EXPONENTIAL ERGODICITY FOR STOCHASTIC BURGERS AND 2D NAVIER-STOKES EQUATIONS

B. GOLDYS AND B. MASLOWSKI

Abstract. It is shown that transition measures of the stochastic Navier-Stokes equation in $2D$ converge exponentially fast to the corresponding invariant measures in the distance of total variation. As a corollary we obtain the existence of spectral gap for a related semigroup obtained by a sort of ground state trasformation. Analogous results are proved for the stochastic Burgers equation.

1. Introduction

In this paper we study ergodic behaviour of two important equations arising in Statistical Physics: the stochastic Burgers equation and the stochastic Navier-Stokes equation in $2D$. In both cases we assume that the random forcing is correlated in space and white in time. The problem of ergodicity and the rate of convergence to invariant measure in various norms for those two equations was an object of intense research in recent years. In the paper [15] the existence and uniqueness of invariant measure for the stochastic Navier-Stokes equation in $2D$ was proved in the case when the random force is sufficiently close to the space-time white noise. The exponential rate of convergence of transition measures to the invariant measure μ of the stochastic Navier-Stokes equation was proved for the first time in [1] for μ-almost every initial condition and subsequently for every square integrable initial condition in [22] and [26] (see also [10]). In all those papers various versions of coupling technique were applied to prove the convergence properties in metrics equivalent to the topology of weak convergence of measures (or an intermediate metric, cf. [26]). The coupling method proved also to be useful to handle random forces which are degenerated in space. For Navier-Stokes equation perurbed by finite-dimensional Wiener process, uniqueness of invariant measure has been proven in a recent paper [18]. Let us note also that similar result were obtained for the forcing consisting of a sequence of random excitations arising in discrete moments of time (random kicks), see e.g. [21].

In this work we continue the approach initiated in [15] assuming that the random force is sufficiently nondegenerate. In particular, we are using results of [11], [12], [3] and [9], where the strong Feller property and irreducubility have been proven in an appropriately chosen state space for particular cases of the stochastic Navier-Stokes equation and stochastic Burgers equation. Our main result may be described as follows. Let $\{u(t,\zeta) : t \geqslant 0, \zeta \in \mathscr{O}\}$ be a solution to either the stochastic Navier-Stokes equation (in which case $\mathscr{O}$ is a bounded domain in $\mathbb{R}^2$) or a solution to the stochastic Burgers equation (and then $\mathscr{O} = (0,1)$) and let μ be the corresponding invariant measure. Then for any initial distribution ν of the $L^2(\mathscr{O})$-valued

Key words and phrases. Navier-Stokes equation, Burgers equation, additive noise, invariant measure, uniform exponential ergodicity, spectral gap.

This work was partially supported by the ARC Discovery Grant DP0346406, the UNSW Faculty Research Grant PS05345 and the GAČR Grant 201/04/0750.

random variable $u(0,\cdot)$ the probability distribution $P_t^*\nu$ of the random variable $u(t,\cdot)$ enjoys the property

$$\|P_t^*\nu-\mu\|_{var}\leqslant\|P_t^*\nu-\mu\|_V\leqslant Ce^{-\beta t}\|\nu\|_V, \tag{1.1}$$

where $\|\cdot\|_{var}$ denotes the norm of total variation of measures and $\|\nu\|_V$ stands for the norm of total variation of the measure $Vd\nu$ considered on $L^2(\mathscr{O})$. The function $V:L^2(\mathscr{O})\to[1,\infty)$ in (1.1) is an appropriate Lyapunov function. A class of functions V is found for which (1.1) holds is also provided and shown to include $V(x)=1+|x|_{L^2}^p$ (for $p>0$) and $V(x)=\exp\left(|x|_{L^2}^{2\alpha}\right)$ (for $\alpha\in(0,1)$). Finally, we derive from (1.1) the spectral gap property of the V-transform $\left(P_t^V\right)$

$$P_t^V\phi(x)=V^{-1}(x)\mathbb{E}\left(V\left(u(t,\cdot)\right)\phi(u(t,\cdot))\right)$$

of the semigroup (P_t). Namely, we show that for $\phi\in L^p(H,V\mu)$, $p\in(1,\infty)$

$$\int_{L^2}\left|P_t^V\phi-V^{-1}\int_{L^2}\phi Vd\mu\right|^pVd\mu\leqslant C_pe^{-\beta t/p}\int_{L^2}|\phi|^pVd\mu. \tag{1.2}$$

Exponential convergence to the invariant measure in the distance of total variation and the spectral gap property (1.2) seem to be new for both equations studied in this paper. The main idea of the proof consists in verifying the V-uniform ergodicity for a skeleton process and to this end a geometric drift towards a nontrivial small set must be shown. It is proven that levelsets of the function V are nontrivial small sets and then the corresponding Lyapunov-Foster condition is verified by means of Ito formula. The proof is given for a general Markov process taking values in a Polish space and satisfying conditions (H1)-(H4) (cf. Section 3) and then these "abstract" conditions are verified for Markov processes defined by stochastic Navier-Stokes and Burgers equations, respectively, under suitable assumptions on the correlation of the noise. It may be expected that these assumptions are not optimal, it should be possible to find other sets of conditions implying (H1)-(H4) in particular cases.

The general scheme of the proof of V-uniform ergodicity of the skeleton exploits well-known results from the theory of Markov chains (cf. [23]), similar idea was used in [27] for some stochastic semilinear equations. For stochastic reaction-diffusion equations V-uniform ergodicity has been proven recently by a slightly different method that allows to give some explicit bounds on the convergence rate ([17]).

It may be interesting to note that we also obtain an independent proof of existence of invariant measure for both Navier-Stokes and Burgers equation (it is an easy consequence of (3.2) and (3.9), cf. [24] or [25] for similar results), which however is known in both cases. The paper is divided into five sections including the Introduction. In Section 2 we provide a rigorous framework for our results and formulate the main theorems, separately for stochastic Navier-Stokes and Burgers equations (the Section is divided into two parts). In the case of Navier-Stokes equation only the Dirichlet type boundary conditions are studied in detail, although the case of periodic boundary conditions may be treated as well; only minor changes in the proofs are needed and the noise may be even more "degenerate" (Hypothesis 2.23 and Theorem 2.13). In Section 3 we study V-uniform ergodicity for a general Markov process in in a Polish space. These results are applied in Sections 4 and 5 to the stochastic Navier-Stokes and Burgers equations, respectively.

In order to avoid clumsy notation, the same symbols may sometimes have different meaning in different sections. All results concerning the Navier-Stokes and Burgers equations are given in separate sections, so there is no risk of confusion. For example, H and (X_t) are the state space and solution to the Navier-Stokes equation (in Sections 2.1 and 4) or the Burgers equation (in Sections 2.2 and 5). The symbols (P_t) and P_t^* denote the Markov semigroups of a general Markov process taking values in a Polish space E (in Section 3 and the general Definition 2.4), or of the Markov processes defined by the Navier-Stokes equation (Sections 2.1 and 4) or the Burgers equation (Sections 2.2 and 5). The notation is explained at the beginning of particular sections.

ACKNOWLEDGEMENT. The authors would like to thank to Franco Flandoli for his valuable remarks.

2. Main Results

2.1. **Navier-Stokes Equation.**

In this Section the equation

$$\frac{\partial u}{\partial t}(t,\zeta)-\nu\Delta u(t,\zeta)+(u(t,\zeta)\cdot\nabla)u(t,\zeta)+\nabla p(t,\zeta)=f(\zeta)+\eta(t,\zeta),\quad (t,\zeta)\in(0,\infty)\times\mathscr{O}, \tag{2.1}$$

is considered, where $u(t,\zeta)=(u_1(t,\zeta),u_2(t,\zeta))$ and $p(t,\zeta)$ denote the velocity and pressure fields respectively, of a viscous incompressible fluid in a bounded domain $\mathscr{O}\subset\mathbb{R}^2$ with a smooth boundary $\partial\mathscr{O}$, f is a deterministic external force, η is a random forcing of white noise type and $\nu>0$ denotes the viscosity. Incompressibility condition reads

$$\operatorname{div} u(t,\zeta)=0,\quad (t,\zeta)\in[0,\infty)\times\mathscr{O}, \tag{2.2}$$

and initial and the Dirichlet boundary conditions

$$u(0,\zeta)=u_0(\zeta),\quad \zeta\in\mathscr{O}, \tag{2.3}$$

and

$$u(t,\zeta)=0\quad\text{for}\quad (t,\zeta)\in[0,\infty)\times\partial\mathscr{O}, \tag{2.4}$$

are considered. We will study an abstract version of system (2.1)-(2.4) and its weak solution (cf. [13]). Set

$$H=\left\{x\in\left(L^2(\mathscr{O})\right)^2:\operatorname{div} x=0,\ x\cdot n|_{\partial\mathscr{O}}=0\right\},$$

and

$$V=\left\{x\in\left(H_0^1(\mathscr{O})\right)^2:\ \operatorname{div} x=0\right\},$$

where n is the outward normal to $\partial\mathscr{O}$ (cf. [28] for an interpretation of the condition $x\cdot n=0$). Identifying H^1 with a subspace of V' (the dual space of V) we have $V\subset H\subset V'$ and (if there is no danger of confusion) $\langle\cdot,\cdot\rangle$ stands for the pairing between V and V'. Furthermore, define a closed operator A in H by the formula

$$Ax=-\nu\Pi\Delta x,\quad \operatorname{dom}(A)=\left(H^2(\mathscr{O})\right)^2\cap V,$$

where Π is the orthogonal projection of $\left(L^2(\mathscr{O})\right)^2$ onto H. The space V coincides with $\operatorname{dom}\left(A^{1/2}\right)$ and is endowed with the norm $|x|_V=\left|A^{1/2}x\right|$. The operator A is strictly positive, selfadjoint and its resolvent is compact. For $\beta>0$ we will denote by H_β the domain of fractional power A^β equipped with the norm $|x|_\beta=\left|A^\beta x\right|$.
The bilinear operator $B:V\times V\to V'$ is defined as

$$\langle B(u,v),z\rangle=\int_{\mathscr{O}} z(\zeta)\cdot(u(\zeta)\cdot\nabla)v(\zeta)d\zeta,\quad u,v,z\in V. \tag{2.5}$$

Then we may rewrite system (2.1)-(2.4) in the abstract form

$$\left\{\begin{array}{l} dX_t + (AX_t + B(X_t, X_t))\,dt = f dt + G dW_t, \quad t \geqslant 0, \\ X_0 = x. \end{array}\right. \tag{2.6}$$

In the equation above X_t is identified with $u(t,\cdot)$ and x with $u_0(\cdot)$. The noise in (2.6) is modelled as a standard cylindrical Wiener process (W_t) on H defined on a stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$, $f \in H$ and G is a bounded injective linear operator on H. Following [13] we adopt a definition of solution to equation (2.6) resembling the classical definition of a weak solution as understood in the theory of deterministic PDE's.

Definition 2.1. *A progressively measurable process $(X_t) = (X_t^x)$ is a (generalised) solution to equation (2.6) if $X^x \in C([0,T];H) \cap L^2\left([0,T];H_{1/4}\right)$ $\mathbb{P}$-a.s. and*

$$\langle X_t, y\rangle + \int_0^t \langle X_s, Ay\rangle\, ds - \int_0^t \langle B(X_s, y), X_s\rangle\, ds = \langle x, y\rangle + t\langle f, y\rangle + \langle W_t, G^* y\rangle,$$

$\mathbb{P}$-a.s. for all $x \in H$, $t \in [0,T]$ and $y \in \mathrm{dom}(A)$.

Remark 2.2. By the incompressibility condition we obtain

$$\langle B(u,v), z\rangle = -\langle B(u,z), v\rangle, \quad u, v, z \in V, \tag{2.7}$$

and by the Sobolev embedding theorem there exists a universal constant C such that

$$|\langle B(u,v), u\rangle| \leqslant C|v|_V |u|^2_{L^4(\mathscr{O})} \leqslant C|v|_V |u|^2_{1/4},$$

which justifies Definition 2.1.

Assume that

$$\mathrm{im}(G) \subset \mathrm{dom}\left(A^{\frac{1}{4}+\epsilon}\right), \tag{2.8}$$

for some $\epsilon > 0$ and consider an Ornstein-Uhlenbeck process defined by the equation

$$\left\{\begin{array}{l} dZ_t + AZ_t dt = G dW_t, \\ Z_0 = 0. \end{array}\right. \tag{2.9}$$

It is well known, (see e.g. [8]) that under assumption (2.8) equation (2.9) has a unique progressively measurable mild solution (Z_t) taking values in $\mathrm{dom}\left(A^{1/4}\right)$ $\mathbb{P}$-a.s. The following result has been proven in [13].

Proposition 2.3. *Assume (2.8). Then for each initial condition $x \in H$ there exists a unique solution X^x to equation (2.6), which additionally enjoys the property*

$$X^x - Z \in L^2([0,T];V) \quad \mathbb{P} - a.s. \tag{2.10}$$

Moreover, the family of solutions $\{X^x : x \in H\}$ forms a Markov family which satisfies the Feller property and has an invariant measure μ_{NS}.

Our next aim is to define the concept of V-uniform ergodicity of a Markov semigroup. The definition is formulated for a general Markov process (X_t) with values in a Polish space E. Let $b\mathscr{B}$ denote the space of bounded Borel functions on E and let (P_t), (P_t^*) and $(P(t,x,\cdot)$ denote the Markov semigroup on $b\mathscr{B}$, the adjoint Markov semigroup on the space $\mathscr{P}$ of probability measures on E, and the transition probability measures, respectively, associated to the process (X_t^x). More precisely, for any $t \geqslant 0$

$$P_t : b\mathscr{B} \to b\mathscr{B}, \quad P_t\phi(x) = \mathbb{E}_x \phi(X_t), \tag{2.11}$$

$$P_t\phi(x) = \int_E \phi(y)P(t,x,dy), \quad \phi \in b\mathscr{B}, \quad x \in E, \tag{2.12}$$

and

$$P_t^* : \mathscr{P} \to \mathscr{P}, \quad P_t^*\nu = \int_E P(t,x,\cdot)\nu(dx), \tag{2.13}$$

where $\mathbb{E}_x$ denotes the expectation corresponding to the initial condition $X_0 = x$. Obviously, $P_t^*\nu$ may be interpreted as the probability distribution of X_t, ν being the initial distribution.

Let $V : E \to [1,\infty)$ be a measurable function and let $b_V\mathscr{B}$ denote the space of Borel functions on E endowed with the norm

$$\|\phi\|_V = \sup_{x\in E} \frac{|\phi(x)|}{V(x)}.$$

Definition 2.4. *The Markov semigroup (P_t) is said to be V-uniformly ergodic if (P_t) extends to operator $P_t : b_V\mathscr{B} \to b_V\mathscr{B}$ and there exist $C > 0$ and $\omega > 0$ such that*

$$\sup_{|\phi|_V \leqslant 1} |P_t\phi(x) - \langle\phi,\mu\rangle| \leqslant CV(x)e^{-\omega t}, \quad t \geqslant 0 \;\; x \in E, \tag{2.14}$$

where $\mu \in \mathscr{P}$ is the invariant measure.

Let $\|\nu\|_{var}$ denote the norm of total variation of a signed measure ν and let $\|\nu\|_V$ denote the so-called V-variation:

$$\|\nu\|_V = \sup_{\|\phi\|_V \leqslant 1} |\langle\nu,\phi\rangle| = \|V\nu\|_{var}, \tag{2.15}$$

where we denoted by $V\nu$ the measure $d\rho = V d\nu$. Obviously $\|\nu\|_{var} \leqslant \|\nu\|_V$. In terms of the adjoint Markov semigroup (2.14) implies

$$\|P_t^*\nu - \mu\|_{var} \leqslant \|P_t^*\nu - \mu\|_V \leqslant C\,\|\nu\|_V\, e^{-\omega t}, \quad t \geqslant 0,\; \nu \in \mathscr{P}. \tag{2.16}$$

Note that one can have $\|\nu\|_V = \langle\nu, V\rangle = \infty$.

In the rest of Section 2.1 (P_t) and (P_t^*) will denote the Markov semigroups defined by the solution of the Navier-Stokes equation (2.6) on the space $E = H$. Under suitable nondegeneracy conditions on G it has been proven (cf. [15], [11]) that $P_t^*\nu \to \mu_{NS}$ as $t \to \infty$ in the metric of total variation of measures for each initial measure $\nu \in \mathscr{P}$. Building upon these results we aim at proving the V-uniform ergodicity under suitable assumptions on G and V.

Hypothesis 2.5. *The operator G is Hilbert-Schmidt on H and there exist $\alpha \in (\frac{1}{4}, \frac{1}{2})$ and $\epsilon > 0$ such that*

$$\mathrm{dom}\left(A^{2\alpha}\right) \subset \mathrm{im}(G) \subset \mathrm{dom}\left(A^{\frac{1}{4}+\frac{\alpha}{2}+\epsilon}\right). \tag{2.17}$$

The second inclusion in (2.17) is slightly stronger than (2.8) because we need more regularity of the solution to (2.6). The first inclusion is a nondegeneracy condition.

Hypothesis 2.6. *The function $V : H \to [1,\infty)$ is measurable and*

$$c_1\Phi\left(|x|^2\right) \leqslant V(x) \leqslant c_2\Phi\left(|x|^2\right), \quad x \in H, \tag{2.18}$$

where $\Phi \in C^2(\mathbb{R}_+)$, $\Phi \geqslant 1$, $\Phi' \geqslant 0$,

$$\lim_{r\to\infty} \Phi(r) = \infty,$$

and for any $\alpha, k > 0$ there exist $\beta, C > 0$ such that

$$-\alpha r\Phi(r) + k\left(r\left|\Phi''(r)\right| + r^{1/2}\Phi'(r)\right) \leqslant -\beta\Phi(r) + C, \tag{2.19}$$

for $r>0$ large enough.

Example 2.7. It is easy to see that the functions

$$V(x)=1+|x|^p, \quad p>0, \tag{2.20}$$

and

$$V(x)=e^{|x|^{2\alpha}}, \quad \alpha\in(0,1), \tag{2.21}$$

satisfy conditions of Hypothesis 2.6.

Remark 2.8. (i) Without loss of generality we may suppose that (2.19) holds for all $r>0$ changing perhaps the constant C.
(ii) If two functions U and V satisfy Hypothesis 2.6 then $\alpha U+\beta V$ and $\alpha V(\beta x)$ satisfy this hypothesis for any $\alpha,\beta>0$ as well.

The following is our main result on the stochastic Navier-Stokes equation with the Dirichlet boundary conditions.

Theorem 2.9. *Let the operator G and the function V satisfy Hypotheses 2.5 and 2.6 respectively. Then the Markov semigroup associated to the Navier-Stokes equation (2.6) is V-uniformly ergodic, i.e. (2.14) and (2.16) hold true with $E=H$ and $\mu=\mu_{NS}$, where (P_t) and (P_t^*) denote the Markov semigroups defined by the equation (2.6). In particular, V may be defined by (2.20) or (2.21).*

We will denote by $\|\phi\|_p$ the norm

$$\|\phi\|_p^p=\int_H|\phi(x)|^pV(x)\mu_{NS}(dx)$$

of a function ϕ in the space $L^p(H,V\mu_{NS})$. Let

$$\left(\frac{1}{V}P_tV\right)\phi(x)=\frac{1}{V(x)}P_t(V\phi)(x).$$

Theorem 2.10. *The family of operators $\left(\frac{1}{V}P_tV\right)$ defines a C_0-semigroup on $L^p(H,V\mu_{NS})$ for $p\in(1,\infty)$ and $V\mu_{NS}$ is an invariant measure for this semigroup. Moreover, there exist $\beta,C_p>0$ such that*

$$\left\|\frac{1}{V}P_tV\phi-\langle V\mu_{NS},\phi\rangle\frac{1}{V}\right\|_p\leqslant C_Pe^{-\beta t/P}\|\phi\|_p, \quad \phi\in L^p(H,V\mu_{NS}). \tag{2.22}$$

Remark 2.11. **(Stochastic Navier-Stokes equation with periodic boundary conditions)** Using the results from the B.Ferrario's paper [12] it is possible to make similar conclusions in case when the Dirichlet boundary conditions (2.4) are replaced by periodic ones. It may be interesting that in such case the noise may be "more degenerate". More specifically, consider the problem 2.1)-(2.3), where $\mathscr{O}=(0,L_1)\times(0,L_2)$, endowed with the periodic boundary conditions

$$u(t,\xi+L_i\eta_i)=u(t,\xi), \quad t>0,\ \xi\in\mathbb{R}^2,\ i=1,2, \tag{2.23}$$

where (η_1,η_2) is the canonical basis of $\mathbb{R}^2$ and L_i is the period in the $i$the direction. The system (2.1)-(2.3), (2.23), may be formalized in terms of the abstract equation of the form

(2.6) (see e.g. [12] for details), where the state spaces are defined by means of the space $(\mathring{H}_p^m(\mathscr{O}))^2$ of functions from $(H_{loc}^m(\mathbb{R}^2))^2$ with zero average and the period (L_1, L_2), e.g.

$$H = \{u \in (\mathring{H}_p^0(\mathscr{O}))^2;\ \operatorname{div} u = 0\}, \quad V = \{u \in (\mathring{H}_p^1(\mathscr{O}))^2;\ \operatorname{div} u = 0\}.$$

Then $A = -\nu\Delta$ with $\operatorname{dom}(A) = (\mathring{H}_p^2((O)))^2 \cap H$ and the spaces H_α are defined again as domains of fractional powers of the operator A equipped with the graph norm. The Wiener process W_t is standard cylindrical on H and G is a bounded injective linear operator on H. The following is our main condition on the noise term:

Hypothesis 2.12. *There exist $\alpha > 1$ and $\epsilon > 0$ such that*

$$\operatorname{dom}(A^{\frac{\alpha}{2}+\frac{1}{2}}) \subset \operatorname{im}(G) \subset \operatorname{dom}(A^{\frac{\alpha}{2}+\epsilon}). \tag{2.24}$$

The existence and uniqueness of solutions, the Markov property and existence of an invariant measure may be shown as in Proposition 2.3, cf. [13] and [12]. Our result on Navier-Stokes equation with periodic boundary conditions is formulated as follows:

Theorem 2.13. *Let $f \in H_{\frac{\alpha}{2}-\frac{1}{2}}$ and assume that Hypotheses 2.6 and 2.12 are satisfied. Then the conclusions of Theorem 2.9 (V-uniform ergodicity) and Theorem 2.10 (the spectral gap) hold true for the Markov semigroups (P_t) and (P_t^*) defined by the system (2.1)-(2.3), (2.23).*

2.2. **Stochastic Burgers Equation.** In this section we study the stochastic Burgers equation

$$\frac{\partial u}{\partial t}(t,\zeta) - \nu\frac{\partial^2 u}{\partial \zeta^2}(t,\zeta) = \frac{1}{2}\frac{\partial\left(u^2\right)}{\partial\zeta}(t,\zeta) + \eta(t,\zeta), \quad (t,\zeta) \in (0,\infty)\times(0,1), \tag{2.25}$$

with viscosity $\nu > 0$, the Dirichlet boundary conditions

$$u(t,0) = u(t,1) = 0, \quad t \geqslant 0, \tag{2.26}$$

and the initial condition

$$u(0,\zeta) = u_0(\zeta), \quad \zeta \in [0,1]. \tag{2.27}$$

Following a well known approach e.g. [5], [6], [9], we will rewrite system (2.25)-(2.27) as an evolution equation

$$\begin{cases} dX_t + AX_t dt = \frac{1}{2}D_\zeta\left(X_t^2\right)dt + GdW_t, \\ X_0 = x, \end{cases} \tag{2.28}$$

in the space $H = L^2(0,1)$, where $X_t = X_t^x$ is identified with $u(t,\cdot)$ and x with $u_0(\cdot) \in H$. In equation (2.28), (W_t) stands for a standard cylindrical Wiener process in H defined on a stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$, $G \in H \to H$ is a bounded operator, D_ζ is the distributional derivative operator and

$$A = -\nu\frac{\partial^2}{\partial\zeta^2}, \quad \operatorname{dom}(A) = H^2(0,1) \cap H_0^1(0,1).$$

Denote by $\left(e^{-tA}\right)$ a symmetric C_0-semigroup generated by A in H. Similarly as in the previous Section we define the Ornstein-Uhlenbeck process

$$Z_t = \int_0^t e^{-(t-s)A}GdW_s, \quad t \geqslant 0. \tag{2.29}$$

It is well known that $Z \in C\left([0,T];C(0,1)\right)$ $\mathbb{P}$-a.s. (cf. p. 14 of [8]). The difference $Y_t^x = X_t^x - Z_t$ satisfies formally the equation

$$\begin{cases} \frac{d}{dt}Y_t^x + AY_t^x = \frac{1}{2}D_\zeta\left(\left(Y_t^x + Z_t\right)^2\right), & t > 0, \\ Y_0^x = x. \end{cases} \tag{2.30}$$

We will define (Y_t^x) as a solution to the integral equation

$$Y_t^x = e^{-tA}x + \frac{1}{2}\int_0^t e^{-(t-s)A}D_\zeta\left(\left(Y_s^x + Z_s\right)^2\right)ds, \quad t \geqslant 0. \tag{2.31}$$

Definition 2.14. *A process $X^x \in C([0,T];H)$ is said to be a mild solution of equation (2.28) if and only if $Y^x = X^x - Z$ a solution to equation (2.31).*

Proposition 2.15. *(cf.* [9]*, pp. 262 and 273) For any $x \in H$ there exists a unique mild solution to equation (2.28) and there exists an invariant measure μ_B for the Markov semigroup associated to equation (2.28).*

Hypothesis 2.16. *The operator $Q = GG^*$ has the following properties:* $\operatorname{tr}(Q) < \infty$ *and for some $\delta \in \left(\frac{1}{2}, 1\right)$*

$$\operatorname{im}\left(A^{\delta/2}\right) \subset \operatorname{im}\left(Q^{1/2}\right). \tag{2.32}$$

Under the nondegeneracy condition (2.32) Da Prato and Debussche proved in [3] that the Markov semigroup associated to (2.28) is strongly Feller in H. Irreducibility has been proven for $Q = I$ by Da Prato and Gątarek in [6] (cf. also [9]), it is however easy to adapt their proof to the present case (see Proposition 5.1 for a sketch of the proof). Therefore, for each initial measure $\nu \in \mathscr{P}$ the probability distributions $P_t^*\nu$ converge to μ_B, as $t \to \infty$, in the norm of total variation. We shall prove a stronger result.

Theorem 2.17. *Let the operator Q and the function V satisfy Hypotheses 2.32 and 2.6, respectively. Then the V-uniform ergodicity (2.14) and (2.16) holds true, where the transition semigroup (P_t) and its adjoint (P_t^*) are associated to equation (2.28), $E = H$ and $\mu = \mu_B$. In particular, V may be defined as in (2.20) or (2.21).*

We will denote by $\|\phi\|_p$ the norm

$$\|\phi\|_p^p = \int_H |\phi(x)|^p V(x)\mu_B(dx)$$

of a function ϕ in the space $L^p\left(H, V\mu_B\right)$. Let

$$\left(\frac{1}{V}P_tV\right)\phi(x) = \frac{1}{V(x)}P_t(V\phi)(x).$$

Theorem 2.18. *The family of operators $\left(\frac{1}{V}P_tV\right)$ defines a C_0-semigroup on $L^p\left(H, V\mu_B\right)$ for all $p \in (1,\infty)$ and $V\mu_B$ is an invariant measure for this semigroup. Moreover, there exist $\beta, C_p > 0$ such that*

$$\left\|\frac{1}{V}P_tV\phi - \langle V\mu_B, \phi\rangle\frac{1}{V}\right\|_p \leqslant Ce^{-\beta t/p}\|\phi\|_p, \quad \phi \in L^2\left(H, V\mu_B\right). \tag{2.33}$$

3. Results on Markov Processes

In this section some results on V-uniform ergodicity and L^p-ergodicity are stated for general time homogeneous Markov processes. These results are applied to Markov processes defined by the stochastic Navier-Stokes and Burgers equations in the next sections.

Throughout the present section we assume that (X_t) a time-homogeneous Markov process in a Polish space E and (P_t), (P_t^*) and $(P(t,x,\cdot))$ are the respective Markov semigroup, its adjoint and the transition kernel as defined in (2.11)-(2.13). We also assume that there exists an invariant measure $\mu \in \mathscr{P}$:

$$P_t^* \mu = \mu, \quad t \geqslant 0.$$

The following hypotheses are supposed to be satisfied, where $V : E \to [1, \infty)$ is a measurable function.

(H1) $\{P(t,x,\cdot) : t > 0,\ x \in E\}$ is a family of equivalent measures.

(H2) There exists a measurable subspace $E_1 \subset E$ such that for each $t > 0$ and $\Gamma \in \mathscr{B}$ the mapping $x \to P(t,x,\Gamma)$ is continuous in E_1.

(H3) For each $r > 1$ there exist $T_0 > 0$ and a compact $K \subset E_1$ such that

$$\inf_{x \in V_r} P(T_0, x, K) > 0, \tag{3.1}$$

where $V_r = \{y \in E : V(y) \leqslant r\}$.

(H4) For certain $k, \alpha, c > 0$

$$\mathbb{E}_x V\left(X_t\right) \leqslant kV(x) e^{-\alpha t} + c, \quad t \geqslant 0. \tag{3.2}$$

Theorem 3.1. *Assume (H1)-(H4). Then the Markov semigroup (P_t) is V-uniformly ergodic, i.e. (2.14) and (2.16) hold true.*

Proof. Take $t_0 > 0$ such that $ke^{-\alpha t_0} \leqslant \frac{1}{4}$. By (H4) we have

$$\mathbb{E}_x V\left(X_t\right) - V(x) \leqslant \left(ke^{-\alpha t_0} - 1\right) V(x) + c \leqslant -\frac{3}{4} V(x) + c, \tag{3.3}$$

for $t \geqslant t_0$, $x \in E$. Therefore,

$$\mathbb{E}_x V\left(X_t\right) - V(x) \leqslant c, \quad x \in E,\ t \geqslant t_0. \tag{3.4}$$

Taking $r \geqslant 4c$ we find that

$$-\frac{1}{4} V(x) + c < -\frac{1}{4} r + c \leqslant 0, \quad \text{for} \quad x \in E \setminus V_r,$$

and thereby

$$\mathbb{E}_x V\left(X_t\right) - V(x) \leqslant -\frac{1}{2} V(x) + c I_{V_r}(x), \quad x \in E,\ t \geqslant t_0. \tag{3.5}$$

The last inequality implies that each skeleton chain $(X_{nt})_{n \geqslant 0}$ with $t \geqslant t_0$ has a geometric drift toward V_r. We will show that there exists a skeleton for which V_r is a nontrivial small set. By (H1) each skeleton $(X_{n\tau})$, $\tau > 0$, is ψ-irreducible where $\psi(\cdot) = P\left(1, x_0, \cdot\right)$ and $x_0 \in E$ is arbitrary and fixed. Hence, (cf. Lemma 2 in [19] or Theorem 5.2.2 in [23]) there exists a small set $\Pi \in \mathscr{B}$, $\psi(\Pi) > 0$, that is

$$P\left(1, x_0, \Pi\right) > 0, \tag{3.6}$$

and

$$\inf_{x \in \Pi} P\left(T, x, \Gamma\right) \geqslant \lambda(\Gamma), \quad \Gamma \in \mathscr{B}, \tag{3.7}$$

for some $T > 0$ and a nonnegative measure λ such that $\lambda(\Pi) > 0$. By the Chapman-Kolmogorov equation we have

$$\inf_{x\in V_r} P\left(2T+T_0,x,\Gamma\right) \geqslant \inf_{x\in V_r}\int_\Pi P\left(T,y,\Gamma\right)P\left(T+T_0,x,dy\right)$$

$$\geqslant \lambda(\Gamma)\inf_{x\in V_r} P\left(T+T_0,x,\Pi\right) \geqslant \lambda(\Gamma)\inf_{x\in V_r}\int_K P(T,y,\Pi)P(T_0,x,dy), \tag{3.8}$$

where t and K are given in (H3). By (H1) and (3.6) the function $y \to P(T,y,\Pi)$ is positive on E and by (H2) it is continuous with respect to the topology of E_1, in which K is compact. Therefore, by (H3)

$$\inf_{x\in V_r} P(2T+T_0,x,\Gamma) \geqslant \delta_1\lambda(\Gamma)\inf_{x\in V_r} P(T_0,x,K) \geqslant \delta_2\lambda(\Gamma), \quad \Gamma\in\mathscr{B}, \tag{3.9}$$

for some $\delta_1,\delta_2>0$. It follows that V_r is a nontrivial small set for each skeleton $\left(X_{nm(2T+T_0)}\right)_{n\geqslant 0}$, where $m\geqslant 1$. Taking m large enough so that $\tau = m(2T+T_0)\geqslant t_0$ we obtain the skeleton $(X_{n\tau})$ which is V-uniformly ergodic, i.e.

$$\sup_{\|\phi\|_V\leqslant 1} |P_{n\tau}\phi(x) - \langle\mu,\phi\rangle| \leqslant C_0 e^{-n\tau\omega}V(x), \quad x\in E,\ n\geqslant 0, \tag{3.10}$$

for some $C_0,\omega>0$. Therefore, by (H4)

$$\sup_{\|\phi\|_V\leqslant 1} |P_{n\tau+s}\phi(x)-\langle\mu,\phi\rangle| \leqslant \sup_{\|\phi\|_V\leqslant 1} \left|P_s\left(P_{n\tau}\phi - \langle\mu,\phi\rangle\right)(x)\right|$$

$$\leqslant C_0e^{-n\tau\omega}\mathbb{E}_x V\left(X_s\right) \leqslant C_0 e^{-n\tau\omega}\left(kV(x)e^{-\alpha s}+c\right)$$

$$\leqslant C_0V(x)e^{-(n\tau+s)\omega}e^{\omega s}\left(ke^{-\alpha s}+c\right) \leqslant CV(x)e^{-(n\tau+s)\omega}, \quad x\in E,\ s\in[0,\tau], \tag{3.11}$$

for some $C>0$, which completes the proof. □

The following simple lemma will be useful to verify condition (H1).

Lemma 3.2. *Let $E_1\subset E$ with continuous and dense embedding and suppose that the process (X_t) is strongly Feller with respect to the topology of E_1 (i.e. (H2) holds) and let (X_t) be E_1-topologically irreducible (i.e. $P(t,x,U)>0$ for each $t>0$, $x\in E_1$ and nonempty open $U\subset E_1$) and $P\left(t,x,E_1\right)=1$ for all $x\in E$ and $t>0$. Then condition (H1) holds as well, i.e. the measures $P(t,x,\cdot)$ are equivalent for all $t>0$ and $x\in E$.*

Proof. The proof based on a modification of an earlier result by Khasminskii cf. [20] is given in [15] for a special choice of spaces E and E_1 but it can be easily extended to the present case. □

In the last part of this Section we will consider the problem of existence of the spectral gap in the weighted space $L^p\left(E,V\mu\right)$, $p\in\{1,\infty)$, with the norm

$$\|\phi\|_p^p = \int_E |\phi(x)|^pV(x)\mu(dx), \quad \phi\in L^p(E,V\mu).$$

For any Radon measure ν on E and $g\in L^1(E,\nu)$ we denote by $g\nu$ the measure

$$(g\nu)(B) = \int_B g(x)\nu(dx), \quad B\in\mathscr{B}(E).$$

Let

$$\left(\frac{1}{V}P_tV\right)\phi(x) = \frac{1}{V(x)}P_t(V\phi)(x).$$

If (2.14) holds then

$$\left\| \frac{1}{V} P_t V \phi - \frac{1}{V} \langle V\mu, \phi \rangle \right\|_\infty \leqslant C e^{-\omega t} \|\phi\|_\infty .$$

We will denote by (Q_t) the semigroup of bounded operators

$$Q_t = \frac{1}{V} P_t V : b\mathscr{B} \to b\mathscr{B}.$$

Let us recall that if the semigroup (P_t) satisfies the condition (H1) then the spaces $L^p(E, \mu)$ are invariant for (P_t) and

$$P_t \phi(x) = \int_E p_t(x, y) \phi(y) \mu(dy). \tag{3.12}$$

where $p_t(x, \cdot)$ the the density $dP(t, x, \cdot)/d\mu$. The next lemma is essentially known. It collects the facts necessary to prove Theorem 3.4.

Lemma 3.3. *Assume that (H1)-(H4) are satisfied. Then the following holds.*
(a) The measure $V\mu$ is invariant for the positive semigroup (Q_t).
(b) Let $\nu = \psi V \mu$ with $\psi \in L^1(E, V\mu)$. Then

$$Q_t^* \nu = (G_t \psi) V \mu,$$

where

$$G_t \psi(y) = \int_E p_t(x, y) \psi(x) \mu(dx) \tag{3.13}$$

with $p_t(\cdot, \cdot)$ given by (3.12). Moreover,

$$\|Q_t^*\|_{var} = \|G_t\|_{L^1(E, V\mu) \to L^1(E, V\mu)} .$$

(c) The space $L^p(E, V\mu)$ is invariant for the semigroup (Q_t^*) for each $p \in [1, \infty]$ and

$$\sup_{t \geqslant 0} \|Q_t^*\|_{L^p(E, V\mu) \to L^p(E, V\mu)} < \infty.$$

Moreover, (Q_t^*) is a C_0-semigroup on $L^p(E, V\mu)$ for $p \in [1, \infty)$ and for $p > 1$ it may be, in fact, identified as the dual of the extension of $Q_t : L^q(E, V\mu) \to L^q(E, V\mu)$ for $q = \frac{p}{p-1}$.

Proof. (a) Clearly, (Q_t^*) is a positive semigroup on $\mathscr{M}_b(E)$ with the invariant measure $V\mu$:

$$\left(\frac{1}{V} P_t V \right)^* (V\mu) = V P_t^* \frac{1}{V} (V\mu) = V\mu.$$

(b) For $\psi \in b\mathscr{B}$, $\psi \geqslant 0$, we define

$$G_t \psi(y) = \int_E p_t(x, y) \psi(x) \mu(dx),$$

where $G_t \psi \geqslant 0$ is a well defined measurable function. Let $\nu = \psi V \mu$. For $\phi \in L^\infty(E, V\mu)$ such that $\phi \geqslant 0$ (3.13) and the Fubini Theorem yield

$$\left\langle V P_t^* V^{-1} (\psi V \mu), \phi \right\rangle = \langle \psi \mu, P_t (V\phi) \rangle$$

$$= \int_E \int_E p_t(x, y) V(y) \phi(y) \mu(dy) \psi(x) \mu(dx) = \int_E \left(\int_E p_t(x, y) \psi(x) \mu(dx) \right) \phi(y) V(y) \mu(dy)$$

$$= \langle (G_t \psi) V\mu, \phi \rangle . \tag{3.14}$$

Hence (3.14) holds for arbitrary $\psi \in L^1(E, V\mu)$ and $\phi \in L^\infty(E, V\mu)$ and $VP_t^* V^{-1}(\psi V\mu) = (G_t\psi)\, V\mu$. Moreover,

$$\|(G_t\psi)\, V\mu\|_{var} = \sup_{\|\phi\| \leqslant 1} |\langle (G_t\psi)\, V\mu, \phi\rangle|$$

$$= \sup_{\|\phi\|\leqslant 1} \left| \left\langle \psi V\mu, \left(\frac{1}{V} P_t V\right) \phi \right\rangle \right| \leqslant C_T \|\psi\|_1 , \quad t \in [0, T].$$

Since

$$\langle (G_t\psi)\, V\mu, \phi\rangle =_{L^1} \langle G_t\psi, \phi\rangle_{L^\infty} ,$$

we find that G_t is bounded on $L^1(E, V\mu)$ and is fact the restriction of $VP_t^* \frac{1}{V}$ from $\mathscr{M}_b(E)$ to this space and (3.13) holds.
(b) If $\psi \in L^\infty(E, V\mu)$ is nonnegative then for $f \geqslant 0$ from $L^1(E, V\mu)$ the arguments similar as in (a) imply

$$\left\langle VP_t^* \frac{1}{V}(\psi V\mu), f \right\rangle = \langle (G_t\psi)\, V\mu, f\rangle \leqslant C_T \|\psi\|_\infty \|f\|_1 , \quad t \in [0, T],$$

since $\left\|\frac{1}{V} P_t V\right\|_{1\to 1} \leqslant C_T$, and therefore $VP_t^* V^{-1}\psi \in L^\infty(E, V\mu)$. All those arguments extend immediately to an arbitrary $\psi \in L^1(E, V\mu)$ and (b) follows by a standard density argument.
(c) This part follows again by the density argument. □

Theorem 3.4. *Assume (H1)-(H4). Then the family of operators $\left(\frac{1}{V} P_t V\right)$ defines a C_0-semigroup on $L^p(E, V\mu)$, $p \in (1, \infty)$ with the invariant measure $V\mu$. Moreover, there exist $\beta, C_p > 0$ such that*

$$\left\| \frac{1}{V} P_t V\phi - \langle V\mu, \phi\rangle \frac{1}{V} \right\|_p \leqslant C_p e^{-\beta t/p} \|\phi\|_p , \quad \phi \in L^p(E, V\mu). \tag{3.15}$$

Proof. By Theorem 3.1 and Lemma 3.3 (2.16) holds and therefore

$$\|Q_t^*\nu - V\mu\|_{var} \leqslant Ce^{-\omega t} \|\nu\|_{var} .$$

By Lemma 3.3 we find that

$$\|Q_t^*\phi - \langle \mu, \phi\rangle\|_1 \leqslant Ce^{-\omega t} \|\phi\|_1$$

and

$$\|Q_t^*\phi - \langle \mu, \phi\rangle\|_\infty \leqslant C \|\phi\|_\infty .$$

Therefore, by interpolation we obtain

$$\|Q_t^*\phi - \langle \mu, \phi\rangle\|_p \leqslant C_p e^{-\omega t/p} \|\phi\|_p ,$$

for all $p \in (1, \infty)$. Finally by (c) of Lemma 3.3

$$\left\| Q_t\phi - \langle V\mu, \phi\rangle \frac{1}{V} \right\|_p \leqslant C_p e^{-\omega t/p} \|\phi\|_p ,$$

for each $p \in (1, \infty)$, which completes the proof. □

4. Proofs: Stochastic Navier-Stokes Equation

The aim of this section is to prove Theorems 2.9 2.10 and 2.13 from Section 2.1. We are going to use the abstract results of Section 3 and to this end we need to verify conditions (H1)-(H4) for the Markov process defined by the stochastic Navier-Stokes equation (2.6) and a function V satisfying Hypothesis 2.6. The full proof is given for the case of Dirichlet boundary conditions (Theorems 2.9 and 2.10). The case of periodic boundary conditions (Theorem 2.13) may be treated very similarly (the sketch of the proof is given at the end of the section).

Throughout the section the notation from Section 2.1 is preserved, Hypotheses 2.5 and 2.6 are supposed to hold true, $E = H$ and the Markov semigroups are those defined by the solution to the Navier-Stokes equation (2.6).

Lemma 4.1. *Let $\beta \in \left[0, \frac{1}{2}\right)$. Then for any $t_0 > 0$ there exists a random variable C_{t_0} depending on t_0 only and such that*

$$\left|X^x_{t_0}\right|^2_\beta \leqslant C_{t_0}|x|^2 e^{|x|^2}, \quad \mathbb{P}-a.s., \tag{4.1}$$

where (X^x_t) denote the solution to equation (2.6) starting from $X^x_0 = x \in H$.

Proof. Recall that (Z_t) is the Ornstein-Uhlenbeck process defined by (2.9). It is well known that the operator A is positive and selfadjoint, with A^{-1} compact and the eigenvalues α_n of A have the property $\alpha_n \sim n^2$. Hence, by the second inclusion of (2.17) Q is of trace class and

$$Z \in C\left([0,T], H_\beta\right), \quad \mathbb{P}-a.s., \tag{4.2}$$

for each $T > 0$ (cf. [7]). Fix $T \geqslant t_0$ and set $Y^x_t = X^x_t - Z_t$ for $x \in H$. Obviously, (Y^x_t) satisfies the equation

$$\begin{cases} \frac{d}{dt}Y^x_t + AY^x_t + B\left(Y^x_t + Z_t, Y^x_t + Z_t\right) = f, & t > 0, \\ Y^x_0 = x, \end{cases} \tag{4.3}$$

and by Proposition 2.3 $Y^x \in C\left([0,T];H\right) \cap L^2\left(0,T;V\right)$ (cf. [13], [14] or [29]). By (4.2) it suffices to show (4.1) with X^x_t replaced with (Y^x_t). Following arguments in the proof of Proposition 4.1 in [13] we find that for each $t \in [0,T]$, $\epsilon > 0$ and sufficiently large $C(\epsilon) > 0$ $\mathbb{P}$-a.s.

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left|Y^x_t\right|^2 + \left|Y^x_t\right|_{1/2} &= -\left\langle B\left(Y^x_t + Z_t, Y^x_t + Z_t\right), Y^x_t\right\rangle + \left\langle f, Y^x_t\right\rangle \\ &\leqslant \epsilon\left|Y^x_t\right|^2_{1/2} + \epsilon\left|Y^x_t\right|^2 + C(\epsilon)\left(\left|Z_t\right|^4_{1/4}\left|Y^x_t\right|^2 + \left|Z_t\right|^4_{1/4} + |f|^2\right), \end{aligned} \tag{4.4}$$

$\mathbb{P}$-a.s. Taking $\epsilon < 1$ and invoking the Gronwall Lemma we obtain

$$\left|Y^x_t\right|^2 \leqslant C_1\left(|x|^2 e^{\int_0^t\left(1+|Z_s|^4_{1/4}\right)ds} + \int_0^t e^{\int_\tau^t\left(1+|Z_s|^4_{1/4}\right)ds}\left(|Z_\tau|^4_{1/4} + |f|^2\right)d\tau\right), \tag{4.5}$$

for all $t \in [0,T]$ and a certain universal constant C_1, which together with (4.2) yields

$$\sup_{t\in[0,T]}\left|Y^x_t\right|^2 \leqslant L_1\left(1 + |x|^2\right), \quad x \in H, \quad \mathbb{P}-a.s., \tag{4.6}$$

where L_1 is a finite random variable. Using (4.6), (4.2) and again (4.4) we find that

$$\begin{aligned} \sup_{0\leqslant t_1\leqslant t_2\leqslant T}\int_{t_1}^{t_2}\left|Y^x_t\right|^2_\beta\,dt &\leqslant C_2\sup_{0\leqslant t_1\leqslant t_2\leqslant T}\int_{t_1}^{t_2}\left|Y^x_t\right|^2_{1/2}\,dt \\ &\leqslant L_2\left(1 + |x|^2\right), \quad x \in H, \quad \mathbb{P}-a.s., \end{aligned} \tag{4.7}$$

where C_2 is a universal constant and L_2 is a finite random variable. Since $B : H_\theta \times H_\rho \to H_\delta$ is bounded for $\theta, \rho > 0$ and $\delta \in [0,1)$ such that $\delta + \theta + \rho \geqslant 1$ and $\delta + \rho > \frac{1}{2}$ (cf. [16]), taking $\theta = \rho = \frac{1}{4} + \frac{\beta}{2}$ and $\delta = \frac{1}{2} - \beta$ we obtain for $v \in H_{\frac{1}{2}+\beta}$ and $z \in H_{\frac{1}{4}+\frac{\beta}{2}}$

$$\begin{aligned}\left|\left\langle A^{-\frac{1}{2}+\beta} B(v+z, v+z), A^{\frac{1}{2}+\beta} v\right\rangle\right| &\leqslant C\left|A^{\frac{1}{4}+\frac{\beta}{2}}(v+z)\right|^2 \left|A^{\frac{1}{2}+\beta} v\right| \\ &\leqslant C\left(|v|^2_{\frac{1}{4}+\frac{\beta}{2}} + |z|^2_{\frac{1}{4}+\frac{\beta}{2}}\right)|v|_{\frac{1}{2}+\beta}, \end{aligned} \tag{4.8}$$

where C stands for a universal constant which may be different on each line. Therefore, by interpolation

$$\begin{aligned}\left|\left\langle A^{-\frac{1}{2}+\beta} B(v+z, v+z), A^{\frac{1}{2}+\beta} v\right\rangle\right| &\leqslant C\left(|v|_\beta |v|_{1/2} |v|_{\frac{1}{2}+\beta} + |z|^2_{\frac{1}{4}+\frac{\beta}{2}} |v|_{\frac{1}{2}+\beta}\right) \\ &\leqslant \epsilon |v|^2_{\frac{1}{2}+\beta} + C(\epsilon)\left(|v|^2_\beta |v|^2_{1/2} + |z|^4_{\frac{1}{4}+\frac{\beta}{2}}\right), \end{aligned} \tag{4.9}$$

for each $\epsilon > 0$ and each constant $C(\epsilon)$ depending on ϵ only. By classical arguments (see e.g. [13] we find that $Y^x \in L^2(0,T;H_\lambda)$ for each $\lambda < 1$, which yields for $t \in (0,T)$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|Y_t^x|^2_\beta + |Y_t^x|^2_{\frac{1}{2}+\beta} &= -\left\langle A^{-\frac{1}{2}+\beta} B\left(Y_t^x + Z_t, Y_t^x + Z_t\right) + f, A^{\frac{1}{2}+\beta} Y_t^x\right\rangle \\ &\leqslant \epsilon |Y_t^x|^2_{\frac{1}{2}+\beta} + C(\epsilon)\left(|Y_t^x|^2_\beta |Y_t^x|^2_{1/2} + |Z_t|^4_{\frac{1}{4}+\frac{\beta}{2}} + |f|^2\right), \end{aligned} \tag{4.10}$$

and therefore for $0 < s \leqslant t \leqslant T$

$$|Y_t^x|^2_\beta \leqslant C|Y_s^x|^2_\beta \exp\left(\int_s^t |Y_r^x|^2_{1/2}\, dr\right) + C\int_s^t |Z_r|^4_{\frac{1}{4}+\frac{\beta}{2}} \exp\left(\int_r^t |Y_\tau^x|^2_{1/2}\, d\tau\right) dr, \tag{4.11}$$

and in virtue of (4.7)

$$\begin{aligned}|Y_t^x|^2_\beta &\leqslant C|Y_s^x|^2_\beta e^{L_2\left(1+|x|^2\right)} + Ce^{L_2\left(1+|x|^2\right)} \int_s^t |Z_r|^4_{\frac{1}{4}+\frac{\beta}{2}}\, dr \\ &\leqslant L_3 |Y_s^x|^2_\beta e^{|x|^2}, \end{aligned} \tag{4.12}$$

where L_3 is a finite random variable independent of $x \in H$ and $0 < s \leqslant t \leqslant T$. Integrating (4.12) with respect to $s \in \left[\frac{1}{2}t_0, t_0\right]$ and invoking (4.7) we obtain

$$\begin{aligned}|Y_t^x|^2_\beta \leqslant \frac{2}{t_0} L_3 e^{|x|^2} \int_{t_0/2}^{t_0} |Y_s^x|^2_\beta\, ds &\leqslant \frac{2}{t_0} L_3 C e^{|x|^2} \int_{t_0/2}^{t_0} |Y_s^x|^2_\beta\, ds \\ &\leqslant L_4 |x|^2 e^{|x|^2}, \end{aligned} \tag{4.13}$$

for a random variable L_4 depending on t_0 only, which together with (4.2) completes the proof. □

Theorem 4.2. *Assume that the function V satisfies Hypothesis 2.6. Then (H4) holds.*

Proof. Let $\{e_n : n \geqslant 1\}$ be the orthonormal basis of H consisting of the eigenvectors of A and let Π_m be the orthogonal projection onto $\mathscr{H}_n = \operatorname{lin}\{e_k : k \leqslant n\}$. Set, for $n \geqslant 1$

$$B_n(x) = \Pi_n B\left(\Pi_n x, \Pi_n x\right), \qquad x_n = \Pi_n x, \quad x \in H,$$

and

$$G_n = \Pi_n G \Pi_n, \quad f_n = \Pi_n f.$$

We will consider the finite dimensional equations

$$\left\{\begin{array}{l} dX_n(t) + AX_n(t)dt + B_n\left(X_n(t)\right)dt = f_n dt + G_n dW_t, \quad t>0, \\ X_n(0) = x_n \in \mathscr{H}_n \end{array}\right. \tag{4.14}$$

Without the loss of generality we may assume that $V(x) = \Phi\left(|x|^2\right)$. Denote by τ_R the exit time of X_n from the ball $B_R = \{y \in H : |y| < R\}$. For a fixed $\lambda > 0$ the Ito formula yields

$$\begin{gathered} \mathbb{E}V\left(X_n\left(t \wedge \tau_R\right)\right)e^{\lambda(t\wedge\tau_R)} \\ \leqslant V\left(x_n\right) + \mathbb{E}\int_0^{t\wedge\tau_R}\Big[\lambda e^{\lambda s}V\left(X_n(s)\right) \\ +e^{\lambda s}\left(-2\Phi'\left(\left|X_n(s)\right|^2\right)\left\langle AX_n(s), X_n(s)\right\rangle + 2\Phi'\left(\left|X_n(s)\right|^2\right)\left\langle B_n\left(X_n(s)\right) + f_n, X_n(s)\right\rangle\right. \\ \left.+2\left|\Phi''\left(\left|X_n(s)\right|^2\right)\right| \cdot \left|X_n(s)\right|^2 \operatorname{tr}\left(G_n G_n^*\right) + \Phi'\left(\left|X_n(s)\right|^2\right)\operatorname{tr}\left(G_n G_n^*\right)\right)]ds \\ \leqslant V\left(x_n\right) + \mathbb{E}\int_0^{t\wedge\tau_R}\Big(\lambda e^{\lambda s}V\left(X_n(s)\right) \\ +e^{\lambda s}\left(-\kappa\Phi'\left(\left|X_n(s)\right|^2\right)\left|X_n(s)\right|^2 + k\left(\Phi''\left(\left|X_n(s)\right|^2\right)\left|X_n(s)\right|^2 + \Phi'\left(\left|X_n(s)\right|^2\right)\left|X_n(s)\right|\right)\right)\Big)ds, \end{gathered} \tag{4.15}$$

for any $t > 0$ and some $\kappa, k > 0$. Hypothesis 2.6 implies that

$$\begin{gathered} \mathbb{E}V\left(X_n\left(t \wedge \tau_R\right)\right)e^{\lambda(t\wedge\tau_R)} \\ \leqslant V\left(x_n\right) + \mathbb{E}\int_0^{t\wedge\tau_R}\Big(\lambda e^{\lambda s}V\left(X_n(s)\right) + e^{\lambda s}\left(-\beta V\left(X_n(s)\right) + C\right)\Big)ds, \end{gathered} \tag{4.16}$$

for some $\beta, C > 0$. Taking $\lambda \in (0, \beta)$ we have

$$\begin{aligned} \mathbb{E}V\left(X_n\left(t \wedge \tau_R\right)\right)e^{\lambda(t\wedge\tau_R)} &\leqslant V\left(x_n\right) + \mathbb{E}\int_0^{t\wedge\tau_R} e^{\lambda s}C ds \\ &\leqslant V\left(x_n\right) + \frac{C}{\lambda}e^{\lambda t}. \end{aligned} \tag{4.17}$$

For $R \to \infty$ we obtain by the Fatou Lemma:

$$\mathbb{E}V\left(X_n(t)\right) \leqslant V\left(x_n\right)e^{-\lambda t} + \frac{C}{\lambda}. \tag{4.18}$$

It remains to justify passing with $n \to \infty$. Set for $m = 1, 2, \ldots$

$$\Phi_m(r) = \left\{\begin{array}{lll} \Phi(r) & \text{if} & 0 \leqslant r \leqslant m, \\ \Phi(m) & \text{if} & r \geqslant m. \end{array}\right.$$

Clearly, $V_m(y) \uparrow V(y)$ as $m \to \infty$ for each $y \in H$ and thereby

$$\mathbb{E}V_m\left(X(t)\right) \nearrow \mathbb{E}V\left(X_t\right), \quad m \to \infty. \tag{4.19}$$

By [2] $X_n(t) \to X_t$ for each $t \geqslant 0$ in distribution on the space $H_{-\delta}$ for each $\delta > 0$ and therefore

$$\mathbb{E}V_m\left(X_n(t)\right) \to \mathbb{E}V_m\left(X_t\right), \quad n \to \infty. \tag{4.20}$$

Moreover, by (4.18)

$$\mathbb{E}V_m\left(X_n(t)\right) \leqslant \mathbb{E}V\left(X_n(t)\right) \leqslant V\left(x_n\right)e^{-\lambda t} + \frac{C}{\lambda},$$

and $V(x_n) \to V(x)$ so (4.19) and (4.20) yield

$$\mathbb{E}V(X_t) \leqslant V(x)e^{-\lambda t} + \frac{C}{\lambda}, \quad t \geqslant 0. \tag{4.21}$$

□

Proof of Theorems 2.9 and 2.10. We need to verify conditions (H1)-(H4) for the Markov process defined by equation (2.6). Under Hypothesis 2.5 it has been proven in [11] that the Markov semigroup (P_t) is strongly Feller and irreducible in the space $E_1 = H_\alpha$. Furthermore, since $\alpha < \frac{1}{2}$, (4.2) and (2.10) imply that $P(t, x, E_1) = 1$ for $x \in E = H$ and $t > 0$. Hence by Lemma 3.2 conditions (H1) and (H2) are satisfied with the above choice of spaces E and E_1. Condition (H4) has been verified in Proposition 4.2 and it remains to check condition (H3). Let $\beta \in \left(\alpha, \frac{1}{2}\right)$ and for $R > 0$ put

$$K_R = \{z \in H_\beta : |z|_\beta \leqslant R\}$$

By compactness of the operator A^{-1} each K_R is relatively compact in $E_1 = H_\alpha$. Since $\Phi(r) \to \infty$ for $r \to \infty$, for each $T > 0$ there exists $L > 0$ such that

$$\begin{aligned}\inf_{x \in V_r} P(T, x, K_R) \geqslant \inf_{|x| \leqslant L} P(T, x, K_R) &\geqslant 1 - \sup_{|x| \leqslant L} P(T, x, H \setminus K_R) \\ &= 1 - \sup_{|x| \leqslant L} \mathbb{P}\left(|X_t^x|_\beta^2 \geqslant R^2\right),\end{aligned} \tag{4.22}$$

hence by Lemma 4.1

$$\begin{aligned}\inf_{x \in V_r} P(T, x, K_R) &\geqslant 1 - \mathbb{P}\left(C_T |x|^2 e^{|x|^2} \geqslant R^2\right) \\ &\geqslant 1 - \mathbb{P}\left(C_T \geqslant \frac{R^2 e^{-L^2}}{L^2}\right) > 0,\end{aligned} \tag{4.23}$$

for R sufficiently large which completes the proof of (H3).

Proof of Theorem 2.13.

The proof almost exactly follows the lines of preceding one, however, the verification of our general condition (H1)-(H3) is based on the results from [12], Theorem 3.2, where it was shown that the Markov semigroup is strongly Feller and irreducible in the space $E_1 = H_{\frac{\alpha}{2}}$. Also, it is standard to check that $P(t, x, E_1) = 1$ for all $x \in H$ and $t > 0$, so Hypotheses (H1) and (H2) are satisfied. Condition (H3) may be verified as in Lemma 4.1 by induction (cf. Proposition 3.1 in [12] for a similar proof). Finally, condition (H4) follows from Proposition 4.2, which applies to the present case without change (note that the second inclusion in Hypothesis 2.23 implies that the operator G is Hilbert-Schmidt on H).

5. Proofs: Stochastic Burgers Equation

In this section Theorems 2.17 and 2.18 are proven. By Lemma 3.2 it suffices to show that conditions (H1)-(H4) are satisfied for the Markov process defined by the Burgers equation (2.28) and a function V satisfying Hypothesis 2.6. Throughout this section we preserve the notation from Section 2.2 and we assume that Hypotheses 2.16 and 2.6 hold true.

Proposition 5.1. *The Markov semigroup defined by equation (2.28) is strongly Feller and irreducible.*

Proof. The strong Feller property has been proven in [3]. Irreducibility was shown in [6] (cf. also [9]) for $Q=I$, the proof may be however, easily adapted to the case of more general Q. For the reader's convenience we provide a sketch of the proof. Let

$$\begin{cases} \dot{z}^u = Az^u + Q^{1/2}u, \\ z^u(0) = x. \end{cases} \tag{5.1}$$

where $u \in L^2(0,T;H)$. The system (5.1) is approximately controllable which follows from the fact that $\overline{\operatorname{im}(Q^{1/2})} = H$. That is, for any $x, y \in H$ and $\epsilon > 0$ we can find $u \in L^2(0,T;H)$ such that $|z^u(T) - y| < \epsilon$. Assume now that $x, y \in H_0^1$. Then $z^u \in C\left(0,T;H_0^1\right)$ and $B\left(z^u(\cdot)\right) \in C(0,T;H)$. Let

$$\psi(t) = -B\left(z^u(t)\right) + Q^{1/2}u(t).$$

and

$$\psi_n(t) = -nQ^{1/2}\left(I + nQ^{1/2}\right)^{-1} B\left(z^u(t)\right) + Q^{1/2}u(t) = Q^{1/2}\phi_n(t)$$

Clearly, $\psi, \psi_n, \phi_n \in L^2(0,T;H)$ and $\psi_n \to \psi$ in $L^1\left(0,T;H\right)$. Let us rewrite equation (5.1) in the form

$$\begin{cases} \dot{z}^u = Az^u + B\left(z^u\right) + \psi, \\ z^u(0) = x, \end{cases}$$

and consider also the equation

$$\begin{cases} \dot{z}_n^u = Az_n^u + B\left(z_n^u\right) + Q^{1/2}\phi_n, \\ z_n^u(0) = x. \end{cases} \tag{5.2}$$

It is known that the corresponding integral equation

$$z_n^u(t) = S_t x + \int_0^t S_{t-s} B\left(z_n^u(s)\right) ds + \int_0^t S_{t-s} Q^{1/2}\phi_n(s) ds,$$

has a unique solution $z_n^u \in C\left(0,T;H_0^1\right)$. We claim that

$$z_n^u \to z^u \quad \text{in} \quad C\left(0,T;H_0^1\right). \tag{5.3}$$

It follows that for any ϵ we can find n big enough, such that $z_n^u(0) = x$ and $|z_n^u(T) - y| < \epsilon$. Since H_0^1 is dense in H we find that the last estimate can be obtained for any $y \in H$. Then using the same arguments as in [9] we can prove that that for any $t > 0$, $y \in H$ and $r > 0$

$$P_t I_{\{z \in H; |z-y| < r\}}(x) > 0, \quad x \in H_0^1. \tag{5.4}$$

If $x \notin H_0^1$ then we can find a sequence $(x_n) \subset H_0^1$ such that $x_n \to x$ and then by the result in [3] we find that

$$\mathbb{E}\left|X_t^x - X_t^{x_n}\right|^2 \longrightarrow 0$$

uniformly in $t \in [0,T]$. Therefore, (5.4) holds for any $x \in H$ and the irreducibility follows. $\square$

Lemma 5.2. *Let $\beta \in \left(0, \frac{1}{4}\right)$. then for each $t_0 > 0$ there exists a random variable C_{t_0} depending on t_0 only and such that*

$$\left|X_{t_0}^x\right|_\beta \leqslant C_{t_0}\left(1 + |x|^2\right), \quad \mathbb{P} - a.s., \tag{5.5}$$

where (X_t^x) denotes the solution starting from $X_0^x = x \in H$.

Proof. the proof is based on a priori estimates given in Chapter 14 of [9]. for $Y_t^x = X_t^x - Z_t$ (where Z_t is defined in (2.29) we have for a fixed $T \geqslant t_0$

$$|Y_t^x|^2 \leqslant |x|^2 \exp\left(8\int_0^t |Z_s|_\infty^2\, ds\right) + 2\int_0^t |Z_s|_\infty^4 \exp\left(8\int_s^t |Z_r|_\infty^2\, dr\right) ds, \quad t \in [0,T],\ x \in H, \tag{5.6}$$

where $|\cdot|_\infty$ stands for the norm in $L^\infty(0,1)$. Since $\operatorname{tr}(Q) < \infty$ it follows that for each $\lambda < \frac{1}{2}$

$$Z \in C\left([0,T]; H_\lambda\right) \cap C\left([0,T]; C(0,1)\right), \tag{5.7}$$

and by (5.6) there exists a random variable C_1 such that

$$|Y_t^x|^2 \leqslant C_1\left(1 + |x|^2\right), \quad \mathbb{P} - a.s., \quad x \in H. \tag{5.8}$$

Furthermore, by Lemma 14.2.1 of [9] there exists a constant $C_\beta > 0$ such that

$$\left|\int_0^t e^{-(t-s)A} D_\zeta u(s) ds\right|_\beta^2 \leqslant C_\beta \sup_{s\leqslant t} |u(s)|_{L^1(0,1)}, \quad t \leqslant T, \tag{5.9}$$

for $\beta \in \left(0, \frac{1}{4}\right)$ and $u \in C\left([0,T]; L^1(0,1)\right)$ and by the analyticity of the semigroup $\left(e^{-tA}\right)$ it follows that

$$\begin{aligned}
|X_t^x|_\beta &\leqslant |Y_t^x|_\beta + |Z_t|_\beta \\
&\leqslant \left|e^{-tA}x\right|_\beta + \left|\frac{1}{2}\int_0^t e^{-(t-s)A} D_\zeta\left(Y_s^x + Z_s\right) ds\right|_\beta + |Z_t|_\beta \\
&\leqslant \frac{c_1}{t^\beta}|x| + \frac{1}{2} C_\beta \sup_{s\leqslant t}\left|\left(Y_t^x + Z_t\right)^2\right|_{L^1(0,1)} + |Z_t|_\beta \\
&\leqslant \frac{c_1}{t^\beta}|x| + C_\beta \sup_{s\leqslant t}\left(|Y_s^x|^2 + |Z_s|^2\right) + |Z_t|_\beta, \quad t \leqslant T,
\end{aligned}$$

for a universal constant $c_1 > 0$, which together with (5.7) and (5.8) yields (5.5). □

Proposition 5.3. *Let the function V satisfy Hypothesis 2.6. then (H4) holds true.*

Proof. Following [3] we consider a sequence of approximating equations

$$\begin{cases} dX_n(t) + AX_n(t)dt = B_n\left(X_n(t)\right) dt + \Pi_n G dW_t, \\ X_n(0) = x_n = \Pi_n x, \end{cases}$$

where Π_n are defined as orthogonal projections on the spans of the first n eigenvectors of the operator A and

$$B_n(x) = \Pi_n D_\zeta\left(\left(\Pi_n x\right)^2\right), \quad x \in H_0^1.$$

Taking into account that $\langle B_n(x), x\rangle = 0$ and $X_n^x \to X^x$ in $L^2\left(\Omega, \mathscr{F}, \mathbb{P}; H\right)$(cf. [3]) we may repeat word by word the proof of Proposition 4.2 i.e. apply the Ito formula to the process $t \to e^{\lambda t} V\left(X_n^x(t)\right)$ with $\lambda > 0$ small enough, estimate the drift term using Hypothesis 2.6 and pass with $n \to \infty$. □

Proof of Theorem 2.17. We have to verify conditions (H1)-(H4) for the Markov process defined by equation (2.28) with $E = E_1 = H$. By Lemma 3.2 and Proposition 5.1 (H1) and (H2) hold true while (H4) follows from Proposition 5.3. It remains to verify (H3). To this end we proceed in a similar way as in the proof of Theorem 2.9. For a fixed $\beta \in \left(0, \frac{1}{4}\right)$ set

$$K_R = \left\{z \in H_\beta : |z|_\beta \leqslant R\right\}, \quad R > 0.$$

Obviously, K_R is relatively compact in $E_1 = H$ for each $R > 0$ and for each $r > 0$ there exists $L > 0$ such that

$$\inf_{x\in V_r} P\left(T,x,K_R\right) \geqslant \inf_{|x|\leqslant L} P\left(T,x,K_R\right) \geqslant 1 - \sup_{|x|\leqslant L} P\left(T,x,H\setminus K_R\right), \tag{5.10}$$

so by Lemma 5.2

$$\begin{aligned}\inf_{x\in V_r} P\left(T,x,K_R\right) &\geqslant 1 - \sup_{|x|\leqslant L} \mathbb{P}\left(C_T\left(1+|x|^2\right)\geqslant R\right)\\ &\geqslant 1-\mathbb{P}\left(C_T \geqslant \frac{R}{1+L^2}\right),\end{aligned} \tag{5.11}$$

which is positive for R large enough and (H3) holds.

References

[1] Bricmont J., Kupiainen A. and Lefevere R.: Exponential mixing of the 2D stochastic Navier-Stokes dynamics, *Comm. Math. Phys.* 230 (2002), 87-132

[2] Capinski M. and Gatarek D.: Stochastic equations in Hilbert space with applicaton to Navier-Stokes equations in Any dimension, *J.Funct. Anal.* 126 (1994), 26-35

[3] Da Prato G. and Debussche A.: Differentiability of the transition semigroup of the stochastic Burgers equation, and application to the corresponding Hamilton-Jacobi equation, *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei* (9) Mat. Appl. 9 (1998), 267-277

[4] Da Prato G. and Debussche A.: Maximal dissipativity of the Dirichlet operator corresponding to the Burgers equation, in Stochastic processes, physics and geometry: new interplays, I (Leipzig, 1999), 85-98, CMS Conf. Proc., 28, Amer. Math. Soc., Providence, RI, 2000

[5] Da Prato G., Debussche A. and Temam R.: Stochastic Burgers' equation, *NoDEA Nonlinear Differential Equations Appl.* 1 (1994), 389-402

[6] Da Prato G. and Gatarek D.: Stochastic Burgers equation with correlated noise. Stochastics Stochastics Rep. 52 (1995), 29-41

[7] Da Prato G., Kwapien S. and Zabczyk J.: Regularity of solutions of linear stochastic equations in Hilbert spaces, *Stochastics* 23 (1987), 1-23

[8] Da Prato G. and Zabczyk J.: Stochastic Equations in Infinite Dimensions, Cambridge University Press, Cambridge 1992

[9] Da Prato G. and Zabczyk J.: Ergodicity for Infinite Dimensional Systems, Cambridge University Press, Cambridge 1996

[10] E W., Mattingly J. C. and Sinai Ya.: Gibbsian dynamics and ergodicity for the stochastically forced Navier-Stokes equation, *Comm. Math. Phys.* 224 (2001), 83-106

[11] Ferrario B.: Ergodic results for stochastic Navier-Stokes equation, *Stochastics Stochastics Rep.* 60 (1997) 271-288

[12] Ferrario B.: Stochastic Navier-Stokes equations: Analysis of the noise to have a unique invariant measure, *Annali Mat. Pura Appl.* 177 (1999), 331-347

[13] Flandoli F.: Dissipativity and invariant measures for stochastic Navier-Stokes equations, *NoDEA* 1 (1994), 252-286

[14] Flandoli F. and Gątarek D.: Martingale and stationary solutions for stochastic Navier-Stokes equations, *Probab. Theory Related Fields* 102 (1995), 367-391

[15] Flandoli F. and Maslowski B.: Ergodicity of the 2-D Navier-Stokes equation under random perturbations, *Comm. Math. Phys.* 172 (1995), 119-141

[16] Giga Y. and Miyakawa T.: Solution in L_r of the Navier-Stokes initial value problem, *Arch.Rational Mech. Anal.* 89 (1985), 267-281

[17] Goldys B. and Maslowski B.: Lower estimates of transition densities and bounds on exponential ergodicity for stochastic PDE's, *preprint* of the School of Mathematics, The University of New South Wales, 2004, http://www.maths.unsw.edu.au/statistics/pubs/statspreprints2004.html

[18] Hairer M. and Mattingly J.C.: Ergodicity of the 2D Navier-Stokes equations with degenerate stochastic forcing, *preprint* 2004, http://front.math.ucdavis.edu/author/Hairer-M*c=PR

[19] Jain N. and Jamison B.: Contributions to Doeblin's theory of Markov processes *Z. Wahrscheinlichkeitstheorie Verw. Geb.* 8 (1967), 19-40

[20] Khasminskii R.Z.: Ergodic properties of recurrent diffusions and stabilization of the Cauchy problem for parabolic equations, *Teoriya Veroyat. Primen.* 5 (1960), 7-28 (in Russian)

[21] Kuksin S. and Shirikyan A.: A coupling approach to randomly forced nonlinear PDE's I, *Comm. Math. Phys.* 221 (2001), 351-366

[22] Kuksin S. and Shirikyan A.: Coupling approach to white-forced nonlinear PDE's, *J. Math. Pures Appl.* 81 (2002), no. 6, 567-602

[23] Meyn S.P. and Tweedie R.L.: Markov Chains and Stochastic Stability, Springer-Verlag, Berlin 1993

[24] Maslowski B. and Simao I.: Asymptotic properties of stochastic semilinear equations by the method of lower measures, *Colloquium Math.* 72 (1997), 147-171

[25] Maslowski B. and Simao I.: Long-time behaviour of non-autonomous SPDE's, *Stochastic Processes Appl.*, 95 (2001), 285-309

[26] Mattingly J. C., Exponential convergence for the stochastically forced Navier-Stokes equations and other partially dissipative dynamics, *Comm. Math. Phys.* 230 (2002), 421-462

[27] Shardlow T.: Geometric ergodicity for stochastic PDE's, *Stochastic Anal. Appl.* , 17 (1999), 857-869

[28] Temam R.: Navier-Stokes Equations, North-Holland, Amsterdam, 1984

[29] Vishik M.J. and Fursikov A.V.: Mathematical Problems of Statistical Hydromechanics, Kluwer, Dordrecht, 1980

School of Mathematics, The University of New South Wales, Sydney 2052, Australia
E-mail address: B.Goldys@unsw.edu.au

Mathematical Institute, Academy of Sciences of Czech Republic, Žitná 25, 11567 Prague 1, Czech Republic
E-mail address: maslow@math.cas.cz