A NONLINEAR OPTIMIZATION PROBLEM IN HEAT CONDUCTION

EDUARDO V. TEIXEIRA

Abstract. In this paper we study the existence and geometric properties of an optimal configuration to a nonlinear optimization problem in heat conduction. The quantity to be minimized is $\int_{\partial D} \Gamma(x, u_\mu) d\sigma$, where D is a fixed domain. A nonconstant temperature distribution is prescribed on ∂D and a volume constraint on the set where the temperature is positive is imposed. Among other regularity properties of an optimal configuration, we prove analyticity of the free boundary.

1. Introduction

In this paper we study a classical optimization problem in heat conduction, which may briefly be described as follows: given a surface ∂D in $\mathbb{R}^n$, and a positive function φ defined on it (the temperature distribution), we want to surround ∂D with a prescribed volume of insulating material so as to minimize the loss of heat in a stationary situation.

Mathematically speaking, we want to find a function u, which corresponds to the temperature in D^C. The function u is harmonic whenever it is positive and the volume of the support of u is equal to 1. The quantity to be minimized, the flow of heat, is a continuous family of convex function of u_μ along ∂D.

Our paper was motivated by a series of remarkable papers [1], [2] and [3]. The first two articles study the constant temperature distribution, i.e., $\varphi \equiv C$ on ∂D. All of them treated the linear case, i.e, $\Gamma(x, t) = t$. The linear setting allows, in [1] and [2], to reduce the quantity to be minimized to the Dirichlet integral. Even in the linear case the nonconstant temperature distribution, problem studied in [3], presents several new difficulties. The ultimate goal of this article is to study the nonlinear case with nonconstant temperature distribution. The nonlinearity treated in this article has physical importance: problems with a monotone operator like the type we study in this paper arise in questions of domain optimization for electrostatic configurations.

The nonlinearity over u_μ presents several new difficulties as well. For instance, even to provide a reasonable mathematical model, one faces the problem that it does not make sense to compute normal derivatives of H^1-functions. In [3], this problem could be overcame by reducing the quantity to be minimized to the total mass of Δu. The later quantity can be thought as a nonnegative measure, whenever u is subharmonic. In the case studied here, there is no integral representation for $\int_{\partial D} \Gamma(x, u_\mu) d\sigma$. To grapple with this difficulty one has to be careful in balancing the correct regularity of the constraint set; otherwise, classical functional analysis methods might not work anymore. Typical arguments used in [2] such as, changing the minimizer in a small ball by a harmonic function with boundary data equal to u, is not conclusive anymore. Indeed near ∂D, u and the new function agree; therefore, they have the same normal derivative. To overcome this difficulty, we solve suitable auxiliary obstacle problems and compare them with the minimizer. Moreover we also inherit all the difficulties intrinsic to the nonconstant temperature distribution. These difficulties appear in the results concerning fine regularity results of the free boundary. As noticed in [3], this is due to the fact that the free boundary condition has a nonlocal character. Inspired by the approach used in [3], we overcome such problems by making use of the powerful results on the behavior of harmonic functions in non-tangentially accessible domains provided in [7].

Our paper is organized as follows: in Section 2 we present the physical problem we are concerned with. Afterwards, we formulate a penalized version of the variational problem for the temperature u. As part of our strategy we define suitable constraint sets. These will be fundamental to overcome some difficulties caused by the nonlinearity. For instance, we shall initially solve the optimization problem over a weakly closed subset of H^1 (the sets V_δ). Unlike in [3], we shall need to establish all the optimal regularity properties of the minimizers of these auxiliary problems, i.e. Lipschitz regularity, to be able to prove the existence of a optimal configuration of the original penalized problem. This is the content of Section 3. Some basic geometric-measure properties of the optimal configuration such as: linear growth from the free boundary and uniformly positive density, are contained in Section 4. These geometric-measure properties allows us to establish a representation theorem in the sense of [2]. Such a representation theorem turns out to be the right starting point to the journey of proving fine regularity results to the free boundary. Section 5 is reserved for the optimal regularity of the free boundary. We initially show the normal derivative of the minimizer over the free boundary is a Hölder continuous function. This allows us to conclude the free boundary is a $C^{1,\alpha}$ surface. Furthermore, using the free boundary condition found in the proof of Hölder continuity of the normal derivative, we shall conclude that the free boundary is an analytic surface, up to a small singular set. In the last section we recover the original physical problem from the penalized problem. The strategy here is to show that for ε small enough, the volume of $\{u_\varepsilon > 0\}$ automatically adjusts to be 1.

2. Statement of the physical problem

In this section we shall state the physical problem we are interested in. Afterwards, we will present a penalized version of the original problem, which turns out to be more suitable from the mathematical point of view. In the last section we shall recover the initial problem from its penalized version. The (real) problem we are concerned with is:

Let $D \subset \mathbb{R}^n$ be a given smooth bounded domain and $\varphi \colon \partial D \to \mathbb{R}_+$ a positive continuous function. For each domain Ω surrounding D such that

$$\text{Vol.}(\Omega \setminus D) = 1,$$

we solve the problem

$$\begin{cases} \Delta u &= 0 \text{ in } \Omega \setminus D \\ u &= \varphi \text{ on } \partial D \\ u &= 0 \text{ on } \partial \Omega \end{cases}$$

and compute

$$J(\Omega) := \int_{\partial D} \Gamma(x, u_\mu(x)) d\sigma,$$

where μ is the inward normal vector defined on ∂D and $\Gamma \colon \partial D \times \mathbb{R} \to \mathbb{R}$ satisfies:

(1) For each $x \in \partial D$ fixed, $\Gamma(x, \cdot)$ is convex and
$$\lim_{t \to +\infty} \int_{\partial D} \Gamma(x,t) d\sigma(x) = +\infty,$$
(2) For each $x \in \partial D$ fixed $\partial_t \Gamma(x,t) > 0$ is nondecreasing in t,
(3) For each $t \in \mathbb{R}$ fixed, $\partial_t \Gamma(\cdot, t)$ is continuous,
(4) If $\Gamma(x_0, t_0) = 0$ then $\Gamma(y, t_0) = 0$ $\forall y \in \partial D$, otherwise, $\dfrac{\Gamma(y,t)}{\Gamma(x,t)} \le L$, for a universal constant $L > 0$.

Remark 2.1. Notice that if we define h_0 to be the harmonic function in D^C taking boundary values equal to φ on ∂D and $\lim_{|x|\to\infty} h_0(x) = 0$ (see Lemma 3.4), and $c_0 := \inf_{\partial D}(h_0)_\mu$, the nonlinearity Γ has only to fulfill the above conditions on $\partial D \times (c_0, +\infty)$. It follows from the Hopf Lemma that, in the constant temperature distribution, $c_0 > 0$. In such a case, the natural nonlinearity to consider is $\Gamma(t) = t^p$, for $p \geq 1$. Typical nonlinearities in a general case is of the form $\Gamma(x,t) = \psi(x)\gamma(t)$, where ψ is a positive continuous map and γ is a coercive and convex function fulfilling condition 2.

Our goal is to study the existence and geometric properties of an optimal configuration related to the functional J. In other words, our purpose is to study the problem:

$$\text{minimize} \left\{ \begin{array}{c} J(u) := \displaystyle\int_{\partial D} \Gamma(x, u_\mu(x))d\sigma : \ u\colon D^C \to \mathbb{R}, \ u = \varphi \text{ on } \partial D, \\ \Delta u = 0 \text{ in } \{u > 0\} \text{ and Vol.(supp } u) = 1 \end{array} \right\} \tag{2.1}$$

2.1. **The Penalized Problem.** Instead of working directly on problem (2.1) we shall study a penalized version of it. This grapples with the difficulty of volume constraint. Our first step toward the right mathematical statement of the penalized problem is to find a suitable (metric) space to look for minimizers.

Definition 2.2. Let $\delta > 0$ be a fixed small positive number. We shall denote by $D_\delta := \{x \in D^C : \text{dist}(x, \ \partial D) < \delta\}$. We define

$$V_\delta := \big\{u \in H^1(D^C) : u \geq 0, \ \Delta u \geq 0, \ \Delta u = 0 \text{ in } D_\delta, \text{ and } u = \varphi \text{ on } \partial D\big\}.$$

We then define

$$V := \bigcup_{\delta \searrow 0} V_\delta.$$

The penalized problem is stated as follows: Let $\varepsilon > 0$ be fixed. We consider the function

$$f_\varepsilon := \left\{ \begin{array}{ll} 1 + \dfrac{1}{\varepsilon}(t-1) & \text{if } t \geq 1 \\ 1 + \varepsilon(t-1) & \text{otherwise.} \end{array} \right.$$

We shall be interested in minimizing

$$J_\varepsilon(u) := \int_{\partial D} \Gamma(x, u_\mu(x))d\sigma + f_\varepsilon\Big(|\{u > 0\}|\Big), \tag{2.2}$$

among V.

Notice that u is harmonic near ∂D; therefore it makes complete sense to compute normal derivative of functions in V.

3. Existence of a solution to the penalized problem

In this section we shall find a minimizer for the problem (2.2). The strategy is to study, for each $\delta > 0$ fixed, the minimizing problem

$$\text{minimize } J_\varepsilon(u) \text{ over } V_\delta. \tag{3.1}$$

Afterwards we shall pass the limit as δ goes to zero. The limiting function will be a minimizer for problem (2.2). In the end of this section we shall not only guarantee the existence of a minimizer but also show the minimizer u_ε is a Lipschitz function. This is the most one should hope, since ∇u_ε jumps among $\partial\{u = 0\}$.

Lemma 3.1. *V_δ is a weakly closed set of $H^1(D^C)$.*

Proof. Let $u_n \rightharpoonup u$ in the H^1-sense. We might suppose, up to a subsequence, that

(1) $\nabla u_n \rightharpoonup \nabla u$ in $[L^2(D^C)]^n$.
(2) $u_n(x) \to u(x)$ for almost every point $x \in D^C$.

First of all, $u \geq 0$ and $u \equiv \varphi$ on ∂D in the sense of trace. Indeed, the former is due to the a.e. convergence. The latter is justified as follows: Let $T\colon H^1(D^C) \to L^2(\partial D)$ be the trace map. We have

$$u - \varphi = T(u - u_n) \rightharpoonup 0, \text{ as } n \to \infty,$$

since T is a continuous linear map. Let $\psi \in C_0^\infty(D^C, \mathbb{R}_+)$ be fixed. We compute

$$\int_{D^C} u\Delta\psi = -\int_{D^C} \nabla u \nabla\psi = -\lim_{n\to\infty} \int_{D^C} \nabla u_n \nabla\psi \geq 0.$$

This proves u is subharmonic. Furthermore a same computation as above, for $\psi \in C_0^\infty(D_\delta)$, yields $\Delta u = 0$ in D_δ. This finishes the proof. □

We recall that for each $u \in V_\delta$, Δu is a positive Radon measure supported in D_δ^C.

Lemma 3.2. *For each $u \in V_\delta$, there holds*

$$\int_{D^C} \Delta u dx = \int_{\partial D} u_\mu d\sigma.$$

Proof. Let $D_k := \{x \in D^C : \operatorname{dist}(x, \partial D) < 1/k\}$. We build $\xi_k \in C^\infty(D^C)$ such that

$$\begin{aligned} \xi_k &\equiv 1 \text{ in } D_k^C \\ \xi_k &\equiv 0 \text{ on } \partial D \end{aligned}$$

Let $u \in V_\delta$ be fixed and k be large enough such that $1/k < \delta$. We compute

$$\begin{aligned} \int_{D^C} \nabla\xi_k \nabla u &= \int_{D_k} \nabla\xi_k \nabla u = \int_{D_k} \nabla\xi_k \nabla u + \xi_k \Delta u \\ &= \int_{\partial D_k} u_\eta dA(D_k). \end{aligned}$$

Finally,

$$\lim_{k\to\infty} \int_{D^C} \nabla\xi_k \nabla u = \int_{D^C} \Delta u dx = \lim_{k\to\infty} \int_{\partial D_k} u_\eta dA(D_k) = \int_{\partial D} u_\mu d\sigma.$$

□

Lemma 3.3. *The functional J_ε is lower semicontinuous with respect to the H^1 weak convergence.*

Proof. Let $\{u_n\}_{n=1}^\infty \subset V_\delta$ be such that $u_n \rightharpoonup u$ in $H^1(D^C)$. We first deal with

$$J(v) = \int_{\partial D} \Gamma(x, u_\mu) d\sigma.$$

Consider for the moment the functional

$$\Phi(v) = \int_{\partial D} \phi(x, u_\mu) d\sigma,$$

where $\phi(x,\xi) = \max\limits_{1\leq j\leq m} (a_j(x)|\xi| + b_j(x))$, $\forall \xi \in \mathbb{R}$. We denote by $A_j := \{x \in \partial D : \phi(x, u_\mu(x)) = a_j(x)u_\mu(x) + b_j(x)\}$. Then $\partial D = \bigcup\limits_{j=1}^m A_j$, and we may assume that this union is disjoint. Moreover, due to the weak convergence assumed, we have that $\Delta u_n \rightharpoonup \Delta u$ in H^{-1}. Therefore

$$\int_{\partial D} u_\mu d\sigma = \int_{D^C} \Delta u dx \leq \liminf_{n\to\infty} \int_{D^C} \Delta u_n dx = \liminf_{n\to\infty} \int_{\partial D} (u_n)_\mu d\sigma$$

We compute

$$\begin{aligned}\Phi(u)=\int_{\partial D}\phi(u_\mu)d\sigma &= \sum_{j=1}^{m}\int_{\Gamma_j} a_j(x)u_\mu+b_j(x)d\sigma\\ &\le \liminf_{n\to\infty}\sum_{j=1}^{m}\int_{\Gamma_j} a_j(x)(u_n)_\mu+b_j(x)d\sigma\\ &\le \liminf_{n\to\infty}\sum_{j=1}^{m}\int_{\Gamma_j}\phi((u_n)_\mu)d\sigma\\ &= \liminf_{n\to\infty}\Phi(u_n).\end{aligned}$$

In the general case, since $\Gamma(x,\cdot)$ is convex for each $x\in\partial D$, $\Gamma(x,\xi)=\lim_{k\to\infty}\phi^k(x,\xi)$ where $\phi^k(x,\xi)=\max_{1\le j\le k}(a_j(x)|\xi|+b_j(x))$. Finally the weak lower semicontinuity of Φ follows by applying the monotone convergence Theorem.

The weak lower semicontinuity of $f_\varepsilon\big(|\{u>0\}|\big)$ follows easily by the general fact that, up to a subsequence, $u_n\to u$ a.e and then

$$|\{u>0\}|\le\liminf_{n\to\infty}|\{u_n>0\}|.$$

To finish, we observe that f_ε is a increasing continuous function, therefore

$$f_\varepsilon\big(|\{u>0\}|\big)\le f_\varepsilon\big(\liminf_{n\to\infty}|\{u_n>0\}|\big)=\liminf_{n\to\infty}f_\varepsilon\big(|\{u_n>0\}|\big).$$

□

Lemma 3.4. *Let h_0 be the harmonic function in D^C taking boundary values equal to φ on ∂D and $\lim_{|x|\to\infty}h_0(x)=0$ and $u\in V_\delta$ be fixed. Then*

$$\int_{D^C}|\nabla u|^2dx\le\int_{D^C}|\nabla h_0|^2dx+\max_{\partial D}\varphi\int_{\partial D}u_\mu d\sigma.$$

Proof. Easily we check that

$$\int_{D^C}\nabla u\nabla(u-h_0)=\int_{D^C}(h_0-u)\Delta u$$

and that

$$\int_{D^C}\nabla h_0\cdot\nabla(h_0-u)=0.$$

Moreover, by the maximum principle we know $0\le u\le h_0\le\max_{\partial D}$. Hence,

$$\begin{aligned}\int_{D^C}|\nabla u|^2dx &= \int_{D^C}\nabla u\nabla h_0dx+\int_{D^C}(h_0-u)\Delta udx\\ &= \int_{D^C}|\nabla h_0|^2dx+\int_{D^C}(h_0-u)\Delta udx\\ &\le \int_{D^C}|\nabla h_0|^2dx+\max_{\partial D}\varphi\int_{D^C}\Delta udx.\\ &= \int_{D^C}|\nabla h_0|^2dx+\max_{\partial D}\varphi\int_{\partial D}u_\mu d\sigma,\end{aligned}$$

by Lemma 3.2. This finishes the proof. □

Theorem 3.5. *There exists a minimizer $u_\varepsilon^\delta \in V_\delta$ for J_ε over V_δ.*

Proof. Let $\{u_n\}_{n=1}^\infty \subset V_\delta$ be a minimizing sequence. Let us denote by $\alpha := H^{n-1}(\partial D)$ and $M = \max\limits_{\partial D} \varphi$. From Lemma 3.4, there holds

$$\frac{1}{2M\alpha}\int_{D^C} |\nabla u_n|^2 dx \le \frac{1}{2M\alpha}\int_{D^C} |\nabla h_0|^2 dx + \frac{1}{2\alpha}\int_{\partial D} (u_n)_\mu d\sigma.$$

Thus, from the fact that $\Gamma(y,\cdot)$ is increasing and convex for each $y \in \partial D$, we obtain

$$\begin{aligned}
2 \quad & \Gamma\left(y, \frac{1}{2M\alpha}\int_{D^C} |\nabla u_n|^2 dx\right) \\
\le \quad & \Gamma\left(y, \frac{1}{\alpha}\int_{D^C} |\nabla h_0|^2 dx\right) + \Gamma\left(y, \int_{\partial D} (u_n)_\mu \frac{d\sigma}{\alpha}\right) \\
\le \quad & \Gamma\left(y, \frac{1}{\alpha}\int_{D^C} |\nabla h_0|^2 dx\right) + \frac{1}{\alpha}\int_{\partial D} \Gamma\Big(y, (u_n)_\mu(x)\Big) d\sigma(x).
\end{aligned}$$

The last inequality follows from Jensen's inequality. We now integrate the above inequality with respect to y and get:

$$\begin{aligned}
2 \quad & \int_{\partial D} \Gamma\left(y, \frac{1}{2M\alpha}\int_{D^C} |\nabla u_n|^2 dx\right) d\sigma(y) \\
\le \quad & \int_{\partial D} \Gamma\left(y, \frac{1}{\alpha}\int_{D^C} |\nabla h_0|^2 dx\right) d\sigma(y) + \frac{1}{\alpha}\int_{\partial D}\int_{\partial D} \Gamma\Big(y, (u_n)_\mu(x)\Big) d\sigma(y) d\sigma(x) \\
\le \quad & \int_{\partial D} \Gamma\left(y, \frac{1}{\alpha}\int_{D^C} |\nabla h_0|^2 dx\right) d\sigma(y) + L\int_{\partial D} \Gamma\Big(x, (u_n)_\mu(x)\Big) d\sigma(x)
\end{aligned} \tag{3.2}$$

The above together with the coercivity of the map $t \mapsto \int_{\partial D} \Gamma(y,t) d\sigma(y)$ implies $\|\nabla u_n\|_{L^2(D^C)}$ is bounded. Lemma 3.1 and Lemma 3.3 complete the proof. $\square$

Now we turn our attention to the minimizing problem (2.2). The idea is to pass from the minimizers of (3.1) to a minimizer of (2.2). In what follows we shall need some lemmas.

Lemma 3.6. *For each $w \in V_\delta$, $\int_{D^C} w\Delta w$ is meaningful and there holds*

$$\int_{D^C} \Big(w\Delta w + |\nabla w|^2\Big) dx = \int_{\partial D} \varphi w_\mu d\sigma.$$

Proof. For any compact set Ξ of D^C, it follows from the mean value theorem that w can be approximated by a decreasing sequence of smooth functions and therefore uniformly in Ξ. Hence $\int_\Xi w\Delta w$ has a meaning. Let ξ_k be like in Lemma 3.2. We have that

$$\int_{D^C} \xi_k w\Delta w = -\int_{D^C} \nabla(\xi_k w)\nabla w.$$

and

$$\int_{D^C} \nabla(\xi_k w)\nabla w = \int_{D_k^C} \nabla(\xi_k w)\nabla w + \int_{D_k} \nabla(\xi_k w)\nabla w$$

If k is big enough such that $1/k < \delta$, we find

$$\begin{aligned}\int_{D_k} \nabla(\xi_k w)\nabla w &= \int_{D_k} \nabla(\xi_k w)\nabla w + \xi_k w \Delta w \\ &= \int_{\partial D_k} w \cdot w_\eta dA(D_k) \\ &\overset{k\to\infty}{\longrightarrow} -\int_{\partial D} w_\mu d\sigma.\end{aligned}$$

Furthermore,

$$\int_{D_k^C} \nabla(\xi_k w)\nabla w \overset{k\to\infty}{\longrightarrow} \int_{D^C} |\nabla w|^2.$$

This finishes the proof. □

Lemma 3.7 (An auxiliary obstacle problem)**.** *Let $u = u_\varepsilon^\delta$ be a minimizer of problem (3.1) and B a ball in D^C. Then there exists a unique $v \in H^1(D^C)$ minimizing the energy functional*

$$\int_{D^C} |\nabla v|^2 dx,$$

such that $v = \varphi$ on ∂D and $v = 0$ in $u^{-1}(0) \setminus B$. Such a function satisfies

(1) $v \in V_\delta$,
(2) $0 \le u \le v \le \sup_{\partial D} \varphi$.
(3) $\displaystyle\int_{D^C} v\Delta v = 0$.

Proof. Let $K := \{w \in H^1(D^C) : w = \varphi \text{ on } \partial D \text{ and } w \le 0 \text{ in } u^{-1}(0)\setminus B\}$. One easily verifies that K is a closed convex subset of $H^1(D^C)$. The energy functional is strictly convex and by the Poincaré inequality it is coercive over K. This implies there exists a unique minimal energy point $v \in K$. Moreover its variational characterization is:

$$\int_{D^C} \nabla(0 - v) \cdot \nabla(w - v) dx \le 0 \qquad \forall w \in K. \tag{3.3}$$

For every $\zeta \in H_0^1(D^C, \mathbb{R}_+)$, we have that $v - \zeta \in K$, so inequality (3.3) says that

$$\int_{D^C} \nabla v \cdot \nabla \zeta dx \le 0 \qquad \forall \zeta \in H_0^1(D^C, \mathbb{R}_+).$$

It means $\Delta v \ge 0$ in the sense of distribution.
Claim: $\Delta v = 0$ in $(u^{-1}(0) \setminus B)^C \supset D_\delta$.

Indeed, let $B(y, \epsilon) \subset (u^{-1}(0) \setminus B)^C$. For all $\psi \in H_0^1(B(y, \epsilon), \mathbb{R}_+)$, we may think it as an element of $H_0^1(D^C)$ just by extending it by zero outside of $B(y, \epsilon)$. Since supp $\psi \cap (u^{-1}(0) \setminus B)^C = \emptyset$, we conclude that $v + \psi$ as well as $v - \psi$ lie in K. Then inequality (3.3) implies

$$\int_{B(y,\epsilon)} \nabla\psi \cdot \nabla v = 0.$$

This shows that $\Delta v = 0$ in $(u^{-1}(0) \setminus B)^C$. Once u is subharmonic, we apply the maximum principle we obtain $0 \le u \le v \le \sup_{\partial D} \varphi$. This proves (2). Finally, let us verify item (3). To this end, let $\psi \in H_0^1(D^C, \mathbb{R}^+)$ and $|\tau|$ be small. Hence $v + \tau\psi v$ is non positive in $(u^{-1}(0) \setminus B)$

and takes the same boundary values as v. That is, $v+\tau\psi v$ competes against v in the energy problem. Thus

$$\int_{D^C} |\nabla v|^2 dx \leq \int_{D^C} |\nabla v|^2 dx + 2\tau \int_{D^C} \nabla(v\psi)\nabla v + \tau^2 \int_{D^C} |\nabla(\psi v)|^2 dx,$$

and once τ is arbitrary,

$$0 = \int_{D^C} \nabla(v\psi)\cdot\nabla v dx = -\int_{D^C} \psi v \Delta v dx.$$

Taking $\psi \to 1$ yields $\displaystyle\int_{D^C} v\Delta v dx = 0$, as desired. $\square$

We shall need the following result from [2].

Lemma 3.8. *Suppose $w \in H^1(\Omega)$ is a non-negative semicontinuous function. There exists a constant $c > 0$, depending only on dimension, such that, whenever $\overline{B}(x,r) \subset \Omega$ there holds*

$$\left(\frac{1}{r}\fint_{\partial B(x,r)} w d\sigma\right)^2 \cdot |\{y \in B(x,r) : w(y) = 0\}| \leq c \int_{B(x,r)} |\nabla(w-h)|^2 dy,$$

where h is the harmonic function in $B(x,r)$ taking boundary values equal to w on $\partial B(x,r)$.

Lemma 3.8 is the final ingredient we needed to prove:

Theorem 3.9. *Let $u = u_\varepsilon^\delta$ be a minimizer to problem (3.1). There exists a constant $M = M(\varepsilon) > 0$ independent of δ, such that if*

$$\frac{1}{r}\fint_{\partial B(x,r)} u d\sigma \geq M,$$

then $B(x,r) \subset \{u > 0\}$.

Proof. Let v be the function given by Lemma 3.7. Such a function is admissible for problem (3.1), thus

$$J_\varepsilon(u) \leq J_\varepsilon(v).$$

We recall that $0 \leq u \leq v \leq h_0$, where h_0 is the harmonic function defined on Lemma 3.4. Then, for each $\chi \in \partial D$, there holds

$$c_0 \leq (h_0)_\mu(\chi) \leq v_\mu(\chi) \leq u_\mu(\chi).$$

Therefore,

$$\int_{\partial D} \Gamma(x,u_\mu) - \Gamma(x,v_\mu) d\sigma \geq \min_{\partial D} \Gamma^{'}(x,c_0) \int_{\partial D} (u_\mu - v_\mu) d\sigma. \tag{3.4}$$

We also have, from Lemma 3.6 and Lemma 3.7 that

$$\begin{aligned} \sup_{\partial D} \varphi \cdot \int_{\partial D} (u_\mu - v_\mu) d\sigma &\geq \int_{D^C} u\Delta u + \int_{D^C} |\nabla u|^2 - \int_{D^C} v\Delta v - \int_{D^C} |\nabla v|^2 \\ &\geq \int_{D^C} |\nabla u|^2 - \int_{D^C} |\nabla v|^2. \end{aligned} \tag{3.5}$$

We consider now the harmonic function h in $B(x,r)$ taking boundary values equal to u. We extend h by u outside of $B(x,r)$. In this way, $h \in V_\delta$ and $0 \leq u \leq h \leq v$. Hence, h is admissible for problem (3.1) as well as for the energy problem in Lemma 3.7. Then using the minimality property of v, we can replace, in the right hand side of (3.5), ∇v by ∇h. That is,

$$\sup_{\partial D} \varphi \cdot \int_{\partial D} (u_\mu - v_\mu) d\sigma \geq \int_{D^C} |\nabla u|^2 - |\nabla h|^2 dx = \int_{B(x,r)} |\nabla (u-h)|^2 dx.$$

Plugging these inequalities into (3.4) we obtain

$$\int_{\partial D} \Gamma(u_\mu) - \Gamma(v_\mu) d\sigma \geq c(\Gamma, \varphi) \int_{B(x,r)} |\nabla (u-h)|^2 dx. \tag{3.6}$$

We recall that f_ε is a Lipschitz function with Lipschitz constant equal to $\frac{1}{\varepsilon}$. Using this together with the key fact that $J_\varepsilon(u) \leq J_\varepsilon(v)$, we end up with

$$\begin{aligned} \int_{\partial D} \Gamma(u_\mu) - \Gamma(v_\mu) d\sigma &\leq f_\varepsilon(|\{v>0\}|) - f_\varepsilon(|\{u>0\}|) \\ &\leq \frac{1}{\varepsilon} |\{y \in B(x,r) : u(y) = 0\}|. \end{aligned} \tag{3.7}$$

Finally, by Lemma 3.8 we get

$$\begin{aligned} |\{B(x,r) \cap u^{-1}(0)\}| &\geq \varepsilon c(\Gamma, \varphi) \int_{B(x,r)} |\nabla (u-h)|^2 dx \\ &\geq \varepsilon c(\Gamma, \varphi) \frac{1}{c} \left(\frac{1}{r} \fint_{\partial B(x,r)} u d\sigma \right)^2 |\{B(x,r) \cap u^{-1}(0)\}|. \end{aligned} \tag{3.8}$$

Hence, if

$$\frac{1}{r} \fint_{\partial B(x,r)} u d\sigma > \sqrt{\frac{c}{\varepsilon c(\Gamma, \varphi)}},$$

$|\{y \in B(x,r) : u(y) = 0\}|$ has to be equal to zero. Observe that in this case, $u \equiv h$ in $B(x,r)$ and hence u is harmonic in such a ball. □

Corollary 3.10. *There exists a constant K_ε, independent of δ, such that all minimizers u_ε^δ are Lipschitz functions with $\|u_\varepsilon^\delta\|_{\mathrm{Lip}} \leq K_\varepsilon$. Moreover $\Delta u_\varepsilon^\delta = 0$ in $\{u_\varepsilon^\delta > 0\}$.*

Proof. Let $u = u_\varepsilon^\delta$. We will first show that $\{u > 0\}$ is an open set. To this end, let $z \in D^C$ be such that $u(z) > 0$. Since u is subharmonic, for a small r

$$\fint_{B(z,r)} u dx \geq u(z) > 0.$$

Now we take $r_0 > 0$ small enough such that

$$\frac{1}{r_0} \fint_{B(z,r_0)} u dx \geq M_\varepsilon.$$

Hence Theorem 3.9 implies $B(z, r_0) \subset \{u > 0\}$ and $\Delta u = 0$ in $B(z, r_0)$. Let $x \in \Omega \subset\subset \Omega' \subset\subset D^C$, with $u(x) > 0$. Let $d = \mathrm{dist}(x, \partial\Omega' \cap \{u > 0\})$ and consider the ball $B = B(x, d)$. Suppose ∂B touches $\partial\{u = 0\}$. Then from Theorem 3.9, for each $\gamma > 0$, there holds

$$\frac{1}{r+\gamma} \fint_{\partial B(x, r+\gamma)} u d\sigma \leq M.$$

Letting $\gamma \to 0$, we get

$$\frac{1}{r} \fint_{\partial B(x,r)} u d\sigma \leq M.$$

Once u is harmonic in B, by the interior estimate of derivatives, we obtain

$$|\nabla u(x)| \leq C(N)\frac{1}{r}\fint_{\partial B(x,r)} u d\sigma \leq C(N,\varepsilon).$$

On the other hand, if ∂B touches $\partial\Omega'$, then again by the interior estimate of derivatives for harmonic functions, we find

$$|\nabla u(x)| \leq \frac{N}{\mathrm{dist}(\Omega,\Omega')}.$$

□

Theorem 3.11. *There exists a minimizer $u_\varepsilon \in V$ for the problem (2.2). Moreover it is a Lipschitz function and $\Delta u_\varepsilon = 0$ in $\{u_\varepsilon > 0\}$.*

Proof. Let $\widetilde{D}$ be a smooth domain such that $D \subset \widetilde{D}$, with $|\widetilde{D} \setminus D| = 1$ and u_0 the harmonic function on $\widetilde{D} \setminus D$, such that $u_0 \equiv \varphi$ on ∂D, $u_0 \equiv 0$ on $\partial\widetilde{D}$. In this way, u_0 competes against u_ε^δ in (3.1) for all $\varepsilon > 0$ and $\delta > 0$. Thus

$$C = J_\varepsilon(u_0) \geq J_\varepsilon(u_\varepsilon) \geq \int_{\partial D} \Gamma\big(x, (u_\varepsilon^\delta)_\mu(x)\big) d\sigma, \qquad \forall \varepsilon > 0,\ \delta > 0.$$

Combining the above with estimate (3.2) implies that, up to a subsequence, we might assume that $u_\varepsilon^\delta \rightharpoonup u_\varepsilon$ in the H^1-sense, as $\delta \to 0$. Furthermore, by Corollary 3.10, we might also assume that $u_\varepsilon^\delta \to u_\varepsilon$ uniformly over compacts. In this way, for each $\overline{B}(x,r) \subset \{u_\varepsilon > 0\}$, there exists a $\delta_0 > 0$ such that, for all $\delta < \delta_0$, $B(x,r) \subset \{u_\varepsilon^\delta > 0\}$. This shows that $\Delta u_\varepsilon = 0$ in $\{u_\varepsilon > 0\}$. Finally, Lemma 3.3 implies

$$J_\varepsilon(u_\varepsilon) = \min_V J_\varepsilon,$$

and thus, since in particular u is a minimizer of a problem (3.1) for any δ such that $D_\delta \subset \{u > 0\}$, u_ε is Lipschitz, and its Lipschitz constant depends only on ε. □

4. Regularity properties of solutions to the penalized problem

In this section we start the journey of showing regularity properties of an optimal configuration to problem (2.2). Optimal regularity of the minimizer has already been obtained in the previous section. In this section, as well as in the next section, we shall be concerned with regularity properties of the free boundary. Throughout this section we will denote u_ε by u.

Theorem 4.1. *For $0 < \tau < 1$, there exists a constant $m_\varepsilon(\tau)$ such that if*

$$\frac{1}{r}\fint_{\partial B(x,r)} u d\sigma \leq m_\varepsilon(\tau),$$

then $B(x,\tau r) \subset \{u = 0\}$

Proof. Following the same idea of Lemma 3.7, we assure the existence of a minimizer to the energy functional, $\int_{D^C} |\nabla v|^2 dx$, subject to the constraints: $v = \varphi$ on ∂D and $v \leq 0$ in $\overline{B}(x,\tau r) \cup \{u = 0\}$. As done in Lemma 3.7, one can show that $\Delta v \geq 0$, $0 \leq v \leq u$ and $\int v\Delta v = 0$. In particular v competes with u in problem (2.2); therefore

$$\int_{\partial D} \Gamma(x, v_\mu) - \Gamma(x, u_\mu) d\sigma \geq \varepsilon\big(|\{u > 0\} \cap B(x,\tau r)|\big), \tag{4.1}$$

where we have used that f_ε^{-1} is Lipschitz with Lipschitz constant equal to ε. Also from Lemma 3.6 we obtain

$$\inf_{\partial D} \varphi \cdot \int_{\partial D} v_\mu - u_\mu d\sigma \leq \int_{D^C} \Big(|\nabla v|^2 - |\nabla u|^2 \Big) dx. \tag{4.2}$$

Let $\delta_0 > 0$ be a fixed small number. We may assume $B(x,r) \subset D^C \setminus D_{\delta_0}$. Let w be the harmonic function in D_{δ_0} taking boundary values equal to φ on ∂D and 0 on ∂D_{δ_0}. For each $\chi \in \partial D$, there holds

$$u_\mu(\chi) \leq v_\mu(\chi) \leq w_\mu(\chi) \leq C_0.$$

Therefore,

$$\int_{\partial D} \Gamma(x, v_\mu) - \Gamma(x, u_\mu) d\sigma \leq \max_{\partial D} \Gamma^{'}(x, C_0) \int_{\partial D} v_\mu - u_\mu d\sigma.$$

Combining the above inequalities we end up with

$$\varepsilon \big(|\{u > 0\} \cap B(x, \tau r)| \big) \leq C(\Gamma, \varphi) \int_{D^C} \Big(|\nabla v|^2 - |\nabla u|^2 \Big) dx. \tag{4.3}$$

Let us consider the auxiliary functions

$$g(\rho) := \begin{cases} \log\left(\dfrac{\rho}{\tau r}\right) & \text{if} \quad N = 2 \\[2ex] \dfrac{1}{(\tau r)^{N-2}} - \dfrac{1}{\rho^{N-2}} & \text{if} \quad N \geq 3, \end{cases}$$

and $h : B(x, \sqrt{\tau} r) \to \mathbb{R}$,

$$h(y) = \min \left\{ u(y), \frac{s}{g(\sqrt{\tau} r)} \big(g(|y - x|) \big)^+ \right\},$$

where $s := \max\limits_{\overline{B}(x,\tau r)}$. Extending h by u outside of $B(x, \sqrt{\tau} r)$ we see that $h = 0$ in $\{u = 0\} \cap \overline{B}(x, \tau r)$. Hence h competes with v in the energy problem. So we can exchange v by h in inequality (4.3) and we get

$$\frac{\varepsilon}{C(\Gamma, \varphi)} |\{u > 0\} \cap B(x, \tau r)| \leq \int_{B(x, \sqrt{\tau} r)} \big(|\nabla h|^2 - |\nabla u|^2 \big) dy. \tag{4.4}$$

Since $h \equiv 0$ on $B(x, \tau r)$ we may rewrite inequality (4.4) as

$$\int_{B(x,\tau r)} |\nabla u|^2 + \frac{\varepsilon}{C(\Gamma, \varphi)} |\{u > 0\} \cap B(x, \tau r)| \leq \int_{B(x, \sqrt{\tau} r) \setminus B(x, \tau r)} \big(|\nabla h|^2 - |\nabla u|^2 \big) dy. \tag{4.5}$$

Notice that $|\nabla h|^2 - |\nabla u|^2 = -2 \nabla h \cdot \nabla(u - h) - |\nabla(u - h)|^2$. In this way we can estimate

$$\begin{aligned} \int_{B(x, \sqrt{\tau} r) \setminus B(x, \tau r)} \big(|\nabla h|^2 - |\nabla u|^2 \big) dy &\leq -2 \int_{B(x, \sqrt{\tau} r)} \nabla \big((u - h)^+ \big) \cdot \nabla h dy \\ &= 2 \int_{\partial B(x, \sqrt{\tau} r)} u \nabla h \cdot \nu dA \\ &\leq \frac{C(N, \tau)}{r} \cdot s \int_{\partial B(x, \sqrt{\tau} r)} u dA. \end{aligned}$$

Hence,

$$\int_{B(x,\tau r)} |\nabla u|^2 + \frac{\varepsilon}{C(\Gamma, \varphi)} |\{u > 0\} \cap B(x, \tau r)| \leq \frac{C(N, \tau)}{r} \cdot s \int_{\partial B(x, \tau r)} u dA. \tag{4.6}$$

On the other hand,

$$\int_{\partial B(x,\tau r)} udA \le C(N,\tau)\left(\int_{B(x,\tau r)} udy + \int_{B(x,\tau r)} |\nabla u| dy\right). \tag{4.7}$$

We observe that, being u subharmonic, we have from the mean value theorem that

$$s := \max_{\overline{B}(x,\tau r)} u \le \overline{c}(N,\tau) \fint_{\partial B(x,r)} udA. \tag{4.8}$$

Finally combining inequalities (4.6), (4.7) and (4.8) we see that if

$$\frac{1}{r}\fint_{\partial B(x,r)} ud\sigma \le m_\varepsilon(\tau),$$

with $m_\varepsilon(\tau)$ depending only on dimension, ε and τ, then necessarily $B(x,\tau r) \subset \{u = 0\}$. □

We shall denote $U := \{x \in D^C : u(x) > 0\}$ and $F = \{x \in D^C : u(x) = 0\}$.

Corollary 4.2. *Let $x \in U$. There exist constants $0 < c,\ C < \infty$ such that*

$$c \cdot dist(x, \partial F) \le u(x) \le C \cdot dist(x, \partial F)$$

Proof. Let us denote by $d = \mathrm{dist}(x, \partial F)$. It follows from Theorem 3.9, Theorem 4.1 and mean value theorem that

$$m_\varepsilon(\frac{1}{2}) \cdot d \le \fint_{\partial B(x,d)} udA = u(x) \le M_\varepsilon \cdot d.$$

□

Corollary 4.3. *There exists a constant $0 < c = c_\varepsilon < 1$, such that for any $x \in \partial F$ there holds*

$$c \le \frac{|F \cap B(x,r)|}{|B(x,r)|} \le 1 - c,$$

for each $B(x,r) \subset D^C$.

Proof. Follows from Theorem 4.1 that there exists a point $y \in B(x, r/2)$ such that $u(y) \ge m_\varepsilon \cdot r$. Furthermore, since u is subharmonic, we have, for τ is small enough

$$\frac{1}{\tau r}\fint_{\partial B(x,d)} udA \ge \frac{1}{\tau r} u(y) \ge \frac{m_\varepsilon}{\tau} > M_\varepsilon,$$

where M_ε is the constant given by Theorem 3.9. Thus, Theorem 3.9 implies $B(y,\tau r) \subset U$. We have obtained the estimate from above. Let us turn our attention to the lower bound estimate. We shall use the construction made in Theorem 3.9. Let h be be harmonic function in $B(x,r)$, with boundary value data equal to u. The same type of computation done in Theorem 3.9 yields

$$\int_{B(x,r)} |\nabla(u-h)|^2 dy \le \frac{1}{\varepsilon}|F \cap B(x,r)|. \tag{4.9}$$

By Poisson's integral formula, we may write, for $|y - x| \le \tau r$, $0 < \tau < 1$,

$$h(y) \le (1 - c(N,\tau)) \fint_{\partial B(x,r)} udA.$$

Furthermore, since $x \in F$, $u(y) = |u(y) - u(x)| \le K \cdot \tau r$, where K is the Lipschitz norm of u. Invoking now Theorem 4.1, we find

$$h(y) - u(x) \ge (1 - c(N,\tau)) \fint_{\partial B(x,r)} udA - K \cdot \tau r \ge \Big[(1 - c(N,\tau)) m_\varepsilon(\tau) - K\tau\Big] \cdot r.$$

Therefore, for τ small enough, we obtain

$$h(y) - u(y) \geq cr, \qquad \forall y \in B(x, \tau r), \tag{4.10}$$

where c depends only on the minimizer u. The classical Poincaré inequality tells us

$$\frac{c_N}{r^2}\int_{B(x,r)} |u-h|^2 dy \leq \int_{B(x,r)} |\nabla(u-h)|^2 dy.$$

Combining the Poincaré inequality, (4.9) and (4.10) we finally get

$$\frac{c_N}{r^2} c^2 r^2 |B(x,\tau r)| \leq \frac{1}{\varepsilon}|F \cap B(x,r)|,$$

which finishes the proof. □

We have fulfilled all the hypothesis of the results in [2] section 4. Hence we can state:

Theorem 4.4. *Let $u = u_\varepsilon$ be a minimizer for the problem (2.2). Then*

1. *The $n-1$ Hausdorff measure of ∂F is locally finite, i.e., $\mathcal{H}^{n-1}(\Omega \cap \partial F) < \infty$, for every $\Omega \subset\subset D^C$. Moreover there exists positive constants c_ε, C_ε, depending on N, D, Ω and ε, such that for all ball $B(x,r) \subset \Omega$ with $x \in \partial F$, there holds*
$$c_\varepsilon r^{n-1} \leq \mathcal{H}^{n-1}(F \cap B(x,r)) \leq C_\varepsilon r^{n-1}.$$
2. *There exists a Borel function $q = q_\varepsilon$ such that $\Delta u = q\mathcal{H}^{n-1}\lfloor \partial F$, that is, for any $\zeta \in C_0^\infty(D^C)$, there holds*
$$-\int_{D^C} \nabla u \cdot \nabla \zeta dx = \int_{\partial F} \zeta q d\mathcal{H}^{n-1}.$$
3. *There exists positive constants c_ε and C_ε such that*
$$c_\varepsilon \leq q(x) \leq C_\varepsilon,$$
for $\mathcal{H}^{n-1}$ almost all points $x \in \partial F$.
4. *For $\mathcal{H}^{n-1}$ almost all points in ∂F, an outward normal $\nu = \nu(x)$ is defined and furthermore*
$$u(x+y) = q(x)(y\cdot\nu)^+ + O(y),$$
where $\frac{O(y)}{|y|} \to 0$ as $|y| \to 0$. This allows us to define $q(x) = u_\nu(x)$ at those points.
5. $\mathcal{H}^{n-1}(\partial F \setminus \partial_{\mathrm{red}} F) = 0.$

5. Regularity of the Free Boundary

In this section, we shall prove that our free boundary is a analytic surface. Our strategy is to initially show that the normal derivative of the minimizer is a Höder continuous function along the free boundary. This allow us to conclude that the free boundary is a $C^{1,\alpha}$ surface. Afterwards, due to a free boundary condition, we shall obtain the analyticity of the free boundary. This section is based on sections 4 and 5 on [3]. The main tool in our analysis will be the notion of non-tangentially accessible domains. Our motivation lies in the results of [7].

Theorem 5.1 (Jerison-Kenig [7]). *Let Ω be a non-tangentially accessible domain and let V be an open set, V and Ω contained in $\mathbb{R}^n$. For any compact set K, $K \subset V$, there exists a constant $\alpha > 0$ such that for any positive harmonic functions v and w which vanish continuously on $\partial D \cap V$, the quotient $\dfrac{v}{w}$ is a Hölder continuous function of order α in $K \cap \big|\partial D$. In particular for any $x_0 \in K \cap \partial D$ the limit $\lim\limits_{x\to x_0} \dfrac{v(x)}{w(x)}$ exists.*

We now can state the following powerful result:

Theorem 5.2. *Let $u = u_\varepsilon$ be a solution to the problem P_ε. Then the set $U := \{x \in D^C : u(x) > 0\}$ is a non-tangentially accessible domain.*

Proof. This result follows from the same analysis as in Theorem 4.8 in [3]. Indeed one should notice that all the ingredients used to show Theorem 4.8 in [3] were proven to our nonlinear case. We now follow section 4 in [3] and conclude the proof of Theorem 5.2. □

Corollary 5.3. *Let u be a solution to the problem P_ε. Let $U := \{x \in D^C : u(x) > 0\}$. Then there exists a (negative) Green's function G for the Dirichlet problem in U. Moreover, there exists an exponent $\alpha > 0$ such that for any fixed $y \in U$ the quotient*

$$\frac{G(x,y)}{u(x)}$$

is a C^α function of x up to the boundary, taking values

$$\frac{G_\nu(x,y)}{u_\nu(x)}$$

at the regular points of ∂U where the normal vector ν is defined. Thus, for any smooth function ψ, we have

$$\psi(y) = \int_{\partial U} G_\nu(x,y)\psi(x)dH^{n-1}(x) + \int_U G(x,y)\Delta\psi(x)dx$$

Let us move toward the $C^{1,\alpha}$ regularity of the free boundary. The idea is to use suitable perturbations of the free boundary. These perturbations are motivated by the Hadamard variational formula. To fix the ideas, consider a function ρ defined in $\mathbb{R}^n$ such that

(1) ρ is radial
(2) $\rho(r)$ is non-increasing
(3) $\rho(r) \equiv 1$ if $r < \frac{1}{4}$, $\rho(r) \equiv 0$, if $r > \frac{1}{2}$
(4) $\rho \in C^\infty(\mathbb{R}^n)$.

We denote by I the integral $I := \int_{x_n=0} \rho(x)d\sigma$. For δ positive and small real number we consider the domains

(1) $\Sigma := \{x \in \mathbb{R}^n : x_n > 0,\ |x| < 1\}$
(2) $\Sigma^+ := \{y \in \mathbb{R}^n : y = x - \delta\rho(x)e_n$ for some $x \in \Sigma\}$
(3) $\Sigma^- := \{y \in \mathbb{R}^n : y = x + \delta\rho(x)e_n$ for some $x \in \Sigma\}$

The following Lemma is a variant of the Hadamard variational formula. Its proof can be found in [3].

Lemma 5.4. *Let v denote the harmonic function in Σ^+ (respectively Σ^-) taking boundary values x_n on $|x| = 1$ and zero otherwise. Then*

$$\frac{1}{\delta} \int_{\Sigma^+ \cap \{x : x_n = 0\}} v d\sigma \quad \to I \qquad \text{and}$$
$$\frac{1}{\delta} \int_{\partial\Sigma^- \cap \Sigma x_n} v_\nu d\sigma \quad \to I,$$

as $\delta \searrow 0$, where v_ν is the inward normal derivative at $\partial\Sigma^-$.

We shall denote by R the reduced boundary of ∂F, i.e., the subset of ∂F for which (3) and (4) in Theorem 4.4 hold, furthermore

$$\frac{1}{r^{n-1}} \int_{\partial F \cap B(x,r)} |\nu(y) - \nu(x)| dH^{n-1}(y) \to 0,$$

as $r \to 0$. We know R can be chosen so that $H^{n-1}(\partial F \setminus R) = 0$. For $x \in R$, it is possible to find a function $\phi = \phi(r)$ so that ϕ is non-decreasing, and if $\nu = \nu(x)$ is the outward normal direction to F at x,

(1) $|u(x+y) - u_\nu(x)(y \cdot \nu)^+| \le \phi(r)$ if $|y| \le r$,
(2) If $y \in B(x,r)$ and either $y \cdot \nu < 0$ and $u(y) > 0$, or $y \cdot \nu > 0$ and $u(y) = 0$, then $|y \cdot \nu| \le \phi(r)$
(3) $\displaystyle \frac{1}{r^{n-1}} \int_{\partial F \cap B(x,r)} |\nu(y) - \nu(x)| dH^{n-1}(y) \le \frac{1}{r}\phi(r)$
(4) $\displaystyle \frac{1}{r^{n-1}} \int_{\partial F \cap B(x,r)} |u_\nu(y) - u_\nu(x)| dH^{n-1}(y) \le \frac{1}{r}\phi(r)$
(5) $\displaystyle \frac{1}{r}\phi(r) \to 0$ as $r \to 0$.

Suppose now $x \in R$ and $r > 0$. Without loss of generality we may assume $x = 0$ and $\nu(x) = e_n$. We define the sets:

$$\Sigma^+(x,r) := \left\{ y : \frac{y}{r} - 2\frac{\phi(r)}{r} e_n \in \Sigma^+ \right\}$$

$$\Sigma^-(x,r) := \left\{ y : \frac{y}{r} + 2\frac{\phi(r)}{r} e_n \in \Sigma^- \right\},$$

where Σ^+ and Σ^- were defined above and we take $\delta = \delta(r) = \left(\frac{\phi(r)}{r}\right)^{1/2}$. Note that

$$\frac{1}{\delta}\frac{\phi(r)}{r} = \delta \to 0 \text{ as } r \to 0.$$

The next two lemmas can also be found in [3].

Lemma 5.5. *Let w be the harmonic function in $S := \big(\Sigma^+(x,r) \cup U\big) \cap B(x,r)$, taking boundary values u in $\partial S \cap \partial B(x,r)$ and zero otherwise. Then*

$$\frac{1}{\delta}\frac{1}{r^n} \int_{S \cap \partial U} w \, dH^{n-1} \to I u_\nu(x),$$

as $r \to 0$.

Lemma 5.6. *Let w be the harmonic function in $S := U \cap \Sigma^-(x,r)$, taking boundary values u in $\partial S \cap \partial B(x,r)$ and zero otherwise. Then*

$$\frac{1}{\delta}\frac{1}{r^n} \int_{U \cap \partial \Sigma^-(x,r)} u w_\nu \, dH^{n-1} \to I u_\nu^2(x),$$

as $r \to 0$, where ν is the inward normal to $\Sigma^-(x,r)$.

Finally we can state the main result of this section.

Theorem 5.7. *u_ν is a Hölder continuous function on R.*

Proof. Let x_1 and x_2 be two generic points in R. Associated to x_1 and x_2 we have functions ϕ_1 and ϕ_2 defined above. Without loss of generality we may assume $\phi_1 = \phi_2 = \phi$. Suppose then $0 < r < \frac{1}{10}|x_1 - x_2|$ and $\phi(r) < 1$. Consider the sets $\Sigma^+(x_1, r)$ and $\Sigma^-(x_2, r)$. We denote by v, v_1, v_2 the following functions respectively:

$$
\begin{aligned}
\Delta v &= 0 \quad \text{in} \quad A_0 := \Big(\big(U \cup \Sigma^+(x_1, r)\big) \setminus B(x_2, r) \Big) \cup \big(U \cap \Sigma^-(x_2, r)\big) \\
v &= \varphi \quad \text{on} \quad \partial D \\
v &= 0 \quad \text{on} \quad \partial A_0 \setminus \partial D \\
\Delta v_1 &= 0 \quad \text{in} \quad A_1 := U \cup \Sigma^+(x_1, r) \\
v_1 &= \varphi \quad \text{on} \quad \partial D \\
v_1 &= 0 \quad \text{on} \quad \partial A_0 \setminus \partial D \\
\Delta v_2 &= 0 \quad \text{in} \quad A_2 := \big(U \setminus B(x_2, r)\big) \cup \big(U \cap \Sigma^-(x_2, r)\ \big) \\
v_2 &= \varphi \quad \text{on} \quad \partial D \\
v_2 &= 0 \quad \text{on} \quad \partial A_0 \setminus \partial D
\end{aligned}
\tag{5.1}
$$

By the maximum principle: $v_2 \le v$ and $u \le v_1$. By Corollary 5.3, for any $x \in U$ we can write:

$$u(x) = \int_{\partial D} G_\nu(x, y) dH^{n-1}(y).$$

It follows also from Corollary 5.3 that

$$v_1(x) - u(x) = \int_{\Lambda_1} G_\nu(x, y) v_1(y) dH^{n-1}(y) \tag{5.2}$$

and

$$v_2(x) - u(x) = -\int_{\Lambda_2} G(x, y) (v_2)_\nu(y) dH^{n-1}(y) \tag{5.3}$$

where $\Lambda_1 = \Sigma^+(x_1, r) \cap \partial U$, $\Lambda_2 = U \cap \partial \Sigma^-(x_2, r)$ and ν is the outward normal. We also find

$$v(x) = u(x) + \int_{\Lambda_1} G_\nu(x, y) v(y) dH^{n-1}(y) - \int_{\Lambda_2} G(x, y) v_\nu(y) dH^{n-1}(y) \tag{5.4}$$

We now fix $x \in \partial D$. For each $h > 0$ consider the point $x + h\mu(x) \in U$. Consider the sequence functions $\mathcal{H}_h = \mathcal{H}_h(x)$ defined by:

$$\mathcal{H}_h(y) = \frac{G(x + h\mu(x), y)}{h}.$$

Notice, For each y fixed, $\mathcal{H}_h(y)$ converges pointwise to $G_\mu(x, y)$. This observation allow us to guarantee, up to a subsequence, the existence of a harmonic function $H(x) \colon U \to \mathbb{R}$ such that:

$$v_\mu(x) = u_\mu(x) + \int_{\Lambda_1} H_\nu(x, y) v(y) dH^{n-1}(y) - \int_{\Lambda_2} H(x, y) v_\nu(y) dH^{n-1}(y). \tag{5.5}$$

¿From (5.3), for $x \in B(x_1, r) \cap U$, $y \in B(x_2, r) \cap U$, we obtain

$$v_2(x) - u(x) \ge -\sup \left| \frac{G(x, y)}{u(y)} \right| \int_{\Lambda_2} u (v_2)_\nu dH^{n-1} \ge -c \int_{\Lambda_2} u (v_2)_\nu dH^{n-1}, \tag{5.6}$$

by Corollary 5.3. If w_2 denotes the harmonic function of Lemma 5.6 (with $x = x_2$), we have $v_2 < w_2$ in $U \cap \Sigma^-(x_2, r)$. Therefore, $(v_2)_\nu \le (w_2)_\nu$ on Λ_2, so from (5.6) and Lemma 5.5 we obtain

$$v_2(x) - u(x) \ge -cI\delta r^n u_\nu^2(x_2) \ge -c\delta r^n.$$

If w_1 denotes the harmonic function of Lemma 5.5 (with $x = x_1$), we have

$$v \ge w_1 - c\delta r^n \text{ in } \Sigma(x_1, r) \cup (U \cap B(x_1, r)),$$

once $v \geq v_2$ and $w_1 = u$ on $(\partial B(x_1, r)) \cap U$. Therefore,

$$\begin{aligned}
\int_{\Lambda_1} H_\nu(x,y)v(y)dH^{n-1}(y) &\leq \int_{\Lambda_1} H_\nu(x,y)w_1(y)dH^{n-1}(y) \\
&- c\delta r^n \int_{\Lambda_1} H_\nu(x,y)dH^{n-1}(y) \\
&= \int_{\Lambda_1} \big[H_\nu(x,y) - H_\nu(x,x_1)\big] w_1(y)dH^{n-1}(y) \\
&- H_\nu(x,x_1)u_\nu(x_1)I\delta r^n + O(\delta r^n).
\end{aligned}$$

We also have from Lemma 5.5 that $w_1 \leq c\phi(r)$ on Λ_1, thus

$$\int_{\Lambda_1} H_\nu(x,y)v(y)dH^{n-1}(y) \leq -H_\nu(x,x_1)u_\nu(x_1)I\delta r^n + O(\delta r^n). \tag{5.7}$$

Let us turn our attention to estimate $\int_{\Lambda_2} H(x,y)v_\nu(y)dH^{n-1}(y)$ from below. Since $v(x) \leq v_1(x) \leq 1$, we obtain from (5.2) that

$$v(x) < u(x) + cr^{n-1} \text{ in } U \cap \Sigma^-(x_2, r).$$

If follows therefore that

$$v(x) < w_2(x) + cr^{n-1}\widetilde{w}(x) \text{ in } U \cap \Sigma^-(x_2, r),$$

where w_2 denotes the harmonic function of Lemma 5.6 (with $x = x_2$) and $\widetilde{w}$ is a non-negative harmonic function in $S := U \cap \Sigma^-(x_2, r)$ taking smooth non-negative boundary values equal to 1 on $\partial S \cap \partial B(x_2, r)$ and 0 on $\partial S \cap \partial B(x_2, r/2)$. Then, by the maximum principle, $v_\nu \geq (w_2)_\nu + cr^{n-1}\widetilde{w}_\nu$ on Λ_2. Hence

$$\begin{aligned}
\int_{\Lambda_2} H(x,y)v_\nu(y)dH^{n-1}(y) &\geq \int_{\Lambda_2} H(x,y)(w_2)_\nu(y)dH^{n-1}(y) \\
&+ cr^{n-1} \int_{\Lambda_2} H(x,y)\widetilde{w}_\nu dH^{n-1}(y) \\
&= \int_{\Lambda_2} H(x,y)(w_2)_\nu(y)dH^{n-1}(y) + O(\delta r^n).
\end{aligned} \tag{5.8}$$

Applying Theorem 5.1 to $H(x, \cdot)$ and u, we may write

$$H(x,y) = \frac{H_\nu(x,x_2)}{u_\nu(x_2)} u(y) + O(r^\alpha)u(y).$$

Plugging the above into (5.8) and using Lemma 5.6 again we end up with

$$\int_{\Lambda_2} H(x,y)v_\nu(y)dH^{n-1}(y) \geq -H_\nu(x,x_2)u_\nu(x_2)I\delta r^n + O(\delta r^n). \tag{5.9}$$

Finally combining (5.5) with inequalities (5.7) and (5.9) we obtain

$$v_\mu(x) = u_\mu(x) - I\delta r^n \Big[H_\nu(x,x_1)u_\nu(x_1) - H_\nu(x,x_2)u_\nu(x_2)\Big] + O(\delta r^n),$$

and then,

$$\begin{aligned}
\Gamma(x, v_\mu(x)) &= \Gamma(x, u_\mu(x)) \\
&+ \Gamma_t(x, u_\mu(x))I\delta r^n \Big[H_\nu(x,x_2)u_\nu(x_2) - H_\nu(x,x_1)u_\nu(x_1)\Big] + O(\delta r^n).
\end{aligned}$$

Since the volume added to U with $\Sigma^+(x_1, r)$ is $I\delta r^n$ with error $O(\delta r^n)$, and the volume taken away from U with $\Sigma^-(x_2, r)$ is $I\delta r^n$ with the same error, we conclude by integrating the above inequality over ∂D that

$$\begin{aligned} 0 &\leq J_\varepsilon(v) - J_\varepsilon(u) \\ &\leq \int_{\partial D} \partial_t \Gamma(x, u_\mu(x)) I\delta r^n \Big[H_\nu(x, x_2) u_\nu(x_2) - H_\nu(x, x_1) u_\nu(x_1) \Big] + O(\delta r^n) \end{aligned}$$

Dividing by δr^n, letting $r \to 0$ and afterwards reversing the roles of x_1 and x_2 gives

$$\int_{\partial D} \partial_t \Gamma(x, u_\mu(x)) \Big[H_\nu(x, x_2) u_\nu(x_2) - H_\nu(x, x_1) u_\nu(x_1) \Big] = 0. \tag{5.10}$$

It provides us the free boundary condition

$$\int_{\partial D} \partial_t \Gamma(x, u_\mu(x)) \Big[H_\nu(x, \cdot) u_\nu(\cdot) \Big] \equiv C \tag{5.11}$$

in R. Let us conclude the Hölder continuity of u_ν on R. We can rewrite (5.10) as

$$\int_{\partial D} \Gamma_t(x, u_\mu(x)) \left\{ \frac{H_\nu(x, x_1)}{u_\nu(x_1)} \Big(u_\nu^2(x_1) - u_\nu^2(x_2) \Big) + u_\nu^2(x_2) \left(\frac{H_\nu(x, x_1)}{u_\nu(x_1)} - \frac{H_\nu(x, x_2)}{u_\nu(x_2)} \right) \right\} d\sigma.$$

and then

$$u_\nu^2(x_1) - u_\nu^2(x_2) \cdot \int_{\partial D} \Gamma_t(x, u_\mu(x)) \frac{H_\nu(x, x_1)}{u_\nu(x_1)} d\sigma = u_\nu(x_2) \int_{\partial D} \frac{H_\nu(x, x_1)}{u_\nu(x_1)} - \frac{H_\nu(x, x_2)}{u_\nu(x_2)} d\sigma \tag{5.12}$$

Let us first analyze the term $\displaystyle\int_{\partial D} \Gamma_t(x, u_\mu(x)) \frac{H_\nu(x, x_1)}{u_\nu(x_1)} d\sigma$:

Notice $H(x, x_1) = 0$ for all $x_1 \in R$ and $x \in \partial D$. Thus $H_\nu(x, x_1) > 0$ on R. Since the maps $x \mapsto H_\nu(x, x_1)$ and $x_1 \mapsto H_\nu(x, x_1)$ are continuous, there exits a constant c such that $H_\nu(x, x_1) > c > 0$. Moreover, from Theorem 4.4 there exist constants c_ε and C_ε such that $0 < c_\varepsilon \leq u_\nu(x_1) \leq C_\varepsilon$. Furthermore, $\Gamma_t(x, u_\mu(x)) \geq \Gamma_t(x, c_0) \geq \min_{\partial D} \Gamma_t(x, c_0) > 0$, where c_0 is as in Lemma 3.9. We have concluded there exists a constant $m_\varepsilon > 0$ such that

$$\int_{\partial D} \Gamma_t(x, u_\mu(x)) \frac{H_\nu(x, x_1)}{u_\nu(x_1)} d\sigma \geq m_\varepsilon > 0. \tag{5.13}$$

Let us now analyze the term $\dfrac{H_\nu(x, \cdot)}{u_\nu(\cdot)}$:

We know from Theorem 5.1 for each $x \in \partial D$ fixed, the map $\dfrac{H_\nu(x, \cdot)}{u_\nu(\cdot)}$ is α-Hölder continuous. We want to argue that there exists a constant M_ε such that $\left[\dfrac{H_\nu(x, \cdot)}{u_\nu(\cdot)} \right]_\alpha \leq M_\varepsilon$. Going back into the proof of Theorem 5.1, we notice that as long as the positive harmonic functions agree at x_0, the C^α norm of the quotient is universally bounded. This fact is due to the *Boundary Harnack Principle* (Theorem 5.1 in [7]). Thus we conclude that if a family of positive harmonic functions satisfying the hypothesis of Theorem 5.1 are comparable in the sense that they are uniformly bounded below and above, the C^α norm of the quotient of any two elements of the family is uniformly bounded. In our specific case, let $x_1 \in R$ and consider $V = B(x_1, 2r)$ and $K = \overline{B(x_1, r)}$. Fix $X_0 \in \partial K \cap U$. All we have to show is that $\left\| \dfrac{H(x, X_0)}{u(X_0)} \right\|_\infty \leq M_\varepsilon$. Corollary 4.2 assures $u(X_0) \geq c_\varepsilon r > 0$. Furthermore, as we have observed before, one can assure the existence of a universal constant C, depending only on

ε such that $H(x, X_0) \leq C$ for all $x \in \partial D$. Hence, we finally conclude

$$\left[\frac{H_\nu(x,\cdot)}{u_\nu(\cdot)}\right]_\alpha \leq M_\varepsilon.$$

We now come back to expression (5.12) with these facts discussed above and conclude u_ν^2 is a α-Hölder continuous and thus u_ν is $\frac{\alpha}{2}$-Hölder continuous. □

It follows now from [2] that the free boundary is a $C^{1,\alpha}$ surface in a neighborhood of any point of R. We observe furthermore that, if we call

$$h(y) := \int_{\partial D} \Gamma_t(x, u_\mu(x)) H(x,y) d\sigma(x),$$

one easily verifies that h is a positive harmonic function in U. Moreover h vanishes on ∂F and for any $y \in R$,

$$h_\nu(y) = \int_{\partial D} \Gamma_t(x, u_\mu(x)) H_\nu(x,y) d\sigma(x).$$

Finally we observe that it follows from our free boundary condition (5.11) that

$$h_\nu \cdot u_\nu \equiv C \text{ on } R.$$

We have verified all the hypothesis of Theorem 7.1 in [3] which provides the analyticity of the free boundary.

6. Recovering the original physical problem

In this section we shall relate a solution to the penalized problem (2.2) to a (possible) solution to our initial problem (2.1). The idea is that for $\varepsilon > 0$ small enough, any minimizer of J_ε actually satisfies $|\{u > 0\}| = 1$. Hence, any solution of problem (2.2) is a solution to our original problem.

Lemma 6.1. *There exist positive constants c and C, independent of ε, such that*

$$c \leq |\{u_\varepsilon > 0\}| \leq 1 + C\varepsilon$$

Proof. As we have already done before, let $\widetilde{D}$ be a smooth domain such that $D \subset \widetilde{D}$, with $|\widetilde{D} \setminus D| = 1$ and u_0 the harmonic function on $\widetilde{D} \setminus D$, such that $u_0 \equiv 1$ on ∂D, $u_0 \equiv 0$ on $\partial\widetilde{D}$. Therefore

$$C = J_\varepsilon(u_0) = \int_{\partial D} \Gamma\big(x, (u_0)_\mu\big) d\sigma + 1 \geq J_\varepsilon(u_\varepsilon), \qquad \forall \varepsilon > 0. \tag{6.1}$$

Thus

$$\frac{1}{\varepsilon}(|\{u_\varepsilon > 0\}| - 1) \leq f_\varepsilon(|\{u_\varepsilon > 0\}| \leq C.$$

This proves the estimate from above. Let us turn our attention to the estimate from below. It also follows from (6.1) that

$$\int_{\partial D} \Gamma\big(x, (u_\varepsilon)_\mu\big) d\sigma \leq C.$$

This together with Lemma 3.4 yields

$$\int_{D^C} |\nabla u_\varepsilon|^2 dx \leq C.$$

As usual let us denote $D_\delta := \{y \in D^C : \operatorname{dist}(y, \partial D) < \delta\}$. If δ is small enough, we can integrate along lines from ∂D and get

$$|\partial D|^2 \leq C(\delta) |D_\delta \cap \{u_\varepsilon > 0\}| \cdot \int_{D_\delta} \big(|\nabla u_\varepsilon|^2 + u_\varepsilon^2\big) dx.$$

This gives an estimate of $|\{u_\varepsilon > 0\}|$ from below. $\square$

Lemma 6.2. *There exists a universal constant C, such that $\inf_{R_\varepsilon}(u_\varepsilon)_\nu \le C$, for all $\varepsilon > 0$.*

Proof. As we have shown in the previous Lemma, there exists a universal constant C such that $C \ge J_\varepsilon(u_\varepsilon)$, for all $\varepsilon > 0$. In particular, using Jensen's inequality we get like in Theorem 3.5

$$\int_{\partial D} \Gamma\left(y, \fint_{\partial D}(u_\varepsilon)_\mu(x)d\sigma(x)\right)d\sigma(y) \le L\int_{\partial D}\Gamma\big(x,(u_\varepsilon)_\mu(x)\big)d\sigma(x). \tag{6.2}$$

Moreover

$$\int_{\partial D}(u_\varepsilon)_\mu d\sigma = -\int_{\partial F}(u_\varepsilon)_\nu dH^{n-1} \tag{6.3}$$

We recall that the isoperimetric inequality together with Lemma 6.1 gives a universal bound by below for $H^{n-1}(\partial F_\varepsilon)$. Combining this with (6.2) and (6.3) we conclude

$$\inf_{R_\varepsilon}(u_\varepsilon)_\nu \le C.$$

$\square$

Lemma 6.3. *There exists a universal positive constant $c > 0$, such that $(u_\varepsilon)_\nu \ge c$, for all $\varepsilon > 0$.*

Proof. Let $x_0 \in \partial F$. Going back into the proof of Theorem 4.1 we conclude, by balancing τ and ε, that there exists a universal constant $\kappa > 0$ such that

$$\fint_{\partial B(x_0,r)} u_\varepsilon d\sigma \ge \kappa \cdot r,$$

for all $\varepsilon > 0$. Let us, hereafter, write u instead of u_ε. Consider the harmonic function, v_0, in $B(x_0,r)$, taking boundary values equal to u. We extend v_0 by u outside of $B(x_0,r)$. Applying Lemma 3.8 we find

$$\int_{B_r(x_0)}\big(|\nabla u|^2 - |\nabla v_0|^2\big)dx \ge c|B(x_0,r)\cap\{u=0\}|,$$

where c is universal. Let x_1 be a regular free boundary point away from x_0, i.e., ∂F is smooth in $B(x_1,r_0)$, for some $r_0 > 0$. Following the idea of the Hadamard variational principle, near x_1 we make an inward smooth perturbation of the set $\{u > 0\}$, decreasing its volume by δ_r, where

$$\delta_r := |B(x_0,r)\cap\{u=0\}|.$$

Let P denote the perturbed set. Let v_1 be the harmonic function in P vanishing on its boundary and equal to u on $\partial B(x_1,r_0)$. Then by the Hadamard variational principle,

$$\int_{B(x_1,r_0)}\big(|\nabla v_1|^2 - |\nabla u|^2\big)dx = u_\nu^2(x_1)\delta_r + o(\delta_r).$$

Let v be the minimizer of the energy functional, subject to the constraints: $v = 1$ on ∂D and $v \le 0$ in $\big(\{u=0\}\setminus B_r(x_0)\big)\cup\big(P\cap\{u>0\}\big)$. In this way, $|\{v>0\}| = |\{u>0\}|$ and it competes with u in problem (2.2). Also we consider the function

$$\widehat{v} := \begin{cases} v_0 & \text{in } B(x_0,r) \\ v_1 & \text{in } B(x_1,r_0) \\ u & \text{elsewhere.}\end{cases}$$

We observe that $\widehat{v}$ competes against v in the energy problem. Moreover, the balls $B(x_0, r)$ and $B(x_1, r_0)$ are far from ∂D. Thus,

$$\begin{aligned} 0 &\leq \int_{\partial D} \Gamma(x, v_\mu) - \Gamma(x, u_\mu) d\sigma \leq C_\Gamma \int_{\partial D} v_\mu - u_\mu d\sigma \\ &\leq C_\Gamma \int_{D^C} |\nabla v|^2 - |\nabla u|^2 dx \\ &\leq \int_{B(x_0,r)} \left(|\nabla v_0|^2 - |\nabla u|^2\right) + \int_{B(x_1,r_0)} \left(|\nabla v_1|^2 - |\nabla u|^2\right) \\ &\leq -c\delta_r + u_\nu^2(x_1)\delta_r + o(\delta_r). \end{aligned}$$

This implies a universal lower bound for $u_\nu^2(x_1)$, i.e., $u_\nu \geq c$. □

Combining the two previous results we obtain

Theorem 6.4. *If ε is small enough, then any solution to problem (2.2) is a solution to problem (2.1).*

Proof. Suppose $|\{u_\varepsilon > 0\}| > 1$. We can make a inward perturbation of the set $\{u_\varepsilon > 0\}$ with volume change V, in such a way that the set of positivity of the new function, $\widetilde{u}_\varepsilon$ is still bigger than 1. Thus

$$f_\varepsilon(|\{\widetilde{u}_\varepsilon > 0\}|) - f_\varepsilon(|\{u_\varepsilon > 0\}|) = -\frac{1}{\varepsilon} V.$$

Such a inward perturbation is made around a point $x \in R$ such that $u_\nu(x) < 2 \inf_R u_\nu$. By Hadamard's variational principle and Lemma 6.2 we have

$$\begin{aligned} \int_{D^C} |\nabla \widetilde{u}_\varepsilon|^2 - |\nabla u|^2 &= u_\nu^2(x) V + o(V) \\ &\leq C^2 V + o(V). \end{aligned}$$

Hence,

$$\begin{aligned} 0 &\leq \int_{\partial D} \Gamma\big(x, (\widetilde{u}_\varepsilon)_\mu\big) - \Gamma(x, u_\mu) d\sigma + f_\varepsilon(|\{\widetilde{u}_\varepsilon > 0\}|) - f_\varepsilon(|\{u_\varepsilon > 0\}|) \\ &\leq C_\Gamma \int_{D^C} |\nabla \widetilde{u}_\varepsilon|^2 - |\nabla u|^2 dx - \frac{1}{\varepsilon} V \\ &\leq C_\Gamma^2 V + o(V) - \frac{1}{\varepsilon} V. \end{aligned}$$

Therefore, $\varepsilon > \varepsilon_\Gamma$. If $|\{u_\varepsilon > 0\}| < 1$, we argue similarly, and again we get an lower bound for ε. Thus if ε is small enough $|\{u_\varepsilon > 0\}|$ automatically adjusts to be equal to 1. □

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References

[1] N. Aguilera, H. Alt and L. Caffarelli, *An optimization problem with volume constraint*, SIAM J. Control Optim. **24** (1986), no. 2, 191–198.

[2] H. Alt and L. Caffarelli, *Existence and regularity for a minimum problem with regularity*, J. Reine Angew. Math. **325** (1981), 105-144.

[3] N. E. Aguilera, L. A. Caffarelli and J. Spruck, *An optimization problem in heat conduction*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **14** (1987), no. 3, 355–387 (1988).

[4] L. A. Caffarelli, *The obstacle problem revisited*, J. Fourier Anal. Appl. **4** (1998), no. 4-5, 383–402.

[5] Luis A. Caffarelli and Xavier Cabré, *Fully nonlinear elliptic equations*, American Mathematical Society Colloquium Publications, 43. American Mathematical Society, Providence, RI, 1995.

[6] Luis A. Caffarelli, David Jerison and Carlos E. Kenig, *Some new monotonicity theorems with applications to free boundary problems* Ann. of Math. (2) **155** (2002), no. 2, 369–404.

[7] D. Jerison and C. Kenig, *Boundary behavior of Harmonic functions in non-tangentially accessible domains.* Adv. in Math. **46** (1982), pp. 80-147.

[8] D. Kinderlehrer and L. Nirenberg, *Regularity in free boundary problems.* Ann. Scuola Norm. Sup. Pisa (4) **4** (1977), 373-391.

Department of Mathematics, University of Texas at Austin, RLM 9.136, Austin, Texas 78712-1082.

E-mail address: teixeira@math.utexas.edu