# Inverse spectral results for AKNS systems with partial information on the potentials

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**Abstract**

For the AKNS operator on $L^2([0,1],\mathbb{C}^2)$ it is well known that the data of two spectra uniquely determine the corresponding potential $\varphi$ a.e. on $[0,1]$ (Borg's type Theorem). We prove that, in the case where $\varphi$ is a priori known on $[a,1]$, then only a part (depending on $a$) of two spectra determine $\varphi$ on $[0,1]$.Our results include generalizations for Dirac Systems of classical results obtained by Hochstadt and Lieberman for the Sturm Liouville case, where they showed that half of the potential and one spectrum determine all the potential function. An important ingredient in our strategy is the link between the rate of growth of an entire function and the distribution of is zeros.

## 1 Introduction

We study problems related to classical results by Borg [2] and Hochstadt-Lieberman [7]. For Sturm Liouville operator there exists a vast literature on this type of inverse problems, cf. [3] and references quoted therein. In this note we want to address similar results for AKNS systems. Actually we use a different approach than in [3], in particular we do not use the Titchmarsh-Weyl function and Marchenko's Theorem.

For $\varphi \in L^2([0,1],\mathbb{C})$ we define the AKNS operator on $L^2([0,1],\mathbb{C}^2)$ by

$$H(\varphi) := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} -q(x) & p(x) \\ p(x) & q(x) \end{pmatrix} \tag{1}$$

where $\varphi = q - ip$ and $q$ and $p$ are real valued.

Notice that $H(\varphi)$ is unitarily equivalent to the Zakharov-Shabat operator,

$$L(\varphi) := i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} 0 & \varphi \\ \bar{\varphi} & 0 \end{pmatrix} \tag{2}$$

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where $\bar{\varphi}$ is the complex conjugate of $\varphi$.

For each $\alpha \in [0, \pi)$ we consider $\sigma(\varphi, \alpha)$ the spectrum of the selfadjoint operator $H(\varphi)$ with domain, $F = \binom{Y}{Z} \in H^1([0,1], \mathbb{C}^2)$ such that

$$
\begin{aligned}
& Z(0) = 0 \\
& \cos\alpha \; Y(1) - \sin\alpha \; Z(1) = 0.
\end{aligned} \tag{3}
$$

Following [4] (cf. also [5]), $\sigma(\varphi, \alpha)$ is a sequence of real numbers $\Big(\mu_n(\varphi, \alpha)\Big)_{n \in \mathbb{Z}}$ satisfying $\mu_n < \mu_{n+1} (n \in \mathbb{Z})$ and $\mu_n = \alpha + n\pi - \frac{\pi}{2} + l^2(n)$ [1].

By $\varphi \mid_{[a,b]}$ we shall denote the restriction of $\varphi$ to the interval $[a, b]$. That is $\varphi \mid_{[a,b]} (x) = \varphi(x)$ if $x \in [a, b]$.

Our main result is

**Theorem 1** *Let $\varphi \in L^2([0,1], \mathbb{C})$, $\alpha, \beta \in [0, \pi)$ with $\alpha \neq \beta$, $0 \leq a \leq 1$, $l, k \in \mathbb{N} \cup \{\infty\}$ with $\frac{1}{l} + \frac{1}{k} \geq 2a$. Then $\{\mu_{ln}(\alpha), \mu_{kn}(\beta) \mid n \in \mathbb{Z}\}$* [2] *and $\varphi \mid_{[a,1]}$ uniquely determine $\varphi$ a.e. on $[0, 1]$ and $\alpha, \beta$.*

For particular values of $a, k, l$ we obtain for the AKNS systems

- Borg type Theorem: two spectra uniquely determine $\varphi$ on $[0, 1]$ $(a = 1,\ l = k = 1)$.
- Hochstadt-Liebermann type Theorem: one spectrum and $\varphi$ on $[1/2, 1]$ uniquely determine $\varphi$ on $[0, 1]$ $(a = 1/2,\ l = 1, k = \infty)$. (cf. [1] for an other proof of this result.)

Actually our Theorem includes much more general results as for example:

- Half of one spectrum and $\varphi$ on $[1/4, 1]$ uniquely determine $\varphi$ on $[0, 1]$ $(a = 1/4,\ l = 2, k = \infty)$.
- Half of two spectra and $\varphi$ on $[1/2, 1]$ uniquely determine $\varphi$ on $[0, 1]$ $(a = 1/2,\ l = k = 2)$.

For the shake of simplicity we only consider the case of two different boundary conditions. However the same method of proof applies to more general situations (for instance considering three spectra, with obvious notation the condition would be $\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} \geq 2a$).

In the same way we prove

**Theorem 2** *Let $\varphi \in L^2([0,1], \mathbb{C})$, $\alpha, \beta \in [0, \pi)$ with $\alpha \neq \beta$, $0 \leq a \leq 1$, $l,\ k \in \mathbb{N} \cup \{\infty\}$ with $\frac{1}{l} + \frac{1}{k} \geq 4a$. Then $\{\mu_{ln}(\alpha), \mu_{kn}(\beta) \mid n \geq 0\}$ and $\varphi \mid_{[a,1]}$ uniquely determine $\varphi$ a.e. on $[0, 1]$ and $\alpha, \beta$.*

[1] $a_n = b_n + l^2(n)$ means that $\sum_{n \in \mathbb{Z}} \mid a_n - b_n \mid^2 < \infty$.

[2] if $l$ (resp. $k$) equals $\infty$ we shall understand that $\{\mu_{ln}(\alpha) \mid n \in \mathbb{Z}\}$ (resp. $\{\mu_{kn}(\beta) \mid n \in \mathbb{Z}\}$) is empty.

Roughly speaking, Theorem 2 means that the "positive" part of one spectrum $\left((\mu_n)_{n\geq 0}\right)$ gives the same information than $(\mu_{2n})_{n\in\mathbb{Z}}$.

Of course in Theorems 1 and 2 the data $\varphi\mid_{[a,1]}$ can be replaced by $\varphi\mid_{[0,1-a]}$. Nevertheless the interval where $\varphi$ is a priori known must contain 0 or 1. Actually even the data of $Re\varphi$ on $[0,1]$, $Im\varphi$ on $[1/2-\varepsilon, 1/2+\varepsilon]$ ( $\varepsilon\in(0,1/2)$ arbitray) and one spectrum do not uniquely determine $\varphi$ on $[0,1]$. Let us construct a counter example: From [5, Propositions 1.3 and 1.4] we learn that for any $\varphi\in L^2([0,1],\mathbb{C})$ and $n\in\mathbb{Z}$, $\mu_n(\tilde{\varphi},\frac{\pi}{2})=\mu_n(\varphi,\frac{\pi}{2})$, where $\tilde{\varphi}(x)=\bar{\varphi}(1-x)$. Let $\varphi=q-ip$ with $q$ even, $p(1-x)=-p(x)$ for $x\in[1/2-\varepsilon,1/2+\varepsilon]$ and $p(1-x)\neq p(x)$ for $x\in[0,1/2-\varepsilon)$. Define $\psi$ by $\psi(x):=q(x)+ip(1-x)$.

Then, $\sigma(\varphi,\pi/2)=\sigma(\psi,\pi/2)$, $Re\varphi=Re\psi$ in $[0,1]$, $Im\varphi=Im\psi$ on $[1/2-\varepsilon,1/2+\varepsilon]$ nevertheless $\varphi\not\equiv\psi$.

Similar construction shows that the data of $Re\varphi$ on $[0,1]$, $Im\varphi$ on $[0,1]\setminus(1/2-\varepsilon,1/2+\varepsilon)$ ($\varepsilon\in(0,1/2)$ arbitrary) and one spectrum do not uniquely determine $\varphi$ on $[0,1]$.

Our fundamental strategy can be described as follows. Let $\varphi$ and $\psi$ satisfying the conditions of Theorem 1.

(a) We construct an entire function $f(\lambda,\varphi,\psi)$ (cf. section 2.1) which vanishes at each $\mu_{ln}(\alpha)$ and $\mu_{kn}(\beta)$, $n\in\mathbb{Z}$.

(b) The partial information on the potential allows us to bound the exponential type of $f$ obtaining $\mid f(\lambda)\mid=o(e^{2a|Im\lambda|})$ as $\mid\lambda\mid\to\infty$ (cf. Section 2.2).

(c) We use the general principle that the growth of an entire function is related to the distribution of its zeros [3] to prove that steps (a) and (b) and the condition $\frac{1}{l}+\frac{1}{k}\geq 2a$ imply $f\equiv 0$. (cf. Section 2.3)

(d) $f\equiv 0$ imply $\varphi\equiv\psi$. (cf. Section 2.4)

# 2 Proofs

## 2.1 Construction of the entire function $f$.

For $\varphi\in L^2([0,1],\mathbb{C})$ and $\lambda\in\mathbb{C}$ let $F(\cdot,\lambda,\varphi)\equiv\begin{pmatrix}Y(\cdot,\lambda,\varphi)\\ Z(\cdot,\lambda,\varphi)\end{pmatrix}\in H^1([0,1],\mathbb{C}^2)$ be the unique vector valued function satisfying $H(\varphi)F=\lambda F$ and $F(0,\lambda,\varphi)=\binom{1}{0}$.

For each $0\leq x\leq 1$ and $\varphi\in L^2([0,1])\mathbb{C})$, $\lambda\mapsto F(x,\lambda,\varphi)$ is entire (cf. [4] or [5] for the construction of $F$).

Notice that the spectrum $\sigma(\varphi,\alpha)$ is the root's set of

[3] This is a generalization of the fact that the rate of growth of a polynomial is given by its degree or equivalently its number of roots.

(4) $$\cos\alpha\; Y(1,\lambda,\varphi) - \sin\alpha\; Z(1,\lambda,\varphi) = 0.$$

Let $G \equiv \begin{pmatrix} G_1(x,\lambda,\varphi) \\ G_2(x,\lambda,\varphi) \end{pmatrix}$ with

(5) $$\begin{aligned} G_1(x,\lambda,\varphi) &= Z(x,\lambda,\varphi) - iY(x,\lambda,\varphi) \\ G_2(x,\lambda,\varphi) &= Z(x,\lambda,\varphi) + iY(x,\lambda,\varphi). \end{aligned}$$

One has

(6) $$L(\varphi)G(\cdot,\lambda,\varphi) = \lambda G(\cdot,\lambda,\varphi)$$

and furthermore for $\lambda \in \mathbb{R}$,

(7) $$\bar{G}_1(x,\lambda,\varphi) = G_2(x,\lambda,\varphi).$$

Let $\varphi, \psi \in L^2([0,1],\mathbb{C})$, from (6) the cross product,

$$A(\lambda) := < G(\cdot,\lambda,\psi), (L(\varphi)-\lambda)G(\cdot,\lambda,\varphi) > - < G(\cdot,\lambda,\varphi), (L(\psi)-\lambda)G(\cdot,\lambda,\psi) >$$

vanishes for all $\lambda \in \mathbb{C}$.

On the other hand by direct calculation using (7), one obtains for $\lambda \in \mathbb{R}$

$$\begin{aligned} A(x) &= < G(\psi), L(\varphi)G(\varphi) > - < G(\varphi), L(\psi)G(\psi) > \\ &= \int_0^1 G_2(\psi)G_2(\varphi)(\varphi - \psi)dx + \int_0^1 G_1(\psi)G_1(\varphi)(\bar{\varphi} - \bar{\psi})dx \\ &+ \; i\int_0^1 \frac{d}{dx}\Big( - G_1(\psi)G_2(\varphi) + G_2(\psi)G_1(\varphi)\Big)dx. \end{aligned}$$

Thus defining for $\lambda \in \mathbb{C}$,

(8) $$\begin{aligned} f(\lambda,\varphi,\psi) &:= \int_0^1 G_2(x,\lambda,\psi)G_2(x,\lambda,\varphi)(\varphi(x) - \psi(x))dx \\ &+ \int_0^1 G_1(x,\lambda,\psi)G_1(x,\lambda,\varphi)(\bar{\varphi}(x) - \bar{\psi}(x))dx\ , \end{aligned}$$

one gets for $\lambda \in \mathbb{R}$,

(9) $$f(\lambda,\varphi,\psi) = -i\Big[W\Big(G(\varphi), G(\psi)\Big)\Big]_0^1.$$

**Lemma 3** *Let* $\varphi, \psi \in L^2([0,1], \mathbb{C})$ *and* $\alpha \in [0, \pi)$. *Assume that* $\mu \in \sigma(\varphi, \alpha) \cap \sigma(\psi, \alpha)$. *Then*

$$f(\mu, \varphi, \psi) = 0.$$

*Proof* In view of formula (9) one has to prove that $[W(G(\cdot, \mu, \varphi), G(\cdot, \mu, \psi))]_0^1 = 0$. Since $G(0, \mu) = \binom{-i}{+i}$, one has

$$W(G(0, \mu, \varphi), G(0, \mu, \psi)) = 0.$$

On the other hand

$$W\Big(G(1, \mu, \varphi), G(1, \mu, \psi)\Big) = 2i\Big(Z(1, \mu, \varphi)Y(1, \mu, \psi) - Y(1, \mu, \varphi)Z(1, \mu, \psi)\Big) = 0$$

where we used that (cf. (4))

$$\cos\alpha\ Y(1) - \sin\alpha\ Z(1) = 0.$$

$\blacksquare$

For simplicity we introduce for $i = 1, 2$

$$g_i(x, \lambda) \equiv g_i(x, \lambda, \varphi, \psi) := G_i(x, \lambda, \varphi)G_i(x, \lambda, \psi). \tag{10}$$

Then formula (8) becomes, with $r = \varphi - \psi$,

$$f(\lambda, \varphi, \psi) = \int_0^1 \Big(g_2(x, \lambda)r(x) + g_1(x, \lambda)\bar{r}(x)\Big)dx. \tag{11}$$

Notice that since $\lambda \mapsto F(x, \lambda, \varphi)$ is entire, $\lambda \mapsto f(\lambda, \varphi, \psi)$ is entire too.

## 2.2 Order and type of the entire function $f$.

In this section we shall prove that the entire function $f$ defined in (11) satisfies the following

**Lemma 4** *Let* $0 \le a \le 1$. *Assume* $\varphi(x) = \psi(x)$ *for* $x \in [a, 1]$, *then*

$$f(\lambda, \varphi, \psi) = o(e^{2|Im\lambda|a})$$

*when* $|\lambda| \to \infty$.

*Proof* From [4] (see also [5]) we learn that uniformly for $x \in [0,1]$ one has when $\mid \lambda \mid \to \infty$,

$$\left\| F(x,\lambda) - \begin{pmatrix} \cos \lambda x \\ -\sin \lambda x \end{pmatrix} \right\| = o(e^{|Im\lambda| x}). \tag{12}$$

Therefore we get from (10) and (5)

$$g_2(x,\lambda,\varphi,\psi) = \left[ie^{i\lambda x} + o(e^{|Im\lambda|x})\right]^2 = -e^{2i\lambda x} + o(e^{2|Im\lambda|x})$$

and

$$g_1(x,\lambda,\varphi,\psi) = -e^{-2i\lambda x} + o(e^{2|Im\lambda|x}).$$

Using (11) we obtain (with $r = \varphi - \psi$)

$$\begin{aligned} f(\lambda,\varphi,\psi) &= \int_0^1 \Big( - e^{2i\lambda x} + o(e^{2|Im\lambda|x})\Big) r(x)dx \\ &+ \int_0^1 \Big( - e^{-2i\lambda x} + o(e^{2|Im\lambda|x})\Big) \bar{r}(x)dx. \end{aligned}$$

Thus if $\varphi \equiv \psi$ on $[a,1]$,

$$f(\lambda) = -\int_0^a r(x)e^{2i\lambda x}dx - \int_0^a \bar{r}(x)e^{-2i\lambda x}dx + o(e^{2|Im\lambda|a}) \tag{13}$$

where we used that the error term $o(e^{2|Im\lambda|x})$ is uniform in $x \in [0,1]$ and that $r \in L^1([0,1])$.

For $\alpha \in C^1([0,1])$ one obtains integrating by parts,

$$\left|\int_0^a \alpha(x)e^{2i\lambda x}dx\right| = o\left(\frac{e^{2|Im\lambda|a}}{\mid \lambda \mid}\right).$$

Now for $\alpha \in L^2([0,1])$ and $\varepsilon > 0$ let $\alpha_\varepsilon$ in $C^1([0,1])$ such that $\| \alpha_\varepsilon - \alpha \|_{L^2} < \varepsilon/2$. There exists $A > 0$ such that for $|\lambda| > A$,

$$\left|\int_0^a \alpha_\varepsilon(x)e^{2i\lambda x}dx\right| \leq \frac{\varepsilon}{2}\, e^{2|Im\lambda a|}$$

and thus

$$\left|\int_0^a \alpha(x)e^{2i\lambda x}dx\right| \;\leq\; \left|\int_0^a \alpha_\varepsilon(x)e^{2i\lambda x}dx\right| + \int_0^a \mid \alpha - \alpha_\varepsilon \mid \; e^{2|Im\lambda|x}dx \leq \varepsilon e^{2|Im\lambda a|}.$$

Therefore one has proved (cf. [10, Problem 3, p. 15]) that for $\alpha \in L^2([0,1])$

$$\int_0^a \alpha(x)e^{2i\lambda x}dx = o(e^{2|Im\lambda|a})\ .$$

Thus, using (13),

$$f(\lambda) = o(e^{2a|Im\lambda|}).$$

$\blacksquare$

Remark: We have proved that $f$ is an *entire* fuction of order not greater than 1 and type not greater than $2a$. We are going to use that such a function cannot have "many" zeros. (cf. [8]).

## 2.3 Infinite product representation

We begin with three auxiliary Lemmas on infinite products.

Given a sequence of complex numbers $(a_k)_{k\in\mathbb{Z}}$ we say that the product $\prod_{k\in\mathbb{Z}} a_k$ is convergent if the limit $\lim_{N\to\infty}\prod_{|k|\leq N} a_k$ exists. In such a case we write

$$\prod_{k\in\mathbb{Z}} a_k := \lim_{N\to\infty}\prod_{|k|\leq N} a_k.$$

**Lemma 5** *Let $(z_n)_{n\in\mathbb{Z}}$ a complex sequence satisfying $z_n = n\pi + l^2(n)$. Then the formula*

$$h(\lambda) := (\lambda - z_0)\prod_{n\in\mathbb{Z}\backslash\{0\}} \frac{z_n - \lambda}{n\pi}$$

*defines and entire function satisfying uniformly for $(n+1/6)\pi \leq |\lambda| \leq (n+5/6)\pi$,*

$$h(\lambda) = \sin\lambda(1+o(1)), \quad n\to+\infty. \tag{14}$$

Lemma 5 is proved in [5, Lemma I - 16] (cf. [4] and [10]), but the uniformity of (14) is proved only for $(n+1/4)\pi \leq |\lambda| \leq (n+3/4)\pi$. The generalization to $(n+1/6)\pi \leq |\lambda| \leq (n+5/6)\pi$ is straightforward (actually the only important fact is that $|\lambda|$ is far away from $n\pi$ $(n\geq 0)$ ).

From Lemma 5 follows

**Lemma 6** *Let $(z_n)_{n\in\mathbb{Z}}$ a sequence of complex numbers satisfying $z_n = n\pi + l^2(n)$ and $k\geq 1$. Then the formula*

$$h_k(\lambda) := (\lambda - z_0)\prod_{n\in\mathbb{Z}\backslash\{0\}} \frac{z_{nk} - \lambda}{nk\pi}$$

*defines an entire function satisfying uniformly on $k(n+1/6)\pi \leq |\lambda| \leq k(n+5/6)\pi$*

$$h_k(\lambda) = k\sin\left(\frac{\lambda}{k}\right)(1+o(1)), \quad n\to+\infty.$$

*Proof* For $n \in \mathbb{Z}$ we define, $\tilde{z}_n = \frac{z_{nk}}{k}$. One has

$$h_k(\lambda) = (\lambda - k\tilde{z}_0) \prod_{n \in \mathbb{Z} \setminus \{0\}} \frac{k\tilde{z}_n - \lambda}{nk\pi}.$$

By Lemma 5, the function

$$h(\lambda) = (\lambda - \tilde{z}_0) \prod_{n \in \mathbb{Z} \setminus \{0\}} \frac{\tilde{z}_n - \lambda}{n\pi}$$

satisfies

$$h(\lambda) = \sin\lambda(1 + o(1)), \quad n \to +\infty$$

uniformly on $(n + 1/6)\pi \leq \mid \lambda \mid \leq (n + 5/6)\pi$.

Notice that $h_k(\lambda) = kh(\frac{\lambda}{k})$, hence

$$h_k(\lambda) = k\sin(\frac{\lambda}{k})(1 + o(1)), \quad n \to +\infty$$

uniformly on $k(n + 1/6)\pi \leq \mid \lambda \mid \leq k(n + 5/6)\pi$.

■

As an application of Lemma 6, one has

**Lemma 7** *Let* $-\frac{\pi}{2} \leq a_j < \frac{\pi}{2}$ $(j = 1, 2)$, $k_1 \geq 1$, $k_2 \geq 1$ *and* $(\mu_n)_{n \in \mathbb{Z}}$, $(\nu_n)_{n \in \mathbb{Z}}$ *two sequences of real numbers satisfying* $\mu_n = n\pi + l^2(n)$, $\nu_n = n\pi + l^2(n)$. *Then the function defined by*

$$h(\lambda) := \quad (\lambda - \mu_0 - a_1)(\lambda - \nu_0 - a_2) \prod_{n \in \mathbb{Z} \setminus \{0\}} \frac{\mu_{k_1 n} + a_1 - \lambda}{k_1 n\pi} \times \frac{\nu_{k_2 n} + a_2 - \lambda}{k_2 n\pi}$$

*is entire.*

*Furthermore there exists* $(\gamma_p)_{p \geq 1}$ *a sequence of positive real numbers with* $\gamma_p \longrightarrow \infty$, *and* $C > 0$ *such that uniformly on* $\mid \lambda \mid = \gamma_p$ *and* $p \geq 1$,

$$\mid h(\lambda) \mid \geq C \exp\Big( (\frac{1}{k_1} + \frac{1}{k_2}) \mid Im\lambda \mid \Big).$$

*Proof* By Lemma 6, the functions

$$h_1(\lambda) := (\lambda - \mu_0 - a_1) \prod_{n \in \mathbb{Z} \setminus \{0\}} \frac{\mu_{k_1 n} + a_1 - \lambda}{k_1 n\pi}$$

and

$$h_2(\lambda) := (\lambda - \nu_0 - a_2) \prod_{n \in \mathbb{Z} \setminus \{0\}} \frac{\nu_{k_2 n} + a_2 - \lambda}{k_2 n \pi}$$

are entire and satisfy as $n \to \infty$

$$h_1(\lambda) = k_1 \sin\Big(\frac{\lambda - a_1}{k_1}\Big)(1 + o(1)) \tag{15}$$

uniformly on $k_1(n + 1/6)\pi \leq | \lambda - a_1 | \leq k_1(n + 5/6)\pi$ and

$$h_2(\lambda) = k_2 \sin\Big(\frac{\lambda - a_2}{k_2}\Big)(1 + o(1)) \tag{16}$$

uniformly on $k_2(n + 1/6)\pi \leq | \lambda - a_2 | \leq k_2(n + 5/6)\pi$.

Moreover, for $j = 1, 2$, there exist $C_j > 0$ such that uniformly on

$$I_j := \bigcup_{n \in \mathbb{Z}} \{\lambda \ \mid \ k_j(n + 1/6)\pi \leq | \lambda - a_j | \leq k_j(n + 5/6)\pi\},$$

one has (cf. [10, Lemma 1 p. 27])

$$\left| \sin\left(\frac{\lambda - a_j}{k_j}\right) \right| > C_j \exp\left(\frac{| \, Im(\lambda - a_j) \, |}{k_j}\right) \ . \tag{17}$$

Noticing that $h(\lambda) = h_1(\lambda) h_2(\lambda)$ and in view of (15) - (17), it remains to prove that there exists a sequence $(\gamma_p)_{p \geq 1}$ with $\gamma_p > 0$ $(p \geq 1)$ and $\gamma_p \underset{p \to \infty}{\longrightarrow} \infty$ such that for each $p \geq 1$

$$\gamma_p \in I_1 \cap I_2 \cap \mathbb{R}.$$

As $I_j \cap \mathbb{R}$ is the union of segments whose wide is $2/3 \ k_j \pi$ and the distance between two consecutive segments is $1/3 \ k_j \pi$, the existence of such sequence $(\gamma_p)_{p \geq 1}$ is clear.

$\blacksquare$

We can now state the main result of this section. Recall that $\sigma(\varphi, \alpha) \equiv (\mu_n(\varphi, \alpha))_{n \in \mathbb{Z}}$ is the spectrum of $H(\varphi)$ with domain, $F = \binom{Y}{Z} \in H^1(0, 1)$ such that $Z(0) = 0$ and $\cos\alpha \ Y(1) - \sin\alpha \ Z(1) = 0$.

**Proposition 8** *Let* $\varphi, \ \psi \in L^2([0, 1], \mathbb{C})$, $0 \leq a \leq 1$, $0 \leq \alpha, \ \beta < \pi$ *with* $\alpha \neq \beta$ *and* $k_1, \ k_2 \geq 1$ *with* $\frac{1}{k_1} + \frac{1}{k_2} \geq 2a$. *Assume that*

*(i)* $\varphi \equiv \psi$ *on* $[a, 1]$

*(ii)* $\mu_{k_1 n}(\varphi, \alpha) = \mu_{k_1 n}(\psi, \alpha), n \in \mathbb{Z}$

*(iii)* $\mu_{k_2 n}(\varphi, \beta) = \mu_{k_2 n}(\psi, \beta), n \in \mathbb{Z}$

*Then* $f(\cdot,\varphi,\psi)\equiv 0$.

*Proof* As mentioned in the introduction, following [5] (see also [4]), one deduces from Rouché's Theorem and formula (12)

$$\mu_n(\varphi,\alpha)=n\pi+\alpha-\frac{\pi}{2}+l^2(n) \tag{18}$$

and

$$\mu_n(\varphi,\beta)=n\pi+\beta-\frac{\pi}{2}+l^2(n). \tag{19}$$

Therefore by Lemma 7, the entire function

$$h(\lambda):=(\lambda-\mu_0)(\lambda-\nu_0)\prod_{n\in\mathbb{Z}\setminus\{0\}}\frac{\mu_{k_1n}-\lambda}{k_1n\pi}\,\frac{\nu_{k_2n}-\lambda}{k_2n\pi},$$

with $\mu_j:=\mu_j(\varphi,\alpha)$, $\nu_j:=\mu_j(\varphi,\beta)$ $(j\in\mathbb{Z})$, satisfies for some constant $C>0$

$$\mid h(\lambda)\mid\geq C\exp\Big((\frac{1}{k_1}+\frac{1}{k_2})\mid Im\lambda\mid\Big)\geq C\exp\Big(2a\mid Im\lambda\mid\Big) \tag{20}$$

uniformly on $\mid\lambda\mid=\gamma_p$ and $p\geq 1$ where $(\gamma_p)_{p\geq 1}$ is a sequence of positive real numbers with $\gamma_p\underset{p\to\infty}{\longrightarrow}\infty$.

Furthermore, as $\alpha\neq\beta$, $\sigma(\varphi,\alpha)\cap\sigma(\varphi,\beta)=\emptyset$. Thus $(\mu_{k_1n})_{n\in\mathbb{Z}}$ and $(\nu_{k_2n})_{n\in\mathbb{Z}}$ are simple roots of $h$.

On the other hand, by Lemma 3 and hypothesis (ii) (iii), $f(\mu_{k_1n})=f(\nu_{k_2n})=0$ for all $n\in\mathbb{Z}$. Besides, by Lemma 4 and Hypothesis (i),

$$f(\lambda)=o(e^{2a|Im\lambda|}) \tag{21}$$

when $\mid\lambda\mid\to\infty$.

Therefore $\lambda\mapsto\frac{f(\lambda)}{h(\lambda)}$ is entire and combining (20), (21) we get $\mid\frac{f(\lambda)}{h(\lambda)}\mid=o(1)$ as $p\to\infty$ uniformly on $\mid\lambda\mid=\gamma_p$.

Hence by the maximum principle we conclude that $f\equiv 0$.

$\blacksquare$

## 2.4 Integral representation

The main result of this section is

**Proposition 9** *Let* $\varphi,\psi\in L^2([0,1],\mathbb{C})$. *Assume that* $f(\lambda,\varphi,\psi)=0$ *for* $\lambda\in\mathbb{R}$, *then* $\varphi\equiv\psi$.

We follow the same strategy as in B. Levin [8, Appendix 4].

We first establish an integral representation of $g_1$ and $g_2$ (cf. formula (10)).

**Lemma 10** *The exists a kernel $K \in L^2([-1,1]^2, \mathbb{C})$ such that for $x \in [-1,1]$ and $\lambda \in \mathbb{R}$,*

$$g_1(x,\lambda) = -e^{-2i\lambda x} + \int_{-x}^{x} \bar{K}(x,u)e^{-2i\lambda u}du \tag{22}$$

*and*

$$g_2(x,\lambda) = -e^{2i\lambda x} + \int_{-x}^{x} K(x,u)e^{2i\lambda u}du. \tag{23}$$

*Proof of Lemma 10*

Let $M(\cdot,\lambda,\varphi)$ be the fundamental ($2\times 2$ matrix) solution of $H(\varphi)M = \lambda M$ satisfying $M(0,\lambda,\varphi) = Id_{2\times 2}$ and $R(x,\lambda)$ be given by

$$R(x,\lambda) = \begin{pmatrix} \cos\lambda x & \sin\lambda x \\ -\sin\lambda x & \cos\lambda x \end{pmatrix}.$$

From [9, p. 514][4](cf. also [8]) one learns that there exists a Kernel $A \in L^2([0,1]^2, M_{2\times 2}(\mathbb{R}))$ (where $M_{2\times 2}(\mathbb{R})$ denotes the space of $2\times 2$ matrix with real entries) such that

$$M(x,\lambda) = R(x,\lambda) + \int_0^x R(x-2y,\lambda)A(x,y)dy \tag{24}$$

By definition $F(x,\lambda,\varphi)$ is the first column of $M(x,\lambda,\varphi)$. Therefore by (5)

$$G_2(x,\lambda) = (i,1)M(x,\lambda)\begin{pmatrix}1\\0\end{pmatrix}$$

and by a straightforward calculation one gets

$$G_2(x,\lambda,\varphi) = ie^{i\lambda x} + \int_0^x e^{i\lambda(x-2y)}K_\varphi(x,y)dy \tag{25}$$

where

$$K_\varphi(x,y) = (i,1)A(x,y)\begin{pmatrix}1\\0\end{pmatrix}.$$

Since for $\lambda \in \mathbb{R}$, $G_1(x,\lambda,\varphi) = \bar{G}_2(x,\lambda,\varphi)$, one also has

$$G_1(x,\lambda,\varphi) = -ie^{-i\lambda x} + \int_0^x e^{-i\lambda(x-2y)}\bar{K}_\varphi(x,y)dy. \tag{26}$$

[4]In [9] the authors do not use the same spectral variable and we have to transform $\lambda$ in $\lambda/2$ in our formula (24)

Inserting (25) in (10) one gets

$$g_2(x,\lambda) = -e^{2i\lambda x} + h_1(x,\lambda) + h_2(x,\lambda) \tag{27}$$

where

$$h_1(x,\lambda) = i\int_0^x (K_\varphi(x,t) + K_\psi(x,t))e^{2i\lambda(x-t)}dt \tag{28}$$

and

$$h_2(x,\lambda) = \int_0^x dt \int_0^x ds K_\varphi(x,t)K_\psi(x,s)e^{2i\lambda(x-t-s)}. \tag{29}$$

By the change of variable $u = x - t - s$ and $v = t - s$ in (29) one obtains

$$h_2(x,\lambda) = \frac{1}{2}\int_D K_\varphi(x, \frac{v-u+x}{2})K_\psi(x, \frac{-v-u+x}{2})e^{2i\lambda u}du\, dv$$

where

$$D = D_1 \cup D_2$$

and with

$$D_1 := \{(u,v) \mid -x \le u \le 0; u \le v \le -u\}$$

and

$$D_2 := \{(u,v) \mid 0 \le u \le x; -u \le v \le u\}.$$

Thus

$$h_2(x,\lambda) = \int_{-x}^x e^{2i\lambda u}K_1(x,u)du \tag{30}$$

with

$$K_1(x,u) := \int_{-x}^x K_\varphi(x, \frac{v-u+x}{2})K_\psi(x, \frac{-v-u+x}{2})\, 1\!\!1_D\,(u,v)dv$$

and where$1\!\!1_D$ denotes the characteristic function of the set $D$.
Similarly by the change of variable $u = x - t$ in (28) one has

$$h_1(x,\lambda) = i\int_0^x \Big(K_\varphi(x,x-u) + K_\psi(x,x-u)\Big)e^{2i\lambda u}du.$$

Thus

$$h_1(x,\lambda) = \int_{-x}^x K_2(x,u)e^{2i\lambda u}du. \tag{31}$$

where

$$K_2(x,u) := i\,\mathbb{1}_{[0,x]}(u)\Big(K_\varphi(x,x-u)+K_\psi(x,x-u)\Big).$$

Combinig (27), (30) and (31) one gets (23) with

$$K(x,u) = K_1(x,u) + K_2(x,u).$$

We deduce (22) from (23) recalling that

$$g_1(x,\lambda) = \bar g_2(x,\lambda) \text{ for } \lambda \in \mathbb{R}.$$

■

*Proof of Proposition 9* Recall that, with $r = \varphi - \psi$ (cf. (11))

$$f(\lambda) \equiv f(\lambda,\varphi,\psi) = \int_0^1 \Big(g_2(x,\lambda)r(x) + g_1(x,\lambda)\bar r(x)\Big)dx.$$

By Lemma 10 one gets for $\lambda \in \mathbb{R}$

$$\begin{aligned} f(\lambda) &= \int_0^1 \Big(-e^{2i\lambda x} + \int_{-x}^x K(x,u)e^{2i\lambda u}du\Big)r(x)dx \\ &+ \int_0^1 \Big(-e^{-2i\lambda y} + \int_{-y}^y \bar K(y,v)e^{-2i\lambda v}dv\Big)\bar r(y)dy. \end{aligned} \tag{32}$$

By the change of variable $x = -y$ and $u = -v$ in the second term of the right side of (32) one obtains

$$\begin{aligned} f(\lambda) &= \int_0^1 \Big(-e^{2i\lambda x} + \int_{-x}^x K(x,u)e^{2i\lambda u}du\Big)r(x)dx \\ &+ \int_{-1}^0 \Big(-e^{2i\lambda x} - \int_{-x}^x \bar K(-x,-u)e^{2i\lambda u}du\Big)\bar r(-x)dx. \end{aligned} \tag{33}$$

Thus with $m(x) := r(x)$ for $x \in [0,1]$ and $m(x) := \bar r(-x)$ for $x \in [-1,0]$ and with

$$\begin{aligned} B(x,u) &:= K(x,u) && \text{for } x \in [0,1] \text{ and} \\ B(x,u) &:= \bar K(-x,-u) && \text{for } x \in [-1,0], \end{aligned} \tag{34}$$

(33) leads to

$$\begin{aligned} f(\lambda) &= \int_{-1}^1 \Big(-e^{2i\lambda x} + \int_{-|x|}^{|x|} B(x,u)e^{2i\lambda u}du\Big)m(x)dx \\ &= \int_{-1}^1 e^{2i\lambda x}\Big(-m(x) + \int_{-1}^{-|x|} B(u,x)m(u)du + \int_{|x|}^1 B(u,x)m(u)du\Big)dx. \end{aligned}$$

Since $f(\lambda) = 0$ for all $\lambda \in \mathbb{R}$ and $\{e^{2i\lambda x} \mid \lambda \in \mathbb{R}\}$ spans $L^2([-1,1], \mathbb{C})$, the last formula implies that, for $x \in [-1, 1]$,

$$m(x) = \int_{-1}^{-|x|} B(u,x)m(u)du + \int_{|x|}^{1} B(u,x)m(u)du \ .$$

In particular for $x \in [0,1]$ one gets by defnition of $B$ and $m$,

$$r(x) = \int_x^1 \Big(\bar{K}(u,-x)\bar{r}(u) + K(u,x)r(u)\Big)du.$$

Therefore defining $P \in L^2([-1,1])$ by

$$P(u,v) :=\mid K(u,v) \mid + \mid \bar{K}(u,-v) \mid,$$

one obtains for $x \in [0,1]$

$$\mid r(x) \mid \leq \int_x^1 P(u,x) \mid r(u) \mid du. \tag{35}$$

Iterating formula (35) leads to, for each $n \geq 1$,

$$\mid r(x) \mid \leq \int_x^1 du_1 \int_{u_1}^1 du_2 \ldots \int_{u_{n-1}}^1 du_n \ P(u_1,x)\ldots P(u_n,u_{n-1}) \mid r(u_n) \mid \ \ .$$

Interchanging the order of integration we obtain[5]

$$\mid r(x) \mid \leq \int_x^1 du_n \int_x^{u_n} du_{n-1} \ldots \int_x^{u_2} du_1 \ P(u_1,x)\ldots P(u_n,u_{n-1}) \mid r(u_n) \mid \ \ . \tag{36}$$

Now setting $K_1(u,x) = P(u,x)$ and defining

$$K_j(u,x) := \int_x^u dv \ K_{j-1}(v,x)P(u,v)$$

the inequality (36) can be written as

$$\mid r(x) \mid \leq \int_x^1 duK_n(u,x) \mid r(u) \mid \ \ . \tag{37}$$

Defining for $0 \leq x \leq u \leq 1$

[5] the following argument is analogous to parts of [6, Theorem 6, Ch. 2]

$$
\begin{aligned}
B(x) &= \int_x^1 \mid P(z,x) \mid^2 dz \ , \\
A(u) &= \int_0^u \mid P(u,z) \mid^2 dz \ , \\
\rho(x) &= \int_0^x A(u) du \ ,
\end{aligned}
$$

one obtains by a straigtforward recurrence and the Cauchy-Schwarz inequality

$$
\mid K_n(u,x) \mid^2 \leq A(u)B(x)\frac{\rho(u)^{n-2}}{(n-2)!} \ .
$$

Therefore

$$
\begin{aligned}
\int_x^1 \mid K_n(u,x) \mid^2 du &\leq \frac{B(x)}{(n-2)!}\int_0^1 du A(u)\rho(u)^{n-2} \\
&= \frac{B(x)}{(n-1)!}\,\rho(1)^{n-1} \ .
\end{aligned}
$$

Hence

$$
(38) \int_x^1 \mid K_n(u,x) \mid^2 du \leq \frac{1}{(n-1)!}\int_0^1 \mid P(z,x) \mid^2 dz \cdot \Big( \int_0^1 du \int_0^1 dz \mid P(u,z) \mid^2 \Big)^{n-1} \ .
$$

Using Cauchy–Schwarz in (37) we get

$$
(39) \qquad \mid r(x) \mid^2 \leq \int_x^1 du \mid K_n(u,x) \mid^2 \cdot \int_x^1 \mid r(u) \mid^2 du \ .
$$

By integrating (39) and using (38) we find

$$
\|r\|_{L_2}^2 \leq \frac{1}{(n-1)!}\|P\|_{L_2}^{2n}\|r\|_{L_2}^2 \stackrel{n\to\infty}{\longrightarrow} 0 \ .
$$

It follows then that

$$
r(x) = \varphi(x) - \psi(x) = 0 \ .
$$

■

## 2.5 *Proofs of Theorems 1 and 2*

Theorem 1 is a direct consequence of Proposition 8 and Proposition 9.
The proof of Theorem 2 is similar; we only have to make the following changes:

- We replace $f(\lambda, \varphi, \psi)$ by
$$\tilde{f}(\lambda, \varphi, \psi) = f(-\lambda, \varphi, \psi) f(\lambda, \varphi, \psi)$$
Thus, since $\mu_{ln}(\varphi, \alpha) = \mu_{ln}(\psi, \alpha)$ and $\mu_{kn}(\varphi, \beta) = \mu_{kn}(\psi, \beta)$ for all $n \in \mathbb{N}$ one has by Lemma 3 that for any $n \in \mathbb{N}$
$$\tilde{f}(\mu_{ln}(\varphi, \alpha)) = \tilde{f}(\mu_{kn}(\varphi, \beta)) = 0$$
and
$$\tilde{f}( - \mu_{ln}(\varphi, \alpha)) = \tilde{f}( - \mu_{kn}(\varphi, \beta)) = 0 \ .$$
- By Lemma 4 one has
$$\tilde{f}(\lambda) = o(e^{|Im\lambda|4a}) \text{ as } \mid \lambda \mid \rightarrow +\infty.$$
- In Proposition 8 we replace $h$ by $\tilde{h}$ where
$$\tilde{h}(x) = (\lambda - \mu_0)(\lambda - \nu_0) \prod_{n>0} \frac{(\mu_{ln}^2 - \lambda^2)(\nu_{kn}^2 - \lambda^2)}{(ln\pi)^2 (kn\pi)^2}$$
with $\mu_j = \mu_j(\varphi, \alpha)$ and $\nu_j = \mu_j(\varphi, \beta) (j \in \mathbb{N})$.
The function $\lambda \longmapsto \frac{\tilde{f}(\lambda)}{\tilde{h}(\lambda)}$ is still entire.
- By Lemma 7 there exist $C > 0$ and $(\gamma_p)_{p \geq 1}$ with $\gamma_p \underset{p \to \infty}{\longrightarrow} \infty$ such that uniformly on $\mid \lambda \mid = \gamma_p$
$\mid h(x) \mid \geq C \exp \left( \frac{1}{l} + \frac{1}{k} \right) \mid Im\lambda \mid$.
- As in Proposition 8 we obtain, using $\frac{1}{l} + \frac{1}{k} \geq 4a$, that $\left| \frac{\tilde{f}(\lambda)}{\tilde{h}(\lambda)} \right| = o(1)$ for $p \to \infty$ uniformly on $\mid \lambda \mid = \gamma_p$. Hence by the maximum principle $\tilde{f} \equiv 0$, i. e. $f \equiv 0$.
- We apply Proposition 9 to conclude to $\varphi = \psi$.

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