}) = -\nabla\Sigma(\xi - k^{(n)}) + \nabla\omega(k_j) = 0$, $1 \leq j \leq n$. By rotation invariance, this implies that $\xi - n\Theta$ is a multiple of Θ. This completes the proof. □

Proposition 3.8. *Let $n > 0$, $t \in \mathbb{R}$ and $s \in \mathcal{I}_0^{(n)}(t)$ be such that $\mathrm{Ind}^{(n)}(t; s) \geq 1$. There exist neighbourhoods $\mathcal{O}_t \ni t$ and $\mathcal{O}_s \ni s$, with $\mathcal{O}_s \subset \cup_{t'\in\mathcal{O}_t} \mathcal{I}_0^{(n)}(t')$, such that the following holds*

1) *If* $\mathrm{Ind}^{(n)}(t;s) = 1$*, then there exists an analytic map* $\Theta : \mathcal{O}_t \to \mathcal{O}_s$*, such that:* $\mathrm{Ind}^{(n)}(t';\Theta(t')) = 1$ *and* $\mathrm{Ind}^{(n)}(t';s') = 0$*, if* $s' \neq \Theta(t')$.
2) *If* $\mathrm{Ind}^{(n)}(t;s) = 2$*, then: For* $t' \in \mathcal{O}_t$*,* $s' \to \sigma^{(n)}(t';s')$ *has either one or two local minima in* $\mathcal{O}_s$*. For* $t' \neq t$*, they have index* 1.
3) *If* $\mathrm{Ind}^{(n)}(t;s) = \ell \in [3,\infty)$*, then there exists a countable set* $\mathcal{K} \subset \mathcal{O}_t \backslash \{t\}$*, with* $\mathcal{K} \cup \{t\}$ *closed, such that: For* $t' \in \mathcal{O}_t$*,* $s' \to \sigma^{(n)}(t';s')$ *has between* 1 *and* ℓ *local minima in* $\mathcal{O}_s$*. For* $t' \in \mathcal{O}_t \backslash (\mathcal{K} \cup \{t\})$*, they all have index* 1*. For* $t' \in \mathcal{K}$ *all local minima* $s' \in \mathcal{O}_s$ *satisfies* $\mathrm{Ind}^{(n)}(t';s') \leq \ell - 1$.
4) *If* $\mathrm{Ind}^{(n)}(t;s) = \infty$*, then for* $t' \in \mathcal{O}_t \backslash \{t\}$*, we have* $\mathrm{Ind}^{(n)}(t';s') = 0$*, for all* $s' \in \mathcal{O}_s$.

Proof. 1) follows by analyticity in t and s of $\partial_s^2 \sigma^{(n)}(t;s)$, and the implicit function theorem.

As for 2) and 3), we write $\ell = \mathrm{Ind}^{(n)}(t;s)$. We again invoke the implicit function theorem to construct an analytic function Θ from a neighbourhood $\mathcal{O}_t$ of t, into a neighbourhood $\mathcal{O}_s$ of s, with the property that $\partial_s^{2\ell-1}\sigma^{(n)}(t';\Theta(t')) = 0$, $t' \in \mathcal{O}_t$. Note that by choosing $\mathcal{O}_t$ small enough we have $t' - n\Theta(t') \in \mathcal{I}_0$.

We begin by showing that near t no local minima can disappear to the same order as at t. We note that near t we may have at most ℓ local minima, but there is at least one. Let $\mathcal{O}_t \ni t_j \to t$ and $\mathcal{O}_s \ni s_j \to s$ be such that s_j is a local minimum of $r \to \sigma^{(n)}(t_j, r)$. Assume $\partial_s^k \sigma^{(n)}(t_j, s_j) = 0$ for $k \leq 2\ell - 1$. Then necessarily, we must have $s_j = \Theta(t_j)$. For $1 \leq k \leq 2\ell - 2$, the function $t' \to \partial_s^k \sigma^{(n)}(t', \Theta(t'))$ is analytic in $\mathcal{O}_t$ and vanishes on the sequence $\{t_j\}$, hence it is identically zero in $\mathcal{O}_t$.

We can now compute

$$0 \;=\; \frac{d}{dt'}\big\{\partial_s^{2\ell-2}\sigma^{(n)}(t';\Theta(t'))\big\} \;=\; n^{2\ell-2}\,\partial^{2\ell-1}\sigma(t' - n\Theta(t'))\,.$$

This implies that $\partial^{2\ell-1}\omega(\Theta(t')) = 0$. The function $\partial^{2\ell-1}\omega(s)$ has only isolated zeroes, since it is a analytic (and not identically zero). Hence $\Theta(t') = \Theta_0$ is a constant function on $\mathcal{O}_t$. Since $t' \to \sigma(t' - n\Theta_0) + n\omega(\Theta_0)$ is thus linear near t, we find that σ is linear near $t - ns$. This implies in particular that $\partial^2\sigma^{(n)}(t;s) = n\partial^2\omega(s) = 0$. Recalling that ω is strictly convex we arrive at a contradiction.

The statement 2) is now proved. The statement 3) follows from an induction argument in ℓ, starting with $\ell = 2$.

As for 4) we note that we must have $\sigma^{(n)}(t;s') = C$, for some constant C. In other words: $\sigma(t - ns') = C - n\omega(s')$, for s' near s. Compute $\sigma(t' - ns') + n\omega(s') = \sigma(t - n(s' + (t - t')/n)) + n\omega(s') = C + n\{\omega(s') - \omega(s' + (t - t')/n)\}$. This gives $\partial_s \sigma^{(n)}(t';s') = n\{\nabla\omega(s') - \nabla\omega(s' + (t - t')/n)\}$. This expression can only vanish if $t = t'$. $\square$

Proof of Theorem 1.11: We argue first that for a given t, the set $\mathcal{M}$ of global minima of $s \to \sigma^{(n)}(t;s)$ is finite. Note that by Lemma 1.7 we have $\mathcal{M} \subset \mathcal{I}_0^{(n)}(t)$. Suppose to the contrary that $\mathcal{M}$ is infinite. Then either $\mathcal{M}$ contains a connected component of $\mathcal{I}_0^{(n)}(t)$ or there is a sequence in $\mathcal{M}$ converging to $\partial\mathcal{I}_0^{(n)}(t)$. In either case, this is a contradiction since $\mathcal{M}$ is closed and $\mathcal{I}_0^{(n)}(t)$ is bounded and open. We remark that this also implies that a global minimum has finite index.

By Proposition 3.8 2)-3), we find that the set $\mathcal{T}_0$ of t for which at least one of the global minima for the map $s \to \sigma^{(n)}(t;s)$ has index strictly larger than 1, is closed and countable. It remains to show that the set of t for which there is more than one global minimum, all with index 1, is countable and can accumulate only at $\mathcal{T}_0$.

Suppose t is such that the map $s \to \sigma^{(n)}(t;s)$ has ℓ global minima all with index 1. Note that for t' near t these minima will persist at least as local minima, and any global minima will be found amongst these. There exists ℓ analytic maps $t' \to \Theta_j(t')$, which parameterize these local minima. they are all defined in a neighbourhood of t, and satisfies $\mathrm{Ind}^{(n)}(t';\Theta_j(t')) = 1$.

We estimate the rate of change of the global minima near t, using twice the critical equation $(\partial_s\sigma^{(n)})(t';\Theta_j(t')) = 0$,

$$\frac{d}{dt'}\sigma^{(n)}(t';\Theta_j(t')) \;=\; \partial\sigma(t' - n\Theta_j(t')) \;=\; \partial\omega(\Theta_j(t'))\,. \tag{3.28}$$

Since $\partial\omega$ is monotonically increasing we find that that there exists a neighbourhood $\mathcal{O}_t$ of t such that for $t' \in \mathcal{O}_t\backslash\{t\}$, the map $s \to \sigma^{(n)}(t';s)$ has a unique global minimum, with index 1.

A compactness argument now concludes the proof. Note that (1.19) is implied by (3.28) since $\sigma^{(n)}(t) = \sigma^{(n)}(t;\Theta^{(n)}(t))$, for $t \in \mathcal{T}^{(n)}$. □

We proceed to study the n dependence of local minima, and to prove Proposition 1.11. (This material is in the preprint version of the paper only.)

Proposition 3.9. *Let $n > 0$, $t \in \mathbb{R}$ and $s \in \mathcal{I}_0^{(n)}(t)$ be such that $\mathrm{Ind}^{(n)}(t;s) \geq 1$. There exist neighbourhoods $\mathcal{O}_n \ni n$ and $\mathcal{O}_s \ni s$, with $\mathcal{O}_s \subset \cup_{n'\in\mathcal{O}_n}\mathcal{I}_0^{(n')}(t)$, such that the following holds*

1) *If $\mathrm{Ind}^{(n)}(t;s) = 1$, then there exists an analytic map $\Theta : \mathcal{O}_n \to \mathcal{O}_s$, such that: $\mathrm{Ind}^{(n')}(t;\Theta(n')) = 1$ and $\mathrm{Ind}^{(n')}(t;s') = 0$, if $s' \neq \Theta(n')$.*
2) *If $\mathrm{Ind}^{(n)}(t;s) = 2$, then: For For $n' \in \mathcal{O}_n$, $s' \to \sigma^{(n')}(t;s')$ has either one or two local minima in $\mathcal{O}_s$. For $n' \neq n$, they have index 1.*
3) *If $\mathrm{Ind}^{(n)}(t;s) = \ell \in [3,\infty)$, then there exists a countable set $\mathcal{K} \subset \mathcal{O}_n\backslash\{n\}$, with $\mathcal{K} \cup \{n\}$ closed, such that: For $n' \in \mathcal{O}_n$, $s' \to \sigma^{(n')}(t;s')$ has between 1 and ℓ local minima in $\mathcal{O}_s$. For $n' \in \mathcal{O}_n\backslash(\mathcal{K}\cup\{n\})$, they all have index 1. For $n' \in \mathcal{K}$, all local minima $s' \in \mathcal{O}_s$ satisfies $\mathrm{Ind}^{(n')}(t;s') \leq \ell - 1$.*
4) *If $\mathrm{Ind}^{(n)}(t;s) = \infty$, then for $n' \in \mathcal{O}_n\backslash\{n\}$, $\mathrm{Ind}^{(n')}(t;s') = 0$, for all $s' \in \mathcal{O}_s$.*

Proof. As for 1) and 4), we refer the reader to the proof of the corresponding statements in Proposition 3.8.

As for 2) and 3), again assume there exists a sequence n_j, s_j such that $n_j \to n$, $s_j \to s$, $t - n_j s_j \in I$, and $\mathrm{Ind}^{(n_j)}(t;s_j) = \ell$. Recall $\ell = \mathrm{Ind}^{(n)}(t;s)$. By the implicit function theorem we get an analytic function Θ mapping a neighbourhood of n into a neighbourhood of s with the property that $\partial_s^{2\ell-1}\sigma^{(n')}(t;\Theta(n')) = 0$. Necessarily we must have $s_j = \Theta(n_j)$, and $\mathrm{Ind}^{(n')}(t;\Theta(n')) = \ell$ in a neighbourhood of n. We compute

$$\begin{aligned} 0 &= \frac{d}{dn'}\{\partial_s^{2\ell-2}\sigma^{(n')}(t;\Theta(n'))\} \\ &= -(n')^{2\ell-2}\,\partial^{2\ell-1}\sigma(t-n'\Theta(n'))\,\Theta(n') \;+\; \partial^{2\ell-1}\omega(\Theta(n')) \\ &= -\partial^{2\ell-1}\omega(\Theta(n'))\,\Theta(n') \;+\; \partial^{2\ell-2}\omega(\Theta(n'))\,. \end{aligned}$$

In the last step we used that $\partial_s^{2\ell-1}\sigma^{(n')}(t;\Theta(n')) = 0$. Taking a second derivative we arrive at the equation

$$0 \;=\; -\partial^{2\ell}\omega(\Theta(n'))\,\Theta'(n')\,\Theta(n') \;=\; -\frac{1}{2}\,\partial^{2\ell}\omega(\Theta(n'))\,\frac{d}{dn'}\big(\Theta(n')\big)^2\,.$$

This equation implies that $\Theta(n') = \Theta_0 = s$ is a constant function.

If $\Theta_0 \neq 0$ we argue as in the proof of Proposition 3.8. Then $\sigma(t-r)$ is a linear function of r near ns. Hence $\partial_s^2\sigma^{(n')}(t;s') = n'\partial^2\omega(s') > 0$, which is a contradiction. If $s = \Theta_0 = 0$ then $\sigma(t) = C - n'\omega(s')$ for n' near n and s' near 0. This is also a contradiction.

The statement 2) is now proved. The statement 3) follows from an induction argument in ℓ, starting with $\ell = 2$. □

Proof of Proposition 4.4: As in the proof of Theorem 1.11 we only need to study the ℓ local minima of $s \to \sigma^{(n')}(t;s)$, n' near n, which comes from ℓ global minima at $n' = n$. They are parameterized by ℓ analytic functions $n' \to \Theta^{(n')}$. We compute, using twice the critical equations $(\partial_s\sigma^{(n')})(t;\Theta_j^{(n')}) = 0$,

$$\begin{aligned} \frac{d}{dn'}\sigma^{(n')}\big(t;\Theta_j^{(n')}\big) &= -\partial\sigma\big(t-n'\Theta_j^{(n')}\big)\,\Theta_j^{(n')} \;+\; \omega\big(\Theta_j^{(n')}\big) \\ &= -\partial\omega\big(\Theta_j^{(n')}\big)\,\Theta_j^{(n')} \;+\; \omega\big(\Theta_j^{(n')}\big)\,. \end{aligned} \tag{3.29}$$

Note that the function $s \to \omega(s) - s\partial\omega(s)$ is reflection invariant and, by Condition 1.3, strictly positive. We furthermore note that $s\partial(\omega - s\partial\omega) = -s^2\partial^2\omega$, so the function is monotonically decreasing away from 0.

Suppose first that no pair $i \neq j$ satisfies $\Theta_i^{(n)} = -\Theta_j^{(n)}$. Then the preceding paragraph and (3.29) implies that there exists $\mathcal{O}_n \ni n$ such that we have a unique global minimum for $n' \in \mathcal{O}_n\backslash\{n\}$.

Now suppose that $\Theta_1^{(n)} = -\Theta_2^{(n)} \neq 0$. Since the maps $n' \to \sigma^{(n')}(t;\Theta_j^{(n')})$ are analytic they are either identical or there exists a neighbourhood of n where they differ (for $n' \neq n$). It hence remains to treat the case where $\sigma^{(n')}(t;\Theta_1^{(n')}) = \sigma^{(n')}(t;\Theta_2^{(n')})$, for n' near n. We argue below that this can only occur if $\Theta_1^{(n')} = -\Theta_2^{(n')}$ for n' near n.

Assume that the function $\Theta_1^{(n')}$ differs from $-\Theta_2^{(n')}$. Since they are analytic, there exists $\ell_0 \geq 1$ such that for $0 \leq \ell < \ell_0$ we have $\frac{d^\ell}{d^\ell n'}\Theta_1^{(n')} = -\frac{d^\ell}{d^\ell n'}\Theta_2^{(n')}$ and $\frac{d^{\ell_0}}{d^{\ell_0} n'}\Theta_1^{(n')} \neq -\frac{d^{\ell_0}}{d^{\ell_0} n'}\Theta_2^{(n')}$. By (3.29) this implies that $\frac{d^{\ell_0+1}}{d^{\ell_0+1}n'}\sigma^{(n')}(t;\Theta_1^{(n')}) \neq \frac{d^{\ell_0+1}}{d^{\ell_0+1}n'}\sigma^{(n')}(t;\Theta_2^{(n')})$, contradicting the assumption.

Write $\Theta^{(n')} := \Theta_1^{(n')} = -\Theta_2^{(n')}$. We can now conclude that $\sigma(t - n'\Theta^{(n')}) = \sigma(t + n'\Theta^{(n')})$. It remains to prove that the analytic function $n' \to n'\Theta^{(n')}$ is strictly monotone near n.

We compute first

$$\begin{aligned} 0 &= \frac{d}{dn'}\partial_s\sigma^{(n')}(t;\Theta^{(n')}) \\ &= \partial_s^2\sigma^{(n')}(t;\Theta^{(n')})\,\frac{d}{dn'}\Theta^{(n')} \;+\; n'\,\partial^2\sigma(t - n'\Theta^{(n')})\,\Theta^{(n')}\,. \end{aligned}$$

This implies

$$\frac{d}{dn'}\Theta^{(n')} \;=\; \frac{1}{n'}\,\Phi^{(n')}\,\Theta^{(n')}\,, \;\text{ where }\; \Phi^{(n')} \;=\; \frac{(n')^2\partial^2\sigma(t - n'\Theta^{(n')})}{\partial_s^2\sigma^{(n')}(t;\Theta^{(n')})} \;<\; 1\,.$$

Using this identity we find $\frac{d}{dn'}\{n'\Theta^{(n')}\} = (1 - \Phi^{(n')})\Theta^{(n')}$. Since $\Theta^{(n')} \neq 0$, we conclude that $\sigma(t - r) = \sigma(t + r)$, for r near ns. □

3.5 Local extrema of $t \to \sigma_{\mathrm{ess}}(t)$

We begin with

Proof of Theorem 1.12: Let t_0 be a local minimum of $t \to \sigma_{\mathrm{ess}}(t)$ and let $\mathcal{U} \ni t_0$ be an open set such that $\sigma_{\mathrm{ess}}(t) \geq \sigma_{\mathrm{ess}}(t_0)$, $t \in \mathcal{U}$.

The function $\mathbb{R}^\nu \ni s \to \sigma^{(1)}(t_0; s)$ has finitely many global minima $\Theta_1^{(1)}(t_0) < \cdots < \Theta_\ell^{(1)}(t_0)$, all in $\mathcal{I}^{(1)}(t_0)$ and with finite index, cf. the proof of Theorem 1.11.

Assume there exists $1 \leq j \leq \ell$ such that $s_0 := \Theta_j^{(1)}(t_0) > 0$. By Proposition 3.8 there exist $\mathcal{O}_{t_0}$, $\mathcal{O}_{s_0}$, and $\mathcal{K}$, with $t_0 \in \mathcal{O}_{t_0} \subset \mathcal{U}$, $s_0 \in \mathcal{O}_{s_0} \subset (0,\infty) \cap (\cup_{t \in \mathcal{O}_{t_0}} \mathcal{I}_0^{(1)}(t))$, and $\mathcal{K} \subset \mathcal{U}$ is countable with $\mathcal{K} \cup \{t_0\}$ closed, such that: For $t \in \mathcal{O}_{t_0} \backslash (\mathcal{K} \cup \{t_0\})$ all local minima of $\mathcal{O}_{s_0} \ni s \to \sigma^{(1)}(t;s)$ has index 1 (and at least one such local minimum exist). Furthermore, the set $\mathcal{O}_{t_0} \backslash (\mathcal{K} \cup \{t_0\})$ can be written as a countable union of disjoint open intervals I_λ. On each of these intervals we get from the Implicit Function Theorem, that the number of local minima $\ell_\lambda \geq 1$, is independent of $t \in I_\lambda$, and the local minima, $\Theta_{\lambda,j}(t)$, $1 \leq j \leq \ell_\lambda$, are analytic in I_λ.

As for (3.28) we compute

$$\partial_t\sigma^{(1)}(t;\Theta_{\lambda,j}(t)) \;=\; \partial\omega(\Theta_{\lambda,j}(t))\,, \quad \text{for } \; t \in I_\lambda\,. \tag{3.30}$$

Let $\tau^{(1)}(t) := \inf_{s \in \mathcal{O}_{s_0}} \sigma^{(1)}(t;s)$. Note that $\tau^{(1)}$ is continuous on $\mathcal{O}_{t_0}$ and on any I_λ we have $\tau^{(1)}(t) = \min_{1 \leq j \leq \ell_\lambda} \sigma^{(1)}(t;\Theta_{\lambda,j}(t))$. Since $\Theta_{\lambda,j}(t) > 0$ we conclude from (3.30) that $\tau^{(1)}$ is monotonely strictly increasing on any I_λ and hence by continuity on $\mathcal{O}_{t_0}$.

We now arrive at a contradiction with the assumption that t_0 is local minumum for $\sigma_{\mathrm{ess}} = \sigma^{(1)}$ as follows. Estimate for $t \in (-\infty, t_0) \cap \mathcal{O}_{t_0}$: $\sigma^{(1)}(t) \leq \tau^{(1)}(t) < \tau^{(1)}(t_0) = \sigma^{(1)}(t_0)$.

We conclude from the argument above that any global minimum $\Theta_j^{(1)}(t_0)$ must be less than or equal to zero. Similarly one can show that $\Theta_j^{(1)}(t_0) \geq 0$, thus leaving only the possibility: $\ell = 1$ and $\Theta^{(1)}(t_0) \equiv \Theta_1^{(1)}(t_0) = 0$. This implies the first part of the theorem, namely that $\sigma_{\mathrm{ess}}(t_0) = \sigma^{(1)}(t_0; 0) = \sigma(t_0) + m$.

Since the gap is m at t_0, and σ_{ess} has a local minimum at t_0, we find from (1.12) that σ also has a local minimum at t_0. In particular σ has a critical point at t_0, with $\partial^2\sigma(t_0) \geq 0$, which yields the bound $\partial_s^2\sigma^{(1)}(t_0; s)_{|s=0} \geq \partial^2\omega(0)$. Hence $\mathrm{Ind}^{(1)}(t_0; 0) = 1$. By Proposition 3.8 1), this implies that σ_{ess} is analytic near t_0 and $\partial\sigma_{\mathrm{ess}}(t) = \partial\omega(\Theta^{(1)}(t))$ near t_0, cf. (3.28). Computing $0 = \partial_t(\partial_s\sigma^{(1)}(\cdot; \Theta^{(1)}(\cdot)))$, near t_0, yields the formula

$$\frac{d}{dt}\Theta^{(1)}(t) \;=\; \frac{\partial^2\sigma(t \,-\, \Theta^{(1)}(t))}{\partial_s^2\sigma^{(1)}(t; \Theta^{(1)}(t))}\,.$$

From this identity, the equation for $\partial^2\sigma(t_0)$ now follows. □

In the rest of this subsection we describe the shape of local maxima of σ_{ess}, and of points with maximal gap, i.e. t with $\sigma_{\mathrm{ess}}(t) - \sigma(t) = m$. We give no formal proofs, but the reader can consult the proof above where most of the needed ingredients are put to use.

Let t_0 be a local maximum of $t \to \sigma_{\mathrm{ess}}(t)$. Then we are in one of the following situations:

I) σ_{ess} forms a wedge at t_0. That is, it is the maximum of two Lipschitz functions with slopes bounded away from 0 (and coinciding at t_0). This occurs if and only if there is at least one negative and one positive global minimum of $s \to \sigma^{(1)}(t_0; s)$.
II) σ_{ess} forms a half-wedge at t_0. That is σ_{ess} has its derivative bounded away from 0 on one side of t_0 and is bounded from below by $\sigma_{\mathrm{ess}}(t_0) - C(t_0 - t)^2$, for t near t_0 and on the other side of t_0. This occurs if and only if: The spectral gap is m, and besides 0, the function $s \to \sigma_0(t_0; s)$ has at least one more global minimum, all with the same sign. Furthermore t_0 is either a saddle point for σ or a local maximum. (this includes the case where σ_{ess} is constant on one side of t_0.)
III) σ_{ess} does not form a wedge (or a half-wedge) at t_0. That is, σ_{ess} is bounded from below by $\sigma_{\mathrm{ess}}(t_0) - C(t_0 - t)^2$, for t near t_0. This occurs if and only if: The spectral gap is m and the function $s \to \sigma^{(1)}(t_0; s)$ has $s = 0$ as a unique global minimum. Furthermore σ has a local maximum at t_0. (This includes the possibility that σ_{ess} is locally constant, in which case t_0 is also a local minimum.)

In both cases **II)** and **III)** we must necessarily have $\partial^2\sigma(t_0) \geq -\partial^2\omega(t_0)$, in order to have a local minimum at $s = 0$.

Now suppose t_0 is such that $\sigma_{\mathrm{ess}}(t_0) - \sigma(t_0) = m$. We have already discussed how this can occur at local extrema. But there are two other possibilities where this may occur.

IV) σ_{ess} is the minimum of two curves, intersecting at t_0. One with slope bounded away from 0 (increasing or decreasing), and one which is analytic, non-decreasing away from t_0, and bounded from above by $\sigma_{\mathrm{ess}}(t_0) + C(t_0 - t)^2$. This occurs if and only if: the function $s \to \sigma^{(1)}(t_0; s)$ has a global minimum at 0 and at least one more global minimum, all with the same sign. Furthermore, σ has a saddle point or a local minimum at t_0.

We end this section with a comment on jump discontinuities of the bounded function $\partial\sigma_{\mathrm{ess}}(t) = \partial\omega(\Theta^{(1)}(t))$. When t increases (away from 0), global minima are a priori not monotone, but when they jump, they jump from large s to smaller s. Passing to larger s, can only happen analytically (where $\partial^2\sigma(t-s) \geq 0$, and hence a local minimum has index 1). This implies that

V) Jump discontinuities of $\partial\sigma_{\mathrm{ess}}$ always decrease the derivative.

We note that a wedge can only coincide with a maximal spectral gap, at a local maximum for σ_{ess}, i.e, where the derivative jumps from being positive to negative.

4 Additional results

In this section we collect some additional results, most of which have appeared elsewhere in some form. They serve to give a more complete picture of the bottom of the joint energy momentum spectrum. In addition we explain how to extend the results described in this paper to models with a number cutoff in the interaction.

4.1 Complimentary results

In this section we recall some known and partly known related results on the structure of the ground state mass shell. The first is due to Gross [31, (6.30)].

Lemma 4.1. (Gross) *Let $v \in L^2(\mathbb{R}^\nu)$ and $\omega(k) = \sqrt{k^2+m^2}$, $m > 0$. Assume Condition 1.1 and that, for any $t > 0$, the map $p \to e^{-t\Omega(p)}$ is positive definite. Then*

$$\Sigma_0(\xi) \ \geq \ \Sigma_0(0)\,.$$

The second result we mention is an extension of a result of Hiroshima and Spohn. See [36, Lemma 3.1] and its proof.

Lemma 4.2. *Let $v \in L^2(\mathbb{R}^\nu)$ satisfy $v > 0$ a.e., and assume Conditions 1.1 and 1.2. Let $\xi \in \mathcal{I}_0$, write ψ_ξ for a normalized ground state eigenfunction, and $\overline{P}_\xi := \mathbb{1}_{\mathcal{F}} - |\psi_\xi\rangle\langle\psi_\xi|$. Then*

$$\begin{aligned}\{\nabla^2\Sigma_0(\xi)\}_{ij} \ &= \ \langle\psi_\xi, \partial_i\partial_j\Omega(\xi - d\Gamma(k))\psi_\xi\rangle \\ &\quad - \big\langle \overline{P}_\xi\, \partial_i\Omega(\xi - d\Gamma(k))\,\psi_\xi, (H(\xi)\, -\, \Sigma_0(\xi))^{-1}\,\overline{P}_\xi\, \partial_j\Omega(\xi - d\Gamma(k))\,\psi_\xi\big\rangle\,.\end{aligned}$$

In particular $\nabla^2\Sigma_0(\xi) \leq \sup_p \sigma(\nabla^2\Omega(p))\,\mathbb{1}_{\mathbb{R}^\nu}$.

Note that by Theorem 1.3, $H(\xi) - \Sigma_0(\xi)$ is bounded invertible on the range of $\overline{P}_\xi$. If $\xi \in \mathcal{I}_0$ is a critical point for $\xi \to \Sigma_0(\xi)$, then $\partial_j\Sigma_0(\xi) = \langle\psi_\xi, \partial_j\Omega(\xi - d\Gamma(k))\psi_\xi\rangle = 0$, $1 \leq j \leq \nu$, and hence the $\overline{P}_\xi$ in the formula above for the Hessian is superfluous. This is the case considered in [36] (see also [53]). We leave the proof to the reader. In the case $\Omega(p) = p^2/2M$, Lemma 4.2 implies a lower bound $M_{\rm eff} \geq M$ on the effective mass, where $M_{\rm eff}^{-1} := \partial^2\sigma(0)$ (assuming rotation invariance). In [53] an upper bound for the effective mass is derived, implying in particular that $\partial^2\sigma(0) > 0$. This is still an open problem for $\Omega(p) \neq p^2/2M$.

We note that similarly one can prove the following statement: Replace v by gv, where $g \in \mathbb{R}$ is a coupling constant. Let g and ξ be such that $\xi \in \mathcal{I}_0$, which is a g-dependent set. Then $\Sigma_0(\xi)$ is an analytic function of the coupling constant in a neighbourhood of g, $\frac{d}{dg}\Sigma_0(\xi) = \langle \psi_\xi, \Phi(v)\psi_\xi\rangle$, and

$$\frac{d^2}{d^2g}\Sigma_0(\xi) \;=\; -\left\langle \overline{P}_\xi\, \Phi(v)\,\psi_\xi, (H(\xi)-\Sigma_0(\xi))^{-1}\,\overline{P}_\xi\,\Phi(v)\,\psi_\xi\right\rangle .$$

In particular, the function $g \to \Sigma_0(\xi)$ is concave in the set $\{g : \xi \in \mathcal{I}_0\}$.

Thirdly we formulate a result, which follows from the proof of [20, Theorem 3.2]. We give a short proof of the statement here because Fröhlich concentrated on the massless case, and the proof simplifies for massive bosons. We remark that the infrared cutoff $\sigma > 0$ in [20] can be viewed as a mass.

Theorem 4.3. *Let $v \in L^2(\mathbb{R}^\nu)$. Assume Conditions 1.1, 1.2, and that the following bounds hold for all $p, k \in \mathbb{R}^\nu$*

$$|\nabla\Omega(p)| \leq 1 \;\; \textit{and} \;\; \omega(k) \;-\; |k| \;>\; 0\,. \tag{4.1}$$

Then $\mathcal{I}_0 = \mathbb{R}^\nu$.

Remark. This theorem implies in particular that in the case of relativistic electrons, i.e. $\Omega(p) = \sqrt{p^2+M^2}$ ($M > 0$), and $\omega(k) = \sqrt{k^2+m^2}$ ($m > 0$), we have an isolated ground state mass shell for all total momenta. This type of result was an important ingredient in [21].

Proof. Suppose $\mathcal{I}_0 \neq \mathbb{R}^\nu$, and let $\xi \in \mathbb{R}\backslash\mathcal{I}_0$.

Define, for $\xi, k \in \mathbb{R}^\nu$ with $k \neq 0$,

$$F(\xi,k) := |k|^{-1}\left\{\Omega(\xi - d\Gamma(k)) - \Omega(\xi - k - d\Gamma(k)\right\} .$$

This self adjoint operator extends from $\mathcal{C}_0^\infty$ to a bounded operator on $\mathcal{F}$, and by (4.1) it satisfies the bound

$$\|F(\xi,k)\|_{\mathcal{B}(\mathcal{F})} \;\leq\; 1\,. \tag{4.2}$$

Let

$$n := \max\left\{n' \;\geq\; 1 \;:\; \Sigma^{(n')}(\xi) \;=\; \Sigma_{\rm ess}(\xi)\right\} . \tag{4.3}$$

By Theorem 1.2 and (1.15) this choice of n is well defined. For $\underline{k} \in \mathcal{I}_0^{(n)}(\xi)$, we write $\psi_{\xi-k^{(n)}} \in \mathcal{D}$ for the ground state eigenfunction at total momentum $\xi - k^{(n)}$. Note that

$k^{(n)} \neq 0$. For such $\underline{k}$ we use (4.2) and the Rayleigh-Ritz variational principle to estimate

$$\begin{aligned} \Sigma_0(\xi) &\leq \langle \psi_{\xi - k^{(n)}}, H(\xi)\, \psi_{\xi - k^{(n)}} \rangle \\ &= \Sigma_0(\xi - k^{(n)}) \;+\; |k^{(n)}| \, \big\langle \psi_{\xi - k^{(n)}}, F(\xi, k^{(n)})\, \psi_{\xi - k^{(n)}} \big\rangle \\ &\leq \Sigma_0(\xi - k^{(n)}) \;+\; |k^{(n)}| \,. \end{aligned} \tag{4.4}$$

Let $\mathcal{U} := \mathcal{I}_0^{(n)}(\xi) \cap \{\underline{\eta} \in \mathbb{R}^{n\nu} : \Sigma_0(\xi - \eta^{(n)}) \leq \Sigma_0(\xi)\}$. The bound (4.4), Lemma 1.7, and the choice (4.3) of n, implies

$$\begin{aligned} \Sigma_0^{(n)}(\xi) &= \inf_{\underline{k} \in \mathbb{R}^{n\nu}} \Sigma_0^{(n)}(\xi; \underline{k}) \;=\; \inf_{\underline{k} \in \mathcal{U}} \big\{ \Sigma_0(\xi \,-\, k^{(n)}) \;+\; \sum_{j=1}^n \omega(k_j) \big\} \\ &\geq \Sigma_0(\xi) \;+\; \inf_{k \in \mathcal{U}} \big\{ \sum_{j=1}^n \omega(k_j) \;-\; |k^{(n)}| \big\} \,. \end{aligned}$$

By (1.14) there exists $C_{\mathcal{U}} > 0$, independent of n, such that $|k^{(n)}| \leq C_{\mathcal{U}}$, $\underline{k} \in \mathcal{U}$. Now choose R such that $\omega(k) \geq C_{\mathcal{U}} + 1$ for $|k| \geq R$. Since $|k^{(n)}| \leq |k_1| + \cdots + |k_n|$, we arrive at the following estimate, cf. (4.3),

$$\Sigma_0(\xi) \;=\; \Sigma_{\rm ess}(\xi) \;=\; \Sigma_0^{(n)}(\xi) \;\geq\; \Sigma_0(\xi) \;+\; \min\big\{ 1 \,,\, \inf_{k : |k| \leq R} (\omega(k) \,-\, |k|) \big\} \,.$$

By (4.1) this is a contradiction. $\square$

In addition to Theorem 1.11 we have a complimentary result which is concerned with the regularity of $\sigma^{(n)}(t)$ as a function of n. The proof is at the end of Subsect. 3.4

Proposition 4.4. *Assume Conditions 1.1, 1.2, 1.3, and 1.4. Let $t \in \mathbb{R}$. There exists a closed countable set $\mathcal{T}(t) \subset (0,\infty)$, and an analytic map $(0,\infty) \backslash \mathcal{T}(t) \ni n \to \Theta^{(n)}(t) \in \mathcal{I}_0^{(n)}(t)$, with the property that the maps $s \to \sigma^{(n)}(t; s)$, $n \in (0,\infty) \backslash \mathcal{T}(t)$, has a global minimum at $s = \Theta^{(n)}(t)$, with $\mathrm{Ind}^{(n)}(t; \Theta^{(n)}(t)) = 1$. Let $(a,b) \subset (0,\infty) \backslash \mathcal{T}(t)$. The global minimum is either unique for all $n \in (a,b)$, or it is accompanied by another global minimum sitting at $s = -\Theta^{(n)}(t)$, for all $n \in (a,b)$. The case of two global minima can occur if and only if $\sigma(t - r) = \sigma(t + r)$ for r in a neighbourhood of $n\Theta^{(n)}(t)$. We furthermore have*

$$\frac{d}{dn}\, \sigma^{(n)}(t) \;=\; \omega\big(\Theta^{(n)}(t)\big) \;-\; \partial\omega\big(\Theta^{(n)}(t)\big)\, \Theta^{(n)}(t) \,, \quad \textit{for} \;\; n \,\in\, (0,\infty) \setminus \mathcal{T}(t) \,. \tag{4.5}$$

The function $x \to \omega(x) - x\partial\omega(x)$ appearing on the right-hand side of (4.5), is the one from Lemma 1.5. The identity (4.5) can be used to estimate the splitting $\Sigma_0^{(n+1)}(\xi) - \Sigma_0^{(n)}(\xi)$. (In the submitted version of this paper, the proof of Proposition 4.4 is left to the reader.)

4.2 Interactions with a number cutoff

In this subsection and the next we consider models of the form, cf. (1.5),

$$H_{\mathcal{N}} \;:=\; H_0 \;+\; 1\!\!1_{\mathcal{K}} \otimes 1\!\!1(N \le \mathcal{N})\, V\, 1\!\!1_{\mathcal{K}} \otimes 1\!\!1(N \le \mathcal{N})\,.$$

Here $\mathcal{N} \in \mathbb{Z}$ is the cutoff parameter. Clearly these operators also commute with the total momentum and The corresponding fiber Hamiltonians are, cf. (1.6),

$$H_{\mathcal{N}}(\xi) \;:=\; H_0(\xi) \;+\; \Phi_{\mathcal{N}}(v), \quad \text{where} \;\; \Phi_{\mathcal{N}}(v) \;:=\; 1\!\!1(N \le \mathcal{N})\, \Phi(v)\, 1\!\!1(N \le \mathcal{N})\,.$$

Note that the notation is consistent since $\Phi_0(v) = 0$. For $\mathcal{N} < 0$ we clearly also have $H_{\mathcal{N}}(\xi) = H_0(\xi)$.

We remark that for $\mathcal{N} = 1$ a complete picture can be obtained, cf. [23], (mass zero case). We note that the spin-boson model has been studied perturbatively for $\mathcal{N} = 2$ in [44]. See also [25, 38, 39].

We now formulate our main results from Subsection 1.3 in the context of the cutoff models. We impose for brevity of exposition Conditions 1.1, 1.2, 1.3, and 1.4 throughout this subsection. We furthermore impose the following additional condition

Condition 4.1. *The form factor satisfies that $v > 0$ a.e. locally uniformly.*

Let $\mathcal{N} \ge 1$. We introduce some notation. First the bottom of the spectrum of the full operator:

$$\Sigma_{\mathcal{N},0} \;:=\; \inf_{\xi\in\mathbb{R}^\nu} \Sigma_{\mathcal{N},0}(\xi)\,, \quad \text{where} \;\; \Sigma_{\mathcal{N},0}(\xi) \;:=\; \inf \sigma(H_{\mathcal{N}}(\xi))\,.$$

For $n \ge 1$ and $\underline{k} = (k_1, \ldots, k_n) \in \mathbb{R}^{n\nu}$ we introduce

$$\Sigma^{(n)}_{\mathcal{N},0}(\xi;\underline{k}) \;:=\; \Sigma_{\mathcal{N}-n,0}\big(\xi - k^{(n)}\big) \;+\; \sum_{j=1}^{n} \omega(k_j)$$

and

$$\Sigma^{(n)}_{\mathcal{N},0}(\xi) \;:=\; \inf_{\underline{k}\in\mathbb{R}^{n\nu}} \Sigma^{(n)}_{\mathcal{N},0}(\xi;\underline{k})\,.$$

The bottom of the essential spectrum is

$$\Sigma_{\mathrm{ess},\mathcal{N}}(\xi) \;:=\; \Sigma^{(1)}_{\mathcal{N},0}(\xi) \;=\; \inf_{k\in\mathbb{R}^\nu} \Sigma^{(1)}_{\mathcal{N},0}(\xi;k)\,.$$

We furthermore write

$$\begin{aligned} \mathcal{I}_{\mathcal{N},0} \;&:=\; \big\{\xi \in \mathbb{R}^\nu \,:\, \Sigma_{\mathcal{N},0}(\xi) \;<\; \Sigma_{\mathrm{ess},\mathcal{N}}(\xi)\,\big\}\,,\\ \mathcal{I}^{(n)}_{\mathcal{N},0}(\xi) \;&:=\; \big\{\underline{k} \in \mathbb{R}^{n\nu} \,:\, \xi - k^{(n)} \in \mathcal{I}_{\mathcal{N}-n,0}\,\big\}\,. \end{aligned}$$

The energies $\Sigma^{(n)}_{\mathcal{N},0}(\xi)$, $n \ge 1$, are bottoms of branches of essential spectrum corresponding to having stripped of n photons to infinity, and having the interacting systems in a groundstate. Lemma 1.5, Condition 4.1, and the Rayleigh-Ritz variational principle ensures that the thresholds are ordered:

$$\Sigma^{(n)}_{\mathcal{N},0}(\xi) \;>\; \Sigma^{(n')}_{\mathcal{N},0}(\xi)\,,$$

for all $n > n' \geq 1$. This is where the assumption $v \geq 0$ comes in. It ensures that the thresholds appear in an ordered fashion as in the full model.

Note that the properties (1.12) and (1.13) do not hold for the cutoff model. The gap $\Sigma_{\mathrm{ess},\mathcal{N}}(\xi)-\Sigma_{\mathcal{N},0}(\xi)$ may exceed m. However, we do have that $\Sigma_{\mathrm{ess},\mathcal{N}}(\xi)-\Sigma_{\mathcal{N}-1,0}(\xi) \leq m$ (it may be negative).

We introduce, as in Subsect. 1.3, the following notation. Let $\vec{u}$ be a unit vector in $\mathbb{R}^\nu$. We write $\sigma_{\mathcal{N}}(t) = \Sigma_{0,\mathcal{N}}(t\vec{u})$, for $t \in \mathbb{R}$. By rotation invariance, $\sigma_{\mathcal{N}}$ is independent of $\vec{u}$. Similarly we write, for $n \in \mathbb{N}$, $\sigma^{(n)}_{\mathcal{N}}(t;s) := \sigma_{\mathcal{N}-n}((t-ns)\vec{u}) + n\omega(s\vec{u})$, $\sigma^{(n)}_{\mathcal{N}}(t) := \Sigma^{(n)}_{0,\mathcal{N}}(t\vec{u})$, and $\sigma_{\mathrm{ess},\mathcal{N}}(t) := \Sigma_{\mathrm{ess}}(t\vec{u})$.

With a slight abuse of notation, we use the same symbol $\mathcal{I}_{0,\mathcal{N}}$ to denote the set of t's such that $t\vec{u} \in \mathcal{I}_{0,\mathcal{N}}$. We furthermore use the symbol $\mathcal{I}^{(n)}_{0,\mathcal{N}}(t)$, $n \in \mathbb{N}$, to denote the set $\{s \in \mathbb{R} : t - ns \in \mathcal{I}_{0,\mathcal{N}}\}$.

We now list a number of results, which we do not prove here. See however the following subsection. In each case the reader can readily mimic the proofs, given in Section 3, of the corresponding results for the full model.

- For each $\mathcal{N} \geq 1$ and $\xi \in \mathbb{R}^\nu$, $\Phi_{\mathcal{N}}(v)$ is $H_0(\xi)$ bounded with relative bound zero. In particular $H_{\mathcal{N}}(\xi)$ is essentially self-adjoint on $\mathcal{C}_0^\infty$, and $\mathcal{D}(H_{\mathcal{N}}(\xi))$ is independent of ξ.

- (HVZ) The bottom of the essential spectrum of $H_{\mathcal{N}}(\xi)$ is $\Sigma_{\mathrm{ess},\mathcal{N}}(\xi)$. Eigenvalues below $\Sigma_{\mathrm{ess},\mathcal{N}}(\xi)$ have finite multiplicity and can only accumulate at $\Sigma_{\mathrm{ess},\mathcal{N}}(\xi)$. See also [25, 38] for the cutoff spin-boson model.

- The ground state is non-degenerate, and in addition: If $1 \leq \nu \leq 2$ then $\mathcal{I}_{\mathcal{N},0} = \mathbb{R}^\nu$. If $3 \leq \nu \leq 4$ then the bottom of the spectrum $\Sigma_{\mathcal{N},0}(\xi)$ is an eigenvalue if and only if $\xi \in \mathcal{I}_{\mathcal{N},0}$. As a consequence of the non-degeneracy, the map $\mathcal{I}_{\mathcal{N},0} \ni t \to \sigma_{\mathcal{N}}(t)$ is analytic.

- Let $n \in \mathbb{N}$. There exists a closed countable set $\mathcal{T}^{(n)}_{\mathcal{N}} \subset \mathbb{R}$, and an analytic map $\mathbb{R}\backslash\mathcal{T}^{(n)}_{\mathcal{N}} \ni t \to \Theta^{(n)}_{\mathcal{N}}(t) \in \mathcal{I}^{(n)}_{\mathcal{N},0}(t)$ with the property that the maps $s \to \sigma^{(n)}_{\mathcal{N}}(t;s)$, $t \in \mathbb{R}\backslash\mathcal{T}^{(n)}_{\mathcal{N}}$, has a unique global minimum at the point $s = \Theta^{(n)}_{\mathcal{N}}(t)$, with index $\mathrm{Ind}^{(n)}(t;\Theta^{(n)}_{\mathcal{N}}(t)) = 1$. In particular $\mathbb{R}\backslash\mathcal{T}^{(n)}_{\mathcal{N}} \ni t \to \sigma^{(n)}_{\mathcal{N}}(t)$ is analytic and $\frac{d}{dt}\sigma^{(n)}_{\mathcal{N}}(t) = \partial\omega(\Theta^{(n)}_{\mathcal{N}}(t))$, for $t \in \mathbb{R}\backslash\mathcal{T}^{(n)}_{\mathcal{N}}$. Recall $\sigma^{(1)}_{\mathcal{N}}(t) = \sigma^{(1)}_{\mathrm{ess},\mathcal{N}}(t)$.

- Let t_0 be a local minimum of $t \to \sigma_{\mathrm{ess},\mathcal{N}}(t)$. Then the 'spectral gap' at t_0 is maximal, i.e. $\sigma_{\mathrm{ess},\mathcal{N}}(t_0) - \sigma_{\mathcal{N}-1}(t_0) = m$, the map $t \to \sigma_{\mathcal{N}-1}(t)$ has a local minimum at t_0, the map $t \to \sigma_{\mathrm{ess},\mathcal{N}}(t)$ is analytic near t_0, and

$$\partial^2\sigma_{\mathrm{ess},\mathcal{N}}(t_0) \;=\; \frac{\partial^2\omega(0)\,\partial^2\sigma_{\mathcal{N}-1}(t_0)}{\partial^2\omega(0) \,+\, \partial^2\sigma_{\mathcal{N}-1}(t_0)}\,.$$

4.3 Comments on proofs

The key difference between the cutoff models and the full model, lies in the self-similarity of the full model. By self-similarity we mean that after removing a number of bosons to infinity, the remaining interacting system has the same Hamiltonian as the original system, albeit at a different total momentum. For the cutoff model the interacting system, after removing bosons to infinity, has a different cutoff. This is manifested in two instances, in the extended Hamiltonian and in the pull-through formula.

For the cutoff model(s) one should replace the extended Hamiltonian, cf. (2.31) and (2.34), by

$$H_{\mathcal{N}}^{\mathrm{ext}}(\xi) \ := \ H_{\mathcal{N}}(\xi) \, \oplus \, \Big\{ \bigoplus_{\ell=1}^{\infty} H_{\mathcal{N}}^{(\ell)}(\xi) \Big\} \, ,$$

where $H_{\mathcal{N}}^{(\ell)}(\xi) = \int_{\mathbb{R}^{\ell\nu}} H_{\mathcal{N}}^{(\ell)}(\xi; \underline{k}) d^{\ell\nu}\underline{k}$ and

$$H_{\mathcal{N}}^{(\ell)}(\xi; \underline{k}) \ = \ H_{\mathcal{N}-\ell}(\xi - k^{(\ell)}) + \sum_{j=1}^{\ell} \omega(k_j) \, .$$

With this choice of extended Hamiltonian, the localization estimates derived in Subsect. 3.1 applies. This is one of the inputs to the HVZ theorem.

The second manifestation of the lack of self-similarity is in the pull-through formula which should be replaced by

$$\begin{aligned} \mathbf{a}(k) \, (H_{\mathcal{N}}(\xi) \, - \, z) \, \psi \quad &= \quad \big(H_{\mathcal{N}-1}(\xi \, - \, k) \, + \, \omega(k) \, - \, z \big) \mathbf{a}(k) \, \psi \\ &\quad + \, v(k) \, \mathbb{1}(N \leq \mathcal{N} - 1) \, \psi \, . \end{aligned}$$

It is now left as an exercise to the reader to verify that the proofs go through. We just remark that when applying the Perron Frobenius argument, as in Subsect. 3.3, one should work only in the sub Hilbert space $\oplus_{j=0}^{\mathcal{N}} \Gamma^{(j)}(\mathfrak{h}_{\mathrm{ph}})$ of $\mathcal{F}$. Any eigenfunction will vanish in n-particle sectors with $n > \mathcal{N}$, which is reflected in the fact that the cutoff resolvents, $(H_{\mathcal{N}}(\xi) + \mu)^{-1}$, are not positivity improving in the full Hilbert cone.

Acknowledgments: The author thanks Z. Ammari, V. Bach, J. Fröhlich, and C. Gérard, for useful discussions, and Dokuz Eylül University for hospitality. This work was supported in parts by Carlsbergfondet and by a Marie-Curie individual fellowship from the European Union.

A Mathematical tools

A.1 Almost analytic extension

In this subsect. we briefly recall the functional calculus provided by almost analytic extensions. In particular we will use a version which handles functions of a vector of commuting operators. See the monographs by Davies [12] and Dimassi and Sjöstrand [16] for details.

Below α will denote multi-indices. Let $s \in \mathbb{R}$ and $f \in C^\infty(\mathbb{R}^\nu)$ satisfy

$$\forall \alpha : \exists C_\alpha \text{ such that } |\partial^\alpha f(x)| \ \le \ C_\alpha \, \langle x \rangle^{s-|\alpha|} \, . \tag{A.1}$$

We define an almost analytic extension $\tilde{f} \in C^\infty(\mathbb{C}^\nu)$ of f, through a Borel construction. Fix a function $\chi \in C_0^\infty(\mathbb{R})$ to be equal to 1 in a neighbourhood of 0, and a sequence $\{\lambda_k\}_{k \in \mathbb{N}_0}$, going sufficiently fast to infinity. The following choice will do: $\lambda_k := \max\{\max_{|\alpha|=k} C_\alpha, \lambda_{k-1} + 1\}$, for $k \ge 1$, and $\lambda_0 = C_0$. Here the constants C_α are coming from (A.1). Then, writing $z = u + \mathrm{i} v \in \mathbb{R}^\nu \oplus \mathrm{i}\mathbb{R}^\nu$,

$$\tilde{f}(z) \ := \ \sum_\alpha \frac{\partial^\alpha f(u)}{\alpha!} (\mathrm{i} v)^\alpha \prod_{j=1}^{\nu} \chi\Big(\frac{\lambda_{|\alpha|} v_j}{\langle u \rangle} \Big) \, . \tag{A.2}$$

Note that there exists $C > 0$ such that

$$\mathrm{supp}(\tilde{f}) \ \subset \ \{ u + \mathrm{i} v \, : \, u \in \mathrm{supp}(f), \, |v| \ \le \ C \langle u \rangle \} \, .$$

We furthermore have the property that

$$\forall \ell \ge 0 : \exists C_\ell \text{ such that } |\bar{\partial} \tilde{f}(z)| \ \le \ C_\ell \, \langle z \rangle^{s-\ell-1} \, |\mathrm{Im} z|^\ell \, . \tag{A.3}$$

Here $\bar{\partial} = (\bar{\partial}_1, \dots, \bar{\partial}_\nu)$, $\bar{\partial}_j := \partial_{u_j} + \mathrm{i} \partial_{v_j}$, and $\mathrm{Im} z = (v_1, \dots, v_\nu)$.

If $s < 0$ we have the following representation,

$$f(x) \ = \ 2 \, |S^{2\nu-1}|^{-1} \int_{\mathbb{C}^\nu} \big\langle \bar{\partial} \tilde{f}(z), \frac{(x+z)}{|x-z|^{2\nu}} \big\rangle \, d^{2\nu} z \, ,$$

where $d^{2\nu} z = \Pi_{j=1}^\nu du_j dv_j$ is the Lebesgue measure on $\mathbb{C}^\nu$, and $|S^{2\nu-1}|$ is the volume of the unit ball in $\mathbb{R}^{2\nu}$. (Note that for $s < 0$ the integral is absolutely convergent.)

For a vector of pairwise commuting self-adjoint operators $A = (A_1, \dots, A_\nu)$, and a function f satisfying (A.1) with $s < 0$, the almost analytic extension thus provides a functional calculus via the formula

$$f(A) \ = \ 2 \, |S^{2\nu-1}|^{-1} \sum_{j=1}^{\nu} \int_{\mathbb{C}^\nu} \bar{\partial}_j \tilde{f}(z) \, (A_j \, + \, z_j) \, |A - z|^{-2\nu} \, d^{2\nu} z \, . \tag{A.4}$$

In the case $\nu = 1$ this reduces to

$$f(A) \ = \ \frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial} \tilde{f}(z) \, (A \, - \, z)^{-1} \, du \, dv \, .$$

A.2 Invariant cones

In this subsect. we recall a result of Faris, cf. [17], which will be used to show non-degeneracy of the ground state. It is an abstract version of the Perron-Frobenius Theorem in L^2-spaces, cf. [49, Theorem XIII.43], which together with the Q-space representation of Fock-space, has been used frequently to show non-degeneracy of the ground state, cf. [5, 28, 31].

Definition A.1. *Let $\mathcal{H}_\mathbb{R}$ be a real Hilbert space. We say $\mathbf{C} \subset \mathcal{H}_\mathbb{R}$, $\mathbf{C} \neq \{0\}$, is a Hilbert cone if:*

i) $u, v \in \mathbf{C}$ *implies* $u + v \in \mathbf{C}$.

ii) $u \in \mathbf{C}, \lambda \geq 0$ *implies* $\lambda u \in \mathbf{C}$.

iii) $\mathbf{C} \cap (-\mathbf{C}) = \{0\}$.

iv) $\mathbf{C}$ *is closed.*

v) $u, v \in \mathbf{C}$ *implies* $\langle u, v\rangle \geq 0$.

vi) For all $w \in \mathcal{H}_\mathbb{R}$ *there exists* $u, v \in \mathbf{C}$ *s. t.* $w = u - v$ *and* $\langle u, v\rangle = 0$.

An important example of a Hilbert cone is, as mentioned above, the subset of real non-negative functions in $L^2(Q, d\mu)$, where Q is a measure space.

Definition A.2. *Let $\mathcal{H}_\mathbb{R}$ be a real Hilbert space, $\mathbf{C} \subset \mathcal{H}_\mathbb{R}$ a Hilbert cone and A a bounded operator on $\mathcal{H}_\mathbb{R}$.*

i) We say $u \in \mathbf{C}$ is strictly positive if $\langle u, v\rangle > 0$ for any $v \in \mathbf{C}\backslash\{0\}$.

ii) A is positive preserving if $A\mathbf{C} \subset \mathbf{C}$.

iii) A is positivity improving if Au is strictly positive for all $u \in \mathbf{C}\backslash\{0\}$.

iv) A is ergodic if for any $u, v \in \mathbf{C}\backslash\{0\}$ there exists $n \geq 0$ s. t. $\langle A^n u, v\rangle > 0$.

Note that a positivity improving operator is in particular ergodic. The following theorem is due to Faris

Theorem A.3. **(Faris)** *Let $\mathcal{H}_\mathbb{R}$ be a real Hilbert space, $\mathbf{C} \subset \mathcal{H}_\mathbb{R}$ a Hilbert cone and A a bounded positive self-adjoint operator on $\mathcal{H}_\mathbb{R}$. Suppose furthermore that A is positivity preserving and that $\|A\|$ is an eigenvalue for A. Then A is ergodic if and only if $\|A\|$ is an eigenvalue of multiplicity one and there exists a strictly positive $u \in \mathbf{C}$ with $Au = \|A\|u$.*

The lemma below follows from the identities $e^{-s} = \lim_{n\to\infty}(\frac{s}{n} + 1)^{-n}$ and $s^{-1} = \int_0^\infty e^{-ts} ds$, for $s > 0$, in conjunction with the first resolvent formula.

Lemma A.4. *Let A be a bounded from below self-adjoint operator on a real Hilbert space. Assume that there exists a $\lambda_0 < \inf \sigma(A)$ such that $(A - \lambda)^{-1}$ is positivity preserving (improving) for all $\lambda < \lambda_0$. Then $(A - \lambda)^{-1}$ is positivity preserving (improving) for all $\lambda < \inf \sigma(A)$.*

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