Boundary Conditions for Singular Perturbations of Self-Adjoint Operators

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Abstract. Let $A : D(A) \subseteq \mathcal{H} \to \mathcal{H}$ be an injective self-adjoint operator and let $\tau : D(A) \to \mathfrak{X}$, $\mathfrak{X}$ a Banach space, be a surjective linear map such that $\|\tau\phi\|_{\mathfrak{X}} \le c\,\|A\phi\|_{\mathcal{H}}$. Supposing that $\text{Range}\,(\tau') \cap \mathcal{H}' = \{0\}$, we define a family A^τ_Θ of self-adjoint operators which are extensions of the symmetric operator $A_{|\{\tau=0\}}$. Any ϕ in the operator domain $D(A^\tau_\Theta)$ is characterized by a sort of boundary conditions on its univocally defined regular component ϕ_{reg}, which belongs to the completion of $D(A)$ w.r.t. the norm $\|A\phi\|_{\mathcal{H}}$. These boundary conditions are written in terms of the map τ, playing the role of a trace (restriction) operator, as $\tau\phi_{\text{reg}} = \Theta\, Q_\phi$, the extension parameter Θ being a self-adjoint operator from $\mathfrak{X}'$ to $\mathfrak{X}$. The self-adjoint extension is then simply defined by $A^\tau_\Theta\phi := A\,\phi_{\text{reg}}$. The case in which $A\phi = T * \phi$ is a convolution operator on $L^2(\mathbb{R}^n)$, T a distribution with compact support, is studied in detail.

1. Introduction

Let

$$A : D(A) \subseteq \mathcal{H} \to \mathcal{H}$$

be a self-adjoint operator on the complex Hilbert space $\mathcal{H}$. As usual $D(A)$ inherits a Hilbert space structure by introducing the scalar product leading to the graph norm

$$\|\phi\|_A^2 := \langle \phi, \phi\rangle_{\mathcal{H}} + \langle A\phi, A\phi\rangle_{\mathcal{H}}\,.$$

Considering then a linear operator

$$\tau : D(A) \to \mathfrak{X}\,, \qquad \tau \in \mathcal{B}(D(A), \mathfrak{X})\ ,$$

$\mathfrak{X}$ a complex Banach space, we are interested in describing the self-adjoint extensions of the symmetric operator $A_{|\{\tau=0\}}$. In tipical situations A is a (pseudo-)differential operator on $L^2(\mathbb{R}^n)$ and τ is a trace (restriction) operator along some null subset $F \subset \mathbb{R}^n$ (see e.g. [**1**]-[**4**], [**7**], [**8**], [**15**]-[**18**], [**20**], [**21**] and references therein).

Denoting the resolvent set of A by $\rho(A)$, we define $R(z) \in \mathcal{B}(\mathcal{H}, D(A))$, $z \in \rho(A)$, by

$$R(z) := (-A + z)^{-1}$$

and we then introduce, for any $z \in \rho(A)$, the operators $\breve G(z) \in \mathcal{B}(\mathcal{H},\mathfrak{X})$ and $G(z) \in \widetilde{\mathcal{B}}(\mathfrak{X}',\mathcal{H})$ by

$$\breve G(z) := \tau \cdot R(z)\,, \qquad G(z) := C_{\mathcal{H}}^{-1} \cdot \breve G(z^*)'\ , \tag{1}$$

(we refer to section 2 below for definitions and notations). As an immediate consequence of the first resolvent identity for $R(z)$ we have (see [**18**, lemma 2.1])

$$(z-w)\,R(w)\cdot G(z) = G(w) - G(z) \tag{2}$$

and so

$$\forall\, w,z \in \rho(A), \qquad \text{Range}\,(\,G(w)-G(z)\,) \subseteq D(A)\,. \tag{3}$$

In [**18**, thm. 2.1], by means of a Kreĭn-like formula, and under the hypotheses

$$\tau \ \text{ is surjective} \tag{h1}$$

$$\text{Range}\,(\tau')\cap \mathcal{H}' = \{0\} \tag{h2}$$

(the hypothesis (h1) could be weakened, see [**18**], but here we prefer to use a simpler framework), we constructed a family A^τ_Θ of self-adjoint extension of $A_{|\{\tau=0\}}$ by giving its resolvent family. The advantage of the formula given in [**18**] over other approaches (see e.g. [**19**], [**9**]-[**11**] and references therein) is its relative simplicity, being expressed directly in terms of the map τ; moreover the domain of definition of A^τ_Θ can be described, interpreting the map τ as a trace (restriction) operator, in terms of a sort of boundary conditions (see [**18**, remark 2.10]). In the case $0 \notin \sigma(A)$, $\sigma(A)$ denoting the spectrum of A, this description becomes particularly expressive since $A^\tau_\Theta\phi$ can be simply defined by the original operator applied to the regular component of ϕ. Such a regular component $\phi_0 \in D(A)$ is univocally determined by the natural decomposition which enter in the definition of $D(A^\tau_\Theta)$ and it has to satisfy the boundary condition

$$\tau\phi_0 = \Theta\, Q_\phi\,.$$

More precisely, by (h1), (h2) and by [**18**, lemma 2.2, thm. 2.1, prop. 2.1, remarks 2.10, 2.12], we have the following

Theorem 1. *Let $A : D(A) \subseteq \mathcal{H} \to \mathcal{H}$ be self-adjoint with $0 \notin \sigma(A)$, let $\tau : D(A) \to \mathfrak{X}$ be continuous and satisfy* (h1) *and* (h2)*. If $\Theta \in \widetilde{\mathsf{L}}(\mathfrak{X}',\mathfrak{X})$ is self-adjoint, $G := G(0)$ and*

$$D(A^\tau_\Theta) := \{\phi \in \mathcal{H}\,:\, \phi = \phi_0 + GQ_\phi,\ \phi_0 \in D(A),\ Q_\phi \in D(\Theta),\ \tau\phi_0 = \Theta\,Q_\phi\}\,,$$

then the linear operator

$$A^\tau_\Theta : D(A^\tau_\Theta) \subseteq \mathcal{H} \to \mathcal{H}\,, \qquad A^\tau_\Theta\phi := A\phi_0$$

is self-adjoint and coincides with A on the kernel of τ; the decomposition entering in the definition of its domain is unique. Its resolvent is given by

$$R^{\tau}_{\Theta}(z) := R(z) + G(z)\cdot(\Theta+\Gamma(z))^{-1}\cdot \breve{G}(z)\,, \qquad z \in W^{-}_{\Theta}\cup W^{+}_{\Theta}\cup \mathbb{C}\backslash\mathbb{R}\,,$$

where

$$\Gamma(z) := \tau\cdot(G - G(z))$$

and

$$W^{\pm}_{\Theta} := \{\, \lambda\in\mathbb{R}\cap\rho(A) \ :\ \gamma(\pm\Gamma(\lambda)) > -\gamma(\pm\Theta)\,\}\ .$$

Remark 2. By (h1) one has $\mathcal{X}\simeq D(A)/\text{Kernel}\,(\tau)\simeq(\text{Kernel}\,(\tau))^{\perp}$ and so

$$D(A)\simeq \text{Kernel}\,(\tau)\oplus\mathcal{X}\,.$$

This implies that $\mathcal{X}$ inherits a Hilbert space structure and we could then use the identification $\mathcal{X}'\simeq\mathcal{X}$. Even if this gives some advantage (see [**18**, remarks 2.13-2.16, lemma 2.4]) here we prefer to use only the Banach space structure of $\mathcal{X}$.

The purpose of the present paper is to extend the above theorem to the case in which A is merely injective. Thus, denoting the pure point spectrum of A by $\sigma_{pp}(A)$, we require $0\notin\sigma_{pp}(A)$ but we do not exclude the case $0\in\sigma(A)\backslash\sigma_{pp}(A)$; this is a tipical situation when A is a differential operator on $L^2(\mathbb{R}^n)$. In order to carry out this program we will suppose that the map τ has a continuos extension to $\widehat{D}(A)$, the completion of $D(A)$ with respect to the norm $\|A\phi\|_{\mathcal{H}}$ (note that $\widehat{D}(A)=D(A)$ when $0\notin\sigma(A)$). Such a further hypothesis allows then to perform the limit $\lim_{\epsilon\to 0}G(i\epsilon)-G(z)$ (see lemma 3); thus an analogue on the above theorem 1 is obtained (see theorem 5). Such an abstract construction is successively specialized to the case in which $A\phi = T*\phi$ is a convolution operator on $L^2(\mathbb{R}^n)$, where T is a distribution with compact support (so that this comprises the case of differential-difference operators). In this situation the results obtained in theorem 5 can be made more appealing (see theorem 11). The case in which $A=\Delta : H^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)$ and τ is the trace (restriction) operator along a d-set with a compact closure of zero Lebesgue measure, $0<n-d<4$, is explicitly studied (see example 14).

2. Definitions and notations

- Given a Banach space $\mathcal{X}$ we denote by $\mathcal{X}'$ its strong dual;
- $\mathsf{L}(\mathcal{X},\mathcal{Y})$, resp. $\widetilde{\mathsf{L}}(\mathcal{X},\mathcal{Y})$, denotes the space of linear, resp. conjugate linear, operators from the Banach space $\mathcal{X}$ to the Banach space $\mathcal{Y}$; $\mathsf{L}(\mathcal{X}) := \mathsf{L}(\mathcal{X},\mathcal{X})$, $\widetilde{\mathsf{L}}(\mathcal{X}) := \widetilde{\mathsf{L}}(\mathcal{X},\mathcal{X})$.

- $\mathcal{B}(\mathcal{X},\mathcal{Y})$, resp. $\widetilde{\mathcal{B}}(\mathcal{X},\mathcal{Y})$, denotes the (Banach) space of bounded, everywhere defined, linear, resp. conjugate linear, operators on the Banach space $\mathcal{X}$ to the Banach space $\mathcal{Y}$.
- Given $A \in \mathsf{L}(\mathcal{X},\mathcal{Y})$ and $\widetilde{A} \in \widetilde{\mathsf{L}}(\mathcal{X},\mathcal{Y})$ densely defined, the closed operators $A' \in \mathsf{L}(\mathcal{Y}',\mathcal{X}')$ and $\widetilde{A}' \in \widetilde{\mathsf{L}}(\mathcal{Y}',\mathcal{X}')$ the are the adjoints of A and $\widetilde{A}$ respectively, i.e.
$$\forall\, x \in D(A) \subseteq \mathcal{X}, \quad \forall \ell \in D(A') \subseteq \mathcal{Y}', \qquad (A'\ell)(x) = \ell(Ax)\,,$$
$$\forall\, x \in D(\widetilde{A}) \subseteq \mathcal{X}, \quad \forall \ell \in D(\widetilde{A}') \subseteq \mathcal{Y}', \qquad (\widetilde{A}'\ell)(x) = (\,\ell(\widetilde{A}x)\,)^*$$
where * denotes complex conjugation.
- $J_{\mathcal{X}} \in \mathcal{B}(\mathcal{X},\mathcal{X}'')$ indicates the injective map (an isomorphism when $\mathcal{X}$ is reflexive) defined by $(J_{\mathcal{X}}\,x)(\ell) := \ell(x)$.
- A closed, densely defined operator $A \in \mathsf{L}(\mathcal{X}',\mathcal{X}) \cup \widetilde{\mathsf{L}}(\mathcal{X}',\mathcal{X})$ is said to be self-adjoint if $J_{\mathcal{X}} \cdot A = A'$.
- For any self-adjoint $A \in \mathsf{L}(\mathcal{X}',\mathcal{X}) \cup \widetilde{\mathsf{L}}(\mathcal{X}',\mathcal{X})$ we define
$$\gamma(A) := \inf\{\,\ell(A\ell),\ \ell \in D(A),\ \|\ell\|_{\mathcal{X}'} = 1\,\}\,.$$
- If $\mathcal{H}$ is a complex Hilbert space with scalar product (conjugate linear w.r.t. the first variable) $\langle\cdot,\cdot\rangle$, then $C_{\mathcal{H}} \in \widetilde{\mathcal{B}}(\mathcal{H},\mathcal{H}')$ denotes the isomorphism defined by $(C_{\mathcal{H}}\,y)(x) := \langle y, x\rangle$. The Hilbert adjoint of the densely defined linear operator A is then given by $A^* = C_{\mathcal{H}}^{-1} \cdot A' \cdot C_{\mathcal{H}}$.
- $\mathcal{F}$ and $*$ denote Fourier transform and convolution respectively.
- $\mathcal{D}'(\mathbb{R}^n)$ denotes the space of distributions and $\mathcal{E}'(\mathbb{R}^n)$ is the subspace of distributions with compact support.
- $H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, is the usual scale of Sobolev-Hilbert spaces, i.e. $H^s(\mathbb{R}^n)$ is the space of tempered distributions with a Fourier transform which is square integrable w.r.t. the measure with density $(1+|x|^2)^s$.
- c denotes a generic strictly positive constant which can change from line to line.

3. Singular Perturbations and Boundary Conditions

Given the injective self-adjoint operator $A : D(A) \subseteq \mathcal{H} \to \mathcal{H}$, we denote by $\widehat{D}(A)$ the Banach space given by the completion of $D(A)$ with respect to the norm
$$\|\phi\|_{(A)} := \|A\phi\|_{\mathcal{H}}\,.$$
As usual $D(A)$ will be treated as a (dense) subset of $\widehat{D}(A)$ by means of the canonical embedding $\mathcal{I} : D(A) \to \widehat{D}(A)$ which associates to ϕ the set of all Cauchy sequences converging to ϕ.

As in the introduction we consider then a continuous linear map

$$\tau : D(A) \to \mathcal{X}\,,$$

$\mathcal{X}$ is a Banach space, and we will suppose that it satisfies, besides (h1) and (h2), the further hypothesis

$$\|\tau\phi\|_{\mathcal{X}} \le c\,\|A\phi\|_{\mathcal{H}}\,. \tag{h3}$$

By (h3) τ admits an extension belonging to $\mathcal{B}(\widehat{D}(A),\mathcal{X})$; analogously A admits an extension belonging to $\mathcal{B}(\widehat{D}(A),\mathcal{H})$. By abuse of notation we will use the same simbols τ and A to denote these extensions.

Let us now take a sequence $\{\epsilon_n\}_1^\infty \subset \mathbb{R}$ converging to zero. By functional calculus one has

$$\|(-A\cdot R(i\epsilon_n)-I)\phi\|_{\mathcal{H}}^2 = \int_{\sigma(A)} d\mu_\phi(\lambda)\,\frac{\epsilon_n^2}{\lambda^2+\epsilon_n^2}$$

with $\mu_\phi(\{0\})=0$ since $0\notin\sigma_{pp}(A)$. Thus

$$1 \ge \frac{\epsilon_n^2}{\lambda^2+\epsilon_n^2} \longrightarrow 0\,, \qquad \mu_\phi\text{-a.e.}$$

and, by dominated convergence theorem,

$$\mathcal{H}\text{-}\lim_{n\uparrow\infty} -A\cdot R(i\epsilon_n)\phi = \phi\,.$$

So $\{R(i\epsilon_n)\phi\}_1^\infty$ is a Cauchy sequence in $D(A)$ with respect to the norm $\|\cdot\|_{(A)}$. We can therefore define $R\in\mathcal{B}(\mathcal{H},\widehat{D}(A))$ by

$$R\phi := \widehat{D}\text{-}\lim_{n\uparrow\infty} R(i\epsilon_n)\phi\,,$$

and then $K(z)\in\widetilde{\mathcal{B}}(\mathcal{X}',\widehat{D}(A))$ by

$$K(z) := zR\cdot G(z)\,.$$

Alternatively, using (2), $K(z)$ can be defined by

$$K(z)\phi := \widehat{D}\text{-}\lim_{n\uparrow\infty} \left(G(i\epsilon_n)-G(z)\right)\phi\,.$$

This immediately implies, using (3),

$$\forall\, w,z\in\rho(A), \qquad \text{Range}\,(\,K(w)-K(z)\,)\subseteq D(A)$$

and

$$\forall\, w,z\in\rho(A), \qquad K(w)-K(z) = G(z)-G(w)\,. \tag{4}$$

Also note that

$$-A\cdot K(z) = zG(z)\,. \tag{5}$$

LEMMA 3. *The map*

$$\Gamma : \rho(A) \to \widetilde{\mathcal{B}}(\mathcal{X}', \mathcal{X})\,, \qquad \Gamma(z) := \tau \cdot K(z)$$

satisfies the relations

$$\Gamma(z) - \Gamma(w) = (z - w)\, \breve{G}(w) \cdot G(z) \tag{6}$$

and

$$J_{\mathcal{X}} \cdot \Gamma(z^*) = \Gamma(z)'\,, \tag{7}$$

PROOF. Since $K(z)$ is the strong limit of $G(\pm i\epsilon_n) - G(z)$, one has

$$\forall\, \ell \in \mathcal{X}'\,, \qquad \Gamma(z)\ell = \lim_{n\uparrow\infty} \hat{\Gamma}_n(z)\ell\,,$$

where

$$\hat{\Gamma}_n(z) : \mathcal{X}' \to \mathcal{X}\,, \qquad \hat{\Gamma}_n(z) := \tau \cdot \left(\frac{G(i\epsilon_n) + G(-i\epsilon_n)}{2} - G(z) \right)\,.$$

Thus $\Gamma(z)$ satisfies (6) and

$$\forall\, \ell_1, \ell_2 \in \mathcal{X}'\,, \qquad \ell_1(\Gamma(z^*)\ell_2) = (\ell_2(\Gamma(z)\ell_1))^*$$

(which is equivalent to (7)) since $\hat{\Gamma}_n(z)$ does (see [**18**, lemma 2.2]). □

Before stating the next lemma we introduce the following definition:
Given $\phi \in \mathcal{H}$ and $\psi \in \widehat{D}(A)$, the writing $\phi = \psi$ will mean that ϕ is in $D(A)$ and $\mathcal{I}\phi = \psi$.

LEMMA 4. *Given $\phi \in \mathcal{H}$, $z \in \rho(A)$, suppose there exist $\psi \in \widehat{D}(A)$ and $Q \in \mathcal{X}'$ such that*

$$\phi - G(z)Q = \psi + K(z)Q\,. \tag{8}$$

Then the couple (ψ, Q) is unique and z-independent.

PROOF. Let (ψ_1, Q_1), (ψ_2, Q_2) both satisfy (8). Then

$$G(z)\,(Q_2 - Q_1) = (\psi_1 - \psi_2) + K(z)\,(Q_1 - Q_2)\,.$$

By (h2) and the definition of $G(z)$ one has Range$\,(G(z)) \cap D(A) = \{0\}$ and so $Q_1 - Q_2 \in$ Kernel$\,(G(z))$. But (h1) implies the injectivity of $G(z)$ (see [**18**, remark 2.1]). Therefore $(\psi_1, Q_1) = (\psi_2, Q_2)$. The proof is then concluded observing that z-independence follows by (4). □

We now can extend theorem 1 to the case in which A is injective:

THEOREM 5. *Let $A : D(A) \subseteq \mathcal{H} \to \mathcal{H}$ be self-adjoint and injective, let $\tau : D(A) \to \mathfrak{X}$ satisfy* (h1)-(h3). *Given $\Theta \in \widetilde{\mathsf{L}}(\mathfrak{X}', \mathfrak{X})$ self-adjoint, let $D(A^\tau_\Theta)$ be the set of $\phi \in \mathcal{H}$ for which there exist $\phi_{\rm reg} \in \widehat{D}(A)$, $Q_\phi \in D(\Theta)$ such that*

$$\phi - G(z)Q_\phi = \phi_{\rm reg} + K(z)Q_\phi$$

and

$$\tau\phi_{\rm reg} = \Theta\, Q_\phi\,.$$

Then

$$A^\tau_\Theta : D(A^\tau_\Theta) \subseteq \mathcal{H} \to \mathcal{H}\,, \qquad A^\tau_\Theta\, \phi := A\, \phi_{\rm reg}\,.$$

is a self-adjoint operator which coincides with A on the kernel of τ and its resolvent is given by

$$R^\tau_\Theta(z) := R(z) + G(z) \cdot (\,\Theta + \Gamma(z)\,)^{-1} \cdot \breve{G}(z)\,, \qquad z \in W^-_\Theta \cup W^+_\Theta \cup \mathbb{C}\backslash\mathbb{R}\,.$$

where

$$\Gamma(z) := \tau \cdot K(z)\,.$$

PROOF. For brevity we define

$$\phi_z := \phi - G(z)Q_\phi\,, \qquad \phi \in D(A^\tau_\Theta)$$

and

$$\Gamma_\Theta(z) := \Theta + \Gamma(z)\,.$$

Then one has

$$\Gamma_\Theta(z)Q_\phi = \tau\phi_{\rm reg} + \tau \cdot K(z)Q_\phi = \tau\phi_z\,.$$

Since $\Gamma(z)$ is a bounded operator satisfying (6) and (7), and Θ is self-adjoint, by (h1) and [**18**, prop. 2.1, remark 2.12], $\Gamma_\Theta(z)$ has a bounded inverse for any $z \in W^-_\Theta \cup W^+_\Theta \cup \mathbb{C}\backslash\mathbb{R}$. Therefore

$$Q_\phi = \Gamma_\Theta(z)^{-1} \cdot \tau\phi_z$$

and

$$D(A^\tau_\Theta) \subseteq \left\{\phi \in \mathcal{H} \,:\, \phi = \phi_z + G(z) \cdot \Gamma_\Theta(z)^{-1} \cdot \tau\,\phi_z,\, \phi_z \in D(A)\right\}\,.$$

Let us now prove the reverse inclusion.

Given $\phi \in \mathcal{H}$,

$$\phi = \phi_z + G(z) \cdot \Gamma_\Theta(z)^{-1} \cdot \tau\,\phi_z\,, \qquad \phi_z \in D(A)\,,$$

we define $\phi_{\rm reg} \in \widehat{D}(A)$ by

$$\phi_{\rm reg} := \phi_z - K(z) \cdot \Gamma_\Theta(z)^{-1} \cdot \tau\,\phi_z\,.$$

Thus one has

$$\begin{aligned}\tau\phi_{\rm reg} &= \tau\phi_z - \tau \cdot K(z) \cdot \Gamma_\Theta(z)^{-1} \cdot \tau\,\phi_z\\ &= \tau\phi_z - \tau\phi_z + \Theta \cdot \Gamma_\Theta(z)^{-1} \cdot \tau\,\phi_z = \Theta\, Q_\phi\,,\end{aligned}$$

with

$$Q_\phi := \Gamma_\Theta(z)^{-1} \cdot \tau\, \phi_z\,.$$

In conclusion

$$D(A^\tau_\Theta) = \left\{\phi \in \mathcal{H} \,:\, \phi = \phi_z + G(z) \cdot \Gamma_\Theta(z)^{-1} \cdot \tau\, \phi_z,\ \phi_z \in D(A)\right\}\,.$$

Since, by (5),

$$A\phi_{\text{reg}} = A\phi_z - A \cdot K(z)Q_\phi = A\phi_z + zG(z)Q_\phi$$

we have

$$(-A^\tau_\Theta + z)\phi = (-A + z)\phi_z\,.$$

Thus A^τ_Θ coincides with the operator constructed in [**18**, thm. 2.1]; therefore, by (h2), this operator is self-adjoint, has resolvent given by $R^\tau_\Theta(z)$ and is equal to A on the kernel of τ. □

4. Singular perturbations of convolution operators

Let $T \in \mathcal{E}'(\mathbb{R}^n)$. By Paley-Wiener theorem we know that $\mathcal{F}T$ is a smooth function which is, together with its derivatives of any order, polynomially bounded. Then we define the continuous convolution operator

$$T* : \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)\,, \qquad \phi \mapsto T * \phi\,,$$

and, supposing $\mathcal{F}T$ real-valued, the restriction of this operator to the dense subspace

$$D(C) := \left\{\phi \in L^2(\mathbb{R}^n) \ :\ T * \phi \in L^2(\mathbb{R}^n)\right\}$$

provide us with the self-adjoint convolution operator

$$C : D(C) \subseteq L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)\,, \qquad C\phi := T * \phi\,.$$

Evidently C is injective if and only if the set of real zeroes of $\mathcal{F}T$ is a null set. From now on we will therefore suppose that $\mathcal{F}T$ is real and has a null set of real zeroes. This implies, since $(\text{Range}\,(C))^\perp = \text{Kernel}\,(C^*)$, that C has a dense range and, using (h3), the following lemma becomes then obvious:

Lemma 6. *The map $\tau : D(C) \to \mathfrak{X}$ can be extended to*

$$\mathfrak{D}(C) := \left\{\varphi \in \mathcal{D}'(\mathbb{R}^n) \ :\ T * \varphi \in L^2(\mathbb{R}^n)\right\}\,,$$

by defining

$$\tau\varphi := \lim_{n\uparrow\infty} \tau\phi_n,\,,$$

where $\{\phi_n\}_1^\infty \subset D(C)$ is any sequence such that

$$L^2\text{-}\lim_{n\uparrow\infty} T * \phi_n = T * \varphi\,.$$

Now we will moreover suppose that $\mathcal{F}T$ is slowly decreasing, i.e. (see [**5**]) we will suppose that there exist $k>0$ such that for any $\zeta\in\mathbb{R}^n$ we can find a point $\xi\in\mathbb{R}^n$ such that

$$|\zeta-\xi|\le k\log(1+|\zeta|)\,,$$
$$|\mathcal{F}T(\xi)|\ge(k+|\xi|)^{-k}\,.$$

By [**5**, thm. 1] we know that $T*:\mathcal{D}'(\mathbb{R}^n)\to\mathcal{D}'(\mathbb{R}^n)$ is surjective if and only if $\mathcal{F}T$ is slowly decreasing. This is certainly true when C is a differential operator, i.e. when $\mathcal{F}T$ is a polynomial. The hypotheses we made on $\mathcal{F}T$ permit us to state the following

LEMMA 7. *Given T as above, one has the identification*

$$\widehat{D}(C)\,\simeq\,\mathcal{D}(C)/\sim\,,$$

where

$$\varphi_1\sim\varphi_2\qquad\Longleftrightarrow\qquad T*\varphi_1=T*\varphi_2\,.$$

This identification is given by the isometric maps which to the equivalence class of Cauchy sequences $[\{\phi_n\}_1^\infty]\in\widehat{D}(C)$ associates the equivalence class of distributions $[\varphi]\in\mathcal{D}(C)/\sim$ such that

$$L^2\text{-}\lim_{n\uparrow\infty}T*\phi_n=T*\varphi\,.$$

PROOF. Given $[\{\phi_n\}_1^\infty]\in\widehat{D}(C)$, the sequence $\{T*\phi_n\}_1^\infty$ is a Cauchy one in $L^2(\mathbb{R}^n)$ and so it converges to some $f\in L^2(\mathbb{R}^n)$. Then, by [**5**, thm. 1], there exists $\varphi\in\mathcal{D}(C)$ such that $T*\varphi=f$. Conversely let $\varphi\in\mathcal{D}(C)$; since C has a dense range there exists a (unique in $\widehat{D}(C)$) sequence $\{\phi_n\}_1^\infty\subset D(C)$ such that $T*\phi_n$ converges in $L^2(\mathbb{R}^n)$ to $T*\varphi$. □

Defining

$$\widetilde{\mathcal{D}}'(\mathbb{R}^n):=\mathcal{D}'(\mathbb{R}^n)/\sim$$

and then the sum of $\phi\in L^2(\mathbb{R}^n)\subset\mathcal{D}'(\mathbb{R}^n)$ plus $\psi=[\varphi]\in\widehat{D}(C)\,\simeq\,\mathcal{D}(C)/\sim$ $\subseteq\widetilde{\mathcal{D}}'(\mathbb{R}^n)$ by

$$\phi+\psi:=[\phi+\varphi]\in\widetilde{\mathcal{D}}'(\mathbb{R}^n)\,,$$

we can introduce the linear operator

$$G:\mathcal{X}'\to\widetilde{\mathcal{D}}'(\mathbb{R}^n)\,,\qquad G:=G(z)+K(z)\,.$$

According to lemma 4, theorem 5 and the definition of G, for any $\phi\in D(C^\tau_\Theta)$ we can give the unique decomposition

$$\phi=\phi_{\text{reg}}+GQ_\phi\,.$$

Thus we can define $D(C^\tau_\Theta)$ as the set of $\phi \in L^2(\mathbb{R}^n)$ for which there exists $Q_\phi \in D(\Theta)$ such that

$$\phi - GQ_\phi =: \phi_{\text{reg}} \in \widehat{D}(C)$$

and

$$\tau\phi_{\text{reg}} = \Theta\, Q_\phi \,.$$

LEMMA 8. *The definition of G is z-independent and*

$$\forall\, \ell \in \mathcal{X}'\,, \qquad G\ell = [G_*\ell]\,,$$

where

$$G_* : \mathcal{X}' \to \mathcal{D}'(\mathbb{R}^n)\,,$$

is any conjugate linear operator such that

$$-T * G_*\ell = \tau^*\ell\,. \tag{9}$$

Here $\tau^* : \mathcal{X}' \to \mathcal{D}'(\mathbb{R}^n)$ is defined by

$$\tau^*\ell(\varphi) := (\,\ell(\tau\varphi^*)\,)^*\,, \qquad \varphi \in C_0^\infty(\mathbb{R}^d)\,.$$

PROOF. z-independence is an immediate consequence of (4). By the definition of G and by (5) there follows

$$-T * G_*\ell = (-T * + z)G(z)\ell$$

and the proof is concluded by the relation

$$(-T * + z)\cdot G(z) = \tau^* \tag{10}$$

which can be obtained proceeding as in [**18**, remark 2.4]. □

REMARK 9. By [**5**, thm. 1], as $\mathcal{F}T$ is slowly decreasing, the equation (9) is always resoluble; in particular, denoting the fundamental solution of $-T*$ by $\mathcal{G}$, when the convolution $\mathcal{G} * \tau^*\ell$ is well defined (e.g. when $\tau^*\ell \in \mathcal{E}'(\mathbb{R}^n)$), one has

$$G_* : \mathcal{X}' \to \mathcal{D}'(\mathbb{R}^n)\,, \qquad G_*\ell = \mathcal{G} * \tau^*\ell\,.$$

Analogously, denoting the fundamental solution of $-T * + z$ by $\mathcal{G}_z$, one has

$$G(z) : \mathcal{X}' \to L^2(\mathbb{R}^n)\,, \qquad G(z)\ell = \mathcal{G}_z * \tau^*\ell\,.$$

REMARK 10. Note that $C * \phi_{\text{reg}} = T * \varphi$ and $\tau\phi_{\text{reg}} = \tau\varphi$ for any $\varphi \in \mathcal{D}(C)$ such that $\phi_{\text{reg}} = [\varphi]$. Here we implicitly used the extension given in lemma 6 and the identification given in lemma 7. This also implies that $\Gamma(z)$ in lemma 3 can be re-written as

$$\Gamma(z) = \tau \cdot (G_* - G(z))\ .$$

By remark 9, when the convolution is well defined, one can also write

$$\Gamma(z)\ell = \tau\left((\mathcal{G} - \mathcal{G}_z) * \tau^*\ell\right)\ .$$

In conclusion, by making use of the previous lemmata and remarks, we can restate theorem 5 in the following way:

THEOREM 11. *Let $T \in \mathcal{E}'(\mathbb{R}^n)$ with $\mathcal{F}T$ real-valued, slowly decreasing and having a null set of real zeroes, let $C : D(C) \subseteq L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, $C\phi := T * \phi$, let $\tau : D(C) \to \mathcal{X}$ satisfy* (h1)-(h3). *Given $\Theta \in \widetilde{\mathsf{L}}(\mathcal{X}', \mathcal{X})$ self-adjoint, let $D(C^\tau_\Theta)$ be the set of $\phi \in L^2(\mathbb{R}^n)$ for which there exists $Q_\phi \in D(\Theta)$ such that*

$$\phi - G_* Q_\phi =: \varphi_{\text{reg}} \in \mathcal{D}(C)$$

and

$$\tau\varphi_{\text{reg}} = \Theta\, Q_\phi\,.$$

Then

$$C^\tau_\Theta : D(C^\tau_\Theta) \subseteq L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)\,, \qquad C^\tau_\Theta\, \phi := T * \varphi_{\text{reg}}\,,$$

is a self-adjoint operator which coincides with C on the kernel of τ and its resolvent is given by

$$R^\tau_\Theta(z) := R(z) + G(z) \cdot (\,\Theta + \Gamma(z)\,)^{-1} \cdot \breve{G}(z)\,, \qquad z \in W^-_\Theta \cup W^+_\Theta \cup \mathbb{C}\backslash\mathbb{R}\,,$$

where

$$\Gamma(z) := \tau \cdot (G_* - G(z))\,.$$

REMARK 12. The boundary conditions and the operators $\Gamma(z)$ and C^τ_Θ appearing in the previous theorem are independent of the choice of the representative (see lemma 8) G_* entering in the definition of φ_{reg}. Indeed any different choice will not change the equivalence class to which φ_{reg} belongs, and both τ and C do not depend on the representative in such a class (see remark 10).

REMARK 13. Proceeding as in [**18**, remark 2.4] one can give the following alternative definition of C^τ_Θ where only $Q_\phi \in D(\Theta)$ appears:

$$C^\tau_\Theta\, \phi := T * \phi + \tau^* Q_\phi\,.$$

This is an immediate consequence of identity (10).

EXAMPLE 14. Let us consider the case $A = \Delta : H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. Obviously A is an injective convolution operator, thus we can apply to it the previous theorem.

A Borel set $F \subset \mathbb{R}^n$ is called a d-set, $d \in (0, n]$, if

$$\exists\, c_1,\, c_2 > 0 \;:\; \forall\, x \in F,\ \forall\, r \in (0,1), \qquad c_1 r^d \le \mu_d(B_r(x) \cap F) \le c_2 r^d\,,$$

where μ_d is the d-dimensional Hausdorff measure and $B_r(x)$ is the closed n-dimensional ball of radius r centered at the point x (see [**10**, §1.1, chap. VIII]). Examples of d-sets are d-dimensional Lipschitz submanifolds and (when d is not an integer) self-similar fractals of Hausdorff dimension d (see [**10**, chap.

II, example 2]). Moreover a finite union of d-sets which intersect on a set of zero d-dimensional Hausdorff measure is a d-set.

In the case $0 < n - d < 4$ we take as the linear operator τ the unique continuous surjective (thus (h1) holds true) map

$$\tau_F : H^2(\mathbb{R}^n) \to B^{2,2}_\alpha(F)\,, \qquad \alpha = 2 - \frac{n-d}{2}$$

such that, for μ_d-a.e. $x \in F$,

$$\tau_F\phi(x) \equiv \left\{\phi_F^{(j)}(x)\right\}_{|j|<\alpha} = \left\{\lim_{r\downarrow 0} \frac{1}{\lambda_n(r)} \int_{B_r(x)} dy\, D^j\phi(y)\right\}_{|j|<\alpha}\,, \tag{11}$$

where $j \in \mathbb{Z}^n_+$, $|j| := j_1 + \cdots + j_n$, $D^j := \partial_{j_1} \cdots \partial_{j_n}$ and $\lambda_n(r)$ denotes the n-dimensional Lebesgue measure of $B_r(x)$. We refer to [**10**, thms. 1 and 3, chap. VII] for the existence of the map τ_F; obviously it coincides with the usual evaluation along F when restricted to smooth functions. The definition of the Besov-like (actually Hilbert, see remark 1) space $B^{2,2}_\alpha(F)$ is quite involved and we will not reproduce it here (see [**10**, §2.1, chap. V]). In the case $0 < \alpha < 1$ (i.e. $2 < n - d < 4$) things simplify and $B^{2,2}_\alpha(F)$ can be defined (see [**10**, §1.1, chap. V]) as the Hilbert space of $f \in L^2(F;\mu_F)$ having finite norm

$$\|f\|^2_{B^{2,2}_\alpha(F)} := \|f\|^2_{L^2(F)} + \int_{|x-y|<1} d\mu_F(x)\, d\mu_F(y)\, \frac{|f(x)-f(y)|^2}{|x-y|^{d+2\alpha}}\,,$$

where μ_F denotes the restriction of the d-dimensional Hausdorff measure μ_d to the set F. When $\alpha > 1$ and F is a generic d-set the functions $\phi_F^{(j)} \in L^2(F;\mu_F)$ are not uniquely determined by $\phi_F^{(0)}$; contrarily we may then identify $\{\phi_F^{(j)}\}_{|j|<\alpha}$ with the single function $\phi_F^{(0)}$. This is possible when F preserves Markov's inequality (see [**10**, §2, chap. II]). Sets with such a property are closed d-sets with $d > n - 1$ (see [**10**, thm. 3, §2.2, chap. II]), a concrete example being e.g. the boundary of von Koch's snowflake domain in $\mathbb{R}^2$ (a d-set with $d = \log 4/\log 3$, see [**23**]). If F has some additional differential structure then $B^{2,2}_\alpha(F) \simeq H^\alpha(F)$, where $H^\alpha(F)$ denotes the usual (fractional) Sobolev-Slobodeckiĭ space. Some known cases where $B^{2,2}_\alpha(F) \simeq H^\alpha(F)$ (for any value of $\alpha > 0$) are the following:

- F is the graph of a Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}^{n-d}$ (see [**6**, §20]);
- F is a bounded manifold of class C_γ, $\gamma > \min(3, \max(1,\alpha))$, i.e. F has an atlas where the transition maps are of class C^k, $k < \gamma \le k+1$, and have derivatives of order less or equal to k which satisfy Lipschitz conditions of order $\gamma - k$ (see [**12**] for the case $\gamma > \max(1,\alpha)$ and see [**6**, §24] for the case $\gamma > 3$);

- F is a connected complete Riemannian manifold with positive injectivity radius and bounded geometry, in particular a connected Lie group (see [**22**, §7.4.5, §7.6.1]).

Supposing now that F has a compact closure, let B_1 and B_2 be two n-dimensional balls such that $F\subset B_1\subset B_2$; given $\phi\in H^2(\mathbb{R}^n)$, let $\tilde\phi\in H^2(\mathbb{R}^n)$ coincide with ϕ on B_1, have support contained in B_2 and satisfy $\|\Delta\tilde\phi\|_{L^2}\le c\,\|\Delta\phi\|_{L^2}$. Then one has, by Poincaré inequality applied to $\tilde\phi\in H^2_0(B_2)$,

$$\|\tau\phi\|_{B^{2,2}_\alpha(F)}=\|\tau\tilde\phi\|_{B^{2,2}_\alpha(F)}\le c\,\|\tilde\phi\|_{H^2(\mathbb{R}^n)}\le c\,\|\Delta\tilde\phi\|_{L^2(\mathbb{R}^n)}\le c\,\|\Delta\phi\|_{L^2(\mathbb{R}^n)}\,,$$

and so τ_F satisfies (h3). Moreover, since

$$\mathcal{D}(\Delta)=\left\{\varphi\in\mathcal{D}'(\mathbb{R}^n)\,:\,\Delta\phi\in L^2(\mathbb{R}^n)\right\}\subseteq H^2_{\text{loc}}(\mathbb{R}^n)$$

and $\bar F$ is supposed to be compact, the extension of τ_F to $\mathcal{D}(\Delta)$ is again defined by (11). Denoting the dual of $B^{2,2}_\alpha(F)$ by $B^{2,2}_{-\alpha}(F)$ (the space $B^{2,2}_{-\alpha}(F)$ can be explicitly characterized in the case $0<\alpha<1$ or when F preserves Markov's inequality, see [**14**]), hypothesis (h2) is equivalent to $\tau'_F\ell\notin L^2(\mathbb{R}^n)$ for any $\ell\in B^{2,2}_{-\alpha}(F)\backslash\{0\}$, where $\tau'_F\ell\in H^{-2}(\mathbb{R}^n)$ is defined by

$$\tau'_F\ell(\phi):=\ell(\tau_F\phi)\,.$$

Therefore, as the support of $\tau'_F\ell$ is given by $\bar F$, (h2) is certainly verified when $\bar F$ has zero Lebesgue measure. Considering the fundamental solution of $-\Delta$, given by

$$\mathcal{G}(x)=\begin{cases}\frac{1}{2\pi}\log\frac{1}{|x|}, & \text{for } n=2\\ \frac{1}{(n-2)\sigma_n}\,\frac{1}{|x|^{n-2}}, & \text{for } n>2\,,\end{cases}$$

σ_n the measure of the unitary sphere in $\mathbb{R}^n$, the convolution $\mathcal{G}*\tau'_F\ell$ is well distribution defined as τ'_F is in $\mathcal{E}'(\mathbb{R}^n)$. Therefore, by lemma 8 we can choose G_* to be the map

$$G_*:B^{2,2}_{-\alpha}(F)\to\mathcal{D}'(\mathbb{R}^n)\,,\qquad G_*\ell:=\mathcal{G}*\tau^*_F\ell\,.$$

Thus, by the previous theorem (and remark 13), supposing that the d-set F has a compact closure of zero Lebesgue measure, given any self-adjoint operator $\Theta\in\widetilde{\mathsf{L}}(B^{2,2}_{-\alpha}(F),B^{2,2}_\alpha(F))$, $\Theta\in\mathsf{L}(B^{2,2}_\alpha(F))$ if one uses the identification $B^{2,2}_{-\alpha}(F)\simeq B^{2,2}_\alpha(F)$, we have then the self-adjoint operator

$$\Delta^F_\Theta\,\phi:=\Delta\varphi_{\text{reg}}\equiv\Delta\phi+\tau^*_FQ_\phi\,,$$

where

$$\phi=\varphi_{\text{reg}}+\mathcal{G}*\tau^*_FQ_\phi\,,\qquad\varphi_{\text{reg}}\in\mathcal{D}(\Delta)\,,\quad Q_\phi\in D(\Theta)\,,$$

and

$$\left\{\lim_{r\downarrow0}\frac{1}{\lambda_n(r)}\int_{B_r(x)}dy\,D^j\left(\phi-\mathcal{G}*\tau^*_FQ_\phi\right)(y)\right\}_{|j|<\alpha}=\Theta\,Q_\phi(x)\,,$$

$|j|=0$ if F preserves Markov's inequality or $B^{2,2}_\alpha(F)\simeq H^\alpha(F)$. When $F=M$, M a connected complete Riemannian manifold with positive injectivity radius and bounded geometry, a natural choice for Θ is given by $\Theta=(-\Delta_M+\lambda^2)^{-\alpha}$ ($\lambda=0$ if M is compact), where Δ_M denotes the Laplace-Beltrami operator. A particular case of this situation, when $A=(\Delta-\epsilon):H^2_0(\Omega)\to L^2(\Omega)$, $\epsilon>0$, $\Omega=\mathbb{R}^2\times[0,\pi]$, (note that here $0\notin\sigma(A)$, thus theorem 1 applies), and $F=[0,\pi]$, is treated, without giving boundary conditions, in [**15**] (also see [**18**, example 3.2] for connections with Birman-Kreĭn-Vishik theory).

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