C^*-Algebras of Energy Observables: II. Graded Symplectic Algebras and Magnetic Hamiltonians

Vladimir GEORGESCU[**] and Andrei IFTIMOVICI[*]

8 March 2001

Abstract

To each symplectic space Ξ we associate a C^*-algebra $\mathfrak{C}^\Xi$ graded by the lattice of all linear subspaces of Ξ. The hamiltonians of N-body systems in constant magnetic fields are affiliated to C^*-subalgebras of $\mathfrak{C}^\Xi$ for certain choices of Ξ, and this allows one to study their spectral properties. The algebra generated by the hamiltonians corresponding to a fixed magnetic field can also be described as the crossed product of an abelian algebra (of "classical potentials") by the action of a non-abelian group. This point of view is generalized to the case of non-constant magnetic fields.

Contents

[*]CNRS and [*]Department of Mathematics, University of Cergy-Pontoise 2, avenue Adolphe Chauvin 95302 Cergy-Pontoise Cedex, France, e-mail: Vladimir.Georgescu@math.u-cergy.fr and Andrei.Iftimovici@math.u-cergy.fr

1 Introduction

The present paper is devoted to a detailed study of a C^*-algebra of energy observables $\mathfrak{C}^\Xi$ naturally associated to a symplectic space Ξ. We use the terminology and the methods introduced in [GI2], where this example has been briefly mentioned in §4.6. The main difference with respect to the other examples considered in [GI2] is that the algebra $\mathfrak{C}^\Xi$ is not a crossed product, so its quotient with respect to the ideal of compact operators has to be computed by other means. We do this by using a grading of the algebra by the lattice of all subspaces of Ξ and applying a method specific to this context.

The graded algebra that we consider here is a generalization of the N-body algebra (see §4.6 in [GI2]) to a situation where the position and momentum observables Q, P play a symmetric role. A physical system covered by our formalism is that of N particles in a constant external magnetic field: indeed, we show in Section 4 that the C^*-algebra of energy observables corresponding to this system is a subalgebra of a symplectic graded algebra associated to the phase space of the system. It is worthwhile to note that this subalgebra is quite sensitive to the value of the magnetic field.

As we said above, the algebra $\mathfrak{C}^\Xi$ is not a crossed product, at least in the usual meaning of this notion. However, we show in §5.1 that *the subalgebra of energy observables of an N-body system in a given constant magnetic field is the crossed product of a C^*-algebra of bounded uniformly continuous functions on the configuration space with a (non-commutative) group G determined by the field.* This point of view allows us in the last part (Section 5) of the paper to go beyond the case of constant fields and to give a procedure of defining the algebra corresponding to an arbitrary magnetic field as a crossed product (in such a generality the group G could be not locally compact). These algebras will be studied in a later publication.

A symplectic space is a real finite dimensional vector space Ξ equipped with a real antisymmetric nondegenerate bilinear form σ. By representation of a symplectic space on a Hilbert space $\mathscr{H}$ we understand a strongly continuous map W from Ξ to the set of unitary operators on $\mathscr{H}$ satisfying:

$$W(\xi)W(\eta) = \mathrm{e}^{\frac{i}{2}\sigma(\xi,\eta)}W(\xi+\eta) \quad \text{for all } \xi,\eta\in\Xi. \tag{1.1}$$

This implies $W(0) = 1$, $W(\xi)^* = W(-\xi)$ and

$$W(\xi)W(\eta) = \mathrm{e}^{i\sigma(\xi,\eta)}W(\eta)W(\xi) \tag{1.2}$$

If E is a vector subspace of Ξ we set

$$(1.3)\qquad \begin{aligned} E^{\sigma} &= \{\xi \in \Xi \mid \sigma(\xi,\eta) = 0 \text{ for all } \eta \in E\} \\ &= \{\xi \in \Xi \mid W(\xi)W(\eta) = W(\eta)W(\xi) \text{ for all } \eta \in E\}. \end{aligned}$$

The relations $E^{\sigma\sigma} = E$ and $(E \cap F)^{\sigma} = E^{\sigma} + F^{\sigma}$ are easy to prove.

The purpose of this paper is to study the family of C^*-subalgebras $\mathfrak{C}(E)$ of $B(\mathscr{H})$ defined as follows:

Definition 1.1 *If E is a vector subspace of Ξ, then $\mathfrak{C}(E) \equiv \mathfrak{C}^{\Xi}(E)$ is the set of operators $T \in B(\mathscr{H})$ such that:*
(i) $\| [W(\xi), T] \| \to 0$ *if* $\xi \to 0$ *in* Ξ,
(ii) $[W(\xi), T] = 0$ *if* $\xi \in E$,
(iii) $\| (W(\xi) - 1)T^{(*)} \| \to 0$ *if* $\xi \in E^{\sigma}$ *and* $\xi \to 0$.

The notation $T^{(*)}$ means that the relation must hold for both T and T^*. We mention that when W is of finite multiplicity (this notion is introduced in Section 2) it suffices to require condition (iii) only for T (see the remark after Theorem 3.1).

The algebra $\mathfrak{C}(E)$ depends, of course, of the representation, but this is largely irrelevant; this is why W is not explicitly mentioned in the notation. In fact, the algebras corresponding to two irreducible representations are canonically isomorphic. The structure of $\mathfrak{C}(E)$ in general is studied in Section 3, where we show that it is canonically isomorphic to a tensor product $\mathfrak{C}(E;\mathbb{C}) \otimes \mathfrak{C}(\Xi)$, where $\mathfrak{C}(E;\mathbb{C})$ is the algebra corresponding to an irreducible representation.

Let $\mathbb{G}(\Xi)$ be the *grassmannian* of Ξ, i.e. the set of all vector subspaces of Ξ equipped with the usual order relation (set inclusion) which makes it a complete lattice. More precisely, this means that each set $\mathscr{E} \subset \mathbb{G}(\Xi)$ has a greatest lower bound (given by $\bigcap_{E \in \mathscr{E}} E$) and a least upper bound (namely $\sum_{E \in \mathscr{E}} E$). The least and greatest elements of $\mathbb{G}(\Xi)$ are $\mathbf{O} = \{0\}$ and Ξ respectively.

The *symplectic C^*-algebra* associated to the system $(\Xi, \sigma; \mathscr{H}, W)$ is defined as the C^*-subalgebra $\mathfrak{C}^{\Xi}$ of $B(\mathscr{H})$ generated by $\bigcup_{E \in \mathbb{G}(\Xi)} \mathfrak{C}(E)$. In the next section we show that the symplectic algebra has a canonical structure of graded C^*-algebra with respect to the lattice $\mathbb{G}(\Xi)$ (in the sense defined in [DG1, GI2]). When the representation W is of finite multiplicity this allows us to give an explicit description of the quotient $\mathfrak{C}^{\Xi}/K(\mathscr{H})$, in particular to obtain a formula for the essential spectrum of all the self-adjoint operators affiliated to $\mathfrak{C}^{\Xi}$.

The preceding description of the quotient can also be used in order to prove the Mourre estimate for some classes of operators affiliated to $\mathfrak{C}^{\Xi}$ along the lines of [DG2]. Our initial purpose was to include this treatment in the present paper. Meanwhile, we received (a preprint version of) the monograph [GL3] by C. Gérard and I. Laba, devoted to a thorough study of the spectral and scattering problem for N-particle systems in constant magnetic fields, and in particular to the proof of the Mourre estimate. Since in their work several problems are left open, we decided to deepen our approach to this questions hopping that our formalism will allow us to overcome some of the difficulties which are pointed out in [DG2].

Section 3 is devoted to the proof of the main technical result of this paper: a description of $\mathfrak{C}(E)$ as an algebra of pseudo-differential operators. This gives in rather explicit terms the structure of a generic element of $\mathfrak{C}(E)$. In particular, we make the connection with the approach of [BG].

In Section 4 we consider the specific situation of an N-body system in a constant magnetic field and we explain how it fits into the preceding framework. We begin with a general version of the HVZ theorem, an immediate consequence of the algebraic approach. In order to justify the treatment of non-constant magnetic fields that we give in Section 5, we discuss in §4.2 the geometric origin of the magnetic fields starting with the notion of localizability. This allows us to construct in a natural way the symplectic form suited for a system in a constant magnetic field as a perturbation of the "free" symplectic form and to interpret subalgebras of $\mathfrak{C}^{\Xi}$ as algebras of energy observables corresponding to various physical situations. In particular, we show that the magnetic many-body hamiltonians considered in the literature are affiliated to these C^*-algebras.

Our purpose in Section 5 is to define the C^*-algebra of energy observables in the non-constant magnetic field case. Our point of view, justified by the developments of Section 4, is that these algebras are crossed products of C^*-algebras of bounded uniformly continuous functions on the configuration space X by the action of non-abelian groups G which are extensions of X. This opens the way for a study (that we intend to carry on) of systems in non-constant magnetic fields by algebraic methods specific to the theory of crossed products, which turned out to be efficient in the abelian case $G = X$ considered in [GI2]. On the other hand, we would like to mention an interesting connection between magnetic fields and 2-cocycles on an abelian group X with coefficients in the group of continuous maps $X \to U(1)$ equipped with the action of X given by translations, cf. §5.2.

The Appendix is devoted to two different subjects. First we recall some facts concerning group extensions and homological algebra needed in Sections 4 and 5. Secondly, we consider the case when Ξ is replaced by an abelian locally compact group equipped with a multiplier (2-cocycle), the role of subspaces of Ξ being played by the subgroups of Ξ. For example, let Ξ be the phase space $X \oplus X^*$ of an arbitrary locally compact abelian group X. The case of $X = \mathbb{Z}$ or $X = \mathbb{R}$ is already interesting and nontrivial. Our purpose is only to show which part of the theory from Section 2 extends easily to this context and to point out the new difficulties which appear. We also describe this construction in terms of representations of "Heisenberg groups" which suggests a possible generalization to the case of a unitary representation of a locally compact (non-abelian) group.

Acknowledgements: We are grateful to Christian Gérard for sending us the monograph [GL3] before its publication.

2 The graded C^*-algebra structure

2.1. We say that a family $\{\mathfrak{C}_i\}_{i\in I}$ of subalgebras of $B(\mathscr{H})$ is *linearly independent* if, for each family $\{S_i\}_{i\in I}$ with $S_i \in \mathfrak{C}_i$ and $S_i \neq 0$ for at most a finite number of $i \in I$, one has $\sum_{i\in I} S_i = 0 \Rightarrow S_j = 0$ for all $j \in I$.

Theorem 2.1 *For each vector subspace E of Ξ the set $\mathfrak{C}(E)$ is a nondegenerate C^*-algebra of operators on $\mathscr{H}$. One has:*
(i) $\mathfrak{C}(E) \cdot \mathfrak{C}(F) \subset \mathfrak{C}(E \cap F)$.
(ii) $\{\mathfrak{C}(E)\}_{E\in\mathbb{G}(\Xi)}$ *is a linearly independent family of subalgebras of* $B(\mathscr{H})$.
(iii) *If* $\mathscr{E} \subset \mathbb{G}(\Xi)$ *is finite, then* $\sum_{E\in\mathscr{E}} \mathfrak{C}(E)$ *is a closed subspace of* $B(\mathscr{H})$.

In order to make the proof more transparent, we begin by establishing a couple of technical results which will be used again later on. For integrable Borel measures μ on Ξ, we denote

$$W(\mu) = \int_\Xi W(\xi)\mu(\mathrm{d}\xi). \tag{2.1}$$

If $F \in \mathbb{G}(\Xi)$ we shall say that the measure μ is F*-absolutely continuous* (F-a.c.) if $\operatorname{supp}\mu \subset F$ and the restriction of μ to F is absolutely continuous with respect to a Haar measure on F (i.e. a translation invariant positive Radon measure on F).

It will be convenient for the next proofs to use the following notation. For each $F \in \mathbb{G}(\Xi)$ we choose a Haar measure λ_F on F and we denote also by λ_F its extension to a measure on Ξ given by $\lambda_F(A) = \lambda_F(A \cap F)$ for each Borel set $A \subset \Xi$. In particular, an integrable measure μ on Ξ is F-a.c. if and only if there is $\rho \in L^1(F) = L^1(F, \lambda_F)$ such that $\mu = \rho\lambda_F$.

Lemma 2.2 *If μ is an E^σ-a.c. integrable measure on Ξ, then $W(\mu) \in \mathfrak{C}(E)$.*

Remark: In Section 3 we shall prove that the set of such $W(\mu)$ is a dense $*$-subalgebra of $\mathfrak{C}(E)$.

Proof: For any $\xi \in \Xi$,

$$[W(\xi), W(\mu)] = \int_\Xi [W(\xi), W(\eta)]\mu(\mathrm{d}\eta) = \int_\Xi (\mathrm{e}^{i\sigma(\xi,\eta)} - 1)W(\eta)W(\xi)\mu(\mathrm{d}\eta)$$

Then, by dominated convergence theorem, we have $\|\,[W(\xi), W(\mu)]\,\| \to 0$ when $\xi \to 0$ in Ξ. Since μ is E^σ-a.c. the above commutator is equal to $\int_{E^\sigma} (\mathrm{e}^{i\sigma(\xi,\eta)} - 1)W(\eta)W(\xi)\mu(\mathrm{d}\eta)$, hence it is zero if $\xi \in E$. It remains to show that $W(\mu)$ verifies (iii) of Definition 1.1. If $\xi \in E^\sigma$ we get from (1.1):

$$\begin{aligned}\|\,(W(\xi) - 1)W(\mu)\,\| &= \left\| \int_{E^\sigma} (\mathrm{e}^{\frac{i}{2}\sigma(\xi,\eta)} W(\xi + \eta) - W(\eta))\rho(\eta)\lambda_{E^\sigma}(\,\mathrm{d}\eta) \right\| \\ &\leq \int_{E^\sigma} |\mathrm{e}^{\frac{i}{2}\sigma(\xi,\eta)} - 1|\,|\rho(\eta)|\lambda_{E^\sigma}(\,\mathrm{d}\eta) + \int_{E^\sigma} |\rho(\eta - \xi) - \rho(\eta)|\lambda_{E^\sigma}(\,\mathrm{d}\eta).\end{aligned}$$

By dominated convergence theorem, both integrals tend to zero as $\xi \to 0$ with $\xi \in E^\sigma$. ■

Let now ρ be an integrable function on E^σ such that $\int_{E^\sigma} \rho(\eta)\lambda_{E^\sigma}(\,\mathrm{d}\eta) = 1$. For $\varepsilon > 0$ set $\rho_\varepsilon(\eta) := \varepsilon^{-n}\rho(\eta/\varepsilon)$, where $n = \dim E^\sigma$ and $\mu_\varepsilon = \rho_\varepsilon \lambda_{E^\sigma}$.

Lemma 2.3 (i) *$W(\mu_\varepsilon) \in \mathfrak{C}(E)$ and* $\operatorname{s-lim}_{\varepsilon\to 0} W(\mu_\varepsilon) = 1$ *on $\mathscr{H}$.*
(ii) *If $T \in \mathfrak{C}(E)$ then* $\lim_{\varepsilon \to 0} \|(W(\mu_\varepsilon) - 1)T^{(*)}\| = 0$.

Proof: A change of variable shows that $W(\mu_\varepsilon) = \int_{E^\sigma} W(\varepsilon\eta)\rho(\eta)\lambda_{E^\sigma}(\,\mathrm{d}\eta)$. Hence, by the previous lemma, $W(\mu_\varepsilon) \in \mathfrak{C}(E)$ for any $\varepsilon > 0$. If $f \in \mathscr{H}$

$$\|(W(\mu_\varepsilon) - 1)f\| \leq \int_{E^\sigma} \|\,(W(\varepsilon\eta) - 1)f\,\|\,|\rho(\eta)|\lambda_{E^\sigma}(\,\mathrm{d}\eta) \to 0 \text{ as } \varepsilon \to 0,$$

by dominated convergence and strong continuity of W. Further, if $T \in \mathfrak{C}(E)$

$$\|(W(\mu_\varepsilon) - 1)T^{(*)}\| \leq \int_{E^\sigma} \|\,(W(\varepsilon\eta) - 1)T^{(*)}\,\|\,|\rho(\eta)|\lambda_{E^\sigma}(\,\mathrm{d}\eta) \to 0 \text{ as } \varepsilon \to 0,$$

by dominated convergence, as a consequence of (iii) in Definition 1.1. ■

Lemma 2.4 *If $T \in \mathfrak{C}(E)$ then* $\operatorname{s-lim} TW(\xi) = 0$ *if the image of ξ in the quotient space Ξ/E tends to infinity.*

Proof: Since T is a norm limit of operators of the form $TW(\mu_\varepsilon)$, it suffices to show that, if μ is an E^σ-a.c. integrable measure, then $\| W(\mu)W(\xi)f \| \to 0$ as the image of ξ in the quotient space Ξ/E tends to infinity (which we will abbreviate by $\xi \to \infty (\mathrm{mod}\, E)$). But $\| W(\mu)W(\xi)f \|^2 = \langle f, W(\xi)^* W(\mu)^* W(\mu) W(\xi) f\rangle$ and it is easy to show that $W(\mu)^*W(\mu) = W(\nu)$ for some E^σ-a.c. integrable measure ν. It suffices thus to prove that if ρ is an integrable function on E^σ:

$$\langle f, W(\xi)^* W(\nu) W(\xi) f\rangle = \int_{E^\sigma} \mathrm{e}^{i\sigma(\eta,\xi)} \langle f, W(\eta) f\rangle \rho(\eta) \lambda_{E^\sigma}(\,\mathrm{d}\eta) \to 0$$

as $\xi \to \infty (\mathrm{mod}\, E)$. The map $\xi \mapsto \sigma(\cdot, \xi)$ is an isomorphism of Ξ onto its dual space which allows us to identify these two spaces. Then the polar of E^σ is $E^{\sigma\sigma} = E$ and the dual of E^σ is identified to the quotient space Ξ/E. Hence if $\xi \to \infty (\mathrm{mod}\, E)$ then the above integral tends to zero by the Riemann-Lebesgue lemma. ■

Proof of Theorem 2.1: The fact that $\mathfrak{C}(E)$ is a C^*-subalgebra of $B(\mathscr{H})$ is obvious; it is also nondegenerate because $\{W(\mu_\varepsilon)f \mid f \in \mathscr{H},\ \varepsilon > 0\}$ is a dense subset of $\mathscr{H}$ by (i) of Lemma 2.3. We now prove (i). If $\{\xi_n\}$ is a sequence in $(E\cap F)^\sigma = E^\sigma + F^\sigma$ which converges to zero, then one can write $\xi_n = \eta_n + \zeta_n$ with $\eta_n \in E^\sigma$, $\eta_n \to 0$ and $\zeta_n \in F^\sigma$, $\zeta_n \to 0$. If $T \in \mathfrak{C}(E)$ and $S \in \mathfrak{C}(F)$, it is easy to show that TS satisfy (i) and (ii) from the definition of $\mathfrak{C}(E\cap F)$; moreover, by (1.1)

$$\begin{aligned} \| (W(\xi_n) - 1)TS \| &= \| [\mathrm{e}^{\frac{i}{2}\sigma(\eta_n,\zeta_n)} W(\zeta_n) W(\eta_n) - 1]\, TS \| \\ &\le |\mathrm{e}^{\frac{i}{2}\sigma(\eta_n,\zeta_n)} - 1)|\, \|TS\| + \|[W(\zeta_n), T]\|\, \|S\| \\ &\quad + \|T\|\, \| (W(\zeta_n) - 1)S \| + \| (W(\eta_n) - 1)T \|\, \|S\| \xrightarrow{n\to\infty} 0. \end{aligned}$$

We prove now the last two assertions of the theorem. We will show that for each finite $\mathscr{E} \subset \mathbb{G}(\Xi)$ there is a constant $C > 0$ such that for each family $\{T(E)\}_{E\in\mathscr{E}}$ with $T(E) \in \mathfrak{C}(E)$ and for each $E \in \mathscr{E}$ one has

$$\|T(E)\| \le C \Big\| \sum_{F\in\mathscr{E}} T(F) \Big\|. \tag{2.2}$$

In particular, $\sum_{F\in\mathscr{E}} T(F) = 0$ implies $T(E) = 0$ for all $E \in \mathscr{E}$, so (ii) is true. Moreover, this also implies (iii). Set $E_0 = E \setminus \bigcup_{F\neq E} F = \bigcap_{F\neq E} [E \setminus (E\cap F)]$. Since $\mathscr{E}$ is finite, we have $E_0 = \emptyset$ if and only if there is an $F \in \mathscr{E}$ such that $E \subset F$ strictly. Hence $E_0 \neq \emptyset$ only if E is a maximal element of $\mathscr{E}$ (w.r.t. the inclusion order relation). Denote $\mathscr{E}_0$ the set of maximal elements of $\mathscr{E}$. If we show that the above estimate is true for $E \in \mathscr{E}_0$ then, replacing $\mathscr{E}$ by $\mathscr{E}_1 \equiv$

$\mathscr{E} \setminus \mathscr{E}_0$, we may repeat the above argument inductively until the proof will be completed for all $E \in \mathscr{E}$. Choose then, for any $E \in \mathscr{E}_0$, a point $\omega \in E_0$. By (ii) of Definition 1.1 if $T(E) \in \mathfrak{C}(E)$ then $[W(\lambda\omega), T(E)] = 0$ for all $\lambda \in \mathbb{R}$, thus $T(E) = W(\lambda\omega)^* T(E) W(\lambda\omega) = \text{s-lim}_{\lambda\to\infty} W(\lambda\omega)^* \sum_{F\in\mathscr{E}} T(F) W(\lambda\omega)$, the last equality being a consequence of the Lemma 2.4 if we take into account that $\omega \notin F$ for all $F \neq E$. ∎

2.2. We define the *symplectic C^*-algebra* $\mathfrak{C}^\Xi$ as the norm closure in $B(\mathscr{H})$ of $\sum_{E\in\mathbb{G}(\Xi)} \mathfrak{C}(E)$. The properties (i), (ii) and (iii) from Theorem 2.1 show that $\mathfrak{C}^\Xi$ is a $\mathbb{G}(\Xi)$*-graded C^*-algebra* (in the sense of §4.6 in [GI2]).

For each $E \in \mathbb{G}(\Xi)$ let $\mathfrak{C}^\Xi_E$ be the closure of $\sum_{F\supset E} \mathfrak{C}(F)$ and $\mathfrak{J}^\Xi_E$ the closure of $\sum_{F\not\supset E} \mathfrak{C}(F)$. By Theorem 3.1 in [DG1] one has

$$\mathfrak{C}^\Xi = \mathfrak{C}^\Xi_E + \mathfrak{J}^\Xi_E \quad \text{and} \quad \mathfrak{C}^\Xi_E \bigcap \mathfrak{J}^\Xi_E = \{0\}.$$

Moreover, the projection $\mathcal{P}_E : \mathfrak{C}^\Xi \to \mathfrak{C}^\Xi_E$ given by the preceding decomposition is a morphism.

It follows easily from Lemma 2.4 that we have a rather explicit expression for $\mathcal{P}_E$ when E is a one dimensional subspace: if $\omega \in E$, $\omega \neq 0$, then

$$\mathcal{P}_E[T] = \operatorname*{s\text{-}lim}_{\lambda\to\infty} W(\lambda\omega)^* T\, W(\lambda\omega). \tag{2.3}$$

Let $\mathbb{P}(\Xi)$ be the projective space of Ξ, i.e. the set of its one dimensional subspaces (or atoms of the lattice $\mathbb{G}(\Xi)$). Then we have a natural morphism

$$\mathcal{P} : \mathfrak{C}^\Xi \to \textstyle\prod_{E\in\mathbb{P}(\Xi)} \mathfrak{C}^\Xi_E \ \ , \ \ \mathcal{P}[T] := (\mathcal{P}_E[T])_{E\in\mathbb{P}(\Xi)}. \tag{2.4}$$

By Theorem 3.2 in [DG1] one has $\ker \mathcal{P} = \mathfrak{C}^\Xi(\mathbf{O})$, so we get a canonical embedding:

$$\mathfrak{C}^\Xi \big/ \mathfrak{C}^\Xi(\mathbf{O}) \ \hookrightarrow\ \textstyle\prod_{E\in\mathbb{P}(\Xi)} \mathfrak{C}^\Xi_E. \tag{2.5}$$

We are, of course, interested in computing the quotient of $\mathfrak{C}^\Xi$ with respect to the ideal of the compact operators, so we have to decide when $\mathfrak{C}(\mathbf{O}) = K(\mathscr{H})$. Since $\mathbf{O}$ is the zero dimensional subspace of Ξ, we have

$$\mathfrak{C}(\mathbf{O}) = \{T \in B(\mathscr{H}) \mid \lim_{\xi\to 0} \| (W(\xi) - 1)T^{(*)} \| = 0\}. \tag{2.6}$$

The strong continuity of the representation W gives then $K(\mathscr{H}) \subset \mathfrak{C}(\mathbf{O})$. We shall say that the representation W is of *finite multiplicity* if it is a finite direct sum of irreducible representations. The following proposition will be proved at the end of Section 3.

Proposition 2.5 *W is of finite multiplicity if and only if $K(\mathscr{H}) = \mathfrak{C}(\mathbf{O})$.*

Gathering the various results obtained so far, we obtain:

Theorem 2.6 *Assume that W is of finite multiplicity. If $T \in \mathfrak{C}^{\Xi}$ and $E \in \mathbb{P}(\Xi)$, then for $\omega \in E \setminus \{0\}$*

$$T_E \equiv \mathcal{P}_E[T] := \operatorname*{s\text{-}lim}_{\lambda \to \infty} W(\lambda\omega)^* \, T \, W(\lambda\omega) \tag{2.7}$$

exists and the map $T \mapsto (T_E)_{E \in \mathbb{P}(\Xi)}$ is a morphism $\mathfrak{C}^{\Xi} \to \prod_{E \in \mathbb{P}(\Xi)} \mathfrak{C}^{\Xi}_E$ with kernel equal to $K(\mathscr{H})$. Moreover, for each $T \in \mathfrak{C}^{\Xi}$ the set $\{T_E \mid E \in \mathbb{P}(\Xi)\}$ is compact in $\mathfrak{C}^{\Xi}$.

The proof of the last assertion (which implies that the right hand side of relation (2.9) below is closed) relies only on the lattice properties of $\mathbb{G}(\Xi)$, hence the arguments of Proposition 5.4 from [DG2] work in this context too.

Corollary 2.7 *If H is a self-adjoint operator on $\mathscr{H}$ affiliated to $\mathfrak{C}^{\Xi}$, then for each $\omega \in \Xi$*

$$H_\omega := \operatorname*{s\text{-}lim}_{\lambda \to \infty} W(\lambda\omega)^* \, H \, W(\lambda\omega) \tag{2.8}$$

exists in the strong resolvent sense, and

$$\boldsymbol{\sigma}_{\mathrm{ess}}(H) = \textstyle\bigcup_{\omega \in \Xi \setminus \{0\}} \boldsymbol{\sigma}(H_\omega) \tag{2.9}$$

We stress that the operators H_ω could be non-densely defined, so in general they should be interpreted as observables, as explained in [GI2]. For example, if H is affiliated to $\mathfrak{C}(\mathbf{O})$ then $(H+i)^{-1}$ is a compact operator, so $H_\omega = \infty, D(H_\omega) = \{0\}$.

2.3. The Definition 1.1 describes the algebra $\mathfrak{C}(E)$ in very simple terms. We shall take advantage of this fact in order to point out a class of self-adjoint operators of physical interest affiliated to it. Theorem 3.1, which will be proved in the next section, allows one to obtain similar (and more general) results, but by more involved arguments.

Let $\mathscr{S}$ be the dense linear subspace of $\mathscr{H}$ consisting of the C^∞ vectors of the representation W:

$$\mathscr{S} = \{f \in \mathscr{H} \mid \xi \mapsto W(\xi) f \in \mathscr{H} \text{ is of class } C^\infty\}. \tag{2.10}$$

For each $\xi \in \Xi$ the family $\{W(t\xi)\}_{t\in\mathbb{R}}$ is a strongly continuous unitary representation of $\mathbb{R}$ in $\mathscr{H}$ which leaves $\mathscr{S}$ invariant. We denote $S(\xi)$ the infinitesimal generator of this group, so that

$$W(t\xi) = \mathrm{e}^{itS(\xi)} \quad \text{for all} \quad t \in \mathbb{R}. \tag{2.11}$$

Clearly $\mathscr{S} \subset D(S(\xi))$, $\mathscr{S}$ is stable under $S(\xi)$, and $S(\xi)\big|_{\mathscr{S}}$ is essentially self-adjoint (by Nelson's lemma). From (1.2) we get

$$W(\xi)S(\eta)W(\xi)^* = \sigma(\xi,\eta) + S(\eta), \qquad [S(\xi), S(\eta)] = -i\sigma(\xi,\eta) \tag{2.12}$$

on $\mathscr{S}$ for all $\xi, \eta \in \Xi$.

Proposition 2.8 *Assume that the linear subspace $E \subset \Xi$ is equipped with an euclidean structure. If $\{e_1, \dots, e_n\}$ is an orthonormal basis of E, then the operator $\sum_{k=1}^n S(e_k)^2$ with domain $\mathscr{S}$ is independent of the choice of the basis and its closure Δ_E is a self-adjoint operator affiliated to $\mathfrak{C}(E^\sigma)$.*

Proof: The essential self-adjointness of $\sum_{k=1}^n S(e_k)^2$ on $\mathscr{S}$ is a general fact, see Theorem 1.1 in [Jo1] (better results are easy to obtain in the present situation). To show its independence of the choice of a basis, note that a second orthonormal basis $\{e'_1, \dots, e'_n\}$ is related to the first one by a relation $\mathrm{e}'_j = \sum_k a_{jk} e_k$ with (a_{jk}) an orthogonal matrix, so that $\sum_i a_{ij} a_{ik} = \delta_{jk}$. Then we clearly get $\sum S(e'_i)^2 = \sum S(e_j)^2$.

Now we start the less trivial part of the proof. We shall show that the operator $T = (\Delta_E + 1)^{-1}$ belongs to $\mathfrak{C}(E^\sigma)$; for this we shall check the three conditions of Definition 1.1 with E replaced by E^σ. Note first that the form domain $\mathscr{K}$ of Δ_E is equal to $\bigcap_k D(S(\mathrm{e}_k))$. We embed in the standard way $\mathscr{K} \subset \mathscr{H} = \mathscr{H}^* \subset \mathscr{K}^*$. Then (2.12) shows that the operators $W(\xi)$ leave invariant $\mathscr{K}$, hence they extend to bounded operators in $\mathscr{K}^*$. Below we consider T as an operator $\mathscr{H} \to \mathscr{K}$ and $\mathscr{K}^* \to \mathscr{H}$ (by taking adjoints) and Δ_E as an operator $\mathscr{K} \to \mathscr{K}^*$. Then it is easy to show that $[W(\xi), T] = T[\Delta_E, W(\xi)]T$. From (2.12) we get

$$\begin{aligned}
[\Delta_E, W(\xi)] &= W(\xi) \sum_k \left[\big(S(e_k) + \sigma(e_k,\xi)\big)^2 - S(e_k)^2 \right] \\
&= W(\xi) \sum_k \left[2S(e_k)\sigma(e_k,\xi) + \sigma(e_k,\xi)^2 \right]
\end{aligned}$$

If $\xi \in E^\sigma$ the right hand side above is zero, so the second condition of Definition 1.1 is satisfied. Then

$$\|[W(\xi), T]\| \le \sum_k \left[2|\sigma(e_k,\xi)|\, \|S(e_k)T\| + \sigma(e_k,\xi)^2 \right].$$

So the first condition of Definition 1.1 is satisfied too. Now observe that if $\xi = \sum_{k=1}^{n} \xi_k$ then (1.1) and an induction argument gives

$$W(\xi) = \exp\left(\textstyle\sum_{j<k}\sigma(\xi_j, \xi_k)/2i\right) W(\xi_1) \dots W(\xi_n).$$

If $\xi \in E = E^{\sigma\sigma}$ then we can write it as $\xi = \sum t_k e_k$ with t_k real. By using the preceding formula we see that in order to prove the third condition of Definition 1.1 it suffices to show that $\|[W(te_k) - 1]T\| \to 0$ when $t \to 0$ for each k. But

$$\|[W(te_k) - 1]T\| \le \|[\mathrm{e}^{itS(e_k)} - 1][S(e_k)^2 + 1]^{-1/2}\| \, \|[S(e_k)^2 + 1]^{1/2}T\|$$

so the assertion follows immediately from the estimate $S(e_k)^2 \le \Delta_E$. ■

3 Pseudo-differential operators

3.1. The main result of this section is the following theorem, which gives a description of $\mathfrak{C}(E)$ as a C^*-algebra of pseudo-differential operators. We first assume that W is irreducible; the general case is discussed in the second part of the section.

Theorem 3.1 *If W is an irreducible representation, then $\mathfrak{C}(E)$ coincides with the closure in $B(\mathscr{H})$ of the set of operators $W(\mu)$ with μ an E^σ-a.c. integrable measure.*

Remark: We shall prove more, namely: if $T \in B(\mathscr{H})$ verifies (i), (ii) of Definition 1.1 and if $\|(W(\xi) - 1)T\| \to 0$ for $\xi \to 0$ in E^σ, then T is a norm limit of operators $W(\mu)$ with μ an E^σ-a.c. integrable measure. In particular, this shows that *if W is of finite multiplicity, then it is not necessary to require the validity of (iii) for T^* in the definition of $\mathfrak{C}(E)$.*

We denote by $\mathcal{M}$ the space of integrable measures on Ξ equipped with the $*$-algebra structure defined by the *twisted convolution*:

$$(\mu \times \nu)(\xi) := \int_\Xi \mathrm{e}^{\frac{i}{2}\sigma(\xi,\eta)} \mu(\xi - \eta)\nu(\eta)\, \mathrm{d}\eta \tag{3.1}$$

as product and $\mu^\dagger(\xi) := \overline{\mu(-\xi)}$ as involution. These definition must be interpreted in the sense of distributions, e.g. for $\varphi \in C_0(\Xi)$:

$$\int_\Xi \varphi(\xi)\, (\mu \times \nu)(\mathrm{d}\xi) := \int_{\Xi\times\Xi} \varphi(\xi + \eta)\, \mathrm{e}^{\frac{i}{2}\sigma(\xi,\eta)} \mu(\mathrm{d}\xi)\, \nu(\mathrm{d}\eta).$$

One can easily check then that

$$W(\mu \times \nu) = W(\mu)W(\nu) \quad \text{and} \quad W(\mu^{\dagger}) = W(\mu)^{*}.$$

Moreover, for any $\xi \in \Xi$, $W(\delta_\xi) = W(\xi)$, where δ_ξ is the Dirac measure at ξ.

If $F \subset \Xi$ is a vector subspace we denote by $\mathcal{M}(F)$ the space of $\mu \in \mathcal{M}$ such that μ is F-a.c.; it is easy to see that $\mathcal{M}(F)$ is a $*$-subalgebra of $\mathcal{M}$. Now we can restate the Theorem 3.1 as follows:

(3.2) $\quad \{W(\mu) \mid \mu \in \mathcal{M}(E^{\sigma})\}$ *is a dense $*$-subalgebra of* $\mathfrak{C}^{\Xi}$.

Due to Lemma 2.2 we only have to check the density assertion.

Remark: There are two particular cases for which the Theorem 3.1 is easy to prove. First, if $E = \Xi$, then $E^{\sigma} = \mathbf{O}$ and by the irreducibility of W we have $\mathfrak{C}(E) = \mathbb{C}$, hence this case is trivial. The second case is $E = \mathbf{O}$. Then $E^{\sigma} = \Xi$ and, by Proposition 2.5, $\mathfrak{C}(\mathbf{O})$ is the space of compact operators on $\mathscr{H}$. On the other hand, the set of operators $W(\mu)$, with $\mu \in \mathscr{S}(\Xi)$, is dense in $K(\mathscr{H})$ as a consequence of the fact that $W(\mu)$ is a finite rank operator if μ is a Hermite function on Ξ.

For the rest of the proof we shall use the Weyl pseudo-differential calculus. We summarize below the facts we need and we refer to [Fol] for details and proofs (see also [BG]).

A subspace $E \subset \Xi$ is called *isotropic* if $E \subset E^{\sigma}$ and *involutive* if $E^{\sigma} \subset E$. If both are valid, i.e. $E = E^{\sigma}$, then E is *lagrangian*. An isotropic subspace $X \subset \Xi$ is lagrangian if and only if $2 \dim X = \dim \Xi$. A couple (X, X^{*}) of subspaces of Ξ is called a *holonomic decomposition* of Ξ if both X and X^{*} are lagrangian and $X \cap X^{*} = \mathbf{O}$ or, equivalently, $X + X^{*} = \Xi$. Such decompositions always exist. Note also that (X, X^{*}) and (X^{*}, X) are considered as different holonomic decompositions.

If for $x \in X$ and $k \in X^{*}$ we set $\langle x, k\rangle = \sigma(k, x)$ then we get a non-degenerate bilinear form on $X \times X^{*}$ which allows us to identify X^{*} with the dual of X. Observe that if $\xi, \eta \in \Xi$ and $\xi = x + k$ and $\eta = y + p$ are their decompositions determined by the direct sum decomposition $\Xi = X + X^{*}$, then

$$\sigma(\xi, \eta) = \langle y, k\rangle - \langle x, p\rangle. \tag{3.3}$$

We recall that a symplectic space has only one irreducible representation (modulo unitary equivalence) and that each of its representations is a multiple of this one (see Theorem C.38 in [RW]). To each holonomic decomposition (X, X^*) of Ξ and to each Haar measure dx on X, one associates an irreducible representation, the *Schrödinger representation*, defined as follows: $\mathscr{H} = L^2(X)$ and for $\xi = x + k$ as above, $f \in \mathscr{H}$ and $y \in X$

$$(W(\xi)f)(y) := \mathrm{e}^{i\langle y - \frac{x}{2}, k\rangle} f(y - x). \tag{3.4}$$

Equivalently,

$$W(\xi) = \mathrm{e}^{-\frac{i}{2}\langle x,k\rangle} \mathrm{e}^{i\langle Q,k\rangle} \mathrm{e}^{-i\langle x,P\rangle} = \mathrm{e}^{\frac{i}{2}\langle x,k\rangle} \mathrm{e}^{-i\langle x,P\rangle} \mathrm{e}^{i\langle Q,k\rangle} = \mathrm{e}^{i(\langle Q,k\rangle - \langle x,P\rangle)}.$$

In the rest of the proof of Theorem 3.1 we assume that we are working in the representation described above.

If $f, g \in \mathscr{S}(X)$ then the map $\xi \mapsto \langle f, W(\xi)g\rangle$ is easily seen to belong to $\mathscr{S}(\Xi)$. Hence we can define $W(\mu)$ as a continuous linear operator $\mathscr{S}(X) \to \mathscr{S}^*(X)$ for each temperate distribution $\mu \in \mathscr{S}^*(\Xi)$ by setting

$$\langle f, W(\mu)g\rangle = \int_\Xi \langle f, W(\xi)g\rangle \mu(\xi)\, \mathrm{d}\xi, \tag{3.5}$$

where the integral has to be interpreted in the sense of distributions.

Thus we get a map $\mathscr{S}^*(\Xi) \ni \mu \mapsto W(\mu) \in B(\mathscr{S}(X), \mathscr{S}^*(X))$ which is bijective (this is a consequence of the kernel theorem). We shall denote then by $T \mapsto T^\#$ the inverse of the map $\mu \mapsto W(\mu)$. Thus, in what follows,

$$T = W(\mu) \;\text{ is equivalent to }\; \mu = T^\#.$$

In particular, to each $T \in B(\mathscr{H}) \subset B(\mathscr{S}(X), \mathscr{S}^*(X))$ we associate a distribution $T^\# \in \mathscr{S}^*(\Xi)$ which is not a measure in general. We describe now the connection between $T^\#$ and the *Weyl symbol* of T. Let the *Fourier measure* dη be the unique Haar measure on Ξ such that the (symplectic) Fourier transform

$$(\mathcal{F}_\sigma f)(\xi) \equiv \widehat{f}(\xi) := \int_\Xi \mathrm{e}^{-i\sigma(\xi,\eta)} f(\eta)\, \mathrm{d}\eta \tag{3.6}$$

is involutive (i.e. $\mathcal{F}_\sigma^2 = 1$) and unitary on $L^2(\Xi)$. We still denote by $\mathcal{F}_\sigma$ the extension to $\mathscr{S}^*(X)$ of this (properly normalized) Fourier transform. Then $\mathcal{F}_\sigma T^\#$ is the Weyl symbol of T.

We mention one more fact: although twisted convolution product defined on $\mathcal{M}$ cannot be extended to $\mathscr{S}^*(\Xi)$, it is possible to define $\mu \times \nu$ for $\mu, \nu \in \mathscr{S}^*(\Xi)$ if one of them has compact support (as in the case of the usual convolution). As a consequence, an easy computation gives, for all $f \in \mathscr{S}^*(\Xi)$,

$$\delta_\xi \times f = \mathrm{e}^{\frac{i}{2}\sigma(\xi,\,\cdot\,)}\tau_\xi f, \quad f \times \delta_{-\xi} = \mathrm{e}^{\frac{i}{2}\sigma(\xi,\,\cdot\,)}\tau_{-\xi} f \quad \text{and}$$
$$\delta_\xi \times f \times \delta_{-\xi} = \mathrm{e}^{i\sigma(\xi,\,\cdot\,)} f, \tag{3.7}$$

where τ_ξ is the translation operator, acting as $(\tau_\xi f)(\eta) = f(\eta - \xi)$. Notice that the relation $W(\mu \times \nu) = W(\mu)W(\nu)$ remains valid if ν is compactly supported. Furthermore, $W(\nu)$ leaves $\mathscr{S}(X)$ invariant and extends to an operator in $B(\mathscr{S}^*(X))$.

Proof of Theorem 3.1: We shall divide the proof into several steps. First, we show that it suffices to consider $T \in \mathfrak{C}(E)$ such that $T^\#$ is a distribution on Ξ whose support is a compact subset of E^σ. As a preliminary fact, we note the following result:

Lemma 3.2 *Let $T \in B(\mathscr{H})$ such that $[W(\xi), T] = 0$ for all $\xi \in E$. Then* $\operatorname{supp} T^\# \subset E^\sigma$.

Proof: Denote $T^\# \equiv \mu \in \mathscr{S}^*(\Xi)$. Then we have, by (3.7),

$$\begin{aligned} T = W(\mu) &= W(\xi)W(\mu)W(\xi)^* = W(\xi)W(\mu)W(-\xi) \\ &= W(\delta_\xi)W(\mu)W(\delta_{-\xi}) = W(\delta_\xi \times \mu \times \delta_{-\xi}) = W(\mathrm{e}^{i\sigma(\xi,\,\cdot\,)}\mu). \end{aligned}$$

But $W : \mathscr{S}^*(\Xi) \to B(\mathscr{S}(X), \mathscr{S}^*(X))$ is a bijective map, hence $(\mathrm{e}^{i\sigma(\xi,\,\cdot\,)} - 1)\mu = 0$ for all $\xi \in E$.

Let now $\eta_0 \notin E^\sigma$. Then there is a $\xi_0 \in E$ such that $\mathrm{e}^{i\sigma(\xi_0,\eta_0)} \neq 1$. So there is a whole compact neighborhood V such that $\mathrm{e}^{i\sigma(\xi_0,\eta)} - 1 \neq 0$ for all $\eta \in V$. If $\theta \in C_\mathrm{c}^\infty(V)$ then $\varphi := \theta/(\mathrm{e}^{i\sigma(\xi_0,\,\cdot\,)} - 1) \in C_\mathrm{c}^\infty(V))$, thus, by the previous computation, $\theta\mu = \varphi(\mathrm{e}^{i\sigma(\xi_0,\,\cdot\,)} - 1)\mu = 0$. Hence $\eta_0 \notin \operatorname{supp}\mu$. ■

Let us construct now the following "regularization" of T. Let $\theta \in L^1(\Xi)$ with $\int_\Xi \theta(\xi)\,\mathrm{d}\xi = 1$ ($\mathrm{d}\xi$ is the Fourier measure on Ξ), set $\theta_\varepsilon(\xi) = \varepsilon^{-N}\theta(\xi/\varepsilon)$ (with $N = \dim \Xi$) and define for $\varepsilon > 0$:

$$T_\varepsilon := \int_\Xi W(\xi)\, T\, W(\xi)^*\, \theta_\varepsilon(\xi)\,\mathrm{d}\xi. \tag{3.8}$$

Lemma 3.3 (i) *If $\lim_{\xi\to 0} \|\,[W(\xi), T]\,\| = 0$, then $\lim_{\varepsilon\to 0}\|T_\varepsilon - T\| = 0$.*
(ii) *If $T \in \mathfrak{C}(E)$ then $T_\varepsilon \in \mathfrak{C}(E)$ for all $\varepsilon > 0$.*
(iii) *If $\theta \in \mathscr{S}(\Xi)$, then $T_\varepsilon^\# = \widehat{\theta}_\varepsilon T^\#$.*

Proof: For (i) notice that $T_\varepsilon := \int_\Xi W(\varepsilon\xi)\, T\, W(\varepsilon\xi)^*\, \theta(\xi)\,\mathrm{d}\xi$. Since $\int_\Xi \theta(\xi)\,\mathrm{d}\xi = 1$, we get

$$\|T_\varepsilon - T\| \le \int_\Xi \|\, W(\varepsilon\xi)\, T\, W(\varepsilon\xi)^* - T\,\|\, |\theta(\xi)|\,\mathrm{d}\xi$$

which tends to zero as $\varepsilon \to 0$ by dominated convergence theorem under the condition of (i). For (ii), notice first that T_ε satisfies (i) of Definition 1.1 (for all $T \in B(\mathscr{H})$). Moreover, since $T \in \mathfrak{C}(E)$, for all $\eta \in E$

$$\begin{aligned} W(\eta)W(\xi)\,T\,W(\xi)^* &= \mathrm{e}^{i\sigma(\eta,\xi)}W(\xi)\,T\,W(\eta)W(-\xi) \\ &= \mathrm{e}^{i(\sigma(\eta,\xi)+\sigma(\eta,-\xi))}W(\xi)\,T\,W(\xi)^*W(\eta) \end{aligned}$$

thus (ii) of Definition 1.1 is valid for T_ε too. Finally, if $\eta \in E^\sigma$ and $\eta \to 0$,

$$\begin{aligned} \|\,(W(\eta)-1)\,T_\varepsilon^{(*)}\| &\leq \int_\Xi |1-\mathrm{e}^{i\sigma(\eta,\xi)}|\,\|T\|\,|\theta_\varepsilon(\xi)|\,\mathrm{d}\xi \\ &\quad + \|\,(W(\eta)-1)\,T^{(*)}\,\| \int_\Xi |\theta_\varepsilon(\xi)|\,\mathrm{d}\xi \end{aligned}$$

which tends to zero by dominated convergence. Thus $T_\varepsilon \in \mathfrak{C}(E)$.

Assume now $\theta \in \mathscr{S}(\Xi)$ and let us compute $T_\varepsilon^\#$. For $f, g \in \mathscr{S}(X)$, since the operators $W(\xi)$ leave $\mathscr{S}(X)$ invariant, one has (as in the proof of Lemma 3.2):

$$\begin{aligned} \langle f, T_\varepsilon g\rangle &= \int_\Xi \langle f,\, W(\xi)W(\mu)W(\xi)^*g\rangle\,\theta_\varepsilon(\xi)\,\mathrm{d}\xi = \int_\Xi \langle f, W(\mathrm{e}^{i\sigma(\xi,\,\cdot\,)}\,\mu)g\rangle\theta_\varepsilon(\xi)\,\mathrm{d}\xi \\ &= \left\langle f,\, W\left(\int_\Xi \mathrm{e}^{i\sigma(\xi,\,\cdot\,)}\theta_\varepsilon(\xi)\,\mathrm{d}\xi\mu\right)\,g\right\rangle = \langle f,\, W(\widehat{\theta}_\varepsilon\mu)\,g\rangle. \end{aligned}$$

Then, the bijectivity of W implies $T_\varepsilon^\# = \widehat{\theta}_\varepsilon\,\mu$ (notice that it suffices to prove this for a set of μ which is dense in $\mathscr{S}^*(\Xi)$, e.g. for $\mu \in \mathscr{S}(\Xi)$; in this case the third equality above is easy to justify). ■

The previous two lemmas show that any $T \in \mathfrak{C}(E)$ may be obtained as a norm limit of operators T_ε constructed as in (3.8) with θ such that $\widehat{\theta} \in C_{\mathrm{c}}^\infty(\Xi)$. Then $T_\varepsilon^\#$ are distributions with compact support included in E^σ. So it suffices to show that each of the corresponding T_ε is in the norm closure of (3.2). In what follows, we shall omit the subscript ε and work only with operators T such that $\operatorname{supp} T^\# \subset E^\sigma$ is compact.

We may reduce the problem still further by using a probability measure μ constructed as before Lemma 2.3 under the supplementary assumption that its density ρ belongs to $C_{\mathrm{c}}^\infty(E^\sigma)$. Then Lemma 2.2 shows that $W(\mu_\varepsilon) \in \mathfrak{C}(E)$ for all $\varepsilon > 0$, thus $W(\mu_\varepsilon)T \in \mathfrak{C}(E)$ too. By (ii) of Lemma 2.3, T is norm limit of operators $W(\mu_\varepsilon)T$, thus it suffices to prove the theorem for this type of operators. Fix again ε and denote μ_ε by μ. Then $W(\mu)T = W(\mu)W(T^\#) = W(\mu \times T^\#)$ (notice that $\mu \times T^\#$ makes sense because both

μ and $T^\#$ are compactly supported distributions). It suffices thus to prove

$$\mu \times T^\# = u\lambda_{E^\sigma} \tag{3.9}$$

where $u \in C_c^\infty(\Xi)$.

We begin by proving that the twisted convolution $\mu \times T^\#$ also has compact support contained in E^σ, and we also deduce precise information about its structure. Let M be a vector subspace of Ξ, supplementary to E^σ. Since $T^\#$ is a distribution whose compact support is contained in E^σ, we may use the Theorem 2.3.5 from [Hor] in order to get a representation of $T^\#$ as a finite sum

$$T^\# = \sum_\alpha u_\alpha \otimes \partial_M^\alpha \delta_0.$$

Here the tensor product refers to the identification $\Xi = E^\sigma \times M$, u_α are compactly supported distributions on E^σ, and $\partial_M^\alpha \delta_0$ are derivatives of the Dirac measure on M with support $\{0\}$. Using then the Theorem XXVI from §III.3 in [Sch], we may write each u_α as a finite sum of derivatives (in the directions of E^σ) of continuous functions with compact support on E^σ. Thus, we may write $T^\#$ as a *finite* sum:

$$T^\# = \sum_{\alpha,\beta} \partial_{E^\sigma}^\beta v_{\alpha\beta} \otimes \partial_M^\alpha \delta_0, \tag{3.10}$$

where $v_{\alpha\beta} \in C_c(E^\sigma)$.

Lemma 3.4 *There is a finite family $\{w_\alpha\}_{|\alpha|\le k}$ of functions in $C_c^\infty(E^\sigma)$ such that*

$$\mu \times T^\# = \sum_{|\alpha|\le k} w_\alpha \otimes \partial_M^\alpha \delta_0. \tag{3.11}$$

Proof: Let $\theta \in C_c^\infty(\Xi)$. The action of the distribution $\mu \times T^\#$ on θ is, by (3.1),

$$\begin{aligned}
\langle \theta, \mu \times T^\# \rangle &= \int_\Xi \int_\Xi \theta(\xi+\eta)\, e^{\frac{i}{2}\sigma(\xi,\eta)} \rho(\xi)\, \lambda_{E^\sigma}(d\xi)\, T^\#(\eta)\, d\eta \\
&= \int_{E^\sigma} \rho(\xi)\lambda_{E^\sigma}(d\xi) \int_\Xi e^{\frac{i}{2}\sigma(\xi,\eta)}\, \theta(\xi+\eta)\, T^\#(\eta)\, d\eta \\
&\equiv \int_{E^\sigma} \left\langle e^{\frac{i}{2}\sigma(\xi,\,\cdot\,)}\, \theta(\xi + \cdot\,)\,,\, T^\# \right\rangle \rho(\xi)\lambda_{E^\sigma}(d\xi).
\end{aligned}$$

If $\eta \in \Xi$ we denote by y and z its components in E^σ and M respectively. Thus, using (3.10), the action of the distribution $T^\#$ on $\mathrm{e}^{\frac{i}{2}\sigma(\xi,\cdot)}\,\theta(\xi+\cdot)$ is given by

$$\begin{aligned}
&\sum_{\alpha,\beta}\int_M \lambda_M(\mathrm{d}z)\int_{E^\sigma}\mathrm{e}^{\frac{i}{2}\sigma(\xi,y+z)}\,\theta(\xi+y+z)\ (\partial_y^\beta v_{\alpha\beta}\otimes\partial_z^\alpha\delta_0)(y+z)\,\lambda_{E^\sigma}(\mathrm{d}y)\\
&\quad=\sum_{\alpha,\beta}(-1)^{|\alpha|}\int_{E^\sigma}\mathrm{e}^{\frac{i}{2}\sigma(\xi,y)}\ \left[\partial_z^\alpha(\mathrm{e}^{\frac{i}{2}\sigma(\xi,z)}\theta(\xi+y+z))\right]_{z=0}\ \partial_y^\beta v_{\alpha\beta}(y)\,\lambda_{E^\sigma}(\mathrm{d}y)\\
&\quad=\sum_{\alpha,\beta}\sum_{\gamma\le\alpha}(-1)^{|\alpha|}\int_{E^\sigma}\mathrm{e}^{\frac{i}{2}\sigma(\xi,y)}\,\partial_y^\beta v_{\alpha\beta}(y)\,\varphi_{\alpha-\gamma}(\xi)\,[\partial_z^\gamma\theta(\xi+y+z)]_{z=0}\,\lambda_{E^\sigma}(\mathrm{d}y)
\end{aligned}$$

for some polynomials $\varphi_{\alpha-\gamma}(\xi)$ (coming from the derivatives of order $\alpha-\gamma$ of $\mathrm{e}^{\frac{i}{2}\sigma(\xi,z)}$ with respect to z at $z=0$). Further, one gets

$$\begin{aligned}
\langle\theta,\mu\times T^\#\rangle \;=\; &\sum_{\alpha,\beta}\sum_{\gamma\le\alpha}(-1)^{|\alpha|+|\beta|}\int_{E^\sigma}\lambda_{E^\sigma}(\mathrm{d}\xi)\,\rho(\xi)\,\varphi_{\alpha-\gamma}(\xi)\,\times\\
&\times\int_{E^\sigma}v_{\alpha\beta}(y)\,\partial_y^\beta\left(\mathrm{e}^{\frac{i}{2}\sigma(\xi,y)}\ [\partial_z^\gamma\theta(\xi+y+z)]_{z=0}\right)\lambda_{E^\sigma}(\mathrm{d}y).
\end{aligned}$$

Taking into account

$$\begin{aligned}
&\partial_y^\beta\left(\mathrm{e}^{\frac{i}{2}\sigma(\xi,y)}\ [\partial_z^\gamma\theta(\xi+y+z)]_{z=0}\right)\\
&\qquad=\ \sum_{\lambda\le\beta}\binom{\beta}{\lambda}\left(\partial_y^{\beta-\lambda}\mathrm{e}^{\frac{i}{2}\sigma(\xi,y)}\right)\,\partial_y^\lambda\,[\partial_z^\gamma\theta(\xi+y+z)]_{z=0}
\end{aligned}$$

and since $\partial_y^\lambda\,[\partial_z^\gamma\theta(\xi+y+z)]_{z=0}=\partial_\xi^\lambda\,[\partial_z^\gamma\theta(\xi+y+z)]_{z=0}$, we get

$$\begin{aligned}
\langle\theta,\mu\times T^\#\rangle =&\sum_{\alpha,\beta,\gamma,\lambda}C_{\alpha\beta\gamma\lambda}\int_{E^\sigma}\lambda_{E^\sigma}(\mathrm{d}\xi)\int_{E^\sigma}\partial_\xi^\lambda\left(\rho(\xi)\,\varphi_{\alpha-\beta}(\xi)\partial_y^{\beta-\lambda}\mathrm{e}^{\frac{i}{2}\sigma(\xi,y)}\right)\times\\
&\qquad\times v_{\alpha\beta}(y)\,[\partial_z^\gamma\theta(\xi+y+z)]_{z=0}\,\lambda_{E^\sigma}(\mathrm{d}y)\\
=&\sum_{\alpha,\beta,\gamma,\lambda}C_{\alpha\beta\gamma\lambda}\int_{E^\sigma}\lambda_{E^\sigma}(\mathrm{d}\tau)\int_{E^\sigma}\partial_\tau^\lambda\left(\rho(\tau-y)\,\varphi_{\alpha-\beta}(\tau-y)\partial_y^{\beta-\lambda}\mathrm{e}^{\frac{i}{2}\sigma(\tau,y)}\right)\times\\
&\qquad\times v_{\alpha\beta}(y)\,[\partial_z^\gamma\theta(\tau+z)]_{z=0}\,\lambda_{E^\sigma}(\mathrm{d}y).
\end{aligned}$$

Now it is clear that there is $k\in\mathbb{N}$, and for $|\alpha|\le k$ there are functions $w_\alpha\in C_c^\infty(E^\sigma)$, such that

$$\langle\theta,\mu\times T^\#\rangle=\sum_{|\alpha|\le k}\int_{E^\sigma}w_\alpha(\tau)\ [\partial_z^\alpha\theta(\tau+z)]_{z=0}\,\lambda_{E^\sigma}(\mathrm{d}\tau)=\sum_{|\alpha|\le k}\langle\theta,w_\alpha\otimes\partial_M^\alpha\delta_0\rangle\quad\blacksquare$$

From Lemma 3.4 we see that (3.9) holds if we may take $k = 0$ in the representation (3.11) of $\mu \times T^{\#}$. So the proof of the Theorem 3.1 is finished once we show that *the operator* $W(\mu \times T^{\#})$ *is bounded if and only if* $k = 0$. Before starting this last step of our proof, we shall describe a direct sum decomposition of the symplectic space Ξ, determined by E.

We define the *centralizer* of E by $E^{\mathrm{c}} := E \cap E^{\sigma}$, and set $\overline{E} := E + E^{\sigma}$. A subspace is *symplectic* if its centralizer reduces to $\{0\}$. Using $E^{\sigma\sigma} = E$ it is easy to see that $E^{\mathrm{c}} = E^{\sigma \mathrm{c}} = \overline{E}^{\mathrm{c}} = \overline{E}^{\sigma}$ and $\overline{E} = E^{\mathrm{c}\sigma} = \overline{E^{\mathrm{c}}} = \overline{E^{\sigma}}$. In what follows, we shall denote by $\oplus^{\sigma}$ and $\oplus$ the symplectic (respectively vector) direct sum, and by $\perp$ the symplectic orthogonality between elements of Ξ. Let now $F \subset E$ such that $E = F \oplus E^{\mathrm{c}}$ and $G \subset E^{\sigma}$ such that $E^{\sigma} = G \oplus E^{\mathrm{c}}$. Then F *and* G *are symplectic* (indeed, suppose there is a $\xi \in F$ with $\xi \perp F$; then $E^{\mathrm{c}} \subset E^{\sigma}$ shows that $\xi \perp E^{\mathrm{c}}$, so $\xi \perp (F + E^{\mathrm{c}})$, thus $\xi \in E^{\sigma}$, which means that $\xi \in E \cap E^{\sigma} = E^{\mathrm{c}}$; then $F \cap E^{\mathrm{c}} = \{0\}$ shows $\xi = 0$). Hence $H \equiv F \oplus^{\sigma} G$ *is a symplectic subspace too, and we have* $\overline{E} = H \oplus^{\sigma} E^{\mathrm{c}}$. For, if $\xi \in H \cap E^{\mathrm{c}}$ then there are $\xi_F \in F$ and $\xi_G \in G$ such that $\xi_F = \xi - \xi_G \in E^{\sigma} - E^{\sigma} = E^{\sigma}$ which shows that $\xi_F \perp F \subset E$. Since F is symplectic, $\xi_F = 0$ and the same holds for ξ_G. On the other hand, $H + E^{\mathrm{c}} = F + G + E^{\mathrm{c}} = (F + E^{\mathrm{c}}) + (G + E^{\mathrm{c}}) = E + E^{\sigma} = \overline{E}$. Further, remark that H^{σ} *is also a symplectic space and* E^{c} *is a lagrangian subspace of* H^{σ}. Indeed, $E^{\mathrm{c}} \perp H$ thus $E^{\mathrm{c}} \subset H^{\sigma}$ and is isotropic. If it where not maximal, there would exist some $\xi \in H^{\sigma} \setminus E^{\mathrm{c}}$ with $\xi \perp E^{\mathrm{c}}$. Thus $\xi \perp (E^{\mathrm{c}} + H) = \overline{E}$, i.e. $\xi \in \overline{E}^{\sigma} = E^{\mathrm{c}}$, which is absurd.

Now let K be a lagrangian subspace of H^{σ} such that $H^{\sigma} = E^{\mathrm{c}} \oplus K$ (this will be a holonomic decomposition of the symplectic space H^{σ}). Then Ξ splits as

$$\Xi = H \oplus^{\sigma} H^{\sigma} = F \oplus^{\sigma} G \oplus^{\sigma} (E^{\mathrm{c}} \oplus K) = E^{\sigma} \oplus F \oplus K.$$

Let us go back now to the proof of the fact that $W(\mu \times T^{\#})$ is bounded only if $k = 0$. Note that we may take $M = F \oplus K$. Any $\xi \in \Xi$ may be uniquely written as $\xi = \eta + \zeta_F + \zeta_K$ with $\eta \in E^{\sigma}$, $\zeta_F \in F$ and $\zeta_K \in K$. In the next formulas we abbreviate $\lambda_F(\mathrm{d}\zeta_F) = \mathrm{d}\zeta_F$ and $\lambda_K(\mathrm{d}\zeta_K) = \mathrm{d}\zeta_K$. Then, by (3.5) and (3.11),

$$W(\mu \times T^{\#}) =$$
$$= \sum_{|\alpha| \leq k} \int_{F \oplus K} \int_{E^{\sigma}} W(\eta + \zeta_F + \zeta_K) w_{\alpha}(\eta) \, [\partial^{\alpha}_{F \oplus K} \delta_0](\zeta_F + \zeta_K) \lambda_{E^{\sigma}}(\mathrm{d}\eta) \, \mathrm{d}\zeta_F \, \mathrm{d}\zeta_K$$

which can be further developed as follows:

$$\sum_{|\alpha|\le k}\int_{F\oplus K}\mathrm{d}\zeta_F\,\mathrm{d}\zeta_K\,W(\zeta_F+\zeta_K)\int_{E^\sigma}\mathrm{e}^{\frac{i}{2}\sigma(\eta,\zeta_F+\zeta_K)}\,W(\eta)w_\alpha(\eta)\times$$
$$\times[\partial^\alpha_{F\oplus K}\delta_0](\zeta_F+\zeta_K)\lambda_{E^\sigma}(\mathrm{d}\eta)=$$

$$=\sum_{|\alpha|\le k}\int_{F\oplus K}\mathrm{d}\zeta_F\,\mathrm{d}\zeta_K\,W(\zeta_F)W(\zeta_K)\int_{E^\sigma}\mathrm{e}^{\frac{i}{2}\sigma(\eta,\zeta_K)}\,W(\eta)w_\alpha(\eta)\times$$
$$\times[\partial^\alpha_{F\oplus K}\delta_0](\zeta_F+\zeta_K)\lambda_{E^\sigma}(\mathrm{d}\eta)$$
$$=\sum_{|\alpha|+|\beta|\le k}(-1)^{|\alpha|+|\beta|}[\partial^\alpha_F W(\zeta_F)]_{\zeta_F=0}\times$$
$$\times\int_{E^\sigma}[\partial^\beta_K\{W(\zeta_K)\mathrm{e}^{\frac{i}{2}\sigma(\eta,\zeta_K)}\}]_{\zeta_K=0}\,W(\eta)\,w_\alpha(\eta)\lambda_{E^\sigma}(\mathrm{d}\eta)$$
$$=\sum_{|\alpha|+|\beta|\le k}[\partial^\alpha_F W(\zeta_F)]_{\zeta_F=0}[\partial^\beta_K W(\zeta_K)]_{\zeta_K=0}W(\mu_{\alpha\beta}).$$

The third equality above is a consequence of $F\perp K$ (so $W(\zeta_F+\zeta_K)=W(\zeta_F)\,W(\zeta_K)$) and of $F\perp E^\sigma$ (which implies $\sigma(\eta,\zeta_F)=0$). Further, we took into account that the derivatives with respect to ζ_K at $\zeta_K=0$ of $\mathrm{e}^{\frac{i}{2}\sigma(\eta,\zeta_K)}$ are polynomial functions of $\eta\in E^\sigma$ and $w_\alpha\in C^\infty_{\mathrm{c}}(E^\sigma)$, thus there are E^σ-a.c. measures $\mu_{\alpha\beta}$ with densities with respect to λ_{E^σ} of class $C^\infty_{\mathrm{c}}(E^\sigma)$ such that the last equality be true.

Let $F=\Omega\oplus\Omega^*$ and $G=\Gamma\oplus\Gamma^*$ be some holonomic decompositions of the symplectic subspaces F and G. Thus we get a decomposition

$$\Xi=(\Omega\oplus\Omega^*)\oplus^\sigma(\Gamma\oplus\Gamma^*)\oplus^\sigma(E^{\mathrm{c}}\oplus K)$$

each of the three parentheses being a symplectic space equipped with a holonomic decomposition. We also have $E^\sigma=(\Gamma\oplus\Gamma^*)\oplus^\sigma E^{\mathrm{c}}$. Then $[\partial^\alpha_F W(\zeta_F)]_{\zeta_F=0}\equiv\phi_\alpha(Q_\Omega,P_\Omega)$ and $[\partial^\beta_K W(\zeta_K)]_{\zeta_K=0}\equiv\psi_\beta(P_{E^c})$ with ϕ_α and ψ_β polynomial functions of degrees $|\alpha|$ and $|\beta|$ respectively. Observe that the notation P_{E^c} is justified by the identification $(E^{\mathrm{c}})^*=K$. Moreover, the operators $W(\mu_{\alpha\beta})$ can be written in the form $F_{\alpha\beta}(Q_\Gamma,P_\Gamma,Q_{E^c})$ for some smooth function $F_{\alpha\beta}$, expression which can be rigorously interpreted in terms of the Weyl calculus (this fact, however, does not play any role in what follows). Thus we have

$$W(\mu\times T^\#)=\sum_{|\alpha|+|\beta|\le k}\phi_\alpha(Q_\Omega,P_\Omega)\,\psi_\beta(P_{E^c})\,F_{\alpha\beta}(Q_\Gamma,P_\Gamma,Q_{E^c})$$

and we have to prove that if this operator is bounded then necessarily $k=0$.

Note that we can assume from the beginning that $X = \Omega \oplus \Lambda$, where $\Lambda = \Gamma \oplus E^{\mathrm{c}}$. Then our Hilbert space factorizes as $\mathscr{H} = L^2(\Omega) \otimes L^2(\Lambda)$ and, if we denote $\phi_\alpha(Q_\Omega, P_\Omega)$ by S_α and $\sum_\beta \psi_\beta(P_{E^{\mathrm{c}}}) W(\mu_{\alpha\beta})$ by T_α we have $W(\mu \times T^\#) = \sum_{|\alpha| \leq k} S_\alpha \otimes T_\alpha$. We show first that, for all α, one has $S_\alpha \in B(L^2(\Omega))$, in fact they are complex multiples of the identity operator on $L^2(\Omega)$.

Notice that $T_\alpha \in B(\mathscr{S}(\Lambda))$ and that we may assume that the family $\{T_\alpha\}_{|\alpha| \leq k}$ is linearly independent in $B(\mathscr{S}(\Lambda))$. Let $\mathscr{T}$ denote the vector space of operators $L : \mathscr{S}(\Lambda) \to \mathscr{S}(\Lambda)$ of the form $L = \sum_i |u_i\rangle\langle v_i|$ (finite sum), with $u_i, v_i \in \mathscr{S}(\Lambda)$. Then we may realize $\mathscr{T}$ as a space of linear forms on $B(\mathscr{S}(\Lambda))$ by defining $B(\mathscr{S}(\Lambda)) \ni T \mapsto \mathrm{Tr}\,(LT) = \sum_i \langle v_i, T u_i\rangle$. Notice that $\mathscr{T}$ is a subspace of the dual of $B(\mathscr{S}(\Lambda))$ which separates the points (for, if $\mathrm{Tr}\,(LT) = \mathrm{Tr}\,(LT')$ for all $L \in \mathscr{T}$, then $\langle u, (T - T')v\rangle = 0$ for all $u, v \in \mathscr{S}(\Lambda)$). It follows that we may find a finite family of operators $\{L_\alpha\} \in \mathscr{T}$ such that

$$\mathrm{Tr}\,(L_\alpha T_\beta) = \delta_{\alpha\beta} \text{ for all } \alpha, \beta \text{ with } |\alpha| \leq k,\ |\beta| \leq k.$$

Indeed, if $\mathscr{V}$ is the finite dimensional vector subspace generated by the family $\{T_\alpha\}$ (which is a basis in it) we may find a basis $\{\Phi_\alpha\}$ in the dual space $\mathscr{V}'$ such that $\Phi_\alpha(T_\beta) = \delta_{\alpha\beta}$. Thus we are reduced to prove that for each $\Phi \in \mathscr{V}'$ we may find $L \in \mathscr{T}$ such that $\Phi(T) = \mathrm{Tr}\,(LT)$ for all $T \in \mathscr{V}$, i.e. that the mapping $\mathscr{T} \ni L \mapsto \mathrm{Tr}\,(L\,\cdot\,)|_{\mathscr{V}} \in \mathscr{V}'$ is surjective. Equivalently, the dual mapping $\mathscr{V} \equiv \mathscr{V}'' \ni T \mapsto \mathrm{Tr}\,(\,\cdot\,T) \in \mathscr{T}'$ has to be injective. But this is true, since $\mathscr{T}$ separates points.

Let now $L_\alpha = \sum_i |u_{\alpha i}\rangle\langle v_{\alpha i}|$ with $u_{\alpha i}, v_{\alpha i} \in \mathscr{S}(\Lambda)$ and, for $u, v \in \mathscr{S}(\Omega)$, let us compute

$$\sum_i \langle v \otimes v_{\alpha i}, W(\mu \times T^\#)\, u \otimes u_{\alpha i}\rangle =$$
$$= \sum_i \sum_{|\gamma| \leq k} \langle v, S_\gamma\, u\rangle\langle v_{\alpha i}, T_\gamma\, u_{\alpha i}\rangle = \sum_{|\gamma| \leq k} \langle v, S_\gamma\, u\rangle \mathrm{Tr}\,(L_\alpha T_\gamma) = \langle v, S_\alpha\, u\rangle.$$

Since $W(\mu \times T^\#)$ has been supposed bounded, this shows that all S_α are bounded. But S_α are polynomials $\phi_\alpha(Q_\Omega, P_\Omega)$, so these polynomials have to be of degree zero, i.e. ϕ_α are complex numbers.

Now we can write $W(\mu \times T^\#) = 1 \otimes \sum_\alpha \phi_\alpha T_\alpha$. Thus it remains to show that if an operator of the form $B = \sum_\beta \psi_\beta(P_{E^{\mathrm{c}}}) W(\mu_\beta)$ is bounded in $L^2(\Lambda)$, then the polynomials ψ_β are constants. By rearranging the sum we can assume that $B = \sum_\gamma P^\gamma_{E^{\mathrm{c}}} W(\mu_\gamma)$ (a basis in E^{c} has been chosen). Let

$a = \lambda\omega$, with $\lambda \in \mathbb{R}$ and $\omega \in E^{\mathrm{c}}$ and for any $u, v \in \mathscr{S}(\Lambda)$, denote $u_a = \exp[i\langle Q_{E^c}, a\rangle]u$ and $v_a = \exp[i\langle Q_{E^c}, a\rangle]v$. Then $|\langle u_a, Bv_a\rangle| \leq C\|u\|\|v\|$, and on the other hand

$$\begin{aligned}|\langle u_a, Bv_a\rangle| &= |\langle u, \sum_{|\gamma|\leq m} (P_{E^c} + a)^\gamma W(\mu_\gamma)v\rangle| \\ &= \lambda^m \left| \sum_{|\gamma|=m} \omega^\gamma \langle u, W(\mu_\gamma)v\rangle + \sum_{|\gamma|<m} \sum_{\delta\leq\gamma} \binom{\gamma}{\delta} \lambda^{-m+|\gamma|-|\delta|} \omega^{\gamma-\delta} \langle u, P^\delta_{E^c} W(\mu_\gamma)v\rangle \right|.\end{aligned}$$

By taking $\lambda \to \infty$ we see that we must have $\sum_{|\gamma|=m} \omega^\gamma \langle u, W(\mu_\gamma)v\rangle = 0$ for all $\omega \in E^{\mathrm{c}}$ if $m \geq 1$. But this implies $\langle u, W(\mu_\gamma)v\rangle = 0$ for all $|\gamma| \geq 1$ by a standard argument. This ends the proof of Theorem 3.1. ■

3.2. We end this section with some remarks concerning the case when the representation W is not irreducible. For systems having a finite number of internal degrees of freedom, e.g. systems with spin, W is of finite multiplicity and Theorem 3.1 extends in a trivial way. The case when W is of infinite multiplicity is, however, also interesting (e.g. consider an N-body system coupled with a confined system) and less trivial. Let us define the *commutator algebra* $\mathrm{Com}(\Xi, \mathscr{H})$ by

$$\mathrm{Com}(\Xi, \mathscr{H}) = \{T \in B(\mathscr{H}) \mid [W(\xi), T] = 0\ \forall \xi \in \Xi\}.$$

With the notations of Definition 1.1 this is just $\mathfrak{C}(\Xi)$. Note that W is of finite multiplicity if and only if the commutator algebra is finite dimensional. The extension of Theorem 3.1 to this general context is the following:

Theorem 3.5 *The algebra $\mathfrak{C}(E)$ coincides with the norm closed linear subspace of $B(\mathscr{H})$ generated by the operators of the form $W(\mu)T$, where μ are E^σ-a.c. integrable measures and $T \in \mathrm{Com}(\Xi, \mathscr{H})$.*

We shall sketch the proof of Theorem 3.5 below. For this it is convenient to choose a special form for the representation W. Let $\Xi = X \oplus X^*$ be a holonomic decomposition of the symplectic space Ξ and let W_{s} be the Schrödinger representation defined in (3.4). Then by Theorem C.38 in [RW] there is a Hilbert space $\mathbf{H}$ (the space of internal degrees of freedom) such that W is unitarily equivalent to the representation $W_{\mathrm{s}} \otimes 1$ in the Hilbert space $L^2(X) \otimes \mathbf{H}$. From now on we make the identifications $W \equiv W_{\mathrm{s}} \otimes 1$ and $\mathscr{H} \equiv L^2(X) \otimes \mathbf{H} = L^2(X; \mathbf{H})$. Also, in order to avoid ambiguities, the algebras $\mathfrak{C}(E)$ from the Definition 1.1 will be denoted now by $\mathfrak{C}(E; \mathbf{H})$. In particular, $\mathfrak{C}(E; \mathbb{C})$ are the algebras acting on $L^2(X)$ associated to W_{s}.

It is easy to check that $\mathfrak{C}(\Xi;\mathbf{H}) = \mathrm{Com}\,(\Xi,\mathscr{H}) = 1 \otimes B(\mathbf{H})$. Moreover, we have $\mathfrak{C}(\mathbf{O};\mathbf{H}) = \mathbb{K}(X) \otimes B(\mathbf{H})$ by Theorem 3.8 from [GI1]. In fact, these are particular cases of the following general assertion: for all $E \in \mathbb{G}(E)$,

$$\mathfrak{C}(E;\mathbf{H}) = \mathfrak{C}(E;\mathbb{C}) \otimes B(\mathbf{H}). \tag{3.12}$$

In other terms, $\mathfrak{C}(E;\mathbf{H})$ *is the norm closed linear subspace of* $B(\mathscr{H})$ *generated by the operators of the form* $W_{\mathrm{s}}(\mu) \otimes T$*, where* μ *are* E^{σ}*-a.c. integrable measures and* $T \in B(\mathbf{H})$. The proof of this assertion is a straightforward extension of that of Theorem 3.1 and will not be given here. On the other hand, the equality (3.12) is equivalent to Theorem 3.5 because $\mathrm{Com}\,(\Xi,\mathscr{H}) = 1 \otimes B(\mathbf{H})$.

Since, when $\mathbf{H}$ is infinite dimensional, $\mathfrak{C}(\mathbf{O};\mathbf{H})$ is larger that the algebra of compact operators on $\mathscr{H}$, it appears that the algebras $\mathfrak{C}(E;\mathbf{H})$ are too big for some purposes. We shall single out the following smaller algebras:

$$\mathfrak{C}(E)^{\mathbf{H}} = \mathfrak{C}(E;\mathbb{C}) \otimes K(\mathbf{H}). \tag{3.13}$$

In particular, $\mathfrak{C}(\mathbf{O})^{\mathbf{H}} = K(\mathscr{H})$. It is easy to show that the algebras $\mathfrak{C}(E)^{\mathbf{H}}$ can also be described directly in terms of the initial representation W (so independently of $\mathbf{H}$) as follows: $\mathfrak{C}(E)^{\mathbf{H}}$ is the set of operators $T \in B(\mathscr{H})$ which satisfy (i)-(iii) of Definition 1.1 and verify $W(\mu)T \in K(\mathscr{H})$ for each Ξ-a.c. integrable measure μ; or $\mathfrak{C}(E)^{\mathbf{H}}$ is equal to

$$\mathfrak{C}(E)_{\circ} = \{T \in \mathfrak{C}(E) \mid W(\mu)T \in K(\mathscr{H}) \text{ if } \mu \text{ is a } \Xi\text{-a.c. integrable measure}\}$$

It is clear that if W is of finite multiplicity then $\mathfrak{C}(E)_{\circ} = \mathfrak{C}(E)$. On the other hand, Theorem 2.1 remains valid for the C^*-algebras $\mathfrak{C}(E)_{\circ}$ too. Indeed, in the representation considered above, $\mathfrak{C}(E)_{\circ}$ is equal to the algebra defined in (3.13); thus the assertion follows from general properties of the tensor product operation.

Finally, we give a proof of Proposition 2.5. By the results quoted above we may assume $\mathscr{H} = L^2(X) \otimes \mathbf{H}$ and $W(\xi) = W_{\mathrm{s}}(\xi) \otimes 1$. Then W is of finite multiplicity if and only if $\dim \mathbf{H} < \infty$. If $\dim \mathbf{H} = \infty$ and $T = T_1 \otimes T_2 \neq 0$ with $T_1 \in K(L^2(X))$ and $T_2 \in B(\mathbf{H})$ not compact, then $T \in \mathfrak{C}(\mathbf{O}) \setminus K(\mathscr{H})$. If $\dim \mathbf{H} < \infty$, then $\mathfrak{C}(\mathbf{O}) \subset K(\mathscr{H})$ follows from the Riesz-Kolmogorov compacity criterion (Proposition 3.13 in [GI2]).

4 N-body systems in constant magnetic fields

4.1. Our purpose in this section is to show that the algebraic framework developed until now provides a natural setting for the study of N-particle systems in constant external magnetic fields. We shall first introduce a certain class of subalgebras of $\mathfrak{C}^\Xi$ similar to those which appear in the treatment of the N-body problem in [ABG, BG, DG1, DG2]. Then we consider particular cases which make the connection with the framework of [GL1, GL3].

For simplicity, *we assume from now on that the representation W is of finite multiplicity* so, by Proposition 2.5, we have $K(\mathscr{H}) = \mathfrak{C}(\mathbf{O})$. If W is of infinite multiplicity, one should replace below the algebras $\mathfrak{C}(E)$ by the $\mathfrak{C}(E)_\circ$ introduced at the end of Section 3.

Let $\mathcal{L}$ be a finite semilattice of subspaces of Ξ having a greatest element $\max\mathcal{L}$. We recall that "semilattice" means that $E, F \in \mathcal{L} \Rightarrow E \cap F \in \mathcal{L}$. It is clear that $\mathcal{L}$ will also have a smallest element $\min\mathcal{L}$. From Theorem 2.1 it follows that

$$\mathfrak{C} \equiv \mathfrak{C}(\mathcal{L}) := \sum_{E\in\mathcal{L}} \mathfrak{C}(E) \tag{4.1}$$

is a C^*-algebra of operators on $\mathscr{H}$ which is $\mathcal{L}$-graded. To be precise, this means that the sum in (4.1) is direct (linearly and topologically) and that $\mathfrak{C}(E)\cdot\mathfrak{C}(F) \subset \mathfrak{C}(E\cap F)$ for all $E, F \in \mathcal{L}$.

The algebra $\mathfrak{C}$ is the C^*-algebra of energy observables which we intend to study. It was shown in [BG] that *the case of N-body systems without external magnetic fields corresponds to the situation when* $\max\mathcal{L}$ *is a lagrangian subspace of* Ξ (so the other subspaces in $\mathcal{L}$ are isotropic). We shall see in the next subsections that *the case when a magnetic field is present is characterized by the fact that* $\max\mathcal{L}$ *is a subspace of dimension half of that of* Ξ *but it is not isotropic.*

We recall several consequences of the fact that $\mathfrak{C}$ is $\mathcal{L}$-graded (details can be found in [BG, ABG], but note that our convention concerning the order relation is opposite to theirs). For each $E \in \mathcal{L}$ set

$$\mathfrak{C}_E = \sum_{F\supset E, F\in\mathcal{L}} \mathfrak{C}(F), \qquad \mathfrak{J}_E = \sum_{F\not\supset E, F\in\mathcal{L}} \mathfrak{C}(F). \tag{4.2}$$

Then $\mathfrak{C}_E$ is a C^*-subalgebra of $\mathfrak{C}$, $\mathfrak{J}_E$ is an ideal (closed, self-adjoint) of $\mathfrak{C}$, and

$$\mathfrak{C} = \mathfrak{C}_E + \mathfrak{J}_E \quad\text{and}\quad \mathfrak{C}_E \bigcap \mathfrak{J}_E = \{0\}. \tag{4.3}$$

Denote $\mathcal{P}_E$ the projection of $\mathfrak{C}$ onto $\mathfrak{C}_E$ given by the preceding direct decomposition; then $\mathcal{P}_E$ is a morphism.

If $E \in \mathcal{L}$ let $E^\circ = E \setminus \bigcup G$ where G runs over the set of elements of $\mathcal{L}$ strictly included in E. We have $E^\circ = \mathbf{O}$ if and only if $E = \mathbf{O}$ (because $\mathcal{L}$ is finite).

Proposition 4.1 *Let $E \in \mathcal{L}$, $E \neq \mathbf{O}$, and $\omega \in E^\circ$, $\omega \neq 0$. Then for each $T \in \mathfrak{C}$ one has*

$$\mathcal{P}_E[T] = \operatorname*{s\text{-}lim}_{\lambda\to\infty} W(\lambda\omega)^* T\, W(\lambda\omega). \tag{4.4}$$

Proof: Let $T = \sum_{F\in\mathcal{L}} T(F)$ be the unique decomposition of T as sum of elements $T(F) \in \mathfrak{C}(F)$. If $F \supset E$ then $\lambda\omega \in F$, hence (ii) of Definition 1.1 gives $W(\lambda\omega)^* T(F)\, W(\lambda\omega) = T(F)$. Thus

$$W(\lambda\omega)^* T\, W(\lambda\omega) = \sum_{F\supset E} T(F) + \sum_{F\not\supset E} W(\lambda\omega)^* T(F)\, W(\lambda\omega).$$

If $E \not\subset F$ then $G = E \cap F$ belongs to $\mathcal{L}$ and is strictly included in E; thus $\omega \notin F$. Hence from Lemma 2.4 it follows that the terms in the last sum tend strongly to zero when $\lambda \to \infty$. ∎

Observe that $\mathfrak{C}(\min\mathcal{L})$ is an ideal in $\mathfrak{C}$ and let $\mathcal{M}$ be the set of atoms in $\mathcal{L}$ (minimal elements of $\mathcal{L} \setminus \{\min\mathcal{L}\}$). Then the morphism $\mathfrak{C} \to \bigoplus_{E\in\mathcal{M}} \mathfrak{C}_E$ (direct sum of C^*-algebras) defined by $\mathcal{P}[T] = (\mathcal{P}_E[T])_{E\in\mathcal{M}}$ has kernel $\ker\mathcal{P} = \mathfrak{C}(\min\mathcal{L})$, so we get a canonical embedding:

$$\mathfrak{C}/\mathfrak{C}(\min\mathcal{L}) \hookrightarrow \textstyle\bigoplus_{E\in\mathcal{M}} \mathfrak{C}_E. \tag{4.5}$$

This is the fundamental relation which allows one to compute the essential spectrum and to prove the Mourre estimate for observables affiliated to $\mathfrak{C}$. Here we shall consider only questions related to the essential spectrum. The following easy fact has to be pointed out:

Lemma 4.2 *Assume that the self-adjoint operator H on the Hilbert space $\mathscr{H}$ is affiliated to a C^*-algebra $\mathfrak{C}$ such that $\mathfrak{C} \cap K(\mathscr{H}) = \{0\}$. The H has no discrete spectrum.*

Proof: This is algebraically obvious, because the restriction to $\mathfrak{C}$ of the canonical surjection of $B(\mathscr{H})$ onto the Calkin algebra $B(\mathscr{H})/K(\mathscr{H})$ is injective, hence an isomorphism onto its image. More directly, note that $\varphi(H) \in \mathfrak{C}$ if $\varphi \in C_c(\mathbb{R})$, so $\varphi(H)$ cannot be a compact operator (if $\neq 0$). ∎

Now, by using the relation $\mathfrak{C}(\mathbf{O}) = K(\mathscr{H})$, the general theory presented in [GI2] gives:

Theorem 4.3 *If $\min\mathcal{L} \neq \mathbf{O}$ then the self-adjoint operators affiliated to $\mathfrak{C}$ have no discrete spectrum. If $\min\mathcal{L} = \mathbf{O}$ and H is a self-adjoint operator affiliated to $\mathfrak{C}$, then*

$$\boldsymbol{\sigma}_{\text{ess}}(H) = \textstyle\bigcup_{E\in\mathcal{M}} \boldsymbol{\sigma}(\mathcal{P}_E[H]) \tag{4.6}$$

The observables $\mathcal{P}_E[H]$ can be computed as strong resolvent limits

$$\mathcal{P}_E[H] = \operatorname*{s\text{-}lim}_{\lambda\to\infty} W(\lambda\omega)^* \, H \, W(\lambda\omega) \tag{4.7}$$

where $\omega \in E^\circ \setminus \{0\}$.

Remark: This theorem gives physically interesting examples of observables whose essential spectrum is not determined by localizations at infinity in *configuration* space (see §4.2 in [GI2]). In particular, this is does not hold for systems in constant, nontrivial, external magnetic fields.

4.2. Our purpose in the rest of this section is to show how the systems of N non relativistic particles in constant external magnetic fields fit into the formalism of §4.1. It will be convenient to begin with some general considerations concerning the localizability of a quantum system which explain the geometrical origin of the magnetic fields (at least of the constant ones) and justify our later constructions. The main point of our argument is that there is no reason to think that the "momentum observable" is a genuine observable (X^*-valued, in the sense defined in Chapter 8 of [ABG] for example). We shall see that in fact the momentum is an object of a quite different nature than the position observable. However, the symplectic algebra point of view allows us to treat them on an equal footing. We refer to [Var] (especially Chapters X and XI) for a detailed presentation of the notions involved in our treatment and for references to the fundamental works of G. Mackey and A. Wightman on these questions.

Let $\mathscr{H}$ be the Hilbert space of states of the system and let X be a locally compact abelian group, interpreted as the configuration space of the system[(1)]. The system is called *localizable in* X if a spectral measure E_Q has been given on X: for a Borel set $A \subset X$ the projection $E_Q(A)$ represents the property "the system is in A". The functional calculus associated to E_Q produces a unital morphism $\varphi \mapsto \varphi(Q)$ of the algebra of bounded Borel complex functions φ on X into $B(\mathscr{H})$, which is the *position observable* Q. Note that this morphism is canonically determined by its restriction to $C_0(X)$, so Q is an X-valued observable. If $k \in X^*$ is a character of X (see §3.4 in [GI2]) and if we set $V_k = k(Q)$, we get a strongly continuous unitary

[(1)] Physically speaking, one should consider a locally compact space X on which acts a group of symmetries G. Since we have nothing to say in such a general situation, we assume from the start that X is an abelian group. In fact, even this case is too difficult to treat in general, so we shall soon specialize to the case when X is a vector space. Here we take $G = X$, but an important more general case will be considered in the next section.

representation $\{V_k\}_{k\in X^*}$ of X^* in $\mathscr{H}$ which uniquely determines E_Q and reciprocally, each such representation comes from a position observable.

In order to define the momentum, we assume that the action of X on itself by translations induces symmetries of the system. More precisely, a weakly continuous group morphism $x \mapsto \mathcal{U}_x$ of X into the space of automorphisms of $B(\mathscr{H})$ is given, such that $\mathcal{U}_x[E_Q(A)] = E_Q(x + A)$ for all $x \in X$ and all Borel sets $A \subset X$.

If U is a unitary or anti-unitary operator in $\mathscr{H}$ then $T \mapsto U^*TU$ is any automorphism of $B(\mathscr{H})$. Reciprocally, each automorphism $\mathcal{U}$ of $B(\mathscr{H})$ is of this form, the operator U being uniquely defined modulo a complex factor of modulus 1. Moreover, locally U can be chosen continuously in $\mathcal{U}$ in the natural topologies. The fact that U is unitary or anti-unitary can be easily decided in terms of $\mathcal{U}$ by topological methods or by the following simple observation: if $\mathcal{U}$ is the square of an automorphism, then clearly U is unitary. To apply this below, it suffices to assume that each $x \in X$ is of the form $x = 2y$. In what follows we denote by $U(\mathscr{H})$ the set of unitary operators on $\mathscr{H}$ and by $U(1)$ the group of complex numbers of modulus 1.

To sum up this discussion, in the cases of interest (e.g. if X is a real vector space) the translation covariance of the system can be expressed as follows: there is a strongly continuous map $U : X \to U(\mathscr{H})$ such that $U_x^* E_Q(A) U_x = E_Q(x + A)$ for all $x \in X$ and all Borel sets $A \subset X$ (we set $U_x = U(x)$). Moreover, $U_0 = 1$ and for all $x, y \in X$ there is $b(x, y) \in U(1)$ such that $U_x U_y = b(x, y) U_{x+y}$. Clearly $b : X \times X \to U(1)$ is a continuous function such that $b(x, 0) = b(0, x) = 1$ and $b(x, y)b(x + y, z) = b(x, y + z)b(y, z)$ for all $x, y, z \in X$. Thus we see that *b is a multiplier on X and U is a projective representation of multiplier b of X on the Hilbert space $\mathscr{H}$* (see [Var]).

It is convenient to write the relation $U_x^* E_Q(A) U_x = E_Q(x + A)$ in an equivalent form involving the unitary representation V_k of X^*. Since the preceding relation implies $U_x^* \varphi(Q) U_x = \varphi(Q - x)$ for all bounded Borel functions φ (with the convention $\varphi(Q - x) = \psi(Q)$ if $\psi(y) = \varphi(y - x)$) we see that the localizability in X of the system is expressed by the conditions:

$$U_x U_y = b(x, y) U_{x+y}, \;\; V_k V_p = V_{k+p}, \;\; U_x V_k = k(x) V_k U_x; \;\; x, y \in X, k, p \in X^*.$$

It is important to notice that there is a rather high degree of arbitrariness in the choice of the operators U_x. Indeed, only the $\mathcal{U}_x$ are given, so an arbitrary continuous function $\lambda : X \to U(1)$ gives operators $U'_x = \lambda(x) U_x$ which play the same role as U_x, the only difference being that the corresponding multiplier is $b'(x, y) = \frac{\lambda(x)\lambda(y)}{\lambda(x+y)} b(x, y)$. Two multipliers related by such a relation are called similar. The equivalence classes of multipliers with respect to the equivalence relation given by similarity form a group $M(X)$, the *multiplier group of* X. Physically distinct situations are indexed by this group, so one must have a complete description of $M(X)$ in order to understand all the possibilities covered by the concept of localization.

In order to make the connection with later developments we note that $M(X)$ can be interpreted in cohomological terms (see §6.1). Indeed, a multiplier is a 2-cocycle on X with coefficients in $U(1)$ (which is equipped with the trivial X-module structure). Hence, if we neglect the continuity conditions required above, we have $M(X) = H^2(X, U(1))$.

If X is an arbitrary locally compact abelian group, the group $M(X)$ may be very complicated. Supplementary difficulties come from the fact that one should also consider multipliers which are not continuous but only Borel (see chapter X in [Var]). For these reasons we shall specialize now the class of X by considering the simplest but physically most interesting ones. We could, of course, go on a little further in the general context by using the framework of §3.6 from [GI2].

4.3. From now on X will be a real finite dimensional vector space. Then we have the following explicit description of $M(X)$ (see Theorem 10.38 in [Var]): *let* B *be a real antisymmetric bilinear form on* X (abbreviated below as "antisymmetric form"); *then* $b(x,y) = \mathrm{e}^{-iB(x,y)/2}$ *is a multiplier of* X *and the map which sends* B *in the similarity class of* b *is an isomorphism of the additive group of all antisymmetric forms on* X *onto the multiplier group* $M(X)$. The factor $-1/2$ in the exponent is needed for compatibility with the notations used in most of the literature on magnetic hamiltonians.

Thus there is no loss of generality if we consider only multipliers of the form $b(x,y) = \mathrm{e}^{-iB(x,y)/2}$, where B is an antisymmetric form on X. We identify B with the antisymmetric linear map $B : X \to X^*$ defined by the condition $B(x,y) = \langle x, By\rangle$ (the dual group X^* being identified with the dual vector space X^* by $k(x) = \mathrm{e}^{i\langle x,k\rangle}$, see §3.4 in [GI2]). We stress that our notation B is slightly misleading: for a particle in an external magnetic field, B depends on the product of the charge of the particle and the field. More precisely, let $X = \mathbb{R}^3$, denote $\vec{B} \in \mathbb{R}^3$ the (constant) magnetic field, and let q be the charge of the particle. Then one gets the correct commutation relations (e.g. those from [GL3]) by taking $B(x,y) = \langle q\vec{B}, \vec{x} \times \vec{y}\rangle$ (scalar and vector products in $\mathbb{R}^3$), in particular $\overrightarrow{Bx} = q\vec{x} \times \vec{B}$.

We now fix B and introduce several objects of interest. The notations are chosen so that in the case $B = 0$ we get those of §3.6 from [GI2] (note, however, that there we considered only the Schrödinger representation).

Let $\mathscr{H}$ be a Hilbert space and let $U : X \to U(\mathscr{H})$ and $V : X^* \to U(\mathscr{H})$

be strongly continuous maps such that ($U_x \equiv U(x)$ and $V_k \equiv V(k)$):

$$U_{x+y} = \mathrm{e}^{\frac{i}{2}B(x,y)} U_x U_y, \quad V_{k+p} = V_k V_p, \quad U_x V_k = \mathrm{e}^{i\langle x,k\rangle} V_k U_x \tag{4.8}$$

for all $x, y \in X$ and all $k, p \in X^*$. Observe that

$$U_x U_y = \mathrm{e}^{-iB(x,y)} U_y U_x. \tag{4.9}$$

We define self-adjoint operators $\langle x, P\rangle$ and $\langle Q, k\rangle$ on $\mathscr{H}$ by the rules:

$$U_{tx} = \mathrm{e}^{it\langle x,P\rangle} \text{ and } U_{tk} = \mathrm{e}^{it\langle Q,k\rangle}, \text{ for all } t \in \mathbb{R}.$$

Let $\mathscr{S}$ be the space of those $f \in \mathscr{H}$ such that the maps $x \mapsto U_x f$ and $k \mapsto V_k f$ are of class C^∞. Then the linear combinations of the operators $\langle x, P\rangle$ and $\langle Q, k\rangle$ (and also the polynomials of order ≤ 2) are essentially self-adjoint on $\mathscr{S}$. Notice the non-trivial commutation relations

$$[\langle x,P\rangle, \langle y,P\rangle] = iB(x,y) \tag{4.10}$$

$$[\langle x,P\rangle, \langle Q,k\rangle] = -i\langle x,k\rangle, \tag{4.11}$$

valid at least on $\mathscr{S}$. Moreover, we have $U_x\langle y,P\rangle U_x^* = \langle y,P\rangle - B(x,y)$ or, with a slightly formal notation,

$$U_x P U_x^* = P + Bx. \tag{4.12}$$

We emphasize once again that P is not an (X^*-valued) observable in the usual sense unless $B = 0$, because the components of P along different directions do not commute in general. However, we can define a rather rich functional calculus for P by using the same formula as in the case $B = 0$ namely, if $\psi : X^* \to \mathbb{C}$ and $\widehat{\psi} : X \to \mathbb{C}$ is its Fourier transform, then we set:

$$\psi(P) = \int_X U_x \widehat{\psi}(x)\,\mathrm{d}x. \tag{4.13}$$

The C^*-algebra of operators on $\mathscr{H}$ generated by $\psi(P)$ with $\widehat{\psi} \in L^1(X^*)$ will be denoted by $C_0(X_b^*; W)$; it is abelian if and only if $B = 0$ (or $b = 1$) and in this case $C_0(X_b^*; W_\mathrm{s}) = C_0(X^*)$ if W_s the Schrödinger representation (see §3.4 in [GI2]). We notice that $C_0(X_b^*; W)$ is a representation of the group C^*-algebra $C^*(X_b)$ of the "Heisenberg group" X_b associated to the couple (X, b), see §5.1. The next relations are formally obvious and easy to make rigorous:

$$U_x \varphi(Q) U_x^* = \varphi(Q + x) \text{ and } U_x \psi(P) U_x^* = \psi(P + Bx). \tag{4.14}$$

We now define the symplectic space which is natural in this context. As vector space $\Xi = X \oplus X^*$ is the same as in the case $B = 0$, but we equip it with the symplectic form:

$$\sigma(\xi, \eta) = \sigma((x, k), (y, p)) := \langle y, k\rangle - \langle x, p\rangle - B(x, y). \tag{4.15}$$

The fact that σ is non-degenerate for any B is easy to check. Now we define for each $\xi = (x, k) \in \Xi$ a unitary operator $W(\xi)$ on $\mathscr{H}$ by the same rule as in the case $B = 0$ (cf. §3.6 in [GI2]):

$$W(\xi) := \mathrm{e}^{i(\langle Q,k\rangle - \langle x,P\rangle)} = \mathrm{e}^{iS(\xi)}. \tag{4.16}$$

The operator $S(\xi) = \langle Q, k\rangle - \langle x, P\rangle$ should be interpreted as the closure of its restriction to $\mathscr{S}$, so it is self-adjoint and the exponential well defined. Straightforward computations give the same relations as in the case $B = 0$ (because B does not appear in (4.11)):

$$W(\xi) := \mathrm{e}^{\frac{i}{2}\langle x,k\rangle} U_{-x} V_k = \mathrm{e}^{-\frac{i}{2}\langle x,k\rangle} V_k U_{-x}. \tag{4.17}$$

A new simple computation based on (4.8) gives

$$W(\xi + \eta) = \mathrm{e}^{-\frac{i}{2}\sigma(\xi,\eta)} W(\xi) W(\eta), \tag{4.18}$$

hence W is a representation of the symplectic space (Ξ, σ) on $\mathscr{H}$ in the sense defined in the introduction. In terms of W, the relations of (4.14) become

$$W(\xi)\varphi(Q)W(\xi)^* = \varphi(Q - x), \ W(\xi)\psi(P)W(\xi)^* = \psi(P - k - Bx). \tag{4.19}$$

The next proposition contains a new characterization of certain algebras $\mathfrak{C}(E)$ which will prove to be of physical interest. Let Y be a subspace of X. We recall the notation $Y^\perp = \{k \in X^* \mid \langle y, k\rangle = 0\}$ and the canonical identification $(X/Y)^* = Y^\perp$. The symplectic form (4.15) gives

$$Y^B := (X \oplus Y^\perp)^\sigma = \{(y, -By) \mid y \in Y\} \subset \Xi. \tag{4.20}$$

Equivalently, $(Y^B)^\sigma = X \oplus Y^\perp$. If $\varphi \in C_0(X/Y)$ let[(2)] $\varphi(Q_Y) = \varphi \circ \pi_Y(Q)$, where π_Y is the canonical surjection $X \to X/Y$. Below, the C^*-algebra on $\mathscr{H}$ consisting of these operators $\varphi(Q_Y)$ is denoted $C_0(X/Y; W)$.

[(2)] Since $\{V_k\}$ is a unitary representation of X^*, the functional calculus associated to Q is very rich: $\phi(Q)$ is a well defined bounded operator for $\phi : X \to \mathbb{C}$ bounded and Borel.

Proposition 4.4 *Let Y be a subspace of X. Then $\mathfrak{C}(Y^B)$ is the norm closure in $B(\mathscr{H})$ of the linear subspace generated by the operators of the form $\varphi(Q_Y)\psi(P)$, with $\varphi \in C_0(X/Y)$ and $\psi \in C_c^\infty(X)$. In other terms:*

$$\mathfrak{C}(Y^B) = [\![C_0(X/Y;W) \cdot C_0(X_b^*;W)]\!]. \tag{4.21}$$

Proof: Theorem 3.1 shows that $\mathfrak{C}(Y^B)$ is the closure in $B(\mathscr{H})$ of the set of operators of the form $W(\mu)$ where μ is an $X \oplus Y^\perp$-a.c. integrable measure. More explicitly, we have to take the closure of the set of operators of the form

$$\int_{Y^\perp}\int_X \mathrm{e}^{i(\langle Q,k\rangle - \langle x,P\rangle)}\phi(k,x)\,\mathrm{d}k\,\mathrm{d}x,$$

where $\phi \in C_c^\infty(Y^\perp \times X)$. Using (4.17) and an approximation argument we see that $\mathfrak{C}(Y^B)$ is the norm linear subspace generated by

$$\int_{Y^\perp}\int_X \mathrm{e}^{i\langle Q,k\rangle}\mathrm{e}^{i\langle x,P\rangle)}u(k)v(x)\,\mathrm{d}k\,\mathrm{d}x$$

with $u \in C_c^\infty(Y^\perp), v \in C_c^\infty(X)$. Since $\langle z,k\rangle = \langle \pi_Y(z),k\rangle$ for $k \in Y^\perp = (X/Y)^*$, we easily get the desired result. ■

Remark 4.5 In particular, if $Y = X$ then $\mathfrak{C}(X^B) = \mathfrak{C}(X^\sigma) = C_0(X_b^*;W)$, with the identification $X = X \oplus \{0\}$.

The representations W for different values of B can be realized on the same Hibert space (so the C^*-algebras associated to them can be compared). The next formulas also allow us to make the connection with the magnetic fields formalism encountered in the literature (see [GL1, GL3] and references therein). Denote σ_0 the symplectic form σ in the case $B = 0$, and $\Xi_0 \equiv (\Xi,\sigma_0)$. Assume that a representation W_0 of the symplectic space Ξ_0 in $\mathscr{H}$ has been given. Then one gets a representation of Ξ in $\mathscr{H}$ by setting

$$W(x,k) = W_0(x,k+\frac{1}{2}Bx) = \mathrm{e}^{-\frac{i}{2}B(x,Q)}W_0(x,k). \tag{4.22}$$

In particular, in the Schrödinger representation of (Ξ,σ_0), one has:

$$(W(\xi)f)(y) = \mathrm{e}^{i\langle y-\frac{x}{2},k\rangle}\mathrm{e}^{-\frac{i}{2}B(x,y)}f(y-x).$$

If Q_0 and P_0 are the operators corresponding to W_0, then $Q = Q_0$ and $P = P_0 + \frac{1}{2}BQ$. For example, in the case of a particle in $\mathbb{R}^3$ considered above, and if W_0 is the Schrödinger representation, then one gets $\vec{P} = \vec{D} + \frac{1}{2}q\,\vec{Q}\times\vec{B}$

where $\vec{D} = -i\vec{\nabla}$. Since $\vec{A} = \frac{1}{2}\vec{B} \times \vec{x}$ is a magnetic potential such that $\vec{B} = \vec{\nabla} \times \vec{A}$, we get the usual formula $\vec{P} = \vec{D} - q\vec{A}$.

4.4. We keep the setting of the preceding subsection and introduce the formalism specific to the case of N-body systems. We adapt our notations to those of [GL3], where one can find a detailed description of the physical context.

Let $\mathcal{A}$ be a finite lattice, i.e. a finite partially ordered set in which each pair a, b has a greatest lower bound $a \wedge b$ and a least upper bound $a \vee b$; in particular $\mathcal{A}$ has a least element $\min \mathcal{A}$ and a greatest element $\max \mathcal{A}$ (denoted $a_{\min}$ and $a_{\max}$ in [GL3]). Let $a \mapsto X_a$ be an injective map of $\mathcal{A}$ into $\mathbb{G}(X)$ such that $X_{a\vee b} = X_a \cap X_b$ and $X_{\min \mathcal{A}} = X$. The range of this map is a semilattice of subspaces of X containing X. We define

$$X_a^B := \{(x, -Bx) \mid x \in X_a\} \subset \Xi. \tag{4.23}$$

We take as semilattice $\mathcal{L}$ in §4.1 the set $\mathcal{L} = \{X_a^B \mid a \in \mathcal{A}\}$. Observe that $a \mapsto X_a^B$ is a decreasing bijection of $\mathcal{A}$ onto $\mathcal{L}$ such that $X_{a\vee b}^B = X_a^B \cap X_b^B$ and $X_{\min \mathcal{A}} = X^B$. Thus the C^*-algebra associated to $\mathcal{L}$ is

$$\mathfrak{C}^B := \sum_{a\in\mathcal{A}} \mathfrak{C}(X_a^B). \tag{4.24}$$

In the present context it is convenient to modify the notations introduced in §4.1 as follows: set $\mathfrak{C}^B(a) = \mathfrak{C}(X_a^B)$, let $\mathfrak{C}_a^B$ be the subalgebra $\sum_{b\le a} \mathfrak{C}^B(b)$ and $\mathfrak{J}_a^B$ the ideal $\sum_{b\not\le a} \mathfrak{C}^B(b)$ of $\mathfrak{C}^B$. Then

$$\mathfrak{C}^B = \mathfrak{C}_a^B + \mathfrak{J}_a^B \quad \text{and} \quad \mathfrak{C}_a^B \bigcap \mathfrak{J}_a^B = \{0\}. \tag{4.25}$$

Observe that now $\mathfrak{C}^B(a) \cdot \mathfrak{C}^B(b) \subset \mathfrak{C}^B(a \vee b)$. The projection of $\mathfrak{C}^B$ onto $\mathfrak{C}_a^B$ given by the preceding direct decomposition is denoted $\mathcal{P}_a^B$; we know that $\mathcal{P}_a^B$ is a morphism and for a self-adjoint operator (or observable) H affiliated to $\mathfrak{C}^B$ we set $H_a = \mathcal{P}_a^B[H]$. For the operators considered in [GL3] this symbol has the usual meaning, see below.

Let us denote $X^a = X/X_a$. As observed in (4.20), we have

$$(X_a^B)^\sigma = X \oplus X_a^\perp = X \oplus (X^a)^*. \tag{4.26}$$

From Proposition 4.4 it follows that $\mathfrak{C}^B(a)$ is the closed linear subspace generated by the operators of the form $\varphi(Q^a)\psi(P)$, where $\varphi \in C_0(X^a)$ and $\psi \in C_c^\infty(X)$. Here $\varphi(Q^a) = \varphi \circ \pi^a(Q)$ where π^a is the canonical surjection

$X \to X^a$. This fact implies that $\mathfrak{C}^B$ is a representation of the crossed product of the C^*-algebra $\sum_a C_0(X^a) \subset C_{\mathrm{b}}^{\mathrm{u}}(X)$ equipped with the action of a certain "Heisenberg group" associated to X and B, see the next section.

In the particular case $a = \min \mathcal{A}$ we will have $\mathfrak{C}^B(\min \mathcal{A}) = \mathfrak{C}(X^\sigma) = C_0(X_b^*; W)$, cf. Remark 4.5. If X is equipped with an euclidean structure then Δ_X is the operator associated to it as explained in Proposition 2.8. For the notion of strict affiliation used in the next theorem, see [DG2] and Section 2 in [GI2].

Theorem 4.6 *Assume that X is equipped with an euclidean structure and let $H_0 = \Delta_X$. For each $a \in \mathcal{A}$ let $H(a)$ be a symmetric continuous sesquilinear form on $D(H_0^{1/2})$ such that $H(\min \mathcal{A}) = 0$ and $H(a) \geq -\mu_a H_0 - \delta_a$ for some positive numbers μ_a, δ_a with $\sum_a \mu_a < 1$. Let $H = H_0 + \sum_a H(a)$ and $H_a = H_0 + \sum_{b \leq a} H(b)$ be the self-adjoint operators defined as form sums. Assume that there is $\alpha > 1/2$ such that $(H_0+1)^{-\alpha} H(a)(H_0+1)^{-1/2} \in \mathfrak{C}^B(a)$ for all a. Then the self-adjoint operator H is strictly affiliated to $\mathfrak{C}^B$ and $\mathcal{P}_a[H] = H_a$ for all $a \in \mathcal{A}$.*

Proof: In Proposition 2.8 it was proved that the operator H_0 is affiliated to $\mathfrak{C}^B(\min \mathcal{A})$. A straightforward argument shows that the affiliation is strict. Thus the theorem is a consequence of Theorem 2.11 from [DG2]. ■

The preceding result shows that the magnetic hamiltonians considered in [GL3] are affiliated to the algebra $\mathfrak{C}^B$. Indeed, the forms $H(a)$ considered there are operators of the form $V^a(Q^a)$, with $V^a : X^a \to \mathbb{R}$. If $V^a \in C_0(X^a)$ then $V^a(Q^a)(H_0+1)^{-1/2} \in \mathfrak{C}^B(a)$ by Remark 4.5. In general, the operators $(H_0 + 1)^{-\alpha} V^a(Q^a)(H_0 + 1)^{-1/2}$ are norm limits of similar operators with $V^a \in C_0(X^a)$, so that they belong to the same algebra.

5 Non-constant magnetic fields

5.1. In this section we consider the problem of defining the C^*-algebra of energy observables for a system placed in an arbitrary external magnetic field. The construction we propose is based on the notion of crossed product and extends the techniques introduced in [GI2]. In order to make natural the transition from constant to non-constant magnetic fields which, as we shall see, involves the consideration of general symmetry groups, we shall first reformulate the conclusions of our discussion of localizability from §4.2.

We assume that we are in the setting of §4.2, in particular X is an abelian locally compact group and b a multiplier on it, and we recall the definition of the "Heisenberg group" (3) X_b associated to the couple (X, b). This is the set $X_b = U(1) \times X$ equipped with the product topology and the multiplication rule $(\lambda, x) \cdot (\mu, y) = (\lambda\mu b(x, y), x + y)$. The multiplication is chosen such that if U is as in §4.2 then the map which sends (λ, x) into λU_x is a unitary representation of X_b. In fact, there is a bijective correspondence between projective representations of X of multiplier b and unitary representations of X_b which send $(\lambda, 0)$ into λ times the identity operator. Note that X_b is a step 2 nilpotent group if $b \neq 1$. The group X_b acts on X by the rule $(\lambda, x).y = x + y$ and this induces an action on $C_{\mathrm{b}}^{\mathrm{u}}(X)$ by translations: $(\alpha_{(\lambda,x)}\varphi)(y) = \varphi(y - x)$. Clearly $C_{\mathrm{b}}^{\mathrm{u}}(X)$ becomes a X_b-algebra, cf. [GI2].

A translation invariant C^*-algebra $\mathscr{A} \subset C_{\mathrm{b}}^{\mathrm{u}}(X)$ is X_b-stable, so we may define the C^*-crossed product $\mathscr{A} \rtimes X_b$. If $b = 1$ we get the algebras considered in detail in [GI2]. If $b \neq 1$ we get analogues of these algebras in which the components of the "momentum observable" do not commute. These correspond to systems subject to interactions of type $\mathscr{A}$ and immersed in an external constant magnetic field. For example, the C^*-algebra of energy observables of an N-body system placed in the constant magnetic field B, as constructed in §4.4, is of the form $\mathscr{A} \rtimes X_b$, where X is a real finite dimensional vector space (the configuration space of the system), $\mathscr{A} = \sum_Y C_0(X/Y)$ (Y runs over a finite semilattice of subspaces of X) and b is the multiplier associated to B.

In order to treat non-constant magnetic fields, we shall replace above X_b by more general groups G which act on X. The difficulty is, of course, to isolate the right class of groups G which have to be considered. The conclusion of our analysis will be that it is natural to assume that G is an extension of X by an abelian group. Thus, if $\mathscr{A}$ is as above, *$\mathscr{A} \rtimes G$ will be the C^*-algebra of energy observables of systems having X as configuration space, subject to internal interactions of type $\mathscr{A}$, and whose "momentum observable" derives from the symmetry group G which is determined by the external magnetic field.*

An extension of X is a surjective group morphism $\pi : G \to X$ with a commutative kernel (this notion is discussed in the Appendix, §6.1). Then $g.x = \pi(g) + x$ defines a transitive action of G on X and $(\alpha_g\varphi)(x) = \varphi(g^{-1}.x) = \varphi(x - \pi(g))$ gives an action α of G on functions on X. If G is a locally compact group and π is continuous, α induces a continuous action of G on the C^*-algebra $C_{\mathrm{b}}^{\mathrm{u}}(X)$ which thus becomes a G-algebra (this

(3) This is not a standard terminology. If (Ξ, σ) is a symplectic space and $\omega(\xi, \eta) = \exp[i\sigma(\xi, \eta)/2]$ then the universal cover of the group Ξ_ω is the Heisenberg group as defined in [Fol] for example. See [Var] and the references to G. Mackey therein, but also [Wei].

being a natural extension of the notion of X-algebra introduced in §3.1 of [GI2]). The action of G on X looks like that of X on itself by translations, hence a G-stable C^*-subalgebra $\mathscr{A}$ of $C_{\mathrm{b}}^{\mathrm{u}}(X)$ is the same thing as a translation invariant C^*-subalgebra. However, the crossed product[(4)] $\mathscr{A} \rtimes G$ of $\mathscr{A}$ by the action of G is quite different from the crossed product $\mathscr{A} \rtimes X$ of $\mathscr{A}$ by the action of X. Now for each of the translation invariant C^*-subalgebras of $C_{\mathrm{b}}^{\mathrm{u}}(X)$ considered in [GI2] for example, one can consider the crossed product by the action of the larger group G. The fact that this has something to do with magnetic fields will be shown later on.

Two particular cases of this construction cover situations that we studied before. First, we take $G = X$ and get the algebras considered in [GI2]. Second, one may take $G = X_b$ and $\pi(\lambda, x) = x$; this gives the case of constant magnetic fields, as explained above.

To see what happens in more general cases, it is useful to think in terms of the universal property of crossed products [Ped, Rae]. Namely, the representations of $\mathscr{A} \rtimes G$ are in bijective correspondence with *covariant representations* of $(\mathscr{A}, G, \alpha)$, i.e. couples consisting of a non-degenerate representation $\varphi \mapsto \varphi(Q)$ of $\mathscr{A}$ and a strongly continuous unitary representation $g \mapsto U_g$ of G on the same Hilbert space $\mathscr{H}$, such that $U_g \varphi(Q) U_g^* = \varphi(Q + \pi(g))$. Set $U(\theta) = \int_G U_g \theta(g)\, \mathrm{d}g$ for $\theta \in C_{\mathrm{c}}(G)$ and $\mathrm{d}g$ a Haar measure on G. Then the range of the representation of $\mathscr{A} \rtimes G$ associated to this covariant representation is the closed linear subspace of $B(\mathscr{H})$ generated by the operators $\varphi(Q)U(\theta)$. Or, if we denote $\mathscr{A}^\flat$ and $C^*(G)^\flat$ the representations of $\mathscr{A}$ and of the group C^*-algebra $C^*(G)$ on $\mathscr{H}$, then $[\![\mathscr{A}^\flat \cdot C^*(G)^\flat]\!]$ is a representation of the abstract crossed product $\mathscr{A} \rtimes G$ on $\mathscr{H}$. Thus the C^*-algebras of energy observables considered here naturally extend those considered in [GI2].

The fact that systems placed in arbitrary external magnetic fields can be naturally treated in this framework will be shown in §5.3. For this purpose, some elementary facts concerning the construction of extensions of X are useful: we recall them in the Appendix. Observe that the kernel of an extension $G \to X$ is a normal subgroup N of G. The main result is that the extensions can be classified in terms of N, actions of X on N by automorphisms, and elements of a second order cohomology group of X with coefficients in N. In order to cover the case of an arbitrary magnetic field we choose $N = U(1)^X$, or rather the subset $C(X; U(1))$ of continuous functions $X \to U(1)$, equipped with the group structure given by usual multiplication of functions and with the natural action of X by translation. We

[(4)] Since $\ker \pi$ is a closed abelian normal subgroup and $G/\ker \pi$ is abelian, the group G is amenable, see 7.3.5 in [Ped]. Thus $\mathscr{A} \rtimes G$ coincides with the reduced crossed product.

provide $U(1)^X$ with the topology of uniform convergence on compact subsets of X; we get a topological group, but not a locally compact one. Particular classes of magnetic fields are obtained by considering translation invariant subgroups which could be locally compact for the induced topology: for example, if X is a vector space, one can consider elements of the form e^{ia} with a belonging to a finite dimensional linear space of real polynomials on X (constant magnetic fields correspond to polynomials of degree zero). We mention that nilpotent Lie groups appear often as extensions of X by such subgroups; magnetic hamiltonians which are quadratic in the generators of a nilpotent Lie algebra have been studied in [Jo2, JoK].

Remark: The choice $N = U(1)^X$ is justified a posteriori by our results. However, an a priori justification is the following. Let $\pi : G \to X$ be an extension of X, assume that $\mathscr{A}$ contains $C_0(X)$, and let $(\varphi \mapsto \varphi(Q), g \mapsto U_g)$ be a covariant representation of $(\mathscr{A}, G)$ on a Hilbert space $\mathscr{H}$. Let $\mathscr{B}(X)$ be the algebra of bounded Borel complex functions on X and let $\widetilde{\mathscr{A}}$ be the Von Neumann algebra generated by the operators $\varphi(Q)$. Then $\varphi \mapsto \varphi(Q)$ uniquely extends to a σ-normal surjective morphism of $\mathscr{B}(X)$ onto $\widetilde{\mathscr{A}}$. In particular, $\widetilde{\mathscr{A}}$ is a quotient of $\mathscr{B}(X)$. If $u \in N = \ker \pi$, then $U_u\varphi(Q)U_u^* = \varphi(Q + \pi(u)) = \varphi(Q)$ for all φ. Hence, *if $\widetilde{\mathscr{A}}$ is maximal abelian*, we get $U_u \in \widetilde{\mathscr{A}}$. Then it can be shown that there is a group morphism $u \mapsto u(\cdot)$ of N into the set of Borel functions $X \to U(1)$ such that $U_u = u(Q)$ for all $u \in N$. This realizes N as a subgroup of $U(1)^X$, which explains why we are looking after extensions of X by $U(1)^X$.

Before going on with the details of the construction we make a remark valid in the general context of abelian groups. From now on we shall freely use the notations and results of §6.1 of the Appendix.

5.2. Fix an additive abelian group X and let $N = U(1)^X$, the set of all maps $X \to U(1)$, equipped with the group structure given by usual multiplication of functions and with the X-module structure defined by the action $(x.u)(y) = u(y + x)$.

Lemma 5.1 *The second cohomology group of X with coefficients in $U(1)^X$ is trivial: $H^2\left(X, U(1)^X\right) = \{1\}$.*

Proof: If $b : X \times X \to N$ and if we set $b_{x,y} = b(x, y)$, then the cocycle condition (6.1) becomes

$$b_{x,y}(t)b_{x+y,z}(t) = b_{x,y+z}(t)b_{y,z}(t + x) \text{ and } b_{x,0}(t) = b_{0,y}(t) = 1$$

for all $x, y, z, t \in X$. If $t = 0$ we get $b_{x,y}(0)b_{x+y,z}(0) = b_{x,y+z}(0)b_{y,z}(x)$ hence, after a change of notation,

$$b_{x,y}(z) = b_{z,x}(0)b_{z+x,y}(0)b_{z,x+y}(0)^{-1}.$$

If we set $\lambda_x(z) = b_{z,x}(0)$ then this relation can be rewritten as

$$b_{x,y}(z) = \lambda_x(z)\lambda_y(z + x)\lambda_{x+y}(z)^{-1}. \tag{5.1}$$

Thus, if we denote $\lambda(x)$ the map $z \mapsto \lambda_x(z)$, then $\lambda(x) \in U(1)^X$, $\lambda(0) = 1$ and $b(x, y) = \lambda(x)\, x.\lambda(y)\, \lambda(x + y)^{-1}$, so b is a coboundary. ∎

As a consequence of the preceding lemma, we get an explicit description of the 2-cocycles on X with coefficients in $U(1)^X$:

Lemma 5.2 *One has $b \in Z^2(X, U(1)^X)$ if and only if there is a function $\beta : X \times X \to U(1)$ such that $\beta(x, x) = 1$ for all $x \in X$ and*

$$b(x, y) = \beta(Q, Q + x)\beta(Q + x, Q + x + y)\beta(Q, Q + x + y)^{-1}. \tag{5.2}$$

One has $b(x, -x) = 1$ for all x if and only if $\beta(y, x) = \beta(x, y)^{-1}$ for all x, y, in which case

$$b(x, y) = \beta(Q, Q + x)\beta(Q + x, Q + x + y)\beta(Q + x + y, Q). \tag{5.3}$$

Proof: If $b(x, y) = \lambda(x)\, x.\lambda(y)\, \lambda(x + y)^{-1}$ and if we set $\beta(x, y) = \lambda_{y-x}(x)$ (the index notation is as in the proof of Lemma 5.1), then $\lambda_s(t) = \beta(t, t + s)$ and (5.1) gives (5.2). Reciprocally, (5.2) clearly implies (5.1). The normalization condition $\lambda(0) = 1$ is equivalent to $\beta(x, x) = 1$ for all x. The last assertion of the lemma is obvious. ∎

5.3. In this subsection we first show that magnetic fields on X can be identified with certain 2-cocycles with coefficients in $U(1)^X$. This point of view will allow us to consider the extension of X associated to the 2-cocycle and then to introduce the C^*-algebras of energy observables for systems in external magnetic fields.

Let X be a finite dimensional real vector space and $A : X \to X^*$ a continuous map (thus A is a differential 1-form on X; what follows can be generalized to the case when A is locally integrable). Physically speaking, A is the *magnetic potential* (multiplied by the charge). According to the usual theory of integration of 1-forms, if γ is a C^1 map of a real interval $[a, b]$ into X we set $\int_\gamma A = \int_a^b \langle \frac{d}{dt}\gamma(t), A(\gamma(t))\rangle\, \mathrm{d}t$, where $\langle \cdot, \cdot \rangle$ is the duality map on $X \times X^*$. Since the result is independent of an orientation preserving reparameterization of γ, this defines $\int_\Gamma A$ for each one dimensional oriented submanifold Γ of X (Γ can have boundary and angular points). If $x, y \in X$ then the segment with origin x and end y is denoted $[x, y]$. Thus we will have $\int_{[x,y]} A = \int_\gamma A$ with $\gamma(t) = (1-t)x + ty$, $t \in [0,1]$. We set

$$\beta(x,y) = \exp\left(-i\int_{[x,y]} A\right). \tag{5.4}$$

Then $\beta : X \times X \to U(1)$ is a continuous function such that $\beta(x,x) = 1$ and $\beta(y,x) = \beta(x,y)^{-1}$ for all x, y. Lemma 5.2 shows that *the function $b(x,y)$ associated to β by the formula (5.3) is a 2-cocycle on the additive group X with coefficients in $U(1)^X$*. From the preceding definitions it follows that

$$\left[\frac{d}{dt}\int_{[x,x+ty]} A\right]_{t=0} = \langle y, A(x)\rangle, \tag{5.5}$$

hence

$$\left[\frac{d}{dt}\beta(x, x+ty)\right]_{t=0} = -i\langle y, A(x)\rangle. \tag{5.6}$$

If x, y, z are three points of X then we denote by $[x,y,z]$ the union of the three segments $[x,y], [y,z], [z,x]$ with the orientation given by this order. Thus $[x,y,z]$ is a closed oriented one dimensional submanifold and is the boundary of an oriented (possibly degenerated) triangle $\Delta(x,y,z)$. If A is of class C^1 then Stokes theorem gives

$$\int_{[x,y,z]} A = \int_{\Delta(x,y,z)} \mathrm{d}A \equiv \int_{\Delta(x,y,z)} B \tag{5.7}$$

where $B = \mathrm{d}A$ is a 2-form on X interpreted as *magnetic field* and the last integral above is the *flux of the magnetic field through the triangle $\Delta(x,y,z)$*. In particular, the product

$$\beta(x,y)\beta(y,z)\beta(z,x) = \exp\Big(-i\int_{\Delta(x,y,z)} B\Big) \tag{5.8}$$

depends only on the magnetic field, not on the magnetic potential. If b is the 2-cocycle associated to β (see (5.3)) and $b(x,y)(z) \equiv b_{x,y}(z)$, we have

$$\begin{aligned} b_{x,y}(z) &= \beta(z, z+x)\beta(z+x, z+x+y)\beta(z+x+y, z) \\ &= \exp\Big(-i\int_{\Delta(z,z+x,z+x+y)} B\Big) \end{aligned} \tag{5.9}$$

Note that $b_{x,y}$ is well defined even if the magnetic field B does not exist as a function (but only as a distribution) and that it has the following physical interpretation: *$b_{x,y}(z)$ is the flux of the magnetic field through the oriented triangle $\Delta(z, z+x, z+x+y)$.*

The following consequence of relation (5.9) will be useful below: *if the vectors x, y are collinear, then $b(x, y) = 1$.* Indeed, the points $z, z+x, z+x+y$ will be collinear, hence the triangle $\Delta(z, z+x, z+x+y)$ will be degenerate for each z.

In the next proposition we work in the Schrödinger representation and we define what is usually considered as the "momentum observable" of a system placed in the magnetic field B. This is the object given by the formal relation $P = D - A(Q)$, where $D = -i\nabla$ is the momentum observable in the Schrödinger representation.

Proposition 5.3 *Let $A : X \to X^*$ be a continuous map (the magnetic potential), define $\beta : X \times X \to U(1)$ by (5.4), and let $b : X \times X \to U(1)^X$ be the 2-cocycle associated to β, as defined in (5.9). For each $x \in X$ let U_x be the unitary operator in $L^2(X)$ given by*

$$U_x = \beta(Q, Q+x)\mathrm{e}^{i\langle x,D\rangle}. \tag{5.10}$$

Then for all $x, y \in X$ one has

$$U_x U_y = b(x, y)U_{x+y}. \tag{5.11}$$

Let $\langle x, P\rangle$ be the closure of the operator $\langle x, D\rangle - \langle x, A(Q)\rangle$ with $C^1_{\mathrm{c}}(X)$ as domain. Then $\langle x, P\rangle$ is a self-adjoint operator and $U_x = \mathrm{e}^{i\langle x,P\rangle}$, $\forall x \in X$.

Proof: Relation (5.11) is an immediate consequence of (5.3). We noticed above that $b(x, y) = 1$ if the vectors x, y are collinear. Hence for each x the family $\{U_{tx}\}_{t\in\mathbb{R}}$ is a strongly continuous one-parameter group of unitary operators on $L^2(X)$. Now *define* $\langle x, P\rangle$ as the self-adjoint operator such that $U_{tx} = \exp(it\langle x, P\rangle)$ for all t. Then (5.10) shows that $C^1_{\mathrm{c}}(X)$ is included in the domain of $\langle x, P\rangle$ and that $\langle x, P\rangle f = \langle x, D\rangle f - \langle x, A(Q)\rangle f$ on this set (use (5.6)). It is clear that $C^1_{\mathrm{c}}(X)$ is stable under the unitary group U_{tx} hence $\langle x, P\rangle$ is essentially self-adjoint on $C^1_{\mathrm{c}}(X)$ (by Nelson's lemma). ∎

Thus (5.10) defines a strongly continuous map of X into the set of unitary operators on $L^2(X)$ which satisfies (5.11). Now let $C(X; U(1))$ be the abelian multiplicative group of continuous functions $u : X \to U(1)$ equipped with the action of X defined by $(x.u)(y) = u(y + x)$ and the topology of uniform convergence on compact sets. Clearly b is a 2-cocycle on X with coefficients in $C(X;U(1))$, so we can construct the extension $C(X;U(1)) \times_b X$ of X. What we have done until now proves that we can define a unitary representation U of $C(X;U(1)) \times_b X$ in $L^2(X)$ by setting

$$U_{(u,x)} = u(Q)U_x = u(Q)\mathrm{e}^{i\langle x,P\rangle}. \tag{5.12}$$

If N is a locally compact subgroup of $C(X;U(1))$ and if $b(x,y) \in N$ for all x,y, we clearly get a strongly continuous unitary representation of the locally compact group extension $G = N \times_b X$ of X. The C^*-algebra generated by the operators of the form $U(\theta) = \int_G U_g \theta(g)\,\mathrm{d}g$, for $\theta \in C_\mathrm{c}(G)$ and $\mathrm{d}g$ a Haar measure on G, is the *C^*-algebra of energy observables of the "magnetic hamiltonians of type N"*. This is a representation of the *group C^*-algebra $C^*(G)$ of G*. Here the system is subject to no other interaction but the external magnetic field. According to the ideas presented in §5.1, *if $\mathscr{A}$ is a translation invariant C^*-subalgebra of $C_\mathrm{b}^\mathrm{u}(X)$, then the C^*-algebra of energy observables of a system subject to internal interactions of type $\mathscr{A}$ and placed in an external magnetic field of type N is the crossed product $\mathscr{A} \rtimes G$*. The closed linear subspace of $B(L^2(X))$ generated by the operators of the form $\varphi(Q)U(\theta)$ with $\varphi \in \mathscr{A}$ and $\theta \in C_\mathrm{c}(G)$ is a representation of this abstract algebra. Most of the algebras considered in [GI2] and in this paper are of this form.

Remark 5.4 One can proceed as in the case of constant magnetic fields and define $\psi(P)$ by formula (4.13) when $\psi : X \to \mathbb{C}$ is the Fourier transform of an integrable measure on X^*. Here U_x is as defined by (5.10) or, more generally, one can start with a topological extension $\pi : G \to X$, a continuous unitary representation U of G, and set $U_x = U_{\eta(x)}$ for a continuous section η of π. Since the magnetic hamiltonians are "functions of P" it could seem natural to take as C^*-algebra of energy observables in the purely magnetic case the C^*-algebra generated by the operators of the form $\psi(P)$ with $\psi \in L^1(X^*)$. However, this algebra is a representation of the group C^*-algebra $C^*(G)$, so the preceding point of view is, if not more natural, at least more general.

6 Appendix

6.1. We shall recall here some facts concerning the theory of group extensions. Our aim is only to present notions and terminology in a way adapted to our needs; for a detailed treatment, see Chapter 6 in [Web]. The framework is purely algebraic, so we can consider groups N which are not locally compact.

Let X be an abelian group. An *extension of* X is a surjective group morphism $G \to X$ with an abelian kernel N. One also says that G *is an extension of* X *by* N. Thus N is a normal abelian subgroup of G and one has a canonical isomorphism $G/N \cong X$. It will be convenient to denote the operation in G multiplicatively and that in X additively (this convention being opposite to that used in most of the references known to us; however, the usual convention is quite unnatural in the situation considered by us in Section 5). The neutral elements of X, G are denoted $0, 1$ respectively. Let $G^{(1)}$ be the subgroup generated by $\{ghg^{-1}h^{-1} \mid g, h \in G\}$; observe that, if a normal subgroup N of G is given, then G/N is abelian if and only if $G^{(1)} \subset N$.

Assume that $\pi : G \to X$ and $\pi' : G' \to X$ are two extensions of X by the same group N (in particular N is a subgroup of both G and G'). One says that the extensions are *equivalent* if there is a group morphism $\phi : G' \to G$ which is the identity on N and such that $\pi' = \pi \circ \phi$. It is easy to prove that ϕ is necessarily an isomorphism. Observe that if one identifies X with the quotients $G/N, G'/N$, then the condition $\pi' = \pi \circ \phi$ says that ϕ induces the identity in the quotient.

The extension $G \to X$ is called *inessential* if it admits a section which is a group morphism. This is the case if and only if G is a semidirect product of N and X. We stress that this situation is of no interest for us. If N is contained in the center of G, the extension is called *central*; such extensions are studied in depth in [Var]. This case corresponds to constant magnetic fields, so not interesting for the general situation considered in Section 5.

Some facts concerning group cohomology will help us to understand the construction of the extensions of an abelian group X by a second abelian group N. Assume that N is equipped with an X-module structure, i.e. a morphism $x \mapsto \nu_x$ of X into the group of automorphisms of N has been given. If we set $x.u \equiv \nu_x(u)$, this means $x.(y.u) = (x+y).u$, $0.u = u$, and $x.(uv) = (x.u)(x.v)$, $x.1 = 1$. Then a (normalized) *2-cocycle on* X *with coefficients in* N is a map $b : X \times X \to N$ such that

$$(6.1) \qquad b(x,y)b(x+y,z) = x.b(y,z)b(x,y+z) \text{ and } b(x,0) = b(0,y) = 1.$$

The set $Z^2(X,N)$ of such 2-cocycles is a group for usual multiplication. A *2-coboundary* is a 2-cocycle of the form $b(x,y) = \lambda(x)x.\lambda(y)\lambda(x+y)^{-1}$, where $\lambda : X \to N$ and $\lambda(0) = 1$. These 2-coboundaries form a subgroup $B^2(X,N) \subset Z^2(X,N)$ and the *second cohomology group of* X *with coefficients in* N is the quotient $H^2(X,N) = Z^2(X,N)/B^2(X,N)$.

Assume that an extension $G \to X$ of kernel N is given. Then there is a canonical X-module structure on N. Indeed, if $x \in X$ and $u \in N$ we may define $x.u = gug^{-1} \in N$, where $g \in G$ has x as image in X, because this expression is independent of the choice of g. Now let η be a section of the map $G \to X$ such that $\eta(0) = 1$ and define $b(x,y) = \eta(x)\eta(y)\eta(x+y)^{-1}$, so that $\eta(x)\eta(y) = b(x,y)\eta(x+y)$. Obviously b maps $X \times X$ into N and the associativity relation $[\eta(x)\eta(y)]\eta(z) = \eta(x)[\eta(y)\eta(z)]$ implies (6.1). Thus $b \in Z^2(X,N)$. If η' is a second section of $G \to X$ such that $\eta'(0) = 1$, then $\eta'(x) = \lambda(x)\eta(x)$ for a map $\lambda : X \to N$ with $\lambda(0) = 1$ and the corresponding cocycle b' is related to b by the formula

$$b'(x,y) = \lambda(x)\, x.\lambda(y)\, \lambda(x+y)^{-1} b(x,y). \tag{6.2}$$

This means that the cocycles b, b' are cohomologous. Hence *to each extension of X by N one can associate a X-module structure on N and an element of the cohomology group $H^2(X,N)$*. A straightforward argument shows that *if the X-module structure on N is fixed, two equivalent extensions give the same cohomology class.*

The reciprocal construction is our main interest and will be described below. It is based on the observations that the map $(u,x) \mapsto u\eta(x)$ is a bijection of $N \times X$ onto G and that the relation

$$u\eta(x)\, v\eta(y) = u\, x.v\, \eta(x)\eta(y) = u\, x.v\, b(x,y)\eta(x+y)$$

allows one to transport the product from G to $N \times X$.

So assume that the abelian group N is equipped with an X-module structure and let $b \in Z^2(X,N)$. We denote $N \times_b X$ the set $N \times X$ provided with the product

$$(u,x) \cdot (v,y) = (u\, x.v\, b(x,y), x+y). \tag{6.3}$$

From (6.1) it follows easily that this is an associative operation, that $(1,0)$ is a neutral element, and that each element (u,x) has an inverse, namely

$$(u,x)^{-1} = \big([(-x).u\, b(-x,x)]^{-1}, -x\big) \tag{6.4}$$

(note that $x.b(-x,x) = b(x,-x)$ or observe that it suffices to check that this is a left inverse). Hence $N \times_b X$ is a group, the map $u \mapsto (u,0)$ is an injective morphism of N into $N \times_b X$, and the map $(u,x) \mapsto x$ is a surjective morphism of $N \times_b X$ onto X. Identifying $u \equiv (u,0)$ we realize N as a subgroup of $N \times_b X$ which coincides with the kernel of $N \times_b X \to X$. Thus $N \times_b X$ is an extension of X by N. Moreover, the X-module structure induced on N by this extension coincides with that already given on it and the 2-cocycle associated to $\eta(x) = (0,x)$ is b.

If the cocycle b' is cohomologous to b, so that they are related as in (6.2), then $N \times_{b'} X$ is identical with $N \times_b X$ as a set but with a different product law. Define $\phi : N \times_{b'} X \to N \times_b X$ by $\phi(u, x) = (\lambda(x)u, x)$. Then ϕ is a group isomorphism which induces the identity on N and in the quotient X. In other terms, the extensions $N \times_b X$ and $N \times_{b'} X$ are equivalent.

We summarize the preceding remarks in the following proposition:

Proposition 6.1 *Let X be an additive abelian group, N a multiplicative abelian group, and let us assume that N is equipped with a X-module structure. For $b \in Z^2(X, N)$ let $N \times_b X$ be the extension of X defined above. Then the map $b \mapsto N \times_b X$ induces a bijection of $H^2(X, N)$ onto the set of equivalence classes of extensions of X by N.*

6.2. In this part of the appendix we mention a possible generalization of the algebra considered in the Introduction. The construction of the symplectic algebra is meaningful in the more general setting of abelian groups. More precisely, let Ξ be an abelian locally compact group (the topology could be the discrete one) and ω a (normalized and continuous) multiplier on it, so $\omega : \Xi \times \Xi \to \mathbb{C}$ is a continuous function such that $|\omega(\xi, \eta)| = 1$, $\omega(\xi, 0) = \omega(0, \xi) = 1$ and $\omega(\xi, \eta)\omega(\xi + \eta, \zeta) = \omega(\xi, \eta + \zeta)\omega(\eta, \zeta)$ for all $\xi, \eta, \zeta \in \Xi$. A projective representation of multiplier (or phase factor) ω of Ξ on the Hilbert space $\mathscr{H}$ is a strongly continuous map $W : \Xi \to U(\mathscr{H})$ such that

$$W(\xi)W(\eta) = \omega(\xi, \eta)W(\xi + \eta). \tag{6.5}$$

This implies[5] $W(0) = 1$ and $W(\xi)^* = W(\xi)^{-1} = \omega(\xi, -\xi)^{-1}W(-\xi)$. If we define the *bicharacter* associated to ω by $\beta(\xi, \eta) = \omega(\xi, \eta)/\omega(\eta, \xi)$ then $W(\xi)W(\eta) = \beta(\xi, \eta)W(\eta)W(\xi)$. Note that β (hence χ, cf. below) depends only on the cohomology class of ω.

A short computation shows that the map χ defined by $\xi \mapsto \beta(\xi, \cdot)$ is a continuous group homomorphism of Ξ into Ξ^*. If E is a subgroup of Ξ we denote E^β the set of $\xi \in \Xi$ such that $\beta(\xi, \eta) = 1$ for all $\eta \in E$ (remark that $\overline{\beta(\xi, \eta)} = \beta(\eta, \xi)$). Clearly $E \subset F \Rightarrow F^\beta \subset E^\beta$. Moreover, $E \subset E^{\beta\beta}$ but the equality does not hold in general (even if E is closed). However, if χ is a homeomorphism[6] then we do have $E = E^{\beta\beta}$ for all closed subgroups E of

[5] Reciprocally, if $W : \Xi \to U(\mathscr{H})$ is strongly continuous, fulfills (6.5) for some function $\omega : \Xi \times \Xi \to \mathbb{C}$, and $W(0) = 1$, then ω is a multiplier in the sense defined above.

[6] This is a version of the non-degeneracy condition from the symplectic case. A more general and useful one (e.g. in the context of CCR or CAR for systems with an infinite number of degrees of freedom) is the requirement of injectivity on χ.

Ξ. Moreover, under the same condition we get $(E+F)^\beta = E^\beta \cap F^\beta$ for all subgroups $E, F \subset \Xi$. Note that if E, F are closed then $E+F$ is not closed in general (for example, consider $\mathbb{Z}$ and $\pi\mathbb{Z}$ as subgroups of $\mathbb{R}$ and similarly in the higher dimensional case, where the cohomology is not trivial). But the relation $(E \bigcap F)^\beta = \overline{E^\beta + F^\beta}$ will hold for all closed subgroups E, F.

The framework of Section 3.6 from [GI2] allows one to construct systems (Ξ, ω) with all the properties required above (including non-degeneracy). Let $\Xi = X \oplus X^*$ be the phase space of a locally compact abelian group X. Then a standard choice for ω is $\omega(\xi, \eta) = k(y)^{-1}$ if $\xi = (x, k), \eta = (y, p)$, hence $\beta(\xi, \eta) = k(y)/p(x)$. One can then take $\mathscr{H} = L^2(X)$ (for some Haar measure on X) and $W(\xi) = U_{-x}V_k$. A second choice (which gives the same bicharacter) is $\omega(\xi, \eta) = p(x)$; then one may take $W(\xi) = V_k U_{-x}$. In the case when X is a vector space a more symmetric choice is described in Section 3.6 from [GI2]. Note that the case when X (hence Ξ) is a *finite group* is interesting: it corresponds to quantum systems with finite dimensional state spaces which are useful as discrete approximations of particle systems.

Now for each closed subgroup E of Ξ one can define a C^*-algebra $\mathfrak{C}(E)$ of operators on $\mathscr{H}$ exactly as in Definition 1.1 (E^σ being replaced by E^β) and then one can consider the C^*-algebra $\mathfrak{C}^\Xi$ obtained as closure of the sum $\sum \mathfrak{C}(E)$ over *all closed subgroups* of Ξ. Lemma 2.2 extends easily to this context. The family of all closed subgroups of Ξ is a lattice for the order relation given by inclusion. Unfortunately, the algebra $\mathfrak{C}^\Xi$ is not graded by this lattice in general: Theorem 2.1 is not true without further restrictions on the lattice or on (Ξ, ω). An interesting situation in which this breaks down and the computations are easy is the following: let $\Xi = \mathbb{R} \oplus \mathbb{R}^*$ with the canonical structure of phase space of $X = \mathbb{R}$ and consider the subgroups $E = \mathbb{Z} \oplus \{0\}, F = \pi\mathbb{Z} \oplus \{0\}$, so that $E^\beta = \mathbb{R} \oplus 2\pi\mathbb{Z}, F^\beta = \mathbb{R} \oplus 2\mathbb{Z}$.

One can reformulate the preceding definitions in terms of the "Heisenberg group" Ξ_ω (see §5.1) associated to the couple (Ξ, ω); then the result extends naturally to general groups. Note that there is a bijective correspondence between irreducible projective representations of multiplier ω^k of Ξ for some $k \in \mathbb{Z}$ and irreducible unitary representations of Ξ_ω (because these send the center $U(1)$ into scalar multiples of the identity operator). It is clear the algebras $\mathfrak{C}(E)$ belong to the following general class.

Let G be a locally compact group with neutral element e and let U be a strongly continuous unitary representation of G in a Hilbert space $\mathscr{H}$. If H is a closed subgroup of G let H' be the set of $g \in G$ such that $gh = hg$ for all $h \in H$. Now let us denote by $\mathfrak{C}(H)$ the set of operators $T \in B(\mathscr{H})$ such that:

(i) $\| [U(g), T] \| \to 0$ if $g \to e$ in G,
(ii) $[W(h), T] = 0$ if $h \in H$,
(iii) $\| (W(g) - 1)T^{(*)} \| \to 0$ if $g \in H'$ and $g \to e$.
We thus get a family of C^*-algebras $\mathfrak{C}(H)$ on $\mathscr{H}$ indexed by the lattice of all closed subgroups of G. It would be interesting to see when the C^*-algebra generated by $\bigcup_H \mathfrak{C}(H)$ is graded by this lattice.

References

[ABG] W. Amrein, A. Boutet de Monvel and V. Georgescu, *C_0-groups, commutator methods and spectral theory of N-body hamiltonians*, Birkhauser Verlag, 1996.

[BG] A. Boutet de Monvel-Berthier, and V. Georgescu, *Graded C^*-algebras associated to symplectic spaces and spectral analysis of many channel Hamiltonians*, Dynamics of complex and irregular systems (Bielefeld, 1991), 22–66, Bielefeld Encount. Math. Phys., VIII, World Sci. Publishing, 1993.

[DG1] M. Damak and V. Georgescu, *C^*-Crossed Products and a Generalized Quantum Mechanical N-Body Problem*, in the Proceedings of the symposium "Mathematical Physics and Quantum Field Theory", Electronic J. of Diff. Equations, Conference 04 (2000), 51–69 (an improved version is available as preprint 99-481 at `http://www.ma.utexas.edu/mp_arc/`).

[DG2] M. Damak and V. Georgescu, *C*-algebras related to the N-body problem and the self-adjoint operators affiliated to them*, available as preprint 99-482 at `http://www.ma.utexas.edu/mp_arc/`.

[Fol] G. B. Folland, *Harmonic analysis in phase space*, Princeton University Press, 1989.

[GI1] V. Georgescu, A. Iftimovici *The Riesz-Kolmogorov and Ruelle Criteria on Locally Compact Abelian Groups*, available as preprint 00-520 at `http://www.ma.utexas.edu/mp_arc/`.

[GI2] V. Georgescu, A. Iftimovici *C^*-Algebras of Energy Observables: I. General theory and bumps algebras*, available as preprint 00-521 at `http://www.ma.utexas.edu/mp_arc/`.

[GL1] C. Gérard and I. Łaba, *Scattering theory for N-particle systems in constant magnetic fields*, Duke Math. J. **76** (1994), no. 2, 433–465.

[GL2] C. Gérard and I. Łaba, *Scattering theory for 3-particle systems in constant magnetic fields: the dispersive case*, Ann Inst. Fourier (Grenoble) **46** (1996), no. 3, 801–876.

[GL3] C. Gérard and I. Łaba, *Magnetic Hamiltonians*, January 17, 2001, manuscript of a book to appear.

[HM] B. Helffer, A. Mohamed, *Caractérisation du spectre essentiel de l'opérateur de Schrödinger avec un champ magnétique*, Ann. Inst. Fourier, Grenoble, **38** (1988), 95–112.

[Hor] L. Hörmander, *The analysis of linear partial differential operators 1*, Springer-Verlag, 1990.

[Jo1] P. E. T. Jorgensen, *Representations of differential operators on a Lie group*, J. Func. Analysis **20** (1975), 105–135.

[Jo2] P. E. T. Jorgensen, *Operators and representation theory*, Mathematics Studies **147**, North-Holland, 1988.

[JoK] P. E. T. Jorgensen, W. H. Klink, *Quantum mechanics and nilpotent groups I. The curved magnetic field*, Publ. RIMS, Kyoto Univ. **21** (1985), 969–999.

[Ped] G. K. Pedersen, *C^*-algebras and their automorphism groups*, Academic Press, 1979.

[Rae] I. Raeburn, *On crossed products and Takai duality*, Proc. Edinburgh Math. Soc. **31** (1988), 321–330.

[RW] I. Raeburn and D. P. Williams, *Morita equivalence and continuous-trace C^*-algebras*, American Mathematical Society, 1998.

[Sch] L. Schwartz, *Théorie des distributions*, Paris, Hermann, 1998.

[Var] V. S. Varadarajan, *Geometry of quantum theory*, Volume II, D. Van Nostrand Company Inc., 1970.

[Web] C. A. Weibel, *An introduction to homological algebra*, Cambridge Stud. in Adv. Math. **38**, Cambridge University Press, 1994.

[Wei] A. Weil, *Sur certains groupes d'opérateurs unitaires*, Acta Math. **111** (1964), 143-211.