Self-adjointness of the Pauli-Fierz Hamiltonian for arbitrary coupling constants

Fumio Hiroshima *†

March 8, 2001

Abstract

The Pauli-Fierz Hamiltonian describes a system of N electrons interacting with a quantized radiation field. The electrons have spin and an ultraviolet cutoff is imposed on the quantized radiation field. For *arbitrary* coupling constants, self-adjointness and essential self-adjointness of the Pauli-Fierz Hamiltonian are proven under a class of ultraviolet cutoffs.

1 Introduction

The purpose of this paper is to establish the self-adjointness and the essential self-adjointness of the Pauli-Fierz Hamiltonian [24] for *arbitrary* coupling constants. The Pauli-Fierz Hamiltonian governs a system of N electrons interacting with a quantized radiation field. The N electrons are assumed to have spin and the quantized radiation field is smeared by an ultraviolet cutoff.

The dynamics of the system is determined by the one-parameter unitary time-evolution generated by the Pauli-Fierz Hamiltonian. So, as a first step, it is necessary to establish the self-adjointness of the Pauli-Fierz Hamiltonian. Generally, the densely defined Hamiltonian has several distinct self-adjoint extensions. Then we need to choose a particular self-adjoint extension of them.

The spectral analysis of the Pauli-Fierz type models (including spin-boson models and the Nelson model [22]) has been obtained in [2, 7, 3, 4, 5, 9, 10, 11, 12, 15, 16,

*Zentrum der Mathematik, Technische Universität München, Gabelsberger straße 49, E-mail: hiro@mathematik.tu-muenchen.de

†Permanent address: Department of Mathematics and Physics, Setsunan University, Ikeda-naka-machi 17-8, 572-8508, Osaka, Japan

17, 19, 20, 23, 27, 28]. The ground state is defined as the vector associated with the bottom of the spectrum of the operator. The existence of the ground state of the Pauli-Fierz type Hamiltonian was proven for weak couplings in [2, 3, 5, 15, 16]. For arbitrary coupling constants, it has been shown by Spohn [28] for the Nelson model, and by Gérard [11] for rather general models. For the Pauli-Fierz model, Griesemer-Lieb-Loss [12] have proven it for arbitrary coupling constants. Moreover the existence of resonances has been established by Bach-Fröhlich-Sigal [3, 4] and Jaksic-Pillet [19] under certain conditions. Recently non-existence of the ground state for the 3D Nelson model with infrared divergence has been shown by Lőrinczi-Minlos-Spohn [21].

We can regard the Nelson Hamiltonian as the sum of a dominating part and an infinitesimally small part for arbitrary coupling constants. Thus the Nelson Hamiltonian is self-adjoint on the dominating part domain by the Kato-Rellich theorem. Moreover Arai [1] proved that the Pauli-Fierz Hamiltonian in the dipole approximation is self-adjoint for arbitrary coupling constants in terms of the Nelson commutator theorem [25, Theorem X.36].

In contrast, the self-adjointness of the full Pauli-Fierz Hamiltonian is not obvious for all values of the coupling constants. For sufficiently small coupling constants, in the same way as the Nelson Hamiltonian, its self-adjointness is established in [3, 5, 6, 23]. More difficulty arises for large coupling constants. Since the Pauli-Fierz Hamiltonian contains a quadratic part, the self-adjoint extension can be defined by the Friedrichs extension. Thus, for arbitrary coupling constants, one can consider the spectral properties of the Friedrichs extension. The Friedrichs extension, however, does not give information of the domain. Moreover there still remains the possibility that several distinct self-adjoint extensions exit for large coupling constants. In particular, distinct self-adjoint operators have distinct spectral properties. We want to resolve such ambiguities in this paper.

In the previous paper [18] we have shown essential self-adjointness of the Pauli-Fierz Hamiltonian for arbitrary coupling constants but with $N = 1$ and too strong integrable conditions on the ultraviolet cutoff. In this paper we show self-adjointness for all $N \geq 1$ and with weaker conditions on the ultraviolet cutoff.

Suppose that the N-electrons move in the d-dimensional space. Let $L^2 := L^2(\mathbb{R}^d)$.

The Hilbert space of state vectors is defined by

$$\mathcal{H} := \left[\mathbb{C}^{2^{[d/2]}} \otimes L^2\right]^N \otimes \mathcal{F} \cong (\otimes^N \mathbb{C}^{2^{[d/2]}}) \otimes L^2(\mathbb{R}^{dN}) \otimes \mathcal{F},$$

where $[k]$ denotes the integer part of k. Here $\mathcal{F}$ is the symmetric Fock space over $\mathbb{C}^{d-1} \otimes L^2 \cong \oplus^{d-1} L^2$, i.e.,

$$\mathcal{F} := \oplus_{n=0}^{\infty} \otimes_s^n \left[\mathbb{C}^{d-1} \otimes L^2\right],$$

$\otimes_s^n$ denoting the symmetric tensor product with $\otimes_s^0[\mathbb{C}^{d-1} \otimes L^2] := \mathbb{C}$. The vacuum vector in $\mathcal{F}$ is defined by $\Omega := \{1, 0, 0, 0, ...\}$. The smeared annihilation and creation operators are denoted by $a^r(f)$ and $a^{\dagger r}(f)$, $f \in L^2$, $r = 1, ..., d-1$, respectively. They are linear in f and satisfy the canonical commutation relations:

$$[a^s(f), a^{\dagger r}(g)] = (\bar{f}, g)_{L^2} \delta_{rs}, \quad [a^r(f), a^s(g)] = 0, \quad [a^{\dagger r}(f), a^{\dagger s}(g)] = 0$$

on the finite particle subspace

$$\mathcal{F}_{\rm fin} := \text{L.H.} \left\{ a^{\dagger r_1}(f_1) \cdots a^{\dagger r_n}(f_n)\Omega, \Omega | f_j \in L^2, r_j = 1, ..., d, j = 1, ..., n \right\}.$$

Here $(f, g)_{\mathcal{K}}$ denotes the scalar product on $\mathcal{K}$ and L.H.$\{\cdots\}$ is the linear hull of the vectors in $\{\cdots\}$. We have that $a^{\dagger r}(f)^* = a^r(\bar{f})$ and formally write $a^r(f) = \int a^r(k) f(k) dk$ and $a^{\dagger r}(f) = \int a^{\dagger r}(k) f(k) dk$. We write the norm on $\mathcal{K}$ as $\|\cdot\|_{\mathcal{K}}$. Unless confusion may arise we write simply (f, g) and $\|f\|$ for $(f, g)_{\mathcal{K}}$ and $\|f\|_{\mathcal{K}}$, respectively. Note that in our notation (f, g) is linear in g and anti-linear in f.

The Pauli-Fierz Hamiltonian with N electrons is given by

$$H_{\rm PF} := \mathbf{1} \otimes \left\{ \frac{1}{2} \sum_{j=1}^N \left(p^j \otimes \mathbf{1} - eA(x^j)\right)^2 + V \otimes \mathbf{1} + \mathbf{1} \otimes H_{\rm f} \right\} + gH_{\rm spin}, \tag{1.1}$$

where we adopt the unit:$c = \hbar = 1$. e denotes the electron charge (the coupling constant), g the gyromagnetic ratio, and p^j and x^j are the-jth electron momentum and position operators in L^2, respectively. Under the identification, $L^2(\mathbb{R}^{dN}) \otimes \mathcal{F} \cong L^2(\mathbb{R}^{dN}; \mathcal{F})$, i.e., the set of $\mathcal{F}$-valued L^2-functions on $\mathbb{R}^{dN}$, we define

$$A_\mu(x) := \frac{1}{\sqrt{2}} \int \left\{ a^{\dagger r}(k) e^{-ikx} e_\mu^r(k) \hat{\lambda}(-k) + a^r(k) e^{ikx} e_\mu^r(k) \hat{\lambda}(k) \right\} dk.$$

The spin term is given by

$$H_{\rm spin} := \frac{-e}{2}\sum_{1\le\mu<\nu\le d}\sigma_\mu^j\sigma_\nu^j\otimes B_{\mu\nu}(x^j),$$

where

$$B_{\mu\nu}(x^j) := i(\partial_\mu^j A_\nu(x^j)-\partial_\nu^j A_\mu(x^j))$$
$$=\frac{1}{\sqrt{2}}\int\left\{(k_\mu e_\nu^r(k)-k_\nu e_\mu^r(k))e^{-ikx^j}\hat\lambda(-k)-(k_\mu e_\nu^r(k)-k_\nu e_\mu^r(k))e^{ikx^j}\hat\lambda(k)\right\}dk.$$

The free Hamiltonian in $\mathcal{F}$ is defined by

$$H_{\rm rad} := \int\omega(k)a^{\dagger r}(k)a^r(k)dk,\quad \omega(k):=|k|.$$

The summation over repeated indices is automatically understood unless otherwise stated. $e^r=(e_1^r,...,e_d^r)$ denotes polarization vectors: $e^r(k)\cdot e^s(k)=\delta_{rs}$ and $e^r(k)\cdot k=0$, a.e. $k\in\mathbb{R}^d$, and

$$\sigma_\mu^j := \underbrace{I\otimes\cdots\otimes\overset{j_{th}}{\sigma_\mu}\otimes\cdots\otimes I}_{N},\quad \mu=1,...,d,\quad j=1,...,N,$$

where σ_μ appears as the j_{th} factor and $\sigma_1,...,\sigma_d$ denote $2^{[d/2]}\times 2^{[d/2]}$ matrices satisfying the anti-commutation relations:

$$\{\sigma_\mu,\sigma_\nu\}=2\delta_{\mu\nu}\mathbf{1},\quad \mu,\nu=1,...,d.$$

$\hat\lambda\in L^2$ is an ultraviolet cutoff function and its physical choice is

$$\hat\lambda(k)=\frac{\hat\rho(k)}{\sqrt{(2\pi)^d\omega(k)}},$$

where $e\rho$ denotes the electron charge density; $\int_{\mathbb{R}^d}\rho(x)dx=1$, and $\hat f$ is the Fourier transform of f. We assume that

$$\hat\lambda(-k)=\overline{\hat\lambda(k)}.$$

This assumption ensures that $H_{\rm PF}$ is a symmetric operator in $\mathcal{H}$. Finally we assume that there exists sufficiently small $a>0$, and $b\ge 0$ such that

$$\|Vf\|\le a\|\Delta f\|+b\|f\|,\quad f\in D(\Delta). \tag{1.2}$$

Here $D(T)$ denotes the domain of T.

Typical example is $d=3$ with V the Coulomb potential of N electrons and M static nuclei:

$$V_{\mathrm{C}}(x_1,...,x_N,y_1,...,y_M):=\sum_{1\le i<j\le N}\frac{e^2}{|x^i-x^j|}-\sum_{j=1}^N\sum_{i=1}^M\frac{|e|Z_j}{|x^j-y^i|},$$

where $y^1,...,y^M$ are constants in $\mathbb{R}^d$, $Z_1,...,Z_M$ positive constants, and $(\sigma_1,\sigma_2,\sigma_3)$ are the 2×2 Pauli matrices:

$$\sigma_1:=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_2:=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \sigma_3:=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

We abbreviate $\mathbf{1}\otimes\mathbf{1}\otimes X$, $\mathbf{1}\otimes X\otimes\mathbf{1}$, $X\otimes\mathbf{1}\otimes\mathbf{1}$ by X unless confusions may arise. Thus we write (1.1) as

$$H_{\mathrm{PF}}=\frac{1}{2}\sum_{j=1}^N\left(p^j-eA(x^j)\right)^2+V+H_{\mathrm{f}}+gH_{\mathrm{spin}}.$$

The Hamiltonian is of the form

$$H_{\mathrm{PF}}=-\Delta/2+H_{\mathrm{rad}}-eH_{\mathrm{int}},$$

where

$$H_{\mathrm{int}}:=\frac{1}{2}\left\{p^jA(x^j)+A(x^j)p^j-eA(x^j)^2\right\}+gH_{\mathrm{spin}}.$$

From the inequalities:

$$\|a^{\dagger r}(f)\Psi\|\le\|f/\sqrt{\omega}\|\|H_{\mathrm{rad}}^{1/2}\Psi\|+\|f\|\|\Psi\|, \tag{1.3}$$

$$\|a^r(f)\Psi\|\le\|f/\sqrt{\omega}\|\|H_{\mathrm{rad}}^{1/2}\Psi\|, \tag{1.4}$$

$$\|a^{\sharp r}(f)a^{\sharp s}(g)\Psi\|\le(\|f/\sqrt{\omega}\|+\|f\|)(\|g/\sqrt{\omega}\|+\|g\|+\|\sqrt{\omega}g\|+\|\omega g\|)(\|H_{\mathrm{rad}}\Psi\|+\|\Psi\|), \tag{1.5}$$

it follows that,

$$\|H_{\mathrm{int}}\Psi\|\le a\|(-\Delta/2+H_{\mathrm{rad}})\Psi\|+b\|\Psi\|$$

with some positive a,b for $\Psi\in D(-\Delta/2+H_{\mathrm{rad}})=D(\Delta)\cap D(H_{\mathrm{rad}})$. Then for sufficiently small $|e|$, H_{PF} is self-adjoint on $D(\Delta)\cap D(H_{\mathrm{rad}})$ by the Kato-Rellich theorem.

In [18] we have shown that for arbitrary coupling constants the Hamiltonian is essentially self-adjoint on $D(\Delta)\cap D(H_{\rm rad})$ in case when

$$\hat\lambda/\sqrt{\omega},\quad \hat\lambda,\quad \omega\hat\lambda,\quad \omega^2\hat\lambda\in L^2,\quad \text{and}\quad N=1.$$

In this paper, we prove that under the weaker condition

$$\hat\lambda/\sqrt{\omega},\quad \hat\lambda,\quad \omega\hat\lambda,\quad \text{and}\quad N\geq 1,$$

the Hamiltonian is self-adjoint on $D(\Delta)\cap D(H_{\rm rad})$ for arbitrary coupling constants. Moreover we remark that the self-adjointness leads to the following relative bound:

$$\|\Delta\Psi\|+\|H_{\rm rad}\Psi\|\leq C(\|H\Psi\|+\|\Psi\|),\quad \Psi\in D(\Delta)\cap D(H_{\rm rad}), \tag{1.6}$$

with some constant C. This relative bound is obvious for small enough coupling constants.

The outline of our argument is from Step 1 to Step 4.

Step 1: Let

$$H:=\frac12\sum_{j=1}^N\left(p^j-eA(x^j)\right)^2+H_{\rm rad}.$$

For sufficiently small coupling constants, the matrix elements of the heat semigroup e^{-tH} can be expressed by a functional integral. We see that for arbitrary coupling constants the functional integral itself is well defined and yields a heat semigroup generated by a self-adjoint operator $\widehat H$.

Step 2: we prove that

$$\widehat H\supset H\lceil_{D(\Delta)\cap D(H_{\rm rad})}.$$

This statements were checked in [14] for only the case when $\hat\lambda/\sqrt{\omega},\hat\lambda,\omega\hat\lambda,\omega^2\hat\lambda\in L^2$. However, $\omega^2\hat\lambda\in L^2$ is a strong assumption, since H is well defined on $D(\Delta)\cap D(H_{\rm rad})$ even for the case when $\hat\lambda/\sqrt{\omega},\hat\lambda,\omega\hat\lambda\in L^2$. In this paper we remove this assumption.

Step 3: we show that $D(\Delta)\cap D(H_{\rm rad})$ is invariant under $e^{-t\widehat H}$, i.e.,

$$e^{-t\widehat H}\left\{D(\Delta)\cap D(H_{\rm rad})\right\}\subset D(\Delta)\cap D(H_{\rm rad}),$$

which implies that $\widehat H$ is essentially self-adjoint on $D(\Delta)\cap D(H_{\rm rad})$ by [25, Theorem X.49].

Step 4: We prove that

$$(-\Delta + H_{\rm rad})(\widehat{H}+E)^{-1}$$

is bounded for every $E>0$. Thus we conclude that $\widehat{H}$ is closed on $D(\Delta)\cap D(H_{\rm rad})$. Hence $\widehat{H}$ is self-adjoint on $D(\Delta)\cap D(H_{\rm rad})$ and bounded from below, and H also is self-adjoint on $D(\Delta)\cap D(H_{\rm rad})$ and bounded from below. Since for arbitrary $\epsilon>0$ there exists $b_\epsilon>0$ such that

$$\|H_{\rm spin}F\| \le \epsilon\|(-\Delta/2 + H_{\rm rad})F\| + b_\epsilon\|F\|, \quad F\in D(\Delta)\cap D(H_{\rm rad}).$$

This, together with (1.6), imply that $H_{\rm PF} = H + V + gH_{\rm spin}$ is self-adjoint on $D(\Delta)\cap D(H_{\rm rad})$.

Next we consider a more subtle case. By (1.3), (1.4), (1.5), for

$$\hat\lambda/\sqrt{\omega}\notin L^2,$$

$H_{\rm PF}$ is a priori *not* defined on $D(\Delta)\cap D(H_{\rm rad})$. Let N be the number operator

$$N := \int a^{\dagger r}(k)a^r(k)dk.$$

For the case when $\hat\lambda, \omega\hat\lambda \in L^2$ but $\hat\lambda/\sqrt{\omega}\notin L^2$, $H_{\rm PF}$ is well defined on $D(\Delta)\cap D(H_{\rm rad})\cap D(N)$ (see Section 2 for details). Under this assumption, we show that $H_{\rm PF}$ is essentially self-adjoint on $C^\infty(\Delta)\cap D(H_{\rm rad})\cap C^\infty(N)$, where $C^\infty(T) := \cap_{n=1}^\infty D(T^n)$.

Finally also it is of mathematical interest to consider self-adjointness or essential self-adjointness of

$$\frac{1}{2}\sum_{j=1}^N \left(p^j - eA(x^j)\right)^2 + V. \tag{1.7}$$

We show that, for the case $\hat\lambda, \omega\hat\lambda\in L^2$, (1.7) is essentially self adjoint on $C^\infty(\Delta)\cap C^\infty(N)$.

The paper is organized as follows: Section 2 is devoted to presenting the relationship between integrable conditions for $\hat\lambda$ and the domain of $H_{\rm PF}$. In Section 3 we state the main theorems. In Section 4 we construct a self-adjoint extension in terms of a functional integral. In Section 5 we show some relative bounds by using the functional integral representation. Finally, in Section 6 we prove the main theorems.

2 Ultraviolet cutoffs and domains

The reader might wonder why we pay attention to the conditions on $\hat{\lambda}$. Now we mention the relationship between $D(H_{\rm PF})$ and $\hat{\lambda}$. Let

$$H_{\rm A} := \frac{1}{2}\sum_{j=1}^{N}\left(p^j - eA(x^j)\right)^2.$$

Let $\hat{\lambda}, k_\mu\hat{\lambda} \in L^2$, $\mu = 1, ..., d$. Then, for $\Psi \in C_0^\infty(\mathbb{R}^{dN})\hat{\otimes}\mathcal{F}_{\rm fin}$ ($\hat{\otimes}$ denotes an algebraic tensor product), it holds that

$$\Psi \in D(pA(x)) \cap D(A(x)p).$$

Taking the Coulomb gauge, we have

$$[p_\mu^j, A_\mu(x^j)]\Psi = \sum_{j=1}^{N} {\rm div} A(x^j)\Psi = 0.$$

Hence, in particular, if $\hat{\lambda}, \omega\hat{\lambda} \in L^2$, then

$$H_{\rm A} = \widetilde{H}_{\rm A} := \frac{1}{2}p^2 - eA(x^j)p^j + \frac{e^2}{2}A^2(x^j)$$

and $\widetilde{H}_{\rm A}$ is symmetric on $\mathcal{H}_0$. Clearly both of $H_{\rm A}$ and $\widetilde{H}_{\rm A}$ are closable. Their closed extensions are denoted by the same symbols $H_{\rm A}$ and $\widetilde{H}_{\rm A}$, respectively. From inequalities:

$$\|a^{\dagger r}(f)\Psi\| \le \|f\|(\|N^{1/2}\Psi\| + \|\Psi\|),$$

$$\|a^r(f)\Psi\| \le \|f\|\|N^{1/2}\Psi\|,$$

$$\|a^{\sharp r}(f)a^{\sharp s}(g)\Psi\| \le 8\|f\|\|g\|(\|N\Psi\| + \|\Psi\|), \quad a^{\sharp r} = a^r \text{ or } a^{\dagger r},$$

it follows that, if $\hat{\lambda}, \omega\hat{\lambda} \in L^2$, then

$$\|\widetilde{H}_{\rm A}\Psi\| \le C\|\hat{\lambda}\|^2(\|\Delta\Psi\| + \|N\Psi\| + \|\Psi\|)$$

with some constant C. Hence $H_{\rm A}$ is well defined on $D(\Delta) \cap D(N)$, While, from (1.3), (1.4), and (1.5), we see that, if $\hat{\lambda}/\sqrt{\omega}, \hat{\lambda}, \omega\hat{\lambda} \in L^2$, then

$$\|\widetilde{H}_{\rm A}\Psi\| \le C'(\|\Delta\Psi\| + \|H_{\rm f}\Psi\| + \|\Psi\|),$$

where C' is a constant depending only on $\|\hat{\lambda}/\sqrt{\omega}\|$, $\|\hat{\lambda}\|$,and $\|\omega\hat{\lambda}\|$. Hence $H_{\rm A}$ is well defined on $D(\Delta) \cap D(H_{\rm rad})$. Thus in order to define $H_{\rm PF}$ we should take care of whether $\hat{\lambda}/\sqrt{\omega} \in L^2$ or not. So we are interested in considering the both cases:

(I) $\hat{\lambda}, \omega\hat{\lambda} \in L^2$,

(II) $\hat{\lambda}/\sqrt{\omega}, \hat{\lambda}, \omega\hat{\lambda} \in L^2$.

In case (I), $H_{\rm PF}$ is well defined on $D(\Delta) \cap D(H_{\rm rad}) \cap D(N)$, and, in case (II), on $D(\Delta) \cap D(H_{\rm rad})$.

3 Main results

Theorem 3.1 *Suppose that $\hat{\lambda}/\sqrt{\omega}, \hat{\lambda}, \omega\hat{\lambda} \in L^2$. Then $H_{\rm PF}$ is self-adjoint on $D(\Delta)\cap D(H_{\rm rad})$ and bounded from below. Moreover $H_{\rm PF}$ is essentially self-adjoint on any core of $-\Delta/2 + H_{\rm rad}$.*

Corollary 3.2 *Let $\hat{\lambda}/\sqrt{\omega}, \hat{\lambda}, \omega\hat{\lambda} \in L^2$. Then there exists a constant C such that*

$$\|\Delta\Psi\| + \|H_{\rm rad}\Psi\| \leq C(\|H_{\rm PF}\Psi\| + \|\Psi\|), \quad \Psi \in D(\Delta) \cap D(H_{\rm rad}).$$

Proof: From Theorem 3.1 and the closed graph theorem,

$$\|(-\Delta/2 + H_{\rm rad})\Psi\| \leq C'(\|H_{\rm PF}\Psi\| + \|\Psi\|), \quad \Psi \in D(\Delta) \cap D(H_{\rm rad}),$$

follows with some constant C'. Thus the corollary holds true. □

Theorem 3.3 *Suppose that $\hat{\lambda}, \omega\hat{\lambda} \in L^2$. Then $H_{\rm PF}$ is essentially self-adjoint on $C^\infty(\Delta) \cap D(H_{\rm rad}) \cap C^\infty(N)$ and bounded from below.*

Theorem 3.4 *Let $\hat{\lambda}, \omega\hat{\lambda} \in L^2$. Then $H_{\rm A} + V$ is essentially self-adjoint on $C^\infty(\Delta)\cap C^\infty(N)$ and bounded from below.*

4 A self-adjoint extension of $H_{\rm PF}$

4.1 A short review of a probabilistic description

For the time being, we study

$$H = \frac{1}{2}\sum_{j=1}^{N}\left(p^j - eA(x^j)\right)^2 + H_{\rm rad}$$

acting in $\mathcal{H} = L^2(\mathbb{R}^{dN}) \otimes \mathcal{F}$. It is useful to take a Schrödinger representation of $\mathcal{F}$ to investigate the semigroup generated by H. Let

$$W := \oplus^d L^2$$

and

$$W_0 := \oplus^d L^2(\mathbb{R}^{d+1}).$$

q and q_0 denote bilinear forms defined by

$$q(f,g) := \frac{1}{2}\int_{\mathbb{R}^d} d_{\mu\nu}(k)\hat{f}_\mu(k)\hat{g}_\nu(k)dk, \quad f,g \in W,$$

$$q_0(f,g) := \int_{\mathbb{R}^{d+1}} d_{\mu\nu}(k)\hat{f}_\mu(k,k_0)\hat{g}_\nu(k,k_0)dkdk_0, \quad f,g \in W_0,$$

where $d_{\mu\nu}(k) := e^r_\mu(k)e^r_\nu(k) = \delta_{\mu\nu} - k_\mu k_\nu/|k|^2$, $k \in \mathbb{R}^d$, is the transverse projection. Let (Q,μ) and (Q_0,μ_0) be probability measure spaces associated with the mean-zero Gaussian random variables $(\phi(f), f \in W)$, and $(\phi_0(g), g \in W_0)$, respectively. The covariances are given by

$$\int_Q \phi(f)\phi(g)\mu(d\phi) = \frac{1}{2}q(f,g), \quad f,g \in W,$$

$$\int_{Q_0} \phi_0(f)\phi_0(g)\mu(d\phi_0) = \frac{1}{2}q_0(f,g), \quad f,g \in W_0.$$

The finite particle subspace $L^2_{\mathrm{fin}}(Q)$ is defined by

$$L^2_{\mathrm{fin}}(Q) := \mathrm{L.H.}\{:\!\phi(f_1)\cdots\phi(f_n)\!:, \mathbf{1} | f_j \in W, j = 1,...,n, n \geq 1\},$$

where $:\!X\!:$ denotes the Wick product of X. Define H_{f} and N_{f} by

$$H_{\mathrm{f}} :\!\phi(f_1)\cdots\phi(f_n)\!: := \sum_{j=1}^n :\!\phi(f_1)\cdots\phi(\hat{\omega}f_j)\cdots\phi(f_n)\!:,$$

$$H_{\mathrm{f}}\mathbf{1} := 0,$$

$$N_{\mathrm{f}} :\!\phi(f_1)\cdots\phi(f_n)\!: := n :\!\phi(f_1)\cdots\phi(f_n)\!:,$$

$$N_{\mathrm{f}}\mathbf{1} := 0,$$

where $\widehat{\omega} : W \to W$ is a self-adjoint operator defined by

$$\widehat{\omega} := \oplus^d \omega(-i\nabla).$$

Let

$$\mathbf{A}_\mu(\lambda) := \phi(\oplus_{\nu=1}^d \delta_{\mu\nu}\lambda), \quad \mu = 1, ..., d.$$

We define a symmetric operator $H_{\rm S}$ on

$$\mathcal{H}_{\rm S} := L^2(\mathbb{R}^{dN}) \otimes L^2(Q) \cong L^2(\mathbb{R}^{dN}; L^2(Q))$$

by

$$H_{\rm S} := H_{\rm SA} + H_{\rm f},$$

where

$$H_{\rm SA} := \frac{1}{2}\sum_{j=1}^N (p^j - e\mathbf{A}(\lambda(\cdot - x^j)))^2.$$

We define a linear operator T of $\mathcal{F}$ to $L^2(Q)$ by

- $T :A_{\mu_1}(\hat{\lambda}_1)\cdots A_{\mu_n}(\hat{\lambda}_n): \Omega :=:\mathbf{A}_{\mu_1}(\lambda_1)\cdots\mathbf{A}_{\mu_n}(\lambda_n):$,
- $T\Omega := \mathbf{1}$.

It is seen that T can be extended to a unitary operator of $\mathcal{F}$ to $L^2(Q)$. We write the extension as the same symbol T. In particular

$$T : D(N^l) \longrightarrow D(N_{\rm f}^l),$$

$$T : D(H_{\rm rad}^l) \longrightarrow D(H_{\rm f}^l),$$

for arbitrary $l \geq 0$. Moreover T implements equivalences: $A_\mu(x) \cong \mathbf{A}_\mu(\lambda(\cdot - x))$ for each $x \in \mathbb{R}^d$, $N \cong N_{\rm f}$, and $H_{\rm rad} \cong H_{\rm f}$. Precisely we proved the following proposition.

Proposition 4.1 *(1)Let $\hat{\lambda}/\sqrt{\omega}, \hat{\lambda}, \omega\hat{\lambda} \in L^2$. Then*

$$THT^{-1}\lceil_{D(\Delta)\cap D(H_{\rm f})} = H_{\rm S}\lceil_{D(\Delta)\cap D(H_{\rm f})}.$$

(2) Let $\hat{\lambda}, \omega\hat{\lambda} \in L^2$. Then

$$THT^{-1}\lceil_{D(\Delta)\cap D(H_{\rm f})\cap D(N_{\rm f})} = H_{\rm S}\lceil_{D(\Delta)\cap D(H_{\rm f})\cap D(N_{\rm f})}.$$

(3)Let $\hat{\lambda}, \omega\hat{\lambda} \in L^2$. Then

$$TH_{\rm A}T^{-1}\lceil_{D(\Delta)\cap D(N_{\rm f})} = H_{\rm SA}\lceil_{D(\Delta)\cap D(N_{\rm f})}.$$

Then, instead of H, we shall study the self-adjointness of $H_{\rm S}$ for the time being.

4.2 A self-adjoint extension

Let $\{b(T)\}_{T\geq 0} = \{b^j_\mu(T)\}_{\mu=1,\ldots,d,j=1,\ldots,N,T\geq 0}$ denote the dN-dimensional Brownian motion starting at the origin on a probability measure space (P, db). We set $X_T := x + b(T)$, $X_T^{j\mu} = x^j_\mu + b^j_\mu(T)$, $M := \mathbb{R}^{dN} \times P$, and $dX := dx \otimes db$. Let $\mathbb{E}$ refer to the expectation value with respect to db. We see that map

$$\Lambda : t \mapsto \lambda(\cdot - t)$$

is an L^2-valued function on $\mathbb{R}^d$. Note that $(\lambda(\cdot - t))\,\hat{}\,(k) = e^{-ik\cdot t}\hat{\lambda}(k)$. Let $\omega^n\hat{\lambda} \in L^2$. Then $\Lambda \in C^n_{\rm b}(\mathbb{R}^d; L^2)$, the set of n-times continuously differentiable bounded (up to n-times derivatives) L^2-valued functions on $\mathbb{R}^d$. Let $\hat{\lambda}, \omega\hat{\lambda} \in L^2$. Then the L^2-valued stochastic integral,

$$\int_0^t \lambda(\cdot - X_s)db^j_\mu(s) \in L^2(P) \otimes L^2,$$

is well defined. Note that, in the case $\hat{\lambda}, \omega\hat{\lambda}, \omega^2\hat{\lambda} \in L^2$, it is proven that

$$s - \lim_{n\to\infty} \frac{1}{2}\sum_{k=1}^{2^n} \left\{\lambda(\cdot - X^j_{tk/2^n}) + \lambda(\cdot - X^j_{t(k-1)/2^n})\right\} \left(X^{j\mu}_{tk/2^n} - X^{j\mu}_{t(k-1)/2^n}\right)$$

$$= \int_0^t \lambda(\cdot - X_s)db^j_\mu(s)$$

in $L^2(P) \otimes L^2$. We define a contractive symmetric operator Q_s as follows. For $F \in \mathcal{H}_{\rm S}$, let $(Q_0F)(x) := F(x)$, and

$$(Q_sF)(x) := \int_{\mathbb{R}^{dN}} p_s(|x-y|)e^{(-ie/2)\mathbf{A}_\mu(\lambda(\cdot - x^j)+\lambda(\cdot - y^j))(x^j_\mu - y^j_\mu)}F(y)dy, \quad s > 0, \quad (4.1)$$

for almost every $x \in \mathbb{R}^d$, where $p_s(T) := (2\pi s)^{-dN/2}e^{-T^2/(2s)}$, $T \in \mathbb{R}$. Let $\hat{\lambda}, \omega\hat{\lambda}, \omega^2\hat{\lambda} \in L^2$ and set $\mathcal{H}^0_{\rm S} := C_0^\infty(\mathbb{R}^{dN})\hat{\otimes}L^2_{\rm fin}(Q)$. For $F \in \mathcal{H}^0_{\rm S}$ and $G \in \mathcal{H}_{\rm S}$, a direct calculation shows that $g(s) := (Q_sF, G)$ is differentiable in $s > 0$ and $\lim_{s\to 0+} g'(s) = -(H_{\rm SA}F, G)$. Hence it obeys that

$$\lim_{n\to\infty} \frac{(Q_{t/n}F, G) - (F, G)}{t/n} = -(H_{\rm SA}F, G), \quad t > 0.$$

Lemma 4.2 *Let $\hat{\lambda} \in L^2$ and $F, G \in \mathcal{H}_{\rm S}$. Put $f_n(s) := (Q_{s/2^n})^{2^n}$. Then*

$$(F, f_n(s)f_m(t)G)_{\mathcal{H}_{\rm S}} = \int_M dX(F(X_0), e^{-ie\phi(L_{mn}(s,t))}G(X_{s+t}))_{L^2(Q)}. \quad (4.2)$$

Here

$$L_{mn}(s,t) := \sum_{j=1}^{N} \oplus_{\mu=1}^{d} l_{mn}^{j\mu}(s,t),$$

$$l_{mn}^{j\mu}(s,t) := \frac{1}{2}\sum_{k=1}^{2^m}\left\{\lambda(\cdot - X^j_{sk/2^m+t}) + \lambda(\cdot - X^j_{s(k-1)/2^m+t})\right\}\left(X^{j\mu}_{sk/2^m+t} - X^{j\mu}_{s(k-1)/2^m+t}\right)$$
$$+\frac{1}{2}\sum_{k=1}^{2^n}\left\{\lambda(\cdot - X^j_{tk/2^n}) + \lambda(\cdot - X^j_{t(k-1)/2^n})\right\}\left(X^{j\mu}_{tk/2^n} - X^{j\mu}_{t(k-1)/2^n}\right)$$

In particular, if $\hat{\lambda}, \omega\hat{\lambda}, \omega^2\hat{\lambda} \in L^2$, *then*

$$\lim_{m,n\to\infty}(F, f_n(s)f_m(t)G) = \int_M dX(F(X_0), e^{-ie\phi(L(s+t))}G(X_{s+t})). \tag{4.3}$$

Here

$$L(T) := \sum_{j=1}^{N} \oplus_{\mu=1}^{d} \int_0^T \lambda(\cdot - X_s)db^j_\mu(s).$$

Proof: See [14, the proof of Lemma 4.6]. □

Lemma 4.3 *Let* $f_n \in L^2(P)\otimes W$. *Assume that*

$$\lim_{n\to\infty}\mathbb{E}\|f_n - f\|^2_W = 0.$$

Then $s-\lim_{n\to\infty} e^{i\phi(f_n)} = e^{i\phi(f)}$ *on* $L^2(P)\otimes L^2(Q)$.

Proof: It is enough to prove the weak convergence of $e^{i\phi(f_n)}G$ for G in a dense domain in $L^2(P)\otimes L^2(Q)$. Note that

$$\|\left(e^{i\phi(f_n)} - I\right)\Phi\|_{L^2(Q)} \le C\|f_n\|_W(\|N_{\rm f}^{1/2}\Phi\|_{L^2(Q)} + \|\Phi\|_{L^2(Q)})$$

for $\Phi \in D(N_{\rm f}^{1/2})$ with some constant C. Thus it follows that, for $F \in L^2(P)\otimes L^2(Q)$ and $G \in L^\infty(P)\hat{\otimes}D(N_{\rm f}^{1/2})$,

$$|\left(F, (e^{i\phi(f_n)} - e^{i\phi(f)})G\right)_{L^2(P)\otimes L^2(Q)}| = \left|\int_P db\left(F(b), \left(e^{i\phi(f_n)} - e^{i\phi(f))}\right)G(b)\right)_{L^2(Q)}\right|$$
$$\le \int_P db\|F(b)\|\|f_n(b) - f(b)\|_W\left(\|N_{\rm f}^{1/2}G(b)\| + \|G(b)\|\right)$$
$$\le \|F\|_{\mathcal{K}}\left(\mathbb{E}\|f_n - f\|^2_W\right)^{1/2}\sup_{b\in\mathbb{R}^d}\left(\|N_{\rm f}^{1/2}G(b)\|_{L^2(Q)} + \|G(b)\|_{L^2(Q)}\right).$$

Hence, lemma follows. □

Lemma 4.4 *Let $\hat{\lambda}, \omega\hat{\lambda} \in L^2$. Then there exists a nonnegative self-adjoint operator $\widehat{H}_{\rm SA}$ in $\mathcal{H}_{\rm S}$ such that, for $F, G \in \mathcal{H}_{\rm S}$,*

$$\int_M dX(F(X_0), e^{-ie\phi(L(t))}G(X_t))_{L^2(Q)} = (F, e^{-t\widehat{H}_{\rm SA}}G)_{\mathcal{H}_{\rm S}}. \tag{4.4}$$

Proof: Let $\hat{\lambda}, \omega\hat{\lambda}, \omega^2\hat{\lambda} \in L^2$. Then it follows from (4.3) that

$$\lim_{n\to\infty}(F, f_n(t)G) = \int_M dX(F(X_0), e^{-ie\phi(L(t))}G(X_t)). \tag{4.5}$$

Since

$$\left|\int_M dX(F(X_0), e^{-ie\phi(L(t))}G(X_t))\right| \leq \|F\|\|G\|,$$

by the Riesz theorem, the right-hand side of (4.5) defines a bounded operator G_t such that $\|G_t\| \leq 1$, and then

$$\lim_{n\to\infty}(F, f_n(t)G) = (F, G_t G).$$

Moreover $\|f_n(t)\| \leq 1$ implies that

$$G_t = s - \lim_{n\to\infty} f_n(t).$$

It is seen that, by (4.3),

$$(F, G_t G_s G) = \lim_{m,n\to\infty}(F, f_n(t) f_m(s) G)$$

$$= \int_M dX(F(X_0), e^{-ie\phi(L(t+s))}G(X_{t+s})) = (F, G_{t+s}G).$$

Hence

$$G_t G_s = G_{t+s}, \quad t, s \geq 0. \tag{4.6}$$

Moreover, by virtue of (4.5), $\lim_{t\to\infty}(F, G_t G) = (F, G)$ holds, which yields that

$$s - \lim_{t\to 0} G_t = I. \tag{4.7}$$

Finally, since $f_n(t)$, $n \geq 0$, is symmetric, we have

$$G_t^* = G_t, \quad t \geq 0. \tag{4.8}$$

Next we let $\hat{\lambda}, \omega\hat{\lambda} \in L^2$ and $\omega^2\hat{\lambda} \notin L^2$. Let $\hat{\lambda}_n$ be such that $\hat{\lambda}_n, \omega\hat{\lambda}_n, \omega^2\hat{\lambda}_n \in L^2$ and that $\hat{\lambda}_n \to \hat{\lambda}$ and $\omega\hat{\lambda}_n \to \omega\hat{\lambda}$ strongly in L^2 as $n \to \infty$. Let $G_t(n)$ and $L_n(t)$ denote G_t and $L(t)$ with $\hat{\lambda}$ replaced by $\hat{\lambda}_n$, respectively. Note that

$$\mathbb{E}\|L(t) - L_n(t)\|_W^2 = tdN\|\hat{\lambda} - \hat{\lambda}_n\|^2. \tag{4.9}$$

By Lemma 4.3 and the Lebesgue dominated convergence theorem,

$$\lim_{n\to\infty}(F, G_t(n)G) = \int_M dX(F(X_0), e^{-ie\phi(L(t))}G(X_t)). \tag{4.10}$$

The right-hand side of (4.10) again defines a bounded operator $\widehat{G}_t$ by the Riesz theorem, i.e.,

$$(F, \widehat{G}_t G) = \int_M dX(F(X_0), e^{-ie\phi(L(t))}G(X_t)). \tag{4.11}$$

Since $\|G_t(n)\| \leq 1$, it obeys that

$$s - \lim_{n\to\infty} G_t(n) = \widehat{G}_t. \tag{4.12}$$

We easily check that

$$\widehat{G}_t\widehat{G}_s = s - \lim_{n\to\infty} G_t(n)G_s(n) = s - \lim_{n\to\infty} G_{t+s}(n) = \widehat{G}_{t+s} \tag{4.13}$$

by (4.6),

$$s - \lim_{t\to\infty} \widehat{G}_t = I \tag{4.14}$$

by (4.11), and, finally

$$\widehat{G}_t^* = \widehat{G}_t \tag{4.15}$$

by (4.8). Then, (4.13), (4.14) and (4.15) imply that $\{\widehat{G}_t\}_{t\geq 0}$ is a symmetric strongly continuous one-parameter semigroup. Thus there exists a nonnegative self-adjoint operator $\widehat{H}_{\rm SA}$ such that

$$e^{-t\widehat{H}_{\rm SA}} = \widehat{G}_t, \quad t \geq 0.$$

Thus we get the desired results. □

Next we investigate the domain of $\widehat{H}_{\rm SA}$.

Lemma 4.5 *(1) Let $\hat{\lambda}, \omega\hat{\lambda} \in L^2$. Then $\widehat{H}_{\rm SA} \supset H_{\rm SA}\lceil_{D(\Delta)\cap D(N_{\rm f})}$.*

(2) Let $\hat{\lambda}/\sqrt{\omega}, \hat{\lambda}, \omega\hat{\lambda} \in L^2$. Then $\widehat{H}_{\rm SA} \supset H_{\rm SA}\lceil_{D(\Delta)\cap D(H_{\rm f})}$.

Proof: Let $G \in \mathcal{H}_{\rm S}^0$, $F \in H_{\rm S}$, and $\hat{\lambda}, \omega\hat{\lambda}, \omega^2\hat{\lambda} \in L^2$. We have, by (4.1),

$$\left(\frac{1}{t}\left(e^{-t\widehat{H}_{\rm SA}} - I\right)G, F\right) = \lim_{n\to\infty}\sum_{j=0}^{2^n-1}\frac{1}{2^n}\left(\frac{Q_{t/2^n} - I}{t/2^n}G, (Q_{t/2^n})^j F\right)$$

$$= \lim_{n\to\infty}\int_0^1\left(\frac{Q_{t/2^n} - I}{t/2^n}G, (Q_{t/2^n})^{[ns]}F\right)ds = \int_0^1 -(H_{\rm SA}G, e^{-ts\widehat{H}_{\rm SA}}F)ds. \tag{4.16}$$

Since we show that, by (4.16) and

$$\|H_{\rm SA}G\| \le C\|\hat\lambda\|(\|\Delta G\| + \|N_{\rm f}G\| + \|G\|), \tag{4.17}$$

equation (4.16) extends to $G \in D(\Delta)\cap D(N_{\rm f})$ and $F \in \mathcal{H}_{\rm S}$. Assume $\omega^2\hat\lambda \notin L^2$. Let $\hat\lambda_n$ be such that $\hat\lambda_n, \omega\hat\lambda_n, \omega^2\hat\lambda_n \in L^2$ and that $\hat\lambda_n \to \hat\lambda$ and $\omega\hat\lambda_n \to \omega\hat\lambda$ strongly as $n\to\infty$. $\widehat{H}_{\rm SA}(n)$ and $H_{\rm SA}(n)$ are defined by $\widehat{H}_{\rm SA}$ and $H_{\rm SA}$ with $\hat\lambda$ replaced by $\hat\lambda_n$. Thus (4.20) holds with $\widehat{H}_{\rm SA}(n)$ and $H_{\rm SA}(n)$ instead of $\widehat{H}_{\rm SA}$ and $H_{\rm SA}$, respectively. Then, by the definition of $\widehat{H}_{\rm SA}$,

$$s-\lim_{n\to\infty} e^{-t\widehat{H}_{\rm SA}(n)} = s-\lim_{n\to\infty} G_t(n) = \widehat{G}_t = e^{-t\widehat{H}_{\rm SA}}. \tag{4.18}$$

It is clear that $H_{\rm SA}(n)F$ converges to $H_{\rm SA}F$ strongly as $n\to\infty$, since

$$\|(H_{\rm SA}(n) - H_{\rm SA})F\| \le C\|\hat\lambda_n - \hat\lambda\|(\|\Delta F\| + \|N_{\rm f}F\| + \|F\|). \tag{4.19}$$

Hence, by a limiting argument

$$\left(\frac{1}{t}\left(e^{-t\widehat{H}_{\rm SA}} - I\right)G, F\right) = \int_0^1 -(H_{\rm SA}G, e^{-ts\widehat{H}_{\rm SA}}F)ds, \quad G\in D(\Delta)\cap D(N_{\rm f}) \quad F\in\mathcal{H}_{\rm S}. \tag{4.20}$$

holds for $\hat\lambda$ with $\hat\lambda, \omega\hat\lambda \in L^2$. Thus taking $t\to\infty$ on the both sides of (4.20), we see that, if $\hat\lambda, \omega\hat\lambda\in L^2$, then

$$\widehat{H}_{\rm SA} \supset H_{\rm SA}\lceil_{D(\Delta)\cap D(N_{\rm f})}.$$

Next we let $\hat\lambda/\sqrt{\omega}, \hat\lambda, \omega\hat\lambda \in L^2$ and $\omega^2\hat\lambda\in L^2$. We have

$$\|H_{\rm SA}G\| \le C_1(\|\Delta G\| + \|H_{\rm f}G\| + \|G\|),$$

where C_1 depends only on $\|\hat\lambda/\sqrt{\omega}\|, \|\hat\lambda\|$, and $\|\omega\hat\lambda\|$. Henceforce (4.20) extends to $G\in D(\Delta)\cap D(H_{\rm f})$. Assume $\omega^2\hat\lambda\notin L^2$. Let $\hat\lambda_n$ be such that $\hat\lambda_n/\sqrt{\omega}, \hat\lambda_n, \omega\hat\lambda_n, \omega^2\hat\lambda_n \in L^2$ and that $\hat\lambda_n/\sqrt{\omega} \to \hat\lambda/\sqrt{\omega}$, $\hat\lambda_n\to\hat\lambda$ and $\omega\hat\lambda_n\to\omega\hat\lambda$ strongly as $n\to\infty$. $\widehat{H}_{\rm SA}(n)$ and $H_{\rm SA}(n)$ are defined in the same way as above. It holds that

$$\|(H_{\rm SA}(n) - H_{\rm SA})G\| \le C_2(n)(\|\Delta G\| + \|H_{\rm f}G\| + \|G\|),$$

where $\lim_{n\to\infty} C_2(n) = 0$. By (4.20) and the limiting argument such as (4.18), we see that, if $\hat\lambda/\sqrt{\omega}, \hat\lambda, \omega\hat\lambda\in L^2$, then

$$\widehat{H}_{\rm SA} \supset H_{\rm SA}\lceil_{D(\Delta)\cap D(H_{\rm f})}$$

holds true. □

Since $D(\widehat{H}_{\rm SA}) \cap D(H_{\rm f})$ is dense,

$$\widehat{H}_{\rm S} := \widehat{H}_{\rm SA} \dot{+} H_{\rm f}$$

is well defined. Next we construct a functional integral representation of $e^{-t\widehat{H}_{\rm S}}$. Let $\Xi_t : L^2(Q) \to L^2(Q_0)$ be the second quantization of isometry $\oplus^d \xi_t : W \to W_0$, which is defined by

$$(\xi_t f)\widehat{\ }(k, k_0) = \frac{e^{-itk_0}}{\sqrt{\pi}} \sqrt{\frac{\omega(k)}{\omega(k)^2 + |k_0|^2}} \hat{f}(k), \quad f \in L^2.$$

Then Ξ_t is also an isometry and satisfies that $\Xi_s^* \Xi_t = e^{-|s-t| H_{\rm f}}$. The following statement is established in [14]:

Proposition 4.6 *Let $\hat{\lambda}, \omega\hat{\lambda}, \omega^2\hat{\lambda} \in L^2$. Then*

$$(F, e^{-t\widehat{H}_{\rm S}} G)_{\mathcal{H}_{\rm S}} = \int_M dX (F_0, \mathbf{J}_t G_t)_{L^2(Q)}, \quad F, G \in \mathcal{H}_{\rm S}, \tag{4.21}$$

where $F_0 := F(X_0)$, $G_t = G(X_t)$ and

$$\mathbf{J}_t := \mathbf{J}_t(X) = \Xi_0^* e^{-ie\phi_0(\mathbf{K}(t))} \Xi_t.$$

Here

$$\mathbf{K}(t) := \mathbf{K}(X, t) := \sum_{j=1}^N \oplus_{\mu=1}^d \int_0^t \xi_s \lambda(\cdot - X_s) db_\mu^j(s).$$

We want to extend (4.21) for the case when $\hat{\lambda}, \omega\hat{\lambda} \in L^2$ but $\omega^2\hat{\lambda} \notin L^2$.

Lemma 4.7 *Let $\hat{\lambda}, \omega\hat{\lambda} \in L^2$. Then (4.21) holds true.*

Proof: By the Trotter-Kato product formula, we have

$$e^{-t\widehat{H}_{\rm S}} = s - \lim_{n\to\infty} \left(e^{-t\widehat{H}_{\rm SA}/2^n} e^{-tH_{\rm f}/2^n} \right)^{2^n}.$$

We put

$$P_n := \left(e^{-t\widehat{H}_{\rm SA}/2^n} e^{-tH_{\rm f}/2^n} \right)^{2^n}$$

and

$$P_n(m) := \left(e^{-t\widehat{H}_{\rm SA}(m)/2^n} e^{-tH_{\rm f}/2^n} \right)^{2^n}.$$

Here $\widehat{H}_{\rm SA}(m)$ is defined by $\widehat{H}_{\rm SA}$ with $\hat\lambda$ replaced by $\hat\lambda_m$. From $s-\lim_{m\to\infty} e^{-t\widehat{H}_{\rm SA}(m)} = e^{-t\widehat{H}_{\rm SA}}$ it follows that $s-\lim_{m\to\infty} P_n(m) = P_n$ follows. It is proven that

$$(F, P_n(m)G) = \int_M dX(\Xi_0 F_0, e^{-ie\phi_0(\mathbf{K}_{nm}(t))}\Xi_t G_t).$$

Here

$$\mathbf{K}_{nm}(t) := \sum_{j=1}^N \sum_{i=0}^{2^n-1} \oplus_{\mu=1}^d \int_{ti/2^n}^{t(i+1)/2^n} \xi_{ti/2^n}\lambda_m(\cdot - X_s)db_\mu^j(s).$$

Let

$$\mathbf{K}_{n}(t) := \sum_{j=1}^N \sum_{i=0}^{2^n-1} \oplus_{\mu=1}^d \int_{ti/2^n}^{t(i+1)/2^n} \xi_{ti/2^n}\lambda(\cdot - X_s)db_\mu^j(s).$$

Since ξ_s is isometry, we have

$$\begin{aligned}
&\mathbb{E}\|\mathbf{K}_{nm}(t) - \mathbf{K}_n(t)\|_{W_0}^2 \\
&= \sum_{j=1}^N\sum_{\mu=1}^d\sum_{i=0}^{2^n-1}\mathbb{E}\left\|\int_{ti/2^n}^{t(i+1)/2^n}\xi_{ti/2^n}\left(\lambda_m(\cdot - X_s) - \lambda(\cdot - X_s)\right)db_\mu^j(s)\right\|_{W_0}^2 \\
&= \sum_{j=1}^N\sum_{\mu=1}^d\sum_{i=0}^{2^n-1}\mathbb{E}\left\|\int_{ti/2^n}^{t(i+1)/2^n}\left(\lambda_m(\cdot - X_s) - \lambda(\cdot - X_s)\right)db_\mu^j(s)\right\|_{W}^2 \\
&= dN\sum_{i=0}^{2^n-1}\frac{t}{2^n}\|\hat\lambda_m - \hat\lambda\|^2 = tdN\|\hat\lambda_m - \hat\lambda\|^2.
\end{aligned}$$

Hence, by Lemma 4.3, we have

$$s-\lim_{m\to\infty}(F, P_n(m)G) = (F, P_nG) = \int_M dX(\Xi_0F_0, e^{-ie\phi_0(\mathbf{K}_n(t))}\Xi_tG_t). \tag{4.22}$$

Moreover

$$\begin{aligned}
&\mathbb{E}\|\mathbf{K}_n(t) - \mathbf{K}(t)\|_{W_0}^2 \\
&\leq \sum_{j=1}^N\sum_{\mu=1}^d\mathbb{E}\left\|\sum_{i=0}^{2^n-1}\left\{\int_{ti/2^n}^{t(i+1)/2^n}\xi_{ti/2^n}\lambda(\cdot - X_s)db_\mu^j(s) - \xi_{ti/2^n}\lambda(\cdot - X_{ti/2^n})\delta b_\mu^j(i)\right\}\right\|_{W_0}^2 \\
&\quad + \sum_{j=1}^N\sum_{\mu=1}^d\mathbb{E}\left\|\sum_{i=0}^{2^n-1}\xi_{ti/2^n}\lambda(\cdot - X_{ti/2^n})\delta b_\mu^j(i) - \int_0^t\xi_s\lambda(\cdot - X_s)db_\nu^j(s)\right\|_{W_0}^2,
\end{aligned} \tag{4.23}$$

where $\delta b_\mu^j(i) := b_\mu^j(t(i+1)/2^n) - b_\mu^j(ti/2^n)$. Since ξ_s is isometry, the first term on the right-hand side of (4.23) is

$$\lim_{n\to\infty}\sum_{j=1}^N\sum_{\mu=1}^d\mathbb{E}\left\|\sum_{i=0}^{2^n-1}\left\{\int_{ti/2^n}^{t(i+1)/2^n}\lambda(\cdot - X_s)db_\mu^j(s) - \lambda(\cdot - X_{ti/2^n})\delta b_\mu^j(t,n)\right\}\right\|_W^2 = 0.$$

Noting that

$$\|\xi_s\hat{\lambda} - \xi_t\hat{\lambda}\|_{L^2}^2 = |(\hat{\lambda}, (1 - e^{-|t-s|\omega})\hat{\lambda})| \leq |t-s|\|\hat{\lambda}\|\|\omega\hat{\lambda}\|,$$

we prove that the second term also converges to zero as $n \to \infty$. Thus, if $\hat{\lambda}, \omega\hat{\lambda} \in L^2$, then

$$\lim_{n\to\infty} \mathbb{E}\|\mathbf{K}_n(t) - \mathbf{K}(t)\|_{W_0}^2 = 0.$$

Taking $n \to \infty$ of the both sides of (4.22), we have (4.21). $\square$

5 Relative bounds and invariant domains

We state a general lemma.

Lemma 5.1 *Let $\mathcal{K}$ be a Hilbert space.*
(1) Assume that T is a densely defined closed operator in $\mathcal{K}$ and $g \in \mathcal{K}$. Let D be a core of T. Suppose that $|(Tf, g)| \leq R_g\|f\|$ for all $f \in D$ with some constant R_g. Then $g \in D(T^*)$ with $\|T^*g\| \leq R_g$.
(2) Assume that S is a densely defined closed operator and U a densely defined linear operator in $\mathcal{K}$. Let D_S be a core of S and D_U a dense subspace of $\mathcal{K}$. Suppose that $|(Uf, Sg)| \leq R\|f\|\|g\|$ for all $f \in D_U$ and $g \in D_S$ with some constant R. Then U^*S has unique bounded operator extension $\overline{U^*S}$ with $\|\overline{U^*S}\| \leq R$. In particular, if S is a bounded operator, then U^*S is a bounded operator.

Proof: It is due to an application of the Riesz theorem. $\square$

5.1 Relative bounds and invariant domains for Δ

Let $H_{\mathrm{d}} := -\Delta/2 + H_{\mathrm{f}}$.

Lemma 5.2 *Let $\hat{\lambda}, \omega\hat{\lambda} \in L^2$. Let $E > 0$ and $t > 0$. Then both of $\Delta^k e^{-t\widehat{H}_{\mathrm{S}}}$, $k \geq 0$, and $\Delta(\widehat{H}_{\mathrm{S}} + E)^{-1}$ are bounded operators. In particular, $e^{-t\widehat{H}_{\mathrm{S}}}$ leaves $C^\infty(\Delta)$ invariant.*

Proof: We fix $k \geq 0$. From functional integral representation (4.21) it follows that

$$|(F, e^{-t\widehat{H}_{\mathrm{S}}}G)| \leq (|F|, e^{-tH_{\mathrm{d}}}|G|). \tag{5.1}$$

Let $F \in \mathcal{H}_{\mathrm{S}}^{00} := C_0^\infty(\mathbb{R}^{dN})\hat{\otimes}[L^2_{\mathrm{fin}}(Q) \cap C^\infty(H_{\mathrm{f}})]$. Let F_{R} denote the real part of F and F_{I} its imaginary part, i.e., $F = F_{\mathrm{R}} + iF_{\mathrm{I}}$. Define subsets of $\mathbb{R}^{dN} \times Q$ by

$$\begin{aligned}
Q_{\mathrm{R}}^+ &:= \{(x,\phi) \in \mathbb{R}^{dN} \times Q | (\Delta^k F_{\mathrm{R}})(x,\phi) > 0\},\\
Q_{\mathrm{R}}^- &:= \{(x,\phi) \in \mathbb{R}^{dN} \times Q | (\Delta^k F_{\mathrm{R}})(x,\phi) < 0\},\\
Q_{\mathrm{I}}^+ &:= \{(x,\phi) \in \mathbb{R}^{dN} \times Q | (\Delta^k F_{\mathrm{I}})(x,\phi) > 0\},\\
Q_{\mathrm{I}}^- &:= \{(x,\phi) \in \mathbb{R}^{dN} \times Q | (\Delta^k F_{\mathrm{I}})(x,\phi) < 0\}.
\end{aligned}$$

Define also

$$F_1 := \begin{cases} F_{\mathrm{R}}(x,\phi), & (x,\phi) \in Q_{\mathrm{R}}^+, \\ 0, & (x,\phi) \notin Q_{\mathrm{R}}^+, \end{cases}$$

$$F_2 := \begin{cases} -F_{\mathrm{R}}(x,\phi), & (x,\phi) \in Q_{\mathrm{R}}^-, \\ 0, & (x,\phi) \notin Q_{\mathrm{R}}^-, \end{cases}$$

$$F_3 := \begin{cases} F_{\mathrm{I}}(x,\phi), & (x,\phi) \in Q_{\mathrm{I}}^+, \\ 0, & (x,\phi) \notin Q_{\mathrm{I}}^+, \end{cases}$$

$$F_4 := \begin{cases} -F_{\mathrm{I}}(x,\phi), & (x,\phi) \in Q_{\mathrm{I}}^-, \\ 0, & (x,\phi) \notin Q_{\mathrm{I}}^-. \end{cases}$$

Note that $F_i \in D(\Delta^k)$ and that $\Delta^k F_i > 0$, $i = 1,2,3,4$. Since $|\Delta^k F_{\mathrm{R}}| = \Delta^k F_1 + \Delta^k F_2$ and $|\Delta^k F_{\mathrm{I}}| = \Delta^k F_3 + \Delta^k F_4$, we have

$$\begin{aligned}
|(\Delta^k F, e^{-t\widehat{H}_{\mathrm{S}}} G)| &\le |(\Delta^k F_{\mathrm{R}}, e^{-t\widehat{H}_{\mathrm{S}}} G)| + |(\Delta^k F_{\mathrm{I}}, e^{-t\widehat{H}_{\mathrm{S}}} G)| \\
&\le |(|\Delta^k F_{\mathrm{R}}|, e^{-tH_{\mathrm{d}}}|G|)| + |(|\Delta^k F_{\mathrm{I}}|, e^{-tH_{\mathrm{d}}}|G|)| \\
&= \sum_{i=1}^4 (\Delta^k F_i, e^{-tH_{\mathrm{d}}}|G|) \le \sum_{i=1}^4 \|F_i\|\|G\|.
\end{aligned}$$

Since $\|F_1\|^2 + \|F_2\|^2 = \|F_{\mathrm{R}}\|^2$, $\|F_3\|^2 + \|F_4\|^2 = \|F_{\mathrm{I}}\|^2$, and $\|F_{\mathrm{R}}\|^2 + \|F_{\mathrm{I}}\|^2 = \|F\|^2$, we have $\sum_{i=1}^4 \|F_i\| \le 2\|F\|$. Hence

$$|(\Delta^k F, e^{-t\widehat{H}_{\mathrm{S}}} G)| \le 2\|F\|\|G\|. \tag{5.2}$$

Since $\mathcal{H}_{\mathrm{S}}^{00}$ is a core of Δ^k, the boundedness of $\Delta^k e^{-t\widehat{H}_{\mathrm{S}}}$ follows from Lemma 5.1. By means of (5.1) we have

$$|(F, (\widehat{H}_{\mathrm{S}} + E)^{-1} G)| \le (|F|, (H_{\mathrm{d}} + E)^{-1}|G|).$$

Note that $\Delta(H_{\mathrm{d}} + E)^{-1}$ is bounded. Thus, $(\widehat{H}_{\mathrm{S}} + E)^{-1}\mathcal{H}_{\mathrm{S}} \subset D(\Delta)$ and

$$|(\Delta F, (\widehat{H}_{\mathrm{S}} + E)^{-1} G)| \le 2\|F\|\|G\|$$

follows in a similar way as (5.2). Hence $\Delta(\widehat{H}_{\mathrm{S}} + E)^{-1}$ is bounded by Lemma 5.1. Thus we get the desired results. $\square$

Lemma 5.3 *Let $\hat\lambda, \omega\hat\lambda \in L^2$. Thus the same statements as Lemma 5.5 hold with $\widehat{H}_{\rm S}$ replaced by $\widehat{H}_{\rm SA}$. In particular $e^{-t\widehat{H}_{\rm SA}}$ leaves $C^\infty(\Delta)$ invariant.*

Proof: We have

$$|(F, e^{-t\widehat{H}_{\rm SA}}G)| \le (|F|, e^{-t(-\Delta/2)}|G|).$$

Thus the same proof of Lemma 5.2 yields the desired results. □

5.2 Relative bounds and invariant domains for $H_{\rm f}$ and $N_{\rm f}$

We identify $W_0 = \oplus^d L^2(\mathbb{R}^{d+1}) \cong L^2(\mathbb{R}) \otimes W$. Let

$$\widehat{\omega}_0 := \mathbf{1} \otimes \widehat{\omega} : W_0 \to W_0.$$

The following proposition is useful to investigate $e^{-t\widehat{H}_{\rm S}}$.

Proposition 5.4 *Let $\hat\lambda, \omega\hat\lambda \in L^2$ and $\omega^{k/2}\hat\lambda \in L^2$. Then $\mathbf{K}(t) \in D(\widehat{\omega}_0^{k/2})$ and*

$$\mathbb{E}\|\widehat{\omega}_0^{k/2}\mathbf{K}(t)\|_{W_0}^{2m} \le \frac{(2m)!}{2^m} t^m d^m N^{2m} \|\omega^{k/2}\hat\lambda\|_{L^2}^{2m}.$$

Proof: See [18, Theorem 4. 6]. □

We define $H_{\rm f}^0 : L^2(Q_0) \to L^2(Q_0)$ by

$$H_{\rm f}^0 :\!\phi_0(f_1)\cdots\phi_0(f_n)\!: := \sum_{j=1}^n :\!\phi_0(f_1)\cdots\phi_0(\widehat{\omega}_0 f_j)\cdots\phi_0(f_n)\!: ,$$

$$H_{\rm f}^0 \mathbf{1} := 0.$$

Then

$$\Xi_t^* H_{\rm f}^0 = H_{\rm f}\Xi_t^*. \tag{5.3}$$

Let

$$\pi_0(f) := i[H_{\rm f}^0, \phi_0(f)].$$

Thus we have

$$[\phi_0(f), \pi_0(g)] = i q_0(\widehat{\omega}_0 \bar f, g). \tag{5.4}$$

Let $\hat\lambda, \omega\hat\lambda \in L^2$. Then it is easily seen that $\mathbf{J}_t$ maps $D(H_{\rm f})$ onto itself. From (5.3) and (5.4) it follows that

$$[H_{\rm f}, \mathbf{J}_t] = \Xi_0^* e^{-ie\phi_0(\mathbf{K}(t))}\left\{-e\pi_0(\mathbf{K}(t)) + \frac{e^2}{2} q_0(\widehat{\omega}_0\mathbf{K}(t), \mathbf{K}(t))\right\}\Xi_t$$

on $D(H_{\rm f})$.

Lemma 5.5 *Let $\hat\lambda, \omega\hat\lambda \in L^2$. Then, for $\Psi \in D(H_{\rm f})$, $e^{-t\widehat{H}_{\rm S}}\Psi \in D(H_{\rm f})$ and there exists a constant C such that*

$$\|[e^{-t\widehat{H}_{\rm S}}, H_{\rm f}]\Psi\| \le C(t+\sqrt{t})\|(H_{\rm f}+\mathbf{1})^{1/2}\Psi\|. \tag{5.5}$$

Proof: Let $F, G \in D(H_{\rm f})$. Note that $(H_{\rm f}F)(x) = H_{\rm f}(F(x))$ for each $x \in \mathbb{R}^{dN}$. Then

$$(H_{\rm f}F, e^{-t\widehat{H}_{\rm S}}G) = \int_M dX(F_0, H_{\rm f}\mathbf{J}_t G_t)$$

$$= \int_M dX(F_0, [H_{\rm f}, \mathbf{J}_t]G_t) + \int_M dX(F_0, \mathbf{J}_t H_{\rm f} G_t)$$

$$= \int_M dX(F_0, [H_{\rm f}, \mathbf{J}_t]G_t) + (F, e^{-t\widehat{H}_{\rm S}} H_{\rm f} G).$$

We have

$$\int_M dX(F_0, [H_{\rm f}, \mathbf{J}_t]G_t)$$

$$= \int_M dX\left(F_0, \Xi_0^* e^{-ie\phi_0(\mathbf{K}(t))}\left\{-e\pi_0(\mathbf{K}(t)) + \frac{e^2}{2}q_0(\widehat\omega_0\mathbf{K}(t), \mathbf{K}(t))\right\}\Xi_t G_t\right).$$

Note that

$$\|\pi_0(\mathbf{K}(t))\Xi_t\Psi\| \le \theta\|(H_{\rm f}+I)^{1/2}\Psi\|.$$

Here

$$\theta := \frac{d-1}{\sqrt{2}}(2\|\widehat\omega_0^{1/2}\mathbf{K}(t)\| + \|\widehat\omega_0\mathbf{K}(t)\|).$$

Moreover

$$|q_0(\widehat\omega_0\mathbf{K}(t), \mathbf{K}(t))| \le \|\widehat\omega_0^{1/2}\mathbf{K}(t)\|^2 := \eta.$$

Using Proposition 5.4, we have

$$\mathbb{E}\theta^2 \le (d-1)^2(4\mathbb{E}\|\widehat\omega_0^{1/2}\mathbf{K}(t)\|^2 + \mathbb{E}\|\widehat\omega\mathbf{K}(t)\|^2)$$

$$\le (d-1)^2(4tN^2d\|\omega^{1/2}\hat\lambda\|^2 + tN^2d\|\omega\hat\lambda\|^2)$$

$$\le 4tN^2d(d-1)^2(\|\omega^{1/2}\hat\lambda\|^2 + \|\omega\hat\lambda\|^2),$$

and

$$\mathbb{E}\eta^4 \le (3!)t^2N^4d^2\|\omega^{1/2}\hat\lambda\|^4.$$

In terms of the Schwartz inequality we have

$$\left|\int_M dX(F_0, [H_{\rm f}, \mathbf{J}_t]G_t)\right| \le |e|\left(\int_M dX\|F_0\|^2\theta^2\right)^{1/2}\left(\int_M dX\|(H_{\rm f}+I)^{1/2}G_t\|^2\right)^{1/2}$$

$$+\frac{e^2}{2}\left(\int_M dX\|F_0\|^2\eta^4\right)^{1/2}\left(\int_M dX\|G_t\|^2\right)^{1/2}.$$

It follows that

$$\left(\int_M dX\|F_0\|^2\theta^2\right)^{1/2} \le \left(\int_{\mathbb{R}^d} dx(\mathbb{E}\theta^2)\|F(x)\|^2_{L^2(Q)}\right)^{1/2}$$
$$\le \sqrt{4tN^2d(d-1)^2}(\|\omega^{1/2}\hat{\lambda}\|+\|\omega\hat{\lambda}\|)\|F\|,$$

and

$$\left(\int_M dX\|F_0\|^2\eta^4\right)^{1/2} \le \left(\int_{\mathbb{R}^d} dx(\mathbb{E}\eta^4)\|F(x)\|^2_{L^2(Q)}\right)^{1/2}$$
$$\le \sqrt{(3!)t^2N^4d^2}\|\omega^{1/2}\hat{\lambda}\|^2\|F\|.$$

Thus

$$\left|\int_M dX(F_0,[H_{\mathrm{f}},\mathbf{J}_t]G_t)\right| \le C(\sqrt{t}+t)\|F\|\|(H_{\mathrm{f}}+I)^{1/2}G\| \tag{5.6}$$

with

$$C := \max\left\{\sqrt{4N^2d(d-1)^2}(\|\omega^{1/2}\hat{\lambda}\|+\|\omega\hat{\lambda}\|),\quad \frac{1}{2}\sqrt{(3!)N^4d^2}\|\omega\hat{\lambda}\|^2\right\}.$$

Hence we proved that

$$|(H_{\mathrm{f}}F, e^{-t\widehat{H}_{\mathrm{S}}}G)| \le \left\{C(\sqrt{t}+t)+1\right\}\|(H_{\mathrm{f}}+\mathbf{1})G\|\|F\|.$$

Then $e^{-t\widehat{H}_{\mathrm{S}}}G \in D(H_{\mathrm{f}})$ by Lemma 5.1. Moreover, since

$$(F,[e^{-t\widehat{H}_{\mathrm{S}}},H_{\mathrm{f}}]G) = \int_M dX(F_0,[H_{\mathrm{f}},\mathbf{J}_t]G_t),$$

(5.5) follows from (5.6). $\square$

Lemma 5.6 *Let* $\hat{\lambda}, \omega\hat{\lambda} \in L^2$. *Let* $\Psi \in D(H_{\mathrm{f}})$ *and* $E > 0$. *Then, for all* $k > 0$, $(\widehat{H}_{\mathrm{S}}+E)^{-k}\Psi \in D(H_{\mathrm{f}})$ *and*

$$H_{\mathrm{f}}(\widehat{H}_{\mathrm{S}}+E)^{-k}\Psi = \frac{1}{\Gamma(k)}\int_0^\infty t^{-1+k}H_{\mathrm{f}}e^{-t\widehat{H}_{\mathrm{S}}}e^{-tE}\Psi dt, \tag{5.7}$$

where $\Gamma(k) := \int_0^\infty e^{-x}x^{k-1}dx$.

Proof: Note that

$$(\widehat{H}_{\rm S}+E)^{-k}\Psi=\frac{1}{\Gamma(k)}\int_0^\infty e^{-t\widehat{H}_{\rm S}}e^{-tE}t^{-1+k}\Psi dt$$

in the strong sense. We have

$$\begin{aligned}\|H_{\rm f}e^{-t\widehat{H}_{\rm S}}\Psi\| &\leq \|[e^{-t\widehat{H}_{\rm S}},H_{\rm f}]\Psi\|+\|e^{-t\widehat{H}_{\rm S}}H_{\rm f}\Psi\| \\ &\leq C(1+\sqrt{t}+t)\|(H_{\rm f}+1)\Psi\| \qquad (5.8)\end{aligned}$$

with some constant C. Let $\Phi\in D(H_{\rm f})$. It obeys that

$$\begin{aligned}(H_{\rm f}\Phi,(\widehat{H}_{\rm S}+E)^{-k}\Psi) &= \frac{1}{\Gamma(k)}\int_0^\infty (H_{\rm f}\Phi,e^{-t\widehat{H}_{\rm S}}e^{-tE}t^{-1+k}\Psi)dt \\ &= \frac{1}{\Gamma(k)}\int_0^\infty (\Phi,H_{\rm f}e^{-t\widehat{H}_{\rm S}}e^{-tE}t^{-1+k}\Psi)dt.\end{aligned}$$

By virtue of (5.8),

$$\begin{aligned}|(H_{\rm f}\Phi,(\widehat{H}_{\rm S}+E)^{-k}\Psi)| &\leq \frac{1}{\Gamma(k)}\int_0^\infty \|\Phi\|\|H_{\rm f}e^{-t\widehat{H}_{\rm S}}e^{-tE}t^{-1+k}\Psi\|dt \\ &\leq \frac{1}{\Gamma(k)}\int_0^\infty C(1+\sqrt{t}+t)e^{-tE}t^{-1+k}dt\|\Phi\|\|(H_{\rm f}+\mathbf{1})\Psi\|.\end{aligned}$$

Hence $(\widehat{H}_{\rm S}+E)^{-k}\Psi\in D(H_{\rm f})$ by Lemma 5.1 and (5.7) holds true by the strong integrability of $\int_0^\infty H_{\rm f}e^{-t\widehat{H}_{\rm S}}e^{-tE}t^{-1+k}\Psi dt$. Then lemma follows. □

Lemma 5.7 *Let $\hat{\lambda},\omega\hat{\lambda}\in L^2$. Then $H_{\rm f}^{1/2}(\widehat{H}_{\rm S}+E)^{-1/2}$ is a bounded operator.*

Proof: Since $(G,H_{\rm f}G)\leq(G,H_{\rm S}G)=(G,\widehat{H}_{\rm S}G)$ for $G\in D(\Delta)\cap D(H_{\rm f})\cap D(N)$, we have

$$\|H_{\rm f}^{1/2}(\widehat{H}_{\rm S}+E)^{-1/2}G\|\leq\|G\|,\quad G\in D(\Delta)\cap D(H_{\rm f})\cap D(N).$$

Thus lemma follows, since $D(\Delta)\cap D(H_{\rm f})\cap D(N)$ is dense. □

Lemma 5.8 *Let $\hat{\lambda},\omega\hat{\lambda}\in L^2$. Then $H_{\rm f}(\widehat{H}_{\rm S}+E)^{-1}$ is a bounded operator.*

Proof: Let $\Psi \in D(H_{\rm f})$. From Lemma 5.6, the following identity,

$$H_{\rm f}(\widehat{H}_{\rm S}+E)^{-1}\Psi = (\widehat{H}_{\rm S}+E)^{-1/2}H_{\rm f}(\widehat{H}_{\rm S}+E)^{-1/2}\Psi + [(\widehat{H}_{\rm S}+E)^{-1/2}, H_{\rm f}](\widehat{H}_{\rm S}+E)^{-1/2}\Psi, \tag{5.9}$$

is well defined and

$$[(\widehat{H}_{\rm S}+E)^{-1/2}, H_{\rm f}](\widehat{H}_{\rm S}+E)^{-1/2}\Psi = \frac{1}{\Gamma(1/2)}\int_0^\infty \frac{1}{\sqrt{t}}[e^{-t\widehat{H}_{\rm S}}, H_{\rm f}]e^{-tE}(\widehat{H}_{\rm S}+E)^{-1/2}\Psi dt$$

holds true. We have

$$\begin{aligned}
&\|[(\widehat{H}_{\rm S}+E)^{-1/2}, H_{\rm f}](\widehat{H}_{\rm S}+E)^{-1/2}\Psi\| \\
&\leq \frac{C}{\Gamma(1/2)}\int_0^\infty \frac{\sqrt{t}+t}{\sqrt{t}}\|(H_{\rm f}+I)^{1/2}(\widehat{H}_{\rm S}+E)^{-1/2}\Psi\|e^{-tE}dt \\
&\leq \left(\frac{C}{\Gamma(1/2)}\int_0^\infty (1+\sqrt{t})e^{-tE}dt\right)\|(H_{\rm f}+I)^{1/2}(\widehat{H}_{\rm S}+E)^{-1/2}\|\|\Psi\|.
\end{aligned}$$

While by Lemma 5.7, $(\widehat{H}_{\rm S}+E)^{-1/2}H_{\rm f}(\widehat{H}_{\rm S}+E)^{-1/2}$ is a bounded operator. Hence, by virtue of (5.9), we have

$$\|H_{\rm f}(\widehat{H}_{\rm S}+E)^{-1}\Psi\| \leq C'\|\Psi\|$$

with some constant C'. Since $D(H_{\rm f})$ is dense, lemma follows. □

Next we show a relative bound with respect to $N_{\rm f}$. The procedure is similar to that of $H_{\rm f}$. We define $N_{\rm f}^0 : L^2(Q_0) \to L^2(Q_0)$ by

$$N_{\rm f}^0 :\phi_0(f_1)\cdots\phi_0(f_n)::= n :\phi_0(f_1)\cdots\phi_0(f_n): .$$

$$N_{\rm f}^0\mathbf{1} := 0.$$

Then

$$\Xi_t^* N_{\rm f}^0 = N_{\rm f}\Xi_t^*. \tag{5.10}$$

Let

$$\pi_0'(f) := i[N_{\rm f}^0, \phi_0(f)].$$

Thus we have

$$[\phi_0(f), \pi_0'(g)] = iq_0(\bar{f}, g). \tag{5.11}$$

For arbitrary $l \geq 0$, $\mathbf{J}_t$ maps $D(N_{\rm f}^l)$ onto itself and it holds that

$$N_{\rm f}^l\mathbf{J}_t = \Xi_0^* e^{-ie\phi_0(\mathbf{K}(t))}\left\{-e\pi_0'(\mathbf{K}(t)) + \frac{e^2}{2}q_0(\mathbf{K}(t),\mathbf{K}(t)) + N_{\rm f}^0\right\}^l \Xi_t \tag{5.12}$$

on $D(N_{\rm f}^l)$.

Lemma 5.9 *Let $\hat{\lambda}, \omega\hat{\lambda} \in L^2$. Then both of $e^{-t\widehat{H}_{\rm S}}$ and $e^{-t\widehat{H}_{\rm SA}}$ leave $C^\infty(N_{\rm f})$ invariant.*

Proof: Let $F, G \in C^\infty(N_{\rm f})$. Then

$$(N_{\rm f}^l F, e^{-t\widehat{H}_{\rm S}} G) = \int_M dX (F_0, N_{\rm f}^l \mathbf{J}_t G_t).$$

By virtue of (5.12), we easily have

$$\|N_{\rm f}^l \mathbf{J}_t G_t\| \le C_1(X) \|(N_{\rm f} + \mathbf{1})^l G_t\|.$$

Here $C_1(X)$ is a constant depending on $\|\mathbf{K}(t)\|^n$, $n = 0, 1, 2, ..., l$, and satisfies that, by (5.4),

$$\mathbb{E} C_1(X)^2 \le P(t),$$

where $P(t) = a_l t^l + \cdots + a_1 t + a_0$ with some $a_j \ge 0$. Hence

$$|(N_{\rm f}^l F, e^{-t\widehat{H}_{\rm S}} G)| \le P(t) \|F\| \|(N_{\rm f} + \mathbf{1})^l G\|.$$

Thus $e^{-t\widehat{H}_{\rm S}} G \in D(N_{\rm f}^l)$. Since l is arbitrary,

$$e^{-t\widehat{H}_{\rm S}} C^\infty(N_{\rm f}) \subset C^\infty(N_{\rm f}).$$

Similarly we have

$$(N_{\rm f}^l G, e^{-t\widehat{H}_{\rm SA}} G) = \int_M dX (N_{\rm f}^l F_0, e^{-ie\phi(L(X))} G_t) = \int_M dX (F_0, N_{\rm f}^l e^{-ie\phi(L(X))} G_t).$$

We also see

$$\|N_{\rm f}^l e^{-ie\phi(L(X))} G_t\| \le C_2(X) \|(N_{\rm f} + \mathbf{1})^l G_t(X)\|$$

with

$$\mathbb{E} C_2(X)^2 \le Q(t),$$

where $Q(t) = b_l t^l + \cdots + b_1 t + b_0$ with some $b_j \ge 0$. Hence

$$|(N_{\rm f}^l F, e^{-t\widehat{H}_{\rm SA}} G)| \le Q(t) \|F\| \|(N_{\rm f} + \mathbf{1})^l G\|.$$

Thus lemma follows in the same way as that of $e^{-t\widehat{H}_{\rm S}}$. $\square$

6 Proofs of theorems

The following lemma is well known.

Lemma 6.1 *(1) Let $\hat{\lambda}, \omega\hat{\lambda} \in L^2$. Then, for arbitrary $g \in \mathbb{R}$, $gH_{\rm spin} + V$ in the Schrödinger representation is relatively bounded with respect to $\widehat{H}_{\rm S}$ with a sufficiently small relative bound. (2) Let $\hat{\lambda}, \omega\hat{\lambda} \in L^2$. Then V is relatively bounded with respect to $\widehat{H}_{\rm SA}$ with a sufficiently small relative bound.*

Proof: See [18, Lemma 5.10]. $\square$

Lemma 6.2 *Let $\hat{\lambda}/\sqrt{\omega}, \hat{\lambda}, \omega\hat{\lambda} \in L^2$. Then $\widehat{H}_{\rm S}$ is self-adjoint on $D(\Delta) \cap D(H_{\rm f})$.*

Proof: Note that $D(\widehat{H}_{\rm S}) \supset D(\Delta) \cap D(H_{\rm f})$. Since $e^{-t\widehat{H}_{\rm S}}$ leaves $D(\Delta) \cap D(H_{\rm f})$ invariant, $D(\Delta) \cap D(H_{\rm f})$ is a core of $\widehat{H}_{\rm S}$. In terms of Lemmas 5.2 and 5.8, we have

$$\|\Delta\Psi\| + \|H_{\rm f}\Psi\| \le C(\|\widehat{H}_{\rm S}\Psi\| + \|\Psi\|) \tag{6.1}$$

with some constant C. Hence $\widehat{H}_{\rm S}$ is closed on $D(\Delta) \cap D(H_{\rm f})$. Thus $\widehat{H}_{\rm S}$ is self-adjoint on $D(\Delta) \cap D(H_{\rm f})$. By Lemma 4.5, it obeys $H_{\rm S}\lceil_{D(\Delta)\cap D(H_{\rm f})} \subset \widehat{H}_{\rm S}$. Then $H_{\rm S}$ is self-adjoint on $D(\Delta) \cap D(H_{\rm f})$. $\square$

Lemma 6.3 *Let $\hat{\lambda}, \omega\hat{\lambda} \in L^2$. Then $\widehat{H}_{\rm S}$ is essentially self-adjoint on $C^\infty(\Delta) \cap D(H_{\rm f}) \cap C^\infty(N_{\rm f})$.*

Proof: Note that $D(\widehat{H}_{\rm S}) \supset D(\Delta) \cap D(N_{\rm f}) \cap D(H_{\rm f})$. Since $e^{-t\widehat{H}_{\rm S}}$ leaves $C^\infty(\Delta) \cap D(H_{\rm f}) \cap C^\infty(N_{\rm f})$ invariant, $\widehat{H}_{\rm S}$ is essentially self-adjoint on $C^\infty(\Delta) \cap D(H_{\rm f}) \cap C^\infty(N_{\rm f})$. By Lemma 4.5, $\widehat{H}_{\rm S} \supset H_{\rm S}\lceil_{D(\Delta)\cap D(N_{\rm f})\cap D(H_{\rm f})}$. Thus $H_{\rm S}$ is essentially self-adjoint on $C^\infty(\Delta) \cap D(H_{\rm f}) \cap C^\infty(N_{\rm f})$. $\square$

Lemma 6.4 *Let $\hat{\lambda}, \omega\hat{\lambda} \in L^2$. Then $\widehat{H}_{\rm SA}$ is essentially self-adjoint on $C^\infty(\Delta) \cap C^\infty(N_{\rm f})$.*

Proof: Note that $D(\widehat{H}_{\rm SA}) \supset D(\Delta) \cap D(N)$. Since $e^{-t\widehat{H}_{\rm SA}}$ leaves $C^\infty(\Delta) \cap C^\infty(N_{\rm f})$ invariant, $\widehat{H}_{\rm SA}$ is essentially self-adjoint on $C^\infty(\Delta) \cap C^\infty(N_{\rm f})$. By Lemma 4.5, $\widehat{H}_{\rm SA} \supset H_{\rm S}\lceil_{D(\Delta)\cap D(N_{\rm f})}$. Thus $\widehat{H}_{\rm SA}$ is essentially self-adjoint on $C^\infty(\Delta) \cap C^\infty(N_{\rm f})$. $\square$

Proof of Theorem 3.1

By Lemmas 6.2 and 6.1, $H_{\rm S} + V + gH_{\rm spin}$ is self-adjoint on $D(\Delta) \cap D(H_{\rm f})$ and bounded from below. By Proposition 4.1, $H_{\rm PF}$ is self-adjoint on $D(\Delta) \cap D(H_{\rm rad})$. The essential self adjointness follows from Corollary 3.2. $\square$

Proof of Theorem 3.3

By Lemmas 6.3 and 6.1, $H_{\rm S} + V + gH_{\rm spin}$ is essentially self-adjoint on $C^\infty(\Delta) \cap D(H_{\rm f}) \cap C^\infty(N)$. Since unitary operator T in Proposition 4.1 maps $T^{-1} : C^\infty(\Delta) \cap D(H_{\rm f}) \cap C^\infty(N_{\rm f}) \to C^\infty(\Delta) \cap D(H_{\rm rad}) \cap C^\infty(N)$, $H_{\rm PF}$ is essentially self-adjoint on $C^\infty(\Delta) \cap D(H_{\rm rad}) \cap C^\infty(N)$. $\square$

Proof of Theorem 3.4

By Lemmas 6.4 and 6.1, $H_{\rm SA}+V$ is essentially self-adjoint on $C^\infty(\Delta) \cap C^\infty(N_{\rm f})$. Since unitary operator T in Proposition 4.1 maps $T^{-1} : C^\infty(\Delta) \cap C^\infty(N_{\rm f}) \to C^\infty(\Delta) \cap C^\infty(N)$, $H_{\rm A} + V$ is essentially self-adjoint on $C^\infty(\Delta) \cap C^\infty(N)$. $\square$

Acknowledgment I thank T. Ichinose for his encouragement and H. Spohn for his hospitality of Technische Universität München. I also thanks Graduiertenkolleg "Mathematik in ihrer Wechselbeziehung zur Physik" for financial support.

References

[1] A. Arai, Rigorous theory of spectra and radiation for a model in quantum electrodynamics, *J. Math. Phys.* **24** (1983), 1896-1910.

[2] A. Arai and M. Hirokawa, On the existence and uniqueness of ground states of a generalized spin-boson model, *J. Funct. Anal.* **151** (1997), 455–503.

[3] V. Bach, J. Fröhlich, I. M. Sigal, Quantum electrodynamics of confined nonrelativistic particles, *Adv. Math.* **137** (1998), 205-298.

[4] V. Bach, J. Fröhlich, I. M. Sigal, Renormalization group analysis of spectral problems in quantum field theory, *Adv. Math.* **137** (1998), 299-395.

[5] V. Bach, J. Fröhlich, I. M. Sigal, Spectral Analysis for systems of atoms and molecules coupled to the quantized radiation field, *Commun. Math. Phys.* **207** (1999), 249-290.

[6] V. Bach, J. Fröhlich, I. M. Sigal, and A. Soffer, Positive commutators and the spectrum of Pauli-Fierz Hamiltonian of atoms and molecules, *Comm. Math. Phys.* **207** (1999), 557-587.

[7] V. Betz, F. Hiroshima, J. Lorinczi, R. Minlos and H. Spohn, Gibbs measure associated with particle-field system, TU München preprint, 1999.

[8] R. Carmona, Pointwise bounds for Schrödinger operators, *Commun. Math. Phys.* **62** (1978), 97-106.

[9] J. Fröhlich, On the infrared problem in a model of scalar electrons and massless, scalar bosons, *Ann. Inst. Henri Poincaré* **19** (1973), 1-103.

[10] J. Fröhlich, Existence of dressed one electron states in a class of persistent models, *Fortschritte der Physik* **22** (1974), 159-198.

[11] C. Gérard, On the existence of ground states for massless Pauli-Fierz Hamiltonians, preprint, 1999.

[12] M. Griesemer, E. Lieb and M. Loss, Ground states in non-relativistic quantum electrodynamics, preprint.

[13] C. Fefferman, J. Fröhlich, G. M. Graf, Stability of ultraviolet-cutoff quantum electrodynamics with non-relativistic matter,*Commun. Math. Phys.* **190** (1997), 309-330.

[14] F. Hiroshima, Functional integral representation of a model in quantum electrodynamics, *Rev. Math. Phys.* **9** (1997), 489-530.

[15] F. Hiroshima, Ground states and spectrum of non-relativistic quantum electrodynamics, to be published in *Trans. Amer. Math. Soc.*

[16] F. Hiroshima, Ground states of a model in nonrelativistic quantum electrodynamics I, *J. Math. Phys.* **40** (1999), 6209-6222.

[17] F. Hiroshima, Ground states of a model in nonrelativistic quantum electrodynamics II, *J. Math. Phys.* **41** (2000), 661-674.

[18] F. Hiroshima, Essential self-adjointness of translation-invariant quantum field models for arbitrary coupling constants, *Commun. Math. Phys.* **211** (2000), 585-613.

[19] V. Jaksic and C. A. Pillet, On a model for quantum friction I. Fermi's golden rule and dynamics at zero temperature, Ann. Inst. Henri Poincaré, **62** (1995), 47-68.

[20] E. Lieb and M. Loss, Self-energy of electrons in non-perturbative QED, preprint, 1999.

[21] J. Lőrinczi, R. Minlos and H. Spohn, The infrared behaviour in Nelson's model of a quantum particle coupled to a massless scalar field, preprint, 2001.

[22] E. Nelson, Interaction of nonrelativistic particles with a quantized scalar field, *J. Math. Phys.* **5** (1964), 1190–1197.

[23] T. Okamoto and K. Yajima, Complex scaling technique in non-relativistic massive QED, *Ann. Int. Henri. Poincaré* **42** (1985), 311-327.

[24] W. Pauli and M. Fierz, Zur Theorie der Emmision langwelliger Lichtquanten, *Nuovo Cimento* **15** (1938), 167-188.

[25] M. Reed and B. Simon, *Method of Modern Mathematical Physics II*, Academic Press, 1975.

[26] B. Simon, *Functional integral and quantum physics*, Academic Press, 1979.

[27] H. Spohn, Ground state(s) of the spin-boson Hamiltonian, *Commun. Math. Phys.* **123** (1989), 277-304.

[28] H. Spohn, Ground state of quantum particle coupled to a scalar boson field, *Lett. Math. Phys.* **44** (1998), 9–16.