Bound States in Curved Quantum Layers

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Abstract

We consider a nonrelativistic quantum particle constrained to a curved layer of constant width built over a non-compact surface embedded in $\mathbb{R}^3$. We suppose that the latter is endowed with the geodesic polar coordinates and that the layer has the hard-wall boundary. Under the assumption that the surface curvatures vanish at infinity we find sufficient conditions which guarantee the existence of geometrically induced bound states.

Key-Words: waveguides, layers, constrained systems, Dirichlet Laplacian, bound states, surface geometry, curvature, integral curvatures, geodesic polar coordinates

1 Introduction

Relations between the geometry of a region Ω in $\mathbb{R}^n$, boundary conditions at $\partial\Omega$, and spectral properties of the corresponding Laplacian are one of the vintage problems of mathematical physics. Recent years brought new motivations and focused attention to aspects of the problem which attracted little attention earlier.

A strong impetus comes from mesoscopic physics, where new experimental techniques make it possible to fabricate semiconductor systems which can be regarded with a reasonable degree of accuracy as waveguides, resonators, etc., for effectively free quantum particles. Often potential barriers at their boundaries can be modeled as a hard wall, in which case it is natural to identify the system Hamiltonian – up to a constant which is usually unimportant – with the Dirichlet Laplacian, $-\Delta_D^\Omega$, defined as the Friedrichs extension – *cf.* Section 3.3. Moreover, the mentioned solid-state physics advances inspired new insights into the classical physics, because analogous problems involving Dirichlet Laplacian arise also in flat electromagnetic waveguides.

For more information about the physical background see [DE, LCM] and references therein.

On the mathematical side a new interesting effect is the binding due to the curvature, supposed to be nonzero and asymptotically vanishing, of an infinitely stretched tubular region in $\mathbb{R}^n$, $n = 2, 3$. Such "trapped modes" may be generated by other local perturbations of a straight tube as well – see, *e.g.*, [BGRS] – but in the bent-tube case they are of a purely quantum origin because there are no classical closed trajectories, apart of a zero measure set of initial conditions in the phase space.

More generally, quantum motion in the vicinity of a manifold with a potential constraint or Dirichlet condition were studied long time ago [JK, dC1, dC2, T] in formal attempts to justify quantization on submanifolds. For a thin neighbourhood one excludes the transverse part of the Hamiltonian which gives rise to normal oscillations and the Hamiltonian is replaced by a tangential operator on the submanifold with the energy appropriately renormalized. Interest to this problem has been renewed recently when time evolution around a compact n-dimensional manifold in $\mathbb{R}^{n+m}$ was treated in a rigorous way and compared with the corresponding classical dynamics [FH]. The confinement was realized by a harmonic potential transverse to the manifold and the thin-neighborhood limit was performed by means of a dilation procedure followed by averaging in the normal direction. If the normal bundle is trivial, which is the case, *e.g.*, for manifolds of codimension one, the resulting tangential Hamiltonian contains two terms; the first is proportional to the Laplace-Beltrami operator on the constraint manifold and the second is an effective potential which depends not only on the intrinsic quantities, but also on the external curvature of the constraint manifold. Notice also that if $\mathbb{R}^{n+m}$ is replaced by a manifold of the same dimension, the effective potential depends also on the curvature of this ambient space [M].

The said potential is important also in the situation when the width of the "fat manifold" is finite and fixed. This was first noticed for bent planar Dirichlet strips in the paper [EŠ] which was followed by numerous studies on which the existence conditions and properties of the geometrically induced discrete spectrum were further investigated – see, in particular, [GJ, DE, RB], the first two papers also for a generalization to curved tubes in $\mathbb{R}^3$. On the other hand, much less is known about other possible generalizations of this problem to higher dimensions starting from the physically interesting case of curved layers in $\mathbb{R}^3$.

This is the question we address in the present paper. While the strategy will be the same as in the work mentioned above, using suitable curvilinear coordinates to transform the Laplacian, the two-dimensional character of the underlying manifold bring new features. To characterize them briefly, recall that in the simplest $(1+1)$-case the effective potential is $-\frac{1}{4}\kappa^2$, where κ is the curvature, which is negative whenever the curvature is nonzero. In case of a layer, $n = 2$ and $m = 1$, which we consider here, the (leading term of the) effective potential is given by $-\frac{1}{4}(k_1 - k_2)^2$ – see the derivation of (3.12) – where k_1, k_2 are the principal curvatures of the surface. This expression may vanish also if the surface is locally spherical, $k_1 = k_2$, but the last relation cannot be valid everywhere at a non-compact surface unless the latter is a plane, $k_1 = k_2 = 0$. Thus the effective potential has again an attractive component, which now combines with a more complicated tangential operator – the surface Laplace-Beltrami – since in distinction to a curve the surface cannot be fully rectified. This makes the layer case richer and more interesting.

2 Survey of the Paper

The ultimate objective of this work is to set a list of sufficient conditions to guarantee the existence of curvature-induced bound states. We restrict ourselves naturally to non-compact layers only, since the spectrum of the Dirichlet Laplacian in a bounded region of $\mathbb{R}^n$ is always discrete [Dav, Chap. 6].

The layer configuration space Ω itself is properly defined in Section 3 as a tubular neighbourhood of width d built over a surface Σ embedded in $\mathbb{R}^3$ which is diffeomorphic to $\mathbb{R}^2$. To make it more visual, we can understand Ω as a part of $\mathbb{R}^3$ between a pair of parallel surfaces. From technical reasons we suppose from the beginning that the surface admits at least one *pole* from which we can parametrize the surface globally by geodesic polar coordinates. We stress already here that the existence of a pole in Σ is a strong geometric assumption and that there may be no poles in general [GM]. We introduce first quantities describing the layer geometry and formulate some basic assumptions. In the subsequent part, the Dirichlet Laplacian, $-\Delta_D^\Omega$, is expressed in terms of the couple $q = (q^1, q^2)$ of the surface (called also longitudinal) coordinates together with the normal (transverse) coordinate u.

In Section 4, we estimate the threshold of the essential spectrum of the Hamiltonian under the assumption $\langle\Sigma 0\rangle$ that the reference surface is *asymptotically planar* in the sense that its Gauss and mean curvatures vanish at large distances. We find that this part of spectrum is bounded from below by $\kappa_1^2 := \left(\frac{\pi}{d}\right)^2$, which is the lowest transverse-mode energy.

Section 5 is dedicated to the analysis of the discrete part of the spectrum. We find here three sufficient conditions and illustrate them on examples. Since these results leave open the existence question for thick layers of positive total Gauss curvature, we present in Section 6 an alternative method, which covers the case of asymptotically planar layers that are cylindrically symmetric. Finally, we conclude in Section 7 by an example of a layer which has no bound states; the reference surface here is not asymptotically planar.

To state here the main results of the paper we need to mention some assumptions which will be discussed in more detail below: $\langle\Sigma 1\rangle$ and $\langle\Sigma 2\rangle$ means respectively the integrability of the Gauss curvature K and the square of $\nabla_g M$, where M is the mean curvature, and $\langle\Omega 1\rangle$ requires the layer half-width to be less than the minimum normal curvature radius of Σ. The integral (total) curvatures corresponding to K and M are defined in (3.3).

Theorem 2.1. *Let Σ be a C^2-smooth complete simply connected non-compact surface with a pole embedded in $\mathbb{R}^3$. Let the layer Ω built over the surface be not self-intersecting. If the surface is not a plane but it is asymptotically planar, then any of the conditions*

- $\langle\Sigma 1\rangle$ *and the* total Gauss curvature is non-positive
- Σ *is C^3-smooth and the layer is* sufficiently thin
- Σ *is C^3-smooth, $\langle\Sigma 1\rangle$, $\langle\Sigma 2\rangle$, and the* total mean curvature is infinite
- $\langle\Sigma 1\rangle$ *and Σ is* cylindrically symmetric

is sufficient for the Laplace operator $-\Delta_D^\Omega$ to have at least one isolated eigenvalue of finite multiplicity below $\inf\sigma_{\mathrm{ess}}(-\Delta_D^\Omega)$ for all the layer half-widths satisfying $\langle\Omega 1\rangle$.

While this theorem covers various wide classes of layers, the list is not exhaustive. For instance, it remains to be clarified whether one can include also thick layers without cylindrical symmetry built over surfaces with strictly positive total Gauss curvature which, however, do not satisfy the assumption $\langle\Sigma 2\rangle$. Another open question is whether one can replace $\langle\Sigma 1\rangle$ by an assumption including the existence of the total Gauss curvature only, defined in the principal value sense. Finally, it is desirable to find existence results also for layers over more general surfaces which do not possess poles or are not diffeomorphic to $\mathbb{R}^2$.

Properties of the obtained curvature-induced bound states will be discussed elsewhere. Let us just mention that in analogy to bent strips [DE] one can perform the Birman-Schwinger analysis for slightly curved planar layers (weak-coupling regime) which yields the first term in the asymptotic expansion for the gap between the eigenvalue and the threshold of the essential spectrum. We also remark that the weak coupling analysis of bent "fat" manifolds is similar to that of a local one-sided deformation of a straight strip [BGRS] or planar layer [BEGK].

We use the standard component notation of the tensor analysis, the range of indices being $1, 2$ for Greek and $1, 2, 3$ for Latin. The indices are associated with the above mentioned coordinates by $(1, 2, 3) \leftrightarrow (q^1, q^2, u) \equiv (s, \vartheta, u)$. The partial derivatives are denoted by *commas*, however, we use also the *dot* notation for the derivatives w.r.t. s.

3 Preliminaries

Let Σ be a C^2-smooth surface in $\mathbb{R}^3$ which has at least one *pole*, *i.e.*, a point $o \in \Sigma$ such that the exponential mapping, $\exp_o : \mathrm{T}_o\Sigma \to \Sigma$, is a diffeomorphism. The existence of a pole in Σ is a nontrivial assumption which has important topological consequences. In particular, Σ is necessarily diffeomorphic to $\mathbb{R}^2$ and as such it is simply connected and non-compact. Using the *geodesic polar coordinates* we can parametrize the surface (with exception of the pole o) by a unique patch $p : \Sigma_0 \to \mathbb{R}^3$, where $\Sigma_0 := (0,\infty) \times S^1$. The tangent vectors $p_{,\mu} := \partial p/\partial q^\mu$ are linearly independent and their cross-product defines a unit normal field n on Σ.

Put $\Omega_0 := \Sigma_0 \times (-a,a)$. We define a *layer* $\Omega := \mathcal{L}(\Omega_0)$ of width $d = 2a > 0$ over the surface Σ by virtue of the mapping $\mathcal{L} : \Omega_0 \to \mathbb{R}^3$ which acts as (*cf.* [Sp3, Prob. 12 of Chap. 3])

$$\mathcal{L}(q,u) := p(q) + un(q). \tag{3.1}$$

3.1 The Surface Geometry

The induced surface metric in the geodesic polar coordinates has the diagonal form, $(g_{\mu\nu}) = \mathrm{diag}(1, r^2)$, where $r^2 \equiv g := \det(g_{\mu\nu})$ is the square of the Jacobian of the exponential mapping which satisfies the classical Jacobi equation

$$\ddot{r}(s,\vartheta) + K(s,\vartheta)\, r(s,\vartheta) = 0 \qquad \text{with} \qquad r(0,\vartheta) = 0,\ \dot{r}(0,\vartheta) = 1. \tag{3.2}$$

The *Gauss curvature* K, together with the *mean curvature* M, can be determined via the Weingarten tensor $h_\mu^{\ \nu}$ – *cf.* [Kli, Prop. 3.5.5].

By means of the invariant surface element, $d\Sigma := g^{\frac{1}{2}} d^2q$, we may introduce some global quantities characterizing Σ, namely the *total Gauss curvature* $\mathcal{K}$ and the *total mean curvature* $\mathcal{M}$ which are defined, respectively, by the integrals

$$\mathcal{K} := \int_\Sigma K d\Sigma \qquad \text{and} \qquad \mathcal{M}^2 := \int_\Sigma M^2 d\Sigma. \tag{3.3}$$

The latter always exists (it may be $+\infty$), while the former is well defined provided

$$\boxed{\langle\Sigma 1\rangle \quad K \in L^1(\Sigma_0, d\Sigma)}$$

If this condition is not satisfied, one can understand the above integral as the principal-value defined through the area restricted by the geodesic circle $p(s,\cdot)$ of radius $s \to \infty$. Assuming $\mathcal{K}$ to be finite, an integration of (3.2) yields the following useful estimate

$$\exists C > 0\ \forall s \in (0,\infty) : \quad \int_0^{2\pi} r(s,\vartheta)\, d\vartheta \le Cs. \tag{3.4}$$

The norm and the inner product in the Hilbert space $L^2(\Sigma_0, d\Sigma)$ will be indicated by the subscript "g".

3.2 The Layer Geometry

It is clear from the definition (3.1) that the metric tensor of the layer (as a manifold with boundary in $\mathbb{R}^3$) has the block form

$$(G_{ij}) = \begin{pmatrix} (G_{\mu\nu}) & 0 \\ 0 & 1 \end{pmatrix} \quad \text{with} \quad G_{\nu\mu} = (\delta_\nu^\sigma - uh_\nu{}^\sigma)(\delta_\sigma^\rho - uh_\sigma{}^\rho)g_{\rho\mu}. \qquad (3.5)$$

This formula is well suited for calculation of the determinant, $G := \det(G_{ij})$, because the eigenvalues of the matrix of the Weingarten map are the *principal curvatures* k_1, k_2, and $K = k_1 k_2$, $M = \frac{1}{2}(k_1 + k_2)$. Hence

$$G = g\left[(1-uk_1)(1-uk_2)\right]^2 = g(1 - 2Mu + Ku^2)^2. \qquad (3.6)$$

In particular, this expression defines through $d\Omega := G^{\frac{1}{2}} d^2q\, du$ the volume element of Ω.

Henceforth, we shall assume

$\boxed{\langle\Omega 0\rangle \quad \Omega \text{ is not self-intersecting}}$ *i.e.*, $\mathcal{L}$ is injective.

We have to require also that $\mathcal{L}$ is a diffeomorphism. In view of the regularity assumptions imposed on Σ and the inverse function theorem, it is equivalent to assuming that $1 - 2Mu + Ku^2$ does not vanish on Ω_0, which can be guaranteed by imposing a restriction on the layer thickness:

$\boxed{\langle\Omega 1\rangle \quad a < \rho_m := \left(\max\left\{\|k_1\|_\infty, \|k_2\|_\infty\right\}\right)^{-1}}$

The number ρ_m is naturally interpreted as the *minimal normal curvature radius* of Σ (for planar surfaces one can put $\rho_m := \infty$). It follows from (3.5) that $C_- \leq 1 - 2Mu + Ku^2 \leq C_+$ holds with $C_\pm := \left(1 \pm a\rho_m^{-1}\right)^2$. The lower bound explains why we assume $\langle\Omega 1\rangle$ (together with $\langle\Omega 0\rangle$) to get the global diffeomorphism. On the other hand, the supremum norms in the definition of ρ_m are necessarily finite since a meaningful layer must have a non-zero width. Another consequence of the considerations is that under the assumption $\langle\Omega 1\rangle$, $G_{\mu\nu}$ can be immediately estimated by the surface metric,

$$C_- g_{\mu\nu} \leq G_{\mu\nu} \leq C_+ g_{\mu\nu} \qquad \text{with} \qquad 0 < C_- \leq 1 \leq C_+ < 4. \qquad (3.7)$$

Remark. We stress the following which will be supposed through all the paper but will not be always referred to hereafter:

- We consider surfaces which can be parametrized by means of the geodesic polar coordinates. This requires the existence of at least one pole.

- ∘ Since Σ is assumed to be of class C^2, the surface curvatures K, M are C^0 and as such bounded locally.
- ∘ Moreover, since we assume layers with non-zero widths, the principal curvatures have to be bounded uniformly on all Σ_0 due to $\langle\Omega 1\rangle$. By virtue of the relation between k_1, k_2 and K, M, the same is true for the latter.

3.3 The Hamiltonian

After these geometric preliminaries let us define the Hamiltonian of our model. We consider a nonrelativistic spinless particle confined to Ω which is free within it and suppose that the boundary of the layer is a hard wall, *i.e.*, the wavefunctions satisfy the Dirichlet boundary condition there. For the sake of simplicity we set Planck's constant $\hbar = 1$ and the mass of the particle $m = \frac{1}{2}$. Then the Hamiltonian can be identified with the Dirichlet Laplacian $-\Delta_D^\Omega$ on $L^2(\Omega)$, which is defined for an open set $\Omega \subset \mathbb{R}^3$ as the Friedrichs extension of the free Laplacian with the domain defined initially on $C_0^\infty(\Omega)$ – *cf.* [RS4, Sec. XIII.15] or [Dav, Chap. 6]. The domain of the closure of the corresponding quadratic form is the Sobolev space $W_0^{1,2}(\Omega)$.

A natural way to investigate this operator is to pass to the coordinates (q, u) in which it acquires the Laplace-Beltrami form ($G_{ij}G^{jk} := \delta_i^k$)

$$H := -G^{-\frac{1}{2}}\partial_i G^{\frac{1}{2}} G^{ij}\partial_j \qquad \text{on} \qquad L^2(\Omega_0, G^{\frac{1}{2}} d^2q\, du). \tag{3.8}$$

This coordinate change is nothing else than the unitary transformation

$$U : L^2(\Omega) \to L^2(\Omega_0, d\Omega) : \{\psi \mapsto U\psi := \psi \circ \mathcal{L}\}$$

which relates the two operators by $H = U(-\Delta_D^\Omega)U^{-1}$. If Σ is not C^3-smooth, the operator H has to be understood in the form sense

$$Q[\psi] := \|H^{\frac{1}{2}}\psi\|_G^2 = (\psi_{,i}, G^{ij}\psi_{,j})_G, \quad \operatorname{Dom} Q = W_0^{1,2}(\Omega_0, d\Omega). \tag{3.9}$$

Here the subscript "G" indicates the norm and the inner product in the Hilbert space of (3.8). Employing the block form (3.5) of G_{ij}, we can split H into a sum of two parts, $H = H_1 + H_2$, given by

$$H_1 \ := \ -G^{-\frac{1}{2}}\partial_\mu G^{\frac{1}{2}} G^{\mu\nu}\partial_\nu = -\partial_\mu G^{\mu\nu}\partial_\nu - 2F_{,\mu}G^{\mu\nu}\partial_\nu \tag{3.10}$$

$$H_2 \ := \ -G^{-\frac{1}{2}}\partial_3 G^{\frac{1}{2}}\partial_3 = -\partial_3^2 - 2\,\frac{Ku - M}{1 - 2Mu + Ku^2}\,\partial_3\,, \tag{3.11}$$

where we have introduced $F := \ln G^{\frac{1}{4}}$ and expressed $F_{,3}$ explicitly for H_2 .

At the same time, it is useful to have an alternative form of the Hamiltonian which has the factor $1 - 2Mu + Ku^2$ removed from the weight $G^{\frac{1}{2}}$ of the inner product. It is obtained by another unitary transformation,

$$\hat{U} : L^2(\Omega_0, d\Omega) \to L^2(\Omega_0, d\Sigma\, du) : \{\psi \mapsto \hat{U}\psi := (1 - 2Mu + Ku^2)^{\frac{1}{2}}\psi\},$$

which leads to the unitarily equivalent operator $\hat{H} := \hat{U} H \hat{U}^{-1}$. This operator makes sense if we impose a stronger regularity assumption on Σ, namely that the latter is piecewise C^4-smooth (or C^3 if $\hat{H}$ is considered in the form sense). The operator $\hat{H}$ can be rewritten by means of an effective potential V using $J := \frac{1}{2} \ln(1 - 2Mu + Ku^2)$ as follows

$$\hat{H} = -g^{-\frac{1}{2}} \partial_i g^{\frac{1}{2}} G^{ij} \partial_j + V, \qquad V = g^{-\frac{1}{2}} (g^{\frac{1}{2}} G^{ij} J_{,j})_{,i} + J_{,i} G^{ij} J_{,j}$$

and again, employing the particular form of G_{ij}, the operator $\hat{H}$ can be split into a sum, $\hat{H}_1 + \hat{H}_2$. The first operator is defined by the part of $\hat{H}$ where one sums over the Greek indices and

$$\hat{H}_2 = -\partial_3^2 + V_2, \qquad V_2 = \frac{K - M^2}{(1 - 2Mu + Ku^2)^2}.$$

To motivate the considerations of the following sections let us look at this transformed operator from a heuristic point if view. While the operator $\hat{H}_1 + V_2$ depends on all the three coordinates, in thin layers ($a \ll \rho_m$) its leading term depends up to an error $\mathcal{O}(a\rho_m^{-1})$ on the longitudinal coordinates q only. One can estimate the former in the form sense by means of (3.7) and use the fact that $C_\pm = 1 + \mathcal{O}(a\rho_m^{-1})$. The transverse coordinate u is isolated in $\hat{H}_2 - V_2 = -\partial_3^2$, so up to higher-order terms in a the Hamiltonian decouples into a sum of the operators

$$H_q := -g^{-\frac{1}{2}} \partial_\mu g^{\frac{1}{2}} g^{\mu\nu} \partial_\nu + K - M^2 \qquad \text{and} \qquad H_u := -\partial_3^2\,, \tag{3.12}$$

the first one being the Laplace-Beltrami operator of Σ, except for the additional potential $K - M^2$ which can be rewritten by means of the principal curvatures as $-\frac{1}{4}(k_1 - k_2)^2$. This is the attractive interaction mentioned in the introduction. Let us remark that similar Laplace-Beltrami operators penalized by a quadratic function of the curvature lead on compact surfaces to interesting isoperimetric problems [H, HL, EHL, F].

In what follows we shall use the family of eigenfunctions $\{\chi_n\}_{n=1}^\infty$ of the transverse operator $(-\partial_3^2)_D$ which is given by

$$\chi_n := \begin{cases} \sqrt{\frac{2}{d}} \cos \kappa_n u & \text{if } n \text{ is odd,} \\ \sqrt{\frac{2}{d}} \sin \kappa_n u & \text{if } n \text{ is even.} \end{cases}$$

Here $\kappa_n^2 := (\kappa_1 n)^2$ with $\kappa_1 := \pi/d$ are the corresponding eigenvalues.

4 Essential Spectrum

The essential spectrum of a planar layer ($K, M \equiv 0$) is clearly $[\kappa_1^2, \infty)$. By a bracketing argument [DEK, Sec. 3.1] and using an appropriate Weyl sequence, it is easy to see that the same remains true if Ω is obtained by a compactly supported deformation of a planar layer. In this section we will prove the inclusion $\sigma_{\mathrm{ess}}(-\Delta_D^\Omega) \subseteq [\kappa_1^2, \infty)$ under the assumption that the surface Σ is *asymptotically planar* in the sense

$$\boxed{\langle\Sigma 0\rangle \quad K, M \to 0 \qquad \text{as} \quad s \to \infty}$$

Theorem 4.1. *Suppose $\langle\Omega 0\rangle$, $\langle\Omega 1\rangle$ and assume that the surface is asymptotically planar $\langle\Sigma 0\rangle$. Then*

$$\inf \sigma_{\mathrm{ess}}(-\Delta_D^\Omega) \geq \kappa_1^2.$$

PROOF: We divide the layer Ω into an exterior and interior part by putting $\Omega_{\mathrm{ext}} := \mathcal{L}(\Omega_{0,s_0})$ and $\Omega_{\mathrm{int}} := \Omega \setminus \overline{\Omega}_{\mathrm{ext}}$, respectively, where $\Omega_{0,s_0} := \Sigma_{0,s_0} \times (-a, a)$, $\Sigma_{0,s_0} := (s_0, \infty) \times S^1$ for some $s_0 > 0$. Imposing the Neumann boundary condition at the common boundary of the two parts, $s = s_0$, we arrive at the decoupled Hamiltonian $H^N = H_{\mathrm{int}}^N \oplus H_{\mathrm{ext}}^N$. More precisely, it is obtained as the operator associated with the quadratic form Q^N acting as (3.9), however with the domain $\operatorname{Dom} Q^N := \operatorname{Dom} Q_{\mathrm{int}}^N \oplus \operatorname{Dom} Q_{\mathrm{ext}}^N$ where

$$\operatorname{Dom} Q_\omega^N := \{\psi \in W^{1,2}(\Omega_\omega, d\Omega) \mid \psi(\cdot, \pm a) = 0\}, \qquad \omega \in \{\mathrm{int}, \mathrm{ext}\}.$$

Since $H \geq H^N$ and the spectrum of H_{int}^N is purely discrete [Dav, Chap. 7], the minimax principle gives the estimate $\inf \sigma_{\mathrm{ess}}(H) \geq \inf \sigma_{\mathrm{ess}}(H_{\mathrm{ext}}^N) \geq \inf \sigma(H_{\mathrm{ext}}^N)$. Hence it is sufficient to find a lower bound on H_{ext}^N. However, by virtue of (3.9) and (3.5), we have for all $\psi \in \operatorname{Dom} Q_{\mathrm{ext}}^N$:

$$\begin{aligned}
Q_{\mathrm{ext}}^N[\psi] \quad &\geq \quad \|\psi_{,3}\|_{G,\mathrm{ext}}^2 \geq \inf_{\Omega_{0,s_0}} \{1 - 2Mu + Ku^2\} \, \|\psi_{,3}\|_{L^2(\Omega_0, d\Sigma du),\mathrm{ext}}^2 \\
&\geq \quad \Big(1 - \sup_{\Sigma_{0,s_0}} \{2a|M| + a^2|K|\}\Big) \, \kappa_1^2 \, \|\psi\|_{L^2(\Omega_0, d\Sigma du),\mathrm{ext}}^2 \\
&\geq \quad \frac{1 - \sup_{\Sigma_{0,s_0}} \{2a|M| + a^2|K|\}}{1 + \sup_{\Sigma_{0,s_0}} \{2a|M| + a^2|K|\}} \, \kappa_1^2 \, \|\psi\|_{G,\mathrm{ext}}^2 \\
&=: \quad (1 + \epsilon(s_0)) \, \kappa_1^2 \, \|\psi\|_{G,\mathrm{ext}}^2,
\end{aligned}$$

where ϵ denotes a function which goes to zero as $s_0 \to \infty$ due to $\langle\Sigma 0\rangle$. The subscript "ext" indicates the restriction of the norm to the exterior part. In the second line we have used $(-\partial_3^2)_D \geq \kappa_1^2$. The claim then easily follows by the fact that s_0 can be chosen arbitrarily large. $\square$

Remark. This threshold estimate is sufficient for the subsequent investigation of the discrete spectrum which is our goal in this paper. In order to show that all energies above κ_1^2 belong to the spectrum, one has to construct an appropriate Weyl sequence to check the opposite conclusion $\sigma_{\mathrm{ess}}(-\Delta_D^\Omega) \supseteq [\kappa_1^2, \infty)$. This can be done under an assumption stronger than $\langle\Sigma 0\rangle$ which involves derivatives of the Weingarten tensor as well.

5 Discrete Spectrum

The aim of this section is to prove three different conditions sufficient for the Hamiltonian to have a non-empty spectrum below κ_1^2. Since we have shown that the essential spectrum does not start below this value for the layers built over asymptotically planar surfaces, the conditions yields immediately the existence of curvature-induced bound states. All the proofs here are based on the variational idea of finding a trial function Ψ from the form domain of H such that

$$\tilde{Q}[\Psi] := Q[\Psi] - \kappa_1^2 \, \|\Psi\|_G^2 < 0.$$

It is convenient to split Q into two parts, $Q = Q_1 + Q_2$, which are associated with H_1 and H_2 of (3.10) and (3.11), respectively.

A powerful method in these situation is to construct a trial function by deforming the transverse-threshold resonance wavefunction separately in the central and tail regions. The idea goes back to Goldstone and Jaffe [GJ], see also [DE, Thm. 2.1], [RB] and [DEK, Sec. 3.2].

Theorem 5.1. *Assume $\langle\Omega 0\rangle$, $\langle\Omega 1\rangle$, $\langle\Sigma 1\rangle$, and suppose that Σ is not planar. If the surface has a non-positive total Gauss curvature,* i.e., $\mathcal{K} \le 0$, *then*

$$\inf \sigma(-\Delta_D^\Omega) < \kappa_1^2.$$

Proof: We begin the construction of Ψ by considering a radially symmetric function $\psi(s,\vartheta,u) := \varphi(s)\chi_1(u)$ where φ is arbitrary for a moment. Employing the explicit form (3.11) of H_2 we get immediately

$$Q_2[\psi] - \kappa_1^2 \|\psi\|_G^2 = (\varphi, K\varphi)_g, \tag{5.1}$$

while the "longitudinal kinetic part" $Q_1(\psi)$ can be estimated by virtue of (3.7) and (3.4) as

$$Q_1[\psi] \le C_1 \int_0^\infty |\dot{\varphi}(s)|^2 s \, ds. \tag{5.2}$$

The r.h.s. of this inequality depends on the surface geometry through the constant $C_1 := (C_+/C_-)^2 C$ only. To make this integral arbitrarily small we replace φ by the family $\{\varphi_\sigma : \sigma \in (0,1]\}$ of elements which are equal to 1 on a compact set, $s \le s_0$, for some $s_0 > 0$, and outside they are given by scaled Macdonald functions [AS, Sec. 9.6]:

$$\varphi_\sigma(s) := \min\left\{1, \frac{K_0(\sigma s)}{K_0(\sigma s_0)}\right\}.$$

Since K_0 is strictly decreasing, the corresponding $\psi_\sigma := \varphi_\sigma \chi_1$ will not be smooth at $s = s_0$ but it remains continuous, hence it is an admissible trial function as an element of $\operatorname{Dom} Q$. Using the properties of the Macdonald function [AS, Sec. 9.6] and [GR, 5.54], it is now easy to verify that for σs_0 small enough

$$\exists C_2 > 0 : \qquad \int_0^\infty |\dot{\varphi}_\sigma(s)|^2 s\, ds < \frac{C_2}{|\ln \sigma s_0|} \tag{5.3}$$

and therefore $Q_1[\psi_\sigma] \to 0+$ as $\sigma \to 0+$. On the other hand, since we assume $\langle \Sigma 1 \rangle$ and $|\varphi_\sigma| \leq 1$ together with $\varphi_\sigma \to 1-$ pointwise as $\sigma \to 0+$, we get by the dominated convergence theorem that (5.1) (after the replacement $\psi \mapsto \psi_\sigma$) converges to $\mathcal{K}$. Thus, by choosing σ small enough, $\tilde{Q}[\psi_\sigma]$ can be made strictly negative if the total Gauss curvature is strictly negative too.

In order to deal with the case $\mathcal{K} = 0$, in analogy to [GJ] we construct the trial function by a small deformation of ψ_σ in the central region. We set $\Psi_{\sigma,\varepsilon} := \psi_\sigma + \varepsilon\Theta$ where $\Theta(q,u) := j(q)u\chi_1(u)$ with $j \in C_0^\infty((0,s_0) \times S^1)$. Since Θ is evidently a function from $\operatorname{Dom} Q$ as well, we can write

$$\tilde{Q}[\Psi_{\sigma,\varepsilon}] = \tilde{Q}[\psi_\sigma] + 2\varepsilon\tilde{Q}(\Theta,\psi_\sigma) + \varepsilon^2\tilde{Q}[\Theta]. \tag{5.4}$$

An explicit calculation where one employs the fact that the scaling acts out of the support of the localization function j yields: $\tilde{Q}(\Theta,\psi_\sigma) = -(j,M)_g$, which can be made non-zero by choosing j supported on a compact where M does not change sign. Let us stress that it is independent of σ, because $\varphi_\sigma = 1$ on $\operatorname{supp} j$; the same is true for $\tilde{Q}[\Theta]$. Now such a compact surely exists because it is supposed that Σ is not a plane and we can take the parameter s_0 arbitrarily large. If we choose now the sign of ε in such a way that the second term on the r.h.s. of (5.4) is negative, then also the sum with the last term will be negative for sufficiently small ε, and we can choose σ so small that $\tilde{Q}(\Psi_{\sigma,\varepsilon}) < 0$ because $\tilde{Q}(\psi_\sigma) \to \mathcal{K} = 0$ as $\sigma \to 0+$ here. $\square$

Remarks. (a) The special choice of the Macdonald function K_0 for the mollifier φ is not indispensable. In analogy to [GJ] or [DE, Thm. 2.1] we need a family of suitable functions scaled exterior to $(0,s_0)$ in such a way that the integral (5.2) tends to zero as $\sigma \to 0+$. However, since this integral contains the extra factor s (the relic of integration in a higher dimension) we have to be more careful about the decay properties. We have adopted for this purpose the mollifier employed in [EV, BCEZ], which is the most natural in a sense, because it employs the Green function kernel of the free 2-dimensional Laplacian at zero energy. Nevertheless, we would have succeeded equally if we had chosen for the scaled tail, *e.g.*, a compactly supported function similar to that of the proof of Theorem 6.2.

(b) In the case $\mathcal{K} = 0$ we have not used the deformation proposed in [DE]: $\tilde{\Theta} := \tilde{\jmath}^2(H - \kappa_1^2)\psi_\sigma$ with $\tilde{\jmath} \in C_0^\infty((0,s_0) \times S^1 \times (-a,a))$, because it requires an extra condition on the surface regularity. The analogous condition in the strip case has been forgotten in [DE, Thm. 2.1]. Moreover, the localization function j used here is simpler since it is independent of u.

A class of layers to which the above theorem applies is represented by those built over *Cartan-Hadamard surfaces*, *i.e.*, geodesically complete simply connected non-compact surfaces with non-positive Gauss curvature. In view of the Cartan-Hadamard theorem [Kli, Thm. 6.6.4] each point is a pole and we can therefore construct infinitely many geodesic polar coordinate systems. Excluding the trivial planar case, the total Gauss curvature is always strictly negative and so all these layers possess at least one bound state provided they are asymptotically planar, $\mathcal{K}$ is finite, and the assumptions $\langle\Omega 0\rangle$, $\langle\Omega 1\rangle$ are satisfied.

EXAMPLE 1 (Hyperbolic Paraboloid). The simple quadric given in $\mathbb{R}^3$ by the equation $z = x^2 - y^2$ is an asymptotically planar surface with $\mathcal{K} = -2\pi$.

EXAMPLE 2 (Monkey Saddle). Take $z = x^3 - 3xy^2$. One can again check that $\langle\Sigma 0\rangle$ holds true and the total Gauss curvature now equals -4π.

A family of layers of the limit case $\mathcal{K} = 0$ was investigated in [DEK]. We consider there compactly supported deformations of a planar layer for which the zero value of $\mathcal{K}$ follows at once by the Gauss-Bonnet theorem. If such a deformed plane contains at least one pole, all the spectral results are trivial consequences of the present Theorems 4.1 and 5.1. On the other hand, the results of [DEK] are more general in the sense that due to the compact support assumption the technique works without the requirement on the existence of a pole.

EXAMPLE 3 (Compactly Perturbed Plane without Poles). Suppose that a plane with a circular hole is connected via a cylindrical tube perpendicular to it with a pierced sphere. Both interfaces can be made as smooth as needed. If the tube is sufficiently long there is only one pole o provided the surface has a cylindrical symmetry w.r.t. the axis of the tube; it coincides with the intersection of the axis with the sphere. If we break now the symmetry by taking an ellipsoid instead of the sphere, we destroy the injectivity of the exponential mapping $\exp_o$ without creating new poles.

The Goldstone-Jaffe trick of choosing the ground state of the transverse operator as the generalized annulator of the shifted energy form $\tilde{Q}$ has proven its usefulness as a robust argument for demonstrating the existence of bound states. However, in the present context it reaches its limits because the above proof does not work for layers built over surfaces with positive total curvature, for instance:

EXAMPLE 4 (Elliptic Paraboloid). The surfaces $z = (x/x_0)^2 + (y/y_0)^2$ with $x_0, y_0 > 0$ are asymptotically planar but $\mathcal{K} = 2\pi > 0$. They always contain two poles given by its umbilics which coincide if it is a paraboloid of revolution.

On the other hand, due to the heuristic argument based on (3.12) one expects existence of bound states in any non-planar layer thin enough. This is indeed true. This fact together with another sufficient condition are established in the next theorem.

Theorem 5.2. *Assume $\langle\Omega 0\rangle$, $\langle\Omega 1\rangle$, and suppose that Σ is C^3-smooth, non-planar and obeys in addition*

$$\boxed{\langle\Sigma 2\rangle \quad \nabla_g M \in L^2(\Sigma_0, d\Sigma)}$$

Then $\inf\sigma(-\Delta_D^\Omega) < \kappa_1^2$ if one of the following two conditions is satisfied:

(a) *the layer is sufficiently thin,* i.e., *d is small enough,*

(b) *$\langle\Sigma 1\rangle$ and the total mean curvature is infinite,* i.e., *$\mathcal{M} = \infty$.*

For brevity we have introduced here the non-component notation ∇_g for the covariant derivative on Σ.

PROOF: We use $\Psi_\sigma(s,\vartheta,u) := (1+M(s,\vartheta)u)\,\psi_\sigma(s,u)$, where $\psi_\sigma = \varphi_\sigma\chi_1$ is the trial function defined in the first part of the proof of Theorem 5.1. Under the stated regularity assumption, Ψ_σ is an admissible trial function, *i.e.*, it belongs to $\operatorname{Dom} Q$. Using (3.7) together with Minkovski's inequality and (3.11), we get

$$Q_1[\Psi_\sigma] \le 2(C_+/C_-)^2 \left((1+a\|M\|_\infty)^2\,\|\dot\varphi_\sigma\|_g^2 + a^2\|\varphi_\sigma\nabla_g M\|_g^2\right)$$

$$Q_2[\Psi_\sigma] - \kappa_1^2\|\Psi_\sigma\|_G^2 = \left(\varphi_\sigma,(K-M^2)\varphi_\sigma\right)_g + \frac{\pi^2-6}{12\kappa_1^2}\left(\varphi_\sigma, KM^2\varphi_\sigma\right)_g.$$

We start by checking the second sufficient condition. We recall that due to $\langle\Omega 1\rangle$, K and M are uniformly bounded. Thus, thanks to $\langle\Sigma 2\rangle$ and the hypotheses assumed in (b), it follows that $\tilde{Q}[\Psi_\sigma] \to -\infty$ as $\sigma \to 0+$.

We pass now to the first sufficient condition. Since $K - M^2$ is negative – *cf.* (3.12) – continuous and the surface is supposed to be non-planar, the first term at the r.h.s. of the second line is strictly negative, say $-c^2$, for sufficiently large value of s_0 (the radius of the disc where $\psi_\sigma = \chi_1$). On the other hand, $\|\dot\varphi_\sigma\|_g$ is estimated by (5.3), so we can choose σ so small that it is less than $c^2/3$. Now we choose the layer half-width a so small that the sum of the remaining terms of the estimated $\tilde{Q}[\Psi_\sigma]$ is less than $c^2/3$ as well. For this we recall that κ_1^{-2} is proportional to a^2. Hence $\tilde{Q}[\Psi_\sigma] \le -c^2/3 < 0$ for σ, d small enough. □

Remark. In order to obtain the first sufficient condition, one can replace $\langle\Sigma 2\rangle$ by an assumption on the boundedness of $\nabla_g M$. Moreover, if we had used the compactly supported function φ_n from the proof of Theorem 6.2 below instead of φ_σ, it would have been sufficient to assume that $\nabla_g M$ was bounded locally only, which is exactly the situation when Σ is of class C^3. This is why $\langle\Sigma 2\rangle$ is not included in the thin layer case of Theorem 2.1.

We believe that the hypothesis $\langle\Sigma 2\rangle$ is technical – *cf.* Example 6. Even with it, however, the class of layers possessing bound states without any restriction on the layer thickness other than $\langle\Omega 1\rangle$ is extended significantly. For instance, it is an easy exercise to verify that all the conditions of Theorem 5.2 (b) are fulfilled for the elliptic paraboloids and many other surfaces with a positive total Gauss curvature. Removing this technical condition is still an open question except for layers endowed with the cylindrical symmetry which we shall discuss below.

6 Cylindrically Symmetric Layers

Consider now layers which are invariant w.r.t. rotations around a fixed axis in $\mathbb{R}^3$. We may thus suppose that Σ is a surface of revolution parametrized by $p:\Sigma_0 \to \mathbb{R}^3$,

$$p(s,\vartheta) := (r(s)\cos\vartheta, r(s)\sin\vartheta, z(s)) , \qquad \text{where} \quad r,z \in C^2((0,\infty)) , \ r>0.$$

It will be the geodesic polar coordinate chart if we impose the following condition on the *canonical* parametrization,

$$\dot{r}^2+\dot{z}^2=1; \qquad \text{then also} \qquad \dot{r}\ddot{r}+\dot{z}\ddot{z}=0. \tag{6.1}$$

An explicit calculation yields the diagonal form of the Weingarten tensor, $(h_\mu{}^\nu) = \operatorname{diag}(k_s,k_\vartheta)$, with the principal curvatures $k_s = \dot{r}\ddot{z}-\ddot{r}\dot{z}$ and $k_\vartheta = \dot{z}r^{-1}$. In fact, it is sufficient to know the function $s \mapsto k_s(s)$ only, since r,z can be constructed from the relations

$$\begin{aligned} r(s) &= \int_0^s \cos b(\xi)\, d\xi \\ z(s) &= \int_0^s \sin b(\xi)\, d\xi \end{aligned} \qquad \text{with} \qquad b(s) := \int_0^s k_s(\xi)\, d\xi. \tag{6.2}$$

Recall that by Theorem 5.1 the spectrum bottom of any layer is strictly less than the first transverse eigenvalue provided $\mathcal{K} \le 0$. However, only the case $\mathcal{K}=0$ is relevant to the present situation of surfaces of revolution, because by the Gauss-Bonnet theorem (see also (3.2))

$$\mathcal{K}+2\pi\dot{r}(\infty) = 2\pi, \qquad \text{where} \quad \dot{r}(\infty) := \lim_{s\to\infty} r(s), \tag{6.3}$$

and $\dot{r}(\infty)>1$ is not allowed because of (6.1). Notice, on the other hand, that $\dot{r}(\infty)$ always exists since the existence of the total Gauss curvature is supposed. Moreover, the positivity of r requires $\mathcal{K}\le 2\pi$.

The goal of this section is to show that in the present special case of symmetric layers $\inf\sigma(-\Delta_D^\Omega) < \kappa_1^2$ holds true also for all admissible strictly positive values of $\mathcal{K}$, irrespective of the layer thickness. Our argument requires to exclude here the extreme case $\mathcal{K}=0$ for which the result is already known, without any symmetry assumption. Hereafter we will therefore assume that $0 \leq \dot{r}(\infty) < 1$. It follows that there exist $0 < \delta' < \frac{1}{2}$ and $s_0 > 0$ such that for all $s \geq s_0$ one has $-\delta' \leq \dot{r}(s) \leq 1 - \delta'$. Using now the explicit dependence of k_ϑ on $r, \dot{z}$ and (6.1), we obtain the essential ingredients of our strategy:

Lemma 6.1. *Assume $\mathcal{K} > 0$. There exist $\delta > 0$ and $s_0 > 0$ such that*

$$\forall s \geq s_0: \quad \frac{\delta}{r(s)} \leq |k_\vartheta(s)| \leq \frac{1}{r(s)} \quad \textit{and} \quad k_\vartheta(s) \textit{ does not change sign.}$$

In particular, employing (3.4), it follows that k_ϑ is not integrable in $L^1(\mathbb{R}^+)$. On the other hand, the meridian curvature k_s is integrable under the assumption $\langle\Sigma 1\rangle$, which is seen by the regularity properties imposed on p and the following estimate

$$\infty > \int_0^\infty |K(s)|\, r(s)\, ds \geq \int_{s_0}^\infty |k_s(s) k_\vartheta(s)|\, r(s)\, ds \geq \delta \int_{s_0}^\infty |k_s(s)|\, ds.$$

This is the essence of what we are going to use in our method. Even if M may decay at infinity it is not negligible in the integral sense there. However, K is supposed to be integrable and it will enable us to eliminate the unpleasant contribution of the corresponding total curvature – *cf.* (5.1) – by going to large distances by means of a family of trial functions supported there.

Theorem 6.2. *Assume $\langle\Omega 0\rangle$, $\langle\Omega 1\rangle$, $\langle\Sigma 1\rangle$, and suppose that Σ is a surface of revolution. Then* $\inf \sigma(-\Delta_D^\Omega) < \kappa_1^2$.

PROOF: Since the result for $\mathcal{K} = 0$ is included in Theorem 5.1, we suppose $\mathcal{K} > 0$ in the following. We use $\Psi_{n,\varepsilon}(s,u) := (\varphi_n(s) + \varepsilon\phi_n(s)u)\chi_1(u)$, where ε will be specified later and φ_n, ϕ_n are functions "localized at infinity" as $n \to \infty$. They are defined in the following way: Consider three sequences $b_1, b_2, b_3 : \mathbb{N} \to \mathbb{N}$ such that $0 < b_1 < b_2 < b_3$ and $b_1(n) \to \infty$ as $n \to \infty$. We set

$$\varphi_n(s) := \frac{\ln(s/b_i)}{\ln(b_j/b_i)}, \quad (i,j) \in \{(1,2),(3,2)\}, \qquad \text{and} \qquad \phi_n(s) := \frac{\varphi_n(s)}{s}$$

if $\min\{b_i, b_j\} < s \leq \max\{b_i, b_j\}$, and assume that φ_n, ϕ_n are zero elsewhere. Defined in this way the functions are not smooth at the matching points, however, $\Psi_{n,\varepsilon}$ still belongs to $\operatorname{Dom} Q$ because they are continuous and of a compact support for each $n \in \mathbb{N}$. Next we note that they are positive and uniformly bounded (the maximum of ϕ_n is even decreasing as $n \to \infty$).

Using (3.7) and (3.4) we can estimate the longitudinal kinetic parts of $\tilde{Q}$ – *cf.* also (5.2) – by one-dimensional integrals

$$Q_1[\varphi_n\chi_1] \leq C_1 \int_0^\infty \dot{\varphi}_n(s)^2 s\, ds, \qquad Q_1[\phi_n u \chi_1] \leq \frac{d^2}{2} C_1 \int_0^\infty \dot{\phi}_n(s)^2 s\, ds,$$

and an explicit calculation yields that both converge to zero as $n \to \infty$ if we demand, in addition, that b_2/b_1 and b_3/b_2 tend to infinity as $n \to \infty$. The same is true for the mixed term $Q_1(\varphi_n \chi_1, \phi_n u \chi_1)$ by the Schwarz inequality. On the other hand, an explicit integration w.r.t. u for the rest of $\tilde{Q}$ yields

$$\begin{aligned} & Q_2[\Psi_{n,\varepsilon}] - \kappa_1^2 \|\Psi_{n,\varepsilon}\|_G \\ & \quad = (\varphi_n, K\varphi_n)_g - 2\varepsilon(\varphi_n, M\phi_n)_g + \varepsilon^2 \left[\|\phi_n\|_g^2 + \frac{\pi^2 - 6}{3\kappa_1^2} (\phi_n, K\phi_n)_g \right]. \end{aligned}$$

For large n the contribution of the Gauss curvature will be negligible because of $\langle \Sigma 1 \rangle$ and the facts that φ_n and ϕ_n are uniformly bounded and the infimum of their support tends to infinity as $n \to \infty$. Summing up the results, we arrive at

$$\lim_{n\to\infty} \tilde{Q}[\Psi_{n,\varepsilon}] = \lim_{n\to\infty} \left[\varepsilon^2 \|\phi_n\|_g^2 - 2\varepsilon(\varphi_n, M\phi_n)_g \right] \tag{6.4}$$

if the limit on the r.h.s. exists.

We put $\varepsilon \equiv \varepsilon_n := (\varphi_n, M\phi_n)_g^{-1}$ which will be seen in a moment as a reasonable choice because the integral tends to infinity as $n \to \infty$ for particular choices of b_j; ε_n is thus well-defined for n large enough. Then the problem turns to comparing the number -2 to the limit

$$\lim_{n\to\infty} \frac{(\phi_n, \phi_n)_g}{(\varphi_n, M\phi_n)_g^2}.$$

In the special case of cylindrically symmetric surfaces when one has the information about the explicit behaviour of M at infinity, it is an easy matter. Indeed, since k_s is integrable in $L^1(\mathbb{R}^+)$ and ϕ_n is chosen in a way to eliminate the weight r with help of (3.4), the meridian curvature does not contribute in the denominator, while in view of Lemma 6.1, $k_\vartheta r$ can be replaced by a constant value near infinity. Using in addition (3.4) in the numerator, one is therefore seeking the zero limit of

$$\frac{\int_0^\infty \phi_n(s)^2 s\, ds}{\left(\int_0^\infty \varphi_n(s)\phi_n(s) ds\right)^2} = \frac{1}{\int_0^\infty \phi_n(s)^2 s\, ds} = \frac{3}{\ln(b_3/b_1)}.$$

One can choose, for instance, $\forall n \geq 2$: $b_1(n) := n$, $b_2(n) := n^2$, $b_3(n) := n^3$, which fulfill also the other properties earlier required about these sequences. We conclude by $\tilde{Q}[\Psi_{n,\varepsilon}] \to -2$ as $n \to \infty$ so we can find a finite n_0 for which the form will be negative. □

Remark. Notice that (6.4) is a general result. We have not supposed anything of the surface symmetry when deriving this relation.

EXAMPLE 5 (Hyperboloid of Revolution). Consider one of the two sheets of the hyperboloid given by the equation $x^2 + y^2 - (z/z_0)^2 = 1$. It is an asymptotically planar surface of revolution and via the parameter $z_0 > 0$ we can get arbitrary value of the total Gauss curvature between 0 and 2π.

EXAMPLE 6 (Surface with Non Square Integrable $\nabla_g M$). Let us construct an asymptotically planar surface of revolution which satisfies $\langle\Sigma 1\rangle$ but contradicts $\langle\Sigma 2\rangle$. We define $k_s(s) := s^{-2}\sin s^2$ and use (6.2) to get the functions r, z and in this way the map p. One can easily check that there is a c such that $r(s) \geq cs$ for all $s \in \mathbb{R}^+$. Therefore $k_\vartheta = \dot{z} r^{-1} \to 0$ as $s \to \infty$ because $|\dot{z}| = |\sin b(s)| \leq 1$; the same limit holds, of course, for k_s. Since K, M are expressed by means of the principal curvatures, it follows that the surface is asymptotically planar $\langle\Sigma 0\rangle$. At the same time, $|K|r = |k_s \dot{z}| \leq |k_s|$ is integrable in $L^1(\mathbb{R}^+)$ which gives $\langle\Sigma 1\rangle$. On the other hand, while it is true that $\dot{k}_\vartheta = k_s r^{-1}\cos b - r^{-2}\sin b\cos b$ belongs to $L^2(\mathbb{R}^+, r(s)ds)$, the same does not hold for $\dot{k}_s$ by its definition. Hence, $\nabla_g M = (\dot{M}, 0)$ does not fulfil $\langle\Sigma 2\rangle$. We note that an explicit calculation together with (6.3) yields $\mathcal{K} = 2\pi\left(1 - \cos\sqrt{\frac{\pi}{2}}\right) \approx 1.38\,\pi$ in this example.

Remark. (*Partial Wave Decomposition*). An alternative approach is to decompose $-\Delta_D^\Omega$ with respect to angular momentum subspaces to investigate the spectral properties of layers endowed with the cylindrical symmetry. The obtained series of partial-wave Hamiltonians have similar form as the pure strip Hamiltonian – *cf.* [EŠ, DE] – except for an additional centrifugal term and different operator domain for the lowest wave. This, however, makes the spectral analysis of layers more complicated than a direct use of the non-decomposed Hamiltonian H. At the same time, it gives an insight into the choice of the trial function in the proof of Theorem 6.2 which has to be supported in the region where the influence of the centrifugal term is negligible.

7 A Layer without Bound States

Consider a semi-cylinder of radius R closed by a hemisphere; the total Gauss curvature is 2π. Since the mean curvature of the cylindrical part is constant, $M = (2R)^{-1} > 0$, such a surface is not asymptotically planar. We shall demonstrate that the Hamiltonian $H := -\Delta_D^\Omega$ of the corresponding layer Ω built over this surface does not possess bound states for any $a < R$.

Imposing the Neumann or Dirichlet boundary condition on the segment of connection of the hemispherical and cylindrical layer, we get the bounds $H^N_{sph} \oplus H^N_{cyl} \leq H \leq H^D_{sph} \oplus H^D_{cyl}$. The spectrum of the hemispherical-segment Hamiltonians is purely discrete. By the minimax principle only the cylindrical part of the estimating operators contributes to the essential spectrum, while a possible eigenvalue of H below the essential spectrum is squeezed between the corresponding eigenvalues of H^N_{sph} and H^D_{sph}. In particular, for our purpose it is sufficient to show that $\inf\sigma(H^N_{sph}) > \inf\sigma_{\mathrm{ess}}(H^D_{cyl})$. The spectral analysis of these operators becomes trivial if they are expressed in the spherical or cylindrical coordinates, respectively.

Due to the mirror symmetry, the ground state energy of H^N_{sph} is the same as the lowest eigenvalue of the entire spherical layer which is κ_1^2. On the other hand, $\sigma(H^j_{cyl}) = \sigma_{\text{ess}}(H^j_{cyl}) = [\epsilon_1, \infty)$ for both the conditions $j \in \{N, D\}$, where the threshold ϵ_1 is given by the first eigenvalue of the radial operator $-\partial_r^2 - (4r^2)^{-1}$ on $L^2(\mathbb{R}^+)$. Since the latter is less than $-\partial_r^2 - (4(R+a)^2)^{-1}$, the Rayleigh principle yields $\epsilon_1 < \kappa_1^2$. It is now easy to conclude that the spectrum of the unified layer satisfies

$$\sigma(H) = \sigma_{\text{ess}}(H) = [\epsilon_1, \infty). \tag{7.1}$$

Remark. The above example shows that without the condition $\langle\Sigma 0\rangle$, or at least without $M \to 0$ at the infinity, one cannot guarantee the existence of bound states. Notice that the reference surface is not C^2-smooth in this counter-example and thus it does not belong to the class of manifolds considered from the beginning. Nevertheless, one can construct a sequence of domains which converges in an appropriate sense to the hemispherical layer and, at the same time, they can be connected to the cylindrical part in a sufficiently smooth way. It follows then from [RT, Thm. 1.5] that the spectral result (7.1) remains preserved for the domains sufficiently close to the limiting layer.

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