ENERGY ASYMPTOTICS FOR TYPE II SUPERCONDUCTORS

S. FOURNAIS AND B. HELFFER

ABSTRACT. We study the Ginzburg-Landau functional in the parameter regime describing 'Type II superconductors'. In the exact regime considered minimizers are localized to the boundary - i.e. the sample is only superconducting in the boundary region. Depending on the relative size of different parameters we describe the concentration behavior and give leading order energy asymptotics. This generalizes previous results by Lu and Pan, Helffer and Pan, and Pan.

CONTENTS

1. INTRODUCTION

In this paper we study the Ginzburg-Landau functional given in (1.1) below. The functional depends on different parameters: H denotes the strength of an external, constant magnetic field and κ is a parameter depending on the material of the superconductor. That the superconductor is of 'Type II' will for us mean that κ is large. We will study the asymptotic regime, where $\kappa, H \to +\infty$. The functional is defined on a domain $\Omega \subset \mathbb{R}^2$, that we assume to be open, bounded, simply connected and with smooth boundary. These assumptions fit general experimental set-ups (though the effect of corners is also interesting, but beyond the scope of the present paper—see [Pan02b] and [Bon03, Bon04] for some of the known results

Date: May 7, 2004.
This research has been partially supported by a Marie Curie Fellowship of the European Community Programme 'Improving the Human Research Potential and the Socio-Economic Knowledge Base' under contract number HPMF-CT-2002-01822, the European Research Network 'Postdoctoral Training Program in Mathematical Analysis of Large Quantum Systems' with contract number HPRN-CT-2002-00277, and the ESF Scientific Programme in Spectral Theory and Partial Differential Equations (SPECT).

in that case). The problem of superconductivity below the critical field H_{C_3} has in the mathematical literature been addressed by several authors starting probably with [BPT98] (see however also [BH93]). See below for further references.

The functional is

$$\mathcal{E}[\psi,\vec{A}] = \mathcal{E}_{\kappa,H}[\psi,\vec{A}] = \int_\Omega \Big\{ |\nabla_{\kappa H\vec{A}}\psi|^2 + \kappa^2 H^2 |\operatorname{curl}\vec{A} - 1|^2 \\ - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 \Big\}\, dx, \quad (1.1)$$

with $(\psi,\vec{A}) \in W^{1,2}(\Omega;\mathbb{C}) \times W^{1,2}(\Omega;\mathbb{R}^2)$. (The notation $W^{j,k}$ denotes the standard Sobolev spaces, see for instance [GT01, Section 7.5]). Here we have introduced the standard notation $\nabla_{\vec{A}} = (-i\nabla - \vec{A})$ for the magnetic momentum operator.

The modulus of the wavefunction $\psi(x)$ is a measure of the concentration of Cooper pairs at the point x. The curl of the vector potential $\vec{A}$ gives the magnetic field at the interior of the superconductor. As always, in problems with magnetic fields, we have 'gauge invariance', i.e. the identity

$$\mathcal{E}[\psi,\vec{A}] = \mathcal{E}[\psi e^{i\kappa H\phi}, \vec{A} + \nabla\phi].$$

Notice that, by 'completion of the square', we get

$$\mathcal{E}[\psi,\vec{A}] \geq -\kappa^2\frac{|\Omega|}{2}, \quad (1.2)$$

where $|\Omega|$ is the Lebesgue measure of Ω.

For notational convenience we fix a notation and a gauge for the vector potential $\vec{F}$ generating the external magnetic field. Let $\vec{F}$ be the unique smooth solution to

$$\operatorname{curl}\vec{F} = 1, \qquad \operatorname{div}\vec{F} = 0, \qquad \vec{F}\cdot\nu = 0 \text{ on } \partial\Omega. \quad (1.3)$$

It is fairly easy to prove that

(1) For fixed κ, H, the functional $\mathcal{E}$ has a (not necessarily unique) minimizer $(\psi,\vec{A}) \in W^{1,2}(\Omega;\mathbb{C}) \times W^{1,2}(\Omega;\mathbb{R}^2)$. By variation around such a pair $(\psi,\vec{A})$, we find that any minimizer satisfies the Ginzburg-Landau equations:

$$\left.\begin{aligned} \nabla^2_{\kappa H\vec{A}}\psi &= \kappa^2(1-|\psi|^2)\psi \\ \operatorname{curl}^2\vec{A} &= -\tfrac{i}{2\kappa}(\overline{\psi}\nabla\psi - \psi\nabla\overline{\psi}) - |\psi|^2\vec{A} \end{aligned}\right\} \text{ in } \Omega\,; \quad (1.4a)$$

$$\left.\begin{aligned} (\nabla_{\kappa H\vec{A}}\psi)\cdot\nu &= 0 \\ \operatorname{curl}\vec{A} - 1 &= 0 \end{aligned}\right\} \text{ on } \partial\Omega\,. \quad (1.4b)$$

Here $\operatorname{curl}(A_1,A_2) = \partial_{x_1}A_2 - \partial_{x_2}A_1$, and

$$\operatorname{curl}^2\vec{A} = (\partial_{x_2}(\operatorname{curl}\vec{A}), -\partial_{x_1}(\operatorname{curl}\vec{A}))\,.$$

(2) For H sufficiently large the only minimizer, up to change of gauge[1], of $\mathcal{E}$ is $(0,\vec{F})$ (see [GP99]).

Notice that

$$\mathcal{E}[0,\vec{F}] = 0. \quad (1.5)$$

Thus a minimizer will always have non-positive energy. When combined with (1.2) this trivial fact is very useful for establishing *a priori* bounds on minimizers.

We define the critical field H_{C_3}, using (2) above, as follows:

[1]We will generally omit the phrase 'up to change of gauge'. Thus, whenever we discuss 'unique minimizers', it will mean unique once the gauge is fixed.

Definition 1.1. For $\kappa > 0$,

$$H_{C_3} = H_{C_3}(\kappa) = \inf\{H \geq 0 \,\big|\, (0, \vec{F}) \text{ is the unique minimizer of } \mathcal{E}_{\kappa,H}\}.$$

One can find the asymptotics of H_{C_3} as $\kappa \to +\infty$. The best known result at present (extending the results from [LP99, dPFS00]) is the two term asymptotics given by Helffer and Pan [HP03][2]:

$$H_{C_3} = \frac{\kappa}{\beta_0} + \frac{C_1}{\beta_0^{3/2}} k_{\max} + \mathcal{O}(\kappa^{-1/3}). \tag{1.6}$$

Here β_0, C_1 are explicit constants, $0.5 < \beta_0 < 0.76$ (that will be defined in (1.12), (1.15) below). The number $k_{\max}$ is the maximal curvature of the boundary $\partial\Omega$.

We will consider field strengths below H_{C_3}. Thus we write

$$H = H_{C_3} - \rho(\kappa), \tag{1.7}$$

for some positive function ρ. The results in this paper will concern ρ's satisfying

$$\rho(\kappa) \geq c > 0 \ , \tag{1.8}$$

and

$$\lim_{\kappa \to +\infty} \rho(\kappa)/\kappa = 0 \ . \tag{1.9}$$

We will need some results concerning the family of reference 1-dimensional spectral problems defined below. For $\xi \in \mathbb{R}$, we define the quadratic form q_ξ on the space $W^{1,2}(\mathbb{R}_+) \cap \langle\tau\rangle^{-1} L^2(\mathbb{R}_+)$ by

$$q_\xi[\phi] = \int_0^\infty |\phi'(\tau)|^2 + |(\tau + \xi)\phi(\tau)|^2 \, d\tau. \tag{1.10}$$

For a given ξ the ground state energy of q_ξ is

$$E_0(\xi) = \inf_{\|\phi\|_2 = 1} q_\xi[\phi], \tag{1.11}$$

where $\| \cdot \|_2$ denotes the L^2-norm. It is a result of Dauge and Helffer [DH93] that there exists a unique $\xi_0 \in \mathbb{R}$, which minimizes $E_0(\xi)$. We define

$$\beta_0 \equiv E_0(\xi_0). \tag{1.12}$$

Furthermore, this minimum is non degenerate

$$E_0''(\xi_0) > 0 \ , \tag{1.13}$$

and there exists a unique positive, normalized (in $L^2(\mathbb{R}_+)$) function u_0 such that

$$q_{\xi_0}[u_0] = \beta_0. \tag{1.14}$$

The constant C_1 introduced in (1.6) is given in terms of u_0 by

$$C_1 = \frac{(u_0(0))^2}{3}. \tag{1.15}$$

The family of quadratic forms, q_ξ will be further analyzed in Section 3.

The first new result that we will prove in the present paper is the following control of the minimal energy :

[2] It is expected that the next term in the asymptotics will be of order $\kappa^{-1/2}$. See [HP03] and [BS98].

Theorem 1.2. *Let the constant c in (1.8) be given and let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying $\lim_{t\to+\infty} f(t) = 0$. Then there exists a function $g : \mathbb{R} \to \mathbb{R}$ satisfying $\lim_{t\to+\infty} g(t) = 0$ such that if (κ, H) is a couple satisfying (1.8) (with the given c),*

$$\frac{\rho(\kappa)}{\kappa^{1/2}} \leq f(\kappa)\,, \tag{1.16}$$

and $(\psi, \vec{A})$ is an associated minimizer of (1.1). Then

$$\begin{aligned} &\left| \mathcal{E}[\psi, \vec{A}] + \frac{1}{2\beta_0^{1/2}\|u_0\|_4^4\kappa} \int_0^{|\partial\Omega|} \left[\beta_0^{3/2}\rho - C_1(k_{\max} - k(s))\right]_+^2 ds \right| \\ &\leq \frac{\rho^2}{\kappa} g(\kappa). \end{aligned} \tag{1.17}$$

Here $k(\cdot)$ denotes the curvature of the boundary (in the path-length parametrization) and $[\,\cdot\,]_+$ is the function

$$[\,t\,]_+ = \begin{cases} t, & t \geq 0 \\ 0, & t < 0. \end{cases}$$

Remark 1.3. In the case when $\rho \to +\infty$, formula (1.17) above becomes

$$\mathcal{E}[\psi, \vec{A}] = -\frac{\beta_0^{5/2}|\partial\Omega| + o(1)}{2\|u_0\|_4^4} \left(\frac{\rho^2}{\kappa}\right) \quad \text{as } \kappa \to +\infty. \tag{1.18}$$

Theorem 1.2 complements a previous result by Lu-Pan [LP99, Thm. 5.1]. They state the same result (1.18) under the assumption

$$\frac{\rho}{\kappa^{\frac{1}{3}}} \to \infty\,. \tag{1.19}$$

Thus our contribution fills, in particular, the gap from $\rho \approx \kappa^{1/3}$ down to ρ of order 1. We take the opportunity of this generalization for clarifying some aspects of their proof by implementing in particular the techniques of [HM01] and [Pan02a].

Our second result expresses the concentration to the boundary. We introduce the parameter ε as

$$\varepsilon = \frac{1}{\sqrt{\kappa H}}. \tag{1.20}$$

This parameter will give the length scale of the boundary concentration. In order to state our result we consider the boundary coordinates defined in Appendix A. Let t_0 be the constant from this appendix. This number is chosen such that the change of coordinates $\Phi : \frac{|\partial\Omega|}{2\pi}\mathbb{S}^1 \times (0, t_0) \to \Omega$ is a diffeomorphism on its range, which is equal to $\{x \in \Omega : d(x, \partial\Omega) \leq t_0\}$. Actually, $t(x) := \operatorname{dist}(x, \partial\Omega)$ satisfies $t(\Phi(s,t)) = t$.

Let $\chi \in C_0^\infty(\mathbb{R})$ be a smooth cut-off function:

$$\chi = 1 \text{ on } [0, t_0/2], \qquad \operatorname{supp}\chi \subset (-t_0, t_0). \tag{1.21}$$

On the support of $\chi(t(x))$ we can use boundary coordinates $(s,t) = \Phi^{-1}(x)$. Thus the function

$$x \mapsto \frac{[\beta_0^{3/2} - C_1\frac{k_{\max} - k(s)}{\rho}]_+}{\beta_0\|u_0\|_4^4} |u_0(\tfrac{t}{\varepsilon})|^2 \chi(t), \tag{1.22}$$

is well-defined (with the dependence $(s,t) = (s(x), t(x)) = \Phi^{-1}(x)$ being tacitly understood).

With this convention, we can now formulate

Theorem 1.4. *Let the assumptions and notations be as in Theorem 1.2. Then*

$$\varepsilon^{-1}\int_{\Omega}\left|\frac{|\psi(x)|^2}{\varepsilon\rho}-\frac{[\beta_0^{3/2}-C_1\frac{k_{\max}-k(s)}{\rho}]_+}{\beta_0\|u_0\|_4^4}|u_0(\tfrac{t}{\varepsilon})|^2\chi(t)\right|^2 dx\to 0. \tag{1.23}$$

Remark 1.5. In case $\rho\to\infty$ the function from (1.22) can be replaced by

$$\frac{\sqrt{\beta_0}}{\|u_0\|_4^4}|u_0(\tfrac{t}{\varepsilon})|^2\chi(t).$$

Combining Theorem 1.2 with the results of Helffer-Pan [HP03], Lu-Pan [LP99], Pan [Pan02a] and Sandier-Serfaty [SS00, SS03], we get the following description of the high κ-high H asymptotic regime. In the regimes considered below—except the first $\frac{H}{\kappa}<1$— the function $|\psi|$ is (exponentially) small outside of a neighborhood of width κ^{-1} of the boundary.

- The region $\frac{H}{\kappa}<1$ has been explored by E. Sandier and S. Serfaty. These authors give in particular a very fine analysis of the transition between superconducting solutions (that is with non vanishing ψ) and solutions with vortices. We refer to [SS00, SS03] and references therein.
- $\frac{H}{\kappa}\to b$ for some $1\leq b<\frac{1}{\beta_0}$ (with an extra condition in case $b=1$). This situation is equivalent to having
$$\frac{\rho}{\kappa}\to\frac{1}{b}-\frac{1}{\beta_0}>0.$$
In [Pan02a] it is proved that in this regime ψ concentrates uniformly along the boundary $\partial\Omega$. Furthermore, the boundary superconductivity is *strong* i.e. ψ is of order 1 on the boundary.
- $\rho\to+\infty$, $\rho/\kappa\to 0$. This is the regime analyzed in the present paper and in [LP99]. Superconductivity is expected to nucleate uniformly at the boundary, but only with a strength
$$|\psi(x)|^2\approx\frac{\rho}{\kappa}\to 0$$
(on the boundary). This result would be consistent with our energy asymptotics (and is the idea used in the construction of our trial functions).
- ρ of order 1. This regime is also treated in the present paper. We see that, as ρ increases, the superconducting part of the boundary becomes larger. When ρ reaches the value
$$\rho_{\text{crit}}=\beta_0^{-3/2}C_1(k_{\max}-k_{\min}), \tag{1.24}$$
(with $k_{\min}$ being the minimum of the curvature of the boundary) the entire boundary carries superconductivity. In [HP03] an Agmon-type decay estimate in the boundary coordinate was proved for $\rho<\rho_{\text{crit}}$ and it was conjectured that for $\rho>\rho_{\text{crit}}$ superconductivity becomes *uniform* in the boundary. From the above result we see that this is only partially true: The entire boundary carries superconductivity but not with the same intensity. The uniformity (to highest order) is only achieved when $\rho\to\infty$.
- $\rho\to 0$. In this case superconductivity nucleates in the vicinity of the points on the boundary with maximum curvature. Precise estimates on the size of the region of localization are given in [HP03]. In Section 5 below we

will complete the picture by giving the leading order energy asymptotics for minimizers of the Ginzburg-Landau functional in this case also.

For convenience (also for the reader when comparing to the works of Lu and Pan) we introduce the parameters that we will use instead of κ, H. The parameter ε has already been defined in (1.20). Furthermore,

$$\delta_\varepsilon = \frac{\kappa}{H} - \beta_0 \,. \tag{1.25}$$

In terms of ρ we get, under the condition (1.9),

$$\varepsilon = \frac{\sqrt{\beta_0}}{\kappa} + O(\frac{\rho}{\kappa^2}), \qquad \delta_\varepsilon = \frac{\beta_0^2 \rho - \beta_0^{-1/2} C_1 k_{\max}}{\kappa} + O(\frac{\rho^2+1}{\kappa^2}). \tag{1.26}$$

Notice that δ_ε does not have a definite sign when ρ is bounded.

2. Basic known results

Let $\vec{F}$ be the vector potential defined in (1.3). We fix the gauge of $\vec{A}$ by demanding also

$$\operatorname{div}\vec{A} = 0, \qquad\qquad \vec{A}\cdot\nu = 0 \text{ on } \partial\Omega. \tag{2.1}$$

Unless otherwise stated, this is the gauge in which we will work in the entire paper. Since $\operatorname{div}\vec{A} = 0$, the equation for $\vec{A}$ in (1.4a) becomes

$$\Delta\vec{A} = \frac{i}{2\kappa}\left(\overline{\psi}\nabla\psi - \psi\overline{\nabla\psi}\right) + |\psi|^2\vec{A}. \tag{2.2}$$

Lemma 2.1. *Let $(\psi, \vec{A})$ be a minimal solution of* (1.4) *and satisfying* (2.1)*. Let $\kappa, H \geq 1$. Then there exists a constant C (independent of κ, H) such that*

$$\|\psi\|_{L^\infty(\Omega)} \leq 1, \qquad \|\nabla_{\kappa H\vec{A}}\psi\|_{L^2(\Omega)} \leq C\kappa, \qquad \|\vec{A}-\vec{F}\|_{W^{1,2}(\Omega)} \leq C/H.$$

We include a short reminder of how these estimates are proved.

Proof.
The first estimate is a consequence of the diamagnetic inequality [LL97, Thm. 7.21] and the maximum principle applied to the equations (1.4). The second and third estimates simply follow from the primitive estimates (1.2) and (1.5). $\square$

From the work of [HM01] and [HP03] we get the following estimate of the bottom of the spectrum of the magnetic Neumann Laplacian (cf [HP03, Theorem 3.1]) :

Theorem 2.2. *There exist $a, c > 0$ such that if $\kappa H \geq a$, $\vec{A}$ is a vector potential satisfying the following estimates:*

$$\|\vec{A}\|_{C^2(\overline{\Omega})} \leq c, \qquad \|\operatorname{curl}\vec{A} - 1\|_{C^1(\overline{\Omega})} \leq c(\kappa H)^{-1/6}, \qquad \operatorname{curl}\vec{A} = 1 \text{ on } \partial\Omega, \tag{2.3}$$

and $Q_{\vec{A}}$ is the quadratic form on $W^{1,2}(\Omega)$ given by

$$Q_{\vec{A}}[\phi] = \int_\Omega |\nabla_{\vec{A}}\phi|^2\,dxdy.$$

Then there exists a constant C, depending only on a, c, such that we have the following lower bound for all $\phi \in W^{1,2}(\Omega)$:

$$Q_{\kappa H\vec{A}}[\phi] \geq \kappa H\left(\beta_0 - \frac{C_1 k_{max}}{\sqrt{\kappa H}} - C(\kappa H)^{-2/3}\right)\|\phi\|_2^2. \tag{2.4}$$

Here k_{max} is the maximum of the curvature of the boundary, and C_1 is the constant defined in (1.15)*.*

It is well known that if we replace Ω by the entire plane $\mathbb{R}^2$, then the right hand side of (2.4) should be replaced by (to leading order) $\kappa H\|\phi\|_2^2$. Thus, since $\beta_0 < 1$, the presence of the boundary lowers the ground state energy[3]. It is therefore no surprise that functions with low energy concentrate near the boundary. The technique of Agmon estimates is an efficient way of quantifying this decay.

It is proved in [HP03] that under condition (1.9) the assumption (2.3) is satisfied. Furthermore they prove exponential localization to the boundary.

Theorem 2.3 (Agmon estimates)**.** *Suppose* (1.9) *is satisfied. Let* (ψ, A) *be a sequence of minimizers of the Ginzburg-Landau functional satisfying the gauge condition* (2.1)*. Let* $t(x) = d(x, \partial\Omega)$*. Then* (2.3) *is satisfied. Furthermore, there exist positive constants* C, c_0, α *such that:*

$$\int_\Omega e^{\alpha\kappa t(x)} \left(|\psi|^2 + (\kappa H)^{-1}|\nabla_{\kappa H\vec{A}}\psi|^2\right) dx \le C \int_{\{t(x)<c_0/\kappa\}} |\psi(x)|^2\, dx.$$

Theorem 2.3 is a summary, adapted to the present context, of [HP03, Lemma 3.2, Prop. 4.2 and Lemma 4.5].

Let us recall the following standard result (see for instance [AHS78]):

Proposition 2.4. *Suppose* $\vec{A}$ *satisfies* $B \equiv \operatorname{curl}\vec{A} > 0$*. Let* ϕ *in* $W_0^{1,2}(\Omega)$*, then, for all* $\varepsilon > 0$*,*

$$\int_\Omega |(-i\nabla - \varepsilon^{-2}\vec{A})\phi|^2\, dx \ge \varepsilon^{-2}\int_\Omega B(x)|\phi(x)|^2\, dx.$$

We also state the following sharpening of Theorem 2.2. It was proved in [HM01, Prop. 10.5], we give the adapted form from [HP03, Prop. 3.7].

Proposition 2.5. *Under the assumptions of Theorem 2.3, there exists* $\varepsilon_0, C_0 > 0$ *such that, if* $W_\varepsilon(x)$ *is the potential defined by*

$$W_\varepsilon(x) = \begin{cases} 1 - C_0\varepsilon^{1/3} & \text{for } \operatorname{dist}(x,\partial\Omega) > 2\varepsilon^{1/3}, \\ \beta_0 - C_1 k(s)\varepsilon - C_0\varepsilon^{4/3}, & \text{for } \operatorname{dist}(x,\partial\Omega) \le 2\varepsilon^{1/3}, \end{cases}$$

then

$$Q_{\varepsilon^{-2}\vec{A}}[u] \ge \frac{1}{\varepsilon^2}\int_\Omega W_\varepsilon(x)|u(x)|^2\, dx,$$

for all $\varepsilon \in (0, \varepsilon_0]$ *and all* $u \in W^{1,2}(\Omega)$*.*

3. The usual suspects

In this section we will study (as is usual in this context) the one-dimensional eigenvalue problem given by the quadratic form

$$q_\xi[\phi] = q_\xi(\phi,\phi) = \int_0^\infty |D_\tau\phi(\tau)|^2 + (\xi+\tau)^2|\phi(\tau)|^2\, d\tau, \tag{3.1}$$

on the maximal domain (i.e. on $W^{1,2}(\mathbb{R}_+) \cap \langle\tau\rangle^{-1}L^2(\mathbb{R}_+)$).

Actually, our starting point is the quadratic form (in two variables)

$$Q_\varepsilon[\phi] = Q_\varepsilon(\phi,\phi) = \int_0^{\varepsilon^{-1}|\partial\Omega|} d\sigma \int_0^\infty |D_\tau\phi(\sigma,\tau)|^2 + |(D_\sigma+\tau)\phi(\sigma,\tau)|^2\, d\tau, \tag{3.2}$$

[3]This would, of course, not be true for Dirichlet boundary conditions.

with periodic boundary conditions in σ and maximal domain. Explicitly, the domain is

$$\Big\{\phi \in L^2([0,\varepsilon^{-1}|\partial\Omega|] \times \mathbb{R}_+) \;\Big|\; D_\tau\phi \in L^2\;,\; (D_\sigma + \tau)\phi \in L^2$$
$$\text{and } \phi(0,\cdot) = \phi(\tfrac{|\partial\Omega|}{\varepsilon},\cdot)\Big\}. \quad (3.3)$$

The introduction of the quadratic form Q_ε is very natural from the following point of view. Suppose ψ is a function localized to a neighborhood of size t_0 (with t_0 small) near the boundary of $\partial\Omega$. We consider the first term of the Ginzburg-Landau functional:

$$\int_{\{t(x)<t_0\}} |(-i\nabla - \kappa H\vec{A})\psi|^2\,dx. \quad (3.4)$$

The quadratic form Q_ε results from this term by the following operations:

- Change to boundary coordinates.
  From the change of coordinates a number of factors $(1 - tk(s))$ will appear. These we replace by 1—using the philosophy that t is small near the boundary, so $(1 - tk(s)) \approx 1$.
- Replace $\vec{A}$ by the vector field (in boundary coordinates (s,t)) $(-t,0)$.
  Using the gauge chosen in (2.1), this is the main term in the Taylor expansion of $\vec{A}$ near the boundary.
- Scale variables by ε^{-1} (new coordinates (σ,τ)) and replace the upper bound $\varepsilon^{-1}t_0$ in the τ integration by $+\infty$.

From Theorem 2.3 we know that minimizers of the Ginzburg-Landau functional concentrate to a region of size ε around the boundary. Therefore, it is reasonable to assume that the quadratic form Q_ε captures the leading behavior of the expression in (3.4).

Upon decomposition in Fourier modes in the σ variable, we get the sequence of quadratic forms q_{ζ_n} from Q_ε, where

$$\zeta_n = \frac{2\pi n\varepsilon}{|\partial\Omega|}. \quad (3.5)$$

and where q_ξ was introduced in (3.1). For $\xi \in \mathbb{R}$ we define

$$E_0(\xi) = \inf_{\|\phi\|_2=1} q_\xi[\phi].$$

From Dauge-Helffer [DH93] (see also [BH93]) we find

Lemma 3.1. *There exists a unique $\xi_0 \in \mathbb{R}$ such that for $\xi \neq \xi_0$ we have $E_0(\xi) > E_0(\xi_0)$. We write $E_0(\xi_0) = \beta_0$. Moreover, there exists a unique normalized, positive function $\phi_\xi \in L^2(\mathbb{R}_+)$ such that $E_0(\xi) = q_\xi[\phi_\xi]$. The eigenfunctions ϕ_ξ satisfy :*
For any $K > 0$, there exist constants $\alpha, C > 0$ such that, for all $\tau > 0$ and $\xi \in]\xi_0 - K, \xi_0 + K[$:

$$\phi_\xi(\tau) \leq Ce^{-\alpha\tau^2}\;.$$

In particular, the lemma can be applied to $u_0 = \phi_{\xi_0}$.

Remark 3.2. The constant β_0 satisfies $\beta_0 \approx 0.59$. Lemma 3.1 provides the mathematical background for the definition of the constants β_0, C_1 used in the introduction. Mathematically, the important point about the numerical value of β_0 is the fact that $0 < \beta_0 < 1$. The localization to the boundary is a consequence of this fact.

Remark 3.3. In the literature there is a bit of confusion as to whether q_ξ should be defined as in (3.1) or we should replace $(\xi+\tau)^2$ by $(\xi-\tau)^2$ in (3.1). This latter convention is taken in [HM01]. However, most works on the Ginzburg-Landau functional (for example [HP03, SS00, LP99]) use our definition (1.1) of $\mathcal{E}$, which naturally leads to the sign convention in (3.1). One should remember this convention when comparing with the literature. In particular, with our definition, one gets $\xi_0 = -\sqrt{\beta_0} < 0$.

In the rest of this section we will prove that if ϕ is a function which almost minimizes Q_ε, i.e. such that $Q_\varepsilon[\phi] \approx \beta_0 \|\phi\|_2^2$, then

$$\phi(\sigma,\tau) \approx f(\sigma)u_0(\tau). \tag{3.6}$$

Proposition 3.11 below summarizes the results of this section.

We want (3.6) to be true in L^p (in particular in L^4 for application to the Ginzburg-Landau functional). In order to control the L^p norms, we need a Sobolev-type imbedding result.

Lemma 3.4 (Sobolev embedding)**.** *For $\varepsilon \le 1$ we define $\Omega^\varepsilon = (\frac{1}{2\pi\varepsilon}\mathbb{S}^1)_\sigma \times (\mathbb{R}_+)_\tau$ with measure $d\sigma\, d\tau$. Then, for all $p \in [2,+\infty)$, there exists a constant C_p, such that for all $\varepsilon \in (0,1]$ and all $u \in \mathcal{D}(Q_\varepsilon)$ we have*

$$\|u\|^2_{L^p(\Omega^\varepsilon)} \le C_p(Q_\varepsilon[\phi] + \|\phi\|^2_{L^2(\Omega^\varepsilon)}).$$

Proof. The uniform control of the constant in the Sobolev embedding Theorem for the scaled domain Ω^ε is probably well-known. We include a simple argument for completeness. Let $\theta \in (-\pi,\pi]$ be the usual parametrization of $\mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})$. In the same way, we have a natural parametrization of $\frac{1}{2\pi\varepsilon}\mathbb{S}^1 = \mathbb{R}/(\varepsilon\mathbb{Z})$ by $\theta \in (-\frac{1}{2\varepsilon}, \frac{1}{2\varepsilon}]$.

Let $\chi_1, \chi_2 \in C^\infty(\mathbb{S}^1)$ be a partition of unity on $\mathbb{S}^1$:

$$\begin{aligned}
&\chi_1^2+\chi_2^2 = 1, \qquad \chi_1(\theta) = 1 \text{ for } \theta \in [-\tfrac{\pi}{4}, \tfrac{\pi}{4}],\\
&\chi_1(\theta) = 0 \text{ for } \theta \in (-\pi,-\pi+\eta)\cup(\pi-\eta,\pi] \text{ for some } \eta>0.
\end{aligned}$$

Finally, define the functions $\chi_{i,\varepsilon}$ $(i=1,2)$ on $\frac{1}{2\pi\varepsilon}\mathbb{S}^1$ by

$$\chi_{i,\varepsilon}(\theta) = \chi_i(2\pi\varepsilon\theta).$$

For sufficiently regular u we can now estimate as follows.
First we use the partition of unity and the triangle inequality

$$\begin{aligned}
\|u\|^2_{L^p(\Omega^\varepsilon)} &\le \big(\|\chi_{1,\varepsilon}u\|_{L^p(\Omega^\varepsilon)} + \|\chi_{2,\varepsilon}u\|_{L^p(\Omega^\varepsilon)}\big)^2\\
&\le 2\big(\|\chi_{1,\varepsilon}u\|^2_{L^p(\Omega^\varepsilon)} + \|\chi_{2,\varepsilon}u\|^2_{L^p(\Omega^\varepsilon)}\big).
\end{aligned}\tag{3.7}$$

One can naturally extend $\chi_{1,\varepsilon}u$ to a function $\tilde\phi_{1,\varepsilon}$ on $\mathbb{R}^2_+$ by defining

$$\tilde\phi_{1,\varepsilon}(\theta,\tau) = \begin{cases}\chi_{1,\varepsilon}(\theta)u(\theta,\tau), & \theta \in (-\frac{1}{2\varepsilon},\frac{1}{2\varepsilon}),\\ 0, & |\theta| \ge \frac{1}{2\varepsilon}.\end{cases}$$

One then obtains a function on $\mathbb{R}^2$ by reflection, i.e.

$$\phi_{1,\varepsilon}(\theta,\tau) = \begin{cases}\tilde\phi_{1,\varepsilon}(\theta,\tau), & \tau \ge 0,\\ \tilde\phi_{1,\varepsilon}(\theta,-\tau), & \tau < 0.\end{cases}$$

We denote by $\phi_{2,\varepsilon}$ the similar extension of $\chi_{2,\varepsilon}u$. We clearly have the relations

$$\|\phi_{i,\varepsilon}\|^p_{L^p(\mathbb{R}^2)} = 2\|\chi_{i,\varepsilon}u\|^p_{L^p(\Omega^\varepsilon)}, \qquad \big\|\nabla|\phi_{i,\varepsilon}|\big\|^2_{L^2(\mathbb{R}^2)} = 2\big\|\nabla|\chi_{i,\varepsilon}u|\big\|^2_{L^2(\Omega^\varepsilon)}. \tag{3.8}$$

By the Sobolev inequality for $\mathbb{R}^2$ (cf. [Ada75])

$$\begin{aligned}\|\phi_{i,\varepsilon}\|^2_{L^p(\mathbb{R}^2)} &\leq S_p\left(\|\nabla|\phi_{i,\varepsilon}|\|^2_{L^2(\mathbb{R}^2)} + \|\phi_{i,\varepsilon}\|^2_{L^2(\mathbb{R}^2)}\right)\\ &= 2S_p\left(\|\nabla|\chi_{i,\varepsilon}u|\|^2_{L^2(\Omega^\varepsilon)} + \|\chi_{i,\varepsilon}u\|^2_{L^2(\Omega^\varepsilon)}\right),\end{aligned}$$

where S_p is the Sobolev constant for $\mathbb{R}^2$. By the diamagnetic inequality (see for example [LL97, Thm. 7.21]), we can estimate the above as

$$\begin{aligned}&\leq 2S_p\left(\|\nabla_{\mathbf{A}}(\chi_{i,\varepsilon}u)\|^2_{L^2(\Omega^\varepsilon)} + \|\chi_{i,\varepsilon}u\|^2_{L^2(\Omega^\varepsilon)}\right)\\ &= 2S_p\left(Q_\varepsilon[\chi_{i,\varepsilon}u] + \|\chi_{i,\varepsilon}u\|^2_{L^2(\Omega^\varepsilon)}\right).\end{aligned}$$

Summing over $i = 1, 2$, and computing the commutator (IMS-formula), we therefore find

$$\|u\|^2_{L^p(\Omega^\varepsilon)} \leq 2S_p\left\{Q_\varepsilon[u] + \left(1 + 2\|\nabla\chi_{1,\varepsilon}\|^2_{L^\infty(\Omega^\varepsilon)}\right)\|u\|^2_{L^2(\Omega^\varepsilon)}\right\}. \tag{3.9}$$

Since $\|\nabla\chi_{1,\varepsilon}\|_{L^\infty(\Omega^\varepsilon)}$ is bounded uniformly in ε ($\in (0,1]$), this finishes the proof of Lemma 3.4. ☐

As a first step towards (3.6) we have the following proposition.

Proposition 3.5. *There exists $\delta_0, \eta_0 > 0$, such that for all $p \in [2, +\infty)$, there exists $C_p > 0$, satisfying that if $\delta \leq \delta_0$, $\eta \leq \eta_0$, $\varepsilon \leq 1$ and if ϕ is a function satisfying,*

$$\|\phi\|_2 = 1; \qquad\qquad Q_\varepsilon[\phi] - (\beta_0 + \delta) < 0, \tag{3.10}$$

then

$$\phi = \sum_{n:|\zeta_n - \xi_0| \leq \eta} c_n e^{i\zeta_n\sigma}\phi_{\zeta_n}(\tau) + \omega_1(\sigma,\tau), \tag{3.11}$$

with ζ_n from (3.5)*, ϕ_{ζ_n} from Lemma 3.1, where*

$$c_n = \frac{\varepsilon}{|\partial\Omega|}\int_0^{\varepsilon^{-1}|\partial\Omega|}\int_0^\infty \phi(\sigma,\tau)\overline{e^{i\zeta_n\sigma}\phi_{\zeta_n}(\tau)}\,d\tau\,d\sigma\ , \tag{3.12}$$

and where

$$\|\omega_1\|_{L^p([0,\varepsilon^{-1}|\partial\Omega|]_\sigma\times(\mathbb{R}_+)_\tau)} \leq C_p\,\eta^{-1}\sqrt{\delta}.$$

For the proof of Proposition 3.5 we need a few preliminary results.

Lemma 3.6. *There exists $C > 0$ such that for $\xi \in \mathbb{R}$, $\phi \in L^2(\mathbb{R}_+)$,*

$$q_\xi[\phi] \geq \left(\beta_0 + \frac{1}{C}\min\{1, |\xi - \xi_0|^2\}\right)\|\phi\|^2.$$

Proof. Since the ground state energy $E_0(\xi)$ is smooth as a function of ξ around ξ_0, the estimate follows from a Taylor expansion to second order around the minimum and the fact that the minimum is non degenerate. ☐

We also need to consider the spectral gap. Define, for $\xi \in \mathbb{R}$,

$$\Delta E_0(\xi) = \inf_{\langle\phi,\phi_\xi\rangle = 0, \|\phi\| = 1} q_\xi[\phi] - E_0(\xi).$$

Lemma 3.7. *There exists $\eta_0, \delta_0 > 0$ such that if $|\xi - \xi_0| < \eta_0$, then*

$$\Delta E_0(\xi) \geq \delta_0 \ . \tag{3.13}$$

Proof. This follows from the existence of a gap at $\xi = \xi_0$ and from the continuity of the eigenvalues with respect to ξ. □

We now return our attention to the functional (in two variables) Q defined in (3.2).

Proof of Proposition 3.5.

Let $\phi \in L^2\big([0, \frac{|\partial\Omega|}{\varepsilon}] \times \mathbb{R}_+\big)$. Write, with $\eta \leq \eta_0$ sufficiently small,

$$\phi = \omega_1 + \phi_{\leq}, \tag{3.14}$$

where

$$\phi_{\leq} = \sum_{n:|\zeta_n - \xi_0| \leq \eta} c_n e^{i\zeta_n \sigma} \phi_{\zeta_n}(\tau),$$

Since the decomposition in (3.14) is an orthogonal projection onto subspaces left invariant by the operator defining Q_ε, we get the relations

$$\langle \omega_1, \phi_{\leq} \rangle = 0, \qquad\qquad Q_\varepsilon(\phi_{\leq}, \omega_1) = 0. \tag{3.15}$$

Using Lemmas 3.6 and 3.7 we find that, for η_0 sufficiently small,

$$\begin{aligned} \delta = \delta\|\phi\|_2^2 \geq & Q_\varepsilon[\phi] - \beta_0\|\phi\|_2^2 = Q_\varepsilon[\omega_1] + Q_\varepsilon[\phi_{\leq}] - \beta_0\|\phi\|_2^2 \\ \geq & C\eta^2\|\omega_1\|_2^2. \end{aligned} \tag{3.16}$$

Thus

$$\|\omega_1\|_2^2 \leq C'\eta^{-2}\delta. \tag{3.17}$$

Furthermore we read from (3.16) that

$$\delta \geq Q_\varepsilon[\omega_1] + Q_\varepsilon[\phi_{\leq}] - \beta_0\|\phi\|_2^2 \geq Q_\varepsilon[\omega_1] - \beta_0\|\omega_1\|_2^2,$$

i.e. using (3.17)

$$Q_\varepsilon[\omega_1] \leq C\,\eta^{-2}\,\delta. \tag{3.18}$$

Thus we find $Q_\varepsilon[\omega_1] + \|\omega_1\|_2^2 = \mathcal{O}(\eta^{-2}\delta)$. Using Lemma 3.4 this implies the desired estimates in $L^p(\Omega_\varepsilon)$ for $p \in [2, +\infty)$. □

We want to replace the functions $\phi_{\zeta_n}(\tau)$ appearing in the expansion (3.11) in Proposition 3.5 by the function (independent of n) $u_0(\tau)$. In order to do so we apply Marcinkiewicz's Theorem [Mar39].

Theorem 3.8. *Let A be a Fourier multiplier on $\mathbb{S}^1$ defined by*

$$Ae^{inx} = \lambda_n e^{inx} \ .$$

for a sequence $\{\lambda_n\}_{n\in\mathbb{Z}} \subset \mathbb{C}$. Then for all $p \geq 2$ there exits $C_p > 0$, such that if $\{\lambda_n\}$ satisfies

$$|\lambda_n| \leq M, \ \textit{for all}\ n \in \mathbb{Z}, \tag{3.19}$$

and

$$\sum_{n=2^\alpha}^{2^{\alpha+1}} (|\lambda_n - \lambda_{n+1}| + |\lambda_{-n} - \lambda_{-n-1}|) \leq M \ \textit{for all}\ \alpha \in \mathbb{N}, \tag{3.20}$$

then A extends to a bounded operator on $L^p(\mathbb{S}^1)$, and

$$\|A\|_{\mathcal{B}(L^p(\mathbb{S}^1),L^p(\mathbb{S}^1))} \leq C_p M.$$

Using Theorem 3.8 we can prove

Proposition 3.9. *For all p in $[2,\infty)$, there exists C_p such that, if we define*

$$\omega_2(\sigma) = \sum_{n:|\zeta_n-\xi_0|\leq\eta} c_n e^{i\zeta_n\sigma}[\phi_{\zeta_n}(\tau) - u_0(\tau)],$$

then, for $\eta \leq \eta_0$ sufficiently small,

$$\|\omega_2\|_{L^p(\Omega_\varepsilon)} \leq C_p\eta\|f\|_{L^p([0,\varepsilon^{-1}|\partial\Omega|])},$$

where

$$f(\sigma) = \sum_{n:|\zeta_n-\xi_0|\leq\eta} c_n e^{i\zeta_n\sigma}. \tag{3.21}$$

For the proof of Proposition 3.9 we will need the following estimates

Lemma 3.10. *There exists $\eta_0 > 0$ and positive continuous functions $w_1, w_2 \in \cap_{p\geq 2} L^p(\mathbb{R}_+)$ such that for $\zeta, \zeta' \in \mathbb{C}$, $|\zeta - \xi_0|, |\zeta' - \xi_0| \leq \eta_0$,*

$$|\phi_\zeta(\tau) - u_0(\tau)| \leq C|\zeta - \xi_0|\, w_1(\tau), \tag{3.22}$$

$$|\phi_\zeta(\tau) - \phi_{\zeta'}(\tau)| \leq C|\zeta - \zeta'|\, w_2(\tau), \tag{3.23}$$

where $\|w_j\|_{L^p(\mathbb{R}_+)}$ is bounded uniformly in ζ, ζ'.

Proof of Lemma 3.10. The first estimate (3.22) is the special case $\zeta' = \xi_0$ of the second. So we only need to prove (3.23). Define

$$\mathcal{D} = \{\phi \in H^2(\mathbb{R}_+) \Big| \phi'(0) = 0 \text{ and } (1+\tau^2)\phi \in L^2(\mathbb{R}_+)\}.$$

We now get by analytic perturbation theory that

$$\zeta \mapsto \phi_\zeta,$$

is an analytic function from a neighborhood of ξ_0 with values in $\mathcal{D}$. Thus, by Taylor's formula, we can write

$$\phi_\zeta - \phi_{\zeta'} = (\zeta - \zeta') \int_0^1 v_{\zeta'+s(\zeta-\zeta')}\, ds, \tag{3.24}$$

where $v_z = \partial_z\phi_z$ is analytic (in particular continuous) in z with values in $\mathcal{D}$. Now

$$\sup_{\zeta,\zeta'\in B(\xi_0,\eta_0)} \frac{|\phi_\zeta(\tau) - \phi_{\zeta'}(\tau)|}{|\zeta - \zeta'|} \leq \sup_{z\in B(\xi_0,\eta_0)} |v_z(\tau)|.$$

We will prove that (for η_0 sufficiently small)

$$w(\tau) = \sup_{z\in B(\xi_0,\eta_0)} |v_z(\tau)| \tag{3.25}$$

belongs to $L^p(\mathbb{R}_+)$.

For $\zeta \in \mathbb{R}$, define the operator $h(\zeta)$ on $L^2(\mathbb{R}_+)$ as the selfadjoint Neumann realization with domain $\mathcal{D}$ of the differential operator

$$\phi \mapsto -\phi'' + (\tau + \zeta)^2\phi\ .$$

The function ϕ_{ζ_n} is the eigenfunction corresponding to the lowest eigenvalue of $h(\zeta_n)$. By choosing η_0 sufficiently small we may assume that there exist unique analytic functions $E(\zeta)$, ϕ_ζ for $\zeta \in B(\xi_0, \eta_0)$ such that

$$h(\zeta)\phi_\zeta = E(\zeta)\phi_\zeta, \tag{3.26}$$

and $\phi_{\xi_0} = u_0$, ϕ_ζ is normalized in $L^2(\mathbb{R}_+)$ and $E(\xi_0) = \beta_0$. By differentiation of (3.26) we get the following equation for v_z:

$$(h(z) - E(z))v_z = (2(\tau + z) + E'(z))\phi_z.$$

By differentiation of the identity $\|\phi_z\|_{L^2(\mathbb{R}_+)} = 1$;, we get that $\langle v_z, \phi_z\rangle = 0$. Define the linear operator $r(z) : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ by

$$r(z)\psi = \begin{cases} (h(z) - E(z))^{-1}\psi, & \text{if } \langle \psi, \phi_z\rangle = 0, \\ 0, & \text{if } \psi \in \operatorname{Span}\{\phi_z\}. \end{cases}$$

We thereby find

$$v_z = r(z)(2(\tau + z) + E'(z))\phi_z. \tag{3.27}$$

Define, for $k \geq 0$, the space $\mathcal{B}^k$ by

$$\mathcal{B}^k = H^k(\mathbb{R}_+) \cap (1 + \tau^2)^{-k/2} L^2(\mathbb{R}_+),$$

with its natural Hilbert space structure.

The desired estimate (3.25) is a consequence of the following three standard facts:

- $r(z)$ defines a bounded operator from $\mathcal{B}^k$ to $\mathcal{B}^{k+2} \cap \{u'(0) = 0\}$.
- $z \mapsto \phi_z$ defines a continuous function from $B(\xi_0, \eta_0)$ to $\mathcal{B}^k$ for all $k \geq 0$.
- The Sobolev embedding theorems permit to recover L^p estimates from the L^2 estimates.

$\square$

Proof of Proposition 3.9. Let f be the function from (3.21), define, for $\tau \in \mathbb{R}_+$,

$$\lambda_n(\tau) = \begin{cases} \phi_{\zeta_n}(\tau) - u_0(\tau), & |\zeta_n - \xi_0| \leq \eta \\ 0, & |\zeta_n - \xi_0| > \eta, \end{cases} \tag{3.28}$$

and let A_τ be the operator associated with the sequence $\{\lambda_n(\tau)\}$ as in Theorem 3.8. Then

$$\begin{aligned} \|\omega_2\|_p^p &= \int_0^\infty \left(\int_0^{2\pi} \Big| \sum_{n:|\zeta_n - \xi_0| \leq \eta} c_n e^{i\zeta_n \frac{|\partial\Omega|}{2\pi\varepsilon} s} [\phi_{\zeta_n}(\tau) - u_0(\tau)] \Big|^p \frac{|\partial\Omega|}{2\pi\varepsilon} ds \right) d\tau \\ &= \int_0^\infty \left(\int_0^{2\pi} |A_\tau \hat{f}|^p(s) \, \frac{|\partial\Omega|}{2\pi\varepsilon} ds \right) d\tau, \end{aligned} \tag{3.29}$$

where we have introduced the 2π-periodic function

$$\hat{f}(s) = \sum_{n:|\zeta_n - \xi_0| \leq \eta} c_n e^{i\zeta_n \frac{|\partial\Omega|}{2\pi\varepsilon} s} = \sum_{n:|\zeta_n - \xi_0| \leq \eta} c_n e^{ins}.$$

We therefore get

$$\|\omega_2\|_p^p \le \int_0^\infty \|A_\tau\|^p_{\mathcal{B}(L^p(\mathbb{S}^1),L^p(\mathbb{S}^1))}\, d\tau \|\hat{f}\|^p_{L^p(\mathbb{S}^1)} \frac{|\partial\Omega|}{2\pi\varepsilon}$$
$$= \|f\|_p^p \int_0^\infty \|A_\tau\|^p_{\mathcal{B}(L^p(\mathbb{S}^1),L^p(\mathbb{S}^1))} d\tau. \tag{3.30}$$

Using Lemma 3.10, we find

$$\sum_{n:|\zeta_n-\xi_0|\le\eta} |\lambda_n - \lambda_{n+1}| \le C\frac{\eta}{\varepsilon}\varepsilon w_2(\tau) = C\eta w_2(\tau),$$
$$|\lambda_n| \le C\eta w_1(\tau). \tag{3.31}$$

Thus, using Theorem 3.8, we get

$$\|A_\tau\|_{\mathcal{B}(L^p(\mathbb{S}^1),L^p(\mathbb{S}^1))} \le C\eta(w_1(\tau)+w_2(\tau)),$$

and therefore, since w_1 and w_2 belong to $L^p(\mathbb{R}_+)$,

$$\int_0^\infty \|A_\tau\|^p_{\mathcal{B}(L^p(\mathbb{S}^1),L^p(\mathbb{S}^1))} d\tau \le C\eta^p. \tag{3.32}$$

□

We can sum up the results of Propositions 3.5 and 3.9, in the form needed for later applications, as follows. This is a precise version of the informal statement (3.6).

Proposition 3.11. *There exist $\delta_0, \eta_0 > 0$ such that if $p \in [2,\infty)$, then there exists $C_p > 0$ such that for all $\delta < \delta_0$, $\eta < \eta_0$, $\varepsilon \in (0,1]$ and all functions ϕ satisfying*

$$Q_\varepsilon[\phi] - (\beta_0 + \delta)\|\phi\|_2^2 < 0, \tag{3.33}$$

we have

$$\phi = f(\sigma)u_0(\tau) + \omega(\sigma,\tau). \tag{3.34}$$

Here

$$f(\sigma) = \sum_{n:|\zeta_n-\xi_0|\le\eta} c_n e^{i\zeta_n\sigma}, \tag{3.35}$$

with ζ_n from (3.5)*, where*

$$c_n = \sqrt{\frac{\varepsilon}{|\partial\Omega|}} \int_0^{\varepsilon^{-1}|\partial\Omega|} \int_0^\infty \phi(\sigma,\tau)\overline{e^{i\zeta_n\sigma}\phi_{\zeta_n}(\tau)}\, d\tau\, d\sigma\ , \tag{3.36}$$

with ϕ_{ζ_n} from Lemma 3.1, and where

$$\|\omega\|_{L^p([0,\varepsilon^{-1}|\partial\Omega|]_\sigma\times(\mathbb{R}_+)_\tau)} \le C_p\, \left(\eta^{-1}\sqrt{\delta}\|f\|_{L^2([0,\varepsilon^{-1}|\partial\Omega|])} + \eta\|f\|_{L^p([0,\varepsilon^{-1}|\partial\Omega|])}\right).$$

Proof. Proposition 3.11 is an immediate consequence of Propositions 3.5 and 3.9. □

4. Energy asymptotics

In this section we will prove the main theorems. First we establish a precise upper bound to the ground state energy of the Ginzburg-Landau functional.

4.1. **Upper bound.**
To get a good upper bound we can use an explicit test configuration. Our choice is very similar to the one used by Lu and Pan in [LP99]. We choose $\vec{A} = \vec{F}$ (the external field). For ψ, we write (in the boundary coordinates defined in Appendix A)

$$\psi(s,t) = e^{i\kappa H \chi_f + i[\xi_0]_\varepsilon s/\varepsilon} \lambda(s) u_0(t/\varepsilon) \chi(t). \tag{4.1}$$

We will proceed to define the different parts of ψ.

The function χ is smooth and localizes to the boundary region. If t_0 is the constant from Appendix A defining the boundary region, the function χ is chosen non-increasing and satisfying

$$\chi \in C^\infty(\mathbb{R}), \qquad \chi(t) = \begin{cases} 0, & t \geq 3t_0/4, \\ 1, & t \leq t_0/2. \end{cases} \tag{4.2}$$

The only reason for introducing χ is that this localization near the boundary allows us to use the boundary coordinates (s,t).

The symbol $[\xi_0]_\varepsilon$ denotes the following number

$$[\xi_0]_\varepsilon = \max\{z \in \tfrac{2\pi\varepsilon}{|\partial\Omega|}\mathbb{Z} \big| z \leq \xi_0\}.$$

Ideally, we would use ξ_0 instead of $[\xi_0]_\varepsilon$, but in order for ψ to be well defined it needs to satisfy the periodicity assumption. This is assured by using $[\xi_0]_\varepsilon$.

For $\lambda(s)$ we would like to make the choice

$$\lambda_{\text{formal}}(s)^2 = \frac{1}{\varepsilon\kappa^2 \|u_0\|_4^4} \Big[\beta_0^{3/2}\rho - C_1(k_{\max} - k(s))\Big]_+ . \tag{4.3}$$

However, the function $t \mapsto [t]_+^{1/2}$ does not have a bounded derivative, so we need to regularize the function slightly. Therefore we introduce, for $\nu > 0$, the smoothed out version

$$[t]_{+,\nu} = \begin{cases} 0, & t \leq 0, \\ \sqrt{t^2 + \nu^2} - \nu, & t > 0. \end{cases}.$$

An elementary analysis gives that $[t]_{+,\nu}^{1/2} \in C^1(\mathbb{R})$ and that

$$\|[t]_+^{1/2} - [t]_{+,\nu}^{1/2}\|_\infty \leq C\sqrt{\nu}, \qquad \left\| \frac{d}{dt} [t]_{+,\nu}^{1/2} \right\|_\infty \leq C \frac{1}{\sqrt{\nu}}. \tag{4.4}$$

We make the choice $\nu = \varepsilon^{1/2}$ and define

$$\lambda(s) = \frac{1}{\sqrt{\varepsilon}\kappa \|u_0\|_4^2} \Big[\beta_0^{3/2}\rho - C_1(k_{\max} - k(s))\Big]_{+,\varepsilon^{1/2}}^{1/2} . \tag{4.5}$$

The function χ_f in (4.1) is the phase of a gauge transformation to be defined below (see (4.6)). Let (using notation from Appendix A)

$$\tilde{A}(s,t) = \begin{pmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{pmatrix} (s,t) = \begin{pmatrix} (1 - tk(s))\vec{F}(\Phi(s,t)) \cdot \gamma'(s) \\ \vec{F}(\Phi(s,t)) \cdot \nu(s) \end{pmatrix},$$

be the vector potential $\vec{F}$ transformed to boundary coordinates. We choose χ_f such that

$$\begin{pmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{pmatrix} + \nabla_{(s,t)}\chi_f = \begin{pmatrix} -t + \mathcal{O}(t^2) \\ 0 \end{pmatrix}. \tag{4.6}$$

Since the choice of gauge (1.3) implies that

$$\tilde{A}_2(s,0) = 0, \qquad\qquad (\partial_t \tilde{A}_2)(s,0) = -1,$$

(4.6) is accomplished by the choice

$$\chi_f(s,t) = \int_0^t \tilde{A}_2(s,t')\,dt'.$$

Let us start by considering the change of gauge χ_f. Using (4.6), we write

$$\begin{pmatrix} \hat{A}_1 \\ 0 \end{pmatrix} = \begin{pmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{pmatrix} + \nabla_{(s,t)}\chi_f = \begin{pmatrix} -t + \mathcal{O}(t^2) \\ 0 \end{pmatrix}. \tag{4.7}$$

Since, by (A.1)

$$-\partial_t \hat{A}_1 = B(s,t)(1-tk(s)) = 1 - tk(s),$$

we get

$$\hat{A}_1 = -t + k(s)\frac{t^2}{2}. \tag{4.8}$$

With all these choices, ψ from (4.1) is defined and we can proceed to calculate $\mathcal{E}[\psi,\vec{F}]$. We will calculate in boundary coordinates, using Appendix A, with the following formula.

$$\begin{aligned}\mathcal{E}[\psi,\vec{F}] = \int (1-tk(s))^{-1}|(-i\partial_s - \kappa H\tilde{A}_1)\psi|^2\,ds\,dt \\ + \int \Big\{|(-i\partial_t - \kappa H\tilde{A}_2)\psi|^2 - \kappa^2|\psi|^2\Big\}(1-tk(s))\,ds\,dt \\ + \tfrac{1}{2}\kappa^2 \int |\psi|^4(1-tk(s))\,ds\,dt\,.\end{aligned}$$

Upon calculating $\mathcal{E}[\psi,\vec{F}]$ we therefore find

$$\begin{aligned}\mathcal{E}[\psi,\vec{F}] = \int (1-tk(s))^{-1}\Big|(-i\partial_s - \kappa H\hat{A}_1 + \varepsilon^{-1}[\xi_0]_\varepsilon)(\lambda(s)u_0(t/\varepsilon)\chi(t))\Big|^2 ds\,dt \\ - \int (1-tk(s))\big|\partial_t\left(\lambda(s)u_0(t/\varepsilon)\chi(t)\right)\big|^2 dsdt \\ - \kappa^2\int (1-tk(s))\big|\lambda(s)u_0(t/\varepsilon)\chi(t)\big|^2(1-tk(s))\,ds\,dt \\ + \frac{\kappa^2}{2}\int \big|\lambda(s)u_0(t/\varepsilon)\chi(t)\big|^4(1-tk(s))\,ds\,dt\,.\end{aligned} \tag{4.9}$$

This we write as

$$\begin{aligned}\mathcal{E}[\psi,\vec{F}] = \int_0^{|\partial\Omega|} |\lambda(s)|^2 S(s)ds \\ + \frac{\kappa^2}{2}\int_0^{|\partial\Omega|}|\lambda(s)|^4\int_0^\infty \big|u_0(t/\varepsilon)\chi(t)\big|^4(1-tk(s))\,dsdt \\ + r_1.\end{aligned} \tag{4.10}$$

Here

$$S(s)=\int_0^\infty (1-tk(s))^{-1}\Big(\kappa H\hat{A}_1-\varepsilon^{-1}[\xi_0]_\varepsilon\Big)^2|u_0(t/\varepsilon)\chi(t)|^2\,dt$$
$$+\int_0^\infty (1-tk(s))\big|\partial_t(u_0(t/\varepsilon)\chi(t))\big|^2\,dt$$
$$-\kappa^2\int_0^\infty (1-tk(s))\big|u_0(t/\varepsilon)\chi(t))\big|^2\,dt\,, \tag{4.11}$$

and (since $\lambda(s)$ is real-valued)

$$r_1=\int (1-tk(s))^{-1}|\lambda'(s)|^2|u_0(t/\varepsilon)\chi(t))|^2\,dsdt=\mathcal{O}(\varepsilon^{3/2}), \tag{4.12}$$

where we used the choice of λ, and consequently that there exists c such that

$$|\lambda'(s)|^2\leq \frac{c}{\varepsilon\kappa^2}\varepsilon^{-1/2}=\mathcal{O}(\varepsilon^{1/2}),$$

to get the last estimate.

Using now the rapid decay of u_0 at $+\infty$, we can 'eliminate' the cut-off function χ in $S(s)$, and get :

$$\mathcal{E}[\psi,\vec{F}]=\int_0^{|\partial\Omega|}|\lambda(s)|^2T(s)ds$$
$$+\frac{\kappa^2\varepsilon}{2}\int_0^{|\partial\Omega|}|\lambda(s)|^4\|u_0\|_4^4(1+\mathcal{O}(\varepsilon))\,ds+\mathcal{O}(\varepsilon^{3/2})\,, \tag{4.13}$$

with

$$T(s)=\int_0^\infty (1-tk(s))^{-1}\Big(\kappa H\hat{A}_1-\varepsilon^{-1}[\xi_0]_\varepsilon\Big)^2|u_0(t/\varepsilon)|^2\,dt$$
$$+\int_0^\infty (1-tk(s))\big|\partial_t(u_0(t/\varepsilon))\big|^2\,dt$$
$$-\kappa^2\varepsilon+\kappa^2\int_0^\infty tk(s)\big|u_0(t/\varepsilon)\big|^2\,dt+\mathcal{O}(\epsilon)\,. \tag{4.14}$$

Let us write, using (4.8),

$$\kappa H\hat{A}_1-\varepsilon^{-1}[\xi_0]_\varepsilon=(-\kappa Ht-\varepsilon^{-1}\xi_0)+\kappa H\frac{t^2k(s)}{2}-\varepsilon^{-1}([\xi_0]_\varepsilon-\xi_0)\,.$$

Then we can rewrite (4.14) as

$$T(s):=T_0(s)+T_1(s)+T_2(s)+T_3(s)+T_4(s)+\mathcal{O}(\varepsilon)$$

with the T_j $(j = 0, \dots, 4)$ defined by :

$$\begin{aligned}
T_0(s) &:= \int_0^\infty \left((\kappa H t + \varepsilon^{-1}\xi_0)^2 |u_0(\frac{t}{\varepsilon})|^2 + |\partial_t (u_0(\frac{t}{\varepsilon}))|^2 \right) dt - \kappa^2 \varepsilon \;, \\
T_1(s) &:= \int_0^\infty t k(s) (\kappa H t + \varepsilon^{-1}\xi_0)^2 |u_0(t/\varepsilon)|^2 \, dt \;, \\
T_2(s) &:= -2 \int_0^\infty \left[\kappa H \frac{t^2 k(s)}{2} - \varepsilon^{-1}([\xi_0]_\varepsilon - \xi_0) \right] (\kappa H t + \varepsilon^{-1}\xi_0) |u_0(t/\varepsilon)|^2 \, dt \;, \\
T_3(s) &:= - \int_0^\infty t k(s) |\partial_t (u_0(t/\varepsilon))|^2 \, dt \;, \\
T_4(s) &:= \kappa^2 \int_0^\infty t k(s) |u_0(t/\varepsilon)|^2 \, dt \;,
\end{aligned} \tag{4.15}$$

which will now be analyzed successively using the formulas from Appendix B.
For T_0 a change of variables gives immediately :

$$T_0(s) = \frac{\beta_0}{\varepsilon} - \kappa^2 \varepsilon \;.$$

Using (1.26), we get :

$$T_0(s) = C_1 k_{\max} - \beta_0^{3/2} \rho + \mathcal{O}(\frac{\rho^2}{\kappa}) \;. \tag{4.16}$$

Next we consider :

$$\begin{aligned}
T_1 &= k(s) \int_0^\infty \tau (\tau + \xi_0)^2 |u_0(\tau)|^2 \, d\tau \\
&= k(s) \int_0^\infty [(\tau + \xi_0)^3 - \xi_0 (\tau + \xi_0)^2] |u_0(\tau)|^2 \, d\tau \\
&= (M_3 - \xi_0 \frac{\beta_0}{2}) k(s) = (\frac{C_1}{2} + \frac{\beta_0^{3/2}}{2}) k(s).
\end{aligned} \tag{4.17}$$

In the last line, we have used (1.15) and (B.2).

$$\begin{aligned}
T_2 &= -k(s) \int_0^\infty \tau^2 (\tau + \xi_0) |u_0(\tau)|^2 \, d\tau \\
&= -k(s) \int_0^\infty \left[(\tau + \xi_0)^2 - 2\tau\xi_0 - \xi_0^2 \right] (\tau + \xi_0) |u_0(\tau)|^2 \, d\tau \\
&= -k(s) \int_0^\infty \left[(\tau + \xi_0)^2 - 2\xi_0 (\tau + \xi_0) + \xi_0^2 \right] (\tau + \xi_0) |u_0(\tau)|^2 \, d\tau \\
&= -(M_3 - 2\xi_0 M_2) k(s) = -(\frac{C_1}{2} + \beta_0^{3/2}) k(s).
\end{aligned} \tag{4.18}$$

Furthermore,

$$
\begin{aligned}
T_3 &= -k(s)\int_0^\infty \tau |u_0'(\tau)|^2\,d\tau \\
&= k(s)\int_0^\infty u_0(\tau)[u_0'(\tau)+\tau u_0''(\tau)]\,d\tau \\
&= k(s)\left(-\frac{u_0(0)^2}{2}+\int_0^\infty \tau\big[(\tau+\xi_0)^2-\beta_0\big]|u_0(\tau)|^2\,d\tau\right) \\
&= -k(s)\frac{u_0(0)^2}{2} \\
&\quad + k(s)\int_0^\infty \big[(\tau+\xi_0)^3-\xi_0(\tau+\xi_0)^2-\beta_0(\tau+\xi_0)+\beta_0\xi_0\big]\,|u_0(\tau)|^2\,d\tau \\
&= k(s)(-3M_3+M_3-\xi_0 M_2-\beta_0\xi_0) = (-C_1-\frac{\beta_0^{3/2}}{2})k(s).
\end{aligned} \tag{4.19}
$$

Here we have used an integration by part (from line 1 to line 2), the equation satisfied by u_0 for line 3 and the formulas from Appendix B. Furthermore, using the same estimates as for T_0, we get

$$
\kappa^2\varepsilon^2 = \frac{\kappa}{H} = \beta_0 + \mathcal{O}(\frac{\rho}{\kappa})\ , \tag{4.20}
$$

and this leads for T_4 to :

$$
\begin{aligned}
T_4(s) &:= \kappa^2\int_0^\infty t k(s)|u_0(t/\varepsilon)|^2\,dt \\
&= \kappa^2\varepsilon^2 k(s)\int_0^\infty \tau|u_0(\tau)|^2\,d\tau = -k(s)\kappa^2\varepsilon^2\xi_0 \\
&= \beta_0^{3/2}k(s)+\mathcal{O}(\frac{\rho}{\kappa}).
\end{aligned} \tag{4.21}
$$

Thus

$$
T_1+T_2+T_3+T_4 = -C_1 k(s) + \mathcal{O}(\frac{\rho}{\kappa}). \tag{4.22}
$$

So the energy estimate (4.13) becomes

$$
\begin{aligned}
\mathcal{E}[\psi,\vec{F}] = \int_0^{|\partial\Omega|} &\Big\{|\lambda(s)|^2[C_1(k_{\max}-k(s)) - \beta_0^{3/2}\rho + \mathcal{O}(\frac{\rho}{\kappa})] \\
&+ \frac{\kappa^2\varepsilon}{2}|\lambda(s)|^4\|u_0\|_4^4(1+\mathcal{O}(\varepsilon))\Big\}\,ds + \mathcal{O}(\varepsilon^{3/2})\ .
\end{aligned} \tag{4.23}
$$

Using the choice of λ and the first inequality in (4.4), we get

$$
\begin{aligned}
\mathcal{E}[\psi,\vec{F}] &= \frac{-1}{2\varepsilon\kappa^2\|u_0\|_4^4}\int_0^{|\partial\Omega|}\Big[\beta_0^{3/2}\rho - C_1(k_{\max}-k(s))\Big]_+^2\,ds \\
&\quad + \mathcal{O}(\varepsilon^{1/4}\frac{\rho}{\kappa}) + \mathcal{O}(\frac{\rho^2}{\kappa^2}).
\end{aligned} \tag{4.24}
$$

If we insert the asymptotics of ε, this upper bound fits the energy asymptotics in (1.17).

4.2. **Lower bound.**
Let $(\psi, \vec{A})$ be a minimizer. First we need to make a localization to the boundary region. Let $1 = f_1^2(t) + f_2^2(t)$ be a standard partition of unity on $[0, \infty)$. We choose f_1 to be non-increasing and satisfying

$$f_1(t) = \begin{cases} 1, & t \leq 1, \\ 0, & t \geq 2. \end{cases} \tag{4.25}$$

Consider $\psi_j(x) = f_j(t(x)/\varepsilon M)\psi(x)$. We will choose $M = M(\varepsilon) \gg 1$ later, under the restriction:

$$\varepsilon M \to 0. \tag{4.26}$$

Therefore, for ε small enough, one can change to boundary coordinates on the support of $f_1(t(x)/\varepsilon M)$. Then, by the IMS-formula,

$$\begin{aligned} \mathcal{E}[\psi, \vec{A}] =& \mathcal{E}[\psi_1, \vec{A}] + \mathcal{E}[\psi_2, \vec{A}] - (\varepsilon M)^{-2} \int (|\nabla f_1|^2 + |\nabla f_2|^2)|\psi(x)|^2 \, dx \\ &+ \frac{\kappa^2}{2} \int (1 - f_1^4 - f_2^4)|\psi(x)|^4 \, dx. \end{aligned} \tag{4.27}$$

Consider first the last term in (4.27). Since

$$1 = (f_1^2 + f_2^2)^2 = f_1^4 + f_2^4 + 2f_1^2 f_2^2,$$

this term is positive. We will therefore discard it for the lower bound.

Remark 4.1. The algebraic fact that the above term is positive is unimportant. By using the Agmon estimates (as will be done for the gradient terms below), one easily finds that the last term in (4.27) is small compared to the main term.

Notice, that Theorem 2.3 tells us that the estimate (2.3) is satisfied. Therefore, Proposition 2.4 implies that

$$\mathcal{E}[\psi_2, \vec{A}] \geq \varepsilon^{-2}(1 - \mathcal{O}(\kappa^{-1/3}))\|\psi_2\|_2^2 \geq 0.$$

So we can ignore this positive term for the lower bound.

The Agmon estimates, combined with the support properties of $(|\nabla f_1|^2 + |\nabla f_2|^2)$ can be used to bound the localization errors as follows:

$$\begin{aligned} (\varepsilon M)^{-2} &\int (|\nabla f_1|^2 + |\nabla f_2|^2)|\psi(x)|^2 \, dx \\ &\leq C(\varepsilon M)^{-2} \int_{\{1 \leq \frac{t(x)}{\varepsilon M} \leq 2\}} e^{-\alpha t(x)/\varepsilon} \left(e^{\alpha t(x)/\varepsilon} |\psi(x)|^2 \right) dx \\ &\leq C(\varepsilon M)^{-2} e^{-\alpha M} \int e^{\alpha t(x)/\varepsilon} |\psi(x)|^2 \, dx \\ &\leq c(\varepsilon M)^{-2} e^{-\alpha M} \int_{\{t(x) < c_0/\kappa\}} |\psi(x)|^2 \, dx \\ &\leq c(\varepsilon M)^{-2} e^{-\alpha M} \|\psi_1\|_2^2. \end{aligned} \tag{4.28}$$

Here we used, in the last line, the fact that $M \to \infty$, so therefore (for ε sufficiently small)

$$\int_{\{t(x) < c_0/\kappa\}} |\psi(x)|^2 \, dx \leq \|\psi_1\|_2^2.$$

We now redefine α in order to absorb the factor of M^{-2} and find

$$(\varepsilon M)^{-2}\int(|\nabla f_1|^2+|\nabla f_2|^2)|\psi(x)|^2\,dx \leq c\varepsilon^{-2}e^{-\alpha M}\|\psi_1\|_2^2.$$

From these estimates and (4.27), we find

$$\mathcal{E}[\psi,\vec{A}] \geq \int |(-i\nabla-\kappa H\vec{A})\psi_1|^2-\kappa^2(1+ce^{-\alpha M})|\psi_1|^2+\frac{\kappa^2}{2}|\psi_1|^4\,dx.$$

Upon changing to boundary coordinates (see Appendix A) this integral becomes:

$$\int_0^{|\partial\Omega|}\int_{\{t\leq 2M\varepsilon\}}\Big\{|D_t\phi|^2+(1-tk(s))^{-2}|(D_s-\kappa H\tilde{A}_1)\phi|^2 \\ -\kappa^2(1+ce^{-\alpha M})|\phi|^2+\frac{\kappa^2}{2}|\phi|^4\Big\}(1-tk(s))\,dt\,ds, \quad (4.29)$$

where

$$\phi=\psi_1(\Phi(s,t)) \quad (4.30)$$

and $\tilde{A}_1$ was defined in Appendix A. Here we used the fact that after possibly effecting a gauge transformation, we may assume that $\tilde{A}_2=0$.

In the gauge where $\tilde{A}_2=0$, we have from (A.1)

$$-\partial_t\tilde{A}_1=(1-tk(s))\tilde{B}.$$

From Theorem 2.3 we know that the estimates (2.3) are satisfied by the minimizing vector potential $\vec{A}$. Thus, by a Taylor expansion near the boundary, we can write

$$\tilde{B}=1+\kappa^{-1/3}tb(s,t),$$

where b is bounded (uniformly in κ) in C^0 in a fixed (i.e. κ-independent) neighborhood of the boundary. So we find

$$\tilde{A}_1=-t+\frac{k(s)t^2}{2}+\mathcal{O}(\kappa^{-1/3}t^2), \quad (4.31)$$

uniformly in κ in a fixed neighborhood of the boundary.

In order to have a simple model operator, we want to replace $\tilde{A}_1(s,t)$ by $-t$. Therefore, we estimate

$$\begin{aligned}|(-i\partial_s-\kappa H\tilde{A}_1)\phi|^2 &\\ &=|(-i\partial_s+\kappa Ht)\phi|^2+(\kappa H)^2|(\tilde{A}_1+t)\phi|^2+\kappa H\Re((-i\partial_s+\kappa Ht)\phi\cdot(\tilde{A}_1+t)\overline{\phi})\\ &\geq(1-\gamma)|(-i\partial_s+\kappa Ht)\phi|^2+(\kappa H)^2(1-\gamma^{-1})|(\tilde{A}_1+t)\phi|^2. \quad (4.32)\end{aligned}$$

Using the Agmon estimates and the simple inequality $|\tilde{A}_1+t|\leq Ct^2$ (valid on $\operatorname{supp}\phi$) deduced from (4.31), we find

$$(\kappa H)^2\int|(\tilde{A}_1+t)\phi|^2\,dsdt\leq c(\kappa H)^2\|t^4e^{-\alpha t/\varepsilon}\|_\infty\int e^{\alpha t/\varepsilon}|\phi|^2\,dsdt\leq c'\|\phi\|_2^2.$$

We use the Agmon estimates (and the boundedness of the curvature $k(s)$) to replace all factors of $(1-tk(s))$ by $1+\mathcal{O}(\varepsilon)$. Upon choosing

$$\gamma=\varepsilon, \qquad M=C_M|\log\varepsilon|, \quad (4.33)$$

(for a big constant C_M), which satisfies (4.26), we get

$$\mathcal{E}[\psi,\vec{A}]\geq(1-c\varepsilon)\tilde{Q}[\phi]-\kappa^2(1+c\varepsilon)\|\phi\|_2^2+\frac{\kappa^2}{2}(1-c\varepsilon)\|\phi\|_4^4,$$

where

$$\tilde{Q}[\phi] = \int_0^{|\partial\Omega|} \int_0^{\infty} |D_t\phi|^2 + |(D_s + \kappa H t)\phi|^2 \, dt \, ds.$$

We finally change coordinates $(s,t) = \varepsilon(\sigma,\tau)$. We introduce

$$\tilde{\phi}(\sigma,\tau) := \phi(\varepsilon\sigma, \varepsilon\tau). \tag{4.34}$$

Thereby the inequality becomes

$$\mathcal{E}[\psi, \vec{A}] \geq (1 - c\varepsilon) Q_\epsilon[\tilde{\phi}] - \frac{\kappa}{H}(1 + c\varepsilon)\|\tilde{\phi}\|_2^2 + \frac{\kappa}{2H}(1 - c\varepsilon)\|\tilde{\phi}\|_4^4. \tag{4.35}$$

Here

$$Q_\varepsilon[\tilde{\phi}] = \int_0^{|\partial\Omega|/\varepsilon} d\sigma \int_0^{+\infty} |D_\tau\tilde{\phi}|^2 + |(D_\sigma + \tau)\tilde{\phi}|^2 \, d\tau,$$

is the quadratic form studied in Section 3.

The boundary concentration (uniform in the case of large ρ) will now essentially be a general feature of functions $\tilde{\phi}$ satisfying

$$Q_\varepsilon[\tilde{\phi}] - (\frac{\kappa}{H} + c\varepsilon)\|\tilde{\phi}\|_2^2 \leq 0.$$

At this point we should recall the definition of δ_ε from (1.25).

The limiting behavior of $\tilde{\phi}$ will follow from Proposition 3.11. Notice that till this point the analysis has been purely *linear*. We have only studied spectral properties of the magnetic quadratic form. The quartic term in (4.35) will only play a role in determining the normalization.

Using Proposition 3.11, we write

$$\tilde{\phi}(\sigma,\tau) = f(\sigma)u_0(\tau) + \omega(\sigma,\tau), \tag{4.36}$$

where the function ω satisfies (since u_0 is normalized)

$$\|\omega\|_2 \leq C\left(\frac{\sqrt{\delta_\varepsilon + c\varepsilon}}{\eta} + \eta\right)\|f\|_2\ , \quad \|\omega\|_4 \leq C'\left(\frac{\sqrt{\delta_\varepsilon + c\varepsilon}}{\eta}\|f\|_2 + \eta\|f\|_4\right)\ . \tag{4.37}$$

The Cauchy-Schwarz inequality gives

$$\|f\|_2^2 = \int_0^{\varepsilon^{-1}|\partial\Omega|} |f(s)|^2 \, ds \leq \|f\|_4^2 \sqrt{\varepsilon^{-1}|\partial\Omega|}.$$

Thus we get from (4.37) (for $\eta \ll 1$)

$$\|\omega\|_4 \leq C\left(\eta^{-1}(\frac{(\delta_\varepsilon + c\varepsilon)^2}{\varepsilon})^{1/4} + \eta\right)\|f\|_4. \tag{4.38}$$

Thus, for $\delta^2 \ll \varepsilon$, we can choose $\eta = (\frac{(\delta_\varepsilon + c\varepsilon)^2}{\varepsilon})^{1/8} \ll 1$ and get

$$\|\omega\|_4 \leq C(\frac{(\delta_\varepsilon + c\varepsilon)^2}{\varepsilon})^{1/8}\|f\|_4 = o(\|f\|_4). \tag{4.39}$$

Remark 4.2. The assumption (1.16) is equivalent to $\delta^2 \ll \varepsilon$. It is clear that, in our approach, (1.16) is needed in order for (4.38) to imply (4.39). This is the only place in the analysis where we need (1.16) and not the weaker (1.9).

We now return to the problem on the entire domain Ω. Using Proposition 2.5, we get

$$\mathcal{E}[\psi,\vec{A}] \geq \frac{1}{\varepsilon^2}\int_\Omega (W_\varepsilon(x) - \varepsilon^2\kappa^2)|\psi(x)|^2\,dx + \frac{\kappa^2}{2}\int_\Omega |\psi(x)|^4\,dx.$$

Applying the Agmon estimates (Theorem 2.3) we find

$$\int_\Omega (W_\varepsilon(x) - \varepsilon^2\kappa^2)|\psi(x)|^2\,dx \geq \int_\Omega (W_\varepsilon(x) - \varepsilon^2\kappa^2 - Ce^{-\alpha\kappa\varepsilon M})|\psi_1(x)|^2\,dx,$$

where M is the length-scale in the partition of unity χ. M has been chosen to be $M = C|\log\varepsilon|$, for some sufficiently large constant C. Thus we may assume that

$$e^{-\alpha\kappa\varepsilon M} = \varepsilon^{\alpha' C} \ll \varepsilon. \tag{4.40}$$

So we find, with $r_1, r_2 = o(1)$,

$$\begin{aligned}
\mathcal{E}[\psi,\vec{A}] &\geq \frac{1}{\varepsilon^2}\int_\Omega (W_\varepsilon(x) - \varepsilon^2\kappa^2 + \varepsilon r_1)|\psi_1(x)|^2\,dx + \frac{\kappa^2}{2}\int_\Omega |\psi_1(x)|^4\,dx \\
&= \int_0^{\varepsilon^{-1}|\partial\Omega|}\int_0^\infty \Big\{(W_\varepsilon(x) - \varepsilon^2\kappa^2 + \varepsilon r_1)|\tilde{\phi}(\sigma,\tau)|^2 + \frac{\kappa}{2H}|\tilde{\phi}(\sigma,\tau)|^4\Big\} \\
&\qquad \times (1-\tau k(\varepsilon\sigma))\,d\sigma d\tau \\
&= \int_0^{\varepsilon^{-1}|\partial\Omega|}\int_0^\infty \Big\{-(\delta_\varepsilon + C_1 k(\varepsilon\sigma)\varepsilon + \varepsilon r_2)|\tilde{\phi}(\sigma,\tau)|^2 + \frac{\beta_0+\delta_\varepsilon}{2}|\tilde{\phi}(\sigma,\tau)|^4\Big\} \\
&\qquad \times (1-\tau k(\varepsilon\sigma))\,d\sigma d\tau
\end{aligned} \tag{4.41}$$

At this point, we can use (4.39) and (4.37) to do the τ-integration and find

$$\begin{aligned}
\mathcal{E}[\psi,\vec{A}] \geq \int_0^{\varepsilon^{-1}|\partial\Omega|} \Big[&-(\delta_\varepsilon + C_1 k(\varepsilon\sigma)\varepsilon + \varepsilon r_3)|f(\sigma)|^2 \\
&+ (\frac{\beta_0}{2} + r_4)|f(\sigma)|^4\|u_0\|_4^4\Big]d\sigma,
\end{aligned} \tag{4.42}$$

where $r_3, r_4 = o(1)$.
By definition of δ_ε we have

$$\delta_\varepsilon + C_1 k(\varepsilon\sigma)\varepsilon + \varepsilon r_3 = \beta_0^{3/2}\rho\varepsilon - C_1(k_{\max} - k(\varepsilon\sigma))\varepsilon + \varepsilon r_3',$$

with $r_3' = o(1)$.
Let $U_{1,\kappa}$ be the set

$$U_{1,\kappa} = \{\sigma : \beta_0^{3/2}\rho(\kappa) - C_1(k_{\max} - k(\varepsilon\sigma)) + r_3' \geq 0\}\,,$$

and

$$U_{2,\kappa} = \complement U_{1,\kappa} = \{\sigma : \beta_0^{3/2}\rho(\kappa) - C_1(k_{\max} - k(\varepsilon\sigma)) + r_3' < 0\}\,.$$

Using a corresponding decomposition for the integration, we get :

$$\mathcal{E}[\psi,\vec{A}] \geq \mathcal{E}_1[\psi,\vec{A}] + \mathcal{E}_2[\psi,\vec{A}]\,,$$

with

$$\mathcal{E}_1[\psi,\vec{A}] := -\varepsilon\int_{U_{1,\kappa}} \left|\beta_0^{3/2}\rho(\kappa) - C_1(k_{\max}-k(\varepsilon\sigma)) + r_3'\right| |f(\sigma)|^2\, d\sigma$$
$$+\int_{U_{1,\kappa}} (\frac{\beta_0}{2}+r_4)|f(\sigma)|^4\|u_0\|_4^4\, d\sigma\ ,$$
$$\mathcal{E}_2[\psi,\vec{A}] := \varepsilon\int_{U_{2,\kappa}} \left|\beta_0^{3/2}\rho(\kappa) - C_1(k_{\max}-k(\varepsilon\sigma)) + r_3'\right| |f(\sigma)|^2 d\sigma$$
$$+\int_{U_{2,\kappa}} (\frac{\beta_0}{2}+r_4)|f(\sigma)|^4\|u_0\|_4^4 d\sigma\ .$$

Then, using the positivity of $\mathcal{E}_2$ and, by completion of the square in $\mathcal{E}_1$, we get

$$\begin{aligned}
&\mathcal{E}[\psi,\vec{A}]\\
\geq&(\frac{\beta_0}{2}+r_4)\|u_0\|_4^4\int_{U_{1,\kappa}}\left[|f(\sigma)|^2-\varepsilon\frac{(\beta_0^{3/2}\rho - C_1(k_{\max}-k(\varepsilon\sigma))+r_3'))}{(\beta_0+2r_4)\|u_0\|_4^4}\right]^2 d\sigma\\
&-\frac{1}{2}\left((\beta_0+2r_4)\|u_0\|_4^4\right)^{-1}\varepsilon^2\int_{U_{1,\kappa}}\left[(\beta_0^{3/2}\rho - C_1(k_{\max}-k(\varepsilon\sigma))+r_3')\right]^2 d\sigma\\
\geq&-\frac{1}{2}\left((\beta_0+2r_4)\|u_0\|_4^4\right)^{-1}\varepsilon^2\int_{U_{1,\kappa}}\left[(\beta_0^{3/2}\rho - C_1(k_{\max}-k(\varepsilon\sigma))+r_3')\right]^2 d\sigma\ .
\end{aligned}$$

Upon changing coordinates $s=\varepsilon\sigma$, we get

$$\mathcal{E}[\psi,\vec{A}] \geq -\frac{1}{2\beta_0\|u_0\|_4^4}\varepsilon\int_0^{|\partial\Omega|}[\beta_0^{3/2}\rho - C_1(k_{\max}-k(s))]_+^2\, ds + o(\epsilon). \tag{4.43}$$

By inserting the definition of ε, this lower bound agrees with the energy asymptotics (1.17). When combined with (4.24), we therefore get (1.17) from (4.43). This finishes the proof of Theorem 1.2.

4.3. **On the asymptotic behavior of the minimizer.**

Coming back to all the "forgotten" terms in the previous proof, we will get the weak localization result from Theorem 1.4. First we observe (reimplementing the upper bound) that

$$\int_{U_{2,\kappa}}\left(\left\{-\beta_0^{\frac{3}{2}}\rho(\kappa)+C_1(k_{\max}-k(\epsilon\sigma))\right\}|f(\sigma)|^2+\frac{1}{\varepsilon}|f(\sigma|^4\right)d\sigma = o(\rho^2)$$

and that

$$\frac{1}{\varepsilon}\int_{U_{1,\kappa}}\left(|f(\sigma)|^2-\frac{\varepsilon\left\{\beta_0^{\frac{3}{2}}\rho(\kappa)-C_1(k_{\max}-k(\epsilon\sigma))\right\}}{\beta_0\|u_0\|_4^4}\right)^2 d\sigma = o(\rho^2)\ .$$

With a little extra work, this leads to :

$$\int_{\partial\Omega}\left(\frac{|f(\epsilon^{-1}s)|^2}{\varepsilon\rho(\kappa)^2}-\frac{\left[\beta_0^{\frac{3}{2}}-\frac{C_1(k_{\max}-k(s))}{\rho(\kappa)}\right]_+}{\beta_0||u_0||_4^4}\right)^2 ds\to 0\ . \tag{4.44}$$

This shows that the concentration at the boundary is not uniform when ρ is bounded, even when

$$\rho(\kappa)\geq C_1\beta_0^{-\frac{3}{2}}(k_{\max}-k_{\min})\ .$$

We can now prove Theorem 1.4.

Proof. Let $M = M(\varepsilon)$ be the parameter of the partition of unity from (4.25). Remember from (4.33) that $M = C_M|\log\varepsilon|$, where C_M is a sufficiently large constant independent of ε. We will use the freedom to choose C_M large.

Consider

$$\varepsilon^{-1}\int_{\{t(x)\geq\varepsilon M\}}\left|\frac{|\psi(x)|^2}{\varepsilon\rho}-\frac{[\beta_0^{3/2}-C_1\frac{k_{\max}-k(s)}{\rho}]_+}{\beta_0\|u_0\|_4^4}|u_0(\frac{t(x)}{\varepsilon})|^2\right|^2\,dx \tag{4.45}$$

We will show that

$$\varepsilon^{-1}(\varepsilon\rho)^{-2}\int_{\{t(x)\geq\varepsilon M\}}|\psi(x)|^4\,dx+\varepsilon^{-1}\int_{\{t(x)\geq\varepsilon M\}}|u_0(\frac{t(x)}{\varepsilon})|^4\,dx\to 0, \tag{4.46}$$

which clearly implies

$$\varepsilon^{-1}\int_{\{t(x)\geq\varepsilon M\}}\left|\frac{|\psi(x)|^2}{\varepsilon\rho}-\frac{[\beta_0^{3/2}-C_1\frac{k_{\max}-k(s)}{\rho}]_+}{\beta_0\|u_0\|_4^4}|u_0(\frac{t(x)}{\varepsilon})|^2\right|^2\,dx\to 0. \tag{4.47}$$

The term with u_0 in (4.46) clearly tends to zero. This is a consequence of the gaussian decay of u_0. For the first term in (4.46), the estimate will follow from the Agmon estimates and the inequality $\|\psi\|_\infty\leq 1$ from Lemma 2.1: First using Lemma 2.1 and the domain of integration

$$\begin{aligned}\varepsilon^{-1}(\varepsilon\rho)^{-2}\int_{\{t(x)\geq\varepsilon M\}}|\psi(x)|^4\,dx &\leq \varepsilon^{-1}(\varepsilon\rho)^{-2}\int_{\{t(x)\geq\varepsilon M\}}|\psi(x)|^2\,dx\\ &\leq \varepsilon^{-1}(\varepsilon\rho)^{-2}e^{-\alpha\varepsilon\kappa M}\int_\Omega e^{\alpha\kappa t(x)}|\psi(x)|^2\,dx.\end{aligned}$$

Here we choose α to be as in Theorem 2.3. We can then continue the estimate as follows

$$\leq C\varepsilon^{-1}(\varepsilon\rho)^{-2}e^{-\alpha\varepsilon\kappa M}\int_\Omega|\psi(x)|^2\,dx\leq C|\Omega|\varepsilon^{-1}(\varepsilon\rho)^{-2}e^{-\alpha\varepsilon\kappa M},$$

where we again used Lemma 2.1. Using that $\varepsilon\kappa\to\sqrt{\beta_0}$ we therefore get

$$\varepsilon^{-1}(\varepsilon\rho)^{-2}\int_{\{t(x)\geq\varepsilon M\}}|\psi(x)|^4\,dx\leq C'\varepsilon^{-1}(\varepsilon\rho)^{-2}e^{-\sqrt{\beta_0}\alpha M}\leq C''\varepsilon^{\sqrt{\beta_0}\alpha C_M-3},$$

where we used that (1.8) to get the last inequality. By choosing C_M sufficiently big ($>\frac{3}{\sqrt{\beta_0}\alpha}$) we get the estimate (4.46).

Thus we only have to prove that

$$\varepsilon^{-1}\int_{\{t(x)\leq\varepsilon M\}}\left|\frac{|\psi_1(x)|^2}{\varepsilon\rho}-\frac{[\beta_0^{3/2}-C_1\frac{k_{\max}-k(s)}{\rho}]_+}{\beta_0\|u_0\|_4^4}|u_0(\frac{t(x)}{\varepsilon})|^2\right|^2\,dx\to 0. \tag{4.48}$$

In boundary coordinates this is equivalent to

$$\begin{aligned}\varepsilon\int_0^{\varepsilon^{-1}|\partial\Omega|}d\sigma\int_0^M\Big|\frac{|\tilde\phi(\sigma,\tau)|^2}{\varepsilon\rho}-&\frac{\left[\beta_0^{3/2}-\frac{C_1(k_{\max}-k(\varepsilon\sigma))}{\rho}\right]_+}{\beta_0\|u_0\|_4^4}|u_0(\tau)|^2\Big|^2\\ &\times(1-\varepsilon\tau k(\varepsilon\sigma)\,d\tau\to 0\,.\end{aligned}\tag{4.49}$$

Applying (4.36) and (4.39) and doing the τ-integration explicitly, reduces (4.49) to (4.44). This finishes the proof of Theorem 1.4. □

5. Energy asymptotics in case $\rho \to 0$

In this section we consider the case

$$\rho \to 0. \tag{5.1}$$

For simplicity we will impose the following Assumption 5.1 on Ω. This assumption is 'generically' satisfied.

Assumption 5.1. The domain Ω is bounded and with smooth boundary. Furthermore the boundary $\partial\Omega$ has only only a finite number of points of maximal curvature and these maxima are non-degenerate. More precisely (using boundary coordinates) there exist $N \in \mathbb{N}$ and $\{s_1, \ldots, s_N\} \in (0, |\partial\Omega|)$ such that

$$k(s_j) = k_{\max} \qquad \text{and} \qquad k''(s_j) < 0, \qquad \text{for all } j \in \{1, \ldots, N\},$$
$$k(s) < k_{\max} \qquad \text{for all } s \notin \{s_1, \ldots, s_N\}.$$

The critical field H_{C_3} has till now only been calculated with limited precision, the best result at present being the asymptotics (1.6) obtained by Helffer and Pan. We expect that the correct asymptotics under Assumption 5.1 is (see Bernoff and Sternberg [BS98])

$$H_{C_3} = \frac{\kappa}{\beta_0} + \frac{C_1}{\beta_0^{3/2}} k_{\max} + \mathcal{O}(\kappa^{-1/2}), \tag{5.2}$$

but that is still work in progress. In order for the result of the present section to be independent of possible improvements on the asymptotics of H_{C_3}, we assume that for the given domain Ω we have

$$H_{C_3} = \frac{\kappa}{\beta_0} + \frac{C_1}{\beta_0^{3/2}} k_{\max} + \mathcal{R}. \tag{5.3}$$

(Of course, we know then, from [HP03], that $\mathcal{R} = \mathcal{O}(\kappa^{-1/3})$.) We then impose the following natural condition on the gap ρ:

$$\rho^{-1} \max(|\mathcal{R}|, \varepsilon) \to 0, \qquad \text{as } \kappa \to \infty. \tag{5.4}$$

We then prove that the energy asymptotics remains formally the same as in the case of large ρ:

Theorem 5.2. *Suppose that Ω satisfies Assumption 5.1. Suppose furthermore that ρ satisfies* (5.1) *as well as* (5.4)*, where $\mathcal{R}$ is defined by* (5.3)*. Let $(\psi, \vec{A})$ be a (sequence of) minimizers of* (1.1)*. Then*

$$\mathcal{E}[\psi, \vec{A}] = -(1 - o(1)) \frac{1}{2\beta_0^{1/2} \|u_0\|_4^4 \kappa} \int_0^{|\partial\Omega|} \left[\beta_0^{3/2}\rho - C_1(k_{\max} - k(s))\right]_+^2 ds, \tag{5.5}$$

as $\kappa \to +\infty$.

Remark 5.3. Since $\rho \to 0$, the integral in (5.5) is not a very explicit asymptotics. In order to better understand the energy asymptotics, let us discuss that integral in detail. Define

$$A = \int_0^{|\partial\Omega|} \left[\beta_0^{3/2}\rho - C_1(k_{\max} - k(s))\right]_+^2 ds \tag{5.6}$$

Let $\{s_1, \ldots, s_N\}$ be the maxima from Assumption 5.1. For each $j = 1, \ldots, N$, we define

$$M_j = -k''(s_j) = |k''(s_j)|.$$

Around s_j we have

$$k(s) = k_{\max} - \frac{M_j}{2}(s-s_j)^2 + \mathcal{O}(|s-s_j|^3).$$

For sufficiently small ρ, the integrand in (5.6) vanishes except in small interval around each point s_j, and (by taking ρ small enough) we can assume these intervals to be disjoint. By elementary calculations we get that the contribution to A from the interval around s_j is

$$\begin{aligned}(1+o(1)) &\int_0^{|\partial\Omega|} \left[\beta_0^{3/2}\rho - C_1 \tfrac{M_j}{2}(s-s_0)^2\right]_+^2 ds \\ &= \frac{16\sqrt{2}}{15}\frac{1}{\sqrt{C_1}}\beta_0^{15/4}\rho^{5/2}\frac{1}{\sqrt{M_j}}.\end{aligned} \tag{5.7}$$

So (5.5) can equivalently be written

$$\mathcal{E}[\psi,\vec{A}] = -(1+o(1))\frac{8\sqrt{2}}{15}\frac{\beta_0^{13/4}}{\sqrt{C_1}\|u_0\|_4^4}\frac{\rho^{5/2}}{\kappa}\sum_{j=1}^{N}\frac{1}{\sqrt{M_j}} \tag{5.8}$$

In particular, this term is of order $\frac{\rho^{5/2}}{\kappa}$.

Proof of Theorem 5.2.
Upper bound.
In the upper bound we use essentially the same calculations as in the case ρ bounded, so we will only indicate the differences. For our test function we use the pair $(\psi,\vec{F})$, with (as usual) $\vec{F}$ being the exterior field. The choice of ψ is as follows:

$$\psi(s,t) = e^{i\kappa H\chi_f + i[\xi_0]_\varepsilon s/\varepsilon}\lambda(s)u_0(t/\varepsilon)\chi(t). \tag{5.9}$$

with

$$\lambda(s) = \frac{1}{\sqrt{\varepsilon}\kappa\|u_0\|_4^2}\left[\beta_0^{3/2}\rho - C_1(k_{\max}-k(s))\right]_{+,\nu}^{1/2}. \tag{5.10}$$

Notice that the only difference to Section 4 is in the choice of regularization of $[\,\cdot\,]_+$. We will choose ν satisfying

$$1 \gg \nu \gg \varepsilon\rho^{-1}. \tag{5.11}$$

Notice that this is possible only if

$$\rho \gg \varepsilon, \tag{5.12}$$

which explains why we impose the second part of the condition (5.4). With the choice of ν from (5.11) we get (using (4.4)) that

$$|\lambda'(s)|^2 \leq \begin{cases} 0, & \beta_0^{3/2}\rho \leq C_1(k_{\max}-k(s)), \\ C\frac{1}{\varepsilon\kappa^2}\nu^{-1}|k'(s)|^2, & \beta_0^{3/2}\rho > C_1(k_{\max}-k(s)). \end{cases}$$

This gives, for the term in (4.12), using Assumption 5.1,

$$\begin{aligned}
r_1 &= \int (1-tk(s))^{-1}|\lambda'(s)|^2|u_0(t/\varepsilon)\chi(t))|^2\,dsdt \\
&\leq C\varepsilon^2\nu^{-1}\int_{\{\beta_0^{3/2}\rho>C_1(k_{\max}-k(s))\}}|k'(s)|^2\,ds \\
&= \mathcal{O}(\varepsilon^2\nu^{-1}\rho^{3/2}). \qquad (5.13)
\end{aligned}$$

Therefore we get, as in (4.13),

$$\begin{aligned}
\mathcal{E}[\psi,\vec{F}] = \int_0^{|\partial\Omega|}&|\lambda(s)|^2T(s)ds \\
&+\frac{\kappa^2\varepsilon}{2}\int_0^{|\partial\Omega|}|\lambda(s)|^4\|u_0\|_4^4(1+\mathcal{O}(\varepsilon))\,ds+\mathcal{O}(\varepsilon^2\nu^{-1}\rho^{3/2})\,. \qquad (5.14)
\end{aligned}$$

The proof proceeds with the calculation of $T(s)$. The terms T_1, T_2, T_3, T_4 are given by (4.17), (4.18), (4.19), and (4.21). Doing the asymptotics in (4.16) with a bit more care, using Assumption 5.1, it becomes

$$T_0(s) = \beta_0^{3/2}\rho\big(1+\mathcal{O}(\mathcal{R}/\rho)\big) - C_1k_{\max}+\mathcal{O}(\varepsilon)\,. \qquad (5.15)$$

Therefore, the energy estimate (4.23) is

$$\begin{aligned}
\mathcal{E}[\psi,\vec{F}] = \int_0^{|\partial\Omega|}&\Big\{|\lambda(s)|^2[C_1(k_{\max}-k(s))-\beta_0^{3/2}\rho+o(\rho)+\mathcal{O}(\varepsilon)] \\
&+\frac{\kappa^2\varepsilon}{2}|\lambda(s)|^4\|u_0\|_4^4(1+\mathcal{O}(\varepsilon))\Big\}\,ds+\mathcal{O}(\varepsilon^2\nu^{-1}\rho^{3/2})\,. \qquad (5.16)
\end{aligned}$$

Using (5.11) and $\rho \gg \varepsilon$, (5.16) is the upper bound in (5.5).

Lower bound.

The proof of the lower bound proceeds exactly as in Section 4 except that we make sure in (4.40) to choose M such that

$$e^{-\alpha\kappa\varepsilon M} \ll \varepsilon^2.$$

This implies that the errors r_1, r_2 in (4.41) can be estimated by $\mathcal{O}(\varepsilon)$ and therefore (4.42) becomes

$$\begin{aligned}
\mathcal{E}[\psi,\vec{A}] \geq \int_0^{\varepsilon^{-1}|\partial\Omega|}&\Big[-(\delta_\varepsilon+C_1k(\varepsilon\sigma)\varepsilon+\varepsilon r_3)|f(\sigma)|^2 \\
&+(\frac{\beta_0}{2}+r_4)|f(\sigma)|^4\|u_0\|_4^4\Big]d\sigma, \qquad (5.17)
\end{aligned}$$

where $r_3 = \mathcal{O}(\varepsilon)$, $r_4 = o(1)$. By the same type of "completion of the square"-argument as before, we therefore get

$$\mathcal{E}[\psi,\vec{A}] \geq -\frac{1}{2\beta_0(1+o(1))\|u_0\|_4^4}\varepsilon\int_0^{|\partial\Omega|}[\beta_0^{3/2}\rho-C_1(k_{\max}-k(s))+r_3']_+^2\,ds, \qquad (5.18)$$

where r_3' is a term which is estimated by $o(\rho)+\mathcal{O}(\varepsilon)$. Since, by assumption, we therefore have $\rho \gg r_3'$, the lower bound (5.18) combined with the upper bound (5.16) implies (5.5). $\square$

Appendix A. Boundary coordinates

Let $\gamma(s)$, $s \in \mathbb{R}/|\partial\Omega|$ be a parametrization of the boundary $\partial\Omega$, with $|\gamma'(s)| = 1$. Let $\nu(s)$ be the inward unit normal vector at the point $\gamma(s)$. We may assume the orientation to be chosen so that

$$\det(\gamma'(s), \nu(s)) = 1 \,.$$

With this orientation, the *curvature* $k(s)$ *of the boundary at the point* $\gamma(s)$ is given by

$$\gamma''(s) = k(s)\nu(s).$$

Define $\Phi : \mathbb{R}/|\partial\Omega| \times [0, t_0) \to \Omega$, by

$$\Phi(s,t) = \gamma(s) + t\nu(s).$$

It is a well-known fact from differential geometry that if t_0 is sufficiently small, then Φ is a diffeomorphism to the neighborhood Ω_{t_0} of the boundary given by

$$\Omega_{t_0} = \{z \in \Omega \big| \operatorname{dist}(z, \partial\Omega) < t_0\}.$$

Let $\vec{A}$ be a vector potential in Ω_{t_0} and let $B = \nabla \times \vec{A}$ be the associated magnetic field. Define

$$\tilde{A}_1(s,t) = (1 - tk(s))\vec{A}(\Phi(s,t)) \cdot \gamma'(s), \qquad \tilde{A}_2(s,t) = \vec{A}(\Phi(s,t)) \cdot \nu(s),$$
$$\tilde{B}(s,t) = B(\Phi(s,t)).$$

With these definitions we get

$$\partial_s \tilde{A}_2 - \partial_t \tilde{A}_1 = (1 - tk(s))\tilde{B}. \tag{A.1}$$

Furthermore, for $u \in W^{1,2}(\Omega)$ with $\operatorname{supp} u \subset \Omega_{t_0}$, we find

$$\int_\Omega |(-i\nabla - \vec{A})u|^2 \, dx =$$
$$\int \Big\{(1 - tk(s))^{-2}|(-i\partial_s - \tilde{A}_1)v|^2 + |(-i\partial_t - \tilde{A}_2)v|^2\Big\}(1 - tk(s))dsdt.$$

Here $v(s,t) = u(\Phi(s,t))$,

$$\int |u|^2 dxdy = \int |v|^2 (1 - tk(s))dsdt.$$

Appendix B. Moments

We now describe some formulas appearing in [BS98] and already used in [BH93, DH93, HM01]. Let M_k denote the centered moment, of order k of the probability measure $u_0^2(x)\,dx$:

$$M_k = \int_0^{+\infty} (\tau + \xi_0)^k u_0^2(\tau) d\tau \,. \tag{B.1}$$

The values of the first few of these moments are used in the calculations in Subsection 4.2.

Lemma B.1. .
The moments can be expressed by the following formulas :

$$M_0 = 1 \,, \qquad M_1 = 0 \,, \qquad M_2 = \frac{\beta_0}{2} \,, \qquad M_3 = \frac{u_0^2(0)}{6} > 0. \tag{B.2}$$

More generally, if $k > 3$*, we have*

$$4kM_k = (k-1)\{4\xi_0^2 M_{k-2} + (k-2)[\xi_0^{k-3}u_0^2(0) + (k-3)M_{k-4}]\}. \tag{B.3}$$

Furthermore, we have the following identity

$$\xi_0 = -\sqrt{\beta_0}. \tag{B.4}$$

References

[Ada75] Robert A. Adams, *Sobolev spaces*, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975, Pure and Applied Mathematics, Vol. 65. MR 56 #9247

[AHS78] J. Avron, I. Herbst, and B. Simon, *Schrödinger operators with magnetic fields. I. General interactions*, Duke Math. J. **45** (1978), no. 4, 847–883. MR **80k:**35054

[BH93] C. Bolley and B. Helffer, *An application of semi-classical analysis to the asymptotic study of the supercooling field of a superconducting material*, Ann. Inst. H. Poincaré Phys. Théor. **58** (1993), no. 2, 189–233. MR **94k:**82120

[Bon03] Virginie Bonnaillie, *On the fundamental state for a Schrödinger operator with magnetic field in a domain with corners*, C. R. Math. Acad. Sci. Paris **336** (2003), no. 2, 135–140. MR 1 969 567

[Bon04] ———, *On the fundamental state energy for a Schrödinger operator with magnetic field in a domain with corners*, To appear in Asymptotic Analysis (2004).

[BPT98] P. Bauman, D. Phillips, and Q. Tang, *Stable nucleation for the Ginzburg-Landau system with an applied magnetic field*, Arch. Rational Mech. Anal. **142** (1998), no. 1, 1–43. MR **99g:**58040

[BS98] Andrew Bernoff and Peter Sternberg, *Onset of superconductivity in decreasing fields for general domains*, J. Math. Phys. **39** (1998), no. 3, 1272–1284. MR **99a:**82099

[DH93] Monique Dauge and Bernard Helffer, *Eigenvalues variation. I. Neumann problem for Sturm-Liouville operators*, J. Differential Equations **104** (1993), no. 2, 243–262. MR **94j:**47097

[dPFS00] Manuel del Pino, Patricio L. Felmer, and Peter Sternberg, *Boundary concentration for eigenvalue problems related to the onset of superconductivity*, Comm. Math. Phys. **210** (2000), no. 2, 413–446. MR **2001k:**35231

[GP99] Tiziana Giorgi and Daniel Phillips, *The breakdown of superconductivity due to strong fields for the Ginzburg-Landau model*, SIAM J. Math. Anal. **30** (1999), no. 2, 341–359 (electronic). MR **2000b:**35235

[GT01] David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin, 2001, Reprint of the 1998 edition. MR **2001k:**35004

[HM01] Bernard Helffer and Abderemane Morame, *Magnetic bottles in connection with superconductivity*, J. Funct. Anal. **185** (2001), no. 2, 604–680. MR **2002m:**81051

[HP03] Bernard Helffer and Xing-Bin Pan, *Upper critical field and location of surface nucleation for superconductivity*, Ann. I.H. Poincaré **20** (2003), no. 1, 145–181.

[LL97] Elliott H. Lieb and Michael Loss, *Analysis*, American Mathematical Society, Providence, RI, 1997. MR **98b:**00004

[LP99] Kening Lu and Xing-Bin Pan, *Estimates of the upper critical field for the Ginzburg-Landau equations of superconductivity*, Phys. D **127** (1999), no. 1-2, 73–104. MR **2000a:**82075

[Mar39] J. Marcinkiewicz, *Sur les multiplicateurs des séries de Fourier.*, Stud. Math. **8** (1939), 78–91 (French).

[Pan02a] Xing-Bin Pan, *Surface superconductivity in applied magnetic fields above* H_{C_2}, Comm. Math. Phys. **228** (2002), no. 2, 327–370. MR **2003i:**82094

[Pan02b] ———, *Upper critical field for superconductors with edges and corners*, Calc. Var. Partial Differential Equations **14** (2002), no. 4, 447–482. MR **2003f:**82105

[SS00] Etienne Sandier and Sylvia Serfaty, *On the energy of type-II superconductors in the mixed phase*, Rev. Math. Phys. **12** (2000), no. 9, 1219–1257. MR **2002f:**58023

[SS03] ———, *The decrease of bulk-superconductivity close to the second critical field in the Ginzburg-Landau model*, SIAM J. Math. Anal. **34** (2003), no. 4, 939–956 (electronic). MR 1 969 609

(S. Fournais) CNRS and Laboratoire de Mathématiques, UMR CNRS 8628, Université Paris-Sud - Bât 425, F-91405 Orsay Cedex, France.
E-mail address: soeren.fournais@math.u-psud.fr

(B. Helffer) Laboratoire de Mathématiques, UMR CNRS 8628, Université Paris-Sud - Bât 425, F-91405 Orsay Cedex, France.
E-mail address: bernard.helffer@math.u-psud.fr