Transfer matrices, hyperbolic geometry and absolutely continuous spectrum for some discrete Schrödinger operators on graphs

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Abstract

We prove the existence of absolutely continuous spectrum for a class of discrete Schrödinger operators on tree like graphs. We consider potentials whose radial behaviour is subject only to an ℓ^∞ bound. In the transverse direction the potential must satisfy a condition such as periodicity. The graphs we consider include binary trees and graphs obtained from a binary tree by adding edges, possibly with weights. Our methods are motivated by the one dimensional transfer matrix method, interpreted as a discrete dynamical system on the hyperbolic plane. This is extended to more general graphs, leading to a formula for the Green's function. Bounds on the Green's function then follow from the contraction properties of the transformations that arise in this generalization. The bounds imply the existence of absolutely continuous spectrum.

1. Introduction

One of the most important open problems in the field of random Schrödinger operators is to prove the existence of absolutely continuous spectrum for weak disorder in the Anderson model in three and higher dimensions (See [CL], [FP] or [CKFS] for general information about the mathematical theory of random Schrödinger operators). One of the few results in this direction is Abel Klein's, for random Schrödinger operators acting on a tree, or Bethe Lattice. Klein [Kl] proves that for weak disorder, almost all potentials will produce absolutely continuous spectrum. This means that there must be many potentials on a tree where the corresponding Schrödinger operator has absolutely continuous spectrum without there being an obvious reason, such as periodicity or decrease at infinity. The goal of this work is to prove a deterministic result for some of these potentials.

Here is an outline of the paper. After this introduction we review, in section 2, the transfer matrix method for a one-dimensional problem. We notice that viewed projectively, the transfer matrices correspond to contractions of the hyperbolic plane. This leads to formula (2.6) for the Green's function. In section 3 we generalize this method to a large class of graphs, including trees and $\mathbb{Z}^n$. The hyperbolic plane is now replaced by a sequence of Siegel upper half spaces. A key issue is the choice of metric for the Siegel upper half spaces. It turns out that the maps that arise in our method are not contractions with respect to the standard Riemannian metric. However, they do contract with respect to a Finsler metric, whose distance function we identify in an appendix as one of a family of naturally occuring distance functions (see [F]). The end result of this section is formula (3.13) for the Green's function. This formula shows that the Green's function at the origin can be approximated by the image of an arbitrary starting point under a series of contractions on Siegel upper half spaces.

Our main results are contained in section 4. Here we restrict to a class of tree like graphs and to potentials satisfying a periodicity or decay condition in the direction transverse to the radial direction. Given such a restriction we can allow arbitrary small fluctuations in the radial direction and still control the image of the maps in (3.13). This results in bounds on the Green's function. We conclude this section with some results on potentials that decay in the radial direction at infinity. We first show how to prove results similar to existing ones (see [AF] [GG]) obtained using the Mourre estimate. Then we show that our method of proving the boundedness of the Green's function allows suitable decreasing perturbations.

Now we fix our notation. Let $G = (V, E)$ be a graph with vertices V and edges E. The discrete Laplacian Δ acting on functions $\varphi : V \to \mathbb{C}$ is defined by

$$(\Delta\varphi)(v) = \sum_{w:\mathrm{dist}(v,w)=1} (\varphi(v) - \varphi(w))\,.$$

Here dist denotes the distance in the graph, so that the sum is over all nearest neighbours of v. Notice that according to this definition $\Delta \geq 0$. There is also a Laplacian associated to each weight function γ defined on the edges of the graph. Given a weight $\gamma_{vw} > 0$ for each edge $vw \in E$ running between the vertices v and w, the corresponding Laplacian is defined by

$$(\Delta_\gamma\varphi)(v) = \sum_{w:\mathrm{dist}(v,w)=1} \gamma_{vw}(\varphi(v) - \varphi(w))\,.$$

A potential is a real valued function q on V acting on functions $\varphi : V \to \mathbb{C}$ diagonally, that is, as a multiplication operator. Thus

$$(q\varphi)(v) = q(v)\varphi(v)\,.$$

A discrete Schrödinger operator is an operator of the form $H = \Delta + q$, or $H = \Delta_\gamma + q$. If the number of edges meeting a vertex is uniformly bounded on the graph, and the potential is bounded, then H is a bounded self-adjoint operator on $\ell^2(V)$.

We are interested in the spectral properties of H. These can be deduced from the Green's function

$$G(v, w; \lambda) = \langle \chi_v, (H - \lambda)^{-1} \chi_w \rangle \,.$$

Here χ_w denotes the characteristic function given by

$$\chi_w(v) = \begin{cases} 1 & \text{if } v = w \\ 0 & \text{otherwise} \end{cases} \,. \tag{1.1}$$

By Stone's formula, the spectral measure μ_{χ_v} for χ_v satisfies

$$\mu_{\chi_v}((a, b)) \leq \lim_{\epsilon \to 0} \frac{1}{\pi} \int_a^b \operatorname{Im}(G(v, v, \mu + i\epsilon)) d\mu \,,$$

with equality unless a or b is an eigenvalue of H. Thus, as is well known, to prove absolute continuity of the spectral measure μ_{χ_v} in some interval (a, b) it suffices to show

$$\limsup_{\epsilon \to 0} \sup_{\mu \in (a,b)} |G(v, v, \mu + i\epsilon))| \leq C \,.$$

For fixed $w \in V$ and λ in the resolvent set of H, the Green's function $G(v, w; \lambda)$ as a function of v can be characterized as the unique ℓ^2 solution to

$$(\Delta + q - \lambda) G(v, w; \lambda) = \chi_w(v) \,, \tag{1.2}$$

where χ_w is given by (1.1). In this paper we consider situations where we can compute all the solutions to (1.2). To find the Green's function we must then identify which one of these solutions decreases rapidly enough to be in ℓ^2. The simplest example of such a situation is the half line.

2. The half line

We will now explain how to compute the Green's function at the origin for the half line with a bounded potential, using the transfer matrix formalism. Since our method for more general graphs is based on these ideas, we review them in some detail. In this one dimensional situation we cannot expect to be able to control the imaginary part of the Green's function as λ approaches the real axis for arbitrary bounded potentials. Such control implies the existence of absolutely continuous spectrum, which is absent for typical bounded potentials in one dimension [KS]. See [LS] for recent work on absolutely continuous spectrum and the transfer matrix method for one dimensional problems.

Consider the graph $\mathbb{Z}^+$ with vertices $0, 1, 2, 3, \ldots$ and edges joining neighbouring integers. Let $q = (q_0, q_1, q_2, \ldots)$ be bounded potential on $\mathbb{Z}^+$. To streamline the notation we redefine Δ by subtracting 2 from the diagonal everywhere except at the first position, where we subtract 1. This amounts to shifting λ by 2 and q_0 by 1. Then

$$(\Delta + q - \lambda)\varphi = \chi_0 \iff \begin{cases} -\varphi_1 + (q_0 - \lambda)\varphi_0 - 1 = 0 \\ -\varphi_{n+1} + (q_n - \lambda)\varphi_n - \varphi_{n-1} = 0 \quad \text{for } n > 0 \end{cases} . \tag{2.1}$$

Here $\varphi = (\varphi_0, \varphi_1, \varphi_2, \ldots)$. This implies that φ solves this equation if and only if

$$\begin{bmatrix} \varphi_{n+1} \\ \varphi_n \end{bmatrix} = A_n A_{n-1} \cdots A_0 \begin{bmatrix} \varphi_0 \\ 1 \end{bmatrix}, \tag{2.2}$$

where, for $n \geq 0$, $A_n \in SL(2, \mathbb{C})$ is the matrix

$$A_n = \begin{bmatrix} q_n - \lambda & -1 \\ 1 & 0 \end{bmatrix}.$$

This formula shows how to compute $\begin{bmatrix} \varphi_{n+1} \\ \varphi_n \end{bmatrix}$ from $\begin{bmatrix} \varphi_0 \\ 1 \end{bmatrix}$. There is only one choice of φ_0 that results in a decreasing solution. With this choice of φ_0 the resulting solution φ_n is $G(n, 0, \lambda)$.

An equivalent formula is

$$\begin{bmatrix} \varphi_0 \\ 1 \end{bmatrix} = A_0^{-1} A_1^{-1} \cdots A_n^{-1} \begin{bmatrix} \varphi_{n+1} \\ \varphi_n \end{bmatrix}. \tag{2.3}$$

This formula shows how to recover $\begin{bmatrix} \varphi_0 \\ 1 \end{bmatrix}$ from $\begin{bmatrix} \varphi_{n+1} \\ \varphi_n \end{bmatrix}$. Although formulas (2.2) and (2.3) are mathematically equivalent, there is a big difference if we were to try actually using them in computation. To see why, consider the case where $q = (0, 0, 0, 0, \ldots)$ and $\lambda \in \mathbb{C} \backslash [-2, 2]$. In this case, each matrix A_i appearing in (2.2) is the same. Lets call this matrix A. For $\lambda \in \mathbb{C} \backslash [-2, 2]$ the matrix A has one eigenvalue inside the unit circle and one outside. Therefore, we must choose $\begin{bmatrix} \varphi_0 \\ 1 \end{bmatrix}$ to lie in the direction of the eigenvector of A corresponding to the small eigenvalue. This choice will result in an exponentially decreasing solution, which must be the Green's function. However, this choice also means that the left side of (2.2) is very sensitive to the input vector $\begin{bmatrix} \varphi_0 \\ 1 \end{bmatrix}$. A small deviation of the input value from its true value will result in a big change in the computed value of $\begin{bmatrix} \varphi_{n+1} \\ \varphi_n \end{bmatrix}$. In contrast to this, the output vector $\begin{bmatrix} \varphi_0 \\ 1 \end{bmatrix}$ in formula (2.3) is the eigenvector corresponding to the *large* eigenvalue of the matrix A^{-1} appearing in that formula. This means that the output vector is insensitive to changes in the input vector. In fact, when n is large, changing the true input vector $\begin{bmatrix} \varphi_{n+1} \\ \varphi_n \end{bmatrix}$ to anything at all (except the other eigenvector) will result in approximately the same output vector, up to a scalar multiple.

This observation can be turned into a method of computing the Green's function. To get rid of the ambiguity of a scalar multiple, consider the equation (2.3) projectively. From now on let φ_n denote the decreasing solution $G(n, 0, \lambda)$ and define $\alpha_{n+1} \in \hat{\mathbb{C}}$ by

$$\alpha_{n+1} = \varphi_{n+1}/\varphi_n .$$

Then, equation (2.3) implies that for any n,

$$\varphi_0 = \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_n(\alpha_{n+1})\,, \tag{2.4}$$

where Φ_n denotes the Möbius transformation associated with the matrix $A_n^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & q_n - \lambda \end{bmatrix}$, that is,

$$\Phi_n(z) = \frac{1}{-z + q_n - \lambda}\,.$$

We now focus on the mapping properties of Φ_n. Let $\mathbb{H} = \{x + iy : y > 0\}$ denote the complex upper half plane equipped with the Poincaré metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}\,.$$

Proposition 2.1

(i) For $\operatorname{Im}(\lambda) \geq 0$, the map Φ_n maps $\mathbb{H}$ into $\mathbb{H}$. When $\operatorname{Im}(\lambda) > 0$, the image lies in the hyperbolic half plane $\{z \in \mathbb{H} : |z| \leq 1/\operatorname{Im}(\lambda)\}$

(ii) When $\lambda \in \mathbb{R}$, the map Φ_n is a hyperbolic isometry. When $\operatorname{Im}(\lambda) > 0$, the map Φ_n is a hyperbolic contraction.

(iii) If z_1 and z_2 lie in the hyperbolic half plane $H = \{z \in \mathbb{H} : |z| \leq C\}$ and $\operatorname{Im}(\lambda) > 0$ then there exists a positive constant $\delta < 1$, depending only on C and $\operatorname{Im}(\lambda)$ such that

$$\operatorname{dist}(\Phi_n(z_1), \Phi_n(z_2)) \leq \delta \operatorname{dist}(z_1, z_2)\,.$$

Proof: It is convenient to factor Φ_n as

$$\Phi_n = \rho \circ \tau_n\,,$$

where $\tau_n : z \mapsto z - q_n + \lambda$ and $\rho : z \mapsto -z^{-1}$. The map τ_n is a translation that maps $\mathbb{H}$ into $\mathbb{H}$ when $\operatorname{Im}(\lambda) \geq 0$. The map ρ is a hyperbolic isometry, a rotation by an angle of π about i. Thus the composition lies in $\mathbb{H}$. Clearly

$$|\Phi_n(z)| = \frac{1}{|-z + q_n - \lambda|} < \frac{1}{\operatorname{Im}(z) + \operatorname{Im}(\lambda)} < \frac{1}{\operatorname{Im}(\lambda)}\,.$$

When $\lambda \in \mathbb{R}$, the translation τ_n, and hence also Φ_n is a hyperbolic isometry. However, when $\operatorname{Im}(\lambda) > 0$ then τ_n is a hyperbolic contraction. To show this, it suffices to show that $d\tau_n$ decreases the length of tangent vectors measured in the hyperbolic metric. If $w \in \mathbb{C}$ represents a tangent vector at $z \in \mathbb{H}$, then its length in the Riemannian norm is $|w|/\operatorname{Im}(z)$. Since $d\tau_n(z)[w] = w$ we have

$$\|d\tau_n(z)[w]\|_{\tau_n(z)} = \frac{|w|}{\operatorname{Im}(\tau_n(z))} = \frac{|w|}{\operatorname{Im}(z - q_n + \lambda)} = \frac{|w|}{\operatorname{Im}(z) + \operatorname{Im}(\lambda)} < \frac{|w|}{\operatorname{Im}(z)} = \|w\|_z\,.$$

This implies that τ_n, and thus Φ_n decreases hyperbolic distances when $\mathrm{Im}(\lambda) > 0$.

When $z \in H$ we have $\mathrm{Im}(z) \le C$. Thus, for $\mathrm{Im}(\lambda) > 0$

$$\begin{aligned}
\delta &= \sup_{z\in H} \frac{\|d\Phi_n(z)[w]\|_{\Phi_n(z)}}{\|w\|_z} \\
&= \sup_{z\in H} \frac{\|d\tau_n(z)[w]\|_{\tau_n(z)}}{\|w\|_z} \\
&= \sup_{z\in H} \frac{\mathrm{Im}(z)}{\mathrm{Im}(z) + \mathrm{Im}(\lambda)} \\
&< C/(C + \mathrm{Im}(\lambda)) < 1\,.
\end{aligned} \tag{2.5}$$

Now suppose that $z_1, z_2 \in H$ and that $\gamma(t)$ for $t \in [0, T]$ is the geodesic joining z_1 and z_2. Then the entire geodesic γ lies in H. Also, the path $\gamma_1(t) = \Phi_n(\gamma(t))$ joins $\Phi_n(z_1)$ and $\Phi_n(z_2)$. Thus we can use (2.5) to estimate

$$\begin{aligned}
\mathrm{dist}(\Phi_n(z_1), \Phi_n(z_2)) &\le \int_0^T \|\dot\gamma_1(t)\|_{\gamma_1(t)} dt \\
&= \int_0^T \|d\Phi_n(\gamma(t))[\dot\gamma(t)]\|_{\Phi_n(\gamma(t))} dt \\
&\le \delta \int_0^T \|\dot\gamma(t)\|_{\gamma(t)} dt \\
&= \delta\, \mathrm{dist}(z_1, z_2)\,.
\end{aligned}$$

□

Remark: The fact that Φ_n is a contraction also follows from the Schwarz Lemma in classical function theory [Kr], which asserts that any analytic map from $\mathbb{H}$ into $\mathbb{H}$ is a contraction. This contraction is strict unless the map is a hyperbolic isometry.

The previous proposition shows that one application of any Φ_n sends $\mathbb{H}$ into a half space. The next proposition shows that two applications send $\mathbb{H}$ into a bounded set. For this proposition, we need the boundedness of the potential.

Proposition 2.2 *Assume* $\mathrm{Im}(\lambda) > 0$*. Given a uniform bound* $\sup_n |q_n| \le C$*, there is a fixed hyperbolic ball* $B \subset \mathbb{H}$*, depending on* $\mathrm{Im}(\lambda)$ *and* C*, such that* $\Phi_n \circ \Phi_{n+1}(\mathbb{H}) \subseteq B$

Proof: Figure 1 illustrates the action of two iterations of Φ on the upper half plane when $\mathrm{Im}(\lambda) > 0$. The first translation shifts the upper half plane horizontally, with no effect, and then up by a vertical Euclidean distance $\mathrm{Im}(\lambda)$. The image is a horosphere at infinity which then is rotated to a horosphere at zero. The second translation moves this horosphere up by a vertical Euclidean distance $\mathrm{Im}(\lambda)$, as before, followed by a horizontal translation whose size depends on the values of q and $\mathrm{Re}(\lambda)$. The diagram shows three possibilities. Under the boundedness assumption on the potential, any possible image can be fit in a fixed hyperbolic ball B, depending on $\mathrm{Im}(\lambda)$ and C, about i. A subsequent rotation about i keeps the images inside this ball. □

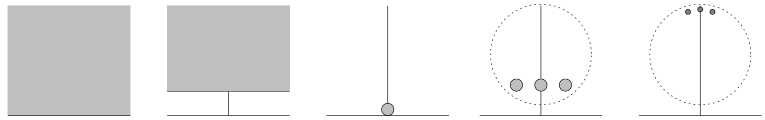

Figure 1

Now we state the promised formula for the Green's function at 0.

Theorem 2.3 *Suppose* $\mathrm{Im}(\lambda) > 0$ *and assume* $\sup_n |q_n| \le C$. *Let* $\gamma_n \in \mathbb{H}$, $n = 1, 2, \ldots$, *be arbitrary. Then*

$$\lim_{n\to\infty} \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_n(\gamma_n) = \varphi_0 = G(0,0;\lambda)\,. \tag{2.6}$$

Proof: Let $w_n = \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_n(\gamma_n)$. We must show that $w_n \to \varphi_0$. We have that

$$w_n = \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_{n-2}(\beta)\,,$$

where $\beta = \Phi_{n-1} \circ \Phi_n(\gamma_n) \in B$. Here B is the hyperbolic ball in Proposition 2.2. Similarly,

$$w_{n+1} = \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_{n-1}(\beta')$$

for some $\beta' \in B$. Now each image of β and of β' under subsequent applications of Φ_k all lie in a fixed hyperbolic half space (in fact, in B). So

$$\begin{aligned}
\mathrm{dist}(w_{n+1}, w_n) &= \mathrm{dist}(\Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_{n-1}(\beta), \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_{n-1}(\beta')) \\
&\le \delta \mathrm{dist}(\Phi_1 \circ \cdots \circ \Phi_{n-2}(\beta), \Phi_1 \circ \cdots \circ \Phi_{n-2}(\beta')) \\
&\le \cdots \\
&\le \delta^{n-1}\mathrm{dist}(\beta, \beta') \\
&= C\delta^n\,.
\end{aligned}$$

In the last step we used that $\beta, \beta' \in B$. This implies that w_n is Cauchy and hence converges. A similar estimate shows that any two such sequences will converge to the same limit. Equation (2.4) shows that this limit must be φ_0. $\square$

Remark 1: Theorem 2.3 shows that the Green's function $\varphi_0 = G(0,0;\lambda)$ must lie in the ball B. Unfortunately this ball expands to fill out all of $\mathbb{H}$ as $\mathrm{Im}(\lambda) \to 0$, so this bound is not very useful. This is not surprising, since we know that in one dimension there typically is no uniform bound on the Green's function.

Remark 2: It is interesting to note for $\mathrm{Im}(\lambda)$ large, the value of the Green's function $G(0,0;\lambda)$ is determined to a good approximation by the values of the potential near 0. The reason for this is that if $\mathrm{Im}(\lambda)$ is large then the maps Φ_n will all be strongly contracting, and the numbers w_n will be close to their limit after only a few steps.

Remark 3: Theorem 2.3 implies that the contracting fixed points of the Möbius transformations $\Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_n$, which all lie in the upper half plane, converge to $G(0,0;\lambda)$. In contrast, the expanding fixed points, corresponding to the "bad" eigenvector of the matrix product in (2.3), all lie in the lower half plane and do not converge.

The numbers α_n are discrete analogues of the Dirichlet to Neumann map. They also can be identified as values of a Green's function for a truncated graph. We digress to explain this, since we will use something similar in the next section. First notice that we could define α_i to be the unique number such that for any choice of φ_{i-1}, the numbers φ_n for $n \geq i$ defined by $\varphi_i = \alpha_i \varphi_{i-1}$ and

$$\begin{bmatrix} \varphi_{n+1} \\ \varphi_n \end{bmatrix} = A_n \cdots A_i \begin{bmatrix} \alpha_i \varphi_{i-1} \\ \varphi_{i-1} \end{bmatrix}$$

are decreasing at infinity. The numbers so defined satisfy

$$-\varphi_{n+1} + (q_n - \lambda)\varphi_n + \varphi_{n-1} = 0 \tag{2.7}$$

for $n \geq i$. We can interpret this as an equation on the truncated graph $\{i, i+1, i+2, \ldots\}$. Let Δ_i be the Laplacian for this graph. Then for $n > i$ equation (2.7) simply says that $((\Delta_i + q - \lambda)\varphi)_n = 0$. However, since there is no $i-1$th site on the truncated graph, equation (2.7) for $n = i$ says $((\Delta_i + q - \lambda)\varphi)_i = \varphi_{i-1}$. In particular, if we have chosen $\varphi_{i-1} = 1$ then φ is the Green's function for the truncated graph, and $\varphi_i = \alpha_i$.

3. A formula for the Green's function on more general graphs

Now we generalize the one dimensional transfer matrix formalism to a more general class of graphs. For this class of graphs we can follow the same strategy as in one dimension to obtain a formula for the Green's function similar to (2.6).

We begin by introducing the graph analogue of polar co-ordinates for a graph (V, E). Pick an origin $0 \in V$ and define the spheres

$$S_n = \{v \in V : \mathrm{dist}(v, 0) = n\},$$

where dist denotes the distance in the graph. Then V is the disjoint union $V = \cup_{n=0}^{\infty} S_n$ and $\ell^2(V) = \bigoplus_{n=0}^{\infty} \ell^2(S_n)$. With respect to this decomposition, the graph Laplacian Δ has the block form

$$\Delta = \begin{bmatrix} D_0 & -E_0^T & 0 & 0 & \cdots \\ -E_0 & D_1 & -E_1^T & 0 & \cdots \\ 0 & -E_1 & D_2 & -E_2^T & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The matrix $D_i = \Delta_i + N_i$, where Δ_i is the Laplacian for S_i and N_i is diagonal matrix whose entry on the diagonal position corresponding to $v \in S_i$ is the number of vertices joining v to neighbouring spheres. The matrix for $E_i : \ell^2(S_i) \to \ell^2(S_{i+1})$ has a 1 in the v, w position if $v \in S_{i+1}$ is connected to $w \in S_i$, and otherwise has a 0 there.

Given a potential q, let q_i denote the restriction of q to S_i. Then, as an operator, $q = \oplus_i q_i$ where q_i is a diagonal operator on $\ell^2(S_i)$.

We will assume the following hypothesis on the graph, the choice of origin and the potential.

Hypothesis 3.1

(i) For every i $\mathrm{Ker}(E_i) = \{0\}$*, or equivalently,* $E_i^T E_i$ *is invertible.*

(ii) $\sup_i \|E_i\| < C$.

(iii) $\sup_i \|D_i\| < C$.

(iv) $\sup_i \|q_i\| < C$.

Remark: Examples for (i), (ii) and (iii) include trees and $\mathbb{Z}^n$ as well as graphs obtained from these by added edges within spheres, subject to (iii).

Now we analyze the solutions to the equation

$$(\Delta + q - \lambda)\varphi = \varphi_{-1}\chi_0 \tag{3.1}$$

by generalizing the transfer matrix formulation of the previous section. Here χ_0 is the characteristic function of the origin, and φ_{-1} is a complex parameter inserted to streamline the notation below. We are most interested in the case where $\varphi_{-1} = 1$ and where the solution $\varphi \in \ell^2(V)$ since, for this solution, $\varphi(\cdot) = G(\cdot, 0, \lambda)$.

To begin we write equation (3.1) in polar co-ordinates. Write $\varphi = (\varphi_0, \varphi_1, \ldots)$ where φ_i is the restriction of φ to S_i. Then φ solves (3.1) if and only if

$$\begin{aligned} (D_0 + q_0 - \lambda)\varphi_0 - E_0^T \varphi_1 &= \varphi_{-1}\,, \\ -E_{i-1}\varphi_{i-1} + (D_i + q_i - \lambda)\varphi_i - E_i^T \varphi_{i+1} &= 0 \quad \text{for} \quad i \geq 1\,. \end{aligned} \tag{3.2}$$

These equations can be rewritten as

$$\begin{bmatrix} E_i^T \varphi_{i+1} \\ \varphi_i \end{bmatrix} = \begin{bmatrix} (D_i + q_i - \lambda)\varphi_i - E_{i-1}\varphi_{i-1} \\ \varphi_i \end{bmatrix}$$

provided we define $E_{-1} = 1$. For $i \geq 1$ we may write the decomposition of φ_i with respect to $\ell^2(S_i) = \mathrm{Ran}(E_{i-1}) \oplus \mathrm{Ker}(E_{i-1}^T)$ as

$$\varphi_i = E_{i-1}(E_{i-1}^T E_{i-1})^{-1} E_{i-1}^T \varphi_i + \psi_i\,,$$

where $\psi_i \in \mathrm{Ker}(E_{i-1}^T)$. This gives

$$\begin{bmatrix} E_i^T \varphi_{i+1} \\ \varphi_i \end{bmatrix} = \begin{bmatrix} (D_i + q_i - \lambda)E_{i-1}(E_{i-1}^T E_{i-1})^{-1} & -E_{i-1} \\ E_{i-1}(E_{i-1}^T E_{i-1})^{-1} & 0 \end{bmatrix} \begin{bmatrix} E_{i-1}^T \varphi_i \\ \varphi_{i-1} \end{bmatrix} + \begin{bmatrix} D_i + q_i - \lambda \\ I \end{bmatrix} \psi_i\,.$$

To write the equations in a compact form, define the matrices

$$A_i = \begin{bmatrix} (D_i + q_i - \lambda) & -I \\ I & 0 \end{bmatrix}$$

and

$$B_i = \begin{bmatrix} E_{i-1}(E_{i-1}^T E_{i-1})^{-1} & 0 \\ 0 & E_{i-1} \end{bmatrix} .$$

for $i = 0, 1, 2, \ldots$. Note that with E_{-1} is defined to be 1, B_0 is the 2×2 identity matrix. Let ψ_i for $i \geq 0$ be the projections of φ_i onto $\mathrm{Ker}(E_{i-1}^T)$. In particular $\psi_0 = 0$. Then the equation for φ is equivalent to

$$\begin{bmatrix} E_i^T \varphi_{i+1} \\ \varphi_i \end{bmatrix} = A_i \left(B_i \begin{bmatrix} E_{i-1}^T \varphi_i \\ \varphi_{i-1} \end{bmatrix} + \begin{bmatrix} \psi_i \\ 0 \end{bmatrix} \right),$$

for $i = 0, 1, 2, \ldots$. The iteration of this equation is the graph analogue of (2.2). When λ is real, both A_i and B_i are symplectic transformations. This is analogous to the transfer matrices in one dimension being in $\mathrm{SL}(2, \mathbb{R})$.

Lemma 3.2 *Assume Hypothesis 3.1. The solutions of* $(\Delta + q - \lambda)\varphi = \varphi_{-1}\chi_0$ *are in one-to-one correspondence with vectors* $(\varphi_0, \psi_1, \psi_2, \ldots)$ *with* $\varphi_0 \in \mathbb{C}$ *and* $\psi_i \in \mathrm{Ker}(E_{i-1}^T)$*.*

Proof: (Sketch) The correspondence is given by first setting $\xi_{-1} = \varphi_0$. Then, starting with $\begin{bmatrix} \xi_{-1} \\ \varphi_{-1} \end{bmatrix}$ we define $\begin{bmatrix} \xi_i \\ \varphi_i \end{bmatrix}$ for $i \geq 0$ by

$$\begin{bmatrix} \xi_i \\ \varphi_i \end{bmatrix} = A_i \left(B_i \begin{bmatrix} \xi_{i-1} \\ \varphi_{i-1} \end{bmatrix} + \begin{bmatrix} \psi_i \\ 0 \end{bmatrix} \right). \tag{3.3}$$

Writing out this equation using the definition of A_i and B_i we find that $\xi_i = E_i^T \varphi_{i+1}$ for every i, and that the solution $\varphi = (\varphi_0, \varphi_1, \ldots)$ generated by this procedure solves the equation (3.1). $\square$

The same procedure can also be used to generate a solution of $(\Delta + q - \lambda)\varphi = 0$ in the exterior of a ball in the graph. To do this, begin with $\begin{bmatrix} \xi_{i-1} \\ \varphi_{i-1} \end{bmatrix} \in \ell^2(S_{i-1}) \oplus \ell^2(S_{i-1})$ and $\psi_n \in \mathrm{Ker}(E_{n-1}^T)$ for $n \geq i$. Then follow the recursive procedure (3.3) above to generate φ_n for $n \geq i$. The resulting vector $\varphi = (\varphi_i, \varphi_{i+1}, \ldots)$ solves the equation

$$(\Delta_i + q - \lambda)\varphi = E_{i-1}\varphi_{i-1} . \tag{3.4}$$

Here Δ_i denotes the Laplacian for the truncated graph with vertices $\cup_{n=i}^{\infty} S_n$, and $E_{i-1}\varphi_{i-1}$ is shorthand for $(E_{i-1}\varphi_{i-1}, 0, 0, \ldots)$. Conversely, given a solution to (3.4), the vectors $\begin{bmatrix} \xi_i \\ \eta_i \end{bmatrix} = \begin{bmatrix} E_{i-1}^T \varphi_i \\ \varphi_i \end{bmatrix}$ satisfy the recursion (3.3).

Now we can define the discrete Dirichlet to Neumann maps for a graph. These are the analogues for more general graphs of the numbers α_i. They will be linear transformations $\Lambda_i = \Lambda_i(\lambda)$ acting on $\ell^2(S_{i-1})$, or on $\mathbb{C}$ for $i = 0$.

Definition: Let $\mathrm{Im}(\lambda) > 0$. For $\varphi_{i-1} \in \ell^2(S_{i-1})$, or $\varphi_{i-1} \in \mathbb{C}$ for $i = 0$, define $\Lambda_i(\lambda)\varphi_{i-1}$ to be the unique vector $\xi \in \ell^2(S_{i-1})$ such that the recursion (3.3) beginning with $\begin{bmatrix} \xi \\ \varphi_{i-1} \end{bmatrix}$ generates an ℓ^2 solution of (3.4) for some choice of $\psi_i, \psi_{i+1} \ldots$.

Just as the numbers α_i represented a direction in $\mathbb{C}^2$, the linear transformations Λ_i can be thought of as representing a subspace of $\ell^2(S_{i-1}) \times \ell^2(S_{i-1})$, namely the graph of Λ_i. This is the subspace of Cauchy data that generates ℓ^2 solutions.

When $\mathrm{Im}(\lambda) > 0$ the numbers α_i are in the upper half plane $\mathbb{H}$. The analogous property for the linear transformations Λ_i is that they take values in the Siegel upper half space, $\mathbb{SH}_{n_{i-1}}$, where $n_{i-1} = \dim(\ell^2(S_{i-1}))$. This is the set of $n_{i-1} \times n_{i-1}$ matrices $Z = X + iY$ where X and Y are symmetric real matrices and Y is positive definite. The facts we will need about the Siegel upper half spaces are collected the Appendix.

Proposition 3.3 *Let* $\mathrm{Im}(\lambda) > 0$. *Then*

(i) Λ_i *is a well defined linear map.*

(ii) Let Δ_i *denote the Laplacian for the truncated graph with vertices* $\cup_{n=i}^{\infty} S_n$. *Then* $\Lambda_0 = G(0, 0; \lambda)$ *while for* $i \geq 1$

$$\Lambda_i = E_{i-1}^T (\Delta_i + q - \lambda)^{-1} E_{i-1}\,. \tag{3.5}$$

(iii) Λ_i *lies in* $\mathbb{SH}_{n_{i-1}}$.

Proof: First we note that the vector ξ in the definition of Λ_i must be unique, if it exists. For if there were two distinct vectors generating ℓ^2 solutions to (3.4) with $\mathrm{Im}(\lambda) > 0$, their difference would be a non-zero ℓ^2 solution to $(\Delta_i + q - \lambda)\varphi = 0$, contradicting the fact that λ is in the resolvent set. For $\mathrm{Im}(\lambda) > 0$, the resolvent gives an ℓ^2 solution

$$\varphi = (\Delta_i + q - \lambda)^{-1} E_{i-1}\varphi_{i-1}$$

to (3.4). This solution is generated by $\begin{bmatrix} E_i^T \varphi_i \\ \varphi_{i-1} \end{bmatrix}$. Thus $\Lambda_i \varphi_{i-1} = E_i^T \varphi_i$, which implies (3.5).

From (3.5) the symmetry of Λ_i follows easily. This formula also implies that $\langle \psi, \Lambda_i \psi \rangle$ equals $\langle E_{i-1}\psi, (\Delta_i + q - \lambda)^{-1} E_{i-1}\psi \rangle$ which has positive imaginary part if $\mathrm{Im}(\lambda) > 0$. Thus $\Lambda_i \in \mathbb{SH}_{n_{i-1}}$.
$\square$

The analogues of the Möbius transformations in the one dimensional case are the maps $\Phi_i : \mathbb{SH}_{n_i} \to \mathbb{SH}_{n_{i-1}}$ defined by

$$\Phi_i(\Lambda) = E_{i-1}^T (-\Lambda + D_i + q_i - \lambda)^{-1} E_{i-1}\,. \tag{3.6}$$

Here E_{-1} is defined to be 1.

Proposition 3.4

$$\Lambda_i = \Phi_i(\Lambda_{i+1})\,.$$

Proof: Let φ_{i-1} be arbitrary and let $(\varphi_i, \varphi_{i+1}, \ldots)$ be the solution generated by $\begin{bmatrix} \Lambda_i \varphi_{i-1} \\ \varphi_{i-1} \end{bmatrix}$. Then $E_i^T \varphi_{i+1} = \Lambda_{i+1}\varphi_i$ so equation (3.2) reads

$$(-\Lambda_{i+1} + D_i + q_i - \lambda)\varphi_i = E_{i-1}\varphi_{i-1}\,,$$

so that

$$\varphi_i = (-\Lambda_{i+1} + D_i + q_i - \lambda)^{-1} E_{i-1}\varphi_{i-1}\,.$$

This is also true when $i = 0$ with the convention $E_{-1} = 1$. Applying E_{i-1}^T to both sides and substituting $\Lambda_i \varphi_{i-1} = E_{i-1}^T \varphi_i$ completes the proof. $\square$

Proposition 3.4 implies that for any n

$$G(0,0;lambda) = \Phi_0 \circ \cdots \circ \Phi_n(\Lambda_{n+1})\,. \tag{3.7}$$

We now must examine the mapping properties of Φ_i on the Siegel upper half space. It turns out that the distance defined by the standard Riemannian metric on $\mathbb{SH}_n$ is not suitable for our estimates. Instead we define a Finsler metric as follows. Let $Z = X + iY \in \mathbb{SH}_n$ and let the complex symmetric matrix W be an element in the tangent space at Z. We define the Finsler norm

$$F_Z(W) = \|Y^{-1/2} W Y^{-1/2}\|\,, \tag{3.8}$$

where $\|\cdot\|$ is the usual matrix (operator) norm. This Finsler norm defines a distance function which we will denote d_∞.

Definition: The distance d_∞ is defined as

$$d_\infty(Z_1, Z_2) = \inf_{Z(t)} \int_0^T F_{Z(t)}(\dot{Z}(t))dt\,,$$

where the infimum is taken over all differentiable paths $Z(t)$ joining Z_1 to Z_2.

The arguments leading to Theorem 3.6 are roughly the same as those in the one dimensional case leading to Theorem 2.3. However, since the geometry of the Siegel half space with the Finsler metric is more complicated, some modifications are needed.

Lemma 3.5 *Assume that Hypothesis 3.1 holds and define $Y_i = E_i^T E_i$.*

(i) For $\mathrm{Im}(\lambda) \geq 0$, the map Φ_0 maps $\mathbb{SH}_{n_0} = \mathbb{H}$ into $\mathbb{H}$, while for $i \geq 1$, the map Φ_i maps $\mathbb{SH}_{n_i}$ into $\mathbb{SH}_{n_{i-1}}$. When $\mathrm{Im}(\lambda) > 0$ the image lies in the set $\{Z \in \mathbb{SH}_{n_{i-1}} : \|Z\| \leq C\}$, where C depends only on λ and the constants in Hypothesis 3.1.

(ii) When $\mathrm{Im}(\lambda) > 0$, the map Φ_i is a contraction with respect to the d_∞ metric.

(iii) $\Phi_i \circ \Phi_{i+1}(\mathbb{SH}_{n_i}) \subseteq B(iY_{i-1}, R)$ *where* $B(iY_{i-1}, R)$ *denotes a ball in the* d_∞ *metric about* iY_{i-1} *whose radius* R *depends only on* λ *and the constants in Hypothesis 3.1.*

(iv) If Z_1 *and* Z_2 *lie in* $B(iY_{i-1}, R)$ *then there exists a positive constant* $\delta < 1$*, depending only on* λ *and the constants in Hypothesis 3.1, such that*

$$d_\infty(\Phi_i(Z_1), \Phi_i(Z_2)) \le \delta\, d_\infty(Z_1, Z_2)\,.$$

Proof: Factor

$$\Phi_i = \pi \circ \rho \circ \tau\,. \tag{3.9}$$

where

$$\begin{aligned} \tau &: Z \mapsto Z + \mathrm{Im}(\lambda) \\ \rho &: Z \mapsto -(Z - D_i - q_i + \mathrm{Re}(\lambda))^{-1} \\ \pi &: Z \mapsto E_{i-1}^T Z E_{i-1}\,. \end{aligned}$$

For $\mathrm{Im}(\lambda) \ge 0$ it is clear that τ maps $\mathbb{SH}_{n_i}$ into $\mathbb{SH}_{n_i}$, while ρ is an isometry of $\mathbb{SH}_{n_i}$ corresponding to the symplectic matrix

$$\begin{bmatrix} 0 & I \\ -I & D_i + q_i - \mathrm{Re}(\lambda) \end{bmatrix}.$$

The map $Z \mapsto E_{i-1}^T Z E_{i-1}$ maps $\mathbb{SH}_{n_i}$ to $\mathbb{SH}_{n_{i-1}}$ since the image is obviously symmetric, and by Hypothesis 3.1, $E_{i-1}^T Y E_{i-1}$ is positive definite if Y is.

When $\mathrm{Im}(\lambda) > 0$ we have $Z - D_i - q_i + \lambda = X + iY$ with $Y \ge \mathrm{Im}(\lambda) I$. Thus $\|Y^{-1/2}\| < C$ and so

$$\begin{aligned} \|E_{i-1}^T (X + iY)^{-1} E_{i-1}\| &= \|E_{i-1}^T Y^{-1/2} (Y^{-1/2} X Y^{-1/2} + iI)^{-1} Y^{-1/2} E_{i-1}\| \\ &\le \|E_{i-1}^T\| \|Y^{-1/2}\| \|(Y^{-1/2} X Y^{-1/2} + iI)^{-1}\| \|Y^{-1/2}\| \|E_{i-1}\| \\ &\le C\,. \end{aligned}$$

This proves (i). Here and throughout the proof, constants depend only on the constants in Hypothesis 3.1 and on λ, and may change in value from line to line.

Since ρ is an isometry, to prove (ii) we must show that τ and π are contractions. The argument for τ is similar to the one dimensional case and follows from the inequality $F_{Z+\mathrm{Im}(\lambda)}(W) < F_Z(W)$ for $Z \in \mathbb{SH}_{n_i}$ and W a symmetric complex matrix representing an element in the tangent space of $\mathbb{SH}_{n_i}$ at Z. We omit the details. Next we must show that $\pi : \mathbb{SH}_{n_i} \to \mathbb{SH}_{n_{i-1}}$ is a contraction with respect to the d_∞ metric. This follows if we can show

$$F_{E^T Z E}(E^T W E) \le F_Z(W)$$

for any matrix W in the tangent space to $\mathbb{SH}_{n_i}$ at $Z = X + iY$. In other words, we need that for any complex symmetric matrix W and positive definite matrix Y,

$$\|(E^T Y E)^{-1/2} E^T W E (E^T Y E)^{-1/2}\| \le \|Y^{-1/2} W Y^{-1/2}\|\,. \tag{3.10}$$

Consider

$$P = (E^T Y E)^{-1/2} E^T Y^{1/2} .$$

Then $PP^* = I$ so that P is a partial isometry. Since the left side of (3.10) can be written $\|PY^{-1/2}WY^{-1/2}P^*\|$, this implies (3.10) and proves (ii).

We already know that $\Phi_{i+1}(\mathbb{SH}_{n_{i+1}}) \subseteq \{Z \in \mathbb{SH}_{n_i} : \|Z\| \le C\}$. Starting with this set we now apply $Z \mapsto Z - D_i - q_i + \lambda$. Given the bounds on $\|D_i\|$ and $\|q_i\|$ the resulting image now lies in the set

$$B_1 = \{Z = X + iY : \mathrm{Im}(\lambda) \le Y \le C \text{ and } \|X\| \le C\} .$$

Now let $Z = X + iY \in B_1$. By the triangle inequality and (5.5)

$$\begin{aligned} d_\infty(iI, Z) &\le d_\infty(iI, iY) + d_\infty(iY, X + iY) \\ &\le \max\{\ln(\|Y\|), \ln(\|Y^{-1}\|)\} + d_\infty(iY, X + iY) \\ &\le C + d_\infty(iY, X + iY) . \end{aligned}$$

We can estimate $d_\infty(iY, X + iY)$ using the path $Z(t) = tX + iY$, for $t \in [0, 1]$. Then $\dot{Z} = X$ and

$$d_\infty(iY, X + iY) \le \int_0^1 F_{Z(t)}(\dot{Z})dt = \int_0^1 \|Y^{-1/2}XY^{-1/2}\|dt \le C\|X\| \le C .$$

Thus $d_\infty(iI, Z) < C$ so $B_1 \subseteq B(iI, R)$. To obtain $\Phi_i \circ \Phi_{i+1}(\mathbb{SH}_{d_i})$ we must apply to B_1 the rotation $Z \mapsto -Z^{-1}$ followed by the map π defined above. The rotation leaves $B(iI, R)$ invariant. Finally, since $Y_{i-1} = \pi(iI)$ and π is a contraction

$$\begin{aligned} d_\infty(\pi(Z), iY_{i-1}) &= d_\infty(\pi(Z), \pi(iI)) \\ &\le d_\infty(Z, iI) \le R . \end{aligned}$$

This completes the proof of (iii).

To prove (iv) we begin with the local version of the desired inequality. Let $Z = X + iY$ and let W be a symmetric matrix tangent at Z. If $\mathrm{Im}(\lambda) > 0$ and $\|Y\| < C$ then

$$\begin{aligned} F_{\tau(Z)}(d\tau(Z)[W]) &= \|(Y + \mathrm{Im}(\lambda))^{-1/2} W (Y + \mathrm{Im}(\lambda))^{-1/2}\| \\ &\le \|(Y + \mathrm{Im}(\lambda))^{-1/2} Y^{1/2}\|^2 \|Y^{-1/2}WY^{-1/2}\| \\ &\le \delta \|Y^{-1/2}WY^{-1/2}\| = \delta F_Z(W) \end{aligned}$$

This inequality will imply (iv) if we can show that if Z_1 and Z_2 lie in $B(iY_{i-1}, R)$ and $Z = X + iY$ is any point along a length minimizing path joining Z_1 and Z_2, then $\|Y\| < C$.

So let Z be such a point. We begin by showing that

$$Z \in B(iY_{i-1}, 2R) . \tag{3.11}$$

If Z lies on a length minimizing path joining Z_1 and Z_2 then $d_\infty(Z_1, Z) + d_\infty(Z, Z_2) = d_\infty(Z_1, Z_2)$. So

$$d_\infty(Z, iY_{i-1}) \le \begin{cases} d_\infty(Z, Z_1) + d_\infty(Z_1, iY_{i-1}) \le d_\infty(Z, Z_1) + R \\ d_\infty(Z, Z_2) + d_\infty(Z_2, iY_{i-1}) \le d_\infty(Z, Z_2) + R \end{cases}$$

implies

$$d_\infty(Z, iY_{i-1}) \leq (1/2)(d_\infty(Z, Z_1) + d_\infty(Z, Z_2) + 2R) \leq 2R\,,$$

proving (3.11). Now we estimate

$$\begin{aligned}
2R &\geq d_\infty(Z, iY_{i-1}) \\
&\geq d_\infty(iY, iY_{i-1}) \\
&= d_\infty(iI, Y^{-1/2}Y_{i-1}Y^{-1/2}) \\
&= \max\{\ln(\|Y^{-1/2}Y_{i-1}Y^{-1/2}\|), \ln(\|Y^{1/2}Y_{i-1}^{-1}Y^{1/2}\|) \\
&\geq \ln(\|Y^{1/2}Y_{i-1}^{-1}Y^{1/2}\|)\,.
\end{aligned} \tag{3.12}$$

Here we used (5.6) in the second line, the fact that $Z \mapsto Y^{-1/2}ZY^{-1/2}$ is a symplectic transformation in the third line, and (5.5) in the fourth line. Now

$$Y_{i-1}^{-1} \geq \|Y_{i-1}\|^{-1} \geq C\,,$$

since by Hypothesis 3.1 (ii) we have the uniform bound $\|Y_{i-1}\| \leq C$. Thus $Y^{1/2}Y_{i-1}^{-1}Y^{1/2} \geq CY$ which implies $\|Y^{1/2}Y_{i-1}^{-1}Y^{1/2}\| \geq C\|Y\|$ and therefore also $\ln(\|Y^{1/2}Y_{i-1}^{-1}Y^{1/2})\| \geq \ln(\|Y\|) + C$. Thus (3.12) implies

$$2R \geq \ln(\|Y\|) + C\,.$$

This provides the desired uniform bound on $\|Y\|$ and completes the proof of (iv). □

Now we can copy the proof of Theorem 2.3 to obtain our formula for the Green's function at the origin.

Theorem 3.6 *Suppose that Hypothesis 3.1 holds. Let $\Gamma_i \in \mathbb{SH}_{n_i}$ be arbitrary, where $n_i = \dim(\ell^2(S_{i-1}))$. Then*

$$\lim_{i\to\infty} \Phi_0 \circ \cdots \circ \Phi_i(\Gamma_i) = \varphi_0 = G(0, 0; \lambda)\,. \tag{3.13}$$

4. Controlling the Green's function near the real axis

In this section we show how formula (3.13) can be used to control the value of $G(0, 0; \lambda)$ in the limit $\mathrm{Im}(\lambda) \to 0$, uniformly for $\mathrm{Re}(\lambda)$ in some interval. According to (3.13), we can compute $G(0, 0; \lambda)$ with the following sequence of maps

$$\mathbb{H} \xleftarrow{\Phi_0} \mathbb{SH}_{n_0} \xleftarrow{\Phi_1} \mathbb{SH}_{n_1} \xleftarrow{\Phi_2} \mathbb{SH}_{n_2} \longleftarrow \cdots$$

by starting at an arbitrary point $\Gamma_i \in \mathbb{SH}_{n_i}$ and following the arrows all the way to the left. The farther to the right the starting point is, the better the approximation to $G(0, 0; \lambda)$. In the previous section we used the contractive properties of the maps Φ_i to control the position of the final image when $\mathrm{Im}(\lambda)$ is fixed and positive. What happens when $\mathrm{Im}(\lambda)$ tends to zero? In the one dimensional case, the maps Φ_i becomes hyperbolic isometries and do not contract. This is reflected in the unbounded variation of $G(0,0;\lambda)$ as $\mathrm{Im}(\lambda) \to 0$. But for the more general graphs in the previous section, the maps Φ_i are still contracting in some directions even for $\lambda \in \mathbb{R}$, due to the non-expanding map π in the factorization (3.9). It is this contraction that we are able to exploit in the graphs and potentials considered in this section.

Our basic idea is to find a sequence of bounded sets $B_{-1}, B_0, B_1, \ldots$ with $B_{-1} \subseteq \mathbb{H}$ and $B_i \subset \mathbb{SH}_{n_i}$ for $i \geq 0$ such that

$$\Phi_i(B_i) \subseteq B_{i-1}\,. \tag{4.1}$$

Since we are always free to choose the starting point Γ_i to lie in B_i, we conclude that $G(0,0;\lambda) \in B_{-1}$.

In the previous section we found sets $B_i = B_i(\lambda)$ that worked for a large collection of graphs and any bounded potential. However as $\mathrm{Im}(\lambda)$ tended to zero, the sets $B_i(\lambda)$ grew to fill out all of $\mathbb{SH}_{n_i}$. For the graphs and potentials in this section we will again find sets $B_i = B_i(\lambda)$ satisfying (4.1). But now we will be able to control $B_{-1}(\lambda)$ as $\mathrm{Im}(\lambda)$ tends to zero. This gives a bound on $G(0,0;\lambda)$ and thereby implies the existence of absolutely continuous spectrum.

Notice that it is not really necessary to find the sets B_i for every sphere. It is enough to find them for i large, say $i \geq N$. Then we can define the rest of the sets as $B_{N-1} = \Phi_N(B_N)$, $B_{N-2} = \Phi_{N-1} \circ \Phi_N(B_N)$, and so on. Since a finite composition of the maps Φ_i maps bounded sets in $\mathbb{SH}_n$ to bounded sets and is continuous in λ, it is enough to check that B_N is bounded for some range of λ values to conclude that B_{-1} and thus $G(0,0;\lambda)$ is also bounded for that range of λ. Moreover, it is not even necessary to find the sets B_i for every large i. It is enough to find them for a subsequence of $i_1, i_2, \ldots$ tending to infinity, provided we can show that $\Phi_{i_{k-1}+1} \circ \Phi_{i_{k-1}+2} \circ \cdots \circ \Phi_{i_k}(B_{i_k}) \subseteq B_{i_{k-1}}$.

The graphs we consider in this section are trees, possibly with some transverse edges added. We are able to handle potentials that have ℓ^∞ fluctuations in the radial direction, while satisfying various conditions restricting the behaviour in the transverse direction. We also briefly consider decreasing potentials.

4.1 Perturbations of potentials with transverse period 2

Our first examples are perturbations of the base graph and potential depicted in Figure 2.

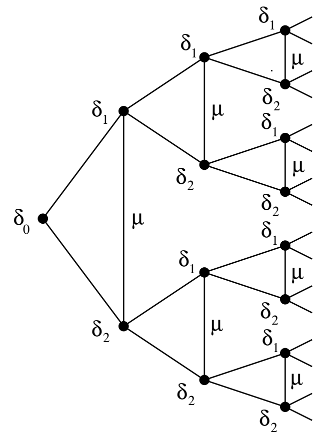

Figure 2: the base graph

On this figure labels δ_1 and δ_2 are the values of the potential at the indicated sites, while μ is a weighting of the corresponding edge. Note that the underlying graph in Figure 2 has many symmetries generated by flipping a subtree about any vertex. This implies that we could flip any pair of potentials δ_1, δ_2 without changing $G(0,0;\lambda)$.

We begin with a spectral analysis of the base graph and potential using our machinery. For this graph $\ell^2(S_i) = \mathbb{C}^{2^i}$. We can compute Φ_i using (3.6). To simplify the expression for Φ_i, we subtract 3 from the diagonal of the graph Laplacian. This makes $D_i = 0$. The map Φ_i preserves diagonal elements of $\mathbb{SH}_{n_i}$. Suppose that $\Gamma_i = \mathrm{diag}[z, z, \ldots, z]$ for some $z \in \mathbb{H}$. Then for $i \geq 1$

$$\Phi_i(\Gamma_i) = \mathrm{diag}[\phi(z,z), \phi(z,z), \ldots, \phi(z,z)]\,, \tag{4.11}$$

where

$$\begin{aligned}\phi(z_1, z_2) &= \phi(z_1, z_2; \delta_1, \delta_2, \mu, \lambda)\\ &= -[1,1]\begin{bmatrix} z_1 - \delta + \lambda & \mu \\ \mu & z_2 + \delta + \lambda \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.\end{aligned}$$

Define the diagonal map

$$\phi_d(z) = \phi_d(z; \delta_1, \delta_2, \mu, \lambda_0) = \phi(z, z; \delta_1, \delta_2, \mu, \lambda_0)\,.$$

Then (3.13) implies

$$G(0,0;\lambda) = \Phi_0 \circ \phi_d \circ \phi_d \circ \cdots \circ \phi_d(z)$$

for any $z \in \mathbb{H}$. Thus, the spectrum for the unperturbed graph can be found explicitly in terms of the fixed points of $\phi_d(z)$.

To simplify notation, let us now assume that

$$\delta_1 = -\delta_2 = \delta\,.$$

This amounts to a real shift in λ. Then

$$\phi_d(z) = \frac{2(-z+\mu-\lambda)}{(z+\lambda)^2 - \delta^2 - \mu^2}\,.$$

Since ϕ_d only depends on δ^2 we will assume that $\delta > 0$. The fixed points of $\phi_d(z)$ are roots of the cubic polynomial

$$p(z) = z^3 + 2\lambda z^2 + (2 + \lambda^2 - \mu^2 - \delta^2)z + 2(\lambda - \mu)\,.$$

When $\lambda \in \mathbb{R}$ there are either three real roots or a pair of conjugate complex roots and one real root. The latter case is the interesting one for us, since the root in the upper half plane will remain there when λ is moved slightly into the upper half plane.

Now suppose that $z_f = z_f(\delta, \mu, \lambda)$ is such a root, that is, a fixed point in $\mathbb{H}$ for the map $\phi_d(z; \delta, -\delta, \mu, \lambda)$. Then $G(0,0;\lambda) = \Phi_0(z_f) = -1/(z_f - q_o + \lambda)$, so the λ dependence of $G(0,0;\lambda)$ as λ approaches the real axis can now be deduced from the behaviour of z_f. In particular, since Φ_0 is a hyperbolic isometry for $\lambda \in \mathbb{R}$, $G(0,0;\lambda)$ has non-zero imaginary part for $\lambda \in \mathbb{R}$ exactly when z_f has.

Here is a plot of the contour $\mathrm{Im}(z_f(\delta, 0, \lambda)) = 0$ in the λ, δ plane. The interior of the contour is the region of spectrum, which is purely absolutely continuous. At the critical value $\delta = \sqrt{2}$ the spectrum splits into two bands which then move out to infinity as δ increases.

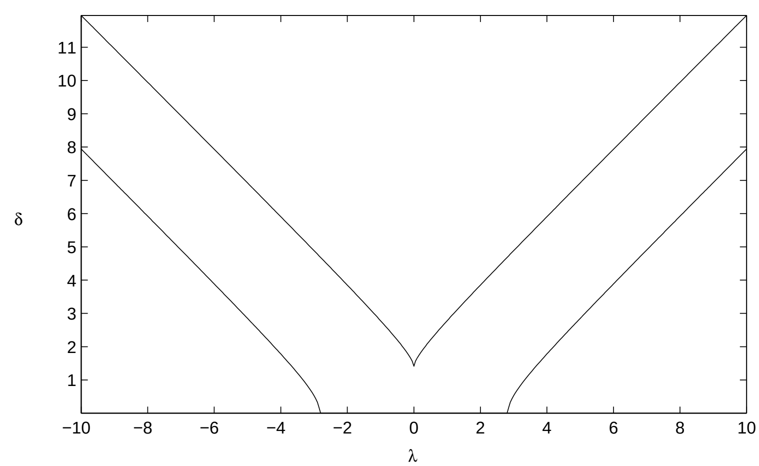

Figure 3: spectra for base potentials

To put this into our general framework, we should define B_i for $i \geq 0$ to be the singleton $\{\mathrm{diag}[z_f, \ldots, z_f]\}$ and $B_{-1} = \{\Phi_0(z_f)\}$. For this sequence of sets we have perfect control as $\mathrm{Im}\,\lambda$ tends to zero. But these sets only work for a single potential and weight function given on the nth sphere as

$$q_n = [\delta, -\delta, \delta, -\delta, \ldots, \delta, -\delta]$$

and

$$w_n = [\mu, \mu, \ldots, \mu]\,.$$

Now we present some ways of fattening the sets B_i to allow for a class of perturbations of this potential and weight. The simplest situation is where the perturbation on each sphere has the same periodicity as the base potential.

Proposition 4.1 *Choose δ, μ and $\lambda_0 \in \mathbb{R}$ with $\delta \neq 0$ and with $z_f(\delta, \mu, \lambda_0)$ in the upper half plane. Let $R > 0$. Then there exists $\epsilon_1, \epsilon_2 > 0$ such that for every perturbation of the form*

$$q_i = [\delta + d_{i,1}, -\delta + d_{i,2}, \delta + d_{i,1}, -\delta + d_{i,2}\,, \ldots, \delta + d_{i,1}, -\delta + d_{i,2}],$$

$$w_n = [\mu + e_i, \mu + e_i, \ldots, \mu + e_i]$$

with

$$|d_{i,1}|, |d_{i,2}|, |e_i| < \epsilon_1 \tag{4.3}$$

and every λ in the upper half plane with

$$|\lambda - \lambda_0| \leq \epsilon_2\,, \tag{4.4}$$

the Green's function for the perturbed potential satisfies

$$G(0, 0; \lambda) \subseteq \Phi_0(B(z_f, R))\,.$$

Here $B(z_f, R)$ is a closed hyperbolic ball of radius R about z_f.

Proof: The diagonal map $\phi_d(z; \delta, -\delta, \mu, \lambda_0)$ is a strict contraction on $\mathbb{H}$. This can be checked directly, but also follows from the Schwarz Lemma [Kr], since (i) ϕ_d is analytic from $\mathbb{H}$ into $\mathbb{H}$ and, (ii) because $\delta \neq 0$, the map ϕ_d is not a Möbius transformation. Since z_f is a fixed point for ϕ_d, this implies that

$$\phi_d(B(z_f, R); \delta, -\delta, \mu, \lambda_0) \subset B(z_f, R)\,,$$

where the inclusion is strict. By continuity, there exists ϵ_1 and ϵ_2 such that (4.3) and (4.4) imply

$$\phi_d(B(z_f, R); \delta + d_{i,1}, -\delta + d_{i,2}, \mu + e_i, \lambda) \subseteq B(z_f, R)\,. \tag{4.5}$$

Now define the sets $B_i = \{\mathrm{diag}[z, z, \ldots, z] : z \in B(z_f, R)\}$ for $i \geq 0$ and $B_{-1} = \Phi_0(B_0)$. Clearly (4.5) implies $\Phi_i(B_i) \subseteq B_{i-1}$ for all potentials and weights in our perturbation class defined by (4.3) and all λ satisfying (4.4). Thus we obtain $G(0, 0; \lambda) \subseteq B_{-1}$, which is what we want to prove. □

Remark: The map Φ_0 is the hyperbolic isometry $z \mapsto (-z + q_0 - \lambda)^{-1}$ so this proposition gives a uniform bound on $\mathrm{Im}\, G(0, 0; \lambda)$ for potentials and weights in our perturbation class defined by (4.3) and all λ satisfying (4.4).

We digress to investigate how large the fluctuations in the potential can be if we want to insure that $G(0,0;\lambda)$ remains bounded. For simplicity, we set $\mu = 0$. Then we pick a base potential value δ and spectral parameter $\lambda \in \mathbb{R}$, so that the corresponding z_f has positive imaginary part. Since for the proof of the existence of absolutely continuous spectrum we are only interested in some bound on $G(0,0;\lambda)$ we will choose R very large. Then we try to find the largest interval $[\delta_-, \delta_+]$ such that $\delta' \in [\delta_-, \delta_+]$ implies $\phi_d(B(z_f, R), \delta', -\delta', 0, \lambda) \subseteq B(z_f, R)$. Fluctuations of the potential in $[\delta_-, \delta_+]$ will result in a bounded Green's function as $\mathrm{Im}(\lambda) \to 0$.

To begin, consider $\lambda = 0$. Choose a base potential value $\delta \in (0, \sqrt{2})$ so that $z_f = i\sqrt{2 - \delta^2}$ has positive imaginary part. For the special value $\lambda = 0$, the map $\phi_d(z) = -2z/(z^2 - \delta^2)$ preserves the positive imaginary axis. Thus we may restrict our attention to z purely imaginary. The intersection of the positive imaginary axis with the hyperbolic ball $B(z_f, R)$ is the interval $i[\sqrt{2-\delta^2}e^{-R}, \sqrt{2-\delta^2}e^{R}]$. We must therefore decide for which δ' the map $\phi_d(z) = -2z/(z^2 - \delta'^2)$ maps this interval to itself. It is an easy calculus problem to compute that the image of the interval under ϕ_d is $i[2\sqrt{2-\delta^2}/((2-\delta^2)e^{-2R} + \delta'^2), 1/\delta']$. If we restrict δ' to lie in $[\epsilon, \sqrt{2} - \epsilon]$ for any positive ϵ, then for R large the image interval lies inside the original interval.

So for $\lambda = 0$, the interval $[\delta_-, \delta_+]$ can be any strict subinterval of $[0, \sqrt{2}]$. What happens when λ moves away from zero? We have done a numerical computation of the interval $[\delta_-, \delta_+]$ about a base value. What we find, as $|\lambda|$ grows, is that the interval $[\delta_-, \delta_+]$ shrinks to a tiny neighbourhood of the base value of δ. It would be interesting to know whether for large λ, comparatively small fluctuations about a base potential given by, say, $\delta = \lambda$, really result in a singular part to the spectral measure $d\mu_{\chi_0}$.

The next situation we consider is where the perturbation is again periodic across each sphere with fixed period. However, now the period can be larger than the period of the base potential. In this situation, the map ϕ_2 below, which plays the same role as ϕ_d above, maps $\mathbb{H}^2$ into $\mathbb{H}^2$. So the Schwarz Lemma is no longer available to show that it is a strict contraction. Instead we will rely on a local analysis at the fixed point.

It is important to realize that the function of two variables $\phi(z_1, z_2)$ is *not* a strict contraction from $\mathbb{H}^2$, with the d_∞ metric, to $\mathbb{H}$. We can see this locally at (z_f, z_f) by calculating the linearization. To simplify the notation let

$$\alpha = \frac{\partial \phi}{\partial z_1}(z_f)$$

and

$$\beta = \frac{\partial \phi}{\partial z_2}(z_f)\,.$$

Lemma 4.2 *Choose δ, μ and $\lambda_0 \in \mathbb{R}$ with $\delta \neq 0$ and with $z_f(\delta, \mu, \lambda_0)$ in the upper half plane. Let*

$\phi(z_1, z_2) = \phi(z_1, z_2; \delta, \mu, \lambda_0)$ *and let* α *and* β *be the partial derivatives evaluated at* z_f*. Then*

$$|\alpha + \beta| < 1\,, \tag{4.6}$$

while

$$|\alpha| + |\beta| = 1\,. \tag{4.7}$$

Proof: Since $\phi(z_1, z_2) = -E^T(Z+Q)^{-1}E$ for $E = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $Z = \begin{bmatrix} z_1 & 0 \\ 0 & z_2 \end{bmatrix}$ and $Q = \begin{bmatrix} -\delta+\lambda & \mu \\ \mu & \delta+\lambda \end{bmatrix}$, we have

$$\frac{\partial \phi}{\partial z_1} = E^T(Z+Q)^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (Z+Q)^{-1}E$$

so that

$$\left|\frac{\partial \phi}{\partial z_1}\right|(z,z) + \left|\frac{\partial \phi}{\partial z_2}\right|(z,z) = E^T(\mathrm{diag}[z,z]+Q)^{-1}(\mathrm{diag}[\overline{z},\overline{z}]+Q)^{-1}E = \frac{\mathrm{Im}(\phi(z,z))}{\mathrm{Im}\, z}.$$

Since $\phi(z_f, z_f) = z_f$ this implies (4.7). Given (4.7) the only way we can have equality in (4.6) is if α is a real multiple of β. A short calculation shows this only happens when $\delta = 0$. $\square$

Now we have $\phi(z_f + w_1, z_f + w_2) \sim z_f + \alpha w_1 + \beta w_2$. So $d(\phi(z_f + w_1, z_f + w_2), z_f) \sim \mathrm{Im}(z_f)^{-1}|\alpha w_1 + \beta w_2|$ while $d_\infty((z_f + w_1, z_f + w_2), (z_f, z_f)) \sim \mathrm{Im}(z_f)^{-1}\max\{|w_1|, |w_2|\}$. So the linearization of the statement that ϕ is a contraction at (z_f, z_f) is

$$|\alpha w_1 + \beta w_2| \le \max\{|w_1|, |w_2|\}\,. \tag{4.8}$$

Although this is true, and follows easily from (4.7), there are always directions, given by $e^{i \arg \alpha} w_1 = e^{i \arg \beta} w_2$, where equality holds, so the contraction is not strict. Notice that in the first example we only considered the diagonal, $w_1 = w_2$. In this case (4.6) confirms that we do get a strict contraction.

Proposition 4.3 *Choose* δ*,* μ *and* $\lambda_0 \in \mathbb{R}$ *with* $\delta \neq 0$ *and with* $z_f(\delta, \mu, \lambda_0)$ *in the upper half plane. Then there exists* $\epsilon_1, \epsilon_2 > 0$ *and a fixed bounded set* $B \subset \mathbb{H}$ *such that for every perturbation of the form*

$$q_i = [\delta + d_{i,1}, -\delta + d_{i,2}, \delta + d_{i,3}, -\delta + d_{i,4}\,, \ldots, \delta + d_{i,2^i-1}, -\delta + d_{i,2^i}],$$

periodic with period 2^k*, that is,* $d_{i,j+2^k} = d_{i,j}$*,*

$$w_i = [\mu + e_{i,1}, \mu + e_{i,2}, \ldots, \mu + e_{i,2^{i-1}}]$$

periodic with period 2^{k-1} *satisfying*

$$|d_{i,1}|, |d_{i,2}|, |d_{i,3}|, |d_{i,4}|, |e_{i,1}|, |e_{i,2}| < \epsilon_1\,, \tag{4.9}$$

and every λ *in the upper half plane with*

$$|\lambda - \lambda_0| \le \epsilon_2\,, \tag{4.10}$$

the Green's function $G(0, 0; \lambda)$ *for the perturbed potential lies in B.*

Proof: We give the proof for period 4. The generalization to higher periods is immediate. For a period 4 perturbation we consider the map

$$\phi_2 : \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \begin{bmatrix} \phi(\phi(z_1, z_2; \delta_1 + d_1, \delta_2 + d_2, \mu + e_1), \phi(z_1, z_2; \delta_1 + d_3, \delta_2 + d_4, \mu + e_2)) \\ \phi(\phi(z_1, z_2; \delta_1 + d_1, \delta_2 + d_2, \mu + e_1), \phi(z_1, z_2; \delta_1 + d_1, \delta_2 + d_2, \mu + e_1)) \end{bmatrix}$$

when $d_1 = d_2 = d_3 = d_4 = e_1 = e_2 = 0$, the map ϕ_2 has fixed point $\begin{bmatrix} z_f \\ z_f \end{bmatrix}$, and it follows easily from (4.6) and (4.7) that ϕ_2 is a strict contraction near this fixed point. Thus there is a small ball in $\mathbb{H}^2$ with the d_∞ distance about $\begin{bmatrix} z_f \\ z_f \end{bmatrix}$ that maps strictly inside itself under ϕ_2. By continuity, this remains true under small perturbations of the potential and weights. Since

$$\Phi \circ \Phi : \mathrm{diag}[z_1, z_2, z_1, z_2, \ldots] \mapsto \mathrm{diag}[\phi_2(\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}), \phi_2(\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}), \ldots]$$

we can construct sets B_i for every second sphere consisting of small balls about the fixed point such that $\Phi_{i-1} \circ \Phi_i(B_i) \subseteq B_{i-2}$. This suffices for the proof. $\square$

Remark: Both Proposition 4.1 and Proposition 4.3 required that $\delta \neq 0$, to insure that ϕ_d was a strict contraction and could be perturbed. In fact, if we set $\delta = 0$ and try to perturb about $q = 0$, we can end up with an unbounded Green's function. To see this, consider the potential that takes the values $[\delta_1, -\delta_1, \delta_1, -\delta_1, \cdots]$ and $[\delta_2, -\delta_2, \delta_2, -\delta_2, \cdots]$ on alternate spheres. Then the fixed point equation is of the form

$$z = \phi_d(\phi_d(z; \delta_2, \lambda); \delta_1, \lambda)\,.$$

When $\lambda = 0$ and $0 < |\delta_1 \delta_2| < 2$ the fixed point is

$$z_f = i\frac{|\delta_2|}{|\delta_1|}\sqrt{2 - |\delta_1 \delta_2|}\,.$$

This point is on the boundary at infinity in hyperbolic space if $\delta_1 = 0$, $\delta_2 \neq 0$ or $\delta_1 \neq 0$, $\delta_2 = 0$, and therefore so is $G(0, 0; 0) = \Phi_0(z_f)$. Of course, this does not mean that the spectrum is not absolutely continuous (in fact, it is), just that the curve $G(0, 0; \lambda) = \Phi_0(z_f(\lambda))$ hits the real axis or goes to infinity when λ tends to zero.

In two previous propositions, the perturbations in each sphere were taken from a subspace of periodic functions whose dimension was fixed. In the following example we consider perturbations that are decreasing across each sphere. As before, the change from one sphere to the next is only subject to an ℓ^∞ bound.

Proposition 4.4 *Choose δ, μ and $\lambda_0 \in \mathbb{R}$ with with $z_f(\delta, \mu, \lambda_0)$ in the upper half plane. Let $g(k)$ be a positive decreasing function satisfying*

$$g(2k - 1) \leq \delta_1 g(k)$$

for some $\delta_1 < 1$. Let q and w be a potential and weight function of the form

$$q_i = [\delta + d_{i,1}, -\delta + d_{i,2}, \delta + d_{i,3}, -\delta + d_{i,4}, \ldots, \delta + d_{i,2^i-1}, -\delta + d_{i,2^i}]\,,$$

and

$$w_i = [\mu + e_{i,1}, \mu + e_{i,2}, \ldots, \mu + e_{i,2^{i-1}}]\,.$$

There exist $\epsilon > 0$ and a bounded set $B \subset \mathbb{H}$ such that for $|d_{i,k}| \le \epsilon g(k)$, $|e_{i,k}| \le \epsilon g(2k-1)$ and $|\lambda - \lambda_0| \le \epsilon$ the Green's function $G(0,0;\lambda)$ for this potential and weight lies in B.

Proof: In this proof we use sets B_i defined by

$$B_i = \{\mathrm{diag}[z_f + w_1, z_f + w_2, z_f + w_3, z_f + w_4, \ldots z_f + w_{2^i}] : |w_k| \le L\epsilon g(k)\}$$

for some constant L. In this case the sets will work only for one fixed value of λ. But the set B_{-1} containing the Green's function is a small ball about z_f which does not move much if we change λ. Thus we may take B in the theorem to be a union of sets B_{-1} for λ close to λ_0. We must show that $\Phi_i(B_i) \subseteq B_{i-1}$. Linearizing about $w_{2k-1} = w_{2k} = d_{i,2k-1} = d_{i,2k} = e_{i,k} = 0$ yields

$$\begin{aligned}\phi(z_f + w_{2k-1}, z_f + w_{2k}; \delta + d_{i,2k-1}, -\delta + d_{i,2k}, \mu + e_{i,k}, \lambda)&\\ = z_f + \alpha w_{2k-1} + \beta w_{2k} + O(\|[w_{2k-1}, w_{2k}]\|^2) + O(\|[d_{i,2k-1}, d_{i,2k}, e_{i,k}]\|)&\end{aligned}$$

Under one iteration of Φ we estimate the value of the new w_k by

$$\begin{aligned}|\alpha w_{2k-1} + \beta w_{2k}| + CL^2\epsilon^2 g(2k-1)^2 + C\epsilon g(2k-1)&\\ \le |\alpha| L\epsilon g(2k-1) + |\beta| L\epsilon g(2k) + CL^2\epsilon^2 g(2k-1)^2 + C\epsilon g(2k-1)&\\ \le (\delta_1 + CL\epsilon\delta_1^2 + C\delta_1/L)L\epsilon g(k)&\\ \le L\epsilon g(k)\,,&\end{aligned}$$

for suitable choice of L large and ϵ small. This is the inequality needed to show that $\Phi_i(B_i) \subseteq B_{i-1}$.
□

Remark 1: In this theorem we did not require $\delta \ne 0$, so it gives results about perturbations about $q = 0$.

Remark 2: The function $g(k) = k^{-\epsilon}$ for $\epsilon > 0$ satisfies the hypothesis in this theorem.

This concludes our list of perturbations of a base potential and weight that are transversally periodic with period two. Our list is certainly not exhaustive. We could, for example begin with a potential of the type in Proposition 4.1 and superimpose a perturbation of the type considered in Proposition 4.4.

4.2 Perturbations of potentials with transverse period 4

In principle, our methods should be able to handle general periodic modifications of the tree. Starting with a block of length 2^k in the sphere of the tree, we can add an arbitrary potential and extra edges with weights within that block, and repeat this block periodically in each sphere. Associated to this base graph and potential is a map of $\mathbb{SH}_{2^k}$ whose fixed point determines the spectrum of the base graph and potential. If this map, or possibly some power of it, is a strict contraction, we may perturb it as in the previous section.

Unfortunately, the fixed point equation for this map is complicated, and we do not have a general way of showing that it has a unique solution in $\mathbb{SH}_{2^k}$, nor do we know a general method for deciding for which parameter values the map, or some power of the map, is a strict contraction at the fixed point. But it perhaps worth pointing out that for a given set of parameter values, both these steps can quite easily be performed numerically.

In this section we will consider a situation where we can perform the necessary calculations. The graph and potential depicted in Figure 4.

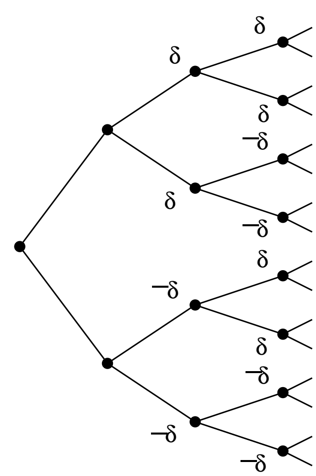

Figure 4: base potentials with period 4

In this figure the unlabeled values of the potential are arbitrary. We will only discuss the base potential

$$q_n = [\delta, \delta, -\delta, -\delta, \ldots, \delta, \delta, -\delta, -\delta]\,.$$

depicted in Figure 4, leaving the perturbation results analogous to Proposition 4.1, Proposition 4.3 and Proposition 4.4 to the reader.

As in the previous section, the map Φ_i preserves diagonal elements of $\mathbb{SH}_{n_i}$. Suppose that

$\Gamma_i = \mathrm{diag}[z_1, z_2, \ldots, z_1, z_2]$ for some $z_1, z_2 \in \mathbb{H}$. Then for $i \geq 2$

$$\Phi_i(\Gamma_i) = \mathrm{diag}[\psi(z_1, z_2, z_1, z_2), \psi(z_1, z_2, z_1, z_2), \ldots, \psi(z_1, z_2, z_1, z_2)]\,, \tag{4.11}$$

where

$$\begin{aligned}&\psi(z_1, z_2, z_3, z_4; \delta, \lambda) \\ &\quad = -\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} z_1 - \delta + \lambda & 0 & 0 & 0 \\ 0 & z_2 - \delta + \lambda & 0 & 0 \\ 0 & 0 & z_3 + \delta + \lambda & 0 \\ 0 & 0 & 0 & z_4 + \delta + \lambda \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}.\end{aligned}$$

The spectrum for the unperturbed graph can be found explicitly in terms of the fixed points of $z \mapsto \psi(z, z; \delta, \lambda)$ with $z = (z_1, z_2) \in \mathbb{H}^2$. These fixed points are roots of a fifth order polynomial. We can solve this explicitly when $\lambda = 0$, and we present this solution now.

First notice that the pair $z = (\zeta, -\zeta)$ with $\zeta \in \mathbb{H}$ is a solution if ζ satisfies

$$\zeta = \frac{1}{\delta - \zeta} + \frac{1}{\delta + \zeta}\,.$$

ζ is thus a solution of the cubic equation, $\zeta^3 - \delta^2\zeta + 2\delta = 0$. We do not need to solve this equation because its solutions do not correspond to a solution in $\mathbb{H}^2$ where both z_1 and z_2 are in the upper half plane.

The other two independent solutions are of the form $(\zeta, -\bar{\zeta})$. So, let $\zeta = x + iy$. Then,

$$x + iy = \frac{1}{\delta - x - iy} + \frac{1}{\delta + x - iy}$$

yields two equations for the real and imaginary parts. They are of the form

$$\begin{aligned} x(\delta^2 - x^2 - y^2) + 2\delta y^2 &= 2\delta\,, \\ y(\delta^2 - x^2 - y^2) - 2\delta xy &= -2y\,. \end{aligned}$$

We may divide the latter equation by y; if $y = 0$ we would be lead to the previous real solution of the equation, $x^3 - \delta^2 x + 2\delta = 0$. Hence we obtain

$$y = \pm\sqrt{2 - x^2 - 2\delta x + \delta^2}\,.$$

We plug this into the first equation and get a *linear* equation in x with the solution $x = \dfrac{\delta(1+\delta^2)}{1+2\delta^2}$. This gives $y = \pm\dfrac{\sqrt{2 + 6\delta^2 + 4\delta^4 - \delta^6}}{1 + 2\delta^2}$. Let

$$\delta^2 < \delta^2_{\max} = \frac{1}{3}\left(199 + 3\sqrt{33}\right)^{1/3} + \frac{34}{3}\left(199 + 3\sqrt{33}\right)^{-1/3} + \frac{4}{3} \approx 5.222262523\,. \tag{4.12}$$

Then, y is real and non-zero. The unique solution (z_1, z_2) in the upper half plane is

$$\left(\frac{\delta(1+\delta^2)}{1+2\delta^2} + i\frac{\sqrt{2 + 6\delta^2 + 4\delta^4 - \delta^6}}{1+2\delta^2}, -\frac{\delta(1+\delta^2)}{1+2\delta^2} + i\frac{\sqrt{2 + 6\delta^2 + 4\delta^4 - \delta^6}}{1+2\delta^2}\right)\,. \tag{4.13}$$

Now we discuss the contraction property of the map $z \mapsto \psi(z) = \psi(z, z; \delta, \lambda) : \mathbb{H}^2 \to \mathbb{H}^2$ at the fixed point, $z_f = (z_1, z_2)$. In this case there is no Schwarz Lemma available. In general, a map $\chi = (\chi_1, \chi_2) : \mathbb{H}^2 \to \mathbb{H}^2$ is a contraction at a point $z = (z_1, z_2)$ iff

$$\mathrm{diag}[\mathrm{Im}(\chi_1(z)), \mathrm{Im}(\chi_2(z))]^{-1}\,(D\chi)(z)\,\mathrm{diag}[\mathrm{Im}(z_1), \mathrm{Im}(z_2)] : \mathbb{C}^2 \to \mathbb{C}^2$$

is a contraction with the ℓ^∞ norm. So let us look at the matrix

$$\begin{aligned} \tilde{D}(z_f) &= \mathrm{diag}[\mathrm{Im}(z_f)]^{-1}\,(D\psi)(z_f)\,\mathrm{diag}[\mathrm{Im}(z_f)] \\ &= \begin{bmatrix} \frac{|\delta-\lambda-z_1|^2}{(\delta-\lambda-z_1)^2}\frac{1}{1+\frac{\mathrm{Im}(z_2)}{\mathrm{Im}(z_1)}\left|\frac{\delta-\lambda-z_1}{\delta-\lambda-z_2}\right|^2} & \frac{|\delta-\lambda-z_2|^2}{(\delta-\lambda-z_2)^2}\frac{1}{1+\frac{\mathrm{Im}(z_1)}{\mathrm{Im}(z_2)}\left|\frac{\delta-\lambda-z_2}{\delta-\lambda-z_1}\right|^2} \\ \frac{|\delta+\lambda+z_1|^2}{(\delta+\lambda+z_1)^2}\frac{1}{1+\frac{\mathrm{Im}(z_2)}{\mathrm{Im}(z_1)}\left|\frac{\delta+\lambda+z_1}{\delta+\lambda+z_2}\right|^2} & \frac{|\delta+\lambda+z_2|^2}{(\delta+\lambda+z_2)^2}\frac{1}{1+\frac{\mathrm{Im}(z_1)}{\mathrm{Im}(z_2)}\left|\frac{\delta+\lambda+z_2}{\delta+\lambda+z_1}\right|^2} \end{bmatrix} \\ &=: \begin{bmatrix} e^{i\alpha_{11}}x & e^{i\alpha_{12}}(1-x) \\ e^{i\alpha_{21}}(1-y) & e^{i\alpha_{22}}y \end{bmatrix}. \end{aligned} \tag{4.14}$$

Notice that $\tilde{D}(z_f)$ has the property that $0 < |\tilde{D}_{ij}(z_f)| < 1$ and $\sum_{j=1}^2 |\tilde{D}_{ij}(z_f)| = 1$. If the eigenvalues of this matrix all have absolute value less than one, then some power of the matrix will be a contraction. Since we are interested in the absolute values of eigenvalues of $\tilde{D}(z_f)$ we may as well look at

$$\hat{D} = e^{iA}\begin{bmatrix} 1 & 0 \\ 0 & e^{iB} \end{bmatrix}\begin{bmatrix} e^{i\alpha_{11}}x & e^{i\alpha_{12}}(1-x) \\ e^{i\alpha_{21}}(1-y) & e^{i\alpha_{22}}y \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^{-iB} \end{bmatrix}.$$

A, B are real. Now we may choose $2A = -\alpha_{11} - \alpha_{22}, 2B = \alpha_{12} - \alpha_{21}$. If we let $2\alpha = \alpha_{11} - \alpha_{22}$ and $2\beta = \alpha_{12} + \alpha_{21} - \alpha_{11} - \alpha_{22}$ then

$$\hat{D} = \begin{bmatrix} e^{i\alpha}x & e^{i\beta}(1-x) \\ e^{i\beta}(1-y) & e^{-i\alpha}y \end{bmatrix}. \tag{4.15}$$

Unfortunately, $\hat{D}$ is not a contraction, a situation we already encountered in Proposition 4.3. However, we shall show that unless $\alpha = \beta = 0$, the square of the matrix $\hat{D}$ is contracting in the ℓ^∞-norm, i.e., $\|\hat{D}^2(w)\|_\infty < \|w\|_\infty$ for non-zero $w \in \mathbb{C}^2$. This can be seen as follows. The non-contracting directions of $\hat{D}$ are $\begin{bmatrix} e^{-i(\alpha-\beta)} \\ 1 \end{bmatrix}$ and $\begin{bmatrix} e^{-i(\alpha+\beta)} \\ 1 \end{bmatrix}$. After one more application of $\hat{D}$, the first direction is mapped into $\dfrac{e^{i\beta}}{e^{i(2\beta-\alpha)}(1-y) + e^{-i\alpha}y}$. Now, this equals $e^{-i(\alpha-\beta)}$ or $e^{-i(\alpha+\beta)}$ iff $\alpha = \beta = 0$. The same applies to the other direction.

Now, the case $\alpha = 0, \beta = 0$ corresponds to $\alpha_{11} = \alpha_{22}, \alpha_{12} + \alpha_{21} = 2\alpha_{11}$, respectively. This implies that

$$\frac{|z_1 - \lambda - \delta|^2}{(z_1 - \lambda - \delta)^2} = \frac{|z_2 + \lambda + \delta|^2}{(z_2 + \lambda + \delta)^2}, \quad \frac{|z_1 + \lambda + \delta|^2 |z_2 - \lambda - \delta|^2}{(z_1 + \lambda + \delta)^2 (z_2 - \lambda - \delta)^2} = \frac{|z_1 - \lambda - \delta|^4}{(z_1 - \lambda - \delta)^4}.$$

Since $z \in \mathbb{H}^2$ we obtain that

$$\frac{|z_1 - \lambda - \delta|}{z_1 - \lambda - \delta} = \frac{|z_2 + \lambda + \delta|}{z_2 + \lambda + \delta}. \tag{4.16}$$

We claim that this condition cannot be satisfied at the fixed point unless $\delta = 0$ but we can prove it only in a neighborhood of $\lambda = 0$. By the symmetry of the fixed point equation, $z_2 = -\bar{z}_1$. Combined with (4.16) we obtain $z_1 = \delta + i\gamma$ for some $\gamma \geq 0$. Then we plug this into the fixed point equation and get that

$$\delta + i\gamma = \frac{i}{\gamma} + \frac{1}{2\delta - i\gamma}\,.$$

A comparison of the real part reveals $\gamma = \sqrt{2 - 4\delta^2}$ which we insert into the equation for the imaginary part. This leads to $\delta = 0$.

Thus, we have shown the twofold iteration of the map $z \mapsto \psi(z) = \psi(z, z; \delta, \lambda)$ is a contraction at the fixed point when $\lambda = 0$. This leaves us in a position to prove results analogous to Proposition 4.1, Proposition 4.3 and Proposition 4.4. We omit the details.

4.3 Decaying potentials

In this section we show how our methods can be used to prove results about potentials that decay at infinity. For binary trees, similar results were proved using the Mourre estimate in [AF]. This work was generalized in [GG], where a more abstract setup in Fock space was studied. We briefly consider the one dimensional situation and the tree, and then prove a result about perturbations by decreasing potentials in a more general situation.

We begin with the following lemma, which exploits formula (3.13) and the contractive property of the maps Φ_i.

Lemma 4.5 *Suppose we are given a graph and potential satisfying Hypothesis 3.1, and that* Φ_k *are the maps defined by (3.6). Suppose that we are given a sequence* $Z_k \in \mathbb{SH}_{n_k}$ *for every* λ *in a compact set* $K \subset \{\mathrm{Im}(\lambda) \geq 0\}$*. This sequence need only be defined for* $k \geq k_0$*, and must satisfy*

$$\sum_{k=k_0}^{\infty} d_\infty(\Phi_{k+1}(Z_{k+1}), Z_k) \leq C_1\,. \tag{4.17}$$

and

$$d_\infty(Z_{k_0}, iI) \leq C_2 \tag{4.18}$$

for every $\lambda \in K$*. Then there exists a constant* C_3 *such that for every* $\lambda \in K$

$$d(G(0,0;\lambda), i) < C_3\,.$$

Proof: According to (3.13) there exists $n \geq k_0$ such that $d(G(0,0;\lambda), \Phi_0 \circ \cdots \circ \Phi_n(Z_n)) \leq 1$. We estimate the distance of $G(0,0;\lambda)$ from i, using the fact that the maps Φ_i and their compositions are contractions.

$$
\begin{aligned}
&d(G(0,0;\lambda),i)\\
&\quad\le d_\infty(\Phi_0\circ\cdots\circ\Phi_n(Z_n),i)+1\\
&\quad\le d_\infty(\Phi_0\circ\cdots\circ\Phi_{n-1}(\Phi_n(Z_n)),\Phi_0\circ\cdots\circ\Phi_{n-1}(Z_{n-1}))+d_\infty(\Phi_0\circ\cdots\circ\Phi_{n-1}(Z_{n-1}),i)+1\\
&\quad\le d_\infty(\Phi_n(Z_n),Z_{n-1})+d_\infty(\Phi_0\circ\cdots\circ\Phi_{n-1}(Z_{n-1}),i)+1\\
&\quad\le\cdots\\
&\quad\le\sum_{k=k_0}^{n} d_\infty(\Phi_{k+1}(Z_{k+1}),Z_k)+d_\infty(\Phi_0\circ\cdots\circ\Phi_{k_0}(Z_{k_0}),i)+1\\
&\quad\le C_1+d_\infty(\Phi_0\circ\cdots\circ\Phi_{k_0}(Z_{k_0}),i)+1\,.
\end{aligned}
$$

The second term on the right is bounded because Z_{k_0} varies in a bounded set and $\Phi_0\circ\cdots\circ\Phi_{k_0}$ is a continuous map. □

In the one dimensional case considered in section 2, Φ_k maps $\mathbb{H}$ to $\mathbb{H}$ and possible choices for Z_k are the fixed points of of Φ_k. These are given explicitly by

$$z_k^f=\frac{q_k-\lambda}{2}+\sqrt{\left(\frac{q_k-\lambda}{2}\right)^2-1}\,, \tag{4.19}$$

where the branch of $\sqrt{\cdot}$ is chosen so that $\mathrm{Im}(z_k^f)\ge 0$. This leads to the following result.

Theorem 4.6 *Let q be a potential on $\mathbb{Z}_+$ such that $q_k\to 0$ as $k\to\infty$ and $\sum_{k=0}^\infty |q_{k+1}-q_k|<\infty$. Then for all $0<\mu<2$ and $\epsilon>0$*

$$\sup_{\lambda\in K_{\mu,\epsilon}} d(i,G(0,0;\lambda))<\infty\,,$$

where $K_{\mu,\epsilon}:=\{\lambda\in\mathbb{H}\,:\,|\,\mathrm{Re}\,\lambda|\le\mu\,,\,0<\mathrm{Im}\lambda\le\epsilon\,\}$.

Proof: When $q_k=0$ and $\lambda\in[-\mu,\mu]$ the corresponding fixed point z_k^f given by (4.19) lies on a portion of the unit circle in the upper half plane. By continuity, the values of z_k^f remain in a bounded set when $\lambda\in\overline{K}_{\mu,\epsilon}$ with ϵ and q_k small. Since $q_k\to 0$ we thus obtain (4.18) for k_0 sufficiently large and ϵ sufficiently small. The estimate (4.17) for k_0 sufficiently large follows from the estimate

$$d(\Phi_{k+1}(z_{k+1}^f),z_k^f)=d(z_{k+1}^f,z_k^f)\le C|q_{k+1}-q_k|\,,$$

which holds for $\lambda\in\overline{K}_{\mu,\epsilon}$ and q_k sufficiently small. The theorem now follows from Lemma 4.5. □

We can modify this argument to treat the case of a binary tree. For a multiplication operator q on $\ell^2(S_k)$ define

$$Z(q,\lambda)=\frac{q-\lambda}{2}+\sqrt{\left(\frac{q-\lambda}{2}\right)^2-2}\,.$$

Then $-(Z(q_k,\lambda)-q_k+\lambda)^{-1}=Z(q_k,\lambda)/2$ so that we find that

$$\Phi_{k+1}(Z(q_{k+1}),\lambda)=-E_k^T(Z(q_{k+1},\lambda)-q_{k+1}+\lambda)^{-1}E_k=\frac{1}{2}E_k^T Z(q_{k+1},\lambda)E_k\,.$$

If we use $Z_k=Z(q_k,\lambda)$ as the sequence in Lemma 4.5 we arrive at the following theorem.

Theorem 4.7 *Let $q = (q_0, q_1, \ldots)$ be a potential on the binary tree such that $\|q_k\|_\infty \to 0$,*

$$\sum_k \|\frac{1}{2}E_k^T(q_{k+1}) - q_k\|_\infty < \infty$$

and

$$\sum_k \|q_{k+1} - E_k(q_k)\|_\infty^2 < \infty.$$

(In these formulas we are thinking of q_{k+1} and q_k as vectors rather than diagonal operators, so that we write $E_k^T(q_{k+1})$ rather than using the operator notation $E_k^T q_{k+1} E_k$.)

Then for all $0 < \mu < 2\sqrt{2}$ and $\epsilon > 0$

$$\sup_{\lambda \in K_{\mu,\epsilon}} d(i, G(0,0;\lambda)) < \infty \ ,$$

where $K_{\mu,\epsilon} := \{\lambda \in \mathbb{H} : |\operatorname{Re}\lambda| \leq \mu \, , \, 0 < \operatorname{Im}\lambda \leq \epsilon\}$.

Proof: Let $Z_k = Z(q_k, \lambda)$. To apply Lemma 4.5, we need to show (4.17) and (4.18). To begin, we point out that Z_k may be regarded either as diagonal operators or vectors. It follows from their definition that $\operatorname{Im}(Z_k)$ is bounded below for k sufficiently large, so that q_k is sufficiently small, and $\lambda \in K_{\mu,\epsilon}$. It follows that $\frac{1}{2}E_k^T(Z_{k+1})$ (or, in operator notation, $\frac{1}{2}E_k^T Z_{k+1} E_k$) is also bounded below. Then, by considering a straight line path joining $\Phi_{k+1}(Z_{k+1}) = \frac{1}{2}E_k^T(Z_{k+1})$ to Z_k we find

$$d_\infty(\Phi_{k+1}(Z_{k+1}), Z_k) \leq C\|\frac{1}{2}E_k^T(Z_{k+1}) - Z_k\|_\infty$$

for k sufficiently large. Now $E_k^T(Z_{k+1})$ can be written by first splitting q_{k+1} into two vectors q_{k+1}^{odd} and q_{k+1}^{even} of half the length containing the values of q_{k+1} on alternate sites in the sphere. We then have

$$E_k^T(Z_{k+1}) = Z(q_{k+1}^{\text{odd}}, \lambda) + Z(q_{k+1}^{\text{even}}, \lambda) \, ,$$

so we obtain

$$\frac{1}{2}E_k^T(Z_{k+1}) - Z_k = \frac{1}{2}\big(Z(q_{k+1}^{\text{odd}}, \lambda) - Z(q_k, \lambda)\big) + \frac{1}{2}\big(Z(q_{k+1}^{\text{even}}, \lambda) - Z(q_k, \lambda)\big) \, .$$

The function $Z(q, \lambda)$ is differentiable for λ in a small neighbourhood of $\overline{K}_{\mu,\epsilon}$ and q sufficiently small. Expanding about $q = q_k$ we have

$$Z(q_{k+1}^{\text{odd}}, \lambda) = Z(q_k, \lambda) + Z'(q_k, \lambda)(q_{k+1}^{\text{odd}} - q_k) + \frac{1}{2}Z''(\xi^{\text{odd}}, \lambda)(q_{k+1}^{\text{odd}} - q_k)^2$$

for some ξ^{odd} which is small for k large. This, and a similar expansion for $Z(q_{k+1}^{\text{even}}, \lambda)$ leads to

$$\begin{aligned} &\frac{1}{2}E_k^T(Z_{k+1}) - Z_k \\ &\quad = Z'(q_k, \lambda)\big(\frac{1}{2}(q_{k+1}^{\text{odd}} + q_{k+1}^{\text{even}}) - q_k\big) + Z''(\xi^{\text{odd}}, \lambda)(q_{k+1}^{\text{odd}} - q_k)^2 + \frac{1}{2}Z''(\xi^{\text{even}}, \lambda)(q_{k+1}^{\text{even}} - q_k)^2 \, . \end{aligned}$$

This implies

$$\|\frac{1}{2}E_k^T(Z_{k+1}) - Z_k\|_\infty \leq C\|\frac{1}{2}E_k^T(q_{k+1}) - q_k\|_\infty + \|q_{k+1} - E_k(q_k)\|_\infty^2 \, .$$

Therefore the assumptions on the potential imply (4.17) for k_0 sufficiently large.

The distance, $d_\infty(Z_{k_0}, iI)$ can be estimated by $\|Z_{k_0} - iI\|_\infty$ using a staight line path. This leads to (4.18) and completes the proof. □

We conclude this section with an ℓ^1 perturbation result. The hypotheses of this theorem are satisfied for the transversally periodic and transversally decaying potentials we considered above.

Theorem 4.8 *Suppose we are given a base graph and potential satisfying Hypothesis 3.1, and define* $Q_{0,k} = D_{0,k} + q_{0,k}$ *for this base potential and graph. Let* $\Phi_{0,k}$ *are the maps defined by (3.6) for this base potential, that is,* $\Phi_{0,k}(Z) = -E_k^T(Z - Q_{0,k} - \lambda)^{-1}E_k$*. Assume that for the base potential and graph we have produced a sequence* $Z_k \in \mathbb{SH}_{n_k}$ *with* $\Phi_{0,k+1}(Z_{k+1}) = Z_k$*,* $Y_k = \mathrm{Im}(Z_k) \geq C > 0$ *and* $Z_0 = G_0(0,0;\lambda)$ *bounded for all* λ *in some compact set* $K \subset \mathbb{H}$*.*

Now consider a perturbation where we change D_k *and* q_k*. Let* $Q_k = D_k + q_k$ *for the perturbed graph and let* $\Phi_k(Z) = -E_k^T(Z - Q_k - \lambda)^{-1}E_k$*. Assume*

$$\sum_k \|Q_{0,k} - Q_k\| < \infty\,.$$

Here $\|\cdot\|$ *is the operator norm. Then* $G(0,0;\lambda)$*, the Green's function for the perturbed graph, is also bounded for* $\lambda \in K$*.*

Proof: We apply Lemma 4.5 using the sequence Z_k in the hypothesis and $k_0 = 0$. The estimate (4.18) is true by hypothesis. To show (4.17) we factor $\Phi_{0,k} = \pi_k \circ \rho \circ \tau_{0,k}$ and $\Phi_k = \pi_k \circ \rho \circ \tau_k$, where $\pi_k(Z) = E_k^T Z E_k$, $\rho(Z) = -Z^{-1}$, $\tau_{0,k}(Z) = Z - Q_0 + \lambda$ and $\tau_k(Z) = Z - Q_k + \lambda$. Now, using that π_k and ρ are contractions, we have

$$\begin{aligned}
d_\infty(\Phi_k(Z_k), Z_{k-1}) &= d_\infty(\Phi_k(Z_k), \Phi_{0,k}(Z_k)) \\
&= d_\infty(\pi_k \circ \rho \circ \tau_{0,k}(Z_k), \pi_k \circ \rho \circ \tau_k(Z_k)) \\
&\leq d_\infty(\tau_{0,k}(Z_k), \tau_k(Z_k)) \\
&= d_\infty(Z_k - Q_{0,k} + \lambda, Z_k - Q_k + \lambda) \\
&\leq \|(Y_k + \lambda)^{-1/2}(Q_{0,k} - Q_k)(Y_k + \lambda)^{-1/2}\| \\
&\leq C\|Q_{0,k} - Q_k\|\,.
\end{aligned}$$

Here we estimated the d_∞ distance using a straight line path. Thus the hypothesis on the potential implies (4.17). □

Appendix: The Siegel upper half space

The Siegel upper half space is a generalization of the hyperbolic plane. In this appendix we collect the facts about this space that we need. For more information we refer to the original work of Siegel [S] and the article of Freitas [F].

Definition: The *Siegel upper half space* $\mathbb{SH}_n$ is the set of $n \times n$ matrices of the form $Z = X + iY$, where X and Y are real symmetric matrices, and Y is positive definite.

There are other models of $\mathbb{SH}_n$ described in [F]. In particular, we will identify the Siegel upper half space $\mathbb{SH}_n$ with a homogeneous space of the group of symplectic matrices.

Definition: The symplectic group $\mathrm{Sp}_{2n}(\mathbb{R})$ is the group of all real $2n \times 2n$ matrices S satisfying

$$S^T J S = J\,, \quad \text{with } J = \begin{bmatrix} 0 & 1_n \\ -1_n & 0 \end{bmatrix}\,,$$

where 1_n is the identity matrix in n dimensions.

To relate the symplectic group to the Siegel upper half space, we write a symplectic $2n \times 2n$ matrix S in terms of four $n \times n$ matrices, A, B, C, D, i.e.,

$$S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\,.$$

It was shown in [S] that for each symplectic matrix S, the map

$$\Phi_S : \mathbb{SH}_n \to \mathbb{SH}_n\,, \quad Z \mapsto (AZ+B)(CZ+D)^{-1}$$

is well defined. Moreover, $\Phi_S \circ \Phi_R = \Phi_{SR}$, $\Phi_{1_{2n}} = \mathrm{id}$, and Φ_S acts transitively on $\mathbb{SH}_n$. A standard calculation shows that the set of elements U in $\mathrm{Sp}_{2n}(\mathbb{R})$ for which $\Phi_U(i1) = i1$ is precisely the subgroup of orthogonal matrices in $\mathrm{Sp}_{2n}(\mathbb{R})$, which we shall denote by K. In particular, this yields the bijective map Ψ

$$\Psi : \mathrm{Sp}_{2n}(\mathbb{R})/K \to \mathbb{SH}_n\,, \quad SK \mapsto \Phi_S(i1)\,. \tag{5.1}$$

For $Z = X + iY \in \mathbb{SH}_n$ we define the symplectic matrix

$$S(X+iY) = \begin{bmatrix} Y^{1/2} & XY^{-1/2} \\ 0 & Y^{-1/2} \end{bmatrix}\,. \tag{5.2}$$

It follows that the map $Z \mapsto S(Z)\,K$ is the inverse of the map (5.1), since $\Phi_{S(Z)}(i1) = Z$. We define the following metric on $\mathrm{Sp}_{2n}(\mathbb{R})/K$

$$d_\infty^S(S_1K, S_2K) = 2\ln \|S_1^{-1}S_2\|\,.$$

It is clear that the expression on the left is independent of the choice of representative in the equivalence class. That this defines indeed a metric, is shown for example in [F]. At the end of this section we will prove the following

Theorem 5.1 *The map Ψ is an isometry for the metrics d_∞^S and d_∞. In particular,*

$$d_\infty(Z_1, Z_2) = 2\ln \|S(Z_1)^{-1}S(Z_2)\|\,. \tag{5.3}$$

Theorem 5.1 implies that for all symplectic matrices S, the map Φ_S acts as an isometry on $\mathbb{SH}_n$ for the metric d_∞. Inserting the expression (5.2) into (5.3), we find

$$d_\infty(X_1+iY_1, X_2+iY_2) = 2\ln \left\| \begin{bmatrix} Y_1^{-1/2}Y_2^{1/2} & Y_1^{-1/2}(X_2-X_1)Y_2^{-1/2} \\ 0 & Y_1^{1/2}Y_2^{-1/2} \end{bmatrix} \right\|\,. \tag{5.4}$$

We shall now show inequalities (5.5) and (5.6), which were used in the proof of Lemma 3.5. As a direct consequence of (5.4) we have

$$d_\infty(iI, iY) = \max\{\ln(\|Y\|), \ln(\|Y^{-1}\|)\} \ . \tag{5.5}$$

Using the explicit expression (5.2), one can show that

$$(S^{-1}_{X_1+iY_1} S_{X_2+iY_2})^T S^{-1}_{X_1+iY_1} S_{X_2+iY_2} \geq (S^{-1}_{iY_1} S_{iY_2})^T S^{-1}_{iY_1} S_{iY_2} \ .$$

This implies, by (5.3), that

$$d_\infty(iY_1, iY_2) \leq d_\infty(X_1 + iY_1, X_2 + iY_2) \ . \tag{5.6}$$

In the case where $n = 1$ the Siegel upper half space $\mathbb{SH}_1 = \mathbb{H}$. For $z_1, z_2 \in \mathbb{H}$ the metric takes the following form

$$d_\infty(z_1, z_2) = d(z_1, z_2) = \cosh^{-1}\left(1 + \frac{|z_1 - z_2|^2}{2\mathrm{Im} z_1 \mathrm{Im} z_2}\right) \ . \tag{5.7}$$

Remark: We note that the above statements can be generalized to p-metrics. For $p \geq 1$, we introduce the trace norm $\|A\|_p = (\mathrm{Tr}|A|^p)^{1/p}$, and define the p-metrics on the homogeneous space $\mathrm{Sp}_{2n}(\mathbb{R})/K$ by

$$d^S_p(S_1K, S_2K) = 2\|(\ln|S_1^{-1}S_2|)_+\|_p \ ,$$

where $(\,\cdot\,)_+$ denotes the positive part. Likewise we define $d_p(Z_1, Z_2)$ on $\mathbb{SH}_n$

$$d_p(Z_1, Z_2) = \inf_{Z(t)} \int_0^1 \|Y^{-1/2}\, dZ\, Y^{-1/2}\|_p \ ,$$

where the infimum is taken over all continuously differentiable curves $Z : [0, 1] \to \mathbb{SH}_n$ with $Z(0) = Z_1$ and $Z(1) = Z_2$. We note that Siegel [S] considered the case $p = 2$, for which he derived analogous statements. We want to point out that the map Ψ defined in (5.1) is an isometry for all p-metrics ($p \geq 1$). The proof of Theorem 5.1 generalizes in a straightforward way to that case.

Proof of Theorem 5.1: Note that $d^S_\infty(S(Z_1)K, S(Z_2)K)$ defines a metric on $\mathbb{SH}_n$. We need to show that this metric agrees with distance d_∞ defined by the Finsler metric (3.8). For notational convenience we set $d^S_\infty(Z_1, Z_2) = d^S_\infty(S(Z_1)K, S(Z_2)K)$

(1) We shall first show that $d^S_\infty(Z_1, Z_2) \leq d_\infty(Z_1, Z_2)$. A calculation reveals that

$$d^S_\infty(Z + \xi, Z) = F_Z(\xi) + r(Z, \xi) \ , \tag{5.8}$$

where $r(Z, \xi)$ is a function such that $\lim_{\|\xi\|\to 0} |r(Z, \xi)|/\|\xi\| = 0$ uniformly in Z, for Z on compact subsets of $\mathbb{SH}_n$. Let γ be a continuously differentiable path with $\gamma(0) = Z_1$ and $\gamma(1) = Z_2$. Then, by the triangle inequality,

$$d^S_\infty(Z_1, Z_2) \leq \sum_{i=1}^n d^S_\infty(\gamma(t_{i-1}), \gamma(t_i)) \ ,$$

where $t_0 = 0 < t_1 < ... < t_n = 1$ is a partition with $t_i - t_{i-1} = 1/n$. By (5.8), in the limit as n tends to infinity, we have

$$d_\infty^S(Z_1, Z_2) \leq \int_0^1 F_{\gamma(t)}(\dot{\gamma}(t))\, dt \ .$$

Since the curve γ is arbitrary,

$$d_\infty^S(Z_1, Z_2) \leq d_\infty(Z_1, Z_2) \ .$$

(2) We shall now show the opposite inequality: $d_\infty^S(Z_1, Z_2) \geq d_\infty(Z_1, Z_2)$. Since $S(Z_1)^{-1}S(Z_2)$ is a symplectic matrix there exist orthogonal and symplectic matrices $R_1, R_2 \in K$, such that $R_1 S(Z_1)^{-1} S(Z_2) R_2$ is diagonal. For $0 \leq t \leq 1$, define the path

$$\gamma(t) = \Psi\big(S(Z_1) R_1^{-1} (R_1 S(Z_1)^{-1} S(Z_2) R_2)^t\big) \, .$$

It is clear that $\gamma(0) = Z_1$ and $\gamma(1) = Z_2$. For $0 \leq s, t \leq 1$,

$$d_\infty^S(\gamma(t), \gamma(s)) = 2 \ln \|(R_1 S(Z_1)^{-1} S(Z_2) R_2)^{s-t}\| = |s - t| d_\infty^S(\gamma(0), \gamma(1)) \ .$$

This implies

$$d_\infty^S(Z_1, Z_2) = \sum_{i=1}^{n} d_\infty^S(\gamma(t_{i-1}), \gamma(t_i)) \ ,$$

for partitions as above. As $n \to \infty$,

$$d_\infty^S(Z_1, Z_2) = \int_0^1 F_{\gamma(t)}(\dot{\gamma}(t))\, dt \ .$$

Taking the infimum over all paths yields the desired inequality. ☐

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