LINEAR RESPONSE THEORY FOR MAGNETIC SCHRÖDINGER OPERATORS IN DISORDERED MEDIA

JEAN-MARC BOUCLET, FRANÇOIS GERMINET, ABEL KLEIN, AND JEFFREY H. SCHENKER

Abstract. We justify the linear response theory for an ergodic Schrödinger operator with magnetic field within the non-interacting particle approximation, and derive a Kubo formula for the electric conductivity tensor. To achieve that, we construct suitable normed spaces of measurable covariant operators where the Liouville equation can be solved uniquely. If the Fermi level falls into a region of localization, we recover the well-known Kubo-Středa formula for the quantum Hall conductivity at zero temperature.

Contents

2000 *Mathematics Subject Classification.* Primary 82B44; Secondary 47B80, 60H25.
A.K. was supported in part by NSF Grant DMS-0200710.
J.S. was supported in part by an NSF postdoctoral research fellowship.

1. Introduction

In theoretical works, the electric conductivity tensor is usually expressed in terms of a "Kubo formula," derived via formal linear response theory. The importance of this Kubo formula is enhanced by its links with the quantum Hall conductivity at zero temperature. During the past two decades a few papers managed to shed some light on these derivations from the mathematical point of view, e.g., [P, Ku, B, NB, AvSS, BES, SB1, SB2, AG, Na, ES, AES]. While a great amount of attention has been brought to the derivation of the quantum Hall conductivity from a Kubo formula, and to the study of this conductivity itself, not much has been done concerning a controlled derivation of the linear response and the Kubo formula itself; only the recent papers [SB2, Na, ES, AES, CoJM] deal with this question.

In this article we consider an ergodic Schrödinger operator with magnetic field, and give a controlled derivation of a Kubo formula for the electric conductivity tensor, validating the linear response theory within the noninteracting particle approximation. For an adiabatically switched electric field, we then recover the expected expression for the quantum Hall conductivity whenever the Fermi energy lies either in a region of localization of the reference Hamiltonian or in a gap of the spectrum.

To perform our analysis we develop an appropriate mathematical apparatus for the linear response theory. We first describe several normed spaces of measurable covariant operators which are crucial for our analysis. We develop certain analytic tools on these spaces, in particular the trace per unit volume and a proper definition of the product of two (potentially unbounded) operators. (Similar spaces and their relevance were already discussed in [BES].) We then use those tools to compute rigorously the linear response of the system forced by a time dependent electric field. This is achieved in two steps. First we set up the Liouville equation which describes the time evolution of the density matrix under the action of a time-dependent electric field, in a suitable gauge with the electric field given by a time-dependent vector potential. In a standard way, this evolution equation can be written as an integral equation, the so-called Duhammel formula. Second, we compute the net current per unit volume induced by the electric field and prove that it is differentiable with respect to the electric field at zero field. This yields the desired Kubo formula for the electric conductivity tensor. We then push the analysis further to recover the expected expression for the quantum Hall conductivity, the Kubo-Středa formula.

Our derivation of the Kubo formula is valid for any initial density matrix $\zeta = f(H)$ with a smooth profile of energies $f(E)$ that has appropriate decay at high energies. In particular, the Fermi-Dirac distributions at positive temperature are allowed. At zero temperature, with the Fermi projection $P^{(E_F)}$ as the initial profile, our analysis is valid whenever the Fermi energy E_F lies either in a gap of the spectrum or in a region of localization of the reference Hamiltonian. The latter is actually one of the main achievements of this article. There is indeed a crucial difference between $P^{(E_F)}$ with E_F in a gap (or similarly $f(H)$, with f smooth with decay at high energies) and $P^{(E_F)}$ with E_F in a region of localization: in the first case the commutator $[x_k, P^{(E_F)}]$ is a bounded operator while it is unbounded in the second case. Dealing with the unbounded commutator $[x_k, P^{(E_F)}]$, which appears naturally in the Kubo-Středa formula, forces us to use the full theory of the normed spaces of measurable covariant operators we develop.

We now sketch the main points of our analysis. We consider a system of non-interacting quantum particles in a disordered background, with the associated one-particle Hamiltonian described by an ergodic magnetic Schrödinger operator

$$H_\omega = (-i\nabla - \mathbf{A}_\omega)^2 + V_\omega \quad \text{on} \quad \mathcal{H} := \mathrm{L}^2(\mathbb{R}^d), \tag{1.1}$$

where the parameter ω runs in the probability space $(\Omega, \mathbb{P})$, and for $\mathbb{P}$-a.e. ω we assign a magnetic potential A_ω and an electric potential V_ω. The precise requirements are described in Assumption 4.1 of Section 4. Briefly, A_ω and V_ω belong to a very wide class of potentials which ensures that H_ω is essentially self-adjoint on $\mathcal{C}_c^\infty(\mathbb{R}^d)$ and uniformly bounded from below for $\mathbb{P}$-a.e. ω. In particular *no smoothness* assumption is required on V_ω. The probability space $(\Omega, \mathbb{P})$ is equipped with an ergodic group $\{\tau(a);\ a \in \mathbb{Z}^d\}$ of measure preserving transformations. The crucial property of the ergodic system is that it satisfies a covariance relation: there exists a unitary projective representation $U(a)$ of $\mathbb{Z}^d$ on $\mathrm{L}^2(\mathbb{R}^d)$, such that for all $a, b \in \mathbb{Z}^d$ and $\mathbb{P}$-a.e. ω we have

$$U(a) H_\omega U(a)^* \;=\; H_{\tau(a)\omega}\,, \tag{1.2}$$

$$U(a) \chi_b U(a)^* \;=\; \chi_{b+a}\,, \tag{1.3}$$

where χ_a denotes the multiplication operator by the characteristic function of a unit cube centered at a. Operators that satisfy the covariance relation (1.2) will be called *covariant operators*. If $\mathbf{A}_\omega = \mathbf{A}$ is the vector potential of a constant magnetic field, the operators $U(a)$ are the magnetic translations. Note that the ergodic magnetic Schrödinger operator may be random, quasi-periodic, or even periodic.

At time $t = -\infty$, which we take as reference, the system is in equilibrium in the state given by a one-particle density matrix $\zeta_\omega = f(H_\omega)$ where f is a non-negative function with fast enough decay at infinity. At zero temperature, we have $\zeta_\omega = P_\omega^{(E_F)} = \chi_{(-\infty, E_F]}(H_\omega)$, the Fermi projection. It is convenient to give the technical statement of the condition on ζ_ω in the language of the normed spaces developed in Section 3. Hence we postpone it to Section 5 where it is stated as Assumption 5.1. We note here, however, that the *key* point in that assumption is that

$$\mathbb{E}\left\{\|x_k\, \zeta_\omega \chi_0\|_2^2\right\} < \infty\,, \text{ or equivalently } \mathbb{E}\left\{\|[x_k,\, \zeta_\omega]\chi_0\|_2^2\right\} < \infty\,, \tag{1.4}$$

for $k = 1, \cdots, d$, where $\|S\|_2$ denotes the Hilbert-Schmidt norm of the operator S. (This is essentially the condition identified in [BES].)

Of course, if $\zeta_\omega = P_\omega^{(E_F)}$ where E_F falls inside a gap of the spectrum of H_ω, or $\zeta_\omega = f(H_\omega)$ with f smooth and appropriately decaying at high energies, then (1.4) is readily fulfilled by general arguments (e.g. [GK2]). The main challenge is to allow for the Fermi energy E_F to be inside a region of localization, as described for random operators in [AG, GK1, GK3, AENSS]. Note that the existence of these regions of localization has been proven for random Landau Hamiltonians with Anderson-type potentials [CH, W, GK4], and that assumption (1.4) holds in these regions of localization [GK1, BoGK].

Under this assumption, as expected, the current is proved to be zero at equilibrium (Lemma 5.7):

$$\mathcal{T}\left\{\mathbf{v}_{j,\omega}\zeta_\omega\right\} = 0, \quad j = 1, 2, \ldots, d\,, \tag{1.5}$$

where the velocity operator $\mathbf{v}_{j,\omega}$ is the self-adjoint closure of $i[H_\omega, x_j]$, initially defined on $C_c^\infty(\mathbb{R}^d)$. Here $\mathcal{T}$ denotes the trace per unit volume, and reads, for suitable covariant operators Y_ω (applying the Birkhoff ergodic theorem),

$$\mathcal{T}(Y_\omega) := \mathbb{E}\left\{\operatorname{tr}\left\{\chi_0 Y_\omega \chi_0\right\}\right\} = \lim_{L\to\infty} \tfrac{1}{|\Lambda_L|}\operatorname{tr}\left\{\chi_{\Lambda_L} Y_\omega \chi_{\Lambda_L}\right\} \quad \text{for } \mathbb{P}\text{-a.e. } \omega\,, \tag{1.6}$$

where Λ_L denotes the cube of side L centered at 0.

We then slowly, from time $t=-\infty$ to time $t=0$, switch on a spatially homogeneous electric field $\mathbf{E}$; i.e., we take (with $t_- = \min\{t,0\}$, $t_+ = \max\{t,0\}$)

$$\mathbf{E}(t) = \mathrm{e}^{\eta t_-}\mathbf{E}\,. \tag{1.7}$$

In the appropriate gauge, the dynamics are now generated by an ergodic time-dependent Hamiltonian,

$$H_\omega(t) = (-i\nabla - \mathbf{A}_\omega - \mathbf{F}(t))^2 + V_\omega(x) = G(t)H_\omega G(t)^*\,, \tag{1.8}$$

where

$$\mathbf{F}(t) = \int_{-\infty}^{t} \mathbf{E}(s)\mathrm{d}s = \left(\tfrac{\mathrm{e}^{\eta t_-}}{\eta} + t_+\right)\mathbf{E}\,, \tag{1.9}$$

and $G(t) = \mathrm{e}^{i\mathbf{F}(t)\cdot x}$ is a gauge transformation on $\mathrm{L}^2(\mathbb{R}^d)$. (Note that, if ψ_t is a solution of $i\partial_t\psi_t = H_\omega(t)\psi(t)$ then, at least formally,

$$i\partial_t G^*(t)\psi_t = (H_\omega + \mathbf{E}(t)\cdot x)G^*(t)\psi_t\ ,$$

which represents $\mathbf{E}(t)$ in a more familiar way via a time dependent scalar potential. This fact is made precise for weak solutions. See Subsection 2.2.)

It turns out that for all t the operators $H_\omega(t)$ are self-adjoint with the common domain $\mathcal{D} = \mathcal{D}(H_\omega)$, and $H_\omega(t)$ is bounded from below uniformly in t. Thanks to these facts, a general theory [Y, Theorem XIV.3.1] of time evolution for time-dependent operators applies: there is a unique unitary propagator $U_\omega(t,s)$, i.e., a unique two-parameters family $U_\omega(t,s)$ of unitary operators, jointly strongly continuous in t and s, and such that $U_\omega(t,r)U_\omega(r,s) = U_\omega(t,r)$, $U_\omega(r,r) = I$, $U_\omega(t,s)\mathcal{D} = \mathcal{D}$, and $i\partial_t U_\omega(t,s)\psi = H(t)U_\omega(t,s)\psi$ for all $\psi\in\mathcal{D}$.

A crucial advantage of our choice of gauge is that $H_\omega(t)$ is a covariant operator for all t, which ensures that the unitary propagator $U_\omega(t,s)$ is also covariant. This is of great importance in calculating the linear response *outside* the trace per unit volume, taking advantage of the centrality of this trace, a key feature of our derivation.

To compute the time evolution of the density matrix $\varrho_\omega(t)$, we shall have to set up and solve the Liouville equation which formally reads

$$\begin{cases} i\partial_t\varrho_\omega(t) = [H_\omega(t),\varrho_\omega(t)] \\ \lim_{t\to-\infty}\varrho_\omega(t) = \zeta_\omega \end{cases}, \tag{1.10}$$

where ζ_ω is the initial density matrix at $t=-\infty$. (Thus $\zeta_\omega = P_\omega^{(E_F)}$ at zero temperature.) We shall also give a meaning to the net current per unit volume (area) in the j-th direction, $j=1,\cdots,d$, induced by the electric field, formally given by

$$J_j(\eta,\mathbf{E};\zeta_\omega) = \mathcal{T}(\mathbf{v}_{j,\omega}(0)\varrho_\omega(0)) - \mathcal{T}(\mathbf{v}_{j,\omega}\zeta_\omega) = \mathcal{T}(\mathbf{v}_{j,\omega}(0)\varrho_\omega(0)), \tag{1.11}$$

with $\mathbf{v}_{j,\omega}(0)$, the self adjoint closure of $i[H_\omega(0),x_j]$ defined on $C_c^\infty(\mathbb{R}^d)$, being the velocity operator in the j-th direction at time t. Note that $\mathbf{v}_{j,\omega}(0) = G(0)\mathbf{v}_{j,\omega}G(0)^* = \mathbf{v}_{j,\omega} - 2\mathbf{F}_j(0)$.

We remark that there is an alternative approach [ES, AES] to a derivation of the Kubo-Středa formula for the quantum Hall current in a two dimensional sample, based on the calculation of a conductance rather than a conductivity. Conductance is the linear response coefficient relating a current to the electric potential difference, whereas conductivity relates a current density to the electric field strength. In [ES, AES] the affect of a finite potential drop is analyzed by considering the affect of adding to the Hamiltonian a term $g(t)\Lambda_1$ with $g(t)$ a time dependent scalar coupling and $\Lambda_1(x) = \Lambda_1(x_1) \to \pm 1$ as $x_1 \to \pm\infty$ a smooth switch function. This term models the effect of a modulated (in time) potential difference between the left and right edges of a physical sample, with the edges formally considered to be located at $x_1 = \pm\infty$. With $g(t)$ of the form $g(t) = \phi(t/\tau)$ with ϕ a fixed function, an expression for the net current across the line $x_2 = 0$ has been derived, which in the adiabatic ($\tau \to \infty$) limit gives the corresponding Kubo-Středa formula for continuum operators with a gap condition [ES] and for discrete operators with a localization assumption [AES].

Let us now briefly describe the normed spaces of measurable covariant operators we construct to carry out this derivation – see Section 3 for their full description. We let $\mathcal{H}_{\mathrm{c}}$ denote the subspace of functions with compact support, and set $\mathcal{L} = \mathcal{L}(\mathcal{H}_{\mathrm{c}}, \mathcal{H})$ to be the vector space of linear operators on $\mathcal{H}$ with domain $\mathcal{H}_{\mathrm{c}}$ (not necessarily bounded). We introduce the vector space $\mathcal{K}_{mc}$ of measurable covariant maps $Y_\omega \colon \Omega \to \mathcal{L}$; where we identify maps that agree $\mathbb{P}$-a.e. We consider the C^*-algebra

$$\mathcal{K}_\infty = \{Y_\omega \in \mathcal{K}_{mc};\ \| Y_\omega \|_\infty < \infty\}\,,\ \text{where}\ \| Y_\omega \|_\infty = \|\, \|Y_\omega\|\, \|_{\mathrm{L}^\infty(\Omega,\mathbb{P})}\,. \tag{1.12}$$

Bounded functions of $H_\omega(t)$ as well as the unitary operators $U_\omega(t,s)$ belong to this algebra.

However, since we must deal with unbounded operators (think of $[x_k, P_\omega^{(E_F)}]$ with E_F in a region of localization), we must look outside $\mathcal{K}_\infty$ and consider subspaces of $\mathcal{K}_{mc}$ which include unbounded operators. We introduce norms on $\mathcal{K}_\infty$ given by

$$\| Y_\omega \|_1 = \mathbb{E}\,\mathrm{tr}\{\chi_0 |Y_\omega| \chi_0\}, \quad \| Y_\omega \|_2 = \left\{\mathbb{E}\, \|Y_\omega \chi_0\|_2^2\right\}^{\frac{1}{2}}\,, \tag{1.13}$$

and consider the normed spaces

$$\mathcal{K}_i^{(0)} = \{Y_\omega \in \mathcal{K}_\infty,\ \| Y_\omega \|_i < \infty\}\,,\ \ i = 1,2\,. \tag{1.14}$$

We denote the (abstract) completion of $\mathcal{K}_i^{(0)}$ in the norm $\|\cdot\|_i$ by $\overline{\mathcal{K}_i}$, $i = 1,2$. In principle, elements of the completion $\overline{\mathcal{K}_i}$ may not be identifiable with elements of $\mathcal{K}_{mc}$: they may not be *covariant operators* defined on the domain $\mathcal{H}_{\mathrm{c}}$. Since it is important for our analysis that we work with operators, we set $\mathcal{K}_i = \mathcal{K}_{mc} \cap \overline{\mathcal{K}_i}$. That is,

$$\mathcal{K}_i = \{Y_\omega \in \mathcal{K}_{mc},\ \| Y_\omega \|_i < \infty\}\,. \tag{1.15}$$

(We are glossing over the technical, but important, detail of defining the norms $\| Y_\omega \|_i$ on $\mathcal{K}_{mc}$. In fact, we shall do this only for *locally bounded* operators Y_ω – see Definition 3.1(iii) – for which the absolute value $|Y_\omega|$ may be defined.)

It turns out that $\mathcal{K}_2 = \overline{\mathcal{K}_2}$ (Proposition 3.7), and the resulting set is a Hilbert space with inner product $\langle\langle Y_\omega, Z_\omega \rangle\rangle = \mathbb{E}\,\mathrm{tr}\{(Y_\omega\chi_0)^*(Z_\omega\chi_0)\}$. However, $\mathcal{K}_1 \neq \overline{\mathcal{K}_1}$ (Proposition 3.13), and the dense subspace $\mathcal{K}_1$ is not complete. Nonetheless, it represents a natural space of unbounded covariant operators on which the trace per unit volume (1.6) is well defined. The trace per unit volume $\mathcal{T}$ is naturally defined on $\mathcal{K}_1$, where it is bounded by the $\mathcal{K}_1$ norm, and hence it extends to a continuous linear functional on $\overline{\mathcal{K}_1}$; but (1.6) is only formal for $Y_\omega \in \overline{\mathcal{K}_1} \setminus \mathcal{K}_1$.

There is a natural norm preserving conjugation on the spaces $\mathcal{K}_i$, given by ${Y_\omega}^\ddagger = ({Y_\omega}^*)_{|\mathcal{H}_c}$, which extends to a conjugation on $\overline{\mathcal{K}_1}$. Moreover, the spaces $\mathcal{K}_i$, $i = 1, 2$, are left and right $\mathcal{K}_\infty$-modules, with left and right multiplication being explicitly defined for $B_\omega \in \mathcal{K}_\infty$ and $Y_\omega \in \mathcal{K}_2$ or $\mathcal{K}_1$ by

$$B_\omega \odot_L Y_\omega = B_\omega Y_\omega\,, \quad Y_\omega \odot_R B_\omega = (B_\omega^* \odot_L {Y_\omega}^\ddagger)^\ddagger = {Y_\omega}^{\ddagger *} B_\omega\,. \tag{1.16}$$

(It is not obvious that the latter equality makes sense!) The properties of left and right multiplication, as well as the fact that they commute, can be read immediately from (1.16). There is also a bilinear map $\Diamond : \mathcal{K}_2 \times \mathcal{K}_2 \to \overline{\mathcal{K}_1}$ with dense range, written $\Diamond(Y_\omega, Z_\omega) = Y_\omega \diamond Z_\omega$, such that $\mathcal{T}(Y_\omega \diamond Z_\omega) = \langle\langle {Y_\omega}^\ddagger, Z_\omega \rangle\rangle$.

Another crucial ingredient is the centrality of the trace per unit volume: if either $Y_\omega, Z_\omega \in \mathcal{K}_2$ or $Y_\omega \in \overline{\mathcal{K}_1}$ and $Z_\omega \in \mathcal{K}_\infty$, we have either

$$\mathcal{T}(Y_\omega \diamond Z_\omega) = \mathcal{T}(Z_\omega \diamond Y_\omega) \quad \text{or} \quad \mathcal{T}(Y_\omega \odot_R Z_\omega) = \mathcal{T}(Z_\omega \odot_L Y_\omega)\,. \tag{1.17}$$

There is a connection with noncommutative integration: $\mathcal{K}_\infty$ is a von Neumann algebra, $\mathcal{T}$ is a faithful normal semifinite trace on $\mathcal{K}_\infty$, $\overline{\mathcal{K}_i} = \mathrm{L}^i(\mathcal{K}_\infty, \mathcal{T})$ for $i = 1, 2$ – see Subsection 3.5. But our explicit construction plays a very important role in our analysis.

The Liouville equation (1.10) will be given a precise meaning and solved in the spaces $\mathcal{K}_1$ and $\mathcal{K}_2$. Note that the assumption (1.4) is equivalent to $[x_j, \zeta_\omega] \in \mathcal{K}_2$ for all $j = 1, 2, \ldots, d$. (We will also have $[x_j, \zeta_\omega] \in \mathcal{K}_1$ for all $j = 1, 2, \ldots, d$. See Remark (i) following Assumption 5.1, and Proposition 4.2.)

If $Y_\omega \in \mathcal{K}_i$, $i = 1, 2, \infty$, is such that $\operatorname{Ran} Y_\omega \subset \mathcal{D} = \mathcal{D}(H_\omega(t))$ and $H_\omega(t) Y_\omega \in \mathcal{K}_i$, and similarly for ${Y_\omega}^\ddagger$, we set

$$[H_\omega(t), Y_\omega]_\ddagger = H_\omega(t) Y_\omega - (H_\omega(t) {Y_\omega}^\ddagger)^\ddagger \in \mathcal{K}_i\,.$$

Our first main result is

Theorem 1.1. *Under Assumptions 4.1 and 5.1, the Liouville equation*

$$\begin{cases} i\partial_t \varrho_\omega(t) = [H_\omega(t), \varrho_\omega(t)]_\ddagger \\ \lim_{t\to-\infty} \varrho_\omega(t) = \zeta_\omega \end{cases} \tag{1.18}$$

has a solution in $\mathcal{K}_1 \cap \mathcal{K}_2$, unique in both $\mathcal{K}_1$ and $\mathcal{K}_2$, given by

$$\varrho_\omega(t) = \lim_{s\to-\infty} \mathcal{U}(t,s)\,(\zeta_\omega) = \lim_{s\to-\infty} \mathcal{U}(t,s)\,(\zeta_\omega(s)) \tag{1.19}$$

$$= \zeta_\omega(t) - i\int_{-\infty}^t \mathrm{d}r\, \mathrm{e}^{\eta r_-} \mathcal{U}(t,r)\,([\mathbf{E}\cdot\mathbf{x}, \zeta_\omega(r)])\,, \tag{1.20}$$

where

$$\mathcal{U}(t,s)(Y_\omega) = U_\omega(t,s) \odot_L Y_\omega \odot_R U_\omega(s,t) \ \text{ for } Y_\omega \in \mathcal{K}_i,\ i = 1, 2\,, \tag{1.21}$$

$$\zeta_\omega(t) = G(t)\zeta_\omega G(t)^* = f(H_\omega(t)) \qquad (\zeta_\omega = f(H_\omega))\,. \tag{1.22}$$

We also have

$$\varrho_\omega(t) = \mathcal{U}(t,s)(\varrho_\omega(s))\,, \ \ |||\varrho_\omega(t)|||_i = |||\zeta_\omega|||_i\,, \tag{1.23}$$

for all t, s and $i = 1, 2, \infty$. Furthermore, $\varrho_\omega(t)$ is non-negative and if $\zeta_\omega = P_\omega^{E_F}$ then $\varrho_\omega(t)$ is an orthogonal projection for all t.

We actually prove a generalization of Theorem 1.1, namely Theorem 5.3, in which the commutator in (1.18) is replaced by the Liouvillian (defined in Corollary 4.12), the closure of $Y_\omega \mapsto [H_\omega(t), Y_\omega]_\ddagger$ as an operator on $\mathcal{K}_i$, $i = 1, 2$. As a by-product of the theorem, we prove that $\operatorname{Ran} \varrho_\omega(t) \in \mathcal{D}$ and $\mathbf{v}_{j,\omega}(t)\varrho_\omega(t) \in \mathcal{K}_1$, and hence the current $\mathcal{T}(\mathbf{v}_{j,\omega}(t)\varrho_\omega(t))$ is well-defined for any time t. In particular, the net current per unit volume $J_j(\eta,\mathbf{E};\zeta_\omega)$ is well defined and, since $\varrho_\omega(t)$ is non-negative, is a real number.

Our next main contribution states the validity of the linear response theory, and provides a Kubo formula.

Theorem 1.2. *Let $\eta > 0$. Under Assumptions 4.1 and 5.1, the map $\mathbf{E} \to \mathbf{J}(\eta,\mathbf{E};\zeta_\omega)$ is differentiable with respect to $\mathbf{E}$ at $\mathbf{E} = 0$ and the derivative $\sigma(\eta;\zeta_\omega)$ is given by*

$$\sigma_{jk}(\eta;\zeta_\omega) = \tfrac{\partial}{\partial \mathbf{E}_k}\mathbf{J}_j(\eta,0;\zeta_\omega) = -\mathcal{T}\left\{\int_{-\infty}^0 \mathrm{d}r\, \mathrm{e}^{\eta r}\mathbf{v}_{j,\omega}\,\mathcal{U}^{(0)}(-r)\left(i[x_k,\zeta_\omega]\right)\right\}, \quad (1.24)$$

where $\mathcal{U}^{(0)}(r)(Y_\omega) = \mathrm{e}^{-irH_\omega} \odot_L Y_\omega \odot_R \mathrm{e}^{irH_\omega}$.

Note that we prove a result stronger than the existence of the partial derivatives of $\mathbf{J}(\eta,\mathbf{E};\zeta_\omega)$ at $\mathbf{E} = 0$: we prove differentiability at $\mathbf{E} = 0$.

Next, taking the limit $\eta \to 0$, we recover the expected form for the quantum Hall conductivity at zero temperature, the Kubo-Středa formula [St, ThKNN, B, NB, BES, AG, Na].

Theorem 1.3. *Under Assumptions 4.1 and 5.1, if $\zeta_\omega = P_\omega^{(E_F)}$, an orthogonal projection, then for all $j,k = 1,2,\ldots,d$, we have*

$$\sigma_{j,k}^{(E_F)} := \lim_{\eta\to 0}\sigma_{jk}(\eta;P_\omega^{(E_F)}) = -i\mathcal{T}\left\{P_\omega^{(E_F)}\odot_L\left[\left[x_j,P_\omega^{(E_F)}\right],\left[x_k,P_\omega^{(E_F)}\right]\right]_\diamond\right\}, (1.25)$$

where $[Z_\omega,Y_\omega]_\diamond = Z_\omega\diamond Y_\omega - Y_\omega\diamond Z_\omega \in \overline{\mathcal{K}_1}$ if $Z_\omega,Y_\omega\in\mathcal{K}_2$. As a consequence, the conductivity tensor is antisymmetric; in particular the direct conductivity is zero in all directions, i.e., $\sigma_{j,j}^{(E_F)} = 0$ for $j = 1,2,\ldots,d$.

If the system is time-reversible the conductivity is zero in the region of localization, as expected.

Corollary 1.4. *Under Assumptions 4.1 and 5.1, if $\mathbf{A}_\omega = 0$ (no magnetic field), we have $\sigma_{j,k}^{(E_F)} = 0$ for all $j,k = 1,2,\ldots,d$.*

We remark that under Assumptions 4.1 and 5.1 $\left[\left[x_j,P_\omega^{(E_F)}\right],\left[x_k,P_\omega^{(E_F)}\right]\right]_\diamond$ is an element of $\overline{\mathcal{K}_1}$, but may not be in $\mathcal{K}_1$. (That is, it may not be representable as a covariant operator with domain $\mathcal{H}_c$). In particular, the product $\odot_L$ in (1.25) is defined via approximation from $\mathcal{K}_1$ and may not reduce to an ordinary operator product. However, under a stronger localization assumption such as

$$\mathbb{E}\,\|\chi_x P_\omega^{(E_F)}\chi_y\|_2^2 \;\le\; C\mathrm{e}^{-|x-y|^\alpha}\,, \quad (1.26)$$

which holds throughout the regime in which (1.4) has been verified [GK1, BoGK], the products in (1.25) reduce to ordinary products of operators, and we have

$$\sigma_{j,k}^{(E_F)} = -i\mathcal{T}\left\{P_\omega^{(E_F)}\left[\left[x_j,P_\omega^{(E_F)}\right],\left[x_k,P_\omega^{(E_F)}\right]\right]\right\}. \quad (1.27)$$

2. Magnetic and time-dependent electromagnetic Schrödinger operators

In this section we review some well known facts about Schrödinger operators incorporating a magnetic vector potential $\mathbf{A}$, and present a basic existence and uniqueness result for associated propagators in the presence of a time-dependent electric field.

2.1. **Magnetic Schrödinger operators.** Let

$$H = H(\mathbf{A}, V) = (-i\nabla - \mathbf{A})^2 + V \quad \text{on} \quad \mathrm{L}^2(\mathbb{R}^d), \tag{2.1}$$

where the magnetic potential $\mathbf{A}$ and the electric potential V satisfy the Leinfelder-Simader conditions:

- $\mathbf{A}(x) \in \mathrm{L}^4_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^d)$ with $\nabla \cdot \mathbf{A}(x) \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d)$.
- $V(x) = V_+(x) - V_-(x)$ with $V_\pm(x) \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d)$, $V_\pm(x) \geq 0$, and $V_-(x)$ relatively bounded with respect to Δ with relative bound < 1, i.e., there are $0 \leq \alpha < 1$ and $\beta \geq 0$ such that

$$\|V_-\psi\| \leq \alpha \|\Delta\psi\| + \beta\|\psi\| \quad \text{for all } \psi \in \mathcal{D}(\Delta). \tag{2.2}$$

Leinfelder and Simader have shown that $H(\mathbf{A}, V)$ is essentially self-adjoint on $C^\infty_{\mathrm{c}}(\mathbb{R}^d)$ [LS, Theorem 3] (see also [CyFKS, Theorem 1.15], [Si2, Theorem B.13.4]), with

$$H\psi = -\Delta\psi + 2i\mathbf{A} \cdot \nabla\psi + \left(i\nabla \cdot \mathbf{A} + \mathbf{A}^2 + V\right)\psi \;\text{ for } \psi \in C^\infty_{\mathrm{c}}(\mathbb{R}^d). \tag{2.3}$$

Note that (2.2) implies that for all $\alpha' > \alpha$ we have [RS2, Proof of Theorem X.18]

$$0 \leq \langle \psi, V_-\psi\rangle \leq \alpha' \langle \psi, -\Delta\psi\rangle + \tfrac{\alpha'}{\alpha'-\alpha}\beta\|\psi\|^2 \,. \tag{2.4}$$

A similar bound holds for $H(\mathbf{A}, V_+)$ [LS, Eq. (4.11)]: for all $\alpha' > \alpha$ we have

$$\|V_-\psi\| \leq \alpha' \|H(\mathbf{A}, V_+)\psi\| + \tfrac{\alpha'}{\alpha'-\alpha}\beta\|\psi\| \quad \text{for all } \psi \in \mathcal{D}(H(\mathbf{A}, V_+))\,, \tag{2.5}$$

from which we immediately get the lower bound [K, Theorem V.4.11][RS2, Theorem X.12]

$$H(\mathbf{A}, V) \geq -\min_{\alpha' \in (\alpha,1)} \frac{\alpha'\beta}{(\alpha'-\alpha)(1-\alpha')} = -\frac{\beta}{(1-\sqrt{\alpha})^2}\,. \tag{2.6}$$

But we can get a better lower bound. We have the a.e. pointwise inequality [LS, Proof of Lemma 2] [BeG]

$$|\nabla(|\psi|)| \leq |(-i\nabla - \mathbf{A})\,\psi| \quad \text{for all } \psi \in C^\infty_{\mathrm{c}}(\mathbb{R}^d)\,. \tag{2.7}$$

Thus it follows for all $\alpha' > \alpha$ that we have (using (2.4))

$$\begin{aligned} \langle \psi, V_-\psi\rangle &\leq \langle |\psi|, V_-|\psi|\rangle \leq \alpha' \langle |\psi|, -\Delta|\psi|\rangle + \tfrac{\alpha'}{\alpha'-\alpha}\beta||\psi||^2 \\ &= \alpha' \left\|\nabla|\psi|\right\|^2 + \tfrac{\alpha'}{\alpha'-\alpha}\beta||\psi||^2 \leq \alpha' \left\|(-i\nabla - \mathbf{A})\,\psi\right\|^2 + \tfrac{\alpha'}{\alpha'-\alpha}\beta||\psi||^2 \\ &\leq \alpha' \langle \psi, H(\mathbf{A}, V_+)\psi\rangle + \tfrac{\alpha'}{\alpha'-\alpha}\beta||\psi||^2 \end{aligned} \tag{2.8}$$

for all $\psi \in C^\infty_{\mathrm{c}}(\mathbb{R}^d)$. We conclude that

$$H(\mathbf{A}, V) \geq -\min_{\alpha' \in (\alpha,1)} \frac{\alpha'\beta}{(\alpha'-\alpha)} = -\frac{\beta}{(1-\alpha)}\,. \tag{2.9}$$

For convenience we write

$$\gamma = \gamma(\alpha, \beta) := \frac{\beta}{1-\alpha} + 1\,, \tag{2.10}$$

and note that

$$H + \gamma \geq 1\,. \tag{2.11}$$

We also have the diamagnetic inequality

$$\left|\mathrm{e}^{-tH(\mathbf{A},V)}\psi\right| \leq \mathrm{e}^{-tH(0,V)}|\psi| \tag{2.12}$$

for all $\psi \in \mathrm{L}^2(\mathbb{R}^d)$ and $t > 0$, see [CyFKS, Proof of Theorem 1.13]. Note that the diamagnetic inequality and (2.9) imply (using $\int_0^\infty t^q \mathrm{e}^{-t(x+\lambda)} \mathrm{d}t = \Gamma(q)(x+\lambda)^{-q}$)

$$\left| (H(\mathbf{A},V)+\lambda)^{-q}\, \psi \right| \le (H(0,V)+\lambda)^{-q}\, |\psi| \tag{2.13}$$

for all $\psi \in \mathrm{L}^2(\mathbb{R}^d)$, $\lambda > \frac{\beta}{(1-\alpha)}$, and $q > 0$.

An important consequence of (2.13) is that the usual trace estimates for $-\Delta + V$ are valid for the magnetic Schrödinger operator $H(\mathbf{A},V)$, with bounds independent of $\mathbf{A}$ and depending on V only through α and β. We state them as in [GK3, Lemma A.4]. (We do not need the Leinfelder-Simader conditions here, just the conditions for the diamagnetic inequality: $\mathbf{A}(x) \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$, $V_+(x) \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$, and $V_-(x)$ relatively form bounded with respect to Δ with relative bound < 1. See [CyFKS, Theorem 1.13] where this is shown for $V_- = 0$. The general case, with V_- relatively bounded as above, is proved by an approximation argument, see [F, Theorems 7.7, 7.9].)

Proposition 2.1. *Let $\nu > \frac{d}{4}$. There is a finite constant $\mathcal{T}_{\nu,d,\alpha,\beta}$, depending only on the indicated constants, such that*

$$\operatorname{tr}\left\{ \langle x \rangle^{-2\nu} \left(H(\mathbf{A},V)+\gamma\right)^{-2[[\frac{d}{4}]]} \langle x \rangle^{-2\nu} \right\} \le \mathcal{T}_{\nu,d,\alpha,\beta}\,, \tag{2.14}$$

where $[[\frac{d}{4}]]$ is the smallest integer bigger than $\frac{d}{4}$ and γ is the constant defined in (2.10)*. Thus, letting*

$$\Phi_{d,\alpha,\beta}(E) = \chi_{\left[-\frac{\beta}{1-\alpha},\infty\right)}(E)\,(E+\gamma)^{2[[\frac{d}{4}]]}\,, \tag{2.15}$$

we have

$$\operatorname{tr}\left(\langle x \rangle^{-2\nu} f(H) \langle x \rangle^{-2\nu} \right) \le \mathcal{T}_{\nu,d,\alpha,\beta} \| f \Phi_{d,\alpha,\beta} \|_\infty < \infty \tag{2.16}$$

for every Borel measurable function $f \ge 0$ on the real line.

Proof. The proposition follows once the estimate (2.13) is converted into an estimate on traces, because then the well known trace estimates for $-\Delta + V$, e.g., [GK3, Lemma A.4], finish the argument. Hence (2.14) follows from the following lemma, with

$$\begin{aligned} A &= \langle x \rangle^{-2\nu} \left(H(\mathbf{A},V)+\gamma\right)^{-2[[\frac{d}{4}]]} \langle x \rangle^{-2\nu}\,, \\ B &= \langle x \rangle^{-2\nu} \left(H(0,V)+\gamma\right)^{-2[[\frac{d}{4}]]} \langle x \rangle^{-2\nu}\,, \end{aligned} \tag{2.17}$$

using the fact that the operator $(H(0,V)+\gamma)^{-2[[\frac{d}{4}]]}$ is positivity preserving. ∎

Lemma 2.2. *Let A and B be bounded positive operators on $\mathrm{L}^2(\mathbb{R}^d)$, with B a positivity preserving operator, such that*

$$\langle \psi, A\psi \rangle \le \langle |\psi|, B|\psi| \rangle \quad \text{for all } \psi \in \mathrm{L}^2(\mathbb{R}^d). \tag{2.18}$$

Then $\operatorname{tr} A \le \operatorname{tr} B$.

Proof. First note that the lemma is obvious if we replace $\mathrm{L}^2(\mathbb{R}^d)$ by $\ell^2(\mathbb{Z}^d)$, since in this case we have a basis of positive functions ($|\delta_x| = \delta_x$). Note also that we may assume $\operatorname{tr} B < \infty$ without loss of generality.

For $\mathrm{L}^2(\mathbb{R}^d)$, let $\mathcal{H}_n$ be the sub-Hilbert space with ortho-normal basis

$$\{\tilde{\chi}_{n,x} = 2^{\frac{nd}{2}} \chi_{\Lambda_{2^{-n}}(2^{-n}x)};\ x \in \mathbb{Z}^d\},$$

where $\Lambda_L(x)$ denotes the cube centered at x and of length L; and let P_n be the orthogonal projection onto $\mathcal{H}_n$. Note that P_n is positivity preserving. Set

$$A_n = P_n A P_n \quad \text{and} \quad B_n = P_n B P_n \tag{2.19}$$

It follows from (2.18) and the fact that both B and P_n are positivity preserving that

$$\langle \psi, A_n \psi\rangle \le \langle |P_n\psi|, B|P_n\psi|\rangle \le \langle |\psi|, B_n|\psi|\rangle \quad \text{for all } \psi \in \mathcal{H}_n. \tag{2.20}$$

Since $\mathcal{H}_n$ has a basis of positive functions, we get

$$\operatorname{tr} A_n \le \operatorname{tr} B_n \le \operatorname{tr} B\,. \tag{2.21}$$

Thus $\sqrt{A}P_n$ is Hilbert-Schmidt, and it follows that

$$\operatorname{tr}\sqrt{A}P_n\sqrt{A} \le \operatorname{tr} B\,. \tag{2.22}$$

Since $P_n \to I$ strongly, we conclude that $\operatorname{tr} A \le \operatorname{tr} B$. □

The velocity operator $\mathbf{v} = i[H, \mathbf{x}]$, where $\mathbf{x}$ is the operator from $\mathrm{L}^2(\mathbb{R}^d)$ to $\mathrm{L}^2(\mathbb{R}^d;\mathbb{C}^d)$ of multiplication by the coordinate vector x, plays an important role in the linear response theory. To give precise meaning to $\mathbf{v}$, we note that on $C^\infty_{\mathrm{c}}(\mathbb{R}^d)$ we have

$$i[H, \mathbf{x}] = 2(-i\nabla - \mathbf{A})\,. \tag{2.23}$$

We let $\mathbf{D} = \mathbf{D}(\mathbf{A})$ be the closure of $(-i\nabla - \mathbf{A})$ as an operator from $\mathrm{L}^2(\mathbb{R}^d)$ to $\mathrm{L}^2(\mathbb{R}^d;\mathbb{C}^d)$ with domain $C^\infty_{\mathrm{c}}(\mathbb{R}^d)$. Each of its components $\mathbf{D}_j = \mathbf{D}_j(\mathbf{A}) = (-i\frac{\partial}{\partial x_j} - \mathbf{A}_j)$, $j = 1, \ldots, d$, is essentially self-adjoint on $C^\infty_{\mathrm{c}}(\mathbb{R}^d)$ since $\mathbf{A}(x) \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$ (see [Si1, Lemma 2.5]). We define

$$\mathbf{v} = \mathbf{v}(\mathbf{A}) = 2\mathbf{D}(\mathbf{A})\ . \tag{2.24}$$

Proposition 2.3. *We have*

(i): $\mathcal{D}(\sqrt{H+\gamma}) \subset \mathcal{D}(\mathbf{D})$. *In fact there exists* $C_{\alpha,\beta} < \infty$ *such that*

$$\left\| \mathbf{D}\,(H+\gamma)^{-\frac{1}{2}} \right\| \ \le\ C_{\alpha,\beta}. \tag{2.25}$$

(ii): *For all* $\chi \in C^\infty_{\mathrm{c}}(\mathbb{R}^d)$ *we have* $\chi\mathcal{D}(H) \subset \mathcal{D}(H)$ *and*

$$H\chi\psi = \chi H\psi - (\Delta\chi)\psi - 2i(\nabla\chi)\cdot\mathbf{D}\psi \quad \text{for all } \psi \in \mathcal{D}(H). \tag{2.26}$$

(iii): *Let*

$$\widetilde{\Phi}_{d,\alpha,\beta}(E) \ :=\ (E+\gamma)^{\frac{1}{2}}\Phi_{d,\alpha,\beta}(E) \ =\ \chi_{[-\frac{\beta}{1-\alpha},\infty)}(E)\,(E+\gamma)^{2[[\frac{d}{4}]]+\frac{1}{2}}\ . \tag{2.27}$$

If f *is Borel measurable function on the real line with* $\|f\widetilde{\Phi}_{d,\alpha,\beta}\|_\infty < \infty$, *the bounded operator* $|\mathbf{D}f(H)| = \left\{\overline{f}(H)\mathbf{D}^*\mathbf{D}f(H_\omega)\right\}^{\frac{1}{2}}$ *satisfies*

$$\operatorname{tr}\left\{\langle x\rangle^{-2\nu}\,|\mathbf{D}f(H)|\,\langle x\rangle^{-2\nu}\right\} \ \le\ \widetilde{\mathcal{T}}_{\nu,d,\alpha,\beta}, \tag{2.28}$$

where $\widetilde{\mathcal{T}}_{\nu,d,\alpha,\beta} < \infty$ *for* $\nu > d/4$ *and depends only on the indicated constants.*

Proof. To prove (i), note that $\mathbf{D}^*\mathbf{D} = (-i\nabla - \mathbf{A})^2$ and by (2.8)

$$\delta\alpha'\mathbf{D}^*\mathbf{D} \ \le\ (1+\delta)\alpha'(-i\nabla - \mathbf{A})^2 - V_- + \frac{\alpha'}{\alpha-\alpha'}\beta \ \le\ H + \frac{\alpha'}{\alpha-\alpha'}\beta \tag{2.29}$$

for $\alpha' \in (\alpha, 1)$ and δ such that $(1+\delta)\alpha' < 1$. Choosing α' and δ such that

$$\frac{\alpha'}{\alpha - \alpha'}\beta = \gamma \quad \text{and} \quad (1+\delta)\alpha' = 1\ , \tag{2.30}$$

we have

$$(1-\alpha')\mathbf{D}^*\mathbf{D} \leq \ H + \gamma \tag{2.31}$$

as quadratic forms. Since $\alpha' = \alpha'(\alpha, \beta)$ is strictly less than one, it follows that $\mathcal{D}(\mathbf{D}) \subset \mathcal{D}(\sqrt{H+\gamma})$ and furthermore

$$(H+\gamma)^{-\frac{1}{2}}\, \mathbf{D}^*\mathbf{D}\, (H+\gamma)^{-\frac{1}{2}} \ \leq \ \frac{1}{1-\alpha'}\ , \tag{2.32}$$

which gives (2.25) with $C_{\alpha,\beta} = \sqrt{\frac{1}{1-\alpha'}}$.

Part (ii) follows from (2.25), since the identity holds for $\psi \in C_c^\infty$ by (2.3). Part (iii) is a result of combining Proposition 2.1, and the estimate

$$|\mathbf{D} f(H)| \ \leq \ C_{\alpha,\beta}(H+\gamma)^{\frac{1}{2}}|f|(H)\ , \tag{2.33}$$

which follows from (2.31) and monotonicity of the square root. □

We shall also need to consider commutators $[x, f(H)]$ with functions of H. For smooth functions, the easiest way to do this is to use the Helffer-Sjöstrand formula [HS, D]. Specifically, we restrict our attention to functions which are finite in one of the following norms:

$$|\!|\!| f |\!|\!|_m = \sum_{r=0}^{m} \int_{\mathbb{R}} |f^{(r)}(u)| \langle u \rangle^{r-1} \mathrm{d}u \quad , \quad m = 1, 2, \dots\ . \tag{2.34}$$

If $|\!|\!| f |\!|\!|_m < \infty$ with $m \geq 2$, then we have [HS, D]

$$f(H) = \int d\tilde{f}(z)(z-H)^{-1}\ , \tag{2.35}$$

where the integral converges absolutely in operator norm:

$$\|f(H)\| \ \leq \ \int |d\tilde{f}(z)| \frac{1}{\operatorname{Im} z} \ \leq \ c\ |\!|\!| f |\!|\!|_m < \infty\ , \tag{2.36}$$

with c independent of $m \geq 2$. Here $z = x + iy$, $\tilde{f}(z)$ is an *almost analytic* extension of f to the complex plane, and $d\tilde{f}(z) = -\frac{1}{2\pi}\partial_{\bar{z}}\tilde{f}(z)\, \mathrm{d}x\, \mathrm{d}y$, with $\partial_{\bar{z}} = \partial_x + i\partial_y$. For various purpose it is useful to note that

$$\int |d\tilde{f}(z)| \frac{\langle \operatorname{Re} z \rangle^{p-1}}{|\operatorname{Im} z|^p} \ \leq \ c_p\ |\!|\!| f |\!|\!|_m < \infty\ , \tag{2.37}$$

for $m \geq p+1$. (See [HuS, Appendix B] for details. Notation: $< y >= \sqrt{1+|y|^2}$.) Note that if $f \in \mathcal{S}(\mathbb{R})$ we have $|\!|\!| f |\!|\!|_m < \infty$ for all $m = 1, 2, \dots$.

Proposition 2.4. *Let $f \in C^\infty(\mathbb{R})$ with $|\!|\!| f |\!|\!|_3 < \infty$. Then*

- **(i):** $f(H)\mathrm{L}^2_{\mathrm{c}}(\mathbb{R}^d) \subset D(H) \cap D(\mathbf{x})$.
- **(ii):** *The operator $[\mathbf{x}, f(H)]$ is well defined on $\mathrm{L}^2_{\mathrm{c}}(\mathbb{R}^d)$ and has a bounded closure: there exists a constant $C_{\alpha,\beta} < \infty$ such that*

$$\left\| \overline{[\mathbf{x}, f(H)]} \right\| \leq C_{\alpha,\beta} |\!|\!| f |\!|\!|_3\ . \tag{2.38}$$

Proof. The Combes-Thomas argument [CT] shows that $R(z)\mathcal{H}_c \subset \mathcal{D}(\mathbf{x})$, with $R(z) = (H-z)^{-1}$, whenever $\operatorname{Im} z \neq 0$. In fact, we have $R(z)\mathcal{H}_c \subset \mathcal{D}(\mathrm{e}^{\mu(z)|\mathbf{x}|})$ with the explicit estimate

$$\left\| \mathrm{e}^{\mu(z)|\mathbf{x}-y|} R(z)\chi_y \right\| \;\le\; C_{\alpha,\beta}\, \frac{1}{|\operatorname{Im} z|}\,, \qquad \text{for every unit cube } \chi_y, \tag{2.39}$$

where $\mu(z) = C_{\alpha,\beta}\, |\operatorname{Im} z|/(\langle \operatorname{Re} z\rangle + |\operatorname{Im} z|)$. (See [GK2, Theorem 1] for details in this context. We denote by the same $C_{\alpha,\beta}$ possibly different constants depending only on the parameters α and β given in (2.2).) We conclude that

$$\|\mathbf{x}R(z)\chi_y\| \;\le\; C_{\alpha,\beta,y} \frac{1}{\mu(z)|\operatorname{Im} z|} \;\le\; C_{\alpha,\beta,y} \begin{cases} \frac{\langle \operatorname{Re} z\rangle}{|\operatorname{Im} z|^2}\,, & |\operatorname{Im} z| \le \langle \operatorname{Re} z\rangle\,, \\ \frac{1}{|\operatorname{Im} z|}\,, & |\operatorname{Im} z| \ge \langle \operatorname{Re} z\rangle\,, \end{cases} \tag{2.40}$$

which gives (i) in light of (2.37).

Furthermore, we see that $[\mathbf{x}, R(z)]$ is well defined on $\mathcal{H}_c$. In particular, for $\psi \in \mathcal{H}_c \cap \mathcal{D}$ we have

$$[\mathbf{x}, R(z)]\,(H-z)\psi \;=\; \mathbf{x}\psi - R(z)\mathbf{x}(H-z)\psi\,, \tag{2.41}$$

where $(H-z)\psi \in \mathcal{H}_c$, since H is local. As ψ is compactly supported, the components of $\mathbf{x}\psi$ are in $\mathcal{D}$ by Prop. 2.3ii. Thus

$$(H-z)\,[\mathbf{x}, R(z)]\,(H-z)\psi \;=\; (H-z)\mathbf{x}\psi - \mathbf{x}(H-z)\psi \;=\; 2i\mathbf{D}(\mathbf{A})\psi\,, \tag{2.42}$$

where to obtain the last equality it is useful to consider $\psi \in C_c^\infty$ initially and pass to $\psi \in \mathcal{H}_c \cap \mathcal{D}$ by a limiting argument. Thus

$$[\mathbf{x}, R(z)]\,(H-z)\psi \;=\; 2iR(z)\mathbf{D}(\mathbf{A})R(z)(H-z)\psi\,, \tag{2.43}$$

whenever $\psi \in \mathcal{H}_c \cap \mathcal{D}$, which is a domain of essential self-adjointness for H. Thus $(H-z)\mathcal{H}_c \cap \mathcal{D}$ is dense, and we conclude that $[\mathbf{x}, R(z)]$ is a bounded operator with

$$[\mathbf{x}, R(z)] \;=\; 2iR(z)\mathbf{D}(\mathbf{A})R(z)\,. \tag{2.44}$$

Specifically we have

$$\|[\mathbf{x}, R(z)]\| \;\le\; 2\left\| R(z)\sqrt{H+\gamma}\right\| \cdot \left\| \frac{1}{\sqrt{H+\gamma}}\mathbf{D}(\mathbf{A})\right\| \cdot \|R(z)\|\,, \tag{2.45}$$

with the middle factor bounded by Proposition 2.3(iii), and the first and last factors bounded by $\sqrt{|z+\gamma|}/|\mathrm{Im} z|$ and $1/|\mathrm{Im} z|$ respectively. Plugging these bounds into the Helffer-Sjöstrand formula (2.35), and using (2.37), we find

$$\|[\mathbf{x}, f(H)]\| \;\le C_{\alpha,\beta} \int |d\tilde{f}(z)| \frac{\sqrt{|z|+\gamma}}{|\mathrm{Im} z|^2} \;\le C_{\alpha,\beta}\, \|\!| f |\!\|_3 < \infty\,. \tag{2.46}$$

□

2.2. **Time-dependent electric fields.** Consider a quantum particle in the presence of a background potential $V(x)$, a magnetic vector potential $\mathbf{A}(x)$, and a time dependent spatially uniform electric field $\mathbf{E}(t)$. We will refer to the time-dependent self-adjoint generator of the unitary evolution as the Hamiltonian.

One's initial impulse might be to add the electric potential $\mathbf{E}(t)\cdot x$ to the magnetic Schrödinger operator $H(A,V)$ and consider the Hamiltonian:

$$\widetilde{H}(t) = H(\mathbf{A}, V) + \mathbf{E}(t)\cdot x = (-i\nabla - \mathbf{A}(x))^2 + V(x) + \mathbf{E}(t)\cdot x\,. \tag{2.47}$$

However, *this choice is not dictated by the physics under consideration.* In fact, there is an infinite family of choices for the Hamiltonian, related to one another by time-dependent gauge transformations, all equally valid from the standpoint of the underlying physics.

The operators defined by (2.47) suffer from the fact that they are unbounded from below, and for general $\mathbf{A}, V$ it is not obvious if there is a unitary propagator $\widetilde{U}(t,s)$ obeying

$$\begin{cases} i\partial_t \widetilde{U}(t,s) & = & \widetilde{H}(t)\widetilde{U}(t,s) \\ \widetilde{U}(s,s) & = & I \end{cases} . \tag{2.48}$$

However, there is a physically equivalent choice of Hamiltonian:

$$H(t) = (-i\nabla - \mathbf{A} - \mathbf{F}(t))^2 + V(x) = H(\mathbf{A} + \mathbf{F}(t), V)\,, \tag{2.49}$$

with $\mathbf{F}(t) = \int_{t_0}^t \mathbf{E}(s)\mathrm{d}s$ (with perhaps $t_0 = -\infty$), for which the propagator can be shown to exist for quite general $\mathbf{A}, V$. It turns out that there is a general theory of propagators with a time dependent generator [Y, Theorem XIV.3.1] which applies to $H(t)$ but does not obviously apply to $\widetilde{H}(t)$. Note that $H = H(t_0)$.

What is the justification for taking the Hamiltonian (2.49)? In classical electrodynamics (Maxwell's equations), one expresses the electric and magnetic field $\mathbf{E}(x,t)$ and $\mathbf{B}(x,t)$ in terms of a "scalar potential" $\phi(x,t)$ and a "vector potential" $\mathbf{A}(x,t)$:

$$\begin{array}{rcl} \mathbf{E}(x,t) & = & -\partial_t \mathbf{A}(x,t) - \nabla\phi(x,t)\,, \\ \mathbf{B}(x,t) & = & \nabla \times \mathbf{A}(x,t)\,. \end{array} \tag{2.50}$$

The key observation is that $\mathbf{E}$ and $\mathbf{B}$ are not changed if $\mathbf{A}$ and ϕ are perturbed by a "gauge transformation":

$$\begin{array}{rcl} \mathbf{A}(x,t) & \mapsto & \mathbf{A}(x,t) + \nabla\alpha(x,t)\,, \\ \phi(x,t) & \mapsto & \phi(x,t) - \partial_t\alpha(x,t)\,. \end{array} \tag{2.51}$$

In particular, $\mathbf{A}$ and ϕ are not uniquely determined by the "observable" fields $\mathbf{E}$ and $\mathbf{B}$. Note that a spatially uniform electric field $\mathbf{E}(t)$ may be obtained from the time dependent vector potential $\mathbf{F}(t)$.

This non-uniqueness carries over to one particle quantum mechanics. Consider a Hamiltonian associated to an electron in the presence of the electromagnetic field described by $\mathbf{A}(x,t)$ and $\phi(x,t)$:

$$H(\mathbf{A}(x,t), \phi(x,t)) = (-i\nabla - \mathbf{A}(x,t))^2 + \phi(x,t)\,, \tag{2.52}$$

acting on $\mathrm{L}^2(\mathbb{R}^d)$ (in units with the electric charge equal to one). To implement the gauge transformation (2.51), we must also transform the wave function $\psi(x,t)$ by

$$\psi(x,t) \mapsto \mathrm{e}^{i\alpha(x,t)}\psi(x,t)\,. \tag{2.53}$$

Indeed, if $\psi(x,t)$ obeys the Schrödinger equation

$$i\partial_t\psi(x,t) = H(\mathbf{A}(x,t), \phi(x,t))\psi(x,t) \tag{2.54}$$

then it is easy to check that, *formally*,

$$\begin{aligned} i\partial_t \mathrm{e}^{i\alpha(x,t)}\psi(x,t) &= -(\partial_t\alpha(x,t))\mathrm{e}^{i\alpha(x,t)}\psi(x,t) \;+\; i\mathrm{e}^{i\alpha(x,t)}\partial_t\psi(x,t) \\ &= \left[\mathrm{e}^{i\alpha(x,t)}H(\mathbf{A}(x,t),\phi(x,t))\mathrm{e}^{-i\alpha(x,t)} - \partial_t\alpha(x,t)\right]\mathrm{e}^{i\alpha(x,t)}\psi(x,t) \\ &= H(\mathbf{A}(x,t) + \nabla\alpha(x,t), \phi(x,t) - \partial_t\alpha(x,t))\mathrm{e}^{i\alpha(x,t)}\psi(x,t)\,. \end{aligned} \tag{2.55}$$

Effectively the gauge transformation (2.53) implements a "moving frame" in $\mathrm{L}^2(\mathbb{R}^d)$, and we must transform the Hamiltonian accordingly to account for the shift in the time derivative in Schrödinger's equation.

The possibility always exists to "choose a gauge" with $\phi \equiv 0$ and work only with $\mathbf{A}$: take $\partial_t \alpha(x,t) = \phi(x,t)$, effectively replacing ϕ by zero and $\mathbf{A}$ by $\mathbf{A} + \int_{t_o}^t \nabla\phi(x,s)\mathrm{d}s$. Generally, this gauge transformation is not used in time independent quantum mechanics, since it replaces a *time-independent* scalar potential with a *time-dependent* vector potential, introducing an extra level complexity. However, our Hamiltonian is *intrinsically* time-dependent, and there is not really any greater complexity to be found working with $\mathbf{A}(x,t)$ in place of $\phi(x,t)$.

For the problem at hand, we do not want to take the extreme step of setting the scalar potential identically to zero. Instead it is convenient to fix a time independent scalar potential $\phi(x,t) = V(x)$ and a time dependent vector potential $\mathbf{A}(x,t) = \mathbf{A}(x) + \mathbf{F}(t)$ with $\mathbf{F}(t) = \int_{t_0}^t \mathbf{E}(s)\mathrm{d}s$. This leads to the Hamiltonian $H(t)$ presented in (2.49). Note that on $C_c^\infty(\mathbb{R}^d)$ we have

$$H(t) \;=\; G(t)\left[(-i\nabla - \mathbf{A})^2 + V\right] G(t)^* \;, \tag{2.56}$$

where $G(t)$ denotes the gauge transformation

$$[G(t)\psi](x) = \mathrm{e}^{i\mathbf{F}(t)\cdot x}\psi(x) \;. \tag{2.57}$$

Repeating the formal calculation leading to (2.55), we find that if $\psi(t)$ obeys Schrödinger equation

$$i\partial_t\psi(t) = H(t)\psi(t), \tag{2.58}$$

then, *formally*,

$$i\partial_t G(t)^*\psi(t) = \left[(-i\nabla - \mathbf{A})^2 + V + \mathbf{E}(t)\cdot x\right] G(t)^*\psi(t) = \widetilde{H}(t)G(t)^*\psi(t) \;, \tag{2.59}$$

although this begs the question of whether $G(t)^*\psi(t)$ is in the domain of either $\mathbf{E}(t)\cdot x$ or $\widetilde{H}(t)$.

While there is no physical reason to work with one particular gauge, it is comforting to know that the choice truly does not affect the results. One difficulty is that we do not know (in general) if strong solutions to the Schrödinger equation

$$i\partial_t\psi_t \;=\; \widetilde{H}(t)\psi_t \tag{2.60}$$

exist with $\widetilde{H}(t)$ given by (2.47). Thus we must consider weak solutions. Given a time dependent Hamiltonian $K(t)$ with $C_c^\infty(\mathbb{R}^d) \subset \mathcal{D}(K(t))$ for all $t \in \mathbb{R}$, a *weak solution* to the Schrödinger equation $i\partial\psi_t = K(t)\psi_t$ is a map $t \mapsto \psi_t \in L^2(\mathbb{R}^d)$ such that

$$i\partial_t\langle\phi,\psi_t\rangle \;=\; \langle K(t)\phi,\psi_t\rangle \quad \text{for all} \quad \phi \in C_c^\infty(\mathbb{R}^d) \;. \tag{2.61}$$

It is easy to see that the weak solutions of the Schrödinger equations (2.58) and (2.60) are related by the gauge transformation $G(t)$: ψ_t is a weak solution of (2.58) if and only if the gauge transformed $G(t)^*\psi_t$ is a weak solution of (2.60).

2.3. Time-dependent Hamiltonians and their propagators.

We assume throughout that $\mathbf{A}(x)$ and $V(x)$ satisfy the Leinfelder-Simader conditions and $\mathbf{E}(t) \in C(\mathbb{R};\mathbb{R}^d)$. (If in addition $\mathbf{E}(t) \in \mathrm{L}^1((-\infty,0];\mathbb{R}^d)$ we take $t_0 = -\infty$.)

Proposition 2.5. *$H(t)$, given in (2.49), is essentially self-adjoint on $C_c^\infty(\mathbb{R}^d)$ with*

$$\begin{aligned} H(t) &= H - 2\mathbf{F}(t)\cdot(-i\nabla - \mathbf{A}) + \mathbf{F}(t)^2 \ \text{ on } C_c^\infty(\mathbb{R}^d)\,, & (2.62)\\ &= H - 2\mathbf{F}(t)\cdot\mathbf{D}(\mathbf{A}) + \mathbf{F}(t)^2 \ \text{ on } \mathcal{D}(H)\,. & (2.63)\end{aligned}$$

Hence

$$\mathcal{D} := \mathcal{D}(H) = \mathcal{D}(H(t)) \quad \text{for all } t\in\mathbb{R}, \tag{2.64}$$

and on $\mathcal{D}$ we have that for all t and s,

$$H(t) = H(s) - 2(\mathbf{F}(t)-\mathbf{F}(s))\cdot\mathbf{D}(\mathbf{A}) + (\mathbf{F}(t)^2 - \mathbf{F}(s)^2)\,. \tag{2.65}$$

In addition, all $H(t)$ satisfy the lower bound given in (2.9):

$$H(t) \geq -\frac{\beta}{1-\alpha} \ \text{ for all } t\in\mathbb{R}. \tag{2.66}$$

Proof. Clearly $\mathbf{A}(x) + \mathbf{F}(t)$ and $V(x)$ satisfy the Leinfelder-Simader conditions with the parameters α, β independent of t, hence $H(t)$ is essentially self-adjoint on $C_c^\infty(\mathbb{R}^d)$, (2.62) follows from (2.3), and we have (2.66). The equality (2.63) follows from (2.62) and Proposition 2.3(i), and implies (2.64). □

Lemma 2.6. *Let $G(t)$ be as in (2.57). Then*

$$\begin{aligned} G(t)\mathcal{D} &= \mathcal{D}\,, & (2.67)\\ H(t) &= G(t)HG(t)^*\,, & (2.68)\\ \mathbf{D}(\mathbf{A}+\mathbf{F}(t)) &= \mathbf{D}(\mathbf{A}) - \mathbf{F}(t) = G(t)\mathbf{D}(\mathbf{A})G(t)^*\,. & (2.69)\end{aligned}$$

Moreover, $i[x_j, H(t)] = 2\mathbf{D}(\mathbf{A}+\mathbf{F}(t))$ as quadratic forms on $\mathcal{D}\cap\mathcal{D}(x_j)$, $j = 1,2,\ldots,d$.

Proof. The lemma follows from (2.56) and Propositions 2.5 and 2.3. □

We now discuss the existence of a propagator $U(t,s)$ satisfying

$$i\partial_t U(t,s) = H(t)U(t,s), \quad U(s,s) = I. \tag{2.70}$$

We note that

$$H(t) + \gamma \geq 1 \ \text{ for all } t\in\mathbb{R}, \tag{2.71}$$

where γ is given in (2.10). We also set

$$\begin{aligned} C(t,s) &= (H(t)-H(s))\,(H(s)+\gamma)^{-1} & (2.72)\\ &= (\mathbf{F}(t)-\mathbf{F}(s))\cdot\{-2\mathbf{D}(\mathbf{A}) + (\mathbf{F}(t)+\mathbf{F}(s))\}\,(H(s)+\gamma)^{-1}\,. \end{aligned}$$

By Proposition 2.3(i), we have

$$\left\|\mathbf{D}(\mathbf{A})\,(H(s)+\gamma)^{-1}\right\| \leq \left\|\mathbf{D}(\mathbf{A})\,(H+\gamma)^{-1}\right\| + |\mathbf{F}(s)| \leq C_{\alpha,\beta} + |\mathbf{F}(s)|\,, \tag{2.73}$$

with $C_{\alpha,\beta}$ a finite constant. Since $F(t)\in C^1(\mathbb{R};\mathbb{R}^d)$, we conclude that both $C(t,s)$ and $\frac{1}{t-s}C(t,s)$ (with $t\neq s$) are uniformly continuous and uniformly bounded in operator norm for t,s restricted to a compact interval. Moreover,

$$\begin{aligned} C(t) = \lim_{s\to t}\tfrac{1}{t-s}C(t,s) &= 2\mathbf{E}(t)\cdot(\mathbf{D}(\mathbf{A})-\mathbf{F}(t))\,(H(t)+\gamma)^{-1} & (2.74)\\ &= 2\mathbf{E}(t)\cdot G(t)\mathbf{D}(\mathbf{A})\,(H+\gamma)^{-1}\,G(t)^* \end{aligned}$$

exists and is continuous in operator norm, and

$$\|C(t)\| \leq 2C_{\alpha,\beta}|\mathbf{E}(t)|\,. \tag{2.75}$$

Theorem 2.7. *The time-dependent Hamiltonian $H(t)$ has a unique unitary propagator $U(t,s)$, i.e., there is a unique two-parameter family $U(t,s)$ of unitary operators, jointly strongly continuous in t and s, such that*

$$U(t,r)U(r,s) = U(t,s) \tag{2.76}$$

$$U(t,t) = I \tag{2.77}$$

$$U(t,s)\mathcal{D} = \mathcal{D}\,, \tag{2.78}$$

$$i\partial_t U(t,s)\psi = H(t)U(t,s)\psi \;\; \text{for all } \psi \in \mathcal{D}\,, \tag{2.79}$$

$$i\partial_s U(t,s)\psi = -U(t,s)H(s)\psi \;\; \text{for all } \psi \in \mathcal{D}\,. \tag{2.80}$$

In addition, $W(t,s) = (H(t)+\gamma)\,U(t,s)\,(H(s)+\gamma)^{-1}$ is a bounded operator, jointly strongly continuous in t and s, with

$$\|W(t,s)\| \le \mathrm{e}^{\int_{\min\{s,t\}}^{\max\{s,t\}} \|C(r)\|\,\mathrm{d}r}\,, \tag{2.81}$$

the operators $U(t,s)\,(H(s)+\gamma)^{-1}$ and $(H(t)+\gamma)^{-1}\,U(t,s)$ are jointly continuous in t and s in operator norm, and

$$i\partial_t \left\{ U(t,s)\,(H(s)+\gamma)^{-2} \right\} = H(t)U(t,s)\,(H(s)+\gamma)^{-2}\,, \tag{2.82}$$

$$i\partial_s \left\{ (H(t)+\gamma)^{-2}\,U(t,s) \right\} = -\,(H(t)+\gamma)^{-2}\,U(t,s)H(s)\,, \tag{2.83}$$

in operator norm.

Furthermore, if we define the unitary operators $U_k(t,s)$, $k=1,2,\ldots$, by

$$U_k(t,s) = \mathrm{e}^{-i(t-s)H\left(m+\frac{i-1}{k}\right)} \;\; \text{if } m+\tfrac{i-1}{k} \le s,t < m+\tfrac{i}{k}\,, \tag{2.84}$$

where $m\in\mathbb{Z}$, $i=1,2,\ldots,k$, and

$$U_k(t,r) = U_k(t,s)U_k(s,r) \;\; \text{for all } \; t,s,r\,, \tag{2.85}$$

then

$$U(t,s)\,(H(s)+\gamma)^{-1} = \lim_{k\to\infty} U_k(t,s)\,(H(s)+\gamma)^{-1} \tag{2.86}$$

in operator norm, uniformly for t,s restricted to a compact interval.

Proof. The uniqueness and unitarity of the propagator $U(t,s)$ follows from existence and the fact that $i\partial_t\phi_t = H(t)\phi_t$ with $H(t)$ self-adjoint implies $\partial_t\|\phi_t\|^2 = 0$.

To prove the existence of the propagator we apply [Y, Theorem XIV.3.1] (see also [RS2, Theorem X.70]) with

$$A(t) = -i(H(t)+\gamma)\,. \tag{2.87}$$

Note that

$$C(t,s) = A(t)A(s)^{-1} - I = (A(t)-A(s))A(s)^{-1}\,. \tag{2.88}$$

The hypotheses of [Y, Theorem XIV.3.1] (and [RS2, Theorem X.70]) require that (a) $0\notin\sigma(A(t))$, (b) $A(t)$ have a common domain, and (c) $C(t,s)$ and $C(t) = \lim_{t\to s}(t-s)^{-1}C(t,s)$ are uniformly bounded and strongly continuous for t,s restricted to a compact interval. Clearly $\mathcal{D}(A(t)) = \mathcal{D}(H(t)) = \mathcal{D}$ for all t, and it follows from (2.71) that $0\notin\sigma(A(t))$ for all t. Boundedness and continuity of $C(t,s)$ and $C(t)$ were discussed before the statement of the theorem.

Thus the hypotheses of [Y, Theorem XIV.3.1] are satisfied. If we set

$$U(t,s) = \mathrm{e}^{i(t-s)\gamma}\,\widehat{U}(t,s)\,, \tag{2.89}$$

where $\widehat{U}(t,s)$ is the propagator for the $A(t)$ given in [Y, Theorem XIV.3.1] (and [RS2, Theorem X.70]) if $s \leq t$, and $\widehat{U}(t,s) = \widehat{U}(s,t)^*$ if $s \geq t$, we obtain unitary operators $U(t,s)$, strongly continuous in t and s, satisfying (2.76)-(2.79). To prove (2.80), we use the chain rule: Since $U(t,s)U(s,t) = I$, it follows from (2.78) and (2.79) that for $\varphi \in \mathcal{D}$ we have, with $\psi = U(s,t)\varphi$,

$$\begin{aligned} 0 &= \partial_s U(t,s)U(s,t)\varphi = \partial_s U(t,s)\psi + U(t,s)\partial_s U(s,t)\varphi \\ &= \partial_s U(t,s)\psi - iU(t,s)H(s)U(s,t)\varphi = \partial_s U(t,s)\psi - iU(t,s)H(s)\psi\,, \end{aligned} \tag{2.90}$$

since $\mathcal{D} = U(s,t)\mathcal{D}$.

The estimate (2.81) is given in [Y, Theorem XIV.3.1]. A careful reading of the proof of [Y, Theorem XIV.3.1], using our stronger hypotheses on $C(t,s)$, shows that the operators $U(t,s)\,(H(s)+\gamma)^{-1}$ and $(H(t)+\gamma)^{-1}\,U(t,s)$ are jointly continuous in t and s in operator norm, and we have (2.82). Since the adjoint operation is an isometry in operator norm, (2.83) follows from (2.82). □

To compute the linear response, we shall make use of the following "Duhamel formula".

Lemma 2.8. *Let $U^{(0)}(t) = \mathrm{e}^{-itH}$. For all $\psi \in \mathcal{D}$ and $t,s \in \mathbb{R}$ we have*

$$U(t,s)\psi = U^{(0)}(t-s)\psi + i\int_s^t U^{(0)}(t-r)(2\mathbf{F}(r)\cdot\mathbf{D}(\mathbf{A}) - \mathbf{F}(r)^2)U(r,s)\psi \;\mathrm{d}r\,. \tag{2.91}$$

Moreover,

$$\lim_{\mathbf{E}\to 0} U(t,s) = U^{(0)}(t-s) \;\; \textit{strongly}\,. \tag{2.92}$$

Proof. Eq. (2.91) follows simply by calculating $\partial_t U^{(0)}(s-t)U(t,s)\psi$ with $\psi \in \mathcal{D}$, using (2.78), (2.79), and (2.63). The strong limit in (2.92) follows from (2.91) for vectors in $\mathcal{D}$, and hence everywhere since all the operators are unitary. □

3. Covariant operators and the trace per unit volume

3.1. **Measurable covariant operators.** We fix the notation $\mathcal{H} = \mathrm{L}^2(\mathbb{R}^d)$ and let $\mathcal{H}_\mathrm{c}$ denote the dense linear subspace of functions with compact support. We set $\mathcal{L} = \mathcal{L}(\mathcal{H}_\mathrm{c}, \mathcal{H})$ to be the vector space of linear operators on $\mathcal{H}$ with domain $\mathcal{H}_\mathrm{c}$. Elements of $\mathcal{L}$ need not be bounded.

We also fix "magnetic translations": for each $a \in \mathbb{Z}^d$ we define a unitary operator

$$U(a) = \mathrm{e}^{ia\cdot Sx}T(a), \quad \text{with} \quad (T(a)\psi)(x) = \psi(x-a)\,, \tag{3.1}$$

where S is a given $d \times d$ real matrix. Note that $a \mapsto U(a)$ is a projective representation of the translation group $\mathbb{Z}^d$ since

$$U(a)U(b) = \mathrm{e}^{-ib\cdot Sa}U(a+b), \tag{3.2}$$

and that $U(a)$ leaves $\mathcal{H}_\mathrm{c}$ invariant, in fact

$$U(a)\chi_b U(a)^* = \chi_{b+a}\,. \tag{3.3}$$

Let $(\Omega, \mathbb{P})$ be a probability space equipped with an ergodic group $\{\tau(a);\ a \in \mathbb{Z}^d\}$ of measure preserving transformations. We study operator–valued maps $A\colon \Omega \to \mathcal{L}$, which we will simply call operators A_ω. We identify maps that agree $\mathbb{P}$-a.e., and all properties stated are supposed to hold for $\mathbb{P}$-a.e. ω.

Definition 3.1. *Let $A = A_\omega \colon \Omega \to \mathcal{L}$. Then*

(i): *A_ω is measurable if $\langle \varphi, A_\omega \psi \rangle$ is a measurable function for all $\varphi, \psi \in \mathcal{H}_c$. (Or, equivalently, if A_ω is strongly measurable on $\mathcal{H}_c$, i.e., $A_\omega \psi$ is a measurable $\mathcal{H}$-valued function for all $\psi \in \mathcal{H}_c$.)*

(ii): *A_ω is covariant if*

$$U(a) A_\omega U(a)^* = A_{\tau(a)\omega} \quad \text{for all } a \in \mathbb{Z}^d. \tag{3.4}$$

(iii): *A_ω is locally bounded if*

$$\|A_\omega \chi_x\| < \infty \text{ and } \|\chi_x A_\omega\| < \infty \quad \text{for all } x \in \mathbb{Z}^d. \tag{3.5}$$

We let $\mathcal{K}_{mc}$ denote the vector space of measurable covariant operators A_ω, with $\mathcal{K}_{mc,lb}$ being the subspace of locally bounded operators. We define the Banach space

$$\mathcal{K}_\infty = \{A_\omega \in \mathcal{K}_{mc};\ \|A_\omega\|_\infty < \infty\} \subset \mathcal{K}_{mc,lb}\,, \tag{3.6}$$

where

$$\|A_\omega\|_\infty = \|\, \|A_\omega\| \,\|_{\mathrm{L}^\infty(\Omega,\mathbb{P})}\,. \tag{3.7}$$

If $A_\omega \in \mathcal{K}_\infty$, we identify A_ω with its extension to $\mathcal{H}$ (i.e., with its closure $\overline{A_\omega}$). If we define multiplication in $\mathcal{K}_\infty$ by $A_\omega B_\omega := \overline{A_\omega} B_\omega$, and the adjoint by $(A_\omega)^* := A_\omega^*$, then $\mathcal{K}_\infty$ becomes a C^*-algebra.

Whenever $A_\omega \in \mathcal{K}_{mc,lb}$, we have $\mathcal{D}(A_\omega^*) \supset \mathcal{H}_c$, since $\chi_x A_\omega$ is bounded for all x. We define $A_\omega^\ddagger$ to be the restriction of A_ω^* to $\mathcal{H}_c$. It follows that $A_\omega^\ddagger \in \mathcal{K}_{mc,lb}$, and the map $A_\omega \to A_\omega^\ddagger$ is a conjugation in $\mathcal{K}_{mc,lb}$. (Note that $A_\omega \in \mathcal{K}_{mc,lb}$ if and only if there exist symmetric operators $B_\omega, C_\omega \in \mathcal{K}_{mc}$ such that $\|B_\omega \chi_x\| + \|C_\omega \chi_x\| < \infty$ for all $x \in \mathbb{Z}^d$ and $A_\omega = B_\omega + iC_\omega$. In this case $A_\omega^\ddagger = B_\omega - iC_\omega$.)

Thus, given $A_\omega \in \mathcal{K}_{mc,lb}$, we have that A_ω^* is densely defined and therefore A_ω is closable. The closure of A_ω, denoted $\overline{A_\omega}$, has a polar decomposition and $\mathcal{H}_c$ is a core for the self-adjoint operator $|\overline{A_\omega}|$. *We will abuse notation and denote the restriction of $|\overline{A_\omega}|$ to $\mathcal{H}_c$ by $|A_\omega|$.* It is not hard to see that $|A_\omega|$ is covariant, i.e., it satisfies (3.4). Similarly, local boundedness of $|A_\omega|$ is a simple consequence of the identities

$$\|\, |A_\omega| \chi_x\| = \|A_\omega \chi_x\| \text{ and } \|\chi_x |A_\omega|\,\| = \|\, |A_\omega| \chi_x\|. \tag{3.8}$$

It is also true that $|A_\omega|$ is measurable, so $|A_\omega| \in \mathcal{K}_{mc,lb}$, but this requires a little more work.

Lemma 3.2. *Let $A_\omega \in \mathcal{K}_{mc,lb}$, and consider the polar decomposition $\overline{A_\omega} = U_\omega |\overline{A_\omega}|$. Then $|A_\omega| \in \mathcal{K}_{mc,lb}$ and $U_\omega \in \mathcal{K}_\infty$. We also have $f(|\overline{A_\omega}|) \in \mathcal{K}_\infty$ for any bounded Borel function f on the real line.*

Proof. Let $A_\omega \in \mathcal{K}_{mc,lb}$. We start by proving that $(|\overline{A_\omega}|^2 + 1)^{-1}$ is strongly measurable on $\mathcal{H}$, from which it follows that $g(|\overline{A_\omega}|^2)$ is also strongly measurable for any bounded Borel function g on the real line. It then follows that $f(|\overline{A_\omega}|) \in \mathcal{K}_\infty$ for any bounded Borel function f on the real line (covariance is easy to see). Picking $f_n(t) = t\chi_{[-n,n]}(t)$, it is clear that $f_n(|\overline{A_\omega}|) \to |A_\omega|$ strongly on $\mathcal{H}_c$, and hence $|A_\omega|$ is strongly measurable. We conclude that $|A_\omega| \in \mathcal{K}_{mc,lb}$.

To prove measurability of $(|\overline{A_\omega}|^2 + 1)^{-1}$, we pick an ortho-normal basis $\{\varphi_n\}_{n\in\mathbb{N}}$ for the subspace $\mathcal{H}_0 = \chi_0 \mathcal{H} \cong \mathrm{L}^2(\mathbb{R}^d, \chi_0(x)\mathrm{d}x)$ of $\mathcal{H}$, and set $\varphi_n^{(a)} = T(a)\varphi_n$ for $a \in \mathbb{Z}^d$. Then $\{\varphi_n^{(a)}\}_{n\in\mathbb{N}, a\in\mathbb{Z}^d}$ is a an ortho-normal basis for $\mathcal{H}$, which we relabel as

$\{\phi_n\}_{n\in\mathbb{N}}$, and let $\widehat{\mathcal{H}_c}$ be the subspace of finite linear combinations of the ϕ_n's. Note that $\widehat{\mathcal{H}_c}$ is a dense subspace of $\mathcal{H}_c$ and hence is a core for $\overline{A_\omega}$, since A_ω is locally bounded.

Let P_n be the orthogonal projection onto the finite dimensional subspace spanned by $\phi_1, \phi_2, \dots, \phi_n$. We set

$$M_\omega^{(n)} = (A_\omega P_n)^* A_\omega P_n \,. \tag{3.9}$$

Then $M_\omega^{(n)}$ is a bounded operator since A_ω is locally bounded. Because $\langle \varphi, M_\omega^{(n)}\psi\rangle = \langle A_\omega P_n\varphi, A_\omega P_n \psi\rangle$ for $\varphi,\psi \in \mathcal{H}$, we conclude that $M_\omega^{(n)}$ is weakly, and hence strongly, measurable on $\mathcal{H}$. Proceeding as in [PF, Proof of Lemma 2.8], we see that $(M_\omega^{(n)}+1)^{-1}$ is measurable on $\mathcal{H}$ (basically, because a matrix element of the inverse may be expressed as a ratio of determinants, which are measurable functions). We now show that $(M_\omega^{(n)}+1)^{-1} \to (|\overline{A_\omega}|^2+1)^{-1}$ weakly as $n\to\infty$, and hence $(|\overline{A_\omega}|^2+1)^{-1}$ is measurable on $\mathcal{H}$.

For this purpose, let $\varphi,\psi \in \widehat{\mathcal{H}_c}$. For sufficiently large n we have

$$\begin{aligned} \langle A_\omega\varphi, A_\omega(M_\omega^{(n)}+1)^{-1}\psi\rangle &= \langle A_\omega P_n\varphi, A_\omega P_n(M_\omega^{(n)}+1)^{-1}\psi\rangle \\ &= \langle \varphi, M_\omega^{(n)}(M_\omega^{(n)}+1)^{-1}\psi\rangle \,, \end{aligned} \tag{3.10}$$

and hence

$$\langle A_\omega\varphi, A_\omega(M_\omega^{(n)}+1)^{-1}\psi\rangle + \langle \varphi, (M_\omega^{(n)}+1)^{-1}\psi\rangle = \langle\varphi,\psi\rangle \,. \tag{3.11}$$

Now let $\phi\in\mathcal{D}(\overline{A_\omega})$. Given $\varepsilon>0$ we pick $\varphi\in\widehat{\mathcal{H}_c}$ such that

$$\|(\phi-\varphi)\| + \|\overline{A_\omega}(\phi-\varphi)\| < \varepsilon \,. \tag{3.12}$$

Since

$$\|A_\omega P_n(M_\omega^{(n)}+1)^{-1}\|^2 = \|(M_\omega^{(n)}+1)^{-1}M_\omega^{(n)}(M_\omega^{(n)}+1)^{-1}\| \le \frac{1}{4}\,, \tag{3.13}$$

we have

$$\begin{aligned} \Big|\langle\overline{A_\omega}(\phi-\varphi), A_\omega(M_\omega^{(n)}+1)^{-1}\psi\rangle + \langle \phi-\varphi, (M_\omega^{(n)}+1)^{-1}\psi\rangle - \langle\phi-\varphi,\psi\rangle\Big| \\ \le 3\varepsilon\|\psi\| \,, \end{aligned} \tag{3.14}$$

whenever $\psi\in\widehat{\mathcal{H}_c}$ and n is correspondingly large. Therefore, it follows from (3.11) that for all $\phi\in\mathcal{D}(\overline{A_\omega})$ we have

$$\lim_{n\to\infty}\langle\overline{A_\omega}\phi, A_\omega(M_\omega^{(n)}+1)^{-1}\psi\rangle + \langle\phi,(M_\omega^{(n)}+1)^{-1}\psi\rangle = \langle\phi,\psi\rangle \tag{3.15}$$

for all $\psi\in\widehat{\mathcal{H}_c}$.

Taking $\phi\in\mathcal{D}(A_\omega^*\overline{A_\omega})\subset\mathcal{D}(\overline{A_\omega})$, we get

$$\lim_{n\to\infty}\langle(A_\omega^*\overline{A_\omega}+1)\phi, (M_\omega^{(n)}+1)^{-1}\psi\rangle = \langle\phi,\psi\rangle \tag{3.16}$$

for all $\psi\in\widehat{\mathcal{H}_c}$, and hence for all $\psi\in\mathcal{H}$. Writing $\eta = (|\overline{A_\omega}|^2+1)\phi$, we get

$$\lim_{n\to\infty}\langle\eta, (M_\omega^{(n)}+1)^{-1}\psi\rangle = \langle(|\overline{A_\omega}|^2+1)^{-1}\eta,\psi\rangle \tag{3.17}$$

for all $\eta,\psi\in\mathcal{H}$. We conclude that $(M_\omega^{(n)}+1)^{-1}\to(|\overline{A_\omega}|^2+1)^{-1}$ weakly.

We now turn to the partial isometry U_ω. We recall that

$$U_\omega = \lim_{\varepsilon\to 0}\overline{A_\omega}(|\overline{A_\omega}|+\varepsilon)^{-1} \quad \text{strongly on } \mathcal{H} \,. \tag{3.18}$$

Thus U_ω is clearly covariant and measurable, so $U_\omega\in\mathcal{K}_\infty$. □

Lemma 3.3. *Let $A_\omega \in \mathcal{K}_{mc,lb}$. Then, for each n,*

$$A_\omega^{(n)} = \left(\tfrac{1}{n}|\overline{A_\omega^\ddagger}|^2 + 1\right)^{-\frac{1}{2}} A_\omega \in \mathcal{K}_\infty\,, \tag{3.19}$$

with $\|A_\omega^{(n)}\| \le n$, and $A_\omega^{(n)} \to A_\omega$ strongly on $\mathcal{H}_c$.

Proof. We clearly have $A_\omega^{(n)} \in \mathcal{K}_{mc}$ since $\left(\tfrac{1}{n}|\overline{A_\omega^\ddagger}|^2 + 1\right)^{-\frac{1}{2}} \in \mathcal{K}_\infty$ by Lemma 3.2. As $\left(\tfrac{1}{n}|\overline{A_\omega^\ddagger}|^2 + 1\right)^{-\frac{1}{2}} \to I$ strongly, we conclude that $A_\omega^{(n)} \to A_\omega$ strongly on $\mathcal{H}_c$.

Thus we only need to show that $\|A_\omega^{(n)}\| \le n$. To do so, let

$$\widetilde{A_\omega^{(n)}} = \left(\tfrac{1}{n}|A_\omega^*|^2 + 1\right)^{-\frac{1}{2}} A_\omega\,, \tag{3.20}$$

and recall $\|\widetilde{A_\omega^{(n)}}\| \le n$. Since $A^\ddagger$ is the restriction of A^* to $\mathcal{H}_c$, we have $|A_\omega^*|^2 \le |\overline{A_\omega^\ddagger}|^2$ as quadratic forms (see [RS1, p. 375]) and hence

$$\left(\tfrac{1}{n}|\overline{A_\omega^\ddagger}|^2 + 1\right)^{-1} \le \left(\tfrac{1}{n}|A_\omega^*|^2 + 1\right)^{-1} \tag{3.21}$$

by [RS1, Theorem S.17]. We conclude that

$$\|A_\omega^{(n)}\| \le \|\widetilde{A_\omega^{(n)}}\| \le n\,. \tag{3.22}$$

□

Lemma 3.4. *If $A_\omega \in \mathcal{K}_{mc,lb}$, $B_\omega \in \mathcal{K}_\infty$, and $B_\omega A_\omega \in \mathcal{K}_{mc,lb}$, we have that $\mathcal{D}(A_\omega^*) \supset B_\omega^* \mathcal{H}_c$ and*

$$(B_\omega A_\omega)^\ddagger \varphi = A_\omega^* B_\omega^* \varphi \ \text{ for all } \varphi \in \mathcal{H}_c\,. \tag{3.23}$$

Remark 3.5. *Note that $B_\omega A_\omega$ is not necessarily in $\mathcal{K}_{mc,lb}$, since we have no control on $\|\chi_x B_\omega A_\omega\|$ for $x \in \mathbb{Z}^d$.*

Proof. For any $\varphi, \psi \in \mathcal{H}_c$ we have

$$\langle \varphi, B_\omega A_\omega \psi\rangle = \langle (B_\omega A_\omega)^\ddagger \varphi, \psi\rangle\,. \tag{3.24}$$

On the other hand,

$$\langle \varphi, B_\omega A_\omega \psi\rangle = \langle B_\omega^* \varphi, A_\omega \psi\rangle\,. \tag{3.25}$$

It follows that

$$B_\omega^* \varphi \in \mathcal{D}(A_\omega^*) \text{ for all } \varphi \in \mathcal{H}_c \tag{3.26}$$

and (3.23) holds. □

Let us define

$$\mathcal{K}_\odot \;=\; \left\{A_\omega \in \mathcal{K}_{mc,lb};\; B_\omega A_\omega, B_\omega A_\omega^\ddagger \in \mathcal{K}_{mc,lb} \text{ if } B_\omega \in \mathcal{K}_\infty\right\}. \tag{3.27}$$

Note that $\mathcal{K}_\odot \subset \mathcal{K}_{mc,lb}$ is a vector space, and in $\mathcal{K}_\odot$ we can define left and, using Lemma 3.4, right multiplication by an element of $\mathcal{K}_\infty$:

$$B_\omega \odot_L A_\omega \;=\; B_\omega A_\omega\,, \tag{3.28}$$

$$A_\omega \odot_R B_\omega \;=\; A_\omega^{\ddagger *} B_\omega|_{\mathcal{H}_c}\,, \tag{3.29}$$

where $A_\omega \in \mathcal{K}_\odot$ and $B_\omega \in \mathcal{K}_\infty$. Note that for $B_\omega \in \mathcal{K}_\infty$ we have $B_\omega^{\ddagger *} = B_\omega$ since we identify B_ω with its closure, so (3.28) could also have been written as

$$B_\omega \odot_L A_\omega = B_\omega^{\ddagger *} A_\omega\,. \tag{3.30}$$

Proposition 3.6. *Let $A_\omega \in \mathcal{K}_\odot$ and $B_\omega, C_\omega \in \mathcal{K}_\infty$. We then have $B_\omega \odot_L A_\omega, A_\omega \odot_R B_\omega \in \mathcal{K}_\odot$. Moreover,*

$$A_\omega \odot_R B_\omega = \left(B_\omega^* \odot_L A_\omega^\ddagger\right)^\ddagger , \tag{3.31}$$

$$B_\omega \odot_L A_\omega \odot_R C_\omega := (B_\omega \odot_L A_\omega) \odot_R C_\omega = B_\omega \odot_L (A_\omega \odot_R C_\omega) , \tag{3.32}$$

$$(B_\omega \odot_L A_\omega \odot_R C_\omega)^\ddagger = C_\omega^* \odot_L A_\omega^\ddagger \odot_R B_\omega^* , \tag{3.33}$$

$$\{B_\omega \odot_L A_\omega \odot_R C_\omega\}\varphi = B_\omega A_\omega^{\ddagger *} C_\omega \varphi \ \ \textit{for all } \varphi \in \mathcal{H}_c . \tag{3.34}$$

Proof. The proof is a simple exercise. □

3.2. **The Hilbert space $\mathcal{K}_2$.** Let

$$\mathcal{K}_2 = \{A_\omega \in \mathcal{K}_{mc};\ |||A_\omega|||_2 < \infty\} , \tag{3.35}$$

$$\mathcal{K}_2^{(0)} = \mathcal{K}_2 \cap \mathcal{K}_\infty , \tag{3.36}$$

where

$$|||A_\omega|||_2 = \left\{\mathbb{E}\left(\|A_\omega \chi_0\|_2^2\right)\right\}^{\frac{1}{2}} . \tag{3.37}$$

Proposition 3.7. (i) *$\mathcal{K}_2$ is a Hilbert space with the inner product*

$$\langle\langle A_\omega, B_\omega\rangle\rangle = \mathbb{E}\left\{\mathrm{tr}\ \{(A_\omega\chi_0)^* B_\omega \chi_0\}\right\} , \tag{3.38}$$

and $|||\ |||_2$ is the corresponding norm, i.e.,

$$|||A_\omega|||_2^2 = \langle\langle A_\omega, A_\omega\rangle\rangle . \tag{3.39}$$

(ii) *$\mathcal{K}_2 \subset \mathcal{K}_{mc,lb}$ and the conjugation $A_\omega \to A_\omega^\ddagger$ is antiunitary in $\mathcal{K}_2$, i.e.,*

$$\langle\langle A_\omega, B_\omega\rangle\rangle = \langle\langle B_\omega^\ddagger, A_\omega^\ddagger\rangle\rangle . \tag{3.40}$$

(iii) *For all $A_\omega \in \mathcal{K}_2$ we have*

$$(A_\omega\chi_0)^* = \overline{\chi_0 A_\omega^*} = \overline{\chi_0 A_\omega^\ddagger} , \tag{3.41}$$

and hence

$$\langle\langle A_\omega, B_\omega\rangle\rangle = \mathbb{E}\left\{\mathrm{tr}\ \left\{\overline{\chi_0 A_\omega^\ddagger} B_\omega \chi_0\right\}\right\} , \tag{3.42}$$

$$|||A_\omega|||_2 = \left\{\mathbb{E}\left(\|\chi_0 A_\omega^\ddagger\|_2^2\right)\right\}^{\frac{1}{2}} = \left\{\mathbb{E}\left(\|\chi_0 A_\omega\|_2^2\right)\right\}^{\frac{1}{2}} . \tag{3.43}$$

(iv) *$\mathcal{K}_2^{(0)}$ is dense in $\mathcal{K}_2$.*

Proof. We first note that $\mathcal{K}_2$ is a vector space, since

$$|||A_\omega + B_\omega|||_2^2 \le \mathbb{E}\left\{\left(\|A_\omega\chi_0\|_2 + \|B_\omega\chi_0\|_2\right)^2\right\} \le 2\left(|||A_\omega|||_2^2 + |||B_\omega|||_2^2\right) . \tag{3.44}$$

Since the right hand side of (3.38) is well defined for $A_\omega, B_\omega \in \mathcal{K}_2$, it clearly defines an inner product.

To show that $\mathcal{K}_2$ is complete it suffices to show that every summable series in $\mathcal{K}_2$ converges. So consider the series

$$\sum_{n=1}^{\infty} |||A_{n,\omega}|||_2 < \infty , \quad A_{n,\omega} \in \mathcal{K}_2 . \tag{3.45}$$

It follows that

$$\mathbb{E}\left(\sum_{n=1}^{\infty}\|A_{n,\omega}\chi_0\|_2\right)=\sum_{n=1}^{\infty}\mathbb{E}\left(\|A_{n,\omega}\chi_0\|_2\right)\le\sum_{n=1}^{\infty}\left\|A_{n,\omega}\right\|_2<\infty\,, \tag{3.46}$$

and hence

$$\sum_{n=1}^{\infty}\|A_{n,\omega}\chi_0\|_2<\infty\,. \tag{3.47}$$

Using the completeness of $\mathcal{H}$ and the covariance property we conclude that $\sum_{n=1}^{\infty}A_{n,\omega}$ converges strongly in $\mathcal{H}_c$ to an operator $A_\omega\in\mathcal{K}_{mc}$. Since the Hilbert-Schmidt operators on $\mathcal{H}$ are also complete, we also conclude that $A_\omega\chi_0=\sum_{n=1}^{\infty}A_{n,\omega}\chi_0$ with convergence in Hilbert-Schmidt norm. Thus, using Fatou's lemma,

$$\begin{aligned}\left\|A_\omega\right\|_2^2 &= \mathbb{E}\left(\lim_{N\to\infty}\left\|\sum_{n=1}^{N}A_{n,\omega}\chi_0\right\|_2^2\right)\le\liminf_{N\to\infty}\mathbb{E}\left(\left\|\sum_{n=1}^{N}A_{n,\omega}\chi_0\right\|_2^2\right)\\ &\le \left(\lim_{N\to\infty}\sum_{n=1}^{N}\left\|A_{n,\omega}\right\|_2\right)^2=\left(\sum_{n=1}^{\infty}\left\|A_{n,\omega}\right\|_2\right)^2<\infty\,,\end{aligned} \tag{3.48}$$

and hence $A_\omega\in\mathcal{K}_2$. Since $A_\omega-\sum_{n=1}^{N}A_{n,\omega}=\sum_{n=N+1}^{\infty}A_{n,\omega}$, the same argument gives

$$\left\|A_\omega-\sum_{n=1}^{N}A_{n,\omega}\right\|_2^2\le\left(\sum_{n=N+1}^{\infty}\left\|A_{n,\omega}\right\|_2\right)^2\to 0\;\text{ as } N\to\infty\,, \tag{3.49}$$

and hence $\mathcal{K}_2$ is complete.

To show $\mathcal{K}_2\subset\mathcal{K}_{mc,lb}$ it suffices to show $A_\omega^*\chi_0$ is well defined and almost surely bounded, since $A_\omega\chi_0$ is almost surely Hilbert-Schmidt and thus bounded. Given $A_\omega\in\mathcal{K}_2$, we set $A_{\omega,x,y}=\chi_x A_\omega\chi_y$ for $x,y\in\mathbb{Z}^2$, a Hilbert-Schmidt operator. Then note that $(A_{\omega,x,y})^*=\chi_y(A_{\omega,x,y})^*\chi_x$ and

$$\begin{aligned}\sum_{y\in\mathbb{Z}^2}\mathbb{E}\left\{\operatorname{tr}\left(A_{\omega,x,y}(A_{\omega,x,y})^*\right)\right\} &= \sum_{y\in\mathbb{Z}^2}\mathbb{E}\left\{\operatorname{tr}\left(\chi_x A_{\omega,x,y}\chi_y(A_{\omega,x,y})^*\chi_x\right)\right\}\\ &=\sum_{y\in\mathbb{Z}^2}\mathbb{E}\left\{\operatorname{tr}\left(\chi_{x-y}A_{\tau(y)\omega,x-y,0}\chi_0 A^*_{\tau(y)\omega,x-y,0}\chi_{x-y}\right)\right\}\\ &=\sum_{y\in\mathbb{Z}^2}\mathbb{E}\left\{\operatorname{tr}\left(\chi_0 A^*_{\omega,x-y,0}\chi_{x-y}A_{\omega,x-y,0}\chi_0\right)\right\}=\left\|A_\omega\right\|_2^2\,;\end{aligned} \tag{3.50}$$

we used (3.4), the invariance of the expectation under the transformations $\{\tau(a);\,a\in\mathbb{Z}^d\}$, and cyclicity of the trace, plus the fact that, as all terms in the expressions are positive, we can interchange the sum with the trace and the expectation. Proceeding as in (3.46)-(3.49) we conclude that the operator $B_\omega=\sum_{x,y\in\mathbb{Z}^2}(A_{y,x})^*$ is in $\mathcal{K}_2$. (Note that covariance only holds for the sum over all $x,y\in\mathbb{Z}^2$). It is easy to see that $B_\omega\subset A_\omega^*$, so $\mathcal{D}(A_\omega^*)\supset\mathcal{H}_c$ and $B_\omega=A_\omega^\ddagger$. Thus

$$\left\|A_\omega^\ddagger\right\|_2^2=\sum_{y\in\mathbb{Z}^2}\mathbb{E}\left\{\operatorname{tr}\left(A_{\omega,0,y}(A_{\omega,0,y})^*\right)\right\}=\left\|A_\omega\right\|_2^2 \tag{3.51}$$

by (3.50), and (3.40) follows using the polarization identity.

The equality (3.41) is an easy consequence of $\mathcal{D}(A^*)\supset\mathcal{H}_c$; (3.42) and (3.43) then follow from (3.38) and (3.40).

It remains to show that $\mathcal{K}_2^{(0)}$ is dense in $\mathcal{K}_2$. Let $A_\omega \in \mathcal{K}_2$, then $A_\omega, A_\omega^\ddagger \in \mathcal{K}_{mc,lb}$, and $A_\omega^{(n)}$, defined in (3.19), is clearly in $\mathcal{K}_2^{(0)}$, and $\left\|\left| A_\omega - A_\omega^{(n)} \right\|\right|_2 \to 0$ by a dominated convergence argument. □

Left and right multiplication by elements of $\mathcal{K}_\infty$ leave $\mathcal{K}_2$ invariant.

Proposition 3.8. *$\mathcal{K}_2 \subset \mathcal{K}_\odot$. Moreover, if $A_\omega \in \mathcal{K}_2$ and $B_\omega \in K_\infty$ we have $B_\omega \odot_L A_\omega, A_\omega \odot_R B_\omega \in \mathcal{K}_2$ with*

$$\begin{aligned} \|\!|B_\omega \odot_L A_\omega|\!\|_2 &\leq \|\!|B_\omega|\!\|_\infty \|\!|A_\omega|\!\|_2 \,, &(3.52)\\ \|\!|A_\omega \odot_R B_\omega|\!\|_2 &\leq \|\!|B_\omega|\!\|_\infty \|\!|A_\omega|\!\|_2 \,. &(3.53)\end{aligned}$$

Proof. Since we clearly have $B_\omega \odot_L A_\omega \in \mathcal{K}_2$ with (3.52), Proposition 3.7(ii) gives $\mathcal{K}_2 \subset \mathcal{K}_{mc\odot}$. The estimate (3.53) follows from (3.31), (3.52), and (3.40). □

The following lemma will be very useful.

Lemma 3.9. *Let $B_{n,\omega}$ be a bounded sequence in $\mathcal{K}_\infty$ such that $B_{n,\omega} \to B_\omega$ strongly. Then for all $A_\omega \in \mathcal{K}_2$ we have $B_{n,\omega} \odot_L A_\omega \to B_\omega \odot_L A_\omega$ and $A_\omega \odot_R B_{n,\omega} \to A_\omega \odot_R B_\omega$ in $\mathcal{K}_2$.*

Proof. It suffices to prove the result for left multiplication in view of (3.31). By considering the sequence $B_{n,\omega} - B_\omega$ we may assume $B_\omega = 0$. We have, with $A_\omega \in \mathcal{K}_2^{(0)}$,

$$\|\!|B_{n,\omega} \odot_L A_\omega|\!\|_2^2 = \mathbb{E}\,\mathrm{tr}\{\chi_0 A_\omega^* B_{n,\omega}^* B_{n,\omega} A_\omega \chi_0\} \to 0 \tag{3.54}$$

by dominated convergence. Since $B_{n,\omega}$ is bounded and $\mathcal{K}_2^{(0)}$ is dense in $\mathcal{K}_2$, this extends to general $A_\omega \in \mathcal{K}_2$. □

3.3. **The normed space $\mathcal{K}_1$.** Let

$$\begin{aligned} \mathcal{K}_1 &= \{A_\omega \in \mathcal{K}_{mc,lb};\ \|\!|A_\omega|\!\|_1 < \infty\} \,, &(3.55)\\ \mathcal{K}_1^{(0)} &= \mathcal{K}_1 \cap \mathcal{K}_\infty, &(3.56)\end{aligned}$$

where

$$\|\!|A_\omega|\!\|_1 = \mathbb{E}\left\{\mathrm{tr}\left\{\chi_0 |A_\omega| \chi_0\right\}\right\}. \tag{3.57}$$

Note that $\|\!|A_\omega|\!\|_1$ is well defined (possibly infinite) for $A_\omega \in \mathcal{K}_{mc,lb}$ by Lemma 3.2.

Lemma 3.10. *Let $A_\omega \in \mathcal{K}_1$. Then*

$$\mathbb{E}\left\{\mathrm{tr}\,|\chi_0 A_\omega \chi_0|\right\} \leq \|\!|A_\omega|\!\|_1 < \infty \,, \tag{3.58}$$

and hence $\mathbb{E}\left\{\mathrm{tr}\left\{\chi_0 A_\omega \chi_0\right\}\right\}$ is well defined.

Proof. Let $\overline{A_\omega} = U_\omega |\overline{A_\omega}|$ be the polar decomposition of $\overline{A_\omega}$. We have

$$\chi_0 A_\omega \chi_0 = \chi_0 U_\omega |\overline{A_\omega}|^{\frac{1}{2}} |\overline{A_\omega}|^{\frac{1}{2}} \chi_0 \,. \tag{3.59}$$

Since $A_\omega \in \mathcal{K}_1$, $|\overline{A_\omega}|^{\frac{1}{2}} \in \mathcal{K}_2$ and, by Lemma 3.2, $U_\omega \in \mathcal{K}_\infty$. (More precisely, the restriction $|A_\omega|^{\frac{1}{2}}$ of $|\overline{A_\omega}|^{\frac{1}{2}}$ to $\mathcal{H}_c$ is in $\mathcal{K}_2$. Note that $\mathcal{H}_c$ is a core for $|\overline{A_\omega}|^{\frac{1}{2}}$.) Thus $U_\omega |A_\omega|^{\frac{1}{2}} \in \mathcal{K}_2$, and $\overline{\chi_0 U_\omega |\overline{A_\omega}|^{\frac{1}{2}}}$ is a Hilbert-Schmidt operator by (3.41). Hence it follows from (3.59) that $\chi_0 A_\omega \chi_0$ is trace class. The inequality (3.58) now follows from (3.59), Hölder's inequality, and (3.43). □

Lemma 3.11. *Let $A_\omega \in \mathcal{K}_1$ and $B_\omega \in \mathcal{K}_\infty$. Then $B_\omega A_\omega \in \mathcal{K}_1$ and*

$$\left\| B_\omega A_\omega \right\|_1 \leq \left\| B_\omega \right\|_\infty \left\| A_\omega \right\|_1 . \tag{3.60}$$

Proof. We have

$$|B_\omega A_\omega| = W_\omega^* B_\omega A_\omega = W_\omega^* B_\omega U_\omega |A_\omega| = W_\omega^* B_\omega U_\omega |\overline{A_\omega}|^{\frac{1}{2}} |A_\omega|^{\frac{1}{2}} , \tag{3.61}$$

where W_ω and U_ω are partial isometries coming from the polar decompositions of $B_\omega A_\omega$ and A_ω respectively. Since $|A_\omega|^{\frac{1}{2}} \in \mathcal{K}_2$ and $B_\omega U_\omega |A_\omega|^{\frac{1}{2}} \in \mathcal{K}_2$, we may proceed as in Lemma 3.10 to conclude that $B_\omega A_\omega \in \mathcal{K}_1$ and (3.60) holds. □

Proposition 3.12. **(i)** *$\mathcal{K}_1$ is a normed vector space with the norm $\| \ \|_1$. .*
(ii) *The conjugation $A_\omega \to A_\omega^\ddagger$ is an isometry on $\mathcal{K}_1$, i.e.,*

$$\left\| A_\omega^\ddagger \right\|_1 = \left\| A_\omega \right\|_1 . \tag{3.62}$$

(iii) *$\mathcal{K}_1^{(0)}$ is dense in $\mathcal{K}_1$*

Proof. We first prove the triangle inequality for $\| \ \|_1$. So let $A_\omega, B_\omega \in \mathcal{K}_1$. We have

$$|A_\omega + B_\omega| = W_\omega^*(A_\omega + B_\omega) = W_\omega^* A_\omega + W_\omega^* B_\omega , \tag{3.63}$$

with W_ω a partial isometry. It follows from Lemmas 3.10 and 3.11 that $A_\omega + B_\omega \in K_1$ and $\|A_\omega + B_\omega\|_1 \leq \|A_\omega\|_1 + \|B_\omega\|_1$. We conclude that $\mathcal{K}_1$ is a normed space.

Given $A_\omega \in \mathcal{K}_1$, we have

$$\begin{aligned} \chi_0 |A_\omega^\ddagger| \chi_0 &= \chi_0 V_\omega^* A_\omega^\ddagger \chi_0 = \chi_0 V_\omega^* A_\omega^* \chi_0 = \chi_0 V_\omega^* |A_\omega| U_\omega^* \chi_0 \\ &= \left(\overline{\chi_0 V_\omega^* |A_\omega|^{\frac{1}{2}}}\right) \left(|A_\omega|^{\frac{1}{2}} U_\omega^* \chi_0\right) , \end{aligned} \tag{3.64}$$

where $\overline{A_\omega} = U_\omega |\overline{A_\omega}|$ and $\overline{A_\omega^\ddagger} = V_\omega |\overline{A_\omega^\ddagger}|$, and the operators in parentheses are Hilbert-Schmidt by Propositions 3.7 and 3.8. It also follows that

$$\left\| A_\omega^\ddagger \right\|_1 \leq \left\| A_\omega \right\|_1 . \tag{3.65}$$

Since $A = A^{\ddagger\ddagger}$, the reverse inequality follows, yielding (3.62).

Finally, we prove that $\mathcal{K}_1^{(0)}$ is dense in $\mathcal{K}_1$. Given $A_\omega \in \mathcal{K}_1$, let $A_\omega^{(n)} \in \mathcal{K}_\infty$ be as in (3.19). Since

$$\operatorname{Ran} \left(\tfrac{1}{n} |\overline{A_\omega^\ddagger}|^2 + 1\right)^{-\frac{1}{2}} = \mathcal{D}(|\overline{A_\omega^\ddagger}|) = \mathcal{D}(\overline{A_\omega^\ddagger}) \subset \mathcal{D}(A_\omega^*) , \tag{3.66}$$

we have

$$A_\omega^{(n)^*} = A_\omega^* \left(\tfrac{1}{n} |\overline{A_\omega^\ddagger}|^2 + 1\right)^{-\frac{1}{2}} \tag{3.67}$$

and

$$|A_\omega^{(n)}|^2 = A_\omega^* \left(\tfrac{1}{n} |\overline{A_\omega^\ddagger}|^2 + 1\right)^{-1} A_\omega \leq |A_\omega|^2 , \tag{3.68}$$

and hence $|A_\omega^{(n)}| \leq |A_\omega|$. It follows that $A_\omega^{(n)} \in \mathcal{K}_1^{(0)}$. To prove that $\left\| A_\omega - A_\omega^{(n)} \right\|_1 \to 0$, we first remark that by a similar argument we have

$$|A_\omega - A_\omega^{(n)}| \leq |A_\omega| . \tag{3.69}$$

So let $\{\varphi_k\}_{k\in\mathbb{N}}$ be an ortho-normal basis for the subspace $\chi_0\mathcal{H}$, we have

$$\left\| A_\omega - A_\omega^{(n)} \right\|_1 = \mathbb{E}\left\{ \sum_{k\in\mathbb{N}} \langle \varphi_k, |A_\omega - A_\omega^{(n)}| \varphi_k \rangle \right\} \leq \left\| A_\omega \right\|_1 < \infty, \tag{3.70}$$

since $A_\omega \in \mathcal{K}_1$ and

$$\langle \varphi_k, |A_\omega - A_\omega^{(n)}|\varphi_k\rangle \leq \langle \varphi_k, |A_\omega|\varphi_k\rangle\,. \tag{3.71}$$

On the other hand, using Jensen's inequality we get

$$\begin{aligned} \langle \varphi_k, |A_\omega - A_\omega^{(n)}|\varphi_k\rangle &\leq \langle \varphi_k, |A_\omega - A_\omega^{(n)}|^2\varphi_k\rangle^{\frac{1}{2}} \\ &= \|(A_\omega - A_\omega^{(n)})\varphi_k\| \to 0 \text{ as } k \to \infty\,. \end{aligned} \tag{3.72}$$

Thus $\left|\!\left|\!\left| A_\omega - A_\omega^{(n)} \right|\!\right|\!\right|_1 \to 0$ by the Dominated Convergence Theorem. □

We will denote the (abstract) completion of $\mathcal{K}_1$ by $\overline{\mathcal{K}_1}$.

Proposition 3.13. *The normed space $\mathcal{K}_1$ is not complete, i.e., $\mathcal{K}_1 \neq \overline{\mathcal{K}_1}$.*

Proof. Let us denote by $\mathcal{K}_{mc,lb}^{(\mathrm{cst})}$ and $K_1^{(\mathrm{cst})}$ the subset of *constant* operators in $\mathcal{K}_{mc,lb}$ and $\mathcal{K}_1$, respectively. In view of (3.4), $A \in \mathcal{K}_{mc,lb}^{(\mathrm{cst})}$ can always be written in the form

$$A = \sum_{x,y\in\mathbb{Z}^d} \chi_x U(x) S_{x-y} U(-y)\chi_y\,, \tag{3.73}$$

where $S = \{S_x\}_{x\in\mathbb{Z}^d}$ is a family of bounded operators in $\chi_0\mathcal{H}$ such that the series $\sum_{x\in\mathbb{Z}^d} \chi_x U(x) S_x \chi_0$ converges strongly to a bounded operator. A sufficient condition for the latter is

$$\sum_{x\in\mathbb{Z}^d} \|S_x\|^2 < \infty\,. \tag{3.74}$$

Operators A as in (3.73) can be partially diagonalized by a Floquet transform given by

$$\mathcal{F} = (2\pi)^{-\frac{d}{2}} \sum_{x\in\mathbb{Z}^d} \mathrm{e}^{ik\cdot x} U(-x)\chi_x\,, \tag{3.75}$$

a unitary map from $\mathcal{H} = \mathrm{L}^2(\mathbb{R}^d, \mathrm{d}x)$ to $\mathrm{L}^2(\mathbb{T}^d, \mathrm{d}k; \chi_0\mathcal{H})$, where $\mathbb{T}^d = [-\frac{\pi}{2}, \frac{\pi}{2})^d$ is the d-dimensional torus. Its inverse, $\mathcal{F}^*$, is given by

$$\mathcal{F}^* = (2\pi)^{-\frac{d}{2}} \sum_{x\in\mathbb{Z}^d} \chi_x U(x) \langle \mathrm{e}^{ik\cdot x}, \cdot\rangle_{\mathrm{L}^2(\mathbb{T}^d,\mathrm{d}k)} \tag{3.76}$$

For A as in (3.73) with $\sum_{x\in\mathbb{Z}^d} \|S_x\|^2 < \infty$ we have

$$(\mathcal{F}A\mathcal{F}^*\Phi)(k) = \hat{A}(k)\Phi(k) \quad \text{for all } \Phi \in \mathcal{F}\mathcal{H}_c\ , \tag{3.77}$$

where

$$\hat{A}(k) = (2\pi)^{-\frac{d}{2}} \sum_{x\in\mathbb{Z}^d} \mathrm{e}^{ik\cdot x} S_x\,. \tag{3.78}$$

Since $\mathcal{F}$ is unitary, in this case we also have

$$(\mathcal{F}|A|\mathcal{F}^*\Phi)(k) = |\hat{A}(k)|\Phi(k) \quad \text{for all } \Phi \in \mathcal{F}\mathcal{H}_c\,, \tag{3.79}$$

and

$$|\!|\!| A |\!|\!|_1 = \operatorname{tr} \chi_0 |A| \chi_0 = (2\pi)^{-d} \int_{\mathbb{T}^d} \operatorname{tr} |\hat{A}(k)|\, \mathrm{d}k\,. \tag{3.80}$$

It follows that the completion $\overline{\mathcal{K}_1^{(\mathrm{cst})}}$ of $\mathcal{K}_1^{(\mathrm{cst})}$ is isomorphic to the Banach space

$$\mathrm{L}^1(\mathbb{T}^d, (2\pi)^{-d}\mathrm{d}k; \mathcal{T}_1(\chi_0\mathcal{H}))\ ,$$

where $\mathcal{T}_1(\chi_0\mathcal{H}))$ denotes the Banach space of trace class operators on $\chi_0\mathcal{H}$.

To see that there are elements in $\mathrm{L}^1(\mathbb{T}^d,(2\pi)^{-d}\mathrm{d}k;\mathcal{T}_1(\chi_0\mathcal{H}))$ that do not correspond to operators in $\mathcal{K}_1^{(\mathrm{cst})}$, let us consider A as in (3.73) with $S_x = s_x Y$ for all $x\in\mathbb{Z}^d$, where $Y\in\mathcal{T}_1(\chi_0\mathcal{H}))$ and the scalars $\{s_x\}_{x\in\mathbb{Z}^d}$ are chosen such $\hat{s}(k)\in\mathrm{L}^1(\mathbb{T}^d,\mathrm{d}k)$ but $\hat{s}(k)\notin\mathrm{L}^2(\mathbb{T}^d,\mathrm{d}k)$, where $\hat{s}(k)$ is defined as in (3.78). (This can always be done.) We clearly have $\hat{A}(k)\in\mathrm{L}^1(\mathbb{T}^d,(2\pi)^{-d}\mathrm{d}k;\mathcal{T}_1(\chi_0\mathcal{H}))$, but for each $\varphi\in\chi_0\mathcal{H}$ we have

$$\|A\varphi\|^2=\left(\sum_{x\in\mathbb{Z}^d}|s_x|^2\right)\|Y\varphi\|^2=\|\hat{s}(k)\|^2_{\mathrm{L}^2(\mathbb{T}^d,\mathrm{d}k)}\|Y\varphi\|^2=\infty \tag{3.81}$$

unless $Y\varphi=0$. Thus $A\notin\mathcal{K}_1^{(\mathrm{cst})}$ as it does not contain $\mathcal{H}_c$ in its domain. (In fact, $A\notin\mathcal{K}_{mc,lb}^{(\mathrm{cst})}$.)

Note that we proved that for any $\varphi\in\chi_0\mathcal{H}$ we can find $A\in\overline{K_1^{(\mathrm{cst})}}$ which cannot be represented by an operator with φ in its domain. In fact, we proved more: for appropriate Y the constructed A has the property that its domain is disjoint from $\mathcal{H}_c$. □

Remark 3.14. *More generally, it follows from* (3.4) *that* $A_\omega\in\mathcal{K}_{mc,lb}$ *can always be written in the form*

$$A=\sum_{x,y\in\mathbb{Z}^d}\chi_x U(x)S_{\tau(-y)\omega,x-y}U(-y)\chi_y\,, \tag{3.82}$$

where $S_\omega=\{S_{\omega,x}\}_{x\in\mathbb{Z}^d}$ *is a family of bounded operators on* $\chi_0\mathcal{H}$ *such that the series* $\sum_{x\in\mathbb{Z}^d}\chi_xU(x)S_{\omega,x}\chi_0$ *converges strongly to a bounded operator. As in* (3.74), *we have*

$$\|A_\omega\chi_x\|^2\le\sum_{y\in\mathbb{Z}^d}\|S_{\tau(-x)\omega,y}\|^2,\ \ \textit{and also}\ \ \ \|A_\omega\chi_x\|_2^2=\sum_{y\in\mathbb{Z}^d}\|S_{\tau(-x)\omega,y}\|_2^2\,. \tag{3.83}$$

In particular,

$$\|A_\omega\|_2^2=\sum_{y\in\mathbb{Z}^d}\mathbb{E}\left(\|S_{\omega,y}\|_2^2\right). \tag{3.84}$$

In the constant case we could write $\|A\|_1$ *as in* (3.80), *but we do not have a similarly simple expression for* $\|A_\omega\|_1$.

Although $\mathcal{K}_1$ is not complete, it is closed in the following sense:

Proposition 3.15. *Let* $A_\omega\in\mathcal{K}_{mc,lb}$ *and suppose there exists a Cauchy sequence* $A_{n,\omega}$ *in* $\mathcal{K}_1$ *such that* $A_{n,\omega}\chi_0\to A_\omega\chi_0$ *weakly. Then* $A_\omega\in\mathcal{K}_1$ *and* $A_{n,\omega}\to A_\omega$ *in* $\mathcal{K}_1$.

Proof. Let $\overline{A_\omega}=U_\omega|\overline{A_\omega}|$ be the polar decomposition. It follows that

$$U_\omega^*A_{n,\omega}\chi_0\to|A_\omega|\chi_0\quad\text{weakly.} \tag{3.85}$$

Thus, if $\{\varphi_j\}_{j\in\mathbb{N}}$ is an ortho-normal basis for the subspace $\chi_0\mathcal{H}$, we have, using Fatou's Lemma,

$$\begin{aligned}\|A_\omega\|_1 &= \mathbb{E}\sum_{j\in\mathbb{N}}\langle\varphi_j,|A_\omega|\varphi_j\rangle = \mathbb{E}\sum_{j\in\mathbb{N}}\lim_{n\to\infty}|\langle\varphi_j,U_\omega^*A_{n,\omega}\varphi_j\rangle| \qquad (3.86)\\ &\le \liminf_{n\to\infty}\mathbb{E}\sum_{j\in\mathbb{N}}|\langle\varphi_j,U_\omega^*A_{n,\omega}\varphi_j\rangle| \le \liminf_{n\to\infty}\|A_{n,\omega}\|_1 < \infty\,,\end{aligned}$$

and hence $A_\omega\in\mathcal{K}_1$.

For fixed m we have that $A_{n,\omega}-A_{m,\omega}$ is a Cauchy sequence in $\mathcal{K}_1$, and that $(A_{n,\omega}-A_{m,\omega})\chi_0\to(A_\omega-A_{m,\omega})\chi_0$ weakly as $n\to\infty$. Thus the above argument gives

$$\|A_\omega-A_{m,\omega}\|_1 \le \liminf_{n\to\infty}\|A_{n,\omega}-A_{m,\omega}\|_1 \to 0 \text{ as } m\to\infty. \qquad (3.87)$$

□

Corollary 3.16. *Let $\mathcal{K}_{1,2}=\mathcal{K}_1\cap\mathcal{K}_2$ with the norm $\|\ \|_{1,2}=\|\ \|_1+\|\ \|_2$. Then $\mathcal{K}_{1,2}$ is a Banach space.*

The corollary is an immediate consequence of Propositions 3.7(i) and 3.15. Its value is that given a sequence $A_{n,\omega}\in\mathcal{K}_{mc,lb}$ which converges in $\overline{\mathcal{K}_1}$, if it also converges in $\mathcal{K}_2$ then its limit in $\overline{\mathcal{K}_1}$ is actually in $\mathcal{K}_1$.

Left and right multiplication by elements of $\mathcal{K}_\infty$ leave $\mathcal{K}_1$ invariant.

Proposition 3.17. *$\mathcal{K}_1\subset\mathcal{K}_\odot$. Moreover, if $A_\omega\in\mathcal{K}_1$ and $B_\omega\in K_\infty$ we have $B_\omega\odot_L A_\omega, A_\omega\odot_R B_\omega\in\mathcal{K}_1$ with*

$$\|B_\omega\odot_L A_\omega\|_1 \le \|B_\omega\|_\infty\|A_\omega\|_1\,, \qquad (3.88)$$

$$\|A_\omega\odot_R B_\omega\|_1 \le \|B_\omega\|_\infty\|A_\omega\|_1\,. \qquad (3.89)$$

Proof. We have $B_\omega\odot_L A_\omega\in\mathcal{K}_2$ and (3.52) from Lemma 3.11, so it follows from Proposition 3.12(ii) that $\mathcal{K}_1\subset\mathcal{K}_\odot$. The estimate (3.89) follows from (3.31), (3.88), and (3.62). □

We consider one other sort of multiplication, namely the bilinear map $\Diamond\colon\mathcal{K}_2^{(0)}\times\mathcal{K}_2^{(0)}\to\mathcal{K}_1$ given by

$$A_\omega\diamond B_\omega := \Diamond(A_\omega,B_\omega)=A_\omega B_\omega\,. \qquad (3.90)$$

Proposition 3.18. *We have*

$$\|A_\omega\diamond B_\omega\|_1 \le \|A_\omega\|_2\|B_\omega\|_2 \quad \text{for all } A_\omega,B_\omega\in\mathcal{K}_2^{(0)}. \qquad (3.91)$$

Thus $\Diamond$ extends by continuity to a bilinear map (we do not change notation) $\Diamond\colon\mathcal{K}_2\times\mathcal{K}_2\to\overline{\mathcal{K}_1}$, which satisfies (3.91) and has dense range. In fact,

$$\mathcal{K}_1^{(0)}=\Diamond\left(\mathcal{K}_2^{(0)}\times\mathcal{K}_2^{(0)}\right) \qquad (3.92)$$

and

$$\mathcal{K}_1\subsetneq\operatorname{Ran}\Diamond\,. \qquad (3.93)$$

Moreover, given $A_\omega,B_\omega\in\mathcal{K}_2$, we have

$$A_\omega\diamond B_\omega = A_\omega\odot_L B_\omega \ \text{ if } A_\omega\in\mathcal{K}_2^{(0)}\,, \qquad (3.94)$$

$$A_\omega\diamond B_\omega = A_\omega\odot_R B_\omega \ \text{ if } B_\omega\in\mathcal{K}_2^{(0)} \qquad (3.95)$$

$$(A_\omega\diamond B_\omega)^\ddagger = B_\omega^\ddagger\diamond A_\omega^\ddagger\,. \qquad (3.96)$$

Proof. To prove (3.91) we proceed as in the proof of Lemma 3.11. The inclusion in (3.93) was exhibited in the proof of Lemma 3.10; note that it also gives (3.92). (3.94) is proven by an approximation argument. (3.96) follows from the special case when $A_\omega, B_\omega \in \mathcal{K}_2^{(0)}$ and (3.62). (3.95) follows from (3.94), (3.96) and (3.31).

To show that we do not have equality in (3.93) we proceed as in the proof of Proposition 3.13. Let A be as in (3.73) with $S_x = s_x Z$ for all $x \in \mathbb{Z}^d$, where $Z \in \mathcal{T}_2(\chi_0\mathcal{H})$) and $\hat{s}(k) \in \mathrm{L}^2(\mathbb{T}^d, \mathrm{d}k)$ but $\hat{s}(k) \notin \mathrm{L}^4(\mathbb{T}^d, \mathrm{d}k)$. (This can always be done.) Then $A \in \mathcal{K}_2$ but $A \diamond A \notin \mathcal{K}_1$ since $\hat{s}(k)^2 \notin \mathrm{L}^2(\mathbb{T}^d, \mathrm{d}k)$. □

Lemma 3.19. *Let $B_{n,\omega}$ be a bounded sequence in $\mathcal{K}_\infty$ such that $B_{n,\omega} \to B_\omega$ strongly. Then for all $A_\omega \in \mathcal{K}_1$ we have $B_{n,\omega} \odot_L A_\omega \to B_\omega \odot_L A_\omega$ and $A_\omega \odot_R B_{n,\omega} \to A_\omega \odot_R B_\omega$ in $\mathcal{K}_1$.*

Proof. Again it suffices to prove the result for left multiplication in view of (3.31). Since the sequence $B_{n,\omega}$ is bounded and $\mathcal{K}_1^{(0)}$ is dense in $\mathcal{K}_1$ it suffices to prove the result for $A_\omega \in \mathcal{K}_1^{(0)}$. But then we can write $A_\omega = C_\omega D_\omega = C_\omega \diamond D_\omega$, with $C_\omega, D_\omega \in \mathcal{K}_2^{(0)}$. Since

$$B_{n,\omega} \odot_L A_\omega = B_{n,\omega} C_\omega D_\omega = (B_{n,\omega} C_\omega) D_\omega = (B_{n,\omega} \odot_L C_\omega) \diamond D_\omega \,, \tag{3.97}$$

the desired conclusion follows from Lemma 3.9 and Proposition 3.18. □

3.4. **The trace per unit volume.**

Given $A = A_\omega \in \mathcal{K}_1$ we define

$$\mathcal{T}(A) = \mathbb{E}\left\{\mathrm{tr}\left\{\chi_0 A_\omega \chi_0\right\}\right\}. \tag{3.98}$$

Lemma 3.10 says that $\mathcal{T}$ is a well defined linear functional on $\mathcal{K}_1$ such that

$$|\mathcal{T}(A)| \le |||A|||_1 \ . \tag{3.99}$$

In fact, $\mathcal{T}$ is the *trace per unit volume.*

Proposition 3.20. *Given $A = A_\omega \in \mathcal{K}_1$ we have*

$$\mathcal{T}(A) = \lim_{L\to\infty} \tfrac{1}{|\Lambda_L|} \mathrm{tr}\left\{\chi_{\Lambda_L} A_\omega \chi_{\Lambda_L}\right\} \quad \text{for } \mathbb{P}\text{-a.e. } \omega \,, \tag{3.100}$$

where Λ_L denotes the cube of side $L = 1, 3, 5, \ldots$ centered at 0.

Proof. We have

$$\mathrm{tr}\left\{\chi_{\Lambda_L} A_\omega \chi_{\Lambda_L}\right\} = \sum_{x \in \mathbb{Z}^d \cap \Lambda_L} \mathrm{tr}\left\{\chi_x A_\omega \chi_x\right\} = \sum_{x \in \mathbb{Z}^d \cap \Lambda_L} \mathrm{tr}\left\{\chi_0 A_{\tau(x)\omega} \chi_0\right\} \,. \tag{3.101}$$

Thus (3.100) follows from (3.58) and the ergodic theorem. □

Lemma 3.21. *Let $A_\omega, B_\omega \in \mathcal{K}_2$. Then*

$$\mathcal{T}(A_\omega \diamond B_\omega) = \langle\langle A_\omega^\ddagger, B_\omega \rangle\rangle \,. \tag{3.102}$$

In particular we have centrality for the trace per unit volume:

$$\mathcal{T}(A_\omega \diamond B_\omega) = \mathcal{T}(B_\omega \diamond A_\omega) \,. \tag{3.103}$$

Moreover, given $C_\omega \in \mathcal{K}_\infty$, we have

$$\mathcal{T}((C_\omega \odot_L A_\omega) \diamond B_\omega) = \mathcal{T}(A_\omega \diamond (B_\omega \odot_R C_\omega)) \,. \tag{3.104}$$

Note that if $A_\omega, B_\omega \in \mathcal{K}_2^{(0)}$ equation (3.103) reads

$$\mathcal{T}(A_\omega B_\omega) = \mathcal{T}(B_\omega A_\omega)\,, \tag{3.105}$$

and equation (3.104) reads

$$\mathcal{T}(C_\omega A_\omega B_\omega) = \mathcal{T}(A_\omega B_\omega C_\omega)\,. \tag{3.106}$$

Proof. It suffices to prove the Lemma for $A_\omega, B_\omega \in \mathcal{K}_2^{(0)}$, in which case it follows from Propositions 3.7 and 3.8 □

We also have a "$\mathcal{K}_\infty$, $\mathcal{K}_1$" version of centrality for the trace per unit volume:

Lemma 3.22. *Let $A_\omega \in \mathcal{K}_1$ and $C_\omega \in \mathcal{K}_\infty$, then*

$$\mathcal{T}(C_\omega \odot_L A_\omega) = \mathcal{T}(A_\omega \odot_R C_\omega)\,. \tag{3.107}$$

Proof. Just use $A_\omega = (U_\omega |A_\omega|^{\frac{1}{2}}) \diamond |A_\omega|^{\frac{1}{2}}$, with $U_\omega|\overline{A_\omega}|$ the polar decomposition of $\overline{A_\omega}$, and (3.104). □

We will also use the following lemmas.

Lemma 3.23. *Let $A_\omega \in \mathcal{K}_1$ be such that $\mathcal{T}(C_\omega \odot_L A_\omega) = 0$ for all $C_\omega \in \mathcal{K}_\infty$. Then $A_\omega = 0$.*

Proof. Let $U_\omega|\overline{A_\omega}|$ be the polar decomposition of $\overline{A_\omega}$. Then $U_\omega \in \mathcal{K}_\infty$ and $|||A_\omega|||_1 = \mathcal{T}(U_\omega^* A_\omega) = 0$. □

Lemma 3.24. *Let $B_{n,\omega}$ be a bounded sequence in $\mathcal{K}_\infty$ such that $B_{n,\omega} \to B_\omega$ weakly. Then for all $A_\omega \in \mathcal{K}_1$ we have $\mathcal{T}(B_{n,\omega} \odot_L A_\omega) \to \mathcal{T}(B_\omega \odot_L A_\omega)$ and $\mathcal{T}(A_\omega \odot_R B_{n,\omega}) \to \mathcal{T}(A_\omega \odot_R B_\omega)$.*

Proof. It suffices to consider the case $B_\omega = 0$. If $U_\omega|\overline{A_\omega}|$ is the polar decomposition,

$$\mathcal{T}(B_{n,\omega} \odot_L A_\omega) = \mathcal{T}(|A_\omega|^{\frac{1}{2}} \diamond \{B_{n,\omega} \odot_L (U_\omega |A_\omega|^{\frac{1}{2}})\}) \to 0 \tag{3.108}$$

by dominated convergence. The other limit then follows from Lemma 3.22. □

3.5. **The connection with noncommutative integration.** There is a connection with noncommutative integration: $\mathcal{K}_\infty$ is a von Neumann algebra, $\mathcal{T}$ is a faithful normal semifinite trace on $\mathcal{K}_\infty$, and $\overline{\mathcal{K}_i} = \mathrm{L}^i(\mathcal{K}_\infty, \mathcal{T})$ for $i = 1, 2$. (We assume that $\mathcal{K}_1^{(0)}$ is not trivial, which is guaranteed by Assumption 4.1 in view of Proposition 4.2.) But our explicit construction plays a very important role in our analysis.

That $\mathcal{K}_\infty$ is a von Neumann algebra can be seen a follows. Let $\widetilde{\mathcal{H}} = \mathrm{L}^2((\Omega,\mathbb{P});\mathcal{H}) = \int_\Omega^\oplus \mathcal{H}\, \mathrm{d}\mathbb{P}$ (see [RS4, Section XIII.16] for the notation). Then the collection $\widetilde{\mathcal{K}}_\infty$ of strongly measurable maps $A = A_\omega : \Omega \to \mathcal{B}(\mathcal{H})$ with $|||A_\omega|||_\infty < \infty$, where $|||A_\omega|||_\infty$ is as in (3.7), form the von Neumann algebra of decomposable operators on $\widetilde{\mathcal{H}}$ [RS4, Theorems XIII.83 and XIII.84]. If we define unitary operators $\widetilde{U}(a)$ on $\widetilde{\mathcal{H}}$ for $a \in \mathbb{Z}^d$ by $(\widetilde{U}(a)\Phi)(\omega) = U(a)\Phi(\tau(-a)\omega)$ for $\Phi \in \tilde{\mathcal{H}}$, it follows that $\mathcal{K}_\infty = \{A_\omega \in \widetilde{\mathcal{K}}_\infty;\ [\widetilde{U}(a), A_\omega] = 0 \text{ for all } a \in \mathbb{Z}^d\}$, and hence $\mathcal{K}_\infty$ is a von Neumann algebra.

$\mathcal{T}$ is a faithful normal semifinite trace (e.g., [T, Definition 2.1]) on $\mathcal{K}_\infty$. That $\mathcal{T}$ is faithful is clear; to see that $\mathcal{T}$ is normal note that the condition given in [BrR, Theorem 2.7.11(i)] can be verified using properties of the usual trace and the monotone convergence theorem. To show that $\mathcal{T}$ is semifinite, pick a self-adjoint

$0 \neq B_\omega \in \mathcal{K}_1^{(0)}$, note that we have the orthogonal projections $Q_{n,\omega} := \chi_{[-n,n]}(B_\omega) \in \mathcal{K}_1^{(0)}$ by Lemma 3.2, and hence we conclude that $\mathcal{T}$ is semifinite since $Q_{n,\omega} \nearrow I$ strongly.

Note that if $A_\omega \in \mathcal{K}_{mc,lb}$ then its closure $\overline{A_\omega}$ is affiliated with $\mathcal{K}_\infty$ by Lemma 3.2. The converse cannot be true in view of Proposition 3.13.

4. Ergodic magnetic media

4.1. **The ergodic Hamiltonian.** We now state the technical assumptions on our ergodic Hamiltonian H_ω.

Assumption 4.1. *The ergodic Hamiltonian $\omega \mapsto H_\omega$ is a measurable map from the probability space $(\Omega, \mathbb{P})$ to the self-adjoint operators on $\mathcal{H}$ such that*

$$H_\omega = H(\mathbf{A}_\omega, V_\omega) = (-i\nabla - \mathbf{A}_\omega)^2 + V_\omega \ , \tag{4.1}$$

almost surely, where $\mathbf{A}_\omega$ (V_ω) are vector (scalar) potential valued random variables which satisfy the Leinfelder-Simader conditions (see Subsection 2.1) almost surely. It is furthermore assumed that H_ω is covariant:

$$U(a) H_\omega U(a)^* = H_{\tau(a)\omega} \text{ for all } a \in \mathbb{Z}^d \ . \tag{4.2}$$

Measurable in this context means that $\langle \psi, H_\omega \phi \rangle$ is a Borel measurable function for every $\psi, \phi \in C_c^\infty(\mathbb{R}^d)$. As a consequence $f(H_\omega) \in \mathcal{K}_\infty$ for every bounded Borel function f on the real line. (The only subtle point here is measurability, but that is well known. See [PF].)

Note that it follows from ergodicity that $V_{\omega -}$ satisfies (2.2) almost surely with *the same constants* α, β.

We remark that much more detailed knowledge of H_ω is required to verify Assumption 5.1 below, at least for $\zeta_\omega = P_\omega^{(E_F)}$. In particular, one might require V_ω to be of the form $V_\omega(x) = \sum_{a \in \mathbb{Z}^d} \eta_a u(x-a)$, where η_a are independent, identically, distributed random variables and u is a function of compact support. However, the only fact we need here regarding localization for ergodic Schrödinger operators is (5.2) below for suitable functions h. Thus we prefer to take the general Assumption 4.1 and note that Assumption 5.1 for $\zeta_\omega = P_\omega^{(E_F)}$ follows, for suitable $\mathbf{A}_\omega, V_\omega$ and E_F, by the methods of, for example, [GK1, BoGK, AENSS].

It is absolutely crucial to our analysis that the parameters α, β in the Leinfelder-Simader conditions may be chosen independently of ω. In particular, this allows us to prove:

Proposition 4.2. *Let f be a Borel measurable function on the real line such that $\|f \Phi_{d,\alpha,\beta}\|_\infty < \infty$, where $\Phi_{d,\alpha,\beta}$ is given in* (2.15)*. Then*

- **(i):** *We have $f(H_\omega) \in \mathcal{K}_1^{(0)}$, and if $\|f^2 \Phi_{d,\alpha,\beta}\|_\infty < \infty$ then $f(H_\omega) \in \mathcal{K}_2^{(0)}$.*
- **(ii):** *If $f(H_\omega) = g(H_\omega)$ for some $g \in \mathcal{S}(\mathbb{R})$, we have $[x_j, f(H_\omega)] \in \mathcal{K}_1^{(0)} \cap \mathcal{K}_2^{(0)}$, $j = 1, 2, \ldots, d$.*
- **(iii):** *If $f(H_\omega) = g(H_\omega) h(H_\omega)$ with $g \in \mathcal{S}(\mathbb{R})$ and h a Borel measurable function with $\|h^2 \Phi_{d,\alpha,\beta}\|_\infty < \infty$, and for some $j \in \{1, 2, \ldots, d\}$ we have $[x_j, h(H_\omega)] \in \mathcal{K}_2$, then we also have $[x_j, f(H_\omega)] \in \mathcal{K}_1 \cap \mathcal{K}_2$.*
- **(iv):** *We have $P_\omega^{(E)} \in \mathcal{K}_1^{(0)} \cap \mathcal{K}_2^{(0)}$, where $P_\omega^{(E)} = \chi_{(-\infty,E]}(H_\omega)$, i.e., $P_\omega^{(E)} = f(H_\omega)$ with $f = \chi_{(-\infty,E]}$. If in addition we have $\left[x_j, P_\omega^{(E)}\right] \in \mathcal{K}_2$ for some $j \in \{1, 2, \ldots, d\}$, then we also have $\left[x_j, P_\omega^{(E)}\right] \in \mathcal{K}_1$.*

(v): *If f is as in either (ii), (iii), or (iv), we also have*

$$\mathcal{T}\left\{[x_j, f(H_\omega)]\right\} = 0\,. \tag{4.3}$$

Proof. (i) is an immediate consequence of (2.16). To prove (ii), first note that $[x_j, f(H_\omega)]$ is in $\mathcal{K}_\infty$ by Proposition 2.4(ii). We recall that [GK3, Eq. (3.8)]

$$\|\chi_x f(H_\omega)\chi_0\|_2^2 \le C_{d,\alpha,\beta,\nu,k}\ \|f\Phi_{d,\alpha,\beta}\|_\infty \,|\!|\!|g|\!|\!|_{k+2}\ \langle x\rangle^{-k+2\nu} \tag{4.4}$$

for $\mathbb{P}$-a.e. ω and all $k = 1,2,\ldots$ and $\nu > \frac{d}{4}$, and set $\mathbf{a}$ to be a step function approximation to the operator $\mathbf{x}$; i.e., $\mathbf{a}$ is the operator given by multiplication by the discretized coordinates $a \in \mathbb{Z}^d$: $\mathbf{a} = \sum_{a\in\mathbb{Z}^d} a\chi_a$. Note that multiplication by $x_j - a_j$ is a bounded operator for each $j \in \{1,2,\ldots,d\}$; in fact, $\|x_j - a_j\| \le \frac{1}{2}$. Since

$$[x_j, f(H_\omega)] = [a_j f(H_\omega)] + [x_j - a_j, f(H_\omega)]\,, \tag{4.5}$$

to prove $[x_j, f(H_\omega)] \in \mathcal{K}_2$ it suffices to prove $[a_j, f(H_\omega)] \in \mathcal{K}_2$. This follows from (4.4) with sufficiently large k:

$$\begin{aligned}\|[a_j, f(H_\omega)]\chi_0\|_2^2 &= \Big\|\sum_{a\in Z^d}\chi_a[a_j, f(H_\omega)]\chi_0\Big\|_2^2 = \sum_{a\in Z^d}\|\chi_a[a_j, f(H_\omega)]\chi_0\|_2^2 \\ &= \sum_{a\in Z^d}|a_j|^2\,\|\chi_a f(H_\omega)\chi_0\|_2^2 \le C_{d,\alpha,\beta,\nu,k}\ \|f\Phi_{d,\alpha,\beta}\|_\infty \,|\!|\!|g|\!|\!|_{k+2}\sum_{a\in Z^d}|a_j|^2\,\langle a\rangle^{-k+2\nu}\,.\end{aligned} \tag{4.6}$$

That $[x_j, f(H_\omega)]$ it is also in $\mathcal{K}_1$ follows from (iii), since we can write $g(t) = (\langle t\rangle^n g(t))\langle t\rangle^{-n}$ with $n \in \mathbb{N}$, $(\langle t\rangle^n g(t)) \in \mathcal{S}(\mathbb{R})$ and $h(t) = \langle t\rangle^{-n}$ is as in (iii) for n large.

To prove (iii), we note that $[x_j, g(H_\omega)] \in \mathcal{K}_\infty$ by (2.38) and, since $[x_j, h(H_\omega)] \in \mathcal{K}_2$, $x_j h(H_\omega)\chi_0$ is a bounded operator. Hence

$$\begin{aligned}[x_j, f(H_\omega)]\,\chi_0 &= [x_j, g(H_\omega)h(H_\omega)]\,\chi_0 \\ &= [x_j, g(H_\omega)]\,h(H_\omega)\chi_0 + g(H_\omega)\,[x_j, h(H_\omega)]\,\chi_0\,.\end{aligned} \tag{4.7}$$

Noting that $g(H_\omega), h(H_\omega) \in \mathcal{K}_2$ by (i), we conclude that

$$[x_j, f(H_\omega)] = [x_j, g(H_\omega)] \odot_R h(H_\omega) + g(H_\omega) \odot_L [x_j, h(H_\omega)] \in \mathcal{K}_2\,, \tag{4.8}$$

and, as $[x_j, g(H_\omega)] \in \mathcal{K}_2$ by (ii),

$$[x_j, f(H_\omega)] = [x_j, g(H_\omega)] \diamond h(H_\omega) + g(H_\omega) \diamond [x_j, h(H_\omega)] \in \mathcal{K}_1\,. \tag{4.9}$$

Item (iv) is an immediate consequence of (i) and (iii). To see (v), note $x_j\chi_0 = \chi_0 x_j\chi_0$ is bounded and $\chi_0 f(H_\omega)x_j\chi_0 = (\chi_0 f(H_\omega)\chi_0)(x_j\chi_0)$ is trace class. Since $[x_j, f(H_\omega)] \in \mathcal{K}_1$, we conclude that $\chi_0 x_j f(H_\omega)\chi_0$ is also trace class, and

$$\mathcal{T}\left\{[x_j, f(H_\omega)]\right\} = \mathbb{E}\,\mathrm{tr}\,(\chi_0 x_j f(H_\omega)\chi_0) - \mathbb{E}\,\mathrm{tr}\,(\chi_0 f(H_\omega)x_j\chi_0) = 0 \tag{4.10}$$

using centrality of the ordinary trace tr. □

4.2. **Commutators of measurable covariant operators.** In this subsection, H_ω stands either for the time independent H_ω or for $H_\omega(t)$ incorporating a time-dependent electric field. By $H_\omega A_\omega \in \mathcal{K}_i$ we mean $A_\omega\mathcal{H}_c \subset \mathcal{D}$ and the operator $H_\omega A_\omega$ with domain $\mathcal{H}_c$ is in $\mathcal{K}_i$.

Definition 4.3. *We define the following (generalized) commutators:*

(i): *If $A_\omega \in \mathcal{K}_\odot$ and $B_\omega \in \mathcal{K}_\infty$, then*

$$[B_\omega, A_\omega]_\odot = B_\omega \odot_L A_\omega - A_\omega \odot_R B_\omega \in \mathcal{K}_\odot\,, \tag{4.11}$$

$$[A_\omega, B_\omega]_\odot = A_\omega \odot_R B_\omega - B_\omega \odot_L A_\omega = \left([B_\omega^*, A_\omega^\ddagger]_\odot\right)^\ddagger \in \mathcal{K}_\odot. \tag{4.12}$$

(ii): *If $A_\omega, B_\omega \in \mathcal{K}_2$, then*

$$[B_\omega, A_\omega]_\diamond = B_\omega \diamond A_\omega - A_\omega \diamond B_\omega \in \overline{\mathcal{K}_1}\,. \tag{4.13}$$

(iii): *If $A_\omega \in \mathcal{K}_\odot$ is such that $H_\omega A_\omega$ and $H_\omega A_\omega^\ddagger$ are in $\mathcal{K}_\odot$, then*

$$[H_\omega, A_\omega]_\ddagger = H_\omega A_\omega - (H_\omega A_\omega^\ddagger)^\ddagger \in \mathcal{K}_\odot\,. \tag{4.14}$$

Remark 4.4. *These commutators agree when any two of them make sense. More precisely:*

- **(a):** *If $A_\omega, B_\omega \in \mathcal{K}_\infty$ then $[B_\omega, A_\omega]_\odot = [B_\omega, A_\omega] = B_\omega A_\omega - A_\omega B_\omega$, the usual commutator.*
- **(b):** (4.13) *agrees with either* (4.11) *or* (4.12) *if either B_ω or A_ω are in $\mathcal{K}_\infty$.*
- **(c):** (4.14) *should be interpreted as an extension of* (4.11) *to unbounded B_ω. Note that* (4.11) *can be rewritten as $[B_\omega, A_\omega]_\odot = B_\omega A_\omega - (B_\omega^* A_\omega^\ddagger)^\ddagger$, and the right hand side makes sense as long as $B_\omega A_\omega$ and $B_\omega^* A_\omega^\ddagger$ are in $\mathcal{K}_{mc,lb}$. In addition,* (4.14) *reduces to the usual commutator on $\mathcal{H}_c \cap \mathcal{D}$, as shown in the following lemma.*

Lemma 4.5. *Let $A_\omega \in K_\odot$ be such that $H_\omega A_\omega \in K_\odot$. Then*

$$(H_\omega A_\omega)^\ddagger \psi = A_\omega^\ddagger H_\omega \psi \quad \text{for all } \psi \in \mathcal{H}_c \cap \mathcal{D}\,. \tag{4.15}$$

In addition, we have $\mathcal{D}((H_\omega A_\omega)^*) \cap \mathcal{D} = \mathcal{D}(A_\omega^* H_\omega)$ and

$$(H_\omega A_\omega)^* \psi = A_\omega^* H_\omega \psi \quad \text{for all } \psi \in D((H_\omega A_\omega)^*) \cap \mathcal{D}\,. \tag{4.16}$$

As a consequence, if $H_\omega A_\omega$ and $H_\omega A_\omega^\ddagger$ are in $\mathcal{K}_\odot$, then

$$[H_\omega, A_\omega]_\ddagger \psi = H_\omega A_\omega \psi - A_\omega H_\omega \psi \quad \text{for all } \psi \in \mathcal{H}_c \cap \mathcal{D}\,. \tag{4.17}$$

Proof. If $H_\omega A_\omega \in K_\odot$, for all $\psi \in \mathcal{H}_c \cap \mathcal{D}$ and $\xi \in \mathcal{H}_c$ we have

$$\langle (H_\omega A_\omega)^\ddagger \psi, \xi \rangle = \langle \psi, H_\omega A_\omega \xi \rangle = \langle H_\omega \psi, A_\omega \xi \rangle = \langle A_\omega^\ddagger H_\omega \psi, \xi \rangle\,, \tag{4.18}$$

where we used the fact that $H_\omega \psi \in \mathcal{H}_c$ since H_ω is a local operator. Thus (4.15) follows. A similar argument proves (4.16). □

The following lemma will also be useful.

Lemma 4.6. *Let $A_\omega, B_\omega \in \mathcal{K}_2$, $C_\omega \in \mathcal{K}_\infty$. Then*

$$\mathcal{T}\left\{[C_\omega, A_\omega]_\odot \diamond B_\omega\right\} = \mathcal{T}\left\{C_\omega \odot_L [A_\omega, B_\omega]_\diamond\right\}\,. \tag{4.19}$$

Proof. It follows from (4.11), (4.13), and Lemma 3.21. □

4.3. **Time evolution on spaces of covariant operators.** For $\mathbb{P}$-a.e. ω let $U_\omega(t,s)$ be the unitary propagator given by Theorem 2.7. Note that $U_\omega(t,s) \in \mathcal{K}_\infty$. (Since we apply Theorem 2.7 independently for each ω, there is the subtle question of measurability for $U_\omega(t,s)$. However, measurability follows from the construction (2.86), since the propagator $U_\omega(t,s)$ is expressed as a limit of "Riemmann products," i.e., multiplicative Riemmann sums, each of which is manifestly measurable since it is a product of finitely many propagators $\mathrm{e}^{-i\Delta t H_\omega(t_k)}$)

It will be important at times to keep track of the dependence of $U_\omega(t,s)$ on the electric field $\mathbf{E}$, in which case we will write $U_\omega(\mathbf{E},t,s)$. Note that

$$U_\omega(\mathbf{E}=0,t,s) = U_\omega^{(0)}(t-s) := \mathrm{e}^{-i(t-s)H_\omega}\,. \tag{4.20}$$

We omit $\mathbf{E}$ from the notation in what follows.

Proposition 4.7. *Let*

$$\mathcal{U}(t,s)(A_\omega) = U_\omega(t,s)\odot_L A_\omega \odot_R U_\omega(s,t) \quad \text{for} \quad A_\omega \in \mathcal{K}_\odot\,. \tag{4.21}$$

Then $\mathcal{U}(t,s)$ is a linear operator on $\mathcal{K}_\odot$, leaving $\mathcal{K}_\odot$, $\mathcal{K}_\infty$, $\mathcal{K}_1$, and $\mathcal{K}_2$ invariant, with

$$\mathcal{U}(t,r)\mathcal{U}(r,s) = \mathcal{U}(t,s)\,, \tag{4.22}$$

$$\mathcal{U}(t,t) = I\,, \tag{4.23}$$

$$\{\mathcal{U}(t,s)(A_\omega)\}^\ddagger = \mathcal{U}(t,s)(A_\omega^\ddagger)\,. \tag{4.24}$$

Moreover, $\mathcal{U}(t,s)$ is unitary on $\mathcal{K}_2$ and an isometry in $\mathcal{K}_1$ and $\mathcal{K}_\infty$; it extends to an isometry on $\overline{\mathcal{K}}_1$ with the same properties. In addition, $\mathcal{U}(t,s)$ is jointly strongly continuous in t and s on $\overline{\mathcal{K}}_1$ and $\mathcal{K}_2$.

Proof. The first part of the proposition follows from Propositions 3.6, 3.8, and 3.17. $\mathcal{U}(t,s)$ is clearly an isometry on $\mathcal{K}_\infty$. To see that $\mathcal{U}(t,s)$ is an isometry on $\mathcal{K}_1$ and $\mathcal{K}_2$, note that from Propositions 3.8 and 3.17 we have

$$|||\mathcal{U}(t,s)(A_\omega)|||_i \le |||A_\omega|||_i \le |||\mathcal{U}(t,s)(A_\omega)|||_i \tag{4.25}$$

for $i=1,2$, where we used $A_\omega = \mathcal{U}(s,t)\left(\mathcal{U}(t,s)(A_\omega)\right)$. As for (4.24), it follows from (3.33).

The joint strong continuity of $\mathcal{U}(t,s)$ on $\overline{\mathcal{K}}_1$ and $\mathcal{K}_2$ follows from the joint strong continuity of $U_\omega(t,s)$ on $\mathcal{H}$ and Lemmas 3.9 and 3.19. ◻

Lemma 4.8. *Let $A_\omega \in \mathcal{K}_i$ be such that $H_\omega(r_0)A_\omega \in \mathcal{K}_i$ for some $r_0 \in [-\infty,\infty)$, where $i \in \{\odot,1,2,\infty\}$. Then $H_\omega(r)A_\omega \in \mathcal{K}_i$ for all $r \in [-\infty,\infty)$.*

Proof. In view of (2.65) it suffices to show $\mathbf{D}_{j,\omega}A_\omega \in \mathcal{K}_i$ if $H_\omega(r_0)A_\omega \in \mathcal{K}_i$ for some $r_0 \in [-\infty,\infty)$. But this follows immediately from (2.73). ◻

Proposition 4.9. *Let $A_\omega \in \mathcal{K}_i$ be such that $H_\omega(r_0)A_\omega$ and $H_\omega(r_0)A_\omega^\ddagger$ are in $\mathcal{K}_i$ for some $r_0 \in [-\infty,\infty)$ Then the map $r \to \mathcal{U}(t,r)(A_\omega) \in \mathcal{K}_i$ is differentiable in $\mathcal{K}_i$, and*

$$i\partial_r \mathcal{U}(t,r)(A_\omega) = -\mathcal{U}(t,r)([H_\omega(r),A_\omega]_\ddagger)\,, \tag{4.26}$$

with $[H_\omega(r),A_\omega]_\ddagger$ defined in (4.14).

Proof. Fix $i=1$ or $i=2$. All the expressions make sense as elements of $\mathcal{K}_i$. Write

$$\frac{i}{h}\left(\mathcal{U}(t,r+h)(A_\omega) - \mathcal{U}(t,r)(A_\omega)\right) \tag{4.27}$$

$$= \frac{i}{h}\left(U_\omega(t,r+h) - U_\omega(t,r)\right)\odot_L A_\omega \odot_R U_\omega(r+h,t) \tag{4.28}$$

$$+\, U_\omega(t,r)\odot_L A_\omega \odot_R \frac{i}{h}\left(U_\omega(r+h,t) - U_\omega(r,t)\right). \tag{4.29}$$

We first focus on (4.28). Since $H_\omega(r)A_\omega \in K_i$ by Lemma 4.8, one has

$$\begin{aligned} B_\omega \odot_L A_\omega &= B_\omega A_\omega = B_\omega (H_\omega(r)+\gamma)^{-1}(H_\omega(r)+\gamma)A_\omega \\ &= B_\omega (H_\omega(r)+\gamma)^{-1} \odot_L (H_\omega(r)+\gamma)A_\omega\,. \end{aligned} \tag{4.30}$$

Theorem 2.7 asserts that

$$\frac{1}{h}\left(U_\omega(t,r+h)-U_\omega(t,r)\right)(H_\omega(r)+\gamma)^{-1}\to iU_\omega(t,r)H_\omega(r)(H_\omega(r)+\gamma)^{-1}$$

strongly with uniformly bounded norm, as $h\to 0$. Using either Lemma 3.19 or Lemma 3.9, and the strong continuity of $U_\omega(r,t)$ in r, we get

$$\begin{aligned}&\lim_{h\to 0}\frac{i}{h}\left(U_\omega(t,r+h)-U_\omega(t,r)\right)\odot_L A_\omega\odot_R U_\omega(r+h,t) \qquad (4.31)\\ &= -U_\omega(t,r)H_\omega(r)(H_\omega(r)+\gamma)^{-1}\odot_L (H_\omega(r)+\gamma)A_\omega\odot_R U_\omega(r,t)\\ &= -U_\omega(t,r)\odot_L H_\omega(r)A_\omega\odot_R U_\omega(r,t).\end{aligned}$$

We now turn to (4.29). Note that if $B_\omega\in\mathcal{K}_\infty$ then

$$A_\omega\odot_R B_\omega=(B_\omega^*\odot_L A_\omega^\ddagger)^\ddagger=\left(((H_\omega(r)+\gamma)^{-1}B_\omega)^*\odot_L (H_\omega(r)+\gamma)A_\omega^\ddagger\right)^\ddagger . \qquad (4.32)$$

Since the map $A_\omega\to A_\omega^\ddagger$ is an isometry on $\mathcal{K}_i$, the same argument as above implies that

$$\begin{aligned}&\lim_{h\to 0}U_\omega(t,r)\odot_L A_\omega\odot_R\frac{i}{h}\left(U_\omega(t,r+h)-U_\omega(t,r)\right) \qquad (4.33)\\ &= U_\omega(t,r)\odot_L\left(((H_\omega(r)+\gamma)^{-1}H_\omega(r)U_\omega(r,t))^*\odot_L (H_\omega(r)+\gamma)A_\omega^\ddagger\right)^\ddagger\\ &= U_\omega(t,r)\odot_L (H_\omega(r)A_\omega^\ddagger)^\ddagger\odot_R U_\omega(r,t).\end{aligned}$$

□

Proposition 4.10. *Let $A_\omega\in\mathcal{K}_i$ be such that $H_\omega(r_0)A_\omega$ and $H_\omega(r_0)A_\omega^\ddagger$ are in $\mathcal{K}_i$ for some $r_0\in[-\infty,\infty)$, , where $i\in\{1,2,\infty\}$. Then $H_\omega(t)U_\omega(t,r)A_\omega$, $H_\omega(t)U_\omega(t,r)A_\omega^\ddagger$, $H_\omega(t)\mathcal{U}(t,r)(A_\omega)$, and $H_\omega(t)\mathcal{U}(t,r)(A_\omega^\ddagger)$ are in $\mathcal{K}_i$, and the map $t\to\mathcal{U}(t,r)(A_\omega)\in\mathcal{K}_i$ is differentiable, with*

$$i\partial_t\,\mathcal{U}(t,r)(A_\omega)=[H_\omega(t),\mathcal{U}(t,r)(A_\omega)]_\ddagger\,, \qquad (4.34)$$

with the proviso that in $\mathcal{K}_\infty$ the meaning of the derivative is as a bounded and $\mathbb{P}$-a.e.-weak limit.

Moreover, we have

$$|||(H_\omega(t)+\gamma)\,\mathcal{U}(t,r)(A_\omega)|||_i\le |||W_\omega(t,r)|||_\infty\,|||(H_\omega(r)+\gamma)A_\omega|||_i\ , \qquad (4.35)$$

$$|||[H_\omega(t),\mathcal{U}(t,r)(A_\omega)]_\ddagger|||_i\le |||W_\omega(t,r)|||_\infty\left(|||(H_\omega(r)+\gamma)A_\omega|||_i+\left|\left|\left|(H_\omega(r)+\gamma)A_\omega^\ddagger\right|\right|\right|_i\right)\ , \qquad (4.36)$$

and, for all $\varphi\in\mathcal{H}_c\cap\mathcal{D}$,

$$[H_\omega(t),\mathcal{U}(t,r)(A_\omega)]_\ddagger\varphi=H_\omega(t)U_\omega(t,r)A_\omega^{\ddagger *}U_\omega(r,t)\varphi-U_\omega(t,r)A_\omega^{\ddagger *}U_\omega(r,t)H_\omega(t)\varphi. \qquad (4.37)$$

We need the following lemma. (Recall that $\overline{A_\omega}=A_\omega^{\ddagger *}$ for $A_\omega\in\mathcal{K}_{mc,lb}$.)

Lemma 4.11. *Let $A_\omega\in\mathcal{K}_i$ with $H_\omega(t)A_\omega\in\mathcal{K}_i$ ($i\in\{\odot,1,2,\infty\}$). If $\varphi\in\mathcal{D}(A_\omega^{\ddagger *})\cap\mathcal{D}((H_\omega(t)A_\omega)^{\ddagger *})$, it follows that $A_\omega^{\ddagger *}\varphi\in\mathcal{D}$ and*

$$(H_\omega(t)A_\omega)^{\ddagger *}\varphi=H_\omega(t)A_\omega^{\ddagger *}\varphi. \qquad (4.38)$$

As a consequence, $H_\omega(t)(A_\omega\odot_R C_\omega)\in\mathcal{K}_i$ for any $C_\omega\in\mathcal{K}_\infty$, and

$$(H_\omega(t)A_\omega)\odot_R C_\omega=H_\omega(t)A_\omega^{\ddagger *}C_\omega=H_\omega(t)(A_\omega\odot_R C_\omega)\,. \qquad (4.39)$$

Lemma 4.11 can be seen as a generalization of (3.32), where $B_\omega\in\mathcal{K}_\infty$ is replaced by the unbounded operator $H_\omega(t)$ whose domain does not contain $\mathcal{H}_c$.

Proof of Lemma 4.11. Let $\varphi \in \mathcal{D}(A_\omega^{\ddagger *}) \cap \mathcal{D}((H_\omega(t)A_\omega)^{\ddagger *})$ and $\psi \in \mathcal{H}_c \cap \mathcal{D}$, we have, using Lemma 4.5,

$$\langle (H_\omega(t)A_\omega)^{\ddagger *}\varphi, \psi\rangle = \langle \varphi, (H_\omega(t)A_\omega)^{\ddagger}\psi\rangle = \langle \varphi, A_\omega^{\ddagger} H_\omega(t)\psi\rangle = \langle A_\omega^{\ddagger *}\varphi, H_\omega(t)\psi\rangle \,. \tag{4.40}$$

Since $\mathcal{H}_c \cap \mathcal{D}$ is a core for $H_\omega(t)$, it follows that $A_\omega^{\ddagger *}\varphi \in \mathcal{D}$ and

$$\langle (H_\omega(t)A_\omega)^{\ddagger *}\varphi, \psi\rangle = \langle H_\omega(t)A_\omega^{\ddagger *}\varphi, \psi\rangle. \tag{4.41}$$

Since $\mathcal{D} \cap \mathcal{H}_c$ is dense in $\mathcal{H}$ (it contains $\mathcal{C}_c^\infty(\mathbb{R}^d)$), (4.38) follows. □

Proof of Proposition 4.10. Since $H_\omega(r_0)A_\omega \in K_i$, $A_\omega\mathcal{H}_c \subset \mathcal{D}$. Since $U_\omega(t,r)\mathcal{D} \subset \mathcal{D}$, the operator $H_\omega(t)U_\omega(t,r)A_\omega$ is well-defined on $\mathcal{H}_c$ and (use Lemma 4.8)

$$H_\omega(t)U_\omega(t,r)A_\omega = H_\omega(t)U_\omega(t,r)(H_\omega(r)+\gamma)^{-1} \odot_L (H_\omega(r)+\gamma)A_\omega \in K_i \,, \tag{4.42}$$

as $H_\omega(t)U_\omega(t,r)(H_\omega(r)+\gamma)^{-1} = W_\omega(t,r) - \gamma U_\omega(t,r)(H_\omega(r)+\gamma)^{-1}$ is an element of $\mathcal{K}_\infty$. The estimate (4.35) follows.

Furthermore, as in (4.31), on account of Theorem 2.7 we have

$$\begin{aligned} &\lim_{h\to 0} \frac{i}{h}\left(U_\omega(t+h,r) - U_\omega(t,r)\right) \odot_L A_\omega \odot_R U_\omega(r,t+h) \\ &= H_\omega(t)U_\omega(t,r)(H_\omega(r)+\gamma)^{-1} \odot_L (H_\omega(r)+\gamma)A_\omega \odot_R U_\omega(r,t) \\ &= (H_\omega(t)U_\omega(t,r)A_\omega) \odot_R U_\omega(r,t), \end{aligned} \tag{4.43}$$

where we used associativity of left and right multiplication in $\mathcal{K}_i$ according to Proposition 3.6, and in $\mathcal{K}_\infty$ we took a bounded and $\mathbb{P}$-a.e.-weak limit.

By the same reasoning as above $H_\omega(t)U_\omega(t,r)A_\omega^{\ddagger} \in \mathcal{K}_i$, and we have an estimate similar to (4.35). Thus we can differentiate the second term as in (4.43) simply by using the conjugates:

$$\begin{aligned} &\lim_{h\to 0} A_\omega \odot_R \frac{i}{h}\left(U_\omega(r,t+h) - U_\omega(r,t)\right) \\ &= \left(\lim_{h\to 0} \frac{i}{h}\left(U_\omega(t+h,r) - U_\omega(t,r)\right) \odot_L A_\omega^{\ddagger}\right)^{\ddagger} = (H_\omega(t)U_\omega(t,r)A_\omega^{\ddagger})^{\ddagger}. \end{aligned} \tag{4.44}$$

Combining (4.43) and (4.44) we get

$$i\partial_t\,\mathcal{U}(t,r)(A_\omega) = (H_\omega(t)U_\omega(t,r)A_\omega) \odot_R U_\omega(r,t) - U_\omega(t,r) \odot_L (H_\omega(t)U_\omega(t,r)A_\omega^{\ddagger})^{\ddagger} \,. \tag{4.45}$$

Recalling that $H_\omega(t)U_\omega(t,r)A_\omega \in \mathcal{K}_i$, it follows from Lemma 4.11 that

$$\begin{aligned} (H_\omega(t)U_\omega(t,r)A_\omega) \odot_R U_\omega(r,t) &= H_\omega(t)U_\omega(t,r)A_\omega^{\ddagger *}U_\omega(r,t) \\ &= H_\omega(t)\mathcal{U}_\omega(t,r)(A_\omega) \,. \end{aligned} \tag{4.46}$$

Likewise, since $H_\omega(t)U_\omega(t,r)A_\omega^{\ddagger} \in \mathcal{K}_i$, we conclude that

$$\begin{aligned} U_\omega(t,r) \odot_L (H_\omega(t)U_\omega(t,r)A_\omega^{\ddagger})^{\ddagger} &= \left((H_\omega(t)U_\omega(t,r)A_\omega^{\ddagger}) \odot_R U_\omega(r,t)\right)^{\ddagger} \\ &= \left(H_\omega(t)\mathcal{U}(t,r)(A_\omega^{\ddagger})\right)^{\ddagger} \,. \end{aligned} \tag{4.47}$$

Eq. (4.34) follows. Furthermore, by Lemma 4.5 we have

$$(H_\omega U_\omega(t,r)A_\omega^{\ddagger})^{\ddagger}\varphi = (U_\omega(t,r)A_\omega^{\ddagger})^{\ddagger}H_\omega\varphi = A_\omega^{\ddagger *}U_\omega(r,t)H_\omega\varphi \tag{4.48}$$

for any $\varphi \in \mathcal{D} \cap \mathcal{H}_c$, so (4.37) holds.

The bound (4.36) follows from (4.35) and its counterpart for $A_\omega^{\ddagger}$. □

In the special case when $\mathbf{E} = 0$ we have the following corollary, with

$$\mathcal{U}^{(0)}(t)(A_\omega) = U_\omega^{(0)}(t) \odot_L A_\omega \odot_R U_\omega^{(0)}(-t) \quad \text{for} \quad A_\omega \in \mathcal{K}_\odot \,, \tag{4.49}$$

where $U_\omega^{(0)}(t) = \mathrm{e}^{-itH_\omega}$ as in (4.20). The operator $\mathcal{L}_i$ introduced in the following lemma is usually called the *Liouvillian*.

Corollary 4.12. *$\mathcal{U}^{(0)}(t)$ is a one-parameter group of operators on $\mathcal{K}_\odot$, leaving $\mathcal{K}_i$ invariant for $i = 1, 2, \infty$. $\mathcal{U}^{(0)}(t)$ is unitary on $\mathcal{K}_2$ and an isometry on $\mathcal{K}_1$ and $\mathcal{K}_\infty$, so it extends to an isometry in $\overline{\mathcal{K}_1}$. It is strongly continuous on $\overline{\mathcal{K}_1}$ and $\mathcal{K}_2$; we denote by $\mathcal{L}_i$, $i = 1, 2$, the corresponding infinitesimal generators :*

$$\mathcal{U}^{(0)}(t) = \mathrm{e}^{-it\mathcal{L}_i} \quad \textit{for all } t \in \mathbb{R}\,. \tag{4.50}$$

Let

$$\mathcal{D}_i^{(0)} = \left\{ A_\omega \in \mathcal{K}_i; \;\; H_\omega A_\omega, H_\omega A_\omega^\ddagger \in \mathcal{K}_i \right\}, \quad i = 1, 2, \infty\,. \tag{4.51}$$

Then $\mathcal{D}_i^{(0)}$ is an operator core for $\mathcal{L}_i$, $i = 1, 2$ (note that $\mathcal{L}_2$ is essentially self-adjoint on $\mathcal{D}_2^{(0)}$), and

$$\mathcal{L}_i(A_\omega) = [H_\omega, A_\omega]_\ddagger \quad \textit{for all } A_\omega \in \mathcal{D}_i^{(0)}, \quad i = 1, 2\,. \tag{4.52}$$

Moreover, for every $B_\omega \in \mathcal{K}_\infty$ there exists a sequence $B_{n,\omega} \in \mathcal{D}_\infty^{(0)}$ such that $B_{n,\omega} \to B_\omega$ as a bounded and $\mathbb{P}$-a.e.-strong limit.

Proof. Most of the Corollary follows immediately from Propositions 4.7, 4.9, 4.10, and Stone's Theorem for the Hilbert space $\mathcal{K}_2$, the Hille-Yosida Theorem for the Banach space $\overline{\mathcal{K}_1}$. Since $f(H_\omega)A_\omega g(H_\omega) \in \mathcal{D}_i^{(0)}$ for all $f, g \in C_c^\infty(\mathbb{R})$ and $A_\omega \in \mathcal{K}_i$, $i = 1, 2, \infty$, we conclude that elements in $\mathcal{K}_\infty$ can approximated by sequences in $\mathcal{D}_\infty^{(0)}$ as a bounded and $\mathbb{P}$-a.e.-strong limit, and also that $\mathcal{D}_i^{(0)}$ is a core for $\mathcal{L}_i$ for $i = 1, 2$, as in the usual proofs of Stone's Theorem and the Hille-Yosida Theorem, □

4.4. **Gauge transformations in spaces of measurable operators.** The map

$$\mathcal{G}(t)(A_\omega) = G(t) A_\omega G(t)^* \,, \tag{4.53}$$

with $G(t) = \mathrm{e}^{i \int_{-\infty}^t \mathbf{E}(s)\cdot \mathbf{x}}$ as in (2.57), is an isometry on $\mathcal{K}_\infty$, $\mathcal{K}_1^{(0)}$, and $\mathcal{K}_2^{(0)}$, and hence extends to an isometry on $\overline{\mathcal{K}}_1$ and on $\mathcal{K}_2$. Moreover, since $G(t)$ and χ_x commute, (4.53) holds for A_ω either in $\mathcal{K}_1$ or $\mathcal{K}_2$.

Lemma 4.13. *The map $\mathcal{G}(t)$ is strongly continuous on both $\overline{\mathcal{K}}_1$ and on $\mathcal{K}_2$, and*

$$\lim_{t \to -\infty} \mathcal{G}(t) = I \;\; \textit{strongly} \tag{4.54}$$

on both $\overline{\mathcal{K}}_1$ and on $\mathcal{K}_2$. Moreover, if $A_\omega \in \mathcal{K}_i$, $i = 1$ or 2, with $[x_j, A_\omega] \in \mathcal{K}_i$ for $j = 1, ..., d$, then $\mathcal{G}(t)(A_\omega)$ is continuously differentiable in $\mathcal{K}_i$ with

$$\partial_t \mathcal{G}(t)(A_\omega) = i \left[\mathbf{E}(t) \cdot \mathbf{x}, \mathcal{G}(t)(A_\omega)\right] = i\mathcal{G}(t) \left([\mathbf{E}(t) \cdot \mathbf{x}, A_\omega] \right). \tag{4.55}$$

Proof. We start by proving the lemma on $\mathcal{K}_2$. For $A_\omega \in \mathcal{K}_2$, we have

$$\mathcal{G}(t+h)(A_\omega) - \mathcal{G}(t)(A_\omega) = \mathcal{G}(t)(\mathcal{G}(t+h)\mathcal{G}(-t) - 1)(A_\omega)\,. \tag{4.56}$$

Since $\mathcal{G}(t)$ is an isometry, continuity follows if we show that

$$\lim_{h \to 0} \left|\!\left|\!\left| (\mathcal{G}_t(h) - 1)(A_\omega) \right|\!\right|\!\right|_2 = 0\,, \tag{4.57}$$

where $\mathcal{G}_t(h)(A_\omega) = G_t(h)(A_\omega)G_t(h)^*$, with $G_t(h) = G(t+h)G(-t)$ being the unitary operator given by multiplication by the function $\mathrm{e}^{-i\int_t^{t+h}\mathbf{E}(s)\cdot\mathbf{x}\,\mathrm{d}s}$. Thus

$$(\mathcal{G}_t(h) - 1)(A_\omega) = G_t(h)\left[(1 - G_t(h)^*)A_\omega + A_\omega(G_t(h)^* - 1)\right] \tag{4.58}$$

Since $G_t(h)$ is unitary, we have

$$\begin{aligned}|||(\mathcal{G}_t(h) - 1)(A_\omega)|||_2^2 &\leq 2\left\{\mathbb{E}\,\|(1 - G_t(h)^*)A_\omega\chi_0\|_2^2 + \mathbb{E}\,\|A_\omega(G_t(h)^* - 1)\chi_0\|_2^2\right\}\\ &= 2\left\{\mathbb{E}\,\|(1 - G_t(h)^*)A_\omega\chi_0\|_2^2 + \mathbb{E}\,\|A_\omega\chi_0(G_t(h)^* - 1)\|_2^2\right\}.\end{aligned} \tag{4.59}$$

Although $G_t(h)^* \notin \mathcal{K}_\infty$ because it is not covariant, we can use the argument in the proof of Lemma 3.9 to conclude that both terms in (4.59) go to 0 as $h \to 0$, obtaining (4.57). The limit in (4.54) is just continuity at $t = -\infty$ and is proven in the same way.

The result in $\overline{\mathcal{K}}_1$ now follows from the result in K_2 using the $\Diamond$ map, since for $B_\omega, C_\omega \in \mathcal{K}_2^{(0)}$, we have on $\mathcal{K}_1$ that

$$\mathcal{G}(t)(B_\omega C_\omega) = \mathcal{G}(t)(B_\omega)\mathcal{G}(t)(C_\omega) = (\mathcal{G}(t)(B_\omega)) \diamond (\mathcal{G}(t)(C_\omega)), \tag{4.60}$$

and, as $\mathcal{G}(t)$ are isometries, it suffices to prove strong continuity on a dense subset.

It only remains to prove differentiability and (4.55) assuming $[x_j, A_\omega] \in \mathcal{K}_i$, since continuity of the derivative follows from (4.57) and the strong continuity just obtained for $\mathcal{G}(t)$. We see by (4.56) that it suffices to show

$$\lim_{h\to 0} \tfrac{1}{h}(\mathcal{G}_t(h) - 1)(A_\omega) = i\left[\mathbf{E}(t)\cdot\mathbf{x}, A_\omega\right], \tag{4.61}$$

with convergence in $\mathcal{K}_i$. Since $[\mathbf{x}, A_\omega] \in \mathcal{K}_i$, the (Bochner) integral

$$\Phi(h) \;=\; i\frac{1}{h}\int_0^h \mathrm{d}u\,\mathcal{G}_t(u)\left([\mathbf{E}(t+u)\cdot\mathbf{x}, A_\omega]\right) \tag{4.62}$$

is, for each $h > 0$, a well defined element of $\mathcal{K}_1$. Furthermore, as $\mathcal{G}_t(\cdot)$ is strongly continuous, the integrand is continuous and

$$\lim_{h\to 0}\Phi(h) \;=\; i\left[\mathbf{E}(t)\cdot\mathbf{x}, A_\omega\right]. \tag{4.63}$$

We claim that $\Phi(h) = h^{-1}(\mathcal{G}_t(h) - 1)(A_\omega)$. Indeed it suffices to verify

$$h\chi_x\Phi(h)\chi_y \;=\; (\mathcal{G}_t(h) - 1)(\chi_x A_\omega\chi_y)) \tag{4.64}$$

for each x, y (since χ_x, χ_y commute with $G(t)$). But this identity follows since the derivatives of the two sides are equal, and both expressions vanish at $h = 0$. (Derivation is permitted here because of the cut-off induced by χ_x, χ_y.) □

5. Linear response theory and Kubo formula

In this section we prove our main results. *We assume throughout this section that Assumptions 4.1 and 5.1 (stated below) hold.*

5.1. **Adiabatic switching of the electric field.** We now fix an initial equilibrium state of the system, i.e., we specify a density matrix ζ_ω which is in equilibrium, so $[H_\omega, \zeta_\omega] = 0$. For physical applications, we would generally take $\zeta_\omega = f(H_\omega)$ with f the Fermi-Dirac distribution at inverse temperature $\beta \in (0, \infty]$ and *Fermi energy* $E_F \in \mathbb{R}$, i.e., $f(E) = \frac{1}{1+\mathrm{e}^{\beta(E-E_F)}}$ if $\beta < \infty$ and $f(E) = \chi_{(-\infty,E_F]}(E)$ if $\beta = \infty$; explicitly

$$\zeta_\omega \;=\; \begin{cases} F_\omega^{(\beta,E_F)} \;:=\; \frac{1}{1+\mathrm{e}^{\beta(H_\omega - E_F)}}\,, & \beta < \infty\,, \\ P_\omega^{(E_F)} \;:=\; \chi_{(-\infty,E_F]}(H_\omega)\,, & \beta = \infty\,. \end{cases} \tag{5.1}$$

The fact that we have a Fermi-Dirac distribution is not so important at first, although when we compute the Hall conductivity we will restrict our attention to the zero temperature case with the *Fermi projection* $P^{(E_F)}$.

The key property we need is that the hypothesis of either Proposition 4.2(ii) or Prop. 4.2(iii) holds:

Assumption 5.1. *The initial equilibrium state ζ_ω is non-negative, i.e., $\zeta_\omega \geq 0$, and, either*

(a): *$\zeta_\omega = g(H_\omega)$ with $g \in \mathcal{S}(\mathbb{R})$,*

or

(b): *ζ_ω decomposes as $\zeta_\omega = g(H_\omega)h(H_\omega)$ with $g \in \mathcal{S}(\mathbb{R})$ and h a Borel measurable function which satisfies $\|h^2 \Phi_{d,\alpha,\beta}\|_\infty < \infty$ and*

$$\mathbb{E}\left\{ \|\mathbf{x}\, h(H_\omega)\chi_0\|_2^2 \right\} \;<\; \infty\,. \tag{5.2}$$

(Condition (5.2) is equivalent to $[x_j, h(H_\omega)] \in \mathcal{K}_2$ for all $j = 1, 2, \ldots, d$.)

Remark 5.2. *We make the following observations about Assumption 5.1:*

(i): *By Proposition 4.2, either (ii) or (iii), we have $[x_j, \zeta_\omega] \in \mathcal{K}_1 \cap \mathcal{K}_2$ for all $j = 1, 2, \ldots, d$.*

(ii): *The equivalence between (5.2) and $[x_j, h(H_\omega)] \in \mathcal{K}_2$ for $j = 1, \ldots, d$ follows from the facts that $h(H_\omega) \in \mathcal{K}_2$ by Prop. 4.2(i) and*

$$\|\mathbf{x}\, h(H_\omega)\chi_0\|_2 \;\leq\; \|[\mathbf{x},\, h(H_\omega)]\chi_0\|_2 \;+\; \|h(H_\omega)\chi_0\|_2\;. \tag{5.3}$$

Although $|\mathbf{x}|^2 = \mathbf{x}\cdot\mathbf{x}$ is not covariant, it follows from (5.2) that for any $a \in \mathbb{Z}^d$ we have

$$\mathbb{E}\left\{ \|\mathbf{x}\, h(H_\omega)\chi_a\|_2^2 \right\} < \infty\,, \tag{5.4}$$

and hence the operators $[x_j, h(H_\omega)]$ are well defined on $\mathcal{H}_c$ for $j = 1, \ldots, d$.

(iii): *The Fermi-Dirac distributions $f^{(\beta,E_F)}(E) \;:=\; (1+\mathrm{e}^{\beta(E-E_F)})^{-1}$ with finite β satisfy Assumption 5.1(a). Just take $g(E) = k(E)f^{(\beta,E_F)}(E)$, where $k(E)$ is any C^∞ function which is equal to one for $E \geq -\gamma$ (defined in (2.10)) and equal to 0 for $E \leq -\gamma_1$ for some $\gamma_1 > \gamma$.*

(iv): *For a Fermi projection $P_\omega^{(E_F)}$ ($\beta = \infty$), it is natural to take $h(H_\omega) = P_\omega^{(E_F)}$ and for g any Schwartz function identically 1 on $[-\gamma, E_F]$. Condition (5.2) does not hold automatically in this case; rather it holds only for E_F in the "localization regime," as discussed in the introduction. The existence of a region of localization been established for random Landau Hamiltonians with Anderson-type potentials* [CH, W, GK4].

Let us now switch on, adiabatically, a spatially homogeneous electric field $\mathbf{E}$, i.e., we take (with $t_- = \min\{t, 0\}$, $t_+ = \max\{t, 0\}$)

$$\mathbf{E}(t) = \mathrm{e}^{\eta t_-}\mathbf{E}\,, \tag{5.5}$$

and hence

$$\mathbf{F}(t) = \int_{-\infty}^{t} \mathbf{E}(s)\mathrm{d}s = \left(\tfrac{\mathrm{e}^{\eta t_-}}{\eta} + t_+\right)\mathbf{E}\,. \tag{5.6}$$

The system is now described by the ergodic time dependent Hamiltonian $H_\omega(t)$, as in (2.49). We write

$$\zeta_\omega(t) = G(t)\zeta_\omega G(t)^* = \mathcal{G}(t)(\zeta_\omega), \quad \text{i.e.,} \quad \zeta_\omega(t) = f(H_\omega(t)). \tag{5.7}$$

Assuming the system was in equilibrium at $t = -\infty$ with the density matrix $\varrho_\omega(-\infty) = \zeta_\omega$, the time dependent density matrix $\varrho_\omega(t)$ would be the solution of the following Cauchy problem for the Liouville equation:

$$\begin{cases} i\partial_t \varrho_\omega(t) = [H_\omega(t), \varrho_\omega(t)]_\ddagger \\ \lim_{t\to-\infty} \varrho_\omega(t) = \zeta_\omega \end{cases}, \tag{5.8}$$

where we have written the commutator $[\cdot,\cdot]_\ddagger$ in anticipation of the fact that this is to be understood as an evolution in $\mathcal{K}_i$, $i = 1, 2$. The main result of this subsection is the following theorem on solutions to (5.8), which relies on the ingredients introduced in Sections 2 and 3. In view of Corollary 4.12, we replace the commutator in (5.8) by the Liouvillian at time t:

$$\mathcal{L}_i(t) = \mathcal{G}(t)\mathcal{L}_i\mathcal{G}(-t), \quad i = 1, 2\,. \tag{5.9}$$

Note that $\mathcal{L}_i(t)$ has $\mathcal{D}_i^{(0)}$ as an operator core for all t, since it follows from Lemma 4.8 that $\mathcal{D}_i^{(0)} = \mathcal{G}(t)\mathcal{D}_i^{(0)}$ for $i = 1, 2, \infty$.

We have the following generalization of Theorem 1.1.

Theorem 5.3. *The Cauchy problem*

$$\begin{cases} i\partial_t \varrho_\omega(t) = \mathcal{L}_i(t)(\varrho_\omega(t)) \\ \lim_{t\to-\infty} \varrho_\omega(t) = \zeta_\omega \end{cases}, \tag{5.10}$$

has a unique solution in both $\overline{\mathcal{K}_1}$ and $\mathcal{K}_2$, with $\mathcal{L}_i(t)$, $i = 1, 2$, being the corresponding Liouvillian. The unique solution $\varrho_\omega(t)$ is in $\mathcal{D}_1^{(0)}(t) \cap \mathcal{D}_2^{(0)}(t) \subset \mathcal{K}_1 \cap \mathcal{K}_2$ for all t, solves the stronger Cauchy problem (5.8) in both $\mathcal{K}_1$ and $\mathcal{K}_2$, and is given by

$$\begin{aligned} \varrho_\omega(t) &= \lim_{s\to-\infty} \mathcal{U}(t,s)\,(\zeta_\omega) &(5.11)\\ &= \lim_{s\to-\infty} \mathcal{U}(t,s)\,(\zeta_\omega(s)) &(5.12)\\ &= \zeta_\omega(t) - i\int_{-\infty}^{t} \mathrm{d}r\, \mathrm{e}^{\eta r_-}\mathcal{U}(t,r)\left([\mathbf{E}\cdot\mathbf{x}, \zeta_\omega(r)]\right)\,. &(5.13)\end{aligned}$$

We also have

$$\varrho_\omega(t) = \mathcal{U}(t,s)(\varrho_\omega(s))\,, \quad \left\|\!\left|\varrho_\omega(t)\right|\!\right\|_i = \left\|\!\left|\zeta_\omega\right|\!\right\|_i\,, \tag{5.14}$$

for all t, s and $i = 1, 2, \infty$. Furthermore, $\varrho_\omega(t)$ is non-negative, and if $\zeta_\omega = P_\omega^{E_F}$, then $\varrho_\omega(t)$ is an orthogonal projection for all t.

Before proving the theorem we need a technical but crucial lemma. We write $\mathbf{D}_{j,\omega} = \mathbf{D}_j(\mathbf{A}_\omega)$.

Lemma 5.4. *Let $j = 1, \cdots, d$.*

(i): *For all* $\varphi \in \mathcal{H}_c$ *we have* $x_j \zeta_\omega \varphi \in \mathcal{D}$ *and*

$$2\mathbf{D}_{j,\omega} \zeta_\omega \varphi = i H_\omega x_j \zeta_\omega \varphi - i x_j H_\omega \zeta_\omega \varphi = i[H_\omega, x_j] \zeta_\omega \varphi \,. \tag{5.15}$$

(ii): $H_\omega[x_j, \zeta_\omega] \in \mathcal{K}_1 \cap \mathcal{K}_2$. *In fact, the operators* $H_\omega[x_j, \zeta_\omega]$ *and* $[x_j, H_\omega \zeta_\omega]$ *are well defined (as commutators) on* $\mathcal{H}_c$, *we have*

$$H_\omega[x_j, \zeta_\omega] = [x_j, H_\omega \zeta_\omega] - 2i\mathbf{D}_{j,\omega} \zeta_\omega \ \text{ on } \mathcal{H}_c \,, \tag{5.16}$$

and the two operators in the right hand side of (5.16) *are in* $\mathcal{K}_1 \cap \mathcal{K}_2$.

(iii): $H_\omega[\mathbf{E} \cdot \mathbf{x}, \zeta_\omega] \in \mathcal{K}_1 \cap \mathcal{K}_2$.

Proof. It follows from (2.3) that

$$H_\omega x_j \phi = x_j H_\omega \phi - 2i\mathbf{D}_{j,\omega} \phi \quad \text{for all } \phi \in C_c^\infty(\mathbb{R}^d) \,. \tag{5.17}$$

Thus if $\phi \in \mathcal{D} \cap \mathcal{D}(x_j)$ with $H_\omega \phi \in \mathcal{D}(x_j)$, we conclude by an approximation argument that $x_j \phi \in \mathcal{D}$ and (5.17) holds for ϕ.

That $[x_j, H_\omega \zeta_\omega] \in \mathcal{K}_1 \cap \mathcal{K}_2$ follows from Assumption 5.1 and Proposition 4.2(ii)-(iii) since the function $Eg(E) \in \mathcal{S}(\mathbb{R})$. In particular, this tells us that $H_\omega \zeta_\omega \mathcal{H}_c \subset \mathcal{D}(x_j)$. Thus, given $\varphi \in \mathcal{H}_c$, we set $\phi = \zeta_\omega \varphi \in \mathcal{D}(x_j)$, so we have $H_\omega \phi \in \mathcal{D}(x_j)$ and $\phi \in \mathcal{D}(x_j)$ (because $[x_j, \zeta_\omega] \in \mathcal{K}_2$). We conclude that (5.15) follows from (5.17). This proves (i).

Since $x_j \zeta_\omega \varphi \in \mathcal{D}$ for all $\varphi \in \mathcal{H}_c$, the operator $H_\omega[x_j, \zeta_\omega]$ is well defined on $\mathcal{H}_c$, and (5.16) follows from (5.15). That $\mathbf{D}_{j,\omega} \zeta_\omega \in \mathcal{K}_1 \cap \mathcal{K}_2$ follows from Proposition 2.3(i). Thus (ii) is proven, and (iii) follows immediately. □

We now turn to the proof of Theorem 5.3.

Proof of Theorem 5.3. Let us first apply Proposition 4.9 and Lemma 4.13 to

$$\varrho_\omega(t,s) := \mathcal{U}(t,s)(\zeta_\omega(s)). \tag{5.18}$$

We get

$$\begin{aligned} i\partial_s \varrho_\omega(t,s) &= -\mathcal{U}(t,s)\left([H_\omega(s), \zeta_\omega(s)]_\ddagger\right) + \mathcal{U}(t,s)\left(-[\mathbf{E}(s)\cdot\mathbf{x}, \zeta_\omega(s)]\right) \\ &= -\mathcal{U}(t,s)\left([\mathbf{E}(s)\cdot\mathbf{x}, \zeta_\omega(s)]\right) , \end{aligned} \tag{5.19}$$

where we used (5.7). As a consequence, with $\mathbf{E}(r) = \mathrm{e}^{\eta r_-}\mathbf{E}$,

$$\varrho_\omega(t,t) - \varrho_\omega(t,s) = i \int_s^t \mathrm{d}r\, \mathrm{e}^{\eta r_-} \mathcal{U}(t,r)\left([\mathbf{E}\cdot\mathbf{x}, \zeta_\omega(r)]\right) . \tag{5.20}$$

Since

$$\left\| \mathcal{U}(t,r)\left([\mathbf{E}\cdot\mathbf{x}, \zeta_\omega(s)]\right) \right\|_i = \left\| [\mathbf{E}\cdot\mathbf{x}, \zeta_\omega] \right\|_i , \tag{5.21}$$

the integral is absolutely convergent and the limit as $s \to -\infty$ can be performed. It yields the equality between (5.12) and (5.13). Equality of (5.11) and (5.12) follows from Lemma 4.13 which gives

$$\zeta_\omega = \lim_{s\to-\infty} \zeta_\omega(s) \ \text{ in both } \mathcal{K}_1 \text{ and } \mathcal{K}_2. \tag{5.22}$$

Since $\mathcal{U}(t,s)$ are isometries on $\mathcal{K}_i$, $i = 1, 2, \infty$ (Proposition 4.7), it follows from (5.11) that $\|\varrho_\omega(t)\|_i = \|\zeta_\omega\|_i$. We also get $\varrho_\omega(t) = \varrho_\omega(t)^\ddagger$, and hence $\varrho_\omega(t) = \varrho_\omega(t)^*$ as $\varrho_\omega(t) \in \mathcal{K}_\infty$. Moreover, (5.11) with the limit in both $\mathcal{K}_1$ and $\mathcal{K}_2$ implies that

$\varrho_\omega(t)$ is nonnegative. Furthermore, if $\zeta_\omega = P_\omega^{(E_F)}$ then $\varrho_\omega(t)$ is a projection, since denoting by $\lim^{(i)}$ the limit in $\mathcal{K}_i$, $i = 1, 2$, we have

$$\varrho_\omega(t) = \lim_{s\to-\infty}{}^{(1)}\, \mathcal{U}(t,s)\left(P_\omega^{(E_F)}\right) = \lim_{s\to-\infty}{}^{(1)}\, \mathcal{U}(t,s)\left(P_\omega^{(E_F)}\right) \diamond \mathcal{U}(t,s)\left(P_\omega^{(E_F)}\right)$$
$$= \left\{\lim_{s\to-\infty}{}^{(2)}\, \mathcal{U}(t,s)\left(P_\omega^{(E_F)}\right)\right\} \diamond \left\{\lim_{s\to-\infty}{}^{(2)}\, \mathcal{U}(t,s)\left(P_\omega^{(E_F)}\right)\right\} \;=\; \varrho_\omega(t)^2 . \quad (5.23)$$

To see that $\varrho_\omega(t)$ is a solution of (5.8) in $\mathcal{K}_i$, we differentiate the expression (5.13) using Proposition 4.10 and Lemma 4.13; the hypotheses of Proposition 4.10 are satisfied in view of Lemma 5.4(iii) and the fact that $i[\mathbf{E}\cdot\mathbf{x}, \zeta_\omega(r)]$ is a symmetric operator. Moreover, it follows from (4.36) that

$$|||[H_\omega(t), \mathcal{U}(t,r)([\mathbf{E}\cdot\mathbf{x}, \zeta_\omega(r)])]|||_i \leq \quad (5.24)$$
$$2\|W_\omega(t,r)\|\, |||(H_\omega(r)+\gamma)[\mathbf{E}\cdot\mathbf{x}, \zeta_\omega(r)]|||_i = 2\|W_\omega(t,r)\|\, |||(H_\omega+\gamma)[\mathbf{E}\cdot\mathbf{x}, \zeta_\omega]|||_i ,$$

where

$$\sup_{r;\, r\leq t} \|W_\omega(t,r)\| \leq C_t < \infty \quad (5.25)$$

by (2.81) and (2.75). Recalling (5.13), we therefore get

$$i\partial_t \varrho_\omega(t) = -i\int_{-\infty}^{t} \mathrm{d}r\, \mathrm{e}^{\eta r_-}\, [H_\omega(t), \mathcal{U}(t,r)([\mathbf{E}\cdot\mathbf{x}, \zeta_\omega(r)])]_\ddagger \quad (5.26)$$
$$= -\left[H_\omega(t), \left\{i\int_{-\infty}^{t} \mathrm{d}r\, \mathrm{e}^{\eta r_-}\mathcal{U}(t,r)([\mathbf{E}\cdot\mathbf{x}, \zeta_\omega(r)])\right\}\right]_\ddagger \quad (5.27)$$
$$= \left[H_\omega(t), \left\{\zeta_\omega(t) - i\int_{-\infty}^{t} \mathrm{d}r\, \mathrm{e}^{\eta r_-}\mathcal{U}(t,r)([\mathbf{E}\cdot\mathbf{x}, \zeta_\omega(r)])\right\}\right]_\ddagger$$
$$= [H_\omega(t), \varrho_\omega(t)]_\ddagger , \quad (5.28)$$

the integrals being Bochner integrals in $\mathcal{K}_i$. We justify going from (5.26) to (5.27) as follows: Since $H_\omega(t)(H_\omega(t)+\gamma)^{-1} \in \mathcal{K}_\infty$ and $(H_\omega(t)+\gamma)^{-1} \in \mathcal{K}_\infty$, we have, as operators on $\mathcal{H}_c$,

$$\int_{-\infty}^{t} \mathrm{d}r\, \mathrm{e}^{\eta r_-}\, H_\omega(t)\mathcal{U}(t,r)([\mathbf{E}\cdot\mathbf{x}, \zeta_\omega(r)]) \quad (5.29)$$
$$= \left(H_\omega(t)(H_\omega(t)+\gamma)^{-1}\right) \odot_L \int_{-\infty}^{t} \mathrm{d}r\, \mathrm{e}^{\eta r_-}\, (H_\omega(t)+\gamma)\mathcal{U}(t,r)([\mathbf{E}\cdot\mathbf{x}, \zeta_\omega(r)])$$
$$= H_\omega(t)\left((H_\omega(t)+\gamma)^{-1} \odot_L \int_{-\infty}^{t} \mathrm{d}r\, \mathrm{e}^{\eta r_-}\, (H_\omega(t)+\gamma)\mathcal{U}(t,r)([\mathbf{E}\cdot\mathbf{x}, \zeta_\omega(r)])\right)$$
$$= H_\omega(t)\int_{-\infty}^{t} \mathrm{d}r\, \mathrm{e}^{\eta r_-}\mathcal{U}(t,r)([\mathbf{E}\cdot\mathbf{x}, \zeta_\omega(r)]) .$$

Since the map $A_\omega \to A_\omega^\ddagger$ is an antilinear isometry, we also have the identity conjugate to (5.29). We thus have (5.28).

It remains to show that the solution of (5.10) is unique in both $\overline{\mathcal{K}_1}$ and $\mathcal{K}_2$. It suffices to show that if $\nu_\omega(t)$ is a solution of (5.10) with $\zeta_\omega = 0$ then $\nu_\omega(t) = 0$ for all t. We give the proof for $\overline{\mathcal{K}_1}$, the proof for $\mathcal{K}_2$ being similar and slightly easier. For any $s \in \mathbb{R}$, set $\tilde{\nu}_\omega^{(s)}(t) = \mathcal{U}(s,t)(\nu_\omega(t))$. If $A_\omega \in \mathcal{D}_\infty^{(0)}$, we have, using Lemma 4.10 in

$\mathcal{K}_\infty$ and (5.10), that

$$\begin{aligned} i\partial_t \mathcal{T}\left\{A_\omega \odot_L \tilde{\nu}^{(s)}_\omega(t)\right\} &= i\partial_t \mathcal{T}\left\{\mathcal{U}(t,s)(A_\omega) \odot_L \nu_\omega(t)\right\} \\ &= \mathcal{T}\left\{[H_\omega(t),\mathcal{U}(t,s)(A_\omega)]_\ddagger \odot_L \nu_\omega(t)\right\} + \mathcal{T}\left\{\mathcal{U}(t,s)(A_\omega) \odot_L \mathcal{L}_1(t)(\nu_\omega(t))\right\} \\ &= -\mathcal{T}\left\{\mathcal{U}(t,s)(A_\omega) \odot_L \mathcal{L}_1(t)(\nu_\omega(t))\right\} + \mathcal{T}\left\{\mathcal{U}(t,s)(A_\omega) \odot_L \mathcal{L}_1(t)(\nu_\omega(t))\right\} = 0\,. \end{aligned} \tag{5.30}$$

In the final step we have used the fact that for $A_\omega \in \mathcal{D}^{(0)}_\infty$ and $B_\omega \in \mathcal{D}_1$ we have

$$\mathcal{T}\left\{[H_\omega(t), A_\omega]_\ddagger \odot_L B_\omega\right\} = -\mathcal{T}\left\{A_\omega \odot_L \mathcal{L}_1(t)(B_\omega)\right\}. \tag{5.31}$$

Indeed, since $\mathcal{D}^{(0)}_1$ is a core for $\mathcal{L}_1(t)$ it suffices to consider $B_\omega \in \mathcal{D}^{(0)}_1$. For such B, (5.31) follows by cyclicity of the trace, with some care needed since $H_\omega(t)$ is unbounded:

$$\begin{aligned} &\mathcal{T}\left\{[H_\omega(t), A_\omega]_\ddagger \odot_L B_\omega\right\} \\ &\quad= \mathcal{T}\left\{H_\omega(t)A_\omega \odot_L B_\omega\right\} \;-\; \mathcal{T}\left\{(H_\omega(t)A^\ddagger_\omega)^\ddagger \odot_L B_\omega\right\} \\ &\quad= \mathcal{T}\left\{(H_\omega(t)+\gamma)A_\omega \odot_L ((H_\omega(t)+\gamma)B^\ddagger_\omega)^\ddagger \odot_R (H_\omega(t)+\gamma)^{-1}\right\} \\ &\qquad - \mathcal{T}\left\{((H_\omega(t)+\gamma)A^\ddagger_\omega)^\ddagger \odot_L (H_\omega(t)+\gamma)^{-1}(H_\omega(t)+\gamma)B_\omega\right\} \\ &\quad= -\mathcal{T}\left\{A_\omega \odot_L [H_\omega(t), B_\omega]_\ddagger\right\} \;=\; -\mathcal{T}\left\{A_\omega \odot_L \mathcal{L}_1(t)(B_\omega)\right\}. \end{aligned} \tag{5.32}$$

We conclude that for all t and $A_\omega \in \mathcal{D}^{(0)}_\infty$ we have

$$\mathcal{T}\left\{A_\omega \odot_L \tilde{\nu}^{(s)}_\omega(t)\right\} = \mathcal{T}\left\{A_\omega \odot_L \tilde{\nu}^{(s)}_\omega(s)\right\} = \mathcal{T}\left\{A_\omega \odot_L \nu_\omega(s)\right\}, \tag{5.33}$$

and hence (5.33) holds for all $A_\omega \in \mathcal{K}_\infty$ by Corollary 4.12 and Lemma 3.19 (or Lemma 3.24) . Thus $\tilde{\nu}^{(s)}_\omega(t) = \nu_\omega(s)$ by Lemma 3.23, that is, $\nu_\omega(t) = \mathcal{U}(t,s)(\nu_\omega(s))$. Since $\lim_{s\to-\infty} \nu_\omega(s) = 0$ by hypothesis, we get $\nu_\omega(t) = 0$ for all t. □

5.2. **The current and the conductivity.** ¿From now on $\varrho_\omega(t)$ will denote the unique solution to (5.10), given explicitly in (5.13). We set

$$\mathbf{D}_\omega(t) = \mathbf{D}(\mathbf{A}_\omega + \mathbf{F}(t)) = G(t)\mathbf{D}(\mathbf{A}_\omega)G(t)^* = G(t)\mathbf{D}_\omega G(t)^*. \tag{5.34}$$

Since $H_\omega(t)\varrho_\omega(t) \in \mathcal{K}_{1,2}$ we have $\varrho_\omega(t)\mathcal{H}_c \subset \mathcal{D}$, hence the operators $\mathbf{D}_{j,\omega}(t)\varrho_\omega(t)$ are well-defined on $\mathcal{H}_c$, $j = 1, 2, \dots, d$, and we have

$$\mathbf{D}_{j,\omega}(t)\varrho_\omega(t) = \left(\mathbf{D}_{j,\omega}(t)(H_\omega(t)+\gamma)^{-1}\right) \odot_L ((H_\omega(t)+\gamma)\varrho_\omega(t)) \in \mathcal{K}_{1,2}\,. \tag{5.35}$$

Definition 5.5. *Starting with a system in equilibrium in state ζ_ω, the net current (per unit volume), $\mathbf{J}(\eta, \mathbf{E}; \zeta_\omega) \in \mathbb{R}^d$, generated by switching on an electric field $\mathbf{E}$ adiabatically at rate $\eta > 0$ between time $-\infty$ and time 0, is defined as*

$$\mathbf{J}(\eta, \mathbf{E}; \zeta_\omega) = \mathcal{T}\left(\mathbf{v}_\omega(0)\varrho_\omega(0)\right) - \mathcal{T}\left(\mathbf{v}_\omega\zeta_\omega\right), \tag{5.36}$$

where the velocity operator $\mathbf{v}_\omega(t)$ at time t is as in (2.24), i.e.,

$$\mathbf{v}_\omega(t) = 2\mathbf{D}_\omega(t) = \{2\mathbf{D}_{j,\omega}(t))\}_{j=1,\cdots,d}\,, \tag{5.37}$$

a vector of essentially self-adjoint operators on $\mathcal{D}$ (or $C^\infty_c(\mathbb{R})$).

Remark 5.6. **(a):** *The term $\mathcal{T}\left(\mathbf{v}_\omega\zeta_\omega\right) = \left\{\mathcal{T}\left(\mathbf{v}_{j,\omega}\zeta_\omega\right)\right\}_{j=1,\cdots,d}$ is the current at time $t = -\infty$. Since the system is then at equilibrium one expects this term to be zero, a fact which we prove in Lemma 5.7. It follows that the net current is equal to the first term of* (5.36)*, which is the current at time* 0*. We will simply call this the current.*

(b): *The current* $\mathbf{J}(\eta, \mathbb{E}; \zeta)$ *is a real vector. This follows from the fact that* $0 \leq \varrho_\omega(t) \in \mathcal{K}_1$*, and hence* $\sqrt{\varrho_\omega(t)} \in \mathcal{K}_2$*, the fact that* $\mathbf{D}_{j,\omega}(t)\sqrt{\varrho_\omega(t)} \in \mathcal{K}_2$ *by the same argument as in* (5.35)*, the centrality of* $\mathcal{T}$*, and the essential self-adjointness of the components of* $\mathbf{v}_\omega(t)$*.*

Lemma 5.7. *Let* f *be a Borel measurable function on the real line, such that* $\|f\widetilde{\Phi}_{d,\alpha,\beta}\|_\infty$ *is finite. Then*

$$\mathcal{T}(\mathbf{D}_{j,\omega} f(H_\omega)) = 0\,. \tag{5.38}$$

As a consequence, we have $\mathcal{T}\left(\mathbf{v}_\omega P_\omega^{(E_F)}\right) = 0$*.*

This result appears in [BES], with a detailed proof in the discrete case and some remarks for the continuous case. The latter is treated in [KeS]. Their proof relies on a Duhamel formula and the Fourier transform. We give an alternative proof based on the Helffer-Sjöstrand formula.

Proof of Lemma 5.7. First note that by a limiting argument it suffices to consider $f \in \mathcal{S}(\mathbb{R})$. In fact, we may find a sequence $g_n \in \mathcal{S}(\mathbb{R})$ such that $\sup_n \|g_n \widetilde{\Phi}_{d,\alpha,\beta}\|_\infty < \infty$ and $g_n(H_\omega) \to f(H_\omega)$ strongly. Then

$$\begin{aligned} &\mathbf{D}_{j,\omega}(f(H_\omega) - g_n(H_\omega)) = \\ &\quad \mathbf{D}_{j,\omega}\frac{1}{\sqrt{H_\omega+\gamma}} \odot_L \frac{1}{(H_\omega+\gamma)^{2[[\frac{d}{4}]]}} \odot_R (H_\omega+\gamma)^{2[[\frac{d}{4}]]+\frac{1}{2}}(f(H_\omega) - g_n(H_\omega)) \,, \end{aligned} \tag{5.39}$$

where the left hand factor is in $\mathcal{K}_\infty$ by Proposition 2.3(i), the middle factor is in $\mathcal{K}_1$ by Proposition 2.14, and the right hand factor is a uniformly bound sequence in $\mathcal{K}_\infty$ converging strongly to zero. By dominated convergence, we conclude that the $\mathcal{K}_1$ norm, and thus the trace per unit volume, converges to zero.

Therefore, suppose $f \in \mathcal{S}(\mathbb{R})$. Let $G(t) = \int_t^\infty \mathrm{d}t\, f(t)$, and set $F(t) = b(t)G(t)$, where $b(t) \in C^\infty(\mathbb{R})$ is such that $b(t) = 1$ for $t > -\gamma$ and $b(t) = 0$ for $t < -\gamma - 1$ (so $b(t) = 1$ in a neighborhood of the spectrum of H_ω). We have $F \in \mathcal{S}(\mathbb{R})$, $G(H_\omega) = F(H_\omega)$, and $f(H_\omega) = F'(H_\omega)$.

We now recall the generalization of the Helffer-Sjöstrand formula given in [HuS, Lemma B.2]: given a self-adjoint operator A and $f \in \mathcal{S}(\mathbb{R})$ we have

$$\tfrac{1}{p!} f^{(p)}(A) = \int d\tilde{f}(z)(z - A)^{-p-1} \quad \text{for } p = 0, 1, \dots\,, \tag{5.40}$$

where the integral converges absolutely in operator norm by (2.37). (See [HuS, Appendix B] for details.)

By (2.44) from the proof of Proposition 2.4, we have

$$[x_j, R_\omega(z)] = 2iR_\omega(z)\mathbf{D}_{j,\omega}R_\omega(z) \in \mathcal{K}_\infty\,, \tag{5.41}$$

for $R_\omega(z) = (H_\omega - z)^{-1}$ with $\operatorname{Im} z \neq 0$. By the usual Helffer-Sjöstrand formula (2.35) we have

$$[x_j, F(H_\omega)] \;=\; -\int d\tilde{F}(z)[x_j, R_\omega(z)] \;=\; -2i\int d\tilde{F}(z) R_\omega(z)\mathbf{D}_{j,\omega}R_\omega(z)\,, \tag{5.42}$$

which in particular gives another proof to the fact that $[x_j, F(H_\omega)] \in \mathcal{K}_\infty$, which we already knew by Proposition 4.2(ii).

There is a slight technical difficulty due to the fact that $R_\omega(z)\mathbf{D}_{j,\omega}R_\omega(z)$ may not be in $\mathcal{K}_1$ (although $[x_j, F(H_\omega)]$ is). Thus we introduce a cutoff by picking a sequence $h_n \in C_c^\infty(\mathbb{R})$, $|h_n| \le 1$, $h_n = 1$ on $[-n, n]$, and apply (5.40) with $p = 0$ and $p = 1$ to obtain

$$\begin{aligned}\mathcal{T}\left\{[x_j, F(H_\omega)] \odot_L h_n(H_\omega)\right\} &= -2i \int d\tilde{F}(z) \mathcal{T}\left\{R_\omega(z)\mathbf{D}_{j,\omega} R_\omega(z) \odot_L h_n(H_\omega)\right\} \\ &= -2i \int d\tilde{F}(z) \mathcal{T}\left\{\mathbf{D}_{j,\omega} R_\omega(z)^2 \odot_L h_n(H_\omega)\right\} = -2i \mathcal{T}\left\{\mathbf{D}_{j,\omega} f(H_\omega) \odot_L h_n(H_\omega)\right\} .\end{aligned} \tag{5.43}$$

In the limit $n \to \infty$, we get

$$\mathcal{T}\left\{\mathbf{D}_{j,\omega} f(H_\omega)\right\} = \tfrac{i}{2} \mathcal{T}\left\{[F(H_\omega), x_j]\right\} = 0 \tag{5.44}$$

by Proposition 4.2(v). □

It is useful to rewrite the current (5.36), using (5.13) and the argument in (5.29), as

$$\begin{aligned}\mathbf{J}(\eta, \mathbf{E}; \zeta_\omega) &= \mathcal{T}\left\{2\mathbf{D}_\omega(0)\left(\varrho_\omega(0) - \zeta_\omega(0)\right)\right\} \\ &= -\mathcal{T}\left\{2 \int_{-\infty}^0 \mathrm{d}r\, \mathrm{e}^{\eta r} \mathbf{D}_\omega(0)\, \mathcal{U}(0, r)\left(i[\mathbf{E}\cdot\mathbf{x}, \zeta_\omega(r)]\right)\right\} ,\end{aligned} \tag{5.45}$$

which is justified, since

$$\mathcal{T}\left(\mathbf{D}_\omega(0)\zeta_\omega(0)\right) = \mathcal{T}\left(G(0)\mathbf{D}_\omega \zeta_\omega G(0)^*\right) = \mathcal{T}\left(\mathbf{D}_\omega \zeta_\omega\right) , \tag{5.46}$$

by cyclicity of the trace, and anyway all three terms are zero.

The conductivity tensor $\sigma(\eta; \zeta_\omega)$ is defined as the derivative (or differential) of the function $\mathbf{J}(\eta, \cdot; \zeta_\omega)\colon \mathbb{R}^d \to \mathbb{R}^d$ at $\mathbf{E} = 0$. Note that $\sigma(\eta; \zeta_\omega)$ is a $d \times d$ matrix $\{\sigma_{jk}(\eta; \zeta_\omega)\}$:

Definition 5.8. *For $\eta > 0$ the conductivity tensor $\sigma(\eta; \zeta_\omega)$ is defined as*

$$\sigma(\eta; \zeta_\omega) = \partial_{\mathbf{E}} \mathbf{J}(\eta, 0; \zeta_\omega) , \tag{5.47}$$

if it exists. The conductivity tensor $\sigma(\zeta_\omega)$ is defined by

$$\sigma(\zeta_\omega) := \lim_{\eta \downarrow 0} \sigma(\eta; \zeta_\omega) , \tag{5.48}$$

whenever the limit exists.

5.3. **Computing the linear response: a Kubo formula for the conductivity.** The next theorem gives a "Kubo formula" for the conductivity.

Theorem 5.9. *Let $\eta > 0$. The current $\mathbf{J}(\eta, \mathbf{E}; \zeta_\omega)$ is differentiable with respect to $\mathbf{E}$ at $\mathbf{E} = 0$ and the derivative $\sigma(\eta; \zeta_\omega)$ is given by*

$$\sigma_{jk}(\eta; \zeta_\omega) = -\mathcal{T}\left\{2 \int_{-\infty}^0 \mathrm{d}r\, \mathrm{e}^{\eta r} \mathbf{D}_{j,\omega}\, \mathcal{U}^{(0)}(-r)\left(i[x_k, \zeta_\omega]\right)\right\}, \tag{5.49}$$

where $\mathcal{U}^{(0)}(r)(A_\omega) = \mathrm{e}^{-irH_\omega} \odot_L A_\omega \odot_R \mathrm{e}^{irH_\omega}$.

We also have the analogue of [BES, Eq. (41)] and [SB2, Theorem 1]; $\mathcal{L}_1$ is the Liouvillian on $\overline{\mathcal{K}_1}$ (see Corollary 4.12).

Corollary 5.10. *The conductivity $\sigma_{jk}(\eta; \zeta_\omega)$ is given by*

$$\sigma_{jk}(\eta; \zeta_\omega) \quad = \quad -\mathcal{T}\left\{2\mathbf{D}_{j,\omega}\left(i\mathcal{L}_1 + \eta\right)^{-1}\left(i[x_k, \zeta_\omega]\right)\right\}, \tag{5.50}$$

Proof. Since $H_\omega[x_k, \zeta_\omega] \in \mathcal{K}_1 \cap \mathcal{K}_2$ by Lemma 5.4(ii), we have

$$\begin{aligned} \mathbf{D}_{j,\omega}\, \mathcal{U}^{(0)}(-r)\,(i[x_k, \zeta_\omega]) &= \mathbf{D}_{j,\omega}(H_\omega + \gamma)^{-1} \odot_L (H_\omega + \gamma)\mathcal{U}^{(0)}(-r)\,(i[x_k, \zeta_\omega]) \\ &= \mathbf{D}_{j,\omega}(H_\omega + \gamma)^{-1} \odot_L \mathcal{U}^{(0)}(-r)\,((H_\omega + \gamma)i[x_k, \zeta_\omega])\,, \end{aligned} \tag{5.51}$$

and it follows from (5.49) that

$$\begin{aligned} \sigma_{jk}(\eta; \zeta_\omega) &= -2\,\mathcal{T}\left\{\mathbf{D}_{j,\omega}(H_\omega + \gamma)^{-1} \odot_L (i\mathcal{L}_1 + \eta)^{-1}\,((H_\omega + \gamma)i[x_k, \zeta_\omega])\right\} \\ &= -2\,\mathcal{T}\left\{\mathbf{D}_{j,\omega}(i\mathcal{L}_1 + \eta)^{-1}\,(i[x_k, \zeta_\omega])\right\}, \end{aligned} \tag{5.52}$$

since $(i\mathcal{L}_1 + \eta)^{-1}\,((H_\omega + \gamma)i[x_k, \zeta_\omega])$ and $(i\mathcal{L}_1 + \eta)^{-1}\,(i[x_k, \zeta_\omega])$ are in $\mathcal{K}_1 \cap \mathcal{K}_2$ and hence in $\mathcal{K}_1$ (not just in $\overline{\mathcal{K}_1}$), where

$$(H_\omega + \gamma)^{-1} \odot_L (i\mathcal{L}_1 + \eta)^{-1}\,((H_\omega + \gamma)i[x_k, \zeta_\omega]) = (i\mathcal{L}_1 + \eta)^{-1}\,(i[x_k, \zeta_\omega])\,. \tag{5.53}$$

□

Proof of Theorem 5.9. From (5.45) and $\mathbf{J}_j(\eta, 0; \zeta_\omega) = 0$ (Lemma 5.7), we have

$$\sigma_{jk}(\eta; \zeta_\omega) = -\lim_{E\to 0} 2\mathcal{T}\left\{\int_{-\infty}^{0} \mathrm{d}r\, \mathrm{e}^{\eta r}\mathbf{D}_{j,\omega}(0)\mathcal{U}(0, r)\,(i[x_k, \zeta_\omega(r)])\right\}, \tag{5.54}$$

where $\mathbf{D}_{j,\omega}(0) = \mathbf{D}_{j,\omega}(\mathbf{E}, 0)$ and $\zeta_\omega(r) = \zeta_\omega(\mathbf{E}, r)$ depend on $\mathbf{E}$ through the gauge transformation $\mathcal{G}$ and $U_\omega(0, r) = U_\omega(\mathbf{E}, 0, r)$ also depends on $\mathbf{E}$. (For clarity, in this proof we display the argument $\mathbf{E}$ in all functions which depend on $\mathbf{E}$.)

Let us first understand that we can interchange integration and the limit $\mathbf{E} \to 0$, i.e., that

$$\sigma_{jk}(\eta; \zeta_\omega) \;=\; -2\int_{-\infty}^{0} \mathrm{d}r\, \mathrm{e}^{\eta r} \lim_{E\to 0} \mathcal{T}\left\{\mathbf{D}_{j,\omega}(\mathbf{E}, 0)\mathcal{U}(\mathbf{E}, 0, r)\,(i[x_k, \zeta_\omega(\mathbf{E}, r)])\right\}. \tag{5.55}$$

Note that

$$\begin{aligned} &\mathbf{D}_{j,\omega}(\mathbf{E}, 0)\mathcal{U}(\mathbf{E}, 0, r)\,(i[x_k, \zeta_\omega(\mathbf{E}, r)]) \\ &\quad= \left\{\mathbf{D}_{j,\omega}(\mathbf{E}, 0)(H_\omega(\mathbf{E}, 0) + \gamma)^{-1}(H_\omega(\mathbf{E}, 0) + \gamma)U_\omega(\mathbf{E}, 0, r)(H_\omega(\mathbf{E}, r) + \gamma)^{-1}\right\} \\ &\qquad\qquad \odot_L \left\{(H_\omega(\mathbf{E}, r) + \gamma)\,(i[x_k, \zeta_\omega(\mathbf{E}, r)])\right\} \odot_R U_\omega(\mathbf{E}, r, 0) \qquad (5.56) \\ &\quad= \left\{\mathcal{G}(\mathbf{E}, 0)\left(\mathbf{D}_{j,\omega}(H_\omega + \gamma)^{-1}\right)\right\} \odot_L W_\omega(\mathbf{E}, 0, r) \\ &\qquad\qquad \odot_L \left\{\mathcal{G}(\mathbf{E}, r)\,((H_\omega + \gamma)[ix_k, \zeta_\omega])\right\} \odot_R U_\omega(\mathbf{E}, r, 0)\,. \end{aligned}$$

Using (2.73), (4.35), gauge invariance of the norms, (2.81), (2.75), and Lemma 5.4(ii), we get

$$\begin{aligned} &\sup_{|E|\le 1, r\le 0} \left|\!\left|\!\left| \mathbf{D}_{j,\omega}(\mathbf{E}, 0)\mathcal{U}(\mathbf{E}, 0, r)\,(i[x_k, \zeta_\omega(\mathbf{E}, r)]) \right|\!\right|\!\right|_1 \qquad (5.57) \\ &\quad\le \left|\!\left|\!\left| \mathbf{D}_{j,\omega}(H_\omega + \gamma)^{-1} \right|\!\right|\!\right|_\infty \left\{\sup_{|E|\le 1, r\le 0} \left|\!\left|\!\left| W_\omega(\mathbf{E}, 0, r) \right|\!\right|\!\right|_\infty\right\} \left|\!\left|\!\left| (H_\omega + \gamma)[x_k, \zeta_\omega] \right|\!\right|\!\right|_1 < \infty\,. \end{aligned}$$

Eq. (5.55) follows from (5.54), (5.57), (3.99), and dominated convergence.

Next, we note that for any s we have

$$\lim_{E\to 0} \mathcal{G}(\mathbf{E}, s) = I \;\text{ strongly in } \mathcal{K}_1\,, \tag{5.58}$$

which can be proven by a argument similar to the one used to prove Lemma 4.13. Along the same lines, for $B_\omega \in \mathcal{K}_\infty$ we have

$$\lim_{E\to 0} \mathcal{G}(\mathbf{E},s)(B_\omega) = B_\omega \text{ strongly in } \mathcal{H}, \text{ with } \|\mathcal{G}(\mathbf{E},s)(B_\omega)\|_\infty = \|B_\omega\|_\infty . \quad (5.59)$$

It therefore follows from (5.56) that

$$\begin{aligned}
&\lim_{E\to 0} \mathcal{T}\left\{\mathbf{D}_{j,\omega}(\mathbf{E},0)\mathcal{U}(\mathbf{E},0,r)\left(i[x_k,\zeta_\omega(\mathbf{E},r)]\right)\right\} \qquad (5.60)\\
&\quad= \lim_{\mathbf{E}\to 0} \mathcal{T}\left\{(\mathbf{D}_{j,\omega} - \mathbf{F}_j(0))\, U_\omega(\mathbf{E},0,r)(H_\omega(\mathbf{E},r)+\gamma)^{-1}\odot_L\right.\\
&\qquad\qquad \left.\odot_L (H_\omega+\gamma)[ix_k,\zeta_\omega]\odot_R U_\omega(\mathbf{E},r,0)\right\}\\
&\quad= \lim_{\mathbf{E}\to 0} \mathcal{T}\left\{\mathbf{D}_{j,\omega} U_\omega(\mathbf{E},0,r)(H_\omega(\mathbf{E},r)+\gamma)^{-1}\odot_L (H_\omega+\gamma)[ix_k,\zeta_\omega]\odot_R U_\omega^{(0)}(r)\right\}\\
&\quad= \lim_{\mathbf{E}\to 0} \mathcal{T}\left\{\mathbf{D}_{j,\omega} U_\omega(\mathbf{E},0,r)(H_\omega+\gamma)^{-1}\left\{(H_\omega+\gamma)(H_\omega(\mathbf{E},0)+\gamma)^{-1}\right\}\odot_L\right.\\
&\qquad\qquad \left.\odot_L (H_\omega+\gamma)[ix_k,\zeta_\omega]\odot_R U_\omega^{(0)}(r)\right\}\\
&\quad= \lim_{\mathbf{E}\to 0} \mathcal{T}\left\{\mathbf{D}_{j,\omega} U_\omega(\mathbf{E},0,r)(H_\omega+\gamma)^{-1}\odot_L (H_\omega+\gamma)[ix_k,\zeta_\omega]\odot_R U_\omega^{(0)}(r)\right\},
\end{aligned}$$

where we used (5.58), (2.92), the fact that $\mathbf{D}_{j,\omega}(\mathbf{E},0) = \mathbf{D}_{j,\omega} - \mathbf{F}_j(0)$, (2.72)-(2.73), and Lemma 3.19. (Technically, we have not shown convergence yet. This equation should be read as saying that if any of these limits exists, then they all exist and agree.)

To proceed it is convenient to introduce a cutoff so that we can deal with $\mathbf{D}_{j,\omega}$ as if it were in $\mathcal{K}_\infty$. Thus we pick $f_n \in C_c^\infty(\mathbb{R})$, real valued, $|f_n| \le 1$, $f_n = 1$ on $[-n,n]$. Using Proposition 2.3(i) and Lemma 3.19 we have

$$\begin{aligned}
&\mathcal{T}\left\{\mathbf{D}_{j,\omega} U_\omega(\mathbf{E},0,r)(H_\omega+\gamma)^{-1}\odot_L (H_\omega+\gamma)[ix_k,\zeta_\omega]\odot_R U_\omega^{(0)}(r)\right\} \qquad (5.61)\\
&= \lim_{n\to\infty} \mathcal{T}\left\{\mathbf{D}_{j,\omega} f_n(H_\omega) U_\omega(\mathbf{E},0,r)\odot_L [ix_k,\zeta_\omega]\odot_R U_\omega^{(0)}(r)\right\} \qquad (5.62)\\
&= \lim_{n\to\infty} \mathcal{T}\left\{U_\omega(\mathbf{E},0,r)\odot_L i[x_k,\zeta_\omega]\odot_R \left(U_\omega^{(0)}(r)\mathbf{D}_{j,\omega} f_n(H_\omega)\right)\right\} \qquad (5.63)\\
&= \lim_{n\to\infty} \mathcal{T}\left\{U_\omega(\mathbf{E},0,r)\odot_L ((H_\omega+\gamma)i[x_k,\zeta_\omega])^\ddagger\odot_R\right. \qquad (5.64)\\
&\qquad\qquad \left.\odot_R U_\omega^{(0)}(r)(H_\omega+\gamma)^{-1}\mathbf{D}_{j,\omega} f_n(H_\omega)\right\}\\
&= \mathcal{T}\left\{U_\omega(\mathbf{E},0,r)\odot_L ((H_\omega+\gamma)i[x_k,\zeta_\omega])^\ddagger\odot_R U_\omega^{(0)}(r)(H_\omega+\gamma)^{-1}\mathbf{D}_{j,\omega}\right\}, \qquad (5.65)
\end{aligned}$$

where we used Lemma 3.22 to go from (5.62) to (5.63). The step from (5.63) to (5.64) is justified because $(H_\omega+\gamma)^{-1}$ commutes with $U^{(0)}$. Finally, since $(H_\omega + \gamma)^{-1}\mathbf{D}_{j,\omega} \in \mathcal{K}_\infty$ (that is, its bounded closure is in $\mathcal{K}_\infty$), we can take the limit $n\to\infty$, using Lemma 3.19 again. (Note $(i[x_k,\zeta_\omega])^\ddagger = i[x_k,\zeta_\omega]$.)

Finally, combining (5.60) and (5.61)-(5.65), we get

$$\begin{aligned}
&\lim_{E\to 0} \mathcal{T}\left\{\mathbf{D}_{j,\omega}(\mathbf{E},0)\mathcal{U}(\mathbf{E},0,r)\left(i[x_k,\zeta_\omega(\mathbf{E},r)]\right)\right\} \qquad (5.66)\\
&\quad= \mathcal{T}\left\{U_\omega^{(0)}(-r)\odot_L ((H_\omega+\gamma)i[x_k,\zeta_\omega])^\ddagger\odot_R U_\omega^{(0)}(r)\left(\mathbf{D}_{j,\omega}(H_\omega+\gamma)^{-1}\right)^*\right\}\\
&\quad= \mathcal{T}\left\{\mathbf{D}_{j,\omega}(H_\omega+\gamma)^{-1}U^{(0)}(-r)\odot_L (H_\omega+\gamma)i[x_k,\zeta_\omega]\odot_R U_\omega^{(0)}(r)\right\} \qquad (5.67)\\
&\quad= \mathcal{T}\left\{\mathbf{D}_{j,\omega}\mathcal{U}^{(0)}(-r)\left(i[x_k,\zeta_\omega]\right)\right\}, \qquad (5.68)
\end{aligned}$$

where to obtain (5.67) we used (5.61)-(5.65) in the reverse direction, with $U_\omega^{(0)}(r)$ substituted for $U_\omega(\mathbf{E},0,r)$, and in the last step used again that $(H_\omega+\gamma)^{-1}$ commutes with $U^{(0)}(r)$.

The Kubo formula (5.49) now follows from (5.55) and (5.68). ☐

5.4. **The Kubo-Středa formula for the Hall conductivity.** Following [BES, AG], we now recover the well-known Kubo-Středa formula for the Hall conductivity at zero temperature. We write

$$\sigma_{j,k}^{(E_f)} = \sigma_{j,k}(P_\omega^{(E_F)}) \ , \text{ and } \ \sigma_{j,k}^{(E_f)}(\eta) = \sigma_{j,k}(\eta; P_\omega^{(E_F)}) \ . \tag{5.69}$$

Theorem 5.11. *If $\zeta_\omega = P_\omega^{(E_F)}$ is a Fermi projection satisfying (5.2), we have*

$$\sigma_{j,k}^{(E_F)} = -i\mathcal{T}\left\{P_\omega^{(E_F)} \odot_L \left[\left[x_j, P_\omega^{(E_F)}\right], \left[x_k, P_\omega^{(E_F)}\right]\right]_\diamond\right\} \tag{5.70}$$

for all $j,k=1,2,\dots,d$. As a consequence, the conductivity tensor is antisymmetric; in particular $\sigma_{j,j}^{(E_F)}=0$ for $j=1,2,\dots,d$.

Clearly the direct conductivity vanishes, $\sigma_{jj}^{(E_F)}=0$. Note that, if the system is time-reversible the off diagonal elements are zero in the region of localization, as expected.

Corollary 5.12. *Under the assumptions of Theorem 5.11, if $\mathbf{A}=0$ (no magnetic field), we have $\sigma_{j,k}^{(E_F)}=0$ for all $j,k=1,2,\dots,d$.*

Proof. Let J denote complex conjugation on $\mathcal{H}$, i.e., $J\varphi=\bar\varphi$, an antiunitary operator on $\mathcal{H}$. The time reversal operation is given by $\Theta(S)=JSJ$, where S is a self-adjoint operator (an observable). We have $J\mathcal{H}_c=\mathcal{H}_c$, and hence $\Theta(A_\omega)\varphi = JA_\omega J$ gives a complex conjugation on $\mathcal{K}_i$, $i=1,2,\infty$.

If $\mathbf{A}=0$, we have $\Theta(H_\omega)=H_\omega$, and thus $\Theta(f(H_\omega))=f(H_\omega)$ for any real valued Borel measurable function f. Moreover $\Theta(i[x_j,P_\omega^{(E_F)}]) = -i[x_j,P_\omega^{(E_F)}]$ and $\Theta([A_\omega,B_\omega]_\diamond) = [\Theta(A_\omega),\Theta(B_\omega)]_\diamond$. On the other hand if $A_\omega\in\mathcal{K}_1$ is symmetric, then $\mathcal{T}(\Theta(A_\omega))=\mathcal{T}(A_\omega)$. Since $P_\omega^{(E_F)}\odot_L i\left[i[x_j,P_\omega^{(E_F)}], i[x_k,P_\omega^{(E_F)}]\right]_\diamond \odot_R P_\omega^{(E_F)}$ is symmetric, it follows from Theorem 5.11 and the above remarks that

$$\begin{aligned}\sigma_{j,k}^{(E_F)} &= \mathcal{T}\left\{P_\omega^{(E_F)}\odot_L i\left[i[x_j,P_\omega^{(E_F)}], i[x_k,P_\omega^{(E_F)}]\right]_\diamond \odot_R P_\omega^{(E_F)}\right\} \\ &= -\mathcal{T}\left\{P_\omega^{(E_F)}\odot_L i\left[i[x_j,P_\omega^{(E_F)}], i[x_k,P_\omega^{(E_F)}]\right]_\diamond \odot_R P_\omega^{(E_F)}\right\} = -\sigma_{j,k}^{(E_F)},\end{aligned} \tag{5.71}$$

and hence $\sigma_{j,k}^{(E_F)}=0$. ☐

Before proving Theorem 5.11, we recall that under Assumption 5.1 the operator $[x_k,P_\omega^{(E_F)}]\in\mathcal{K}_1\cap\mathcal{K}_2$ is defined on $\mathcal{H}_c$ as $x_kP_\omega^{(E_F)} - P_\omega^{(E_F)}x_k$ thanks to (5.2).

Lemma 5.13. *We have (as operators on $\mathcal{H}_c$)*

$$\left[P_\omega^{(E_F)}, \left[P_\omega^{(E_F)}, [x_k,P_\omega^{(E_F)}]\right]_\odot\right]_\odot = [x_k,P_\omega^{(E_F)}]. \tag{5.72}$$

Proof. Since $P_\omega^{(E_F)}\in\mathcal{K}_\infty$ and $[x_k,P_\omega^{(E_F)}]\in\mathcal{K}_1\cap\mathcal{K}_2$, the left hand side of (5.72) makes sense in $\mathcal{K}_1$ and $\mathcal{K}_2$, and thus as an operator on $\mathcal{H}_c$.

Note that the orthogonal projection $1-P_\omega^{(E_F)}$ is in $\mathcal{K}_\infty$, although it is *not* in $\mathcal{K}_1$ or $\mathcal{K}_2$. Furthermore $(1-P_\omega^{(E_F)})\mathcal{H}_c\subset\mathcal{H}_c+P_\omega^{(E_F)}\mathcal{H}_c\subset\mathcal{D}(\mathbf{x})$. Thus $P_\omega^{(E_F)}x_k(1-P_\omega^{(E_F)})$

and $(1-P_\omega^{(E_F)})x_k P_\omega^{(E_F)}$ make sense as operators on $\mathcal{H}_c$ (almost surely), and we have

$$\left[x_k, P_\omega^{(E_F)}\right] \;=\; (1-P_\omega^{(E_F)})x_k P_\omega^{(E_F)} - P_\omega^{(E_F)} x_k (1-P_\omega^{(E_F)}) \quad \text{on } \mathcal{H}_c. \tag{5.73}$$

Since $P_\omega^{(E_F)}(1-P_\omega^{(E_F)})=0$, the right hand side of this expression is unchanged if we replace x_k by $[x_k, P_\omega^{(E_F)}]$ in the first term and by $-[x_k, P_\omega^{(E_F)}]$ in the second. As technically $[x_k, P_\omega^{(E_F)}]$ is defined on $\mathcal{H}_c$, we should introduce the products $\odot_{L,R}$ here. Thus,

$$\begin{aligned}\left[x_k, P_\omega^{(E_F)}\right] \;=\; & (1-P_\omega^{(E_F)}) \odot_L [x_k, P_\omega^{(E_F)}] \odot_R P_\omega^{(E_F)} \\ & + P_\omega^{(E_F)} \odot_L [x_k, P_\omega^{(E_F)}] \odot_R (1-P_\omega^{(E_F)}) \,. \end{aligned}\tag{5.74}$$

Now, given any $A_\omega \in \mathcal{K}_\odot$ we have

$$\left[P_\omega^{(E_F)}, A_\omega\right]_\odot \;=\; -\left[1-P_\omega^{(E_F)}, A_\omega\right]_\odot \,, \tag{5.75}$$

and thus

$$\begin{aligned}&\left[P_\omega^{(E_F)}, \left[P_\omega^{(E_F)}, A_\omega\right]_\odot\right]_\odot = \\ &\quad P_\omega^{(E_F)} \odot_L A_\omega \odot_R (1-P_\omega^{(E_F)}) \;+\; (1-P_\omega^{(E_F)}) \odot_L A_\omega \odot_R P_\omega^{(E_F)} \,,\end{aligned}\tag{5.76}$$

using that $P_\omega^{(E_F)} \odot (1-P_\omega^{(E_F)}) = 0$. Finally, (5.72) follows from (5.74) and (5.76). □

Remark 5.14. *(i)Eq.* (5.74) *appears in* [BES] *(and then in* [AG]*) as a key step in the derivation of the expression of the Hall conductivity.*
(ii) In (5.72) *we use crucially the fact that we work at temperature zero, i.e. that the initial density matrix is the orthogonal projection $P_\omega^{(E_F)}$. The argument does not go through at positive temperature.*

Proof of Theorem 5.11. We first regularize the velocity $\mathbf{D}_{j,\omega}$ with a smooth function $f_n \in \mathcal{C}_c^\infty(\mathbb{R})$, $|f_n| \le 1$, $f_n = 1$ on $[-n,n]$, so that $\mathbf{D}_{j,\omega} f_n(H_\omega) \in \mathcal{K}_1 \cap \mathcal{K}_2 \in \mathcal{K}_\infty$. We have, using the centrality of the trace $\mathcal{T}$ (see Lemma 3.22), that

$$\begin{aligned}\widetilde{\sigma}_{jk}^{(E_F)}(r) \;&:=\; -\mathcal{T}\left\{2\mathbf{D}_{j,\omega}\mathcal{U}^{(0)}(-r)(i[x_k, P_\omega^{(E_F)}])\right\} && (5.77)\\ &=\; -\lim_{n\to\infty} \mathcal{T}\left\{(2\mathbf{D}_{j,\omega} f_n(H_\omega)) \odot_L \mathcal{U}^{(0)}(-r)(i[x_k, P_\omega^{(E_F)}])\right\} \\ &=\; -\lim_{n\to\infty} \mathcal{T}\left\{\mathcal{U}^{(0)}(r)(2\mathbf{D}_{j,\omega} f_n(H_\omega)) \odot_L i[x_k, P_\omega^{(E_F)}]\right\}. && (5.78)\end{aligned}$$

Next, it follows from Lemma 3.22 that, for $A_\omega, B_\omega \in \mathcal{K}_\infty$ and $C_\omega \in \mathcal{K}_1$, we have

$$\mathcal{T}\left\{A_\omega \odot_L [B_\omega, C_\omega]_\odot\right\} = \mathcal{T}\left\{[A_\omega, B_\omega] \odot_L C_\omega\right\} \,. \tag{5.79}$$

It follows, on the account of Lemma 5.13, that

$$\begin{aligned}&\mathcal{T}\left\{\mathcal{U}^{(0)}(r)(2\mathbf{D}_{j,\omega} f_n(H_\omega)) \odot_L i[x_k, P_\omega^{(E_F)}]\right\} && (5.80)\\ &\;=\; \mathcal{T}\left\{\mathcal{U}^{(0)}(r)(2\mathbf{D}_{j,\omega} f_n(H_\omega)) \odot_L \left[P_\omega^{(E_F)}, \left[P_\omega^{(E_F)}, i[x_k, P_\omega^{(E_F)}]\right]_\odot\right]_\odot\right\} \\ &\;=\; \mathcal{T}\left\{\mathcal{U}^{(0)}(r)\left(\left[P_\omega^{(E_F)}, \left[P_\omega^{(E_F)}, 2\mathbf{D}_{j,\omega} f_n(H_\omega)\right]\right]\right) \odot_L i[x_k, P_\omega^{(E_F)}]\right\},\end{aligned}$$

where we used that $P_\omega^{(E_F)}$ commutes with $U_\omega^{(0)}$.

We now claim that

$$\left[P_\omega^{(E_F)}, 2\mathbf{D}_{j,\omega} f_n(H_\omega)\right] = \left[H_\omega, i[x_j, P_\omega^{(E_F)}]\right]_\ddagger \odot_R f_n(H_\omega)\,. \tag{5.81}$$

To see this, we use (5.16) to conclude that

$$\begin{aligned}\left[H_\omega, i[x_j, P_\omega^{(E_F)}]\right]_\ddagger \odot_R f_n(H_\omega) &= 2\left(\mathbf{D}_{j,\omega} P_\omega^{(E_F)}\right)^\ddagger \odot_R f_n(H_\omega) - 2\mathbf{D}_{j,\omega} P_\omega^{(E_F)} f_n(H_\omega)\\ &= 2\left(P_\omega^{(E_F)}\mathbf{D}_{j,\omega} f_n(H_\omega) - \mathbf{D}_{j,\omega} P_\omega^{(E_F)} f_n(H_\omega)\right) \\ &= 2\left(P_\omega^{(E_F)}\mathbf{D}_{j,\omega} f_n(H_\omega) - \mathbf{D}_{j,\omega} f_n(H_\omega) P_\omega^{(E_F)}\right)\,,\end{aligned} \tag{5.82}$$

which is just (5.81). Combining (5.78), (5.80), and (5.81), we get after taking $n\to\infty$,

$$\widetilde{\sigma}_{jk}^{(E_F)}(r) \;=\; -\mathcal{T}\left\{\mathcal{U}^{(0)}(r)\left(\left[P_\omega^{(E_F)}, \left[H_\omega, i[x_j, P_\omega^{(E_F)}]\right]_\ddagger\right]_\odot\right) \diamond i[x_k, P_\omega^{(E_F)}]\right\}. \tag{5.83}$$

Here it is useful to note that, by Proposition 2.3(i), the restriction to $\mathcal{H}_c$ of $\overline{\left[P_\omega^{(E_F)}, 2\mathbf{D}_{j,\omega}\right]}$ is in $\mathcal{K}_\infty\cap\mathcal{K}_1\cap\mathcal{K}_2$, and

$$\left[H_\omega, i[x_j, P_\omega^{(E_F)}]\right]_\ddagger = \overline{\left[P_\omega^{(E_F)}, 2\mathbf{D}_{j,\omega}\right]} \in \mathcal{K}_1\cap\mathcal{K}_2\,. \tag{5.84}$$

In addition, on $\mathcal{K}_i$, $i=1,2$, we have

$$P_\omega^{(E_F)} \odot_L (H_\omega i[x_j, P_\omega^{(E_F)}]) = H_\omega(P_\omega^{(E_F)} \odot_L i[x_j, P_\omega^{(E_F)}])\,, \tag{5.85}$$

and, on the account of Lemma 4.11,

$$(H_\omega i[x_j, P_\omega^{(E_F)}]) \odot_R P_\omega^{(E_F)} = H_\omega(i[x_j, P_\omega^{(E_F)}] \odot_R P_\omega^{(E_F)})\,. \tag{5.86}$$

It also follows from (5.85) and (5.86) that

$$H_\omega\left[P_\omega^{(E_F)}, i[x_j, P_\omega^{(E_F)}]\right]_\odot = \left[P_\omega^{(E_F)}, H_\omega i[x_j, P_\omega^{(E_F)}]\right]_\odot\,, \tag{5.87}$$

all terms being well defined in $\mathcal{K}_i$. Therefore,

$$\left[P_\omega^{(E_F)}, \left[H_\omega, i[x_j, P_\omega^{(E_F)}]\right]_\ddagger\right]_\odot = \left[H_\omega, \left[P_\omega^{(E_F)}, i[x_j, P_\omega^{(E_F)}]\right]_\odot\right]_\ddagger\,. \tag{5.88}$$

We thus get

$$\begin{aligned}\widetilde{\sigma}_{jk}^{(E_F)}(r) \;&=\; -\mathcal{T}\left\{\mathcal{U}_\omega^{(0)}(r)\left(\left[H_\omega, \left[P_\omega^{(E_F)}, i[x_j, P_\omega^{(E_F)}]\right]_\odot\right]_\ddagger\right) \diamond i[x_k, P_\omega^{(E_F)}]\right\}\\ &=\; -\left\langle\!\!\left\langle \mathrm{e}^{-ir\mathcal{L}_\omega}\mathcal{L}_2\left(\left[P_\omega^{(E_F)}, i[x_j, P_\omega^{(E_F)}]\right]_\odot\right), i[x_k, P_\omega^{(E_F)}]\right\rangle\!\!\right\rangle\,,\end{aligned} \tag{5.89}$$

where we used (3.102) and Corollary 4.12. Recall that $\langle\langle\cdot,\cdot\rangle\rangle$ is the inner product on $\mathcal{H}_2$ and $\mathcal{L}_2$ is the Liouvillian in $\mathcal{K}_2$ – the self-adjoint generator of the unitary group $\mathcal{U}^{(0)}(t)$. Combining (5.49), (5.77), and (5.89), we get

$$\sigma_{jk}^{(E_F)}(\eta) = -\left\langle\!\!\left\langle i\left(\mathcal{L}_2 + i\eta\right)^{-1}\mathcal{L}_2\left(\left[P_\omega^{(E_F)}, i[x_j, P_\omega^{(E_F)}]\right]_\odot\right), i[x_k, P_\omega^{(E_F)}]\right\rangle\!\!\right\rangle\,. \tag{5.90}$$

It follows from the spectral theorem (applied to $\mathcal{L}_2$) that

$$\lim_{\eta\to 0} (\mathcal{L}_2 + i\eta)^{-1} \mathcal{L}_2 = P_{(\operatorname{Ker}\mathcal{L}_2)^\perp} \text{ strongly in } \mathcal{K}_2 , \tag{5.91}$$

where $P_{(\operatorname{Ker}\mathcal{L}_2)^\perp}$ is the orthogonal projection onto $(\operatorname{Ker}\mathcal{L}_2)^\perp$. Moreover, we have

$$\left[P_\omega^{(E_F)}, i[x_j, P_\omega^{(E_F)}]\right]_\odot \in (\operatorname{Ker}\mathcal{L}_2)^\perp . \tag{5.92}$$

To see this, note that if $A_\omega \in \operatorname{Ker}\mathcal{L}_2$, then for all t we have $\mathcal{U}^{(0)}(r)(A_\omega) = A_\omega$, and hence $\mathrm{e}^{-itH_\omega} \odot_L A_\omega = A_\omega \odot_R \mathrm{e}^{-itH_\omega}$, so it follows that $f(H_\omega) \odot_L A_\omega = A_\omega \odot_R f(H_\omega)$ for all $f \in \mathcal{S}(\mathbb{R})$, i.e., $[A_\omega, f(H_\omega)]_\odot = 0$. An approximation argument using Lemma 3.9 gives $[A_\omega, P_\omega^{(E_F)}]_\odot = 0$. Thus

$$\left\langle\!\!\left\langle A_\omega, \left[P_\omega^{(E_F)}, i[x_j, P_\omega^{(E_F)}]\right]_\odot \right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle [A_\omega, P_\omega^{(E_F)}]_\odot, i[x_j, P_\omega^{(E_F)}] \right\rangle\!\!\right\rangle = 0 , \tag{5.93}$$

and (5.92) follows.

Combining (5.90), (5.91), (5.92), and Lemma 4.6, we get

$$\begin{aligned} \sigma_{j,k}^{(E_F)} &= i \left\langle\!\!\left\langle \left[P_\omega^{(E_F)}, i[x_j, P_\omega^{(E_F)}]\right]_\odot, i[x_k, P_\omega^{(E_F)}] \right\rangle\!\!\right\rangle \\ &= -i\mathcal{T}\left\{ \left[P_\omega^{(E_F)}, i[x_j, P_\omega^{(E_F)}]\right]_\odot \diamond i[x_k, P_\omega^{(E_F)}] \right\} \\ &= -i\mathcal{T}\left\{ P_\omega^{(E_F)} \odot_L \left[i[x_j, P_\omega^{(E_F)}], i[x_k, P_\omega^{(E_F)}]\right]_\diamond \right\} , \end{aligned} \tag{5.94}$$

which is just (5.70). The theorem is proved. □

Acknowledgement. We thank Jean Bellissard for many discussions on the Kubo formula. We are grateful to Daniel Lenz for the proof that $\mathcal{K}_\infty$ is a von Neumann algebra. We thank Vladimir Georgescu for bringing the inequality (2.7) to our attention.

References

[AENSS] Aizenman, M., Elgart, A., Naboko, S., Schenker, J.H., Stolz, G.: Moment Analysis for Localization in Random Schrödinger Operators. 2003 Preprint, math-ph/0308023.

[AES] Aizenman, M., Elgart, A., Schenker, J.H.: Adiabatic charge transport,localization, and the Kubo formula for $2D$ Hall conductance in a gapless state. In preparation.

[AG] Aizenman, M., Graf, G.M.: Localization bounds for an electron gas, J. Phys. A: Math. Gen. **31**, 6783-6806, (1998).

[AvSS] Avron, J., Seiler, R., Simon, B.: Charge deficiency, charge transport and comparison of dimensions. Comm. Math. Phys. **159**, 399-422 (1994).

[B] Bellissard, J.: Ordinary quantum Hall effect and noncommutative cohomology. In *Localization in disordered systems (Bad Schandau, 1986)*, pp. 61-74. Teubner-Texte Phys. **16**, Teubner, 1988.

[BES] Bellissard, J., van Elst, A., Schulz-Baldes, H.: The non commutative geometry of the quantum Hall effect. J. Math. Phys. **35**, 5373-5451 (1994).

[BeG] Berthier, A., Georgescu, V.: On the point spectrum of Dirac operators. J. Funct. Anal. **71**, 309-338 (1987).

[BoGK] Bouclet, J.M., Germinet, F., Klein, A.: Sub-exponential decay of operator kernels for functions of generalized Schrödinger operators. Proc. Amer. Math. Soc. **132** , 2703-2712 (2004).

[BrR] Bratteli, O., Robinson, D.W.: *Operator Algebras and Quantum Statitical Mechanics I.* Springer-Verlag, 1979.

[CH] Combes, J.M., Hislop, P.D.: Landau Hamiltonians with random potentials: localization and the density of states. Commun. Math. Phys. **177**, 603-629 (1996).

[CT] Combes, J.M., Thomas, L.: Asymptotic behaviour of eigenfunctions for multiparticle Schrödinger operators. Comm. Math. Phys. **34**, 251–270 (1973).

[CoJM] Cornean, H.D., Jensen, A., Moldoveanu, V.: A rigorous proof for the Landauer-Büttiker formula. Preprint on mp_arc 04-71.

[CyFKS] Cycon, H.L., Froese, R.G., Kirsch, W., Simon, B.: *Schrödinger operators*. Heidelberg: Springer-Verlag, 1987.

[D] Davies, E.B.: Kernel estimates for functions of second order elliptic operators. Quart. J. Math. Oxford (2) **39**, 37-46 (1988).

[ES] Elgart, A. , Schlein, B.: Adiabatic charge transport and the Kubo formula for Landau Type Hamiltonians. Comm. Pure Appl. Math. 57, 590-615 (2004).

[F] Faris, W.G.: *Self-Adjoint Operators*. Lecture Notes in Mathematics **433**. Springer-Verlag, 1975.

[GK1] Germinet, F., Klein, A.: Bootstrap Multiscale Analysis and Localization in Random Media. Commun. Math. Phys. **222**, 415-448 (2001).

[GK2] Germinet, F., Klein, A.: Operator kernel estimates for functions of generalized Schrödinger operators. Proc. Amer. Math. Soc. 131, 911-920 (2003).

[GK3] Germinet, F., Klein, A.: A characterization of the Anderson metal-insulator transport transition. Duke Math. J. **124** (2004).

[GK4] Germinet, F, Klein, A.: Explicit finite volume criteria for localization in continuous random media and applications. Geom. Funct. Anal. 13, 1201-1238 (2003).

[HS] Helffer, B., Sjöstrand, J.: Equation de Schrödinger avec champ magnétique et équation de Harper. In *Schrödinger Operators*, H. Holden and A. Jensen, eds., pp. 118-197. Lectures Notes in Physics **345**, Springer-Verlag, 1989.

[HuS] Hunziker W., Sigal, I.M.: Time-dependent scattering theory for N-body quantum systems. Rev. Math. Phys. **12**, 1033-1084 (2000).

[K] Kato, T.: *Perturbation Theory for Linear Operators*. Springer-Verlag, 1976.

[KeS] Kellendonk, J., Schulz-Baldes, H., Quantization of edge currents for continuous magnetic operators. J. Funct. Anal. **209**, 388-413 (2004).

[Ku] Kunz, H.: The Quantum Hall Effect for Electrons in a Random Potential. Commun. Math. Phys. **112**, 121-145 (1987).

[LS] H. Leinfelder, C.G. Simader, *Schrödinger operators with singular magnetic potentials*, Math. Z. 176, 1-19 (1981).

[NB] Nakamura, S., Bellissard, J.: Low Energy Bands do not Contribute to Quantum Hall Effect. Commun. Math. Phys. **131**, 283-305 (1990).

[Na] Nakano, F.: Absence of transport in Anderson localization. Rev. Math. Phys. **14**, 375-407 (2002).

[P] Pastur, L., Spectral properties of disordered systems in the one-body approximation. Comm. Math. Phys. **75**, 179-196 (1980).

[PF] Pastur, L., Figotin, A.: *Spectra of Random and Almost-Periodic Operators*. Springer-Verlag, 1992.

[RS1] Reed, M., Simon, B.: *Methods of Modern Mathematical Physics I: Functional Analysis*, revised and enlarged edition. Academic Press, 1980.

[RS2] Reed, M., Simon, B.: *Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness*. Academic Press, 1975.

[RS4] Reed, M., Simon, B.: *Methods of Modern Mathematical Physics IV: Analysis of Operators*. Academic Press, 1978.

[SB1] Schulz-Baldes, H., Bellissard, J.: Anomalous transport: a mathematical framework. Rev. Math. Phys. **10**, 1-46 (1998).

[SB2] Schulz-Baldes, H., Bellissard, J.: A Kinetic Theory for Quantum Transport in Aperiodic Media. J. Statist. Phys. **91**, 991-1026 (1998).

[Si1] Simon, B.: Maximal and minimal Schrödinger forms. J. Operator Theory **1**, 37-47 (1979).

[Si2] Simon, B.: Schrödinger semi-groups. Bull. Amer. Math. Soc. **7**, 447-526 (1982).

[St] Středa, P.: Theory of quantised Hall conductivity in two dimensions. J. Phys. C. **15**, L717-L721 (1982).

[T] Takesaki, M.: *Theory of Operator Algebras I*. Springer-Verlag, 1979.

[ThKNN] Thouless, D. J., Kohmoto, K., Nightingale, M. P., den Nijs, M.: Quantized Hall conductance in a two-dimensional periodic potential. Phys. Rev. Lett. **49**, 405-408 (1982).

[W] Wang, W.-M.: Microlocalization, percolation, and Anderson localization for the magnetic Schrödinger operator with a random potential. J. Funct. Anal. **146**, 1-26 (1997).

[Y] Yosida, K.: *Functional Analysis, 6th edition.* Springer-Verlag, 1980.

(Bouclet) UMR 8524 CNRS, UFR de Mathématiques, Université de Lille 1, F-59655 Villeneuve d'Ascq Cédex, France

E-mail address: Jean-Marc.Bouclet@agat.univ-lille1.fr

(Germinet) Université de Cergy-Pontoise, Département de Mathématiques, Site de Saint-Martin, 2 avenue Adolphe Chauvin, 95302 Cergy-Pontoise cedex, France

E-mail address: germinet@math.u-cergy.fr

(Klein) University of California, Irvine, Department of Mathematics, Irvine, CA 92697-3875, USA

E-mail address: aklein@uci.edu

(Schenker) ETH Zürich,Theoretische Physik, CH-8093 Zürich, Switzerland

E-mail address: jschenker@itp.phys.ethz.ch