Scattering of Dirac Particles by Electromagnetic Fields with Small Support in Two Dimensions and Effect from Scalar Potentials

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Abstract We study the asymptotic behavior of scattering amplitudes for the scattering of Dirac particles in two dimensions when electromagnetic fields with small support shrink to point–like fields. The result is strongly affected by perturbations of scalar potentials and the asymptotic form changes discontinuously at half–integer fluxes of magnetic fields even for small perturbations. The analysis relies on the behavior at low energy of resolvents of magnetic Schrödinger operators with resonance at zero energy. The magnetic scattering of relativistic particles appears in the interaction of cosmic string with matter. We discuss this closely related subject as an application of the obtained results.

1. Introduction

We consider the relativistic massless particle moving in the two dimensional space. We denote by $x = (x_1, x_2)$ a generic point in $\boldsymbol{R}^2$ and write

$$D(A,V) = \sum_{j=1}^{2} \sigma_j \left(-i\partial_j - A_j\right) + V, \qquad \partial_j = \partial/\partial x_j,$$

for the Dirac operator, where $A = (A_1, A_2) : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ and $V : \boldsymbol{R}^2 \to \boldsymbol{R}$ are magnetic and scalar potentials respectively, and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli spin matrices. The magnetic field $b : \boldsymbol{R}^2 \to \boldsymbol{R}$ is defined by

$$b = \nabla \times A = \partial_1 A_2 - \partial_2 A_1.$$

The operator $D(A,V)$ acts on $[L^2]^2 = [L^2(\boldsymbol{R}^2)]^2$. If A and V are bounded, then it is self–adjoint with domain $[H^1(\boldsymbol{R}^2)]^2$, where $H^s(\boldsymbol{R}^2)$ is the Sobolev space of order s. We also write

$$L(A,V) = (-i\nabla - A)^2 + V$$

for the Schrödinger operator. If A has further bounded derivatives, then $L(A, V)$ is self–adjoint with domain $H^2(\boldsymbol{R}^2)$ in L^2. If $L(A, V)u = 0$ has a bounded but not square integrable solution, then $L(A, V)$ is said to have a resonance at zero energy.

Let b and V be given magnetic field and scalar potential. We assume that $b,\ V \in C_0^\infty(\boldsymbol{R}^2 \to \boldsymbol{R})$ are smooth functions with compact support. We define $A(x)$ by

$$A(x) = (-\partial_2\varphi(x), \partial_1\varphi(x))\,, \tag{1.1}$$

where

$$\varphi(x) = (2\pi)^{-1} \int \log\, |x - y|\, b(y)\, dy \tag{1.2}$$

and the integration without the domain attached is taken over the whole space. By definition, A satisfies $\nabla \times A = \Delta\varphi = b$, and hence it becomes the potential associated with field b. The function φ obeys $\varphi(x) = \alpha \log\, |x| + O(|x|^{-1})$ as $|x| \to \infty$, where

$$\alpha = (2\pi)^{-1} \int b(x)\, dx$$

is called the flux of b. The magnetic effect strongly appears when $\alpha \notin \boldsymbol{Z}$ is not an integer. We restrict ourselves to the case

$$0 < \alpha < 1. \tag{1.3}$$

We make a brief comment on the the other cases that $\alpha < 0$ and $\alpha > 1$ (Remark 8.1 at the end of section 8). The potential $A(x)$ is not necessarily expected to fall off rapidly and it has the long–range property at infinity even if b is of compact support. In fact, it behaves like

$$A(x) = A_{0\alpha}(x) + O(|x|^{-2}), \tag{1.4}$$

where $A_{0\alpha}$ is defined by

$$A_{0\alpha}(x) = \alpha(-x_2/|x|^2, x_1/|x|^2) = \alpha(-\partial_2 \log |x|, \partial_1 \log |x|) \tag{1.5}$$

and it is often called the Aharonov–Bohm potential in physical articles.

Let $T = D(A, V) = T_0 + V$, where

$$T_0 = D(A, 0) = \sigma_1\nu_1 + \sigma_2\nu_2, \quad (\nu_1, \nu_2) = -i\nabla - A,$$

is the Dirac operator without scalar potential V. We sometimes identify the coordinates $\omega = (\omega_1, \omega_2)$ over the unit circle S with the azimuth angle from the positive x_1 axis. According to this notation, we set

$$\tau(\omega) = {}^{\mathrm{t}}(1, e^{i\omega}), \qquad e^{i\omega} = \cos\omega + i \sin\omega = \omega_1 + i\omega_2. \tag{1.6}$$

We denote by $f(\omega \to \tilde{\omega}; E)$ the scattering amplitude of T for scattering from initial direction $\omega \in S$ to final one $\tilde{\omega}$ at energy $E > 0$. Roughly speaking, it is defined through the behavior at infinity of solution $\psi = \psi(x; E, \omega)$ to equation $T\psi = E\psi$, and the solution takes the asymptotic form

$$\psi(r\tilde{\omega}) \sim \psi_{\mathrm{in}} + f(\omega \to \tilde{\omega}; E)\tau(\tilde{\omega})e^{iEr}r^{-1/2}, \qquad r = |x| \to \infty,$$

along direction $\tilde{\omega} \neq \omega$, where the first term $\psi_{\mathrm{in}} = \tau(\omega)e^{iEx\cdot\omega}$ is the wave incident from ω and the second term denotes the scattering wave. The precise representation of it is given in section 4. We study the scattering by electromagnetic fields with small support. We set

$$A_\varepsilon(x) = \varepsilon^{-1}A(x/\varepsilon), \quad b_\varepsilon(x) = \varepsilon^{-2}b(x/\varepsilon), \quad V_\varepsilon(x) = \varepsilon^{-1}V(x/\varepsilon) \tag{1.7}$$

for $0 < \varepsilon \ll 1$ small enough. Then A_ε satisfies $\nabla \times A_\varepsilon = b_\varepsilon$. Our aim here is to analyze the asymptotic behavior as $\varepsilon \to 0$ of amplitude $f_\varepsilon(\omega \to \tilde{\omega}; E)$ of $T_\varepsilon = D(A_\varepsilon, V_\varepsilon)$.

The problem is closely related to the resonance state at zero energy of magnetic Schrödinger operators in a natural way. Let $R(z; H)$ denote the resolvent $(H - z)^{-1}$ of self–adjoint operator H. We write $T_0 = \sigma_1\nu_1 + \sigma_2\nu_2$ as

$$T_0 = \begin{pmatrix} 0 & \nu_1 - i\nu_2 \\ \nu_1 + i\nu_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \nu_- \\ \nu_+ & 0 \end{pmatrix},$$

where $(\nu_1, \nu_2) = -i\nabla - A$ with $A = (-\partial_2\varphi(x), \partial_1\varphi(x))$, φ being defined by (1.2). Since ν_1 and ν_2 satisfies the commutator relation

$$[\nu_1, \nu_2] = \nu_1\nu_2 - \nu_2\nu_1 = ib,$$

a simple computation yields

$$\nu_\pm\nu_\mp = \nu_1^2 + \nu_2^2 \pm b = L(A, \pm b),$$

so that T_0^2 is diagonalized as

$$T_0^2 = \begin{pmatrix} L(A, -b) & 0 \\ 0 & L(A, b) \end{pmatrix}.$$

The two Schrödinger operators $L(A, \pm b) = \nu_\mp^*\nu_\mp \geq 0$ are non–negative, but the spectral structure at zero energy is different. By (1.1), we have

$$\begin{aligned} \nu_+ &= \nu_1 + i\nu_2 = -i\partial_1 + \partial_2\varphi + i(-\partial_2 - \partial_1\varphi) \\ &= -i\left((\partial_1 + \partial_1\varphi) + i(\partial_2 + \partial_2\varphi)\right) = -ie^{-\varphi}\left(\partial_1 + i\partial_2\right)e^{\varphi}. \end{aligned} \tag{1.8}$$

Hence $L(A, -b)u = 0$ has a bounded solution behaving like

$$\rho(x) = e^{-\varphi(x)} = |x|^{-\alpha}(1 + O(|x|^{-1})), \qquad |x| \to \infty. \tag{1.9}$$

By assumption (1.3), ρ is not in L^2, and hence $L(A,-b)$ has a resonance state at zero energy. On the other hand, $L(A,b)$ does not have a resonance state. The amplitude f_ε is represented in terms of the boundary values

$$R(E+i0;T_\varepsilon)=\lim_{\delta\downarrow 0}R(E+i\delta;T_\varepsilon)$$

to the real axis of resolvent $R(E+i\delta;T_\varepsilon)$. We now define the unitary operator $J_\varepsilon:[L^2]^2\to[L^2]^2$ by

$$(J_\varepsilon u)(x)=\varepsilon^{-1}u(x/\varepsilon), \tag{1.10}$$

then we have $T_\varepsilon=\varepsilon^{-1}J_\varepsilon TJ_\varepsilon^*$ for $T=D(A,V)$, and hence

$$R(E+i0;T_\varepsilon)=\varepsilon J_\varepsilon R(k+i0;T)J_\varepsilon^*,\quad k=\varepsilon E. \tag{1.11}$$

Thus the analysis relies on the behavior at low energy of resolvents

$$R(k+i0;T_0)=(T_0+k)R(k^2+i0;T_0)$$

and $R(k+i0;T)$, and a basic role is played by the zero energy resonance of the magnetic Schrödinger operator $L(A,-b)$. We note that there is no fear of our confusing the operator J_ε with the Bessel function $J_\nu(x)$ in the argument below.

We take the limit $\varepsilon\to 0$ in a formal way. It follows from (1.4) that A_ε is convergent to the Aharonov–Bohm potential $A_{0\alpha}(x)$, and hence

$$T_\varepsilon=D(A_\varepsilon,V_\varepsilon)\to D_\alpha=D(A_{0\alpha},0) \tag{1.12}$$

on $[C_0^\infty(\boldsymbol{R}^2\setminus\{0\})]^2$. However $A_{0\alpha}$ is strongly singular at the origin, and it has the δ–like field $2\pi\alpha\delta(x)$ as a magnetic field. We know ([14, 19, 21]) that D_α is not essentially self–adjoint and it has the deficiency indices (1,1). According to the Krein theory, we can obtain a family of self–adjoint extensions $\{H_\kappa\}$ with one real parameter κ, $-\infty<\kappa\le\infty$. The element $u={}^t(u_1,u_2)$ in the domain $\mathcal{D}(H_\kappa)$ is specified by the boundary condition

$$u_{-1}+i\kappa\, u_{-2}=0 \tag{1.13}$$

at the origin under assumption (1.3), where

$$u_{-1}=\lim_{r\to 0}r^\alpha u_1(x),\qquad u_{-2}=\lim_{r\to 0}r^{1-\alpha}e^{-i\theta}u_2(x) \tag{1.14}$$

in the polar coordinate system (r,θ). If $\kappa=\infty$, then $u_{-2}=0$ and the second component $u_2(x)$ has a weak singularity near the origin for $u\in\mathcal{D}(H_\infty)$, while the first component $u_1(x)$ has a weak singularity for $\kappa=0$. The boundary condition in which both components remain bounded is not in general allowed ([14, 19]). In section 2, we explicitly calculate the amplitude of H_κ after discussing the problem of self–adjoint extension in some detail.

The amplitude f_ε in question is expected to converge to that of H_κ for some κ. We state the obtained results somewhat loosely. All the main theorems are formulated in section 5. We denote by $g_\kappa(\omega \to \tilde{\omega}; E)$ the scattering amplitude of H_κ. As stated above, g_κ can be calculated explicitly. If the scalar potential $V(x)$ vanishes identically, then f_ε is shown to converge to g_∞ (Theorem 5.1). However the situation changes as soon as V is added as a perturbation (Theorem 5.2). It is interesting that this occurs even for small perturbations. We here deal with only the simple but generic case that T has neither bound state nor resonance state at zero energy. The definition of resonance state is given in section 5. Roughly speaking, it means that the equation $Tu = 0$ admits a bounded solution. We note that T does not have a resonance state for V small enough. The obtained result depends on the flux α of field b. The amplitude f_ε is proved to converge to g_∞ for $0 < \alpha < 1/2$ and to g_0 for $1/2 < \alpha < 1$. If $\alpha = 1/2$, then f_ε is convergent to g_κ for some κ determined from the resonance state $\rho = e^{-\varphi}$ of $L(A, -b)$. A similar problem has been studied by the physical literature [2, section 7.10] for the scattering outside the small disk $\{|x| < \varepsilon\}$, and it has shown that the limit takes a different form according as $0 < \alpha < 1/2$, $\alpha = 1/2$ or $1/2 < \alpha < 1$. However the argument there is based on the explicit calculation using the Bessel functions, and the connection with zero energy resonance has not been recognized.

As stated in the beginning, another motivation of this work comes from the study on the scattering of Dirac particles in the interaction of cosmic string with matter. This problem is mathematically formulated as follows (see [7] for the detail on the physical background). Let A_ε, $b_\varepsilon = \nabla \times A_\varepsilon$ and V_ε be defined by (1.7). We consider two kinds of particles (for example, lepton and quark) moving in the magnetic field b_ε and interacting with each other through the scalar potential V_ε. If we denote by $w = {}^{\mathrm{t}}(u, v) = {}^{\mathrm{t}}(u_1, u_2, v_1, v_2)$ the wave function of these two particles, then w obeys the equation

$$\mathbf{T}_\varepsilon w = \mathbf{T}_{0\varepsilon} w + \mathbf{V}_\varepsilon w = Ew \tag{1.15}$$

at energy $E > 0$, where

$$\mathbf{T}_{0\varepsilon} = \begin{pmatrix} T_{0\varepsilon} & 0 \\ 0 & T_{0\varepsilon} \end{pmatrix}, \quad \mathbf{V}_\varepsilon = \begin{pmatrix} 0 & V_\varepsilon \\ V_\varepsilon & 0 \end{pmatrix}, \quad T_{0\varepsilon} = D(A_\varepsilon, 0).$$

We assume that the wave function w has only u–wave as an incident wave. Then w behaves like

$$w \sim {}^{\mathrm{t}}(\tau(\omega), 0) e^{iEx\cdot\omega} + w_{\mathrm{scat}} + o(r^{-1/2}), \qquad r \to \infty,$$

where $\tau(\omega)$ is defined by (1.6), and the scattering wave w_{scat} takes the form

$$w_{\mathrm{scat}} = \left(f_{1\varepsilon}(\omega \to \tilde{\omega}; E)\,{}^{\mathrm{t}}(\tau(\tilde{\omega}), 0) + f_{2\varepsilon}(\omega \to \tilde{\omega}; E)\,{}^{\mathrm{t}}(0, \tau(\tilde{\omega})) \right) e^{iEr} r^{-1/2} \tag{1.16}$$

along direction $\tilde{\omega}$. The amplitude $f_{2\varepsilon}(\omega \to \tilde{\omega}; E)$ describes the v–wave produced by incident u–wave, and it is an important physical quantity in the interaction of cosmic string with matter. We analyze the asymptotic behavior as $\varepsilon \to 0$ of $f_{2\varepsilon}(\omega \to \tilde{\omega}; E)$. The asymptotic form is shown to take the form

$$f_{2\varepsilon}(\omega \to \tilde{\omega}; E) = C_\alpha \varepsilon^{|2\alpha-1|}\left(1 + o(1)\right), \qquad \varepsilon \to 0,$$

for some constant C_α (Theorem 5.3). The constant is independent of incident and final directions ω and $\tilde{\omega}$, but is different according as $0 < \alpha < 1/2$, $\alpha = 1/2$ or $1/2 < \alpha < 1$. A similar asymptotic form has been derived by the earlier work [7] in the special case that $A(x) = A_{0\alpha}(x)$ is the Aharonov–Bohm potential and $V(x)$ is the characteristic function of the unit disk. However the calculation there is again based on the explicit calculation using the Bessel functions, and the important role of zero energy resonance seems to have been completely hidden behind this explicit calculation. In this work we make clear from a mathematical point of view how the leading coefficient C_α is determined and how it is related to the resonance state ρ of $L(A, -b)$ at zero energy.

We confine ourselves to the positive energy case $E > 0$ for notational brevity, and we fix $E > 0$ throughout the whole exposition. The dependence on E does not matter. We end the section by noting that the obtained results easily extend to the operator $\sigma_1\nu_1 + \sigma_2\nu_2 + m\sigma_3 + V$ with mass $m > 0$.

2. Dirac operators with point–like fields

In this section we calculate the scattering amplitude $g_\kappa(\omega \to \tilde{\omega}; E)$ of self–adjoint extension H_κ obtained from D_α defined by (1.12) after explaining briefly the Krein theory on the problem of self–adjoint extension. The problem of self–adjoint extension for two dimensional Dirac operators with singular magnetic fields has already been studied by several authors. We refer to [14, 19, 21] for details, and, in particular, to [21] for the recent references. The argument here follows [23].

The operator

$$D_\alpha = D(A_{0\alpha}, 0) = \begin{pmatrix} 0 & \pi_- \\ \pi_+ & 0 \end{pmatrix}, \qquad \pi_\pm = \pi_1 \pm i\,\pi_2, \tag{2.1}$$

defined over $\left[C_0^\infty(\boldsymbol{R}^2 \setminus \{0\})\right]^2$ is symmetric, where $(\pi_1, \pi_2) = -i\nabla - A_{0\alpha}$. The two operators $\pi_\pm$ are represented as

$$\pi_+ = e^{i\theta}\left(-i\partial_r + r^{-1}(\partial_\theta - i\alpha)\right), \quad \pi_- = e^{-i\theta}\left(-i\partial_r - r^{-1}(\partial_\theta - i\alpha)\right) \tag{2.2}$$

in terms of polar coordinates (r, θ), and we have

$$\pi_+\pi_- = \pi_1^2 + \pi_2^2 = -\partial_r^2 - r^{-1}\partial_r + r^{-2}\left(-i\partial_\theta - \alpha\right)^2$$

for $r = |x| > 0$, and similarly for $\pi_-\pi_+$. We denote by $\overline{D}_\alpha$ and D_α^* the closure and adjoint of D_α respectively, and we set

$$\Sigma_\pm = \{u \in [L^2]^2 : (D_\alpha^* \mp i)\,u = 0\}.$$

The pair (n_+, n_-), $n_\pm = \dim \Sigma_\pm$, is called the deficiency indices of D_α. As is well known, D_α has self–adjoint extensions if and only if $n_+ = n_-$.

We show that $n_+ = n_- = 1$. We denote by $H_\mu(z) = H_\mu^{(1)}(z)$ the Hankel function of first kind, and all the Hankel functions are understood to be of first kind throughout. If $u = {}^{\mathrm{t}}(u_1, u_2) \in [L^2]^2$ solves $(D_\alpha - i)\, u = 0$, then u_2 satisfies $(\pi_+\pi_- + 1)\, u_2 = 0$ in $\boldsymbol{R}^2 \setminus \{0\}$, and u_1 is given by $u_1 = -i\pi_- u_2$. By formula, $H_\mu(z)$ satisfies

$$(d/dz)\left[z^{\pm\mu} H_\mu(az)\right] = \pm a z^{\pm\mu} H_{\mu\mp 1}(az). \tag{2.3}$$

The same formula is still true for $J_\mu(z)$. This formula yields

$$\pi_-\left(H_{1-\alpha}(ir)e^{i\theta}\right) = H_{-\alpha}(ir) = e^{i\alpha\pi} H_\alpha(ir).$$

Hence we see that Σ_+ is the one dimensional space spanned by

$$u_+ = N_\alpha {}^{\mathrm{t}}(-ie^{i\alpha\pi} H_\alpha(ir), H_{1-\alpha}(ir)e^{i\theta}),$$

where u_+ is normalized as $\|u_+\|_{L^2} = 1$. Similarly Σ_- is also the one dimensional space spanned by

$$u_- = N_\alpha {}^{\mathrm{t}}(ie^{i\alpha\pi} H_\alpha(ir), H_{1-\alpha}(ir)e^{i\theta}), \qquad \|u_-\|_{L^2} = 1.$$

All the possible self–adjoint extensions are determined by the Krein theory ([8, 20]). Let $U : \Sigma_+ \to \Sigma_-$ be the unitary mapping defined by multiplication $Uu_+ = e^{i\zeta} u_-$ with $-\pi < \zeta \le \pi$. Then the self–adjoint extension H_U associated with U is realized as the operator

$$H_U u = \overline{D}_\alpha v + icu_+ - ice^{i\zeta} u_-$$

acting on the domain

$$\mathcal{D}(H_U) = \{u \in [L^2]^2 : u = v + cu_+ + ce^{i\zeta} u_-,\ \ v \in \mathcal{D}(\overline{D}_\alpha),\ \ c \in \boldsymbol{C}\}.$$

We examine which boundary condition $u \in \mathcal{D}(H_U)$ satisfies at the origin. The Hankel function $H_\mu(z)$ with non–integer $\mu > 0$ is represented as

$$H_\mu(z) = (i/\sin \mu\pi)\left(e^{-i\mu\pi} J_\mu(z) - J_{-\mu}(z)\right) \tag{2.4}$$

in terms of Bessel functions, and it behaves like

$$H_\mu(z) = (-i/\sin \mu\pi)\,(2^\mu/\Gamma(1-\mu))\, z^{-\mu}\left(1 + O(|z|^{2\mu}) + O(|z|^2)\right) \tag{2.5}$$

as $|z| \to 0$. If $v = {}^{\mathrm{t}}(v_1, v_2) \in \mathcal{D}(\overline{D}_\alpha)$, then v obeys $v_1 = o(|x|^{-\alpha})$ and $v_2 = o(|x|^{-(1-\alpha)})$ as $|x| \to 0$, so that $u = {}^{\mathrm{t}}(u_1, u_2) \in \mathcal{D}(H_U)$ has the limits u_{-1} and u_{-2} in (1.14). If we take account of the above asymptotic formula of Hankel functions, then the ratio

$$\kappa = iu_{-1}/u_{-2} = \left(2^{2\alpha-1}\Gamma(\alpha)/\Gamma(1-\alpha)\right)\tan(\zeta/2)$$

is calculated as a quantity independent of u. Thus we obtain the family of self–adjoint extensions $\{H_\kappa\}$ parameterized by real number κ, $-\infty < \kappa \le \infty$, and the operator has the domain

$$\mathcal{D}(H_\kappa) = \{u = (u_1, u_2) \in [L^2]^2 : D_\alpha u \in [L^2]^2,\ u_{-1} + i\kappa u_{-2} = 0\}, \qquad (2.6)$$

where $D_\alpha u$ is understood in the distribution sense, and u_{-1} and u_{-2} are defined by (1.14).

We move to calculating the scattering amplitude of H_κ. It has already been calculated in the physical articles ([17]) for the special case $\kappa = 0$ or $\kappa = \infty$. We again note that $\omega \in S$ is often identified with the azimuth angle from the positive x_1 axis.

Proposition 2.1 *Let $g_\kappa(\omega \to \tilde{\omega}; E)$, $\tilde{\omega} \ne \omega$, denote the scattering amplitude of H_κ for the scattering from initial direction ω into final one $\tilde{\omega}$ at energy $E > 0$. Then*

$$g_\kappa = -(2\pi i E)^{-1/2} \sin \alpha\pi \left(\frac{e^{i(\tilde{\omega}-\omega)/2}}{\sin((\tilde{\omega}-\omega)/2)} + \frac{2\kappa\tau_\alpha E^{2\alpha-1}}{i(\kappa\tau_\alpha E^{2\alpha-1} - e^{i\alpha\pi})} \right), \qquad (2.7)$$

where

$$\tau_\alpha = 2^{1-2\alpha}\Gamma(1-\alpha)/\Gamma(\alpha). \qquad (2.8)$$

If, in particular, $\kappa = 0$ or $\kappa = \infty$, then

$$\begin{aligned} g_0 &= -(2\pi i E)^{-1/2} \sin \alpha\pi \frac{e^{i(\tilde{\omega}-\omega)/2}}{\sin((\tilde{\omega}-\omega)/2)}, \\ g_\infty &= -(2\pi i E)^{-1/2} \sin \alpha\pi \frac{e^{-i(\tilde{\omega}-\omega)/2}}{\sin((\tilde{\omega}-\omega)/2)}, \end{aligned}$$

and if $\alpha = 1/2$, then

$$g_\kappa = -(2\pi i E)^{-1/2} \left(\frac{e^{i(\tilde{\omega}-\omega)/2}}{\sin((\tilde{\omega}-\omega)/2)} + \frac{2\kappa}{1+i\kappa} \right).$$

We need two lemmas to prove the proposition. Before stating the lemmas, we briefly discuss the problem of self–adjoint extensions for magnetic Schrödinger operator

$$L_\alpha = L(A_{0\alpha}, 0) = (-i\nabla - A_{0\alpha})^2 \qquad (2.9)$$

with Aharonov–Bohm potential $A_{0\alpha}$. We know ([1, 13]) that L_α has the deficiency indices (2,2) as a symmetric operator on $C_0^\infty(\boldsymbol{R}^2 \setminus \{0\})$, and the Krein theory again yields the family of all possible self–adjoint extensions $\{L_U\}$ parameterized by 2×2 unitary mapping U from one deficiency subspace to the other one. The self–adjoint operator L_U is realized as a differential operator with some boundary conditions at the origin. If w is in the domain $\mathcal{D}(L_U)$, then w behaves like

$$w = \left(w_{-0}r^{-\alpha} + w_{+0}r^{\alpha} + o(r^{\alpha})\right) + \left(w_{-1}r^{-(1-\alpha)} + w_{+1}r^{1-\alpha} + o(r^{1-\alpha})\right)e^{i\theta} + o(r)$$

for some coefficients $w_{\pm k},\ k = 0, 1$, and there exist 2×2 matrices $B_\pm$ for which the boundary condition is described as the relation

$$B_- \begin{pmatrix} w_{-0} \\ w_{-1} \end{pmatrix} + B_+ \begin{pmatrix} w_{+0} \\ w_{+1} \end{pmatrix} = 0$$

between these four coefficients. We distinguish the two operators by the following special notation :

$$\begin{aligned} \mathcal{D}(L_{AB}) &= \{w \in L^2 : Lw \in L^2,\ \ w_{-0} = w_{-1} = 0\} \\ \mathcal{D}(L_Z) &= \{w \in L^2 : Lw \in L^2,\ \ w_{+0} = w_{-1} = 0\} \end{aligned} \tag{2.10}$$

among admissible self–adjoint extensions. The first operator L_{AB} is known as the Aharonov–Bohm Hamiltonian ([3]).

We denote by $\gamma(x;\omega)$ the azimuth angle from ω. The operator L_α defined by (2.9) admits the polar coordinate decomposition

$$L_\alpha \simeq \sum_{l \in Z} \oplus\, h_l,$$

where $h_l = -(d/dr)^2 + (\nu^2 - 1/4)r^{-2}$ with $\nu = |l - \alpha|$. If we define

$$\varphi_\pm(x; E, \omega) = \sum_{l \in Z} e^{\mp i\nu\pi/2} e^{il\gamma(x;\mp\omega)} J_\nu(Er) \tag{2.11}$$

for $\nu = |l-\alpha|$, then $\varphi_\pm$ vanishes at the origin and solves $(L_\alpha - E^2)\,\varphi_\pm = 0$. Thus $\varphi_\pm$ becomes the generalized eigenfunction of L_{AB} with eigenvalue E^2. The first lemma is due to [16] (see [3, 10] also).

Lemma 2.1 *Let $\varphi_+(x; E, \omega)$ be as above. Define*

$$\varphi_{\rm in}(x; E, \omega) = e^{iEx\cdot\omega} e^{i\alpha(\gamma(x;\omega)-\pi)} \tag{2.12}$$

for $x = r\theta,\ \theta \neq \omega$. Then $\varphi_+(x; E, \omega)$ obeys

$$\varphi_+(r\theta; E, \omega) = \varphi_{\rm in}(r\theta; E, \omega) + g_+(\omega \to \theta; E)e^{iEr}r^{-1/2}\left(1 + o(1)\right), \quad r \to \infty,$$

along direction θ, where

$$g_+(\omega \to \theta; E) = -\left(2\pi i E\right)^{-1/2} \sin\alpha\pi \frac{e^{i(\theta-\omega)/2}}{\sin((\theta-\omega)/2)}. \tag{2.13}$$

This lemma implies that $\varphi_+(x;E,\omega)$ is the outgoing eigenfunction of L_{AB}, and $g_+(\omega \to \theta; E)$ defines the scattering amplitude. This is known as the Aharonov–Bohm scattering amplitude ([3]). On the other hand, $\varphi_-(x;E,\omega)$ is shown to be the incoming eigenfunction, but its asymptotic form is not required in the argument below. We move to the second lemma. The proof of this lemma uses the following formula for the Bessel functions :

$$\pi_\pm\left(J_\nu(Er)e^{il\theta}\right) = \begin{cases} \pm iEJ_{\nu\pm1}(Er)e^{i(l\pm1)\theta} & (l \ge 1) \\ \mp iEJ_{\nu\mp1}(Er)e^{i(l\pm1)\theta} & (l \le 0) \end{cases} \tag{2.14}$$

for $\nu = |l-\alpha|$ with $0<\alpha<1$. This follows from (2.3) after a direct computation. The same formula remains true for the Hankel $H_\nu(Er)$.

Lemma 2.2 *Let π_+ be as in (2.2) and let g_+ be as in Lemma 2.1. Then*

$$(\pi_+\varphi_+)(r\theta;E,\omega) = Ee^{i\omega}\varphi_{\rm in}(r\theta;E,\omega) + Ee^{i\theta}g_+(\omega\to\theta;E)e^{iEr}r^{-1/2}(1+o(1))$$

as $r\to\infty$ along direction θ, $\theta\ne\omega$.

Proof. We calculate $I = (\pi_+\varphi_+)(x;E,\omega)/E$. Since $e^{il\gamma(x;-\omega)} = e^{il\theta}e^{il(\pi-\omega)}$ for $x = r\theta$, we obtain

$$I = \sum_{l\ge1} ie^{-i\nu\pi/2}J_{\nu+1}(Er)e^{i(l+1)\theta}e^{il(\pi-\omega)} - \sum_{l\le0} ie^{-i\nu\pi/2}J_{\nu-1}(Er)e^{i(l+1)\theta}e^{il(\pi-\omega)}$$

by use of formula (2.14). We use the simple relation

$$e^{i(l+1)\theta}e^{il(\pi-\omega)} = -e^{i(l+1)\gamma(x;-\omega)}e^{i\omega}.$$

If $l\ge1$, then $\nu+1 = |l+1-\alpha|$ and $ie^{-i\nu\pi/2} = -e^{-i|l+1-\alpha|\pi/2}$, and if $l\le-1$, then $\nu-1 = |l+1-\alpha|$ and $ie^{-i\nu\pi/2} = e^{-i|l+1-\alpha|\pi/2}$. If we take account of these relations, then we make a change of variables $l+1\to l$ to obtain that

$$I = e^{i\omega}\sum_{l\ne1} e^{-i\nu\pi/2}e^{il\gamma(x;-\omega)}J_\nu(Er) - e^{-i(\alpha-1)\pi/2}J_{\alpha-1}(Er)e^{i\theta},$$

so that it equals

$$I = e^{i\omega}\varphi_+(x;E,\omega) + \left(e^{-i(1-\alpha)\pi/2}J_{1-\alpha}(Er) - e^{-i(\alpha-1)\pi/2}J_{\alpha-1}(Er)\right)e^{i\theta}.$$

Hence it follows from (2.4) that

$$I = e^{i\omega}\varphi_+(x;E,\omega) + e^{-i\alpha\pi/2}\sin\alpha\pi H_{1-\alpha}(Er)e^{i\theta}. \tag{2.15}$$

The Hankel function $H_\mu(z)$, $\mu>0$, is known to behave like

$$H_\mu(z) = (2/i\pi)^{1/2}e^{-i\mu\pi/2}e^{iz}z^{-1/2}\left(1+O(|z|^{-1})\right) \tag{2.16}$$

as $|z| \to \infty$. This, together with Lemma 2.1, implies that

$$I = e^{i\omega}\varphi_{\rm in}(x;E,\omega) + \tilde{g}(\omega \to \theta; E)e^{iEr}r^{-1/2}\left(1 + o(1)\right),$$

where

$$\tilde{g} = e^{i\omega}g_+(\omega \to \theta; E) - 2i\left(2\pi i E\right)^{-1/2}\sin\alpha\pi e^{i\theta}.$$

A simple computation yields

$$\tilde{g} = \left(2\pi i E\right)^{-1/2}\sin\alpha\pi\left(-e^{-i(\theta-\omega)/2}/\sin((\theta-\omega)/2) + 2/i\right)e^{i\theta} = g_+(\omega \to \theta; E)e^{i\theta}.$$

This proves the lemma. $\square$

Proof of Proposition 2.1. Let $D_\alpha = D(A_{0\alpha}, 0)$ be as in (2.1). We look for the solution $\psi = (\psi_1, \psi_2)$ to equation $(D_\alpha - E)\,\psi = 0$ in the form

$$\psi_1 = \varphi_+(x;E,\omega) + \beta_\kappa H_\alpha(Er), \quad \psi_2 = (1/E)\left(\pi_+\psi_1\right)(x;E,\omega) \tag{2.17}$$

with some constant β_κ. If ψ takes the above form, then it is easy to see that ψ solves the equation. The coefficient β_κ is determined so as to satisfy the boundary condition (1.13) at the origin. Then $\psi = \psi(x;E,\omega)$ becomes the eigenfunction of self–adjoint operator H_κ and the amplitude g_κ is determined through the asymptotic form of $\psi(x;E,\omega)$. We calculate the limits u_{-1} and u_{-2} defined by (1.14). The eigenfunction φ_+ of L_{AB} vanishes at the origin, so that

$$u_{-1} = \lim_{r\to 0} r^\alpha\psi_1 = \beta_\kappa\left(-i/\sin\alpha\pi\right)\left(2^\alpha/\Gamma(1-\alpha)\right)E^{-\alpha}$$

by (2.5). Since

$$\pi_+H_\alpha(Er) = -iEH_{\alpha-1}(Er)e^{i\theta} = iEe^{-i\alpha\pi}H_{1-\alpha}(Er)e^{i\theta}$$

by (2.14), it follows from (2.15) that

$$\psi_2 = e^{i\omega}\varphi_+(x;E,\omega) + \left(e^{-i\alpha\pi/2}\sin\alpha\pi + ie^{-i\alpha\pi}\beta_\kappa\right)H_{1-\alpha}(Er)e^{i\theta} \tag{2.18}$$

and hence

$$u_{-2} = \left(-i/\sin\alpha\pi\right)\left(e^{-i\alpha\pi/2}\sin\alpha\pi + ie^{-i\alpha\pi}\beta_\kappa\right)\left(2^{1-\alpha}/\Gamma(\alpha)\right)E^{-1+\alpha}.$$

Thus β_κ is determined as

$$\beta_\kappa = ie^{i\alpha\pi/2}\sin\alpha\pi\left(\kappa\tau_\alpha E^{2\alpha-1}/(\kappa\tau_\alpha E^{2\alpha-1} - e^{i\alpha\pi})\right), \tag{2.19}$$

where τ_α is defined in (2.8). By Lemmas 2.1 and 2.2 and by (2.16), $\psi(x;E,\omega)$ behaves like

$$\psi = \tau(\omega)\varphi_{\rm in}(x;E,\omega) + g_\kappa(\omega \to \tilde{\omega}; E)\tau(\tilde{\omega})e^{iEr}r^{-1/2} + o(r^{-1/2}) \tag{2.20}$$

as $r \to \infty$ along direction $\tilde{\omega} \neq \omega$, where $\tau(\omega)$ is in (1.6), and

$$g_\kappa = g_+(\omega \to \tilde{\omega}; E) + 2(2\pi i E)^{-1/2} e^{-i\alpha\pi/2} \beta_\kappa.$$

This determines the desired amplitude and the proof is complete. $\square$

We end the section by making some additional comments on the outgoing eigenfunction $\psi_+(x; E, \omega)$ and the incoming one $\psi_-(x; E, \omega)$ of H_∞. These eigenfunctions are used to represent the amplitude $f(\omega \to \tilde{\omega}; E)$ of $T = D(A, V)$ in section 4. The outgoing eigenfunction $\psi_+ = {}^{\mathrm{t}}(\psi_{+1}, \psi_{+2})$ is defined by (2.17) with $\beta_\infty = i e^{i\alpha\pi/2} \sin \alpha\pi$, and we have

$$\psi_{+1} = \varphi_+(x; E, \omega) + \beta_\infty H_\alpha(Er), \qquad \psi_{+2} = e^{i\omega} \varphi_+(x; E, \omega)$$

by (2.18). This is expanded as

$$\begin{aligned} \psi_{+1}(x; E, \omega) &= \sum_{l \neq 0} e^{-i\nu\pi/2} e^{il\gamma(x; -\omega)} J_\nu(Er) + e^{i\alpha\pi/2} J_{-\alpha}(Er), \\ \psi_{+2}(x; E, \omega) &= e^{i\omega} \sum_{l \in Z} e^{-i\nu\pi/2} e^{il\gamma(x; -\omega)} J_\nu(Er). \end{aligned} \tag{2.21}$$

The Hankel function $H_\mu^{(2)}(z)$ of second kind is related to $H_\mu(z)$ through $H_\mu^{(2)}(z) = \overline{H_\mu(z)}$ for $z \in \boldsymbol{R}$, and it satisfies $H_{-\mu}^{(2)}(z) = e^{-i\mu\pi} H_\mu^{(2)}(z)$. If we make use of these relations, a similar argument enables us to construct the incoming eigenfunction $\psi_-(x; E, \omega) = {}^{\mathrm{t}}(\psi_{-1}, \psi_{-2})$ as

$$\psi_{-1} = \varphi_-(x; E, \omega) + \overline{\beta}_\infty \overline{H_\alpha(Er)}, \qquad \psi_{-2} = e^{i\omega} \varphi_-(x; E, \omega)$$

with φ_- defined by (2.11), and it admits the expansion

$$\begin{aligned} \psi_{-1}(x; E, \omega) &= \sum_{l \neq 0} e^{i\nu\pi/2} e^{il\gamma(x; \omega)} J_\nu(Er) + e^{-i\alpha\pi/2} J_{-\alpha}(Er), \\ \psi_{-2}(x; E, \omega) &= e^{i\omega} \sum_{l \in Z} e^{i\nu\pi/2} e^{il\gamma(x; \omega)} J_\nu(Er). \end{aligned} \tag{2.22}$$

3. Resolvent of self–adjoint extensions

We here establish the relation between the two resolvents $R(E + i0; H_\kappa)$ and $R(E + i0; H_\infty)$. We fix several new notation. We denote by $(\ ,\ )$ the scalar product in L^2 or $[L^2]^2$, and write $f \otimes g = (\,\cdot\,, g) f$ for the integral operator with kernel $f(x)\, \overline{g}(y)$. This acts as $(f \otimes g)\, u = (u, g) f$ on $u \in L^2$. We also use a similar notation

$$u \otimes v = (u_j \otimes v_k)_{1 \le j,k \le 2}, \quad u = {}^{\mathrm{t}}(u_1, u_2), \quad v = {}^{\mathrm{t}}(v_1, v_2),$$

for a vector version over $[L^2]^2$. We further define the two basic functions

$$\begin{aligned} \xi_+(x; E) &= {}^{\mathrm{t}}\left(-i e^{i\alpha\pi} H_\alpha(Er), H_{1-\alpha}(Er) e^{i\theta}\right), \\ \xi_-(x; E) &= {}^{\mathrm{t}}\left(-i e^{-i\alpha\pi} \overline{H_\alpha(Er)}, \overline{H_{1-\alpha}(Er)} e^{i\theta}\right) \end{aligned} \tag{3.1}$$

for $E > 0$. The second function may be written as

$$\xi_-(x;E) = {}^{\mathrm{t}}\left(-ie^{-i\alpha\pi}H_\alpha^{(2)}(Er), H_{1-\alpha}^{(2)}(Er)e^{i\theta}\right).$$

If we repeat almost the same argument as in the previous section, then it is easy to see that these two functions solve $(D_\alpha - E)\,u = 0$, and form a pair of linearly independent solutions. The aim here is to prove the following proposition.

Proposition 3.1 *Let $\xi_\pm = \xi_\pm(x;E)$ be as above. Then*

$$R(E + i0; H_\kappa) = R(E + i0; H_\infty) - c_\kappa E\,(\xi_+ \otimes \xi_-)\,,$$

where

$$c_\kappa = \sin\alpha\pi/(4(\kappa\tau_\alpha E^{2\alpha-1} - e^{i\alpha\pi}))$$

with τ_α defined by (2.8). If, in particular, $\alpha = 1/2$, then $c_\kappa = -1/(4(i-\kappa))$.

The proposition is proved at the end of this section. Let L_{AB} and L_Z be defined in (2.10), and let A_ε and $b_\varepsilon = \nabla \times A_\varepsilon$ be as in (1.9). We again set $T_{0\varepsilon} = D(A_\varepsilon, 0)$, which is convergent to $D_\alpha = D(A_{0\alpha}, 0)$ as $\varepsilon \to 0$ on $[C_0^\infty(\boldsymbol{R}^2 \setminus \{0\})]^2$ by (1.12). We represent $R(E + i0; H_\infty)$ in terms of resolvents of L_{AB} and L_Z. We repeat the same argument as used in section 1 to obtain

$$R(z; T_{0\varepsilon}) = (T_{0\varepsilon} + z)\begin{pmatrix} R(z^2; L_{-\varepsilon}) & 0 \\ 0 & R(z^2; L_{+\varepsilon}) \end{pmatrix}, \quad L_{\pm\varepsilon} = L(A_\varepsilon, \pm b_\varepsilon),$$

for z, $\operatorname{Im} z \neq 0$. According to the results in [23, section 3], we have $R(z; T_{0\varepsilon}) \to R(z; H_\infty)$ and

$$R(z; L_{+\varepsilon}) \to R(z; L_{AB}), \qquad R(z; L_{-\varepsilon}) \to R(z; L_Z),$$

as $\varepsilon \to 0$ in norm (in norm resolvent sense). We also have

$$R(E + i0; H_\infty) = \begin{pmatrix} ER(E^2 + i0; L_Z) & \pi_- R(E^2 + i0; L_{AB}) \\ \pi_+ R(E^2 + i0; L_Z) & ER(E^2 + i0; L_{AB}) \end{pmatrix}. \tag{3.2}$$

We now calculate the Green kernels of $R(E^2 + i0; L_{AB})$ and $R(E^2 + i0; L_Z)$. To do this, we decompose $L^2 = L^2(0,\infty) \otimes L^2(S)$, and we define the mapping U_l by

$$(U_l f)(r) = (2\pi)^{-1/2} r^{1/2} \int_0^{2\pi} f(r\theta)e^{-il\theta}\,d\theta : L^2 \to L^2(0,\infty)$$

for $l \in \boldsymbol{Z}$. Then

$$(U_l^* g)(x) = (2\pi)^{-1/2} r^{-1/2} g(r) e^{il\theta} : L^2(0,\infty) \to L^2,$$

and $R(E^2+i0; L_{AB})$ admits the decomposition

$$R(E^2+i0; L_{AB}) = \sum_{l\in Z} \oplus R_l, \qquad R_l = U_l^* R(E^2+i0; h_l) U_l, \tag{3.3}$$

where the domain of self–adjoint operator

$$h_l = -(d/dr)^2 + (\nu^2 - 1/4)r^{-2}, \quad \nu = |l-\alpha|,$$

is specified by the boundary condition $\lim_{r\to 0} r^{-(1/2-\alpha)} g(r) = 0$ at the origin. Similarly we have

$$R(E^2+i0; L_Z) = \tilde{R}_0 \oplus \sum_{l\neq 0} \oplus R_l, \qquad \tilde{R}_0 = U_0^* R(E^2+i0; \tilde{h}_0) U_0, \tag{3.4}$$

and the domain of self–adjoint operator

$$\tilde{h}_0 = -(d/dr)^2 + (\alpha^2 - 1/4)r^{-2}$$

is specified by the condition

$$\lim_{r\to 0} r^{-(1/2+\alpha)} (g(r) - g_0 r^{1/2-\alpha}) = 0$$

with $g_0 = \lim_{r\to 0} r^{-(1/2-\alpha)} g(r)$. The two functions $r^{1/2} J_\nu(Er)$ and $r^{1/2} H_\nu(Er)$ are linearly independent solutions to $(h_l - E^2)g = 0$ for $E > 0$. By formula, we know

$$W(J_\mu, J_{-\mu})(z) = -2\sin \mu\pi/(\pi z)$$

for the Wronskian of Bessel functions, so that

$$W(H_\mu, J_\mu)(z) = -2i/(\pi z), \qquad W(H_\mu, J_{-\mu})(z) = -2ie^{-i\mu\pi}/(\pi z)$$

by (2.4). Thus we can construct the Green kernels

$$\begin{aligned} R_l(x,y) &= (i/4)\, H_\nu(E(r\vee\rho)) J_\nu(E(r\wedge\rho)) e^{il(\theta-\varphi)}, \\ \tilde{R}_0(x,y) &= \left(ie^{i\alpha\pi}/4\right) H_\alpha(E(r\vee\rho)) J_{-\alpha}(E(r\wedge\rho)) \end{aligned} \tag{3.5}$$

in the standard way, where $r \vee \rho = \max(r,\rho)$ and $r\wedge\rho = \min(r,\rho)$ for $(x,y) = (re^{i\theta}, \rho e^{i\varphi})$. We are now in a position to prove Proposition 3.1.

Proof of Proposition 3.1. According to the Krein theory ([8]), the two resolvents are related to each other through the relation in the proposition. We have only to calculate the constant c_κ. We set

$${}^{\mathrm{t}}(u_1, u_2) = R(E+i0; H_\kappa)F$$

for $F = {}^{\mathrm{t}}(f, 0)$ with $f \in C_0^\infty(\boldsymbol{R}^2 \setminus \{0\})$. Then

$$u_1 = v_1 - ie^{i\alpha\pi} c\sigma H_\alpha(Er), \quad u_2 = v_2 + c\sigma H_{1-\alpha}(Er)e^{i\theta}, \quad c = -c_\kappa E,$$

where

$$ {}^{\mathrm{t}}(v_1, v_2) = R(E + i0; H_\infty)F = {}^{\mathrm{t}}(ER(E^2 + i0; L_Z)f, \pi_+ R(E^2 + i0; L_Z)f) $$

by (3.2), and $\sigma = (F, \xi_-)$ is the scalar product between $F = {}^{\mathrm{t}}(f, 0)$ and ξ_-. The constant c_κ is determined by boundary condition $u_{-1} + i\kappa u_{-2} = 0$, where u_{-1} and u_{-2} are defined by (1.14). We calculate the limits u_{-1} and u_{-2}. Since ${}^{\mathrm{t}}(v_1, v_2) = R(E + i0; H_\infty)F$, v_2 obeys $v_2 = o(r^{-(1-\alpha)})e^{i\theta}$, and hence it follows from (2.5) that

$$ u_{-2} = c\sigma\,(-i/\sin\alpha\pi)\left(2^{1-\alpha}/\Gamma(\alpha)\right)E^{\alpha-1}. $$

If we use (3.5) and (3.1), then v_1 behaves like

$$ v_1 = E\tilde{R}_0 f + o(1) = (\sigma E/4)J_{-\alpha}(Er) + o(1), \quad r \to 0, $$

and hence

$$ u_{-1} = \sigma\left(E/4 - ce^{i\alpha\pi}/\sin\alpha\pi\right)(2^\alpha/\Gamma(1-\alpha))\,E^{-\alpha} $$

by (2.5). Then c_κ is determined as in the proposition. $\square$

4. Scattering amplitudes in the presence of scalar potentials

The aim here is to derive the representation (4.6) below for the scattering amplitude $f(\omega \to \tilde{\omega}; E)$ of $T = D(A, V)$ with scalar potential $V \in C_0^\infty(\boldsymbol{R}^2 \to \boldsymbol{R})$, where $A \in C^\infty(\boldsymbol{R}^2 \to \boldsymbol{R}^2)$ is defined by (1.1). The derivation requires two lemmas.

Lemma 4.1 *Write $\psi_-(\omega)$ for the incoming eigenfunction $\psi_-(x; E, \omega)$, defined by* (2.22)*, of H_∞. Let*

$$ F(x) = {}^{\mathrm{t}}(f_1(r)e^{im\theta}, f_2(r)e^{i(m+1)\theta}), \qquad m \in \boldsymbol{Z}, $$

for $f_1,\ f_2 \in C_0^\infty[0, \infty)$. Then

$$ (R(E + i0; H_\infty)F)(r\tilde{\omega}) = (iE/8\pi)^{1/2}\,(F, \psi_-(\tilde{\omega}))\,\tau(\tilde{\omega})e^{iEr}r^{-1/2} + o(r^{-1/2}) $$

as $r \to \infty$ uniformly in $\tilde{\omega} \in S$, where $(F, \psi_-(\omega))$ is the scalar product in $[L^2]^2$ between F and $\psi_-(\omega)$.

Proof. We prove the lemma for the case $m = 0$ only. A similar argument applies to the other cases. Set ${}^{\mathrm{t}}(u_1, u_2) = R(E + i0; H_\infty)F$ for F as in the lemma. Then

$$ u_1 = Ev_1 + \pi_- v_2, \qquad u_2 = \pi_+ v_1 + Ev_2 $$

by (3.2), where

$$ v_1 = R(E^2 + i0; L_Z)f_1, \qquad v_2 = R(E^2 + i0; L_{AB})(f_2 e^{i\theta}). $$

It follows from (3.3) and (3.4) that $v_1 = \tilde{R}_0 f_1$ and $v_2 = R_1(f_2 e^{i\theta})$. The two operators $\tilde{R}_0$ and R_1 have the kernels (3.5). By assumption, f_1 and f_2 have compact support. Hence we have

$$v_1 = (ie^{i\alpha\pi}/4)(f_1, J_{-\alpha})H_\alpha(Er), \quad v_2 = (i/4)(f_2, J_{1-\alpha})H_{1-\alpha}(Er)e^{i\theta}$$

for $|x| \gg 1$. Since

$$\pi_-\left(H_{1-\alpha}(Er)e^{i\theta}\right) = -iEH_{-\alpha}(Er) = -iEe^{i\alpha\pi}H_\alpha(Er)$$

by (2.14), it follows from (2.16) that

$$\begin{aligned} u_1 &= (iE/4)\, e^{i\alpha\pi}\left((f_1, J_{-\alpha}) - i(f_2, J_{1-\alpha})\right) H_\alpha(Er) \\ &= (iE/8\pi)^{1/2}\, e^{i\alpha\pi/2}((f_1, J_{-\alpha}) - i(f_2, J_{1-\alpha}))e^{iEr}r^{-1/2} + o(r^{-1/2}) \end{aligned}$$

as $r \to \infty$. The eigenfunction ψ_- has the expansion (2.22), and we have

$$(F, \psi_-(\tilde{\omega})) = (f_1, \psi_{-1}(\tilde{\omega})) + (f_2 e^{i\theta}, \psi_{-2}(\tilde{\omega})) = e^{i\alpha\pi/2}\left((f_1, J_{-\alpha}) - i(f_2, J_{1-\alpha})\right).$$

This yields the desired asymptotic form for u_1. We can show in a similar way that u_2 also takes the asymptotic form in the theorem. Thus the proof is complete. $\square$

We now introduce the Banach spaces B and B^* with norms

$$\|u\|_B = \sum_{j=0}^{\infty}\left(2^j \int_{\Omega_j} |u(x)|^2\, dx\right)^{1/2}, \quad \|u\|_{B^*} = \sup_{R>0}\left(\frac{1}{R}\int_{|x|<R} |u(x)|^2\, dx\right)^{1/2},$$

where $\Omega_0 = \{|x| \le 1\}$ and $\Omega_j = \{2^{j-1} < |x| \le 2^j\}$ for $j \ge 1$. The two spaces fulfill the inclusion relations

$$L^2_s \subset B \subset L^2_{1/2}, \qquad L^2_{-1/2} \subset B^* \subset L^2_{-s}$$

for $s > 1/2$, where $L^2_s = L^2(\boldsymbol{R}^2; \langle x\rangle^{2s}\, dx)$ with $\langle x\rangle = (1+|x|^2)^{1/2}$. We use the notation $o_*(r^{-1/2})$ as $r = |x| \to \infty$ to denote functions u obeying the bound

$$\frac{1}{R}\int_{|x|<R} |u(x)|^2\, dx \to 0, \qquad R \to \infty.$$

We use the same notation for vector–valued functions. If $u(x) = o(r^{-1/2})$ at infinity, then u is of class $o_*(r^{-1/2})$.

Lemma 4.2 *Assume that $F \in [C_0^\infty(\boldsymbol{R}^2)]^2$. Then*

$$(R(E+i0; H_\infty)F)(x) = (iE/8\pi)^{1/2}\,(F, \psi_-(\tilde{\omega}))\,\tau(\tilde{\omega})e^{iEr}r^{-1/2} + o_*(r^{-1/2}),$$

where the leading term on the right side is regarded as a function of $x = r\tilde{\omega}$.

Proof. According to [5], we know that $R(E+i0;H_\infty):[B]^2\to[B^*]^2$ is bounded. If we expand F as the Fourier series, then the lemma is obtained as a consequence of Lemma 4.1. $\square$

We proceed to calculating the amplitude $f(\omega\to\tilde{\omega};E)$ of $T=D(A,V)$. We assume that b and V have support in $\{|x|<1\}$. According to Lemma 2.1 of [22] (see (2.2) there), we can construct a smooth magnetic potential $a(x):\boldsymbol{R}^2\to\boldsymbol{R}^2$ associated with field b such that

$$a(x)=(a_1,a_2)=A_{0\alpha}=\alpha\left(-x_2/|x|^2,x_1/|x|^2\right),\qquad |x|>2. \tag{4.1}$$

We define the auxiliary operator K as

$$K=D(a,V). \tag{4.2}$$

This is self–adjoint with domain $\mathcal{D}(K)=[H^1(\boldsymbol{R}^2)]^2$, and we know ([11, 15]) that the boundary value $R(E+i0;K)$ to the real axis exists as a bounded operator from $[L^2_s]^2$ into $[L^2_{-s}]^2$ for $s>1/2$. We further introduce a basic cut–off function $\chi_0\in C_0^\infty(\boldsymbol{R}^2\to\boldsymbol{R})$ with the properties

$$\operatorname{supp}\chi_0\subset\{|x|<2\},\qquad \chi_0=1\ \text{ on }\ \{|x|<1\}. \tag{4.3}$$

We set $\chi_+(x)=\chi_0(x/2)$ and $\chi_-(x)=\chi_0(x/4)$.

We study the behavior at infinity of eigenfunction $\psi(x;E,\omega)$ of K. Since $K=D(A_{0\alpha},0)=D_\alpha$ over $\{|x|>2\}$ by (4.1), we have $(1-\chi_+)\,(K-E)\,\psi_+=0$ for the outgoing eigenfunction $\psi_+(\omega)=\psi_+(x;E,\omega)$ of H_∞. Hence the eigenfunction $\psi=\psi(x;E,\omega)$ with incident wave $\varphi_{\rm in}(x;E,\omega)$ as in Lemma 2.2 is written as

$$\psi=(1-\chi_+)\psi_++R(E+i0;K)\Pi_+\psi_+. \tag{4.4}$$

where $\Pi_+=[D_\alpha,\chi_+]$. Similarly $\psi_+(x;E,\omega)$ is represented as

$$\psi_+=(1-\chi_-)\psi+R(E+i0;H_\infty)\Pi_-\psi$$

with $\Pi_-=[D_\alpha,\chi_-]$. Hence it follows from Lemma 4.2 that

$$\psi=\psi_+-(iE/8\pi)^{1/2}\left(\Pi_-\psi,\psi_-(\tilde{\omega})\right)\tau(\tilde{\omega})e^{iEr}r^{-1/2}+o_*(r^{-1/2}). \tag{4.5}$$

We insert (4.4) into ψ on the right side of (4.5). Since $\Pi_-(1-\chi_+)=0$ and $\Pi_-^*=-\Pi_-$, we obtain

$$(\Pi_-\psi,\psi_-(\tilde{\omega}))=-(R(E+i0;K)\Pi_+\psi_+(\omega),\Pi_-\psi_-(\tilde{\omega})).$$

We recall that ψ_+ obeys (2.20) with $\kappa=\infty$. Hence the amplitude $f(\omega\to\tilde{\omega};E)$ of K is given by

$$f=g_\infty(\omega\to\tilde{\omega};E)+(iE/8\pi)^{1/2}(R(E+i0;K)\Pi_+\psi_+(\omega),\Pi_-\psi_-(\tilde{\omega})), \tag{4.6}$$

where g_∞ is the amplitude of H_∞. The amplitude of $T = D(A, V)$ is shown to be represented in the same way. Since A and a have the same field b, we have the relation

$$A = a + \nabla h \tag{4.7}$$

for some function $h \in C^\infty(\boldsymbol{R}^2 \to \boldsymbol{R})$, and $T = e^{ih} K e^{-ih}$. The difference obeys $A - a = O(|x|^{-2})$ at infinity, so that h falls off with $h = O(|x|^{-1})$ and $e^{ih(x)} = 1 + O(|x|^{-1})$. Thus T has the same scattering operator as K and hence the scattering amplitude of T is also represented as (4.6).

To sum up, the amplitude $f(\omega \to \tilde{\omega}; E)$ of $T = D(A, V)$ is defined through the asymptotic form

$$\psi = \tau(\omega)\varphi_{\mathrm{in}}(x; E, \omega) + f(\omega \to \tilde{\omega}; E)\tau(\tilde{\omega})e^{iEr}r^{-1/2} + o_*(r^{-1/2})$$

as $r = |x| \to \infty$ of solution ψ to equation $T\psi = (T_0 + V)\,\psi = E\psi$, and it has the representation (4.6). In the mathematical scattering theory, it is standard to define the scattering amplitudes through integral kernels of scattering matrices after establishing the basic problems such as the existence and completeness of wave operators and the limiting absorption principle [9, 15, 18, 24, 25]. However, K has the special property that it admits the polar coordinate decomposition on $\{|x| > 2\}$. If we make use of this property, the Agmon–Hörmander theory ([5]) enables us to define directly the scattering amplitude through the asymptotic form of eigenfunction. We can show that these two representations defined in a different way coincide with each other, but we do not go into the details here.

5. Scattering by electromagnetic fields with small support

In this section we formulate the results on the asymptotic behavior of amplitudes for the scattering by electromagnetic fields with small support. We obtain the three main theorems and the remaining four sections (sections 6,7,8 and 9) are devoted to the proof of these theorems.

Let A_ε and V_ε be defined by (1.7). We denote by $f_\varepsilon(\omega \to \tilde{\omega}; E)$ the scattering amplitude of $T_\varepsilon = D(A_\varepsilon, V_\varepsilon)$. If we set

$$K_\varepsilon = D(a_\varepsilon, V_\varepsilon), \qquad a_\varepsilon = \varepsilon^{-1}a(x/\varepsilon), \tag{5.1}$$

then $a_\varepsilon(x) = A_{0\alpha}(x)$ over $|x| > 2\varepsilon$, and the amplitude f_ε has the representation

$$f_\varepsilon = g_\infty(\omega \to \tilde{\omega}; E) + (iE/8\pi)^{1/2}(R(E + i0; K_\varepsilon)\Pi_+\psi_+(\omega), \Pi_-\psi_-(\tilde{\omega})), \tag{5.2}$$

where $\Pi_\pm = [D_\alpha, \chi_\pm]$ with $\chi_+ = \chi_0(x/2)$ and $\chi_- = \chi_0(x/4)$ again. We have explicitly calculated the scattering amplitude $g_\kappa(\omega \to \tilde{\omega}; E)$ of H_κ in Proposition 2.1. It admits the representation

$$g_\kappa = g_\infty(\omega \to \tilde{\omega}; E) + (iE/8\pi)^{1/2}(R(E + i0; H_\kappa)\Pi_+\psi_+(\omega), \Pi_-\psi_-(\tilde{\omega})) \tag{5.3}$$

in terms of resolvent $R(E+i0; H_\kappa)$. In fact, this is obtained by repeating almost the same argument as used to derive (4.6). We first deal with the case without electric fields.

Theorem 5.1 *Assume that* $V=0$ *identically. Then*

$$f_\varepsilon(\omega \to \tilde{\omega}; E) \to g_\infty(\omega \to \tilde{\omega}; E), \qquad \varepsilon \to 0,$$

for $\omega \neq \tilde{\omega}$.

Next we discuss the case when $V \in C_0^\infty(\boldsymbol{R}^2 \to \boldsymbol{R})$ does not vanish identically. We assume that

$$V(x) \geq 0, \tag{5.4}$$

so that the scalar product

$$\lambda_0 = (V\rho, \rho) > 0 \tag{5.5}$$

is strictly positive for the resonance function $\rho(x) = e^{-\varphi(x)}$ defined by (1.9). The assumption (5.4) does not matter, but $\lambda_0 \neq 0$ is important to the future argument. Before stating the second theorem, we define the resonance state of Dirac operator $T = D(A, V)$ at zero energy. The definition is different according as $0 < \alpha \leq 1/2$ or $1/2 < \alpha < 1$.

Definition 5.1. (1) Let $0 < \alpha \leq 1/2$. Assume that the equation $Tv = 0$ has a non–trivial solution such that $v = {}^t(v_1, v_2) \in L^2 \times L^\infty$ and $v_2(x) = O(|x|^{-1+\alpha})$ at infinity. If $v_2 \notin L^2$, then T is said to admit a resonance state at zero energy, and if $v_2 \in L^2$, then T has an eigenvalue at zero energy.

(2) Let $1/2 < \alpha < 1$. Assume that $Tv = 0$ has a non–trivial solution such that $v = {}^t(v_1, v_2) \in L^\infty \times L^2$ and $v_1(x) = O(|x|^{-\alpha})$ at infinity. If $v_1 \notin L^2$, then T is said to admit a resonance state at zero energy, and if $v_1 \in L^2$, then T has an eigenvalue at zero energy.

In the present work, we deal with only the case that T has neither eigenstates nor resonance states at zero energy. This case is simple but generic. Thus we always assume that

$$T \text{ has neither eigenstates nor resonance states at zero energy.} \tag{5.6}$$

If $|V| \ll 1$ is small enough, then it can be shown that T fulfills (5.6). The lemma below plays an important role in proving the remaining two main theorems. This basic lemma is proved in section 7.

Lemma 5.1 *Assume that* (5.6) *is fulfilled. Then :*

(1) *Let* $0 < \alpha \leq 1/2$. *Then there exists a unique solution* $e \in L^\infty \times L^\infty$ *to equation* $Te = 0$ *such that* $e = {}^t(e_1, e_2)$ *obeys*

$$e_1 = r^{-\alpha} + O(|x|^{-1-\alpha}), \qquad e_2 = O(|x|^{-1+\alpha}) \tag{5.7}$$

at infinity, and $e_2(x)$ behaves like

$$e_2(x) = i\lambda_2 r^{-1+\alpha} e^{i\theta} + O(|x|^{-2+\alpha}), \qquad |x| \to \infty, \tag{5.8}$$

for some real constant λ_2.

(2) *Let $1/2 < \alpha < 1$. Then there exists a unique solution $e \in L^\infty \times L^\infty$ to $Te = 0$ such that $e = {}^{\mathrm{t}}(e_1, e_2)$ obeys*

$$e_1 = O(|x|^{-\alpha}), \qquad e_2 = ir^{-1+\alpha} e^{i\theta} + O(|x|^{-2+\alpha}) \tag{5.9}$$

at infinity, and $e_1(x)$ behaves like

$$e_1(x) = \lambda_1 r^{-\alpha} + O(|x|^{-1-\alpha}), \qquad |x| \to \infty, \tag{5.10}$$

for some real constant λ_1.

We are now in a position to state the second theorem. When the scalar potential V is added as a perturbation, the situation changes even for small perturbation. The limit heavily depends on the values α of fluxes and it changes discontinuously at half–integer flux $\alpha = 1/2$.

Theorem 5.2 *Let $V \in C_0^\infty(\boldsymbol{R}^2)$ satisfy* (5.4)*, and assume that T fulfills* (5.6)*. If $\omega \neq \tilde{\omega}$ for incident and final directions w and $\tilde{\omega}$, then one has the following asymptotic form as $\varepsilon \to 0$:*

(1) *Let $0 < \alpha < 1/2$. Then*

$$f_\varepsilon(\omega \to \tilde{\omega}; E) \to g_\infty(\omega \to \tilde{\omega}; E).$$

(2) *Let $\alpha = 1/2$ and let λ_2 be as in (5.8) of Lemma 5.1. Then*

$$f_\varepsilon(\omega \to \tilde{\omega}; E) \to g_\kappa(\omega \to \tilde{\omega}; E)$$

for $\kappa = 1/\lambda_2$ ($\kappa = \infty$ provided that $\lambda_2 = 0$).

(3) *Let $1/2 < \alpha < 1$. Then*

$$f_\varepsilon(\omega \to \tilde{\omega}; E) \to g_0(\omega \to \tilde{\omega}; E).$$

The third theorem is concerned with the scattering of Dirac particles appearing in the interaction of cosmic string with matter. We now consider the 2×2 system (1.15) of Dirac equations. The amplitude $f_{2\varepsilon}(\omega \to \tilde{\omega})$ in question is defined through the asymptotic form of solution w to equation (1.15). The solution behaves like

$$\begin{aligned} w \;=\; & {}^{\mathrm{t}}(\tau(\omega), 0)\varphi_{\mathrm{in}}(x; E, \omega) + f_{1\varepsilon}(\omega \to \tilde{\omega}; E)\,{}^{\mathrm{t}}(\tau(\tilde{\omega}), 0) e^{iEr} r^{-1/2} \\ & + f_{2\varepsilon}(\omega \to \tilde{\omega}; E)\,{}^{\mathrm{t}}(0, \tau(\tilde{\omega})) e^{iEr} r^{-1/2} + o_*(r^{-1/2}), \quad r \to \infty, \end{aligned}$$

for incident wave ${}^{\mathrm{t}}(\tau(\omega), 0)\varphi_{\mathrm{in}}(x; E, \omega)$. The aim of the third theorem is to analyze the asymptotic behavior as $\varepsilon \to 0$ of $f_{2\varepsilon}(\omega \to \tilde{\omega}; E)$.

Theorem 5.3 *Let $V \in C_0^\infty(\boldsymbol{R}^2 \to \boldsymbol{R})$ satisfy (5.4), and assume that T fulfills* (5.6). *Then the amplitude $f_{2\varepsilon}(\omega \to \tilde{\omega}; E)$ behaves like*

$$f_{2\varepsilon} = \left(\frac{iE}{8\pi}\right)^{1/2} C_\alpha \varepsilon^{|2\alpha-1|} + o(\varepsilon^{|2\alpha-1|}), \qquad \varepsilon \to 0,$$

where

$$C_\alpha = \begin{cases} \left(2^\alpha E^{-\alpha} i^\alpha / \Gamma(1-\alpha)\right)^2 2\pi\lambda_2, & 0 < \alpha < 1/2, \\ 4E^{-1} i\lambda_2 (1+\lambda_2^2)^{-1}, & \alpha = 1/2, \\ \left(2^{1-\alpha} E^{\alpha-1} i^{1-\alpha} / \Gamma(\alpha)\right)^2 2\pi\lambda_1, & 1/2 < \alpha < 1. \end{cases}$$

We end the section by making some comments on Theorems 5.2 and 5.3.

(1) As stated in section 1, a result similar to Theorem 5.2 has been obtained by Afanasiev [2, section 7.10], where the behavior of amplitude has been analyzed for the scattering by the small obstacle $\{|x| < \varepsilon\}$ under a certain impenetrable boundary condition in the background of the δ–like field $2\pi\alpha\delta(x)$. As $\varepsilon \to 0$, the amplitude f_ε is convergent to $g_\infty,\ g_\kappa$ with $\kappa = -1$ or g_0 according as $0 < \alpha < 1/2,\ \alpha = 1/2$ or $1/2 < \alpha < 1$.

(2) The assumption that $A(x)$ and $V(x)$ are smooth is not essential. The two theorems extend to the case of bounded electromagnetic fields, and the extension is possible even for singular magnetic potentials. For example, the theorems apply to the case that $A(x) = A_{0\alpha}(x)$ is the Aharonov–Bohm potential and $V(x)$ is the characteristic function of unit disk $\{|x| < 1\}$. If we consider (1.13) with $\kappa = \infty$ as the boundary condition at the origin, we can calculate λ_1 and λ_2 explicitly. In fact, if we set $e(x) = {}^{\mathrm{t}}(e_1(r), e_2(r)e^{i\theta})$, then it follows from (2.2) that e solves

$$e_1' + \alpha\, r^{-1} e_1 + iVe_2 = 0, \qquad e_2' + (1-\alpha) r^{-1} e_2 + iVe_1 = 0,$$

where $e' = (d/dr)e$. We use the formula (2.14) to solve the equation above. If we take account of (5.7), then λ_2 is determined as $\lambda_2 = -J_{1-\alpha}(1)/J_{-\alpha}(1)$ for $0 < \alpha \le 1/2$, while (5.9) yields $\lambda_1 = -J_{-\alpha}(1)/J_{1-\alpha}(1)$ for $1/2 < \alpha < 1$.

(3) As a work related to Theorem 5.3, [7] has dealt with the case that the electric potential is $\lambda V(x)$ and $A(x)$ is the Aharonov–Bohm potential $A_{0\alpha}(x)$ with boundary condition (1.13) with $\kappa = \infty$ or $\kappa = 0$, where $\lambda > 0$ is a small coupling constant and V still denotes the characteristic function of the unit disk.

6. Behavior of resolvent at low energy

The proof of all the theorems in the previous section is based on the behavior as $\varepsilon \to 0$ of resolvent $R(E + i0; K_\varepsilon)$. We first follow the idea from [6, chapter I.1.2] to derive the basic representation for $R(E+i0; K_\varepsilon)$. The derivation is done by repeated use of the resolvent identity. If we set $K_{0\varepsilon} = D(a_\varepsilon, 0)$, then $K_\varepsilon = K_{0\varepsilon} + V_\varepsilon$, and we have

$$R(E + i0; K_\varepsilon) = R(E + i0; K_{0\varepsilon}) - R(E + i0; K_\varepsilon) V_\varepsilon R(E + i0; K_{0\varepsilon})$$

by the resolvent identity. We have assumed that $V(x) \ge 0$. If we further define

$$Y_\varepsilon = V_\varepsilon^{1/2} R(E+i0; K_{0\varepsilon}) V_\varepsilon^{1/2} : [L^2]^2 \to [L^2]^2, \tag{6.1}$$

then the resolvent identity yields the relation

$$R(E+i0; K_\varepsilon) V_\varepsilon^{1/2} (1+Y_\varepsilon) = R(E+i0; K_{0\varepsilon}) V_\varepsilon^{1/2}.$$

The operator $1+Y_\varepsilon$ has the bounded inverse $(1+Y_\varepsilon)^{-1} : [L^2]^2 \to [L^2]^2$, which follows from the fact that the outgoing solution to equation $(K_\varepsilon - E)u = 0$ identically vanishes. Thus $R(E+i0; K_\varepsilon)$ is represented as

$$R(E+i0; K_{0\varepsilon}) - R(E+i0; K_{0\varepsilon}) V_\varepsilon^{1/2} (1+Y_\varepsilon)^{-1} V_\varepsilon^{1/2} R(E+i0; K_{0\varepsilon})$$

by the resolvent identity. Let $J_\varepsilon : [L^2]^2 \to [L^2]^2$ be again the unitary operator defined by $(J_\varepsilon u)(x) = \varepsilon^{-1} u(x/\varepsilon)$. We set $X_\varepsilon = J_\varepsilon^* Y_\varepsilon J_\varepsilon$. Since $K_{0\varepsilon} = \varepsilon^{-1} J_\varepsilon K_0 J_\varepsilon^*$ for $K_0 = D(a, 0)$, we have

$$X_\varepsilon = J_\varepsilon^* Y_\varepsilon J_\varepsilon = V^{1/2} R(k+i0; K_0) V^{1/2}, \quad k = \varepsilon E > 0, \tag{6.2}$$

and hence

$$R(E+i0; K_\varepsilon) = R(E+i0; K_{0\varepsilon}) - \varepsilon^{-1} \Gamma_\varepsilon(E+i0)(1+X_\varepsilon)^{-1} \Gamma_\varepsilon(E-i0)^*, \tag{6.3}$$

where

$$\Gamma_\varepsilon(E \pm i0) = R(E \pm i0; K_{0\varepsilon}) J_\varepsilon V^{1/2}. \tag{6.4}$$

This is a basic representation. This section is devoted to the analysis on the behavior as $\varepsilon \to 0$ of X_ε as the first step towards proving the three theorems.

By (4.7), the potential $a : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ takes the form

$$a = (-\partial_2 \varphi(x), \partial_1 \varphi(x)) + \nabla h = A + \nabla h$$

for some $h \in C^\infty(\boldsymbol{R}^2 \to \boldsymbol{R})$ falling off like $h = O(|x|^{-1})$ at infinity, and the field $b = \nabla \times a$ has support in $\{|x| < 1\}$. We set $p = (p_1, p_2) = -i\nabla - a$ and write K_0 as

$$K_0 = \sigma_1 p_1 + \sigma_2 p_2 = \begin{pmatrix} 0 & p_- \\ p_+ & 0 \end{pmatrix}$$

in the matrix form, where $p_\pm = p_1 \pm i p_2$. We define the Schrödinger operators $L_\pm$ by

$$L_\pm = L(a, \pm b) = p_1^2 + p_2^2 \pm b = (-i\nabla - a)^2 \pm b. \tag{6.5}$$

These are self–adjoint with domain $\mathcal{D}(L_\pm) = H^2(\boldsymbol{R}^2)$ in L^2. Since

$$i[p_1, p_2] = i(p_1 p_2 - p_2 p_1) = -b,$$

we have $L_\pm = p_\pm p_\mp = p_\mp^* p_\mp$, and $R(k+i0;K_0)$ is represented as

$$R(k+i0;K_0) = \begin{pmatrix} kR(k^2+i0;L_-) & p_-R(k^2+i0;L_+) \\ p_+R(k^2+i0;L_-) & kR(k^2+i0;L_+) \end{pmatrix}. \tag{6.6}$$

Thus the problem is reduced to the study on the behavior of $R(k^2+i0;L_\pm)$ as $k \to 0$.

The two operators $L_\pm = p_\mp^* p_\mp \geq 0$ are non–negative, and since $0 < \alpha < 1$ by assumption (1.3), it follows by the Aharonov–Casher theorem ([4]) that $L_\pm$ have no bound states at zero energy. However, the spectral structure at zero energy is different in the sense that L_- has a resonance state. The resonance state is defined as a bounded solution u to equation $L_- u = p_- p_+ u = 0$. If u is such a solution, then a simple calculation using integral by parts shows that $p_1 u$ and $p_2 u$ are in L^2, so that $p_+ u = 0$. By (4.7) (see also (1.8)), we have

$$p_+ = -ie^{ih}e^{-\varphi}\,(\,\partial_1 + i\partial_2)\,e^{\varphi}e^{-ih}. \tag{6.7}$$

Thus L_- has the resonance state behaving like

$$u(x) = e^{-\varphi}e^{ih} = |x|^{-\alpha}\left(1 + O(|x|^{-1})\right)$$

at infinity. On the other hand, $L_+ = p_+ p_-$ does not have a resonance state. We note that if $\alpha > 1$, L_- has bound states at zero energy with multiplicity $[\alpha]$ by the Aharonov–Casher theorem again.

We now introduce the following notation : $\eta \in C_0(\boldsymbol{R}^2)$ is a continuous function with compact support and $\eta_0 \in C_0(\boldsymbol{R}^2)$ is a function compactly supported away from the origin. We further use the notation $Op(\varepsilon^\sigma)$ and $op(\varepsilon^\sigma)$ to denote the classes of bounded operators obeying the bound $O(\varepsilon^\sigma)$ and $o(\varepsilon^\sigma)$ in norm respectively.

We make a brief review on the behavior at low energy of $R(k^2+i0;L_\pm)$ obtained by ([23, Propositions 4.2 and 4.3]). We first consider L_-. Let $h(x)$ be as in (6.7). Then

$$\rho_0(x) = e^{-\varphi}e^{ih}, \tag{6.8}$$

solves $L_-\rho_0 = 0$ and behaves like

$$\rho_0(x) = |x|^{-\alpha}\left(1 + O(|x|^{-1})\right) \tag{6.9}$$

at infinity. We know ([23]) that L_- has the one dimensional resonance space spanned by ρ_0 at zero energy.

Proposition 6.1 *Let ρ_0 be as above and let γ_0 be the constant defined by*

$$\gamma_0 = -2^{2(1-\alpha)}\pi\Gamma(1-\alpha)/\Gamma(\alpha). \tag{6.10}$$

Then

$$\eta R(k^2+i0;L_-)\eta = \gamma_-(k)i^{2\alpha}k^{-2\alpha}\eta(\rho_0 \otimes \rho_0)\eta + Op(\varepsilon^0)$$

for some coefficient $\gamma_-(k)$ obeying $\gamma_-(k) = -1/\gamma_0 + o(1)$ as $k \to 0$.

Remark 6.1. (1) The proposition above corresponds to Proposition 4.3 in [23], where the resonance function $\rho_0(x)$ is normalized as $\rho_0(x) = (2\pi\alpha)^{-1/2}e^{-\varphi}e^{ih}$, so that the constant $\gamma_-(k)$ undergoes a suitable change. (2) By elliptic estimate, $\nabla\eta R(k^2 + i0; L_-)\eta$ admits a similar asymptotic form under a natural modification.

Next we move to L_+ which has neither bound states nor resonance states at zero energy. We set

$$L^2_{\mathrm{com}} = \{u \in L^2(\boldsymbol{R}^2) : \operatorname{supp} u \subset B_M\}, \qquad B_M = \{|x| < M\},$$

for $M \gg 1$ fixed arbitrarily but sufficiently large. We have shown in [23] that there exists a limit

$$G_+ = \lim_{k\to 0} R(k^2 + i0; L_+) : L^2_{\mathrm{com}} \to L^2_{-1} \tag{6.11}$$

as a bounded operator from L^2_{com} to $L^2_{-1} = L^2(\boldsymbol{R}^2; \langle x\rangle^{-2}\, dx)$. We further know that the equation $L_+ = p_+p_-u = 0$ has a unique solution behaving like

$$\omega_{+l} = r^\nu e^{il\theta} + O(1), \qquad |x| \to \infty, \tag{6.12}$$

for $l = 0, 1$, where $\nu = |l - \alpha|$ again.

Proposition 6.2 *Let the notation be as above. Then there exists $\gamma_{+l}(k)$ such that*

$$\eta R(k^2 + i0; L_+)\eta = \eta G_+\eta + \sum_{l=0,1} \gamma_{+l}(k) i^{-2\nu} k^{2\nu} \eta(\omega_{+l} \otimes \omega_{+l})\eta + Op(\varepsilon^2),$$

where the two constants $\gamma_{+l}(k)$, $l = 0, 1$, are bounded uniformly in $k = \varepsilon E > 0$.

This proposition has been obtained as Proposition 4.2 in [23]. We can make precise the behavior as $k \to 0$ of the constant $\gamma_{+l}(k)$, but the argument below does not require such an asymptotic form.

By (6.7), $p_+ = -2i\, e^{ih}e^{-\varphi}\overline{\partial}e^{\varphi}e^{-ih}$ with $\overline{\partial} = (1/2)\,(\partial_1 + i\partial_2)$. The Cauchy–Riemann operator $\overline{\partial}$ has the fundamental solution $(1/\pi)\,(x_1 + ix_2)^{-1}$. We denote by $\overline{\partial}^{-1}$ the convolution operator

$$\overline{\partial}^{-1} = (1/\pi)\ (x_1 + ix_2)^{-1} *$$

and we define

$$p_+^{-1} = -(2i)^{-1}e^{ih}e^{-\varphi}\overline{\partial}^{-1}e^{\varphi}e^{-ih}$$

and $p_-^{-1} = \left(p_+^{-1}\right)^*$. By definition, we have $p_\pm p_\pm^{-1} = 1$.

Lemma 6.1 *One has the relations*

$$p_-G_+f = p_+^{-1}f, \qquad G_+p_+f = p_-^{-1}f$$

for any bounded function f with compact support.

Proof. We prove only the first relation. The second one follows by taking the adjoint of both sides. Let f be as in the lemma, and set $w_1 = p_+^{-1} f$. Then $w_1 \in L^2$ and it solves $p_+ w_1 = f$. If, on the other hand, we set $w_2 = p_- G_+ f$, then w_2 satisfies

$$p_+ w_2 = p_+ p_- G_+ f = L_+ G_+ f = f.$$

Since $w_2 \in L^2_{-1}$ by (6.11), it follows that $w_2 \in L^2$. In fact, we have $\|p_- G_+ f\|_{L^2} < \infty$ by a simple use of partial integration. Set $w = e^{\varphi} e^{-ih} (w_1 - w_2)$. Then $\overline{\partial} w = 0$, so that w is an entire function. Note that $e^{\varphi} = O(|x|^{\alpha})$ at infinity for $0 < \alpha < 1$. Since $w_1 - w_2 \in L^2$, we can easily show that $w = 0$, and hence $w_1 = w_2$. Thus the lemma is obtained. $\square$

Lemma 6.2 *Let ω_{+0} be as in* (6.12)*. Then one has $p_- \omega_{+0} = 0$.*

Proof. Set $v_0 = e^{-ih} e^{\varphi}$. Then $p_- v_0 = 0$ and the difference $u = \omega_{+0} - v_0$ is bounded. The function u solves

$$p_+ p_- u = L_+ u = L_+ \omega_{+0} - p_+ p_- v_0 = 0.$$

Hence it follows from Lemma 4.3 of [22] (or by the argument used in its proof) that $p_- u = 0$. This implies that $p_- \omega_{+0} = 0$, and the proof is complete. $\square$

Lemma 6.3 *Let ω_{+1} be also as in* (6.12)*. Then one has $p_- \omega_{+1} = c\rho_0$ for some c.*

Proof. Set $u = p_- \omega_{+1}$. Then u obeys the bound $u = O(|x|^{-\alpha})$ at infinity, and it solves the equation $L_- u = p_- L_+ \omega_{+1} = 0$. This implies that u is in the resonance space of L_- at zero energy. Since the resonance space is one dimensional, the lemma follows at once. $\square$

If we make use of the simple relation

$$p_+ R(k^2 \pm i0; L_-) = R(k^2 \pm i0; L_+) p_+,$$

then we obtain from (6.6) that

$$R(k + i0; K_0) = \begin{pmatrix} kR(k^2 + i0, L_-) & p_- R(k^2 + i0; L_+) \\ R(k^2 + i0; L_+) p_+ & kR(k^2 + i0, L_+) \end{pmatrix}$$

for $k = \varepsilon E > 0$. Thus we combine Propositions 6.1, 6.2 and Lemmas 6.1, 6.2 and 6.3 to get the following proposition.

Proposition 6.3 *As $\varepsilon \to 0$, $\eta R(k+i0;K_0)\eta$ takes the form*

$$\eta R(k+i0;K_0)\eta = \eta\left\{\gamma(\varepsilon)\,(\tilde\rho_0\otimes\tilde\rho_0)\,\varepsilon^{1-2\alpha} + G_0 + O(\varepsilon^{2(1-\alpha)})G_1\right\}\eta + Op(\varepsilon),$$

where $\tilde\rho_0 = {}^{\mathrm{t}}(\rho_0,0)$ and

$$G_0 = \begin{pmatrix} 0 & p_+^{-1} \\ p_-^{-1} & 0 \end{pmatrix}, \qquad G_1 = \begin{pmatrix} 0 & c\rho_0\otimes\omega_{+1} \\ \omega_{+1}\otimes c\rho_0 & 0 \end{pmatrix},$$

c being as in Lemma 6.3, and

$$\gamma(\varepsilon) = i^{2\alpha}E^{1-2\alpha}\gamma_-(\varepsilon E) = -i^{2\alpha}E^{1-2\alpha}\left(1/\gamma_0 + o(1)\right), \quad \varepsilon\to 0. \tag{6.13}$$

In particular, X_ε defined by (6.2) takes the form

$$X_\varepsilon = \gamma(\varepsilon)\,(q_0\otimes q_0)\,\varepsilon^{1-2\alpha} + Z_0 + O(\varepsilon^{2(1-\alpha)})Z_1 + Op(\varepsilon), \tag{6.14}$$

where

$$q_0 = V^{1/2}\tilde\rho_0, \qquad \tilde\rho_0 = {}^{\mathrm{t}}(\rho_0,0), \tag{6.15}$$

and $Z_0 = V^{1/2}G_0V^{1/2}$ and $Z_1 = V^{1/2}G_1V^{1/2}$.

7. Resonance at zero energy: proof of Lemma 5.1

The second step is to analyze the inversion of $(1+X_\varepsilon)^{-1}$ which appears in representation (6.3) for the resolvent $R(E+i0;K_\varepsilon)$ under consideration. We also prove Lemma 5.1 at the end of the section. As is easily seen from assumption (5.6), $K = D(a,V) = K_0+V$ has neither eigenstates nor resonance states at zero energy.

Lemma 7.1 *Assume that $0<\alpha\le 1/2$. Let Z_0 be as in Proposition 6.3. If (5.6) is fulfilled, then $Z_0 : [L^2]^2 \to [L^2]^2$ has the bounded inverse $(1+Z_0)^{-1}$ on $[L^2]^2$.*

Proof. The operator Z_0 is compact. Set $\Phi = \ker(1+Z_0)$. It suffices to show that $\dim\Phi = 0$. The proof is done by contradiction. Assume that $u = {}^{\mathrm{t}}(u_1,u_2)\in\Phi$ does not vanish identically. If we set $v = {}^{\mathrm{t}}(v_1,v_2) = G_0V^{1/2}u$ for u as above, then $V^{1/2}v = Z_0u = -u$, and v satisfies

$$K_0v = V^{1/2}u = -Vv,$$

so that v solves $Kv=0$. We can easily see that v is not identically zero. The first component $v_1 = p_+^{-1}V^{1/2}u_2$ is in L^2. Since $p_-^{-1} = \left(p_+^{-1}\right)^*$ is the integral operator with kernel

$$-(2\pi i)^{-1}e^{\varphi}e^{ih}\left((x_1-ix_2)^{-1}*\right)e^{-\varphi}e^{-ih},$$

the second component $v_2 = p_-^{-1} V^{1/2} u_1$ behaves like

$$v_2(x) = -(2\pi i)^{-1}(u_1, V^{1/2}\rho_0)e^{\varphi}e^{ih}(x_1 - ix_2)^{-1} + O(|x|^{-2+\alpha}) = O(|x|^{-1+\alpha}) \quad (7.1)$$

as $|x| \to \infty$. This implies that K has either eigenstates or resonance states at zero energy. This contradicts the assumption and the proof is complete. $\square$

By assumption (5.5), $\lambda_0 = (V\rho_0, \rho_0) \neq 0$. This enables us to define

$$P = \lambda_0^{-1}(q_0 \otimes q_0), \qquad q_0 = V^{1/2}\tilde{\rho}_0, \quad (7.2)$$

as a projection on $[L^2]^2$.

Lemma 7.2 *Assume that* $1/2 < \alpha < 1$. *Let* $Q = 1 - P$ *and* $\Sigma = \operatorname{Ran} Q$. *If* (5.6) *is fulfilled, then* $QZ_0Q : \Sigma \to \Sigma$ *has the bounded inverse* $(1 + QZ_0Q)^{-1}$ *on* Σ.

Proof. We again show by contradiction that $\dim \Psi = 0$, where $\Psi = \{u \in \Sigma : QZ_0Qu = -u\}$. Assume that u not vanishing identically belongs to Ψ. We set

$$v = {}^{\mathrm{t}}(v_1, v_2) = G_0 V^{1/2} u - d\tilde{\rho}_0,$$

where $d = \lambda_0^{-1}(Z_0 u, V^{1/2}\tilde{\rho}_0) = \lambda_0^{-1}(Z_0 u, q_0)$. Since $K_0\tilde{\rho}_0 = 0$ and since

$$V^{1/2} v = Z_0 u - PZ_0 u = QZ_0 u = -u,$$

we see that v satisfies $K_0 v = V^{1/2} u = -Vv$, and hence v solves $Kv = 0$. We also have that $v \neq 0$. The first component v_1 behaves like

$$v_1(x) = -d\rho_0(x) + O(|x|^{-1-\alpha}) = O(|x|^{-\alpha})$$

at infinity. We claim that $v_2 \in L^2$, which follows from (7.1). In fact, we have only to note that

$$(u_1, V^{1/2}\rho_0) = (u, V^{1/2}\tilde{\rho}_0) = -(Vv, \tilde{\rho}_0) = -(V^{1/2}Z_0 u - dV\tilde{\rho}_0, \tilde{\rho}_0) = 0$$

by the choice of constant d. Thus $v \in L^\infty \times L^2$ becomes either eigenstate or resonance state. This proves the lemma. $\square$

Remark 7.1. The converse statements of the two lemmas above are also true, although we do not prove it here. The proof is easy. Hence, if $|V| \ll 1$ is small enough, then (5.6) is fulfilled.

Lemma 7.3 (1) *Let* $0 < \alpha \leq 1/2$ *and set*

$$q = (1 + Z_0)^{-1} q_0 \in L^2 \times L^2.$$

Then q is represented as $q = V^{1/2}e$ with $e = {}^t(e_1, e_2) \in L^\infty \times L^\infty$, and e uniquely solves $Ke = 0$ under the condition that

$$e_1 = r^{-\alpha} + O(|x|^{-1-\alpha}), \quad e_2 = O(|x|^{-1+\alpha}), \qquad |x| \to \infty. \tag{7.3}$$

(2) *Let $1/2 < \alpha < 1$ and set*

$$q = q_0 - (1 + QZ_0Q)^{-1}QZ_0q_0.$$

Then $q = V^{1/2}e$ for some $e = {}^t(e_1, e_2) \in L^\infty \times L^\infty$, and e uniquely solves $Ke = 0$ under the condition that

$$e_1 = O(|x|^{-\alpha}), \quad e_2 = -i\,(\lambda_0/2\pi)\, r^{-1+\alpha}e^{i\theta} + O(|x|^{-2+\alpha}), \qquad |x| \to \infty. \tag{7.4}$$

Proof. (1) If we set $e = \tilde{\rho}_0 - G_0V^{1/2}q$, then it follows that

$$q = q_0 - Z_0q = V^{1/2}e.$$

We assert that e has the desired properties. By definition, e satisfies

$$Ke = -V^{1/2}q + V\left(\tilde{\rho}_0 - G_0V^{1/2}q\right) = V^{1/2}\,(q_0 - q - Z_0q) = 0$$

and obeys (7.3). Since K has neither eigenstates nor resonance states, it is easy to see that e uniquely solves $Ke = 0$. This proves (1).

(2) This is verified in almost the same way as (1). We set

$$r = -(1 + QZ_0Q)^{-1}QZ_0q_0.$$

Then we have

$$r = -QZ_0r - QZ_0q_0 = -Z_0r - Z_0q_0 + PZ_0r + PZ_0q_0$$

and hence q is represented as $q = q_0 + r = V^{1/2}e$, where

$$e = d_1\tilde{\rho}_0 - G_0V^{1/2}r - G_0V^{1/2}q_0 \tag{7.5}$$

with constant $d_1 = 1 + (Z_0(r + q_0), q_0)/\lambda_0$. A simple calculation yields

$$\begin{aligned} Ke &= V^{1/2}\,(d_1q_0 - (r + q_0) - Z_0(r + q_0)) \\ &= V^{1/2}\,(d_1q_0 - q_0 - PZ_0(r + q_0)) = 0. \end{aligned}$$

It is easy to see that $e_1 = O(|x|^{-\alpha})$. We look at the second component e_2. If we note that

$$(V^{1/2}r, \tilde{\rho}_0) = (Qr, V^{1/2}\tilde{\rho}_0) = (Qr, q_0) = 0,$$

then it follows from (7.1) that the second component of $G_0V^{1/2}r$ obeys $O(|x|^{-2+\alpha})$. The second component $-p_-^{-1}V^{1/2}q_0$ of the term $-G_0V^{1/2}q_0$ behaves like

$$-p_-^{-1}V^{1/2}q_0 = (2\pi i)^{-1}e^{\varphi}e^{ih}r^{-1}e^{i\theta}\lambda_0 + O(|x|^{-2+\alpha}).$$

This yields the coefficient $-i(\lambda_0/2\pi)$ in (7.4). Thus we can show that e has the desired properties and the lemma is proved. □

We end the section by proving Lemma 5.1.

Proof of Lemma 5.1. (1) Assume that $0 < \alpha \le 1/2$. Let

$$q = {}^{\mathrm{t}}(q_1, q_2) = (1+Z_0)^{-1}q_0 = V^{1/2}e$$

be as in Lemma 7.3, where $e = \tilde{\rho}_0 - G_0V^{1/2}q$. Then the second component $e_2 = -p_-^{-1}V^{1/2}q_1$ behaves like

$$e_2 = i\lambda_2 r^{-1+\alpha}e^{i\theta} + O(|x|^{-2+\alpha}), \qquad |x| \to \infty,$$

for some constant λ_2. We show that λ_2 is real. To to this, we compute

$$((1+Z_0)^{-1}q_0, q_0) = (q, q_0) = (Ve, \tilde{\rho}_0) = -(K_0e, \tilde{\rho}_0) = -(p_-e_2, \rho_0).$$

Recall the representation (2.2) for π_- in terms of the polar coordinates. Since $p_- = \pi_-$ on $\{|x| > 2\}$ and since $p_+\rho_0 = 0$, we have

$$((1+Z_0)^{-1}q_0, q_0) = i\lim_{R\to\infty}\int_{|x|=R} e^{-i\theta}e_2\overline{\rho}_0\,ds = -2\pi\lambda_2$$

by partial integration. This yields

$$\lambda_2 = -((1+Z_0)^{-1}q_0, q_0)/2\pi \tag{7.6}$$

and λ_2 is real. This implies that e has all the desired properties.

(2) We proceed to proving (2). Assume that $1/2 < \alpha < 1$. Let e be defined by (7.5) in the proof of Lemma 7.3. We calculate the constant d_1 in (7.5). According to the argument in the proof of Lemma 7.3, we have

$$\begin{aligned} d_1 &= 1 + ((r+q_0), Z_0q_0)/\lambda_0 = 1 + (V^{1/2}e, Z_0q_0)/\lambda_0 \\ &= 1 + (q, Z_0q_0)/\lambda_0 = 1 + (q_0 - (1+QZ_0Q)^{-1}QZ_0q_0, Z_0q_0)/\lambda_0 \\ &= 1 + \Big((q_0, Z_0q_0) - ((1+QZ_0Q)^{-1}QZ_0q_0, QZ_0q_0)\Big)/\lambda_0. \end{aligned}$$

Thus d_1 is real, and e_1 behaves like $e_1(x) = d_1r^{-\alpha} + O(|x|^{-1-\alpha})$. The desired solution is obtained as $-(2\pi/\lambda_0)e$, and then

$$\lambda_1 = -(2\pi/\lambda_0)d_1 \tag{7.7}$$

is also determined as a real number. This completes the proof. $\square$

8. Convergence of resolvent: proof of Theorems 5.1 and 5.2

In this section we prove Theorems 5.1 and 5.2 through a series lemmas. We recall that $\eta_0 \in C_0(\boldsymbol{R}^2)$ has support away from the origin. We also use the notation $o_2(1)$ to denote remainder terms of which the L^2 norm obeys the bound $o(1)$ as $\varepsilon \to 0$. We start by the following two lemmas.

Lemma 8.1 *Let $\xi_\pm = \xi_\pm(x;E)$ be defined by (3.1). Then*

$$\eta_0 R(E \pm i0; H_\infty) J_\varepsilon \eta = \beta_\pm \eta_0 \left(\xi_\pm \otimes \tilde{r}_0\right) \eta \varepsilon^{1-\alpha} + Op(\varepsilon),$$

where $\tilde{r}_0(x) = {}^{\mathrm{t}}(r_0(x), 0)$ with $r_0(x) = |x|^{-\alpha}$, and

$$\beta_\pm = \mp \left(2^{\alpha-2}/\Gamma(1-\alpha)\right) E^{1-\alpha}. \tag{8.1}$$

Lemma 8.2 *Let the notation be as in Lemma 8.1. Then*

$$\eta_0 R(E \pm i0; K_{0\varepsilon}) J_\varepsilon \eta = \beta_\pm \left((\eta_0 \xi_\pm + o_2(1)) \otimes \tilde{\rho}_0\right) \eta \varepsilon^{1-\alpha} + Op(\varepsilon)$$

and, in particular, $\Gamma_\varepsilon(E \pm i0)$ defined by (6,4) takes the form

$$\eta_0 \Gamma_\varepsilon(E \pm i0) = \beta_\pm \left((\eta_0 \xi_\pm + o_2(1)) \otimes q_0\right) \varepsilon^{1-\alpha} + Op(\varepsilon),$$

where $q_0 = {}^{\mathrm{t}}(V^{1/2}\rho_0, 0) \in [L^2]^2$ is defined by (6.15).

Proof of Lemma 8.1. We prove the lemma for the $+$ case only. For brevity, we write

$$\xi_+ = {}^{\mathrm{t}}(\xi_1, \xi_2), \quad \xi_1 = -ie^{i\alpha\pi} H_\alpha(Er), \quad \xi_2 = H_{1-\alpha}(Er)e^{i\theta}.$$

The resolvent $R(E+i0; H_\infty)$ is represented in terms of $R(E^2+i0; L_{AB})$ and $R(E^2+i0; L_Z)$ by (3.2). We first consider $R(E^2+i0; L_Z)$. This admits the decomposition

$$R(E^2 + i0; L_Z) = \tilde{R}_0 \oplus \sum_{l \neq 0} \oplus R_l$$

with respect to angular momentum (see (3.4)), and the Green kernels of $\tilde{R}_0$ and R_l are defined by (3.5). Since η_0 has support away from the origin, we can take ε so small that $|x| > \varepsilon|y|$ when $x \in \operatorname{supp} \eta_0$ and $y \in \operatorname{supp} \eta$, and hence $\eta_0 \tilde{R}_0 J_\varepsilon \eta$ has the kernel

$$G(x,y) = \varepsilon(ie^{i\alpha\pi}/4)\eta_0(x) H_\alpha(E|x|) J_{-\alpha}(\varepsilon E|y|)\eta(y)$$

by a change of variables. This implies that

$$E\eta_0 \tilde{R}_0 J_\varepsilon \eta = \beta_+ \eta_0 (\xi_1 \otimes r_0) \eta \varepsilon^{1-\alpha} + Op(\varepsilon).$$

A similar argument applies to $R_l,\ l \neq 0$, and we obtain $\eta_0 R_l J_\varepsilon \eta = Op(\varepsilon)$ uniformly in l. Thus we have

$$E\eta_0 R(E^2 + i0; L_Z) J_\varepsilon \eta = \beta_+ \eta_0 (\xi_1 \otimes r_0) \eta \varepsilon^{1-\alpha} + Op(\varepsilon).$$

Since $\pi_+ \xi_1 = E\xi_2$ by (2.14), we make use of this relation to obtain that

$$\eta_0 \pi_+ R(E^2 + i0; L_Z) J_\varepsilon \eta = \beta_+ \eta_0 (\xi_2 \otimes r_0) \eta \varepsilon^{1-\alpha} + Op(\varepsilon).$$

Similarly $R(E^2 + i0; L_{AB})$ is shown to obey

$$\eta_0 R(E^2 + i0; L_{AB}) J_\varepsilon \eta = Op(\varepsilon), \quad \eta_0 \pi_- R(E^2 + i0; L_{AB}) J_\varepsilon \eta = Op(\varepsilon).$$

This proves the lemma. $\square$

Proof of Lemma 8.2. We again prove the lemma for the + case only. Set

$$\zeta_\varepsilon(x) = \zeta(x/\varepsilon), \qquad \zeta(x) = 1 - \chi_0(x/2), \tag{8.2}$$

for the basic cut–off function $\chi_0(x)$ with property (4.3). Then we have

$$\operatorname{supp} \zeta_\varepsilon \subset \{|x| > 2\varepsilon\}, \qquad \zeta_\varepsilon = 1 \ \text{ on } \ \{|x| > 4\varepsilon\}.$$

We may assume that $\zeta_\varepsilon \eta_0 = \eta_0$ for ε small enough, and we have

$$\begin{aligned} \eta_0 R(E + i0; K_{0\varepsilon}) J_\varepsilon \eta &= \eta_0 R(E + i0; H_\infty) \zeta_\varepsilon J_\varepsilon \eta \\ &+ \eta_0 R(E + i0; H_\infty) W_\varepsilon R(E + i0; K_{0\varepsilon}) J_\varepsilon \eta \end{aligned}$$

by the resolvent identity, where $W_\varepsilon = H_\infty \zeta_\varepsilon - \zeta_\varepsilon K_{0\varepsilon}$. By (4.1), $H_\infty = K_{0\varepsilon} = D_\alpha$ over $|x| > 2\varepsilon$. If we make use of relations $\zeta_\varepsilon = J_\varepsilon \zeta J_\varepsilon^*$ and $D_\alpha = \varepsilon^{-1} J_\varepsilon D_\alpha J_\varepsilon^*$, W_ε equals the commutator $W_\varepsilon = [D_\alpha, \zeta_\varepsilon] = \varepsilon^{-1} J_\varepsilon [D_\alpha, \zeta] J_\varepsilon^*$. If we further use the relation

$$J_\varepsilon^* R(E + i0; K_{0\varepsilon}) J_\varepsilon = \varepsilon R(k + i0; K_0)$$

with $k = \varepsilon E$, then we obtain

$$\eta_0 R(E + i0; K_{0\varepsilon}) J_\varepsilon \eta = \eta_0 R(E + i0; H_\infty) J_\varepsilon \zeta \eta + F_\varepsilon R(k + i0; K_0) \eta, \tag{8.3}$$

where $F_\varepsilon = \eta_0 R(E + i0; H_\infty) J_\varepsilon [D_\alpha, \zeta]$. It follows from Lemma 8.1 that F_ε is of the form

$$F_\varepsilon = \begin{pmatrix} 0 & \beta_+ \eta_0 \,(\xi_1 \otimes r_0)\, [\pi_-, \zeta] \varepsilon^{1-\alpha} \\ 0 & \beta_+ \eta_0 \,(\xi_2 \otimes r_0)\, [\pi_-, \zeta] \varepsilon^{1-\alpha} \end{pmatrix} + Op(\varepsilon)$$

with $\xi_+ = {}^{\mathrm{t}}(\xi_1, \xi_2)$ as in the proof of Lemma 8.1. Next we evaluate $F_\varepsilon R(k + i0; K_0)\eta$. The operator $\eta R(k + i0; K_0)\eta$ admits the decomposition in Proposition 6.3 for $\eta \in C_0(\boldsymbol{R}^2)$. We calculate :

$$\begin{aligned} &F_\varepsilon \,(\tilde\rho_0 \otimes \tilde\rho_0)\, \eta \varepsilon^{1-2\alpha} = (o_2(1) \otimes \tilde\rho_0)\, \eta, \\ &F_\varepsilon G_0 \eta = \beta_+ \eta_0 \,(\xi_+ \otimes \tilde r_0) [\, \pi_-, \zeta] p_-^{-1} \eta \varepsilon^{1-\alpha} + Op(\varepsilon), \\ &O(\varepsilon^{2(1-\alpha)}) F_\varepsilon G_1 \eta = (o_2(1) \otimes \tilde\rho_0)\, \eta + Op(\varepsilon) \end{aligned}$$

for G_0 and G_1 as in Proposition 6.3. We combine these relations with Lemma 8.1. Then

$$\eta_0(E+i0;K_{0\varepsilon})J_\varepsilon\eta = \beta_+\left((\eta_0\xi_+ + o_2(1))\otimes\tilde r_1\right)\eta\varepsilon^{1-\alpha} + Op(\varepsilon)$$

with $\tilde r_1 = {}^{\mathrm t}(r_1, 0)$, where

$$r_1 = \zeta r_0 + p_+^{-1}[\zeta,\pi_+]r_0, \qquad r_0(x) = |x|^{-\alpha}.$$

Since $\zeta\pi_+ r_0 = 0$, it is easy to see that $p_+ r_1 = 0$, and also $r_1(x)$ behaves like

$$r_1(x) = |x|^{-\alpha} + O(|x|^{-1-\alpha})$$

at infinity. By uniqueness, this implies that $r_1 = \rho_0$, and the proof is complete. □

Theorem 5.1 is obtained as an immediate consequence of the lemma below.

Lemma 8.3 *One has*

$$\eta_0 R(E\pm i0;K_{0\varepsilon})\eta_0 \to \eta_0 R(E\pm i0;H_\infty)\eta_0, \qquad \varepsilon\to 0,$$

in norm.

Proof. We deal with the + case only. Let ζ_ε be defined by (8.2). Since $\zeta_\varepsilon\eta_0 = \eta_0$ for ε small enough, we have

$$\begin{aligned}\eta_0 R(E+i0;K_{0\varepsilon})\eta_0 &= \eta_0 R(E+i0;H_\infty)\eta_0\\ &+ \eta_0 R(E+i0;K_{0\varepsilon})W_\varepsilon^* R(E+i0;H_\infty)\eta_0\end{aligned} \tag{8.4}$$

by the resolvent identity, where

$$W_\varepsilon^* = (H_\infty\zeta_\varepsilon - \zeta_\varepsilon K_{0\varepsilon})^* = \zeta_\varepsilon H_\infty - K_{0\varepsilon}\zeta_\varepsilon = \varepsilon^{-1}J_\varepsilon[\zeta, D_\alpha]J_\varepsilon^*.$$

We decompose the second term on the right side of (8.4) into the product $F_{1\varepsilon}F_{0\varepsilon}F_{2\varepsilon}$ of three operators, where

$$F_{1\varepsilon} = \eta_0 R(E+i0;K_{0\varepsilon})J_\varepsilon\eta, \quad F_{2\varepsilon} = \eta J_\varepsilon^* R(E+i0;H_\infty)\eta_0 = (\eta_0 R(E-i0;H_\infty)J_\varepsilon\eta)^*$$

for some $\eta\in C_0(\boldsymbol{R}^2)$, and $F_{0\varepsilon} = \varepsilon^{-1}[\zeta, D_\alpha]$. By Lemmas 8.1 and 8.2, $F_{1\varepsilon}$ and $F_{2\varepsilon}$ take the form

$$F_{1\varepsilon} = \begin{pmatrix} Op(\varepsilon^{1-\alpha}) & Op(\varepsilon)\\ Op(\varepsilon^{1-\alpha}) & Op(\varepsilon)\end{pmatrix}, \qquad F_{2\varepsilon} = \begin{pmatrix} Op(\varepsilon^{1-\alpha}) & Op(\varepsilon^{1-\alpha})\\ Op(\varepsilon) & Op(\varepsilon)\end{pmatrix}$$

and $F_{0\varepsilon}$ equals

$$F_{0\varepsilon} = \begin{pmatrix} 0 & \varepsilon^{-1}[\zeta,\pi_-]\\ \varepsilon^{-1}[\zeta,\pi_+] & 0\end{pmatrix}.$$

A simple computation yields $F_{1\varepsilon}F_{0\varepsilon}F_{2\varepsilon} = Op(\varepsilon^{1-\alpha})$. This proves the lemma. $\square$

Proof of Theorem 5.1. If we recall that f_ε and g_∞ are represented by (5.2) and (5.3) respectively, then the theorem follows from Lemma 8.3 at once. $\square$

We proceed to the proof of Theorem 5.2. We first accept the lemma below as proved to complete the proof of the theorem.

Lemma 8.4 *Assume that (5.6) is fulfilled. Recall that $P : [L^2]^2 \to [L^2]^2$ is the projection defined by (7.2), and set $Q = 1 - P$. Then $(1+X_\varepsilon)^{-1}$ obeys the following asymptotic form as $\varepsilon \to 0$:*

(1) *If $0 < \alpha < 1/2$, then*

$$(1+X_\varepsilon)^{-1} = (1+Z_0)^{-1} + Op(\varepsilon^{1-2\alpha}).$$

(2) *If $\alpha = 1/2$, then*

$$(1+X_\varepsilon)^{-1} = (1+Z_0)^{-1} + a\,(q \otimes q) + op(\varepsilon^0),$$

where

$$a = -i/(2\pi + i\tau), \quad \tau = (q, q_0), \quad q = (1+Z_0)^{-1} q_0. \tag{8.5}$$

(3) *If $1/2 < \alpha < 1$, then*

$$\begin{aligned} (1+X_\varepsilon)^{-1} &= \delta_+(\varepsilon) P \left(1 + Op(\varepsilon^{2\alpha-1})\right) P \\ &- \delta_+(\varepsilon) Q \left((Q + QZ_0Q)^{-1} QZ_0 + Op(\varepsilon^{2\alpha-1}) + Op(\varepsilon^{2(1-\alpha)})\right) P \\ &- \delta_+(\varepsilon) P \left(Z_0 Q (Q + QZ_0Q)^{-1} + Op(\varepsilon^{2\alpha-1}) + Op(\varepsilon^{2(1-\alpha)})\right) Q \\ &+ Q \left((Q + QZ_0Q)^{-1} + Op(\varepsilon^{2\alpha-1})\right) Q, \end{aligned}$$

where

$$\delta_+(\varepsilon) = 1/\mu_+(\varepsilon), \quad \mu_+(\varepsilon) = 1 + \gamma_-(k) i^{2\alpha} k^{1-2\alpha} \lambda_0, \quad k = \varepsilon E. \tag{8.6}$$

Proof of Theorem 5.2. The proof is based on the relation

$$R(E+i0; K_\varepsilon) = R(E+i0; K_{0\varepsilon}) - \varepsilon^{-1} \Gamma_\varepsilon(E+i0)(1+X_\varepsilon)^{-1} \Gamma_\varepsilon(E-i0)^*$$

derived by (6.3). By Lemma 8.3, we have

$$\eta_0 R(E+i0; K_{0\varepsilon}) \eta_0 \to \eta_0 R(E+i0; H_\infty) \eta_0, \qquad \varepsilon \to 0,$$

in norm for the first operator on the right side.

We analyze the second operator

$$R(\varepsilon) = \varepsilon^{-1} \eta_0 \Gamma_\varepsilon(E+i0)(1+X_\varepsilon)^{-1} \Gamma_\varepsilon(E-i0)^* \eta_0.$$

The behavior as $\varepsilon \to 0$ of $R(\varepsilon)$ takes a different form according as $0 < \alpha < 1/2$, $\alpha = 1/2$ or $1/2 < \alpha < 1$.

(1) Let $0 < \alpha < 1/2$. Then it follows from Lemmas 8.2 and 8.4 that

$$\|R(\varepsilon)\| = O(\varepsilon^{-1})O(\varepsilon^{2(1-\alpha)}) = O(\varepsilon^{1-2\alpha}),$$

so that

$$\eta_0 R(E + i0; K_{0\varepsilon})\eta_0 \to \eta_0 R(E + i0; H_\infty)\eta_0, \qquad \varepsilon \to 0,$$

and hence $f_\varepsilon \to g_\infty$. This proves (1).

(2) If $\alpha = 1/2$, then

$$\beta_\pm = \mp 2^{-3/2}E^{1/2}/\pi^{1/2}$$

by (8.1), so that $\beta_+\beta_- = -E/8\pi$. By Lemmas 8.2 and 8.4 again, we have

$$R(\varepsilon) \to a_0\eta_0(\xi_+ \otimes \xi_-)\eta_0,$$

where

$$a_0 = \beta_+\beta_-\left(\tau + a\tau^2\right) = -(E/4)\,(i + 2\pi/\tau)^{-1}.$$

Since $\lambda_2 = -\tau/2\pi$ by (7.6), it follows from Proposition 3.1 that

$$\eta_0 R(E + i0; K_{0\varepsilon})\eta_0 \to \eta_0 R(E + i0; H_\kappa)\eta_0, \quad \kappa = 1/\lambda_2.$$

This proves (2).

(3) The final case is $1/2 < \alpha < 1$. Recall that

$$\|q_0\|^2 = \|V^{1/2}\rho_0\|^2 = (V\rho_0, \rho_0) = \lambda_0$$

by (5.5). Since $Pq_0 = q_0$ and $Qq_0 = 0$, we have by Lemmas 8.2 and 8.4 that $R(\varepsilon)$ behaves like

$$R(\varepsilon) = a_1(\varepsilon)\eta_0(\xi_+ \otimes \xi_-)\eta_0 + op(\varepsilon^0), \quad a_1(\varepsilon) = \varepsilon^{-1}\beta_+\beta_-\varepsilon^{2(1-\alpha)}\delta_+(\varepsilon)\lambda_0.$$

We calculate $\beta_+\beta_- = -\left(2^{\alpha-2}E^{1-\alpha}/\Gamma(1-\alpha)\right)^2$ by (8.1). Since

$$\gamma_-(k) \to -1/\gamma_0 = \Gamma(\alpha)/\left(2^{2(1-\alpha)}\pi\Gamma(1-\alpha)\right)$$

in Proposition 6.1, it follows that

$$\varepsilon^{1-2\alpha}\delta_+(\varepsilon) \to -\gamma_0 i^{-2\alpha}E^{2\alpha-1}/\lambda_0$$

and hence

$$\begin{aligned} a_1(\varepsilon) &\to \gamma_0 i^{-2\alpha}E^{2\alpha-1}\left(2^{\alpha-2}E^{1-\alpha}/\Gamma(1-\alpha)\right)^2 \\ &= -(E/4)\,(\pi/\Gamma(\alpha)\Gamma(1-\alpha))\,i^{-2\alpha} = -(E/4)\left(\sin\alpha\pi/e^{i\alpha\pi}\right). \end{aligned}$$

This, together with Proposition 3.1, implies that $f_\varepsilon \to g_0$, and (3) is obtained. Thus the proof of the theorem is now complete. $\square$

Proof of Lemma 8.4. By Proposition 6.3, we have

$$1 + X_\varepsilon = 1 + Z_0 + \gamma_-(k) i^{2\alpha} k^{1-2\alpha} (q_0 \otimes q_0) + O(\varepsilon^{2(1-\alpha)}) Z_1 + Op(\varepsilon)$$

for $k = \varepsilon E > 0$, where $\gamma_-(k) = -1/\gamma_0 + o(1)$ as $\varepsilon \to 0$.

(1) Assume that $0 < \alpha < 1/2$. If $K = K_0 + V$ has neither bound nor resonance state at zero energy, then $1 + Z_0 : [L^2]^2 \to [L^2]^2$ admits a bounded inverse by Lemma 7.1, and hence $(1 + X_\varepsilon)^{-1}$ takes the form as in the lemma.

(2) If $\alpha = 1/2$, we have

$$1 + X_\varepsilon = 1 + Z_0 + (i/2\pi) (q_0 \otimes q_0) + op(\varepsilon^0).$$

Let $q = (1 + Z_0)^{-1} q_0$ and $\tau = (q, q_0)$ be as in (8.5). Then

$$1 + X_\varepsilon = (1 + Z_0) (1 + (i/2\pi) (q \otimes q_0)) + op(\varepsilon^0).$$

A simple computation yields

$$(1 + (i/2\pi) (q \otimes q_0))^{-1} = 1 + a (q \otimes q_0)$$

with a as in the lemma. Hence $(1 + X_\varepsilon)^{-1}$ takes the desired form.

(3) We deal with the case $1/2 < \alpha < 1$. We employ the method from [12], which has been applied to the analysis on the behavior at low energy of resolvents of Schrödinger operators $-\Delta + V$ in two dimensions. We write $\mu(\varepsilon)$ and

$$\delta(\varepsilon) = 1/\mu(\varepsilon) = O(\varepsilon^{2\alpha-1}), \qquad \varepsilon \to 0,$$

for $\mu_+(\varepsilon)$ and $\delta_+(\varepsilon)$ respectively. Then

$$1 + X_\varepsilon = \mu(\varepsilon) P + Q + Z_0 + O(\varepsilon^{2(1-\alpha)}) Z_1 + Op(\varepsilon)$$

by Proposition 6.3. If we use the two simple relations

$$(\mu(\varepsilon) P + Q)^{-1} = \delta(\varepsilon) P + Q, \qquad (1 + Q Z_0 P)^{-1} = 1 - Q Z_0 P,$$

then $1 + X_\varepsilon$ takes the form

$$1 + X_\varepsilon = (\mu(\varepsilon) P + Q) (1 + Q Z_0 P) G_\varepsilon,$$

and hence

$$(1 + X_\varepsilon)^{-1} = G_\varepsilon^{-1} (\delta(\varepsilon)(P - Q Z_0 P) + Q), \tag{8.7}$$

where G_ε is represented in the form

$$G_\varepsilon = 1 + Q Z_0 Q + \delta(\varepsilon)(1 - Q Z_0) P Z_0 + Q Op(\varepsilon^{2(1-\alpha)}) + Op(\varepsilon).$$

We now set $\Sigma_0 = \operatorname{Ran} P$ and $\Sigma = \operatorname{Ran} Q$. The second factor on the right side of (8.7) has the matrix representation

$$\delta(\varepsilon)(P - QZ_0P) + Q = \begin{pmatrix} \delta(\varepsilon)P & 0 \\ -\delta(\varepsilon)QZ_0P & Q \end{pmatrix} : \begin{pmatrix} \Sigma_0 \\ \Sigma \end{pmatrix} \to \begin{pmatrix} \Sigma_0 \\ \Sigma \end{pmatrix}, \tag{8.8}$$

while $G_\varepsilon = (G_{jk}(\varepsilon))_{0\le j,k\le 1}$ has the components

$$G_{00} = P(1 + Op(\varepsilon^{2\alpha-1}))P, \quad G_{01} = P(\delta(\varepsilon)Z_0 + Op(\varepsilon))Q,$$

$$G_{10} = Q(-\delta(\varepsilon)Z_0PZ_0 + Op(\varepsilon^{2(1-\alpha)}))P, \quad G_{11} = Q(1 + Z_0 + Op(\varepsilon^{2\alpha-1}))Q.$$

By Lema 7.2, $Q + QZ_0Q : \Sigma \to \Sigma$ has a bounded inverse, so that $G_{11}^{-1} : \Sigma \to \Sigma$ exists for ε small enough. If we take account of this fact, then $G_\varepsilon^{-1} = E_\varepsilon = (E_{jk}(\varepsilon))_{0\le j,k\le 1}$ can be calculated as

$$E_{00} = \left(G_{00} - G_{01}G_{11}^{-1}G_{10}\right)^{-1}, \quad E_{01} = -\left(G_{00} - G_{01}G_{11}^{-1}G_{10}\right)^{-1} G_{01}G_{11}^{-1},$$

$$E_{10} = -\left(G_{11} - G_{10}G_{00}^{-1}G_{01}\right)^{-1} G_{10}G_{00}^{-1}, \quad E_{11} = \left(G_{11} - G_{10}G_{00}^{-1}G_{01}\right)^{-1}.$$

Hence $(1 + X_\varepsilon)^{-1}$ takes the form

$$(1 + X_\varepsilon)^{-1} = \delta(\varepsilon)(E_{00}P - E_{01}QZ_0P) + E_{01}Q + \delta(\varepsilon)(E_{10}P - E_{11}QZ_0P) + E_{11}Q$$

by use of (8.7) and (8.8). Each component $E_{jk}(\varepsilon)$ behaves like :

$$\begin{aligned}
E_{00} &= P(1 + Op(\varepsilon^{2\alpha-1}))P, \\
E_{01} &= P(-\delta(\varepsilon)Z_0Q(Q + QZ_0Q)^{-1} + Op(\varepsilon^{2(2\alpha-1)}) + Op(\varepsilon))Q, \\
E_{10} &= Q(\delta(\varepsilon)(Q + QZ_0Q)^{-1}QZ_0PZ_0 + Op(\varepsilon^{2(2\alpha-1)}) + Op(\varepsilon^{2(1-\alpha)}))P, \\
E_{11} &= Q((Q + QZ_0Q)^{-1} + Op(\varepsilon^{2\alpha-1}))Q.
\end{aligned}$$

If we take account of these relations, $(1 + X_\varepsilon)^{-1}$ can be shown to take the form in the lemma, and the proof is complete. □

We end the section by making a brief comment on the case when $\alpha < 0$ and $\alpha > 1$.

Remark 8.1. If we replace the magnetic potential $A(x)$ by $-A(x)$, the argument here extends to the case $-1 < \alpha < 0$ without any essential change. If $|\alpha| > 1$, then the magnetic Schrödinger operator $L(A, -b)$ has eigenstates at zero energy besides the resonance state by the Aharonov–Casher theorem [4], so that the norm convergence of resolvent $\eta_0 R(E + i0; K_\varepsilon)\eta_0$ can not be expected ([23]). However the strong convergence can be expected, and hence Theorems 5.1 and 5.2 seem to remain true in the case $|\alpha| > 1$ also.

9. Scattering in the interaction of cosmic string with matter

The last section is devoted to proving Theorem 5.3. We begin by representing the amplitude $f_{2\varepsilon}(\omega \to \tilde{\omega}; E)$ in question in terms of the resolvent $R(E+i0; \mathbf{K}_\varepsilon)$ of

$$\mathbf{K}_\varepsilon = \mathbf{K}_{0\varepsilon} + \mathbf{V}_\varepsilon = \begin{pmatrix} K_{0\varepsilon} & 0 \\ 0 & K_{0\varepsilon} \end{pmatrix} + \begin{pmatrix} 0 & V_\varepsilon \\ V_\varepsilon & 0 \end{pmatrix}.$$

If we decompose $\mathbf{V}$ into the product

$$\mathbf{V} = \begin{pmatrix} 0 & V \\ V & 0 \end{pmatrix} = \begin{pmatrix} V^{1/2} & 0 \\ 0 & V^{1/2} \end{pmatrix} \begin{pmatrix} 0 & V^{1/2} \\ V^{1/2} & 0 \end{pmatrix} = \mathbf{V}_1 \mathbf{V}_2,$$

then almost the same argument as used to derive (6.3) enables us to obtain

$$R(E+i0; \mathbf{K}_\varepsilon) = R(E+i0; \mathbf{K}_{0\varepsilon}) - \varepsilon^{-1} \mathbf{\Gamma}_{1\varepsilon}(E+i0)\,(1+\mathbf{X}_\varepsilon)^{-1}\,\mathbf{\Gamma}_{2\varepsilon}(E-i0)^*, \quad (9.1)$$

where $\mathbf{X}_\varepsilon = \mathbf{V}_2 R(k+i0; \mathbf{K}_0)\mathbf{V}_1$ with $k = \varepsilon E > 0$, and

$$\mathbf{\Gamma}_{1\varepsilon}(E+i0) = R(E+i0; \mathbf{K}_{0\varepsilon}) J_\varepsilon \mathbf{V}_1, \quad \mathbf{\Gamma}_{2\varepsilon}(E-i0) = R(E-i0; \mathbf{K}_{0\varepsilon}) J_\varepsilon \mathbf{V}_2.$$

A direct computation yields

$$\mathbf{\Gamma}_{1\varepsilon} = \begin{pmatrix} \Gamma_\varepsilon(E+i0) & 0 \\ 0 & \Gamma_\varepsilon(E+i0) \end{pmatrix}, \quad \mathbf{\Gamma}_{2\varepsilon} = \begin{pmatrix} 0 & \Gamma_\varepsilon(E-i0) \\ \Gamma_\varepsilon(E-i0) & 0 \end{pmatrix},$$

where $\Gamma_\varepsilon(E \pm i0)$ is defined by (6.4). We further have

$$\mathbf{X}_\varepsilon = \begin{pmatrix} 0 & X_\varepsilon \\ X_\varepsilon & 0 \end{pmatrix}, \quad (1+\mathbf{X}_\varepsilon)^{-1} = \begin{pmatrix} (1-X_\varepsilon^2)^{-1} & -X_\varepsilon(1-X_\varepsilon^2)^{-1} \\ -X_\varepsilon(1-X_\varepsilon^2)^{-1} & (1-X_\varepsilon^2)^{-1} \end{pmatrix}.$$

We divide $R(E+i0; \mathbf{K}_\varepsilon)$ into the block form

$$R(E+i0; \mathbf{K}_\varepsilon) = \left(R_{jk}(E+i0; \mathbf{K}_\varepsilon)\right)_{1 \le j,k \le 2},$$

where $R_{jk}(E+i0; \mathbf{K}_\varepsilon)$ acts on $[L^2]^2$. In particular, we have

$$R_{21}(E+i0; \mathbf{K}_\varepsilon) = -\varepsilon^{-1} \Gamma_\varepsilon(E+i0)(1-X_\varepsilon^2)^{-1} \Gamma_\varepsilon(E-i0)^*.$$

We can represent $f_{2\varepsilon}(\omega \to \tilde{\omega}; E)$ as

$$f_{2\varepsilon}(\omega \to \tilde{\omega}; E) = (iE/8\pi)^{1/2} (R_{21}(E+i0; \mathbf{K}_\varepsilon)\Pi_+\psi_+(\omega), \Pi_-\psi_-(\tilde{\omega}))$$

by repeating the same argument as in section 4, and hence we have

$$f_{2\varepsilon} = -\varepsilon^{-1}(iE/8\pi)^{1/2}(\Gamma_\varepsilon(E+i0)(1-X_\varepsilon^2)^{-1}\Gamma_\varepsilon(E-i0)^*\Pi_+\psi_+(\omega), \Pi_-\psi_-(\tilde{\omega})). \quad (9.2)$$

The argument here is based on this representation.

Lemma 9.1 *The operator $K_0 - V$ has a resonance at zero energy if and only if so does $K = K_0 + V$, and the same statement is also true for an eigenstate.*

Proof. The lemma is easy to prove. For brevity, we consider the case $0 < \alpha \le 1/2$ only. A similar argument applies to the case $1/2 < \alpha < 1$. Let $v_+ = (v_1, v_2) \in L^2 \times L^\infty$ be a resonance state of $K_0 + V$. If we set $v_- = (v_1, -v_2)$, then v_- solves $(K_0 - V)\, v_- = 0$, and it becomes a resonance by Definition 5.1. The case of eigenstate is also shown in the same way. $\Box$

We keep the same notation as in the previous sections. The lemma above implies the existence of bounded inverses $(1 - Z_0)^{-1} : [L^2]^2 \to [L^2]^2$ and $(1 - QZ_0Q)^{-1} : \Sigma \to \Sigma$. The following lemma is verified in exactly the same way as in the proof of Lemmas 8.4. We skip the proof.

Lemma 9.2 *If (5.6) is fulfilled, then $(1 - X_\varepsilon)^{-1}$ has the following asymptotic form as $\varepsilon \to 0$:*

(1) *If $0 < \alpha < 1/2$, then*

$$(1 - X_\varepsilon)^{-1} = (1 - Z_0)^{-1} + Op(\varepsilon^{1-2\alpha}).$$

(2) *If $\alpha = 1/2$, then*

$$(1 - X_\varepsilon)^{-1} = (1 - Z_0)^{-1} + a'\,(q' \otimes q') + op(\varepsilon^0),$$

where

$$a' = i/(2\pi - i\tau'), \quad \tau' = (q', q_0), \quad q' = (1 - Z_0)^{-1} q_0. \tag{9.3}$$

(3) *If $1/2 < \alpha < 1$, then*

$$\begin{aligned} (1 - X_\varepsilon)^{-1} &= \delta_-(\varepsilon) P \left(1 + Op(\varepsilon^{2\alpha-1})\right) P \\ &+ \delta_-(\varepsilon) Q \left((Q - QZ_0Q)^{-1} Q Z_0 + Op(\varepsilon^{2\alpha-1}) + Op(\varepsilon^{2(1-\alpha)})\right) P \\ &+ \delta_-(\varepsilon) P \left(Z_0 Q (Q - QZ_0Q)^{-1} + Op(\varepsilon^{2\alpha-1}) + Op(\varepsilon^{2(1-\alpha)})\right) Q \\ &+ Q \left((Q - QZ_0Q)^{-1} + Op(\varepsilon^{2\alpha-1})\right) Q, \end{aligned}$$

where

$$\delta_-(\varepsilon) = 1/\mu_-(\varepsilon), \quad \mu_-(\varepsilon) = 1 - \gamma_-(k) i^{2\alpha} k^{1-2\alpha} \lambda_0, \quad k = \varepsilon E. \tag{9.4}$$

Lemma 9.3 *Let $\xi_\pm$ be defined by (3.1). Set*

$$I_+ = (\xi_+, \Pi_- \psi_-(\tilde{\omega}))\,, \qquad I_- = (\xi_-, \Pi_+ \psi_+(\omega))\,.$$

Then

$$I_+ = -4e^{i\alpha\pi/2}/E, \qquad I_- = 4e^{-i\alpha\pi/2}/E.$$

Proof. We calculate I_+ only. A similar computation applies to I_-. For brevity, we write

$$\xi_+ = \xi = {}^{\mathrm{t}}(\xi_1, \xi_2), \quad \psi_- = \psi = {}^{\mathrm{t}}(\psi_1, \psi_2), \quad \chi_-(x) = \chi_0(x/4) = \chi(x).$$

By (4.3), χ has support in $\{|x| < 8\}$ and $\chi = 1$ on $\{|x| < 4\}$. Since

$$\Pi_- \psi = [D_\alpha, \chi]\psi = [D_\alpha - E, \chi]\psi = (D_\alpha - E)\,\chi\psi$$

for $x \neq 0$, I_+ equals

$$I_+ = \lim_{\delta \to 0} \int_{|x|>\delta} \left(\xi_1 \overline{(\pi_- \chi\psi_2 - E\chi\psi_1)} + \xi_2 \overline{(\pi_+ \chi\psi_1 - E\chi\psi_2)} \right) dx.$$

Note that $(D_\alpha - E)\,\xi = 0$, and π_+ and π_- take the form

$$\pi_+ = e^{i\theta}\,(-i\partial_r + \ldots), \qquad \pi_- = e^{-i\theta}\,(-i\partial_r + \ldots)$$

by (2.2). We integrate by parts to calculate I_+. Since $\chi = 1$ on $\{|x| = \delta\}$, we have

$$I_+ = -i \lim_{\delta \to 0} \int_{|x|=\delta} \left(e^{i\theta} \xi_1 \overline{\psi}_2 + e^{-i\theta} \xi_2 \overline{\psi}_1 \right) ds, \quad ds = \delta\, d\theta.$$

By (2.5) and (2.22), the first term in the integrand obeys

$$e^{i\theta} \xi_1(x) \overline{\psi}_2(x) = O(r^{1-2\alpha}) + O(1), \qquad r = |x| \to 0,$$

and hence

$$\lim_{\delta \to 0} \int_{|x|=\delta} e^{i\theta} \xi_1 \overline{\psi}_2 \, ds = 0,$$

because $0 < \alpha < 1$. On the other hand, the second term behaves like

$$e^{-i\theta} \xi_2 \overline{\psi}_1 = (-i/\sin\alpha\pi)\,(1/\Gamma(\alpha)\Gamma(1-\alpha))\,(Er/2)^{-1}\, e^{i\alpha\pi/2}\,(1 + o(1))$$

as $|x| \to 0$. Since $\Gamma(\alpha)\Gamma(1-\alpha) = \pi/\sin\alpha\pi$ by formula, we have

$$-i \lim_{\delta \to 0} \int_{|x|=\delta} e^{-i\theta} \xi_2(x) \overline{\psi}_1(x)\, ds = -4e^{i\alpha\pi/2}/E.$$

This yields the desired value. $\square$

We now define I_ε by

$$I_\varepsilon = \left((1 - X_\varepsilon^2)^{-1} q_0, q_0 \right) = \left((1 + X_\varepsilon)^{-1} q_0, (1 - X_\varepsilon^*)^{-1} q_0 \right).$$

Lemma 9.4 *Let λ_1 and λ_2 be as in Lemma 5.1. Then one has the following statements :*

(1) *If $0 < \alpha < 1/2$, then*

$$I_\varepsilon = -2\pi\lambda_2 + o(1), \qquad \varepsilon \to 0.$$

(2) *If $\alpha = 1/2$, then*

$$I_\varepsilon = -2\pi\lambda_2(1 + \lambda_2^2)^{-1} + o(1), \qquad \varepsilon \to 0.$$

(3) *If $1/2 < \alpha < 1$, then*

$$I_\varepsilon = -\gamma_0^2(\lambda_1/2\pi) i^{-4\alpha} E^{2(2\alpha-1)} \varepsilon^{2(2\alpha-1)}\,(1 + o(1))\,, \qquad \varepsilon \to 0.$$

We complete the proof Theorem 5.3, accepting this lemma as proved. Throughout the proof of the theorem, we use the notation $O_2(\varepsilon)$ to denote remainder terms of which the L^2 norm obeys $O(\varepsilon)$.

Proof of Theorem 5.3. We set $\eta_\pm = \eta_0\xi_\pm + o_2(1)$ in Lemma 8.2. The amplitude $f_{2\varepsilon}(\omega \to \tilde{\omega}; E)$ is represented as (9.2). If we use Lemma 8.2, then a simple computation enables us to evaluate the amplitude as follows :

$$\begin{aligned} f_{2\varepsilon} &= -(iE/8\pi)^{1/2}\beta_-\beta_+(\Pi_+\psi_+(\omega), \eta_-)(\eta_+, \Pi_-\psi_-(\tilde{\omega}))I_\varepsilon \varepsilon^{1-2\alpha} \\ &+ O(\varepsilon^{-\alpha})((1-X_\varepsilon^2)^{-1}O_2(\varepsilon), q_0) \\ &+ O(\varepsilon^{-\alpha})((1-X_\varepsilon^2)^{-1}q_0, O_2(\varepsilon)) + O(\varepsilon). \end{aligned} \tag{9.5}$$

The leading term comes from the first term on the right side of (9.5).

We first consider the case $1/2 < \alpha < 1$. If $1/2 < \alpha < 1$, then it follows from Lemmas 8.4 and 9.2 that $(1-X_\varepsilon^2)^{-1}$ takes the form

$$\begin{aligned} (1-X_\varepsilon^2)^{-1} &= P\,Op(\varepsilon^{2(2\alpha-1)})P + Q\,Op(\varepsilon^0)Q \\ &+ P\,Op(\varepsilon^{2\alpha-1})Q + Q\,Op(\varepsilon^{2\alpha-1})P \end{aligned}$$

and hence we have

$$|((1-X_\varepsilon^2)^{-1}O_2(\varepsilon), q_0)| + |((1-X_\varepsilon^2)^{-1}q_0, O_2(\varepsilon))| = O(\varepsilon^{2\alpha}),$$

because $Qq_0 = 0$. This implies that the three remainder terms on the right side of (9.5) obey $O(\varepsilon^\alpha) = O(\varepsilon^{2\alpha-1})O(\varepsilon^{1-\alpha}) = o(\varepsilon^{2\alpha-1})$. Thus we have

$$f_{2\varepsilon} = -(iE/8\pi)^{1/2}\beta_-\beta_+(\Pi_+\psi_+(\omega), \eta_-)(\eta_+, \Pi_-\psi_-(\tilde{\omega}))I_\varepsilon \varepsilon^{1-2\alpha} + o(\varepsilon^{2\alpha-1}).$$

If we combine Lemmas 9.3 and 9.4, the desired asymptotic form is obtained after a little tedious computation of the leading constant C_α.

Next we move to the case $0 < \alpha \le 1/2$. By Lemmas 8.4 and 9.2 again, $(1-X_\varepsilon^2)^{-1}$ is bounded uniformly in ε, so that

$$|((1-X_\varepsilon^2)^{-1}O_2(\varepsilon), q_0)| + |((1-X_\varepsilon^2)^{-1}q_0, O_2(\varepsilon))| = O(\varepsilon).$$

Then the remainder terms on the right side of (9.5) obey $O(\varepsilon^{1-\alpha}) = o(\varepsilon^{1-2\alpha})$. Thus we have

$$f_{2\varepsilon} = -(iE/8\pi)^{1/2}\beta_-\beta_+(\Pi_+\psi_+(\omega), \eta_-)(\eta_+, \Pi_-\psi_-(\tilde{\omega}))I_\varepsilon \varepsilon^{1-2\alpha} + o(\varepsilon^{1-2\alpha}).$$

We again combine Lemmas 9.3 and 9.4 to obtain the desired asymptotic form for the case $0 < \alpha \le 1/2$, and the proof is complete. □

It remains to prove Lemma 9.4. The proof requires two auxiliary lemmas. The first lemma below is proved in the same way as Lemma 7.3. We skip the proof.

Lemma 9.5 (1) *If* $0 < \alpha \le 1/2$*, then*

$$q' = (1 - Z_0)^{-1} q_0 = V^{1/2} e$$

for some $e = {}^{\mathrm{t}}(e_1, e_2) \in L^\infty \times L^\infty$*, and* e *uniquely solves* $(K_0 - V)\, e = 0$ *under the condition that*

$$e_1 = r^{-\alpha} + O(|x|^{-1-\alpha}), \quad e_2 = O(|x|^{-1+\alpha}), \qquad |x| \to \infty.$$

(2) *If* $1/2 < \alpha < 1$*, then*

$$q' = q_0 + (Q - QZ_0Q)^{-1} QZ_0 q_0 = V^{1/2} e$$

for some $e = {}^{\mathrm{t}}(e_1, e_2) \in L^\infty \times L^\infty$*, and* e *uniquely solves* $(K_0 - V)\, e = 0$ *under the condition that*

$$e_1 = O(|x|^{-\alpha}), \quad e_2 = i(\lambda_0/2\pi) r^{-1+\alpha} e^{i\theta} + O(|x|^{-2+\alpha}), \qquad |x| \to \infty.$$

Lemma 9.6 *Assume that* $0 < \alpha \le 1/2$*. Let* τ *and* τ' *be the real numbers as in (8.5) and (9.3) respectively. Then one has*

$$\begin{aligned} \tau &= (q, q_0) = \Big((1 + Z_0)^{-1} q_0, q_0\Big) = -2\pi\lambda_2, \\ \tau' &= (q', q_0) = \Big((1 - Z_0)^{-1} q_0, q_0\Big) = -2\pi\lambda_2. \end{aligned}$$

Proof. We write $e_+ = {}^{\mathrm{t}}(e_1, e_2)$ for e in Lemma 7.3 and e_- for e in Lemma 9.5. Then it follows by uniqueness that e_- is given as $e_- = {}^{\mathrm{t}}(e_1, -e_2)$ for $0 < \alpha \le 1/2$. We prove the first relation only. The second relation is obtained in a similar way. By Lemma 7.3, $\tau = (Ve, \tilde{\rho}_0)$ and e solves $Ke = (K_0 + V)\, e = 0$. Hence

$$\tau = -(K_0 e, \tilde{\rho}_0) = -(p_- e_2, \rho_0).$$

Note that $p_-^* \rho_0 = p_+ \rho_0 = 0$, and p_- takes the form $p_- = e^{-i\theta}(-i\partial_r \ldots)$. Hence we have

$$\tau = i \lim_{R \to \infty} \int_{|x|=R} e^{-i\theta} e_2 \overline{\rho}_0 \, ds, \quad ds = R\, d\theta,$$

by partial integration. Since $\rho_0(x) = r^{-\alpha} + O(r^{-1-\alpha})$ as $|x| \to \infty$ and since

$$e_2(x) = i\lambda_2 e^{i\theta} r^{-1+\alpha} + O(r^{-2+\alpha})$$

by Lemma 5.1, the desired relation follows from (7.6). □

Proof of Lemma 9.4. We again write $e_+ = {}^{\mathrm{t}}(e_1, e_2)$ for e in Lemma 7.3 and e_- for e in Lemma 9.5. If $0 < \alpha \le 1/2$, then $e_- = {}^{\mathrm{t}}(e_1, -e_2)$, and if $1/2 < \alpha < 1$, then $e_- = {}^{\mathrm{t}}(-e_1, e_2)$.

(1) Assume that $0 < \alpha < 1/2$. By Lemmas 8.4 and 9.2, it follows that

$$I_\varepsilon = ((1+Z_0)^{-1}q_0, (1-Z_0)^{-1}q_0) + o(1), \qquad \varepsilon \to 0.$$

We further obtain $I_\varepsilon = (Ve_+, e_-) + o(1)$ by Lemmas 7.3 and 9.5. The leading term on the right side equals

$$(Ve_+, e_-) = -((K_0 - V)e_+, e_-)/2, \tag{9.6}$$

because $(K_0 \pm V)e_\pm = 0$. We assert that

$$((K_0 - V)e_+, e_-) = 4\pi\lambda_2, \tag{9.7}$$

which implies that $I_\varepsilon = -2\pi\lambda_2 + o(1)$. We shall show (9.7). By definition,

$$((K_0 - V)e_+, e_-) = ((p_- e_2 - Ve_1), e_1) - ((p_+ e_1 - Ve_2), e_2).$$

We recall that $p_\pm = e^{\pm i\theta}(-i\partial_r \ldots)$ for $|x| \gg 1$. Hence we have

$$((K_0 - V)e_+, e_-) = -i \lim_{R\to\infty} \int_{|x|=R} \left(e^{-i\theta} e_2 \overline{e}_1 - e^{i\theta} e_1 \overline{e}_2\right) ds, \quad ds = R\, d\theta,$$

by integration by parts. Thus Lemma 5.1 yields (9.7).

(2) Assume that $\alpha = 1/2$. According to Lemmas 8.4 and 9.2, we have

$$(1 + X_\varepsilon)^{-1} q_0 = (1 + a\tau)q + o_2(1), \quad (1 - X_\varepsilon^*)^{-1} q_0 = (1 + \overline{a'}\tau')q' + o_2(1).$$

Hence

$$I_\varepsilon = (1 + a\tau)(1 + a'\tau')(q, q') + o(1), \qquad \varepsilon \to 0.$$

We repeat the same argument as used in proving (1) to obtain that

$$(q, q') = ((1+Z_0)^{-1}q_0, (1-Z_0)^{-1}q_0) = -2\pi\lambda_2.$$

On the other hand, Lemma 9.6, together with (8.5), implies that

$$1 + a\tau = 1 - i\tau/(2\pi + i\tau) = 2\pi/(2\pi + i\tau) = (1 - i\lambda_2)^{-1},$$

and similarly $1 + a'\tau' = (1 + i\lambda_2)^{-1}$ (see (9.3)). This proves (2).

(3) Let $1/2 < \alpha < 1$. (3) is verified in almost the same way as (1). Since $Qq_0 = 0$ and $Pq_0 = q_0$, it follows from Lemmas 8.4 and 9.2 that

$$\begin{aligned} (1 + X_\varepsilon)^{-1} q_0 &\sim \delta_+(\varepsilon)\left(q_0 - Q(Q + QZ_0Q)^{-1}QZ_0q_0\right), \\ (1 - X_\varepsilon^*)^{-1} q_0 &\sim \overline{\delta_-(\varepsilon)}\left(q_0 + Q(Q - QZ_0Q)^{-1}QZ_0q_0\right), \end{aligned}$$

and hence we have

$$\begin{aligned} I_\varepsilon &= \delta_+(\varepsilon)\delta_-(\varepsilon)(Ve_+, e_-) + o(\varepsilon^{2(2\alpha-1)}) \\ &= -\delta_+(\varepsilon)\delta_-(\varepsilon)((K_0 - V)e_+, e_-)/2 + o(\varepsilon^{2(2\alpha-1)}) \end{aligned}$$

by Lemmas 7.3 and 9.5. Note that e_1 behaves like

$$e_1(x) = -(\lambda_1\lambda_0/2\pi)r^{-\alpha} + O(|x|^{-1-\alpha}), \qquad |x| \to \infty,$$

for the real number λ_1 as in Lemma 5.1. Hence the scalar product $((K_0 - V)e_+, e_-)$ is calculated as

$$((K_0 - V)e_+, e_-) = -i \lim_{R\to\infty} \int_{|x|=R} \left(-e^{-i\theta}e_2\overline{e}_1 + e^{i\theta}e_1\overline{e}_2\right) ds = -\lambda_1\lambda_0^2/\pi \tag{9.8}$$

by use of partial integration. As is seen from (8.6) and (9.4),

$$\delta_\pm(\varepsilon) = 1/\mu_\pm(\varepsilon) = \mp(\gamma_0/\lambda_0)i^{-2\alpha}E^{2\alpha-1}\varepsilon^{2\alpha-1}(1 + o(1)),$$

because $\gamma_-(k) \to -1/\gamma_0$ as $k = \varepsilon E \to 0$. This, together with (9.8), yields the desired asymptotic form. $\square$

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