LIFSHITZ TAILS FOR RANDOM ACOUSTIC OPERATORS

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Abstract. Using techniques developed in [Kp1], we prove that the integrated density of states of $A_\omega = -\nabla \frac{1}{\varrho_\omega} \nabla$ has Lifshitz behavior in the neighborhood of the internal spectral gaps if and only if the integrated density of states of some periodic operator is non degenerated on the same edge of the spectral gaps.
From this result we obtain the initial estimates necessary for deducing localization near non degenerate band edges.

1. Introduction:

We consider A_ω, the self adjoint operator on $L^2(\mathbb{R}^d)$ formally defined by:

$$A_\omega = A(\varrho_\omega) = -\nabla \frac{1}{\varrho_\omega(x)} \nabla = -\sum_{i=1}^{d} \partial_{x_i} \frac{1}{\varrho_\omega(x)} \partial_{x_i}, \tag{1.1}$$

where the real function $\varrho_\omega(x)$ satisfies

$$0 < \varrho_- \leq \varrho_\omega(x) \leq \varrho_+ < \infty,$$

for some constants ϱ_- and ϱ_+.
We start by defining the main object of our study, the integrated density of states. We consider a cube $\Lambda \subset \mathbb{R}^d$; we note by $A_{\omega,\Lambda}$ the restriction of A_ω to Λ with Dirichlet boundary conditions. As A_ω is an elliptic operator, the resolvent of $A_{\omega,\Lambda}$ is compact and, consequently, the spectrum of $A_{\omega,\Lambda}$ is discrete and is made of isolated eigenvalues of finite multiplicity ([Re,Si4]). We define

$$N_\Lambda(E) = \frac{1}{\mathrm{vol}(\Lambda)} \cdot \#\{\text{eigenvalues of } A_{\omega,\Lambda} \leq E\}. \tag{1.2}$$

Here the volume of the cube Λ is taken in the Lebesgue sense.
It is shown that the limit of $N_\Lambda(E)$ when Λ tends to $\mathbb{R}^d$ exists almost surely and is independent of the boundary conditions. It is called the **integrated density of states** of A_ω (IDS for the short form). See [Kir,Mar1].
The question we are interested in here regards the behavior of N at the edges of the spectrum of A_ω.

1.1. The behavior of the IDS.

We give a brief history of the subject. In 1964, Lifshitz [Lif] argued that, for a Schrödinger operator $H_\omega = -\Delta + V_\omega$, there exists $c_1, c_2 > 0$ such that $N(E)$ satisfies the asymptotic :

$$N(E) \simeq c_1 \exp(-c_2 (E - E_0)^{-\frac{d}{2}}), \quad E \to E_0. \tag{1.3}$$

Here E_0 is the bottom of the spectrum of H_ω. The behavior (1.3) is known as **Lifshitz tails** (for more details see part IV.9.A of [Pas,Fig]). Lifshitz predicted (1.3) not only at the bottom of the spectrum but also at fluctuating edges inside the spectrum. We call this asymptotic “internal Lifshitz tails”.
The principal results known on Lifshitz tails are mainly shown for Schrödinger

operators (in the continuous and discrete cases). (see [Na][Pas], [Kir,Mar2],[Sim1], [Kp1] and others).
Lifshitz tails for an operator of type (1.1), to our knowledge, was never studied until now. However Kozlov in [Koz] gives the Weyl asymptotic of $N(E)$ in the neighborhood of 0.

1.2. **The result.** The essential goal of this work is to study the Lifshitz behavior for the operator defined by (1.1). Using the techniques used in [Kp1], one proves that the IDS has a Lifshitz behavior in a neighborhood of the spectral gaps.
To present our result we will consider the following plan:
In section 2, we define the model to be studied and specify various assumptions. We introduce a reference periodic operator (A_{ω^+}). We state the principal theorem (Theorem 2.1) which gives us the Lifshitz tails. We give an interesting application of our result (Theorem 2.3).
To show Theorem 2.1, we start by locating the integrated density of states of our random operator in the neighborhood of an energy E_+. This will be done in sections 3 and 4.
Section 5 is devoted for the proof of Theorem 2.1. In section 6 we give the proof of Theorem 2.3.
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2. The model

Let $\Gamma = \mathbb{Z}^d$. Let $\varrho_\omega(x)$ a function which satisfies
(**H.0**)

$$\varrho_\omega(x) = \varrho_0(x)\Big(1 + \sum_{\gamma\in\Gamma} \omega_\gamma u_\gamma(x)\Big),$$

where
(i) $\varrho_0(x)$ is measurable with real values and Γ-periodic function i.e.

$$\varrho_0(x) = \varrho_0(x+\gamma),\ \forall x \in \mathbb{R}^d, \gamma \in \Gamma.$$

(ii) There exist constants $\varrho_{0,+} > \varrho_{0,-} > 0$ such that for almost all $x \in \mathbb{R}^d$, we have

$$0 < \varrho_{0,-} \leq \varrho_0(x) \leq \varrho_{0,+}. \tag{2.4}$$

(iii) For $\gamma \in \Gamma$, one sets $u_\gamma(x) = u(x-\gamma)$. We suppose that u is real values function such that there exists $U_+ > 0$: such that for almost all $x \in \mathbb{R}^d$

$$0 \leq U(x) \equiv \sum_{\gamma\in\Gamma} u_\gamma \leq U_+ < \infty. \tag{2.5}$$

(iv) $(\omega_\gamma)_{\gamma\in\Gamma}$ is a family of non constant and positive, independent identically distributed random variables whose common probability messier is noted by $\mathbb{P}_{\omega_0}$. We note the probability space by $(\Omega, \mathcal{F}, \mathbb{P})$.
We define the quadratic form $\mathcal{A}(\varrho_\omega)$ by: for $u \in H^1(\mathbb{R}^d) = \mathcal{D}(\mathcal{A}(\varrho_\omega))$

$$\mathcal{A}(\varrho_\omega)[u,u] = \int_{\mathbb{R}^d} \frac{1}{\varrho_\omega(x)} \nabla u(x)\overline{\nabla u(x)}dx.$$

$\mathcal{A}(\varrho_\omega)$ is a symmetric, closed and positive quadratic form. A_ω given by (1.1) is defined as the self adjoint operator associated to $\mathcal{A}(\varrho_\omega)$ [Re,Si4].

(H.0) implies that A_ω is a family of measurable selfadjoint operators [Fig,Kle] and ergodic. Indeed, if τ_γ refers to the translation by γ, then $(\tau_\gamma)_{\gamma\in\Gamma}$ is a group of unitary operators on $L^2(\mathbb{R}^d)$ and for $\gamma \in \Gamma$ we have

$$\tau_\gamma A_\omega \tau_{-\gamma} = A_{\tau_\gamma \omega}.$$

According to [Kir,Mar3] we know that there exists $\Sigma, \Sigma_{pp}, \Sigma_{ac}$ and Σ_{sc} closed and non random sets of $\mathbb{R}$ such that Σ is the spectrum of A_ω with probability one and such that if σ_{pp} (respectively σ_{ac} and σ_{sc}) indicates the pure point spectrum (respectively the absolutely continuous and singular continuous spectrum) of A_ω, then $\Sigma_{pp} = \sigma_{pp}, \Sigma_{ac} = \sigma_{ac}$ and $\Sigma_{sc} = \sigma_{sc}$ with probability one.

2.1. **Reference operator.** It's convenient to consider A_ω as a perturbation of some periodic operator A_{ω^+}. More precisely, for $\varrho_{\omega^+}(x) = \varrho_0(x)\Big(1 + \sum_{\gamma\in\mathbb{Z}^d} u_\gamma(x)\Big)$, we can write :

$$A_\omega = A_{\omega^+} + \Delta A_\omega,$$

with

$$A_{\omega^+} = A(\varrho_{\omega^+}) \text{ and } \Delta A_\omega = A_\omega - A_{\omega^+} = -\nabla \frac{\varrho_{\omega^+} - \varrho_\omega}{\varrho_{\omega^+}\varrho_\omega} \nabla \geq 0.$$

We now review some standard facts from the Floquet theory for periodic operators. Basic references of this material are [Re,Si4], [Sj].
As ϱ_{ω^+} is Γ-periodic, for any $\gamma \in \Gamma$, we have

$$\tau_\gamma A_{\omega^+} \tau_\gamma^* = \tau_\gamma A_{\omega^+} \tau_{-\gamma} = A_{\omega^+}.$$

If for $\mathbb{T}^* = \mathbb{R}^d/(2\pi\Gamma)$ we define $\mathcal{H}$ by

$$\mathcal{H} = \{u(x,\theta) \in L^2_{loc}(\mathbb{R}^d) \otimes L^2(\mathbb{T}^*); \forall (x,\theta,\gamma) \in \mathbb{R}^d \times \mathbb{T}^* \times \mathbb{Z}^d;\ u(x+\gamma,\theta) = e^{i\gamma\theta}u(x,\theta)\}$$

then, we know that there exists U a unitary isometry from $L^2(\mathbb{R}^d)$ to $\mathcal{H}$ such that A_{ω^+} admits the Floquet decomposition

$$UA_{\omega^+}U^* = \int_{\mathbb{T}^*}^{\oplus} A_{\omega^+}(\theta)d\theta.$$

Here $A_{\omega^+}(\theta)$ is the operator A_{ω^+} acting on $\mathcal{H}_\theta$, defined by

$$\mathcal{H}_\theta = \{u \in L^2_{loc}(\mathbb{R}^d); \forall \gamma \in \Gamma, u(x+\gamma) = e^{i\gamma\theta}u(x)\}.$$

As A_{ω^+} is elliptic, we know that, $A_{\omega^+}(\theta)$ has a compact resolvent; hence its spectrum is discrete. We denote its eigenvalues, called Floquet eigenvalues of A_{ω^+}, by

$$E_0(\theta) \leq E_1(\theta) \leq \cdots \leq E_n(\theta) \leq \cdots.$$

The coresponding eigenfunctions are denoted by $(w(x,\cdot)_j)_{j\in\mathbb{N}}$ The functions $(\theta \to E_n(\theta))_{n\in\mathbb{N}}$ are Lipshitz-continuous, and we have

$$E_n(\theta) \to +\infty \quad \text{as } n \to +\infty \quad \text{uniformly in} \quad \theta.$$

The spectrum $\sigma(A_{\omega^+})$ of A_{ω^+} is made of bands (i.e $\sigma(A_{\omega^+}) = \cup_{n\in\mathbb{N}} E_n(\mathbb{T}^*)$.)
The periodic operator A_{ω^+} has an IDS which will be denoted by n.
We assume that
(**H.1**)
There exist $E_+, a > 0$ and $\delta > 0$ such that $\sigma(A_{\omega^+}) \cap [E_+, E_+ + a) = [E_+, E_+ + a)$ and $\sigma(A_{\omega^+}) \cap (E_+ - \delta, E_+] = \emptyset$.

To prove the principal result, we will also need the following assumptions:
(**H.2**)
- For all $1 \leq i \leq d, (\partial_{x_i}\varrho_\omega)(x)$ and $\varrho_\omega(x)$ are Lipschitiz continuous.

• There exists $g_+(x) \in L^p(C_0)$ with $C_0 = \{x \in \mathbb{R}^d; \forall 1 \le j \le d; -\frac{1}{2} < x_j \le \frac{1}{2}\}$ and ($p = 2$, if $d \le 3, p > 2$, if $d = 4$ and $p > \frac{d}{2}$ if $d \ge 5$) such that for all $\nu > d+2$, $\gamma \in \Gamma$ and a.e $x \in C_0$ we have

$$0 \le u(x+\gamma) \cdot (1+|\gamma|)^\nu \le g_+(x),$$

and for all $1 \le i \le d$,

$$0 \le |(\partial_{x_i} u)(x+\gamma)| \cdot (1+|\gamma|)^\nu \le g_+(x).$$

We assume also that
(H.3)
• $\limsup_{\varepsilon \to 0^+} \dfrac{\log|\log \mathbb{P}_{\omega^0}([\omega^+ - \varepsilon, \omega^+])|}{|\log \varepsilon|} = 0.$
As, $\Delta A_\omega \ge 0$ and ω^+ is in the support of $\mathbb{P}_{\omega_0}$, Σ contains an interval of the form $[E_+, E_+ + a](a > 0)$ (see [Kir, Mar3]).
As we are interested in the behavior of the IDS in the neighborhood of E_+, we ask that E_+ remains always the edge of a gap for Σ, when the perturbation is turned on. More precisely, if for all $t \in [0,1]$, we define $A_{\omega,t} = A_{\omega^+} + t\Delta A_\omega$ and Σ_t is the almost sure spectrum of $A_{\omega,t}$, then one requires that the following assumption is satisfied.
(H.4)
There exists $\delta' > 0$ such that for all $t \in [0,1]$, $\Sigma_t \cap [E_+ - \delta', E_+) = \emptyset$.

The principal result of this work is:

Theorem 2.1. *Let A_ω the operator defined by (1.1). Let us suppose that* **(H.1)**, **(H.2)**, **(H.3)** *and* **(H.4)** *are satisfied, then E_+ is continuity point for N and one has*

$$\liminf_{\varepsilon \to 0^+} \frac{\log|\log\left(N(E_+ + \varepsilon) - N(E_+)\right)|}{\log \varepsilon} \ge -\frac{d}{2},$$

$$\lim_{\varepsilon \to 0^+} \frac{\log|\log\left(N(E_+ + \varepsilon) - N(E_+)\right)|}{\log \varepsilon} = -\frac{d}{2} \Leftrightarrow \lim_{\varepsilon \to 0^+} \frac{\log\left(n(E_+ + \varepsilon) - n(E_+)\right)}{\log \varepsilon} = \frac{d}{2}.$$

Remark 2.2. *The same result is still true for the lower edge of the gap if in the assumptions which precedes ω^+ is replaced by $\omega^- = \inf \operatorname{supp} \mathbb{P}_{\omega_0}$ and if the assumption* **(H.4)** *is replaced by: there exists $\delta' > 0$ such that $\forall t \in [0,1], \Sigma_t \cap (E_-, E_- + \delta'] = \emptyset$.*

Outline the proof To prove Theorem 2.1 we use the method of periodic approximations [Kp1]. Lower and upper bound of the IDS are proven separately.
For the upper bound, we compare the behavior of $N(E) - N(E_+)$ with that of the IDS of well chosen Anderson discrete model. More precisely, one can see that for an energy close to E_+, $N(E) - N(E_+)$ can be upper bounded by $N_{\mathcal{E}_0}(E)$, the IDS of some reduced, bounded, and random operator $A^0_\omega = \Pi_0 A_\omega \Pi_0$. Here Π_0 is the spectral projection for A_{ω^+} on the band starting at E_+.
To study the behavior of $N(E) - N(E_+)$, we study the behavior of $N_{\mathcal{E}_0}(E)$. This represents several advantages: the first is that A^0_ω is equivalent to a random Jacobi matrix acting on $L^2(\mathbb{T}^*) \otimes \mathbb{C}^n$. The second advantage is that, E_+ is an interior edge of a gap for A_ω, it becomes the bottom of the spectrum for A^0_ω. We prove that when $\lim_{\varepsilon \to 0^+} \dfrac{\log\left(n(E_+ + \varepsilon) - n(E_+)\right)}{\log \varepsilon} = \dfrac{d}{2}$, A^0_ω is lower bounded by the usual discrete Laplace operator whose behavior of the IDS at the edges of the spectral gaps is alerady known.

The lower bound is proved by constructing a sufficiently large number of approximate eigenfunctions of $A_{\omega,\Lambda}$ associated with approximate eigenvalues in $[E_+ - \varepsilon, E_+ + \varepsilon]$. This will enable us to lower bound the number of the eigenvalues of $A_{\omega,\Lambda}$ in the interval $[E_+ - \varepsilon, E_+ + \varepsilon]$.

Now we make some remarks about our assumptions.
Let us start with **(H.1)**. Figotin and Kuchment in [Fig,Kuc] studied the existence of open spectral gaps in the spectrum of certain periodic acoustic operators for $d = 2, 3$. In assumption **(H.1)** we asked that $E_+ > 0$ which excludes the spectral gap $(-\infty, 0)$ and one will not have the Lifshitz tails in 0, as the Lifshitz tails is expected in the neighborhood of the fluctuating edges. See [Pas,Fig]. One precise that 0 is not a fluctuating edge of the spectrum. 0 is in the spectrum of A_ω independently of the choice of $\varrho_\omega(x)$.
If the support of $\mathbb{P}_{\omega_0}$ is connected, the assumption **(H.4)** can be replaced by:
(H.4.bis) There exists $\delta' > 0$ such that $\Sigma \cap [E_+ - \delta', E_+) = \emptyset$.
By adding a disorder parameter g in the equation which define $\varrho_\omega(x)$ i.e.

$$\varrho_\omega(x) = \varrho_0(x)\Big(1 + g \sum_{\gamma\in\Gamma} \omega_\gamma u_\gamma(x)\Big),$$

we can choose g sufficiently small so that the spectral gap in $\sigma(A_{\omega^+})$ will be not closed after the perturbation.

2.2. **Application : Localization.** Let us consider the wave equation :

$$\frac{\partial^2 v}{\partial t^2} = -A_\omega v, \; v(0) = v_0, \; \frac{\partial v}{\partial t}\Big|_{t=0} = v_1. \tag{2.6}$$

The solution of the equation (2.6) is given by:

$$v(t) = \cos\left(t\sqrt{A_\omega}\right) v_0 + \sin\left(t\sqrt{A_\omega}\right) w_1,$$

where $v_1 = \sqrt{A_\omega} w_1$ and v_0, w_1 are in the domain of A_ω.
The result which we will present can be summarized in the following way:
under adequate conditions on $\mathbb{P}_{\omega_0}$ and the IDS, n of the periodic operator we will show that
• the spectrum of A_ω is purepoint, in the neighborhood of the edge of $\Sigma\backslash\{0\}$.
• the eigenfunctions corresponding to eigenvalues at the edge of the spectrum, decrease exponentially.
• for a good choice of v_0 and w_1, $v(t)$ dose not disperse in the space.
The last assertion is often called **dynamical localization** and the two first **Andreson localization**.
This phenomena occurs in various cases (large disorder, extreme energies, Lifshitz tails...)

Theorem 2.3. *Let A_ω defined by (1.1). We assume that the assumptions* **(H.1)**, **(H.2)**, **(H.3)** *and* **(H.4)** *are satisfied and that*
• *n is nondegenerated at the edge E_+ and u compactly supported .*
• *the probability mesure $\mathbb{P}_{\omega_0}$ is hölder continus, i.e. there exists $a > 0$ such that $\mathbb{P}_{\omega_0}(I) \leq |I|^a, \forall I \subset \mathbb{R}$ ($|I|$ is the Lebesgue measure of I). There exists $\varepsilon_0 > 0$ such that*
i) $\Sigma \cap [E_+, E_+ + \varepsilon_0] \subset \Sigma_{pp}$,
ii) the eigenfunction corresponding to eigenvalues in $[E_+, E_+ + \varepsilon_0]$ decrease exponentially .
iii) there exist $p > 0$ (dependent on $\mathbb{P}_{\omega_0}$ and ϱ) such that

$$\mathbb{E}\Big\{ \sup_{t>0} \Big|\Big| \Big|X\Big|^p \cos\left(t\sqrt{A_\omega}\right) P_{[E_+, E_+ + \varepsilon_0]}(A_\omega) \chi_K \Big|\Big| \Big\} < \infty$$

and
$$\mathbb{E}\Big\{\sup_{t>0}\Big|\Big|\Big|X\Big|^p \sin\Big(t\sqrt{A_\omega}\Big)P_{[E_+,E_++\varepsilon_0]}(A_\omega)\chi_K\Big|\Big|\Big\}<\infty,$$
where $P_I(A_\omega)$ is the spectral projection on the interval I, χ_K is the characteristic function of K, K is a compact of $\mathbb{R}^d$ and X is the position operator.

From the work of Damanik and Stollman, [Dam,Sto] the proof of Theorem 2.3 is reduced to a simple verification of the two well known initial estimte of the multiscal analysis. From the Lifshitz tails this can be done by following the technique used in the Vesilieć paper [Ves]. We will give the detailed proof in section 6.

3. Localization in energy for the density of states

The goal of this part is to give an estimate of $N(E)-N(E_+)$ for an energy E close to E_+. This will be accomplished by means of the IDS of certain reference operators.

3.1. **Wannier basis.** We recall concepts used in [Kp1]. Let $\mathcal{E}\subset L^2(\mathbb{R}^d)$ a closed subspace invariant by the Γ-translations, i.e. such that $\Pi^{\mathcal{E}}$, the orthogonal projection on $\mathcal{E}$, satisfies
$$\forall\gamma\in\Gamma,\ \Pi^{\mathcal{E}}=\tau_\gamma^*\Pi^{\mathcal{E}}\tau_\gamma.$$
Folwing the same computations done in section 1.2 of [Kp1], we see that there exists an orthonormal system of vector $(\widetilde{w}_{n,0})_{n\in N}$ such that if for $\gamma\in\Gamma$, we set $\widetilde{w}_{n,\gamma}=\tau_\gamma(\widetilde{w}_{n,0})$, then $(\widetilde{w}_{n,\gamma})_{(n\in N;\gamma\in\Gamma)}$ is an orthonormal basis of $\mathcal{E}$. Such system is called **Wannier basis** of $\mathcal{E}$. The vectors $(\widetilde{w}_{n,0})_{n\in N}$ are called **Wannier generators** of $\mathcal{E}$.

We consider $\mathcal{E}\subset L^2(\mathbb{R}^d)$ a space which is invariant under the translations of Γ. $\mathcal{E}$ is said of finite energy for A_ω if $\Pi^{\mathcal{E}}A_\omega\Pi^{\mathcal{E}}$ is a bound operator. In this case, $\mathcal{E}$ admits a finished set of wannier generators.

Let Z, the set of $\theta\in\mathbb{T}^*$ for which there exists $j\in J_0$; such that $E_j(\theta)=E_+$. If n is nondegenerated Z is a set of isolated points, [Kp1]. The Floquet eigenvalues are ordered increasely. So if for $j\in J_0$; we define $Z_j=\{\theta\in\mathbb{T}^*;E_j(\theta)=E_+\}$, the sequence $(Z_j)_{j\in J_0}$ is decreasing and $Z_1=Z$. For $\theta^0\in Z$, N_{θ^0} is the set of indices such that, $E_j(\theta^0)=E_+$. We point out that the Floquet eigenvector associated with the eigenvalue $E_j(\theta)$ is noted by $w_j(\cdot,\theta)$.

Lemma 3.1. [Kp1]There exist $(v_j(\cdot,\theta))_{1\in J_0}$ functions in $\mathcal{H}_\theta$ such that :
1) for $\theta^0\in Z$ and $j\in N_{\theta^0}$, there exists V_{θ^0} a neighbourhood of V_{θ^0} (in $\mathbb{T}^*$) such that,
• the map $\theta\in V_{\theta^0}\mapsto v_j(\cdot,\theta)\in\mathcal{H}_\theta$ is real analytic (i.e. $\theta\to T_{\theta\to\theta^0}v_j(\cdot,\theta)$ is analytic of V_{θ^0} in $\mathcal{H}_{\theta^0}$),
• for $\theta\in V_{\theta^0}$, $vect\langle((v_j(\cdot,\theta))_{j\in N_{\theta^0}})\rangle=vect\langle((w_j(\cdot,\theta))_{j\in N_{\theta^0}})\rangle$,
2) for $\theta\in\mathbb{T}^*$, we have
• the system $((v_j(\cdot,\theta))_{j\in N_{\theta^0}})$ is orthonormal in $\mathcal{H}_\theta$.
• $vect\langle(w_j(\cdot,\theta)_{j\in J_0}\rangle=vect\langle(v_j(\cdot,\theta)_{j\in J_0}\rangle$

Here $vect\langle(w_1,...,w_{n_0})\rangle$ indicates the space vector spanned by $(w_1,...,w_{n_0})$ and for $(\theta,\theta')\in(\mathbb{T}^*)^2$, we define $T_{\theta\to\theta'}:\mathcal{H}_\theta\mapsto\mathcal{H}_{\theta'}$ by: $(T_{\theta\to\theta'}v)(x)=e^{ix\cdot(\theta'-\theta)}v(x)$.

3.2. **Reduction to discrete problem.** We will decompose the operator A_ω according to various translation invariant subspace. The random operators thus obtained are the reference operators. They will be used for the upper bound of the IDS.

Let

* J_0 be the set of indices of the Floquet eigenvalues of A_{ω^+} which take the value E_+ in certain points $\theta\in\mathbb{T}^*$.We identifie J_0 to $\{1...,n_0\}$.
* J_- be the set of indices smaller than $\inf J_0$.

* J_+ be the set of the indices greater than $\sup J_0$.
We denote by $\Pi_0(\theta)$, (respectively $\Pi_-(\theta)$ and $\Pi_+(\theta)$) the orthogonal projections in $\mathcal{H}_\theta$ on the vector space generated by $(w_j(\cdot,\theta))_{j\in J_0}$, (respectively by $(w_j(\cdot,\theta))_{j\in J_-}$ and $(w_j(\cdot,\theta))_{j\in J_+}$). It is clear that these three projections are two by two orthogonal and that their sum is the identity for all $\theta\in\mathbb{T}^*$.
One defines

$$\Pi_\alpha = U^{-1}\Big(\int_{\mathbb{T}^*}\Pi_\alpha(\theta)d\theta\Big)U : L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d),$$

where $\alpha\in\{-,0,+\}$. Π_α is an orthogonal projection on $L^2(\mathbb{R}^d)$ and for all $\gamma\in\Gamma$, we have $\tau_\gamma^*\Pi_\alpha\tau_\gamma=\Pi_\alpha$. It is clear that Π_-,Π_0 and Π_+ are two by two orthogonal and that $\Pi_-+\Pi_0+\Pi_+=Id_{L^2(\mathbb{R}^d)}$. For $\alpha\in\{-,0,+\}$,we set $\mathcal{E}_\alpha=\Pi_\alpha(L^2(\mathbb{R}^d))$. These subspaces are invariant by Γ-translation. Moreover $\mathcal{E}_-$ and $\mathcal{E}_0$ are of finished energies for A_{ω^+}.We prove

Theorem 3.2. *Let A_ω defined by (1.1). We assume that the assumptions* **(H.1)**, **(H.2)**, **(H.3)** *and* **(H.4)** *are satisfied. There exists $E_0>E_+$ and $C>1$ such that, for $E_+\le E\le E_0$ one has*

$$0\le N(E)-N(E_+)\le N_{\mathcal{E}_0}\Big(C\cdot(E-E_+)+E_+\Big) \tag{3.7}$$

where $N_{\mathcal{E}_0}$ is the integrated density of states $\Pi_0A_\omega\Pi_0=A_\omega^0$.

Theorem 3.2 gives immediately the following corollary :

Corollary 3.3. *Under the assumptions of the last theorem, E_+ is a continuity point for N. More precisely, there exists $C>1$ and $E_0>E_+$ such that, for $E_+\le E\le E_0$ we have*

$$0\le N(E)-N(E_+)\le n\Big(C\cdot(E-E_+)+E_+\Big)-n(E_+).$$

Proof As $\Delta A_\omega\ge 0$, this implies $\Pi_0A_\omega\Pi_0\ge\Pi_0A_{\omega^+}\Pi_0$. Thus, if $n_{\mathcal{E}_0}$ is the integrated density of states $\Pi_0A_{\omega^+}\Pi_0$; we have

$$N_{\mathcal{E}_0}(E)\le n_{\mathcal{E}_0}(E).$$

As Π_0 commutes with A_{ω^+}, if E is sufficiently close to E_+ such that $E\ge E_+$, then we have

$$n_{\mathcal{E}_0}(E)=n(E)-n(E_+).$$

The continuity of N in E_+ is an immediate consequence of the continuity of n.♣

4. The proof of Theorem 3.2

To prove Theorem 3.2, we will approach the density of states A_ω by the density of states of periodic approximations. In a neighborhood of E_+, we will control the behavior of the density of states of the periodic approximations via the density of states of periodic approximations of the reference operators. Passing to the limit for the density of states of the reference operators we obtain the sought result.

4.1. **The periodic approximations.**

Let $k\in\mathbb{N}^*$. We define the following periodic operator

$$A_{\omega,k}=-\nabla\frac{1}{\varrho_{\omega,k}(x)}\nabla,$$

where the function $\varrho_{\omega,k}$ is defined by

$$\varrho_{\omega,k}(x)=\varrho_0(x)\left(1+\sum_{\gamma\in C_k\cap\Gamma}\omega_\gamma\sum_{\beta\in(2k+1)\Gamma}u_{\gamma+\beta}\right),$$

C_k is the cube

$$C_k = \{x \in \mathbb{R}^d; \forall 1 \leq j \leq d,\ -\frac{2k+1}{2} < x_j \leq \frac{2k+1}{2}\}.$$

$A_{\omega,k}$ is $(2k+1)\Gamma$-periodic and essentially selfadjoint operator. Let $\mathbb{T}_k^* = (\mathbb{R}^d)/2(2k+1)\pi\Gamma$. We define $N_{\omega,k}$ the IDS of $A_{\omega,k}$ by

$$N_{\omega,k}(E) = \frac{1}{(2\pi)^d} \sum_{n\in\mathbb{N}} \int_{\{\theta\in\mathbb{T}_k^*,\ E_{\omega,k,n}(\theta)\leq E\}} d\theta. \tag{4.8}$$

Let $dN_{\omega,k}$ the derivative of $N_{\omega,k}$ in the distribution sense. As $N_{\omega,k}$ is increasing, $dN_{\omega,k}$ is a positive measure; it is the density of states of $A_{\omega,k}$. We denote by dN the density of states of A_ω. For all $\varphi \in C_0^\infty(\mathbb{R}), dN_{\omega,k}$ verifies (see [Re,Si4] and [Kp1])

$$\begin{aligned}
\langle \varphi, dN_{\omega,k}\rangle &= \frac{1}{(2\pi)^d}\int_{\theta\in\mathbb{T}_k^*} \mathrm{tr}_{\mathcal{H}_\theta}\big(\varphi(A_{\omega,k,\theta})\big)d\theta,\\
&= \frac{1}{\mathrm{vol}(C_k)}\mathrm{tr}\Big(\chi_{C_k}\varphi(A_{\omega,k})\chi_{C_k}\Big),
\end{aligned} \tag{4.9}$$

where for $\Lambda \subset \mathbb{R}^d$, χ_Λ will design the characteristic function of Λ and $\mathrm{tr}(A)$ is the trace of A (we index by $\mathcal{H}_\theta$ if the trace is calculated in $\mathcal{H}_\theta$). The proof of (4.9) is given in [Kp1].

Theorem 4.1. *When k tends to the infinity, for almost all $\omega \in \Omega$, $dN_{\omega,k}$ converge in the distributions sense to dN. Moreover, for all $\lambda \in \mathbb{R}$ continuity point for N, we have $\lim_{k\to\infty} \mathbb{E}(N_{\omega,k}(\lambda)) = N(\lambda)$ almost surely .*

Remark 4.2. *The result of the Theorem 4.1 is close to the one of Theorem 5.1 of* [Kp1]. *The proof also is similair and is based on functional analyse. The unique difference point in the proof comes from the control of the behavior of the resolvent. In* [Kp1]*, the perturbation was a potential; in our case, it is a differential operator of the same order as the background operator. We will give a sketch of the proof.*

proof Let $\varphi \in C_0^\infty(\mathbb{R})$. From [Ca] and [Pas,Fig], we know that

$$\langle \varphi, dN\rangle = \int_{\mathbb{R}} \varphi(\lambda)dN = \mathbb{E}\Big(\frac{1}{\mathrm{vol}(C_0)}\mathrm{tr}(\chi_{C_0}\varphi(A_\omega)\chi_{C_0})\Big).$$

By writing $\chi_{C_k} = \sum_{\gamma\in C_k\cap\Gamma} \tau_\gamma(\chi_{C_0})$, we have,

$$\mathbb{E}\Big(\mathrm{tr}(\chi_{C_k}\varphi(A_{\omega,k})\chi_{C_k})\Big) = \sum_{\gamma\in C_k\cap\Gamma} \mathbb{E}\Big[\mathrm{tr}\Big(\tau_\gamma(\chi_{C_0})\varphi(A_{\omega,k})\Big)\Big].$$

As the family of random variables $(\omega_\gamma)_{\gamma\in\Gamma}$ is independent and identically distributed, we compute

$$\mathbb{E}\Big(\mathrm{tr}(\chi_{C_k}\varphi(A_{\omega,k})\chi_{C_k})\Big) = \#(C_k\cap\Gamma)\mathbb{E}\Big(\mathrm{tr}(\chi_{\Lambda_0}\varphi(A_{\omega,k}))\Big).$$

So from (4.9), we obtain

$$\mathbb{E}\Big(\langle\varphi, dN_{\omega,k}\rangle\Big) = \frac{1}{\mathrm{vol}(C_0)}\mathbb{E}\Big(\mathrm{tr}(\chi_{C_0}\varphi(A_{\omega,k})\chi_{C_0})\Big).$$

The random variables $(\omega_\gamma)_{\gamma\in\Gamma}$ and the functions $\varrho_0(x)$ and u are bounded, then we deduce that $A_{\omega,k}$ is lowerbounded uniformly in ω and k. Let $\lambda_0 \in \mathbb{R}^*$ such that $I \leq \lambda_0 + A_{\omega,k}$ and $I \leq \lambda_0 + A_\omega$, for all $\omega \in \Omega$ and $k \in \mathbb{N}^*$.

Let $\varphi \in C_0^\infty(\mathbb{R})$ and $\widetilde{\varphi}$ an almost analytic extension of $(\lambda_0+x)^{-q}\varphi(x)$, where $q > 2d+1$. The Helffer-Sjöstrand formula [Kp3] gives :

$$\varphi(A_{\omega,k}) = \frac{i}{2\pi}\int_{\mathbb{C}} \frac{\partial\widetilde{\varphi}}{\partial\overline{z}}(z)(\lambda_0+A_{\omega,k})^{-q}(z-A_{\omega,k})^{-1}dz\wedge d\overline{z}.$$

We have the

Lemma 4.3. ([Sim2]) *For $q > \frac{d}{2}$, $\chi_{C_0}(\lambda_0+A_{\omega,k})^{-q}(z-A_{\omega,k})^{-1}\chi_{C_0}$ is trace class.*

So, we can write that

$$\mathrm{tr}\Big(\chi_{C_0}\varphi(A_{\omega,k})\chi_{C_0}\Big) = \frac{i}{2\pi}\int_{\mathbb{C}} \frac{\partial\widetilde{\varphi}}{\partial\overline{z}}(z)\mathrm{tr}\Big(\chi_{C_0}(\lambda_0+A_{\omega,k})^{-q}(z-A_{\omega,k})^{-1}\chi_{C_0}\Big)dz\wedge d\overline{z}.$$

This implies that

$$\Big|(\langle\varphi, dN_{\omega,k}\rangle) - \langle\varphi, dN\rangle\Big| = \Big|\mathbb{E}\Big(\mathrm{tr}(\chi_{C_0}\varphi(A_{\omega,k}))\Big) - \mathbb{E}\Big(\mathrm{tr}(\chi_{C_0}\varphi(A_\omega))\Big)\Big|$$

$$\leq \mathbb{E}\Bigg(\frac{1}{\pi}\int_{\mathbb{C}}\Big|\frac{\partial\widetilde{\varphi}}{\partial\overline{z}}(z)\Big|\Big|\mathrm{tr}\Big[\chi_{C_0}\big\{(\lambda_0+A_{\omega,k})^{-q}(z-A_{\omega,k})^{-1} - (\lambda_0+A_\omega)^{-q}(z-A_\omega)^{-1}\big\}\Big]\Big|dxdy\Bigg). \tag{4.10}$$

Now we state a lemma which proof will be given later,

Lemma 4.4. *There exists $K>0$, such that $\forall\varphi\in C_0^\infty(\mathbb{R})$*

$$(4.10) \leq \mathbb{E}\Bigg(K\int_{\mathbb{R}^2}\Big|\frac{\partial\widetilde{\varphi}}{\partial\overline{z}}(z)\Big|\cdot\sum_{\beta,\gamma\in\Gamma,|\gamma|>k}\frac{e^{-m_z|\beta|_\Gamma/K}}{|Im(z)|^2}\Big|\Big|\tau_{\beta+\gamma}u\Big|\Big|_{L^2(C_0)}dxdy\Bigg).$$

where $m_z = \frac{\eta}{\varrho_-^{-1}+|z|+\eta}$, $\eta = dist(z,\Sigma)$ and $|\alpha|_\Gamma$ is the usual norm in $\mathbb{Z}^d$.

By the use of Lemma 4.4 and the fact that $\sum_{\gamma\in\Gamma}||\tau_\gamma u||_{L^p(C_0)} < \infty$, we obtain that there exists $K>0$ such that for any $\varphi\in C_0^\infty(\mathbb{R})$ we have

$$\Big|\mathbb{E}\Big(\langle\varphi, dN_{\omega,k}\rangle\Big) - \langle\varphi, dN\rangle\Big| \leq$$
$$\mathbb{E}\Bigg(K\int_{\mathbb{R}^2}\Big|\frac{\partial\widetilde{\varphi}}{\partial\overline{z}}(z)\Big|\sum_{\beta,\gamma\in\Gamma,|\gamma|>k}\frac{e^{-m_z|\beta|_\Gamma/K}}{|Im(z)|^2}\Big|\Big|\tau_{\beta+\gamma}u\Big|\Big|_{L^p(C_0)}dxdy\Bigg) < +\infty.$$

As $\forall\ M\in\mathbb{N}$; $\Big|\frac{\partial\widetilde{\varphi}}{\partial\overline{z}}(z)\Big| = O\Big(\Big|Im(z)\Big|^M\Big)$, the dominated convergence implies that :

$$\lim_{k\to\infty}\Big|(\langle\varphi, dN_{\omega,k}\rangle) - \langle\varphi, dN\rangle\Big| = 0.$$

So the first assertion of Theorem 4.1.
The proof of the second assertion is given in [Kp1]. We close this section by the following remark.

Remark 4.5. *If we suppose that u is compactly supported or exponentialy decaying at the infinity then, we obtain the estimate*

$$\left\|\mathrm{tr}\left(\chi_{C_0}(\lambda_0+A_{\omega,k})^{-q}(z-A_{\omega,k})^{-1}\chi_{C_0}-\chi_{C_0}(\lambda_0+A_{\omega})^{-q}(z-A_{\omega})^{-1}\chi_{C_0}\right)\right\|_{\mathrm{tr}} \leq \frac{K}{|Im(z)|^2}e^{-k\cdot|Im(z)|/K}.$$

Lemma 4.4 implies that for any $\varphi\in C_0^\infty(\mathbb{R}^d)$, we have

$$\left|\mathbb{E}\Big(\langle\varphi,dN_{\omega,k}\rangle\Big)-\langle\varphi,dN\rangle\right|=O(k^{-\infty}). \tag{4.11}$$

4.2. **Localization in energy.** In this section we prove a similar result to Theorem 3.2 for the counting functions of the periodic approximations of reference operators. For $k\in\mathbb{N}$, we define the operators

$$A_{\omega^+,0}=\Pi_0A_{\omega^+}\Pi_0 \text{ and } A_{\omega,k,-}=\Pi_-A_{\omega,k}\Pi_-=\Pi_-A_{\omega^+}\Pi_-+\Pi_-\Delta A_{\omega,k}\Pi_-.$$

In the same way we define, $\Delta A_{\omega,k,0},\Delta A_{\omega,k,+},A_{\omega,k,+},A_{\omega^+,+}$ and $A_{\omega,k,0}$. According to the assumption (**H.1**) and as $\Delta A_\omega\geq 0$, we know that there exists $a>0$ such that, for any $k\in\mathbb{N}$, we have

$$A_{\omega,k,+}\geq A_{\omega^+,+}\geq(E_++a)\Pi_+. \tag{4.12}$$

The projector Π_- being Γ-periodic, we can consider it as $(2k+1)\Gamma$-periodic which enables us to consider the following Floquet decomposition :

$$\Pi_-=\int_{\mathbb{T}_k^*}^{\oplus}\Pi_{k,-}(\theta)d\theta.$$

For $\theta\in\mathbb{T}_k^*$, the operator $\Pi_{k,-}(\theta)$ is an orthogonal projection in $\mathcal{H}_\theta$; the same for $\Pi_{k,+}(\theta)$ and $\Pi_{k,0}(\theta)$ are orthogonal projectors. Finally one defines the operator

$$A_{\omega,k,-}(\theta)=\Pi_{k,-}(\theta)A_{\omega,k}(\theta)\Pi_{k,-}(\theta).$$

In the same way we define the operators $A_{\omega,k,0}(\theta)$ and $A_{\omega,k,+}(\theta)$. From (4.12) we obtain that

$$A_{\omega,k,+}(\theta)\geq(E_++a)\Pi_{k,+}(\theta). \tag{4.13}$$

For an operator H with a discrete spectrum, we note his counting function by

$$\mathcal{V}(H,E)=\#\Big\{\text{eigenvalues of } H\leq E\Big\}.$$

One then shows the following lemma

Lemma 4.6. *We suppose that* **(H.1)**, **(H.2)**, **(H.3)** *and* **(H.4)** *are satisfied. There exists $E_0>E_+$ and $C>1$ such that for $E_+\leq E\leq E_0$ we have*

$$0\leq\mathcal{V}(A_{\omega,k}(\theta),E)-\mathcal{V}(A_{\omega,k}(\theta),E_+)\leq\mathcal{V}\Big(A_{\omega,k,0}(\theta),C\cdot(E-E_+)+E_+\Big).$$

Proof We define $\widetilde{A}_{\omega,k}=A_{\omega,k}-E_+$ et $\widetilde{A}_{\omega^+}=A_{\omega^+}-E_+$. As we did previously we define

$$\widetilde{A}_{\omega,k,-}(\theta),\ \widetilde{A}_{\omega,k,+}(\theta) \text{ and } \widetilde{A}_{\omega,k,0}(\theta)$$

(i.e. $\widetilde{A}_{\omega,k,-}(\theta)=\Pi_{k,-}(\theta)\widetilde{A}_{\omega,k}(\theta)\Pi_{k,-}(\theta)$).
To prove Lemma 4.6 it is sufficient to prove

Lemma 4.7. *We suppose that* **(H.1)**, **(H.2)**, **(H.3)** *and* **(H.4)** *is satisfied. There exists $\widetilde{E}_0>0$ and $C>1$ such that for $0\leq\widetilde{E}\leq\widetilde{E}_0$, we have*

$$0\leq\mathcal{V}(\widetilde{A}_{\omega,k}(\theta),\widetilde{E})-\mathcal{V}(\widetilde{A}_{\omega,k}(\theta),0)\leq\mathcal{V}(\widetilde{A}_{\omega,k,0}(\theta),C\widetilde{E}).$$

Proof: If (**H**.4) is satisfied, then for any $t \in [0,1]$ and any $k \in \mathbb{N}^*$, we have $[-\delta, 0) \cap \sigma(\widetilde{A}_{\omega,k,t}) = \emptyset$ where $\widetilde{A}_{\omega,k,t} = \widetilde{A}_{\omega^+} + t\Delta A_{\omega,k}$, [Kir,Mar3]. Thus, for any $t \in [0,1]$ and $\widetilde{E} \in [-\delta', 0)$, we have

$$\mathcal{V}(\widetilde{A}_{\omega,k,t}(\theta), \widetilde{E}) = \mathcal{V}(\widetilde{A}_{\omega,k,t}(\theta), 0^-),$$

Here $\widetilde{A}_{\omega,k,t}(\theta) = \Pi_k(\theta)\widetilde{A}_{\omega^+}(\theta)\Pi_k(\theta) + t\Delta A_{\omega,k}(\theta)$. For $t \in [0,1]$, the function $t \mapsto \mathcal{V}(\widetilde{A}_{\omega,k,t}(\theta), \widetilde{E})$ is continuous, with integer values and constant for $\widetilde{E} \in [-\delta', 0[$. Thus, for $\widetilde{E} \in [-\delta', 0[$, we have

$$\begin{aligned}
\mathcal{V}(\widetilde{A}_{\omega,k}(\theta), \widetilde{E}) &= \mathcal{V}(\widetilde{A}_{\omega,k,t}(\theta), \widetilde{E}) = \mathcal{V}(\widetilde{A}_{\omega,k,t=0}(\theta), \widetilde{E}) \\
&= \mathcal{V}(\widetilde{A}_{\omega^+,k}(\theta), \widetilde{E}) = (2k+1)^d \mathcal{V}(\widetilde{A}_{\omega^+}(\theta), \widetilde{E}) = (2k+1)^d \# J_-.
\end{aligned}$$

Here J_- is the set of indices of the Floquet eigenvalues of $\widetilde{A}_{\omega^+}(\theta)$ smaller than 0. We decompose $\mathcal{H}$ in the following way:

$$\mathcal{H} = \mathcal{H}_- \oplus \mathcal{H}^+, \ \mathcal{H}_- = \Pi_{k,-}(\theta)\mathcal{H} \text{ et } \mathcal{H}^+ = \Pi_k^+(\theta)\mathcal{H} = (\Pi_{k,0}(\theta) + \Pi_{k,+}(\theta))\mathcal{H}.$$

So we write $\widetilde{A}_{\omega,k}(\theta)$ on the following matrix form

$$\begin{bmatrix} \widetilde{A}_{\omega,k,-}(\theta) & \widetilde{A}^{-,+}_{\omega,k}(\theta) \\ \widetilde{A}^{+,-}_{\omega,k}(\theta) & \widetilde{A}^{+}_{\omega,k}(\theta) \end{bmatrix}$$

where $\widetilde{A}_{\omega,k,-}(\theta) = \Pi_{k,-}(\theta)\widetilde{A}_{\omega,k}\Pi_{k,-}(\theta)$ $\widetilde{A}^+_{\omega,k}(\theta) = \Pi^+_k(\theta)\widetilde{A}_{\omega,k}(\theta)\Pi^+_k(\theta)$ $\widetilde{A}^{+,-}_{\omega,k}(\theta) = \Pi^+_k(\theta)\widetilde{A}_{\omega,k}(\theta)\Pi_{k,-}(\theta)$ and $\widetilde{A}^{-,+}_{\omega,k}(\theta)$ is the adjoint of $\widetilde{A}^{+,-}_{\omega,k}(\theta)$. From (4.14), we can see that the number of the Floquet eigenvalues of $\widetilde{A}_{\omega,k}(\theta)$ less than 0 is equal to the rank of $\Pi_{k,-}(\theta)$. We then prove

Lemma 4.8. *For $\widetilde{E} \geq 0$,*

$$0 \leq \mathcal{V}(\widetilde{A}_{\omega,k}(\theta), \widetilde{E}) - \mathcal{V}(\widetilde{A}_{\omega,k}(\theta), 0) \leq \mathcal{V}(\widetilde{A}^+_{\omega,k}(\theta), \widetilde{E}). \tag{4.14}$$

Proof The proof of this lemma is based on the min-max principle and is given in [Kp1] . ♣
We recall that $\mathcal{H}^+$ is decomposed into two orthogonal spaces, $\mathcal{H}^+ = \mathcal{H}_0 \oplus \mathcal{H}_+$; this gives the following matrix

$$\widetilde{A}^+_{\omega,k}(\theta) = \begin{bmatrix} \widetilde{A}_{\omega,k,0}(\theta) & \widetilde{A}_{\omega,k,0,+}(\theta) \\ \widetilde{A}_{\omega,k,+,0}(\theta) & \widetilde{A}_{\omega,k,+}(\theta) \end{bmatrix}.$$

Here $\widetilde{A}_{\omega,k,0}(\theta) = \Pi_{k,0}(\theta)\widetilde{A}_{\omega,k}(\theta)\Pi_{k,0}(\theta)$ $\widetilde{A}_{\omega,k,+}(\theta) = \Pi_{k,+}(\theta)\widetilde{A}_{\omega}\Pi_{k,+}(\theta)$, $\widetilde{A}_{\omega,k,+,0}(\theta) = \Pi_{k,+}(\theta)\widetilde{A}_{\omega,k}(\theta)\Pi_{k,0}(\theta)$ and $\widetilde{A}_{\omega,k,0,+}(\theta)$ is the adjoint of $\widetilde{A}_{\omega,k,+,0}(\theta)$. As $\widetilde{A}_{\omega^+,+,k}(\theta) \geq a\Pi_{k,+}(\theta)$ and $\Delta A_{\omega,k}(\theta) \geq 0$, to prove Lemma 4.7, it is sufficient to show the following lemma :

Lemma 4.9. *There exists $C > 1$ such that*

$$\begin{bmatrix} \widetilde{A}_{\omega,k,0}(\theta) & \widetilde{A}_{\omega,k,0,+}(\theta) \\ \widetilde{A}_{\omega,k,+,0}(\theta) & \widetilde{A}_{\omega,k,+}(\theta) \end{bmatrix} \geq \frac{1}{C} \begin{bmatrix} \widetilde{A}_{\omega,k,0}(\theta) & 0 \\ 0 & \widetilde{A}_{\omega,k,+}(\theta) \end{bmatrix}.$$

Proof Let $\varphi \in \mathcal{H}^+ \cap \mathcal{D}(\widetilde{A}^+_\omega)$. φ is written $\varphi = \varphi_+ + \varphi_0$. As $\mathcal{H}_0$ and $\mathcal{H}_+$ are orthogonal and invariant by $\widetilde{A}_{\omega^+}$, there exists $C > 1$ such that

$$\langle \widetilde{A}_{\omega,k,+}(\theta)\varphi_+, \varphi_+ \rangle \leq C \langle \widetilde{A}_{\omega^+,k,+}(\theta)\varphi_+, \varphi_+ \rangle \leq C \langle \widetilde{A}^+_{\omega,k}(\theta)\varphi, \varphi \rangle. \tag{4.15}$$

We know that

$$\langle \widetilde{A}^+_\omega(\theta)\varphi, \varphi \rangle = \langle \widetilde{A}_{\omega,k,+}(\theta)\varphi_+, \varphi_+ \rangle + \langle \widetilde{A}_{\omega,k,0}(\theta)\varphi_0, \varphi_0 \rangle + 2\Re e(\langle \widetilde{A}_{\omega,k,+,0}(\theta)\varphi_0, \varphi_+ \rangle).$$

This leads to
(4.16)
$$\langle \widetilde{A}_{\omega,k,+}(\theta)\varphi_+,\varphi_+\rangle + \langle \widetilde{A}_{\omega,k,0}(\theta)\varphi_0,\varphi_0\rangle \le \langle \widetilde{A}^+_{\omega}(\theta)\varphi,\varphi\rangle + 2|\langle \widetilde{A}_{\omega,k,+,0}(\theta)\varphi_0,\varphi_+\rangle|.$$

As $\widetilde{A}^+_{\omega}$ is positive we write

$$\begin{aligned} |\langle \widetilde{A}_{\omega,k,+,0}(\theta)\varphi_0,\varphi_+\rangle| &\le |\langle (\widetilde{A}^+_{\omega,k}(\theta))^{\frac12}\varphi, (\widetilde{A}^+_{\omega,k})^{\frac12}\varphi_+\rangle| + |\langle \widetilde{A}_{\omega,k,+}(\theta)\varphi_+,\varphi_+\rangle| \\ &\le \|(\widetilde{A}^+_{\omega,k}(\theta))^{\frac12}\varphi\| \cdot \|(\widetilde{A}^+_{\omega,k}(\theta))^{\frac12}\varphi_+\| + |\langle \widetilde{A}_{\omega,k,+}(\theta)\varphi_+,\varphi_+\rangle|. \end{aligned}$$

Taking (4.15) into account we obtain that there exists $C > 1$ such that

$$|\langle \widetilde{A}_{\omega,k,+,0}(\theta)\varphi_0,\varphi_+\rangle| \le C\langle \widetilde{A}^+_{\omega,k}(\theta)\varphi,\varphi\rangle. \tag{4.17}$$

The proof of Lemma 4.9 is ended by taking (4.16) and (4.17) into account. ♣
We use the (4.14) to finish the proof of Lemma 4.7. As $\widetilde{A}_{\omega,k,+}(\theta) \ge a\Pi_{+,k}(\theta)$, in the neighborhood of 0 the eigenvalues of $\widetilde{A}^+_{\omega,k}(\theta)$ higher than 0 are undervalued by the eigenvalues of $\frac{1}{C}(\widetilde{A}_{\omega,k,0}(\theta))$; we obtain thus the estimate of Lemma 4.7. ♣
Now we have all the necessary tools to show Theorem 3.2. We note by $N(A_{\omega,k,0},E)$ the periodic integrated density of states of the operator $A_{\omega,k,0}$ and by $N(A_{\omega,k},E)$ that of $A_{\omega,k}$. By definition we have

$$N(A_{\omega,k},E) = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^*_k} \mathcal{V}(A_{\omega,k}(\theta),E)d\theta$$

and

$$N(A_{\omega,k,0},E) = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^*_k} \mathcal{V}(A_{\omega,k,0}(\theta),E)d\theta.$$

As in [Kp1], for any $\varphi \in C_0^\infty(\mathbb{R})$, we have

$$\lim_{k\to\infty} \mathbb{E}\left(\int_{\mathbb{R}} \varphi(\lambda)dN(A_{\omega,k,0},\lambda)\right) = \int_{\mathbb{R}} \varphi(\lambda)dN_{\mathcal{E}_0}(\lambda). \tag{4.18}$$

Here $dN_{\mathcal{E}_0}$ is the density of states of $A_{\omega,k,0}$. According to this one can see that to prove Theorem 3.2, it is sufficient to integrate the equation of Lemma 4.6 and use Proposition 4.1 of [Kp1] and Theorem 4.1. For E sufficiently close to E_+, a continuity point for $N(E)$ and $N_{\mathcal{E}_0}(C\cdot E)$, we obtain

$$0 \le N(E) - N(E_+) \le N_{\mathcal{E}_0}\Big(C\cdot(E-E_+) + E_+\Big).$$

So the proof of Theorem 3.2. ♣

5. The proof of Theorem 2.1

We start this section by giving lemma used in [Kp1]. For $v \in \mathcal{H}^2$ one defines the norm:

$$\sup_{\theta\in\ \mathbb{T}^*}\Big(|\langle v(.,\theta), A_{\omega^+}(\theta)v(.,\theta)\rangle_{L^2(C_0)}| + ||v(.,\theta)||^2_{L^2(C_0)}\Big) = ||v||_{A_{\omega^+},\infty}.$$

Let $L^2_{unif}(\mathbb{R}^d)$ the space defined by

$$L^p_{loc,unif}(\mathbb{R}^d) = \{u \text{ mesurables};\ \sup_{\gamma\in\Gamma}||u||_{L^p(\gamma+C_0)} < \infty\},$$

endwed with the norm $\sup_{\gamma\in\Gamma}||u||_{L^p(\gamma+C_0)}$.
Let $0 \le V \in L^p_{loc,unif}(\mathbb{R}^d)$ where p is selected as on the assumption **(H.2)**. We define the operator $Op_{V,v} : L^2(\mathbb{T}^*) \longrightarrow L^2(\mathbb{R}^d)$ by

$$\forall t \in L^2(\mathbb{T}^*),\quad Op_{V,v}(t) = \sqrt{V(x)}\int_{\mathbb{T}^*} v(x,\theta)t(\theta)d\theta.$$

Lemma 5.1. ([Kp1]) *Let $v \in \mathcal{H}^2_\theta$. If $||v||_{A_{\omega+},\infty} < +\infty$, then $Op_{V,v}$ is bounded, precisely there exists $C > 0$ such that,*

$$||Op_{V,v}||_{\mathcal{L}(L^2(\mathbb{T}^*),L^2(\mathbb{R}^d))} \leq C||v||_{A_{\omega+},\infty}\sqrt{||V||_{L^p_{loc,unif}}}.$$

One sets $f(x,\theta) = v_1(x,\theta)$, ($v_1(x,\theta)$ is one of the vectors constructed in Lemme3.1). Let us remark that

Remark 5.2. *For any $1 \leq i \leq d$, we have $||\partial_{x_i} f(x,\theta)||_{A_{\omega+},\infty} < +\infty$.*

5.1. **The upper bound.** In this section we will show the following theorem:

Theorem 5.3. *Let A_ω the operator defined by (1.1). Let us suppose that* **(H.1)**, **(H.2)**, **(H.3)** *and* **(H.4)** *are satisfied, then*

$$\limsup_{\varepsilon \longrightarrow 0^+} \frac{\log|\log(N(E_+ + \varepsilon) - N(E_+))|}{\log \varepsilon} \leq -\frac{d}{2}.$$

Proof To prove the upper bound, it is sufficient to upper bound $N_{\mathcal{E}_0}$ (defined in Theorem 3.2). To do this, we will lower bound $\Pi_0 A_\omega \Pi_0$ by an operator whose behavior of the IDS in the neighborhood of the edges of the gaps is known.
Note that if a vector minimizes $A^0_\omega = \Pi_0 A_\omega \Pi_0$, it necessarily minimize $A^0_{\omega^+} = \Pi_0 A_{\omega^+} \Pi_0$; hence, it has to be concentrated in the quasimomentum θ near the zeros of $(E_j(\theta) - E_+)_{1\leq j \leq n_0}$. So we must consider all the point where the Floquet eigenvalue take the value E_+.
We start by isolating the contributions from the various points for which $E_1(\theta)$ take the value E_+. We recall that the band in E_+ is generated by $(E_j(\theta))_{1\leq j\leq n_0}$. For $1 \leq j \leq n_0$, we define $Z_j = \{\theta \in \mathbb{T}^*; E_j(\theta) = E_+\}$; the sequence $(Z_j)_{1\leq j\leq n_0}$ is decreasing $(Z_{j+1} \subset Z_j)$. Let $\theta^0 \in Z$. We set $j(\theta^0) = \sup N_{\theta^0}$ with $N_{\theta^0} = \{j, 1 \leq j \leq n_0; E_j(\theta^0) = E_+\}$. Let $m = \#Z$. We replace the Floquet eigenvectors $(w_j(\cdot,\theta))_{1\leq j\leq j(\theta^0)}$ associated with $(E_j(\theta))_{1\leq j\leq j(\theta^0)}$ by the vectors $(v_j(\cdot,\theta))_{1\leq j\leq j(\theta^0)}$ constructed in Lemma 3.1, they are analytic in a neiberghod V_{θ^0} of θ^0. Let θ close to θ^0. The operator $A^0_{\omega^+} = \Pi_0 A_{\omega^+} \Pi_0$ is unitarily equivalent to the multiplication operator (acting on $(\mathbb{T}^*) \otimes \mathbb{C}^{n_0}$) by the following matrix :

$$A^0_{\omega^+}(\theta) = \begin{bmatrix} B_{j(\theta^0)}(\theta) & 0 & 0 & \dots & 0 \\ 0 & E_{j(\theta^0)+1}(\theta) & 0 & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & E_{n_0}(\theta) \end{bmatrix}$$

where the matrix $B_{j(\theta^0)}(\theta)$ is $j(\theta^0) \times j(\theta^0)$ blocks given by

$$\begin{bmatrix} \langle v_1(\cdot,\theta), A_{\omega^+}(\theta) v_1(\cdot,\theta)\rangle_{L^2(C_0)} & \dots & \langle v_1(\cdot,\theta), A_{\omega^+}(\theta) v_{j(\theta^0)}(\cdot,\theta)\rangle_{L^2(C_0)} \\ \vdots & \ddots & \vdots \\ \langle v_{j(\theta^0)}(\cdot,\theta), A_{\omega^+}(\theta) v_1(\cdot,\theta)\rangle_{L^2(C_0)} & \dots & \langle v_{j(\theta^0)}(\cdot,\theta), A_{\omega^+}(\theta) v_{j(\theta^0)}(\cdot,\theta)\rangle_{L^2(C_0)} \end{bmatrix}.$$

The matrix $B_{j(\theta^0)}(\theta)$ have eigenvalues $(E_j(\theta))_{1\leq j\leq j(\theta^0)}$.
For $\theta^0 \in Z$, we define

$$\varpi_{\theta^0}(\theta) = E_+ + \sum_{j=1}^{d} \Big(1 - \cos(\theta_j - \theta^0_j)\Big).$$

The eigenvalues $(E_j(\theta))_{1\leq j\leq j(\theta^0)}$ being nondegenerated, there exists $\widetilde{V}_{\theta^0}$ a neighborhood of θ^0 such that, for $\theta \in \widetilde{V}_{\theta^0}$, $\exists C > 1$ such that we have
* for $1 \leq j \leq j(\theta^0)$, $CE_j(\theta) \geq \varpi_{\theta^0}(\theta)$,
* for $j \geq j(\theta^0)$, $CE_j(\theta) \geq 2 + E_+$.

Let $A^b_{\omega+,\theta^0}(\theta)$ the $n_0\times n_0$ diagonal matrix with identical diagonal entries equal to ϖ_{θ^0}. For $\theta\in\widetilde{V}_{\theta^0}$, we have

$$A^b_{\omega+,\theta^0}(\theta)\le CA^0_{\omega+}(\theta). \tag{5.19}$$

Finally, we note that the $(\widetilde{V}_{\theta^0})_{\theta^0\in Z}$ can be chosen so that they cover $\mathbb{T}^*$, (i.e. $\cup_{\theta^0\in Z}\widetilde{V}_{\theta^0}=\mathbb{T}^*$) and such that each one of them contains only one point of Z, (i.e. such that if we have $\theta\in Z,\theta'\in Z$ such that $\theta\neq\theta'$, then $\theta'\notin\overline{\widetilde{V}}_\theta$). We order the points in $Z=\{\theta^k;1\le k\le m\}$. Let $(\chi_k)_{1\le k\le m}$ be $C_0^\infty(\mathbb{T}^*)$ functions which form a partition of the unity on $\mathbb{T}^*$ such that
* for $1\le k\le m,\ \mathrm{supp}(\chi_k)\subset\widetilde{V}_{\theta^k}$,
* for $1\le k\le m,\ \ 0\le\chi_k\le 1$,
* for $1\le k\le m,\ \chi_k\equiv 1$ in a neighborhood of θ^k.
So for any $\theta\in\mathbb{T}^*$, we have

$$\frac{1}{m}\le\sum_{k=1}^m\chi_k^2\le 1 \text{ and } \sum_{k=1}^m A^b_{\omega+,\theta^k}(\theta)\chi_k^2\le CA^0_{\omega+}(\theta). \tag{5.20}$$

For $t\in L^2(\mathbb{T}^*)\otimes\mathbb{C}^{n_0}\otimes\mathbb{C}^m$, we note $t=(t_{j,k})_{1\le j\le n_0;1\le k\le m}$. We consider t as a system of m columns denoted by $(t_{\cdot,k})_{1\le k\le m}$. Each component belongs to $\mathbb{C}^{n_0}$. We endow $L^2(\mathbb{T}^*)\otimes\mathbb{C}^{n_0}\otimes\mathbb{C}^m$ with the scalar product generating the following Euclidean norm:

$$\left\|t\right\|^2_{L^2(\mathbb{T}^*)\otimes\mathbb{C}^{n_0}\otimes\mathbb{C}^m}=\sum_{k=1}\left\|t_{\cdot,k}\right\|^2_{L^2(\mathbb{T}^*)\otimes\mathbb{C}^{n_0}}=\sum_{1\le j\le n_0,1\le k\le m}\left\|t_{j,k}\right\|^2_{L^2(\mathbb{T}^*)}.$$

We define the application $S:L^2(\mathbb{T}^*)\otimes\mathbb{C}^{n_0}\longrightarrow L^2(\mathbb{T}^*)\otimes\mathbb{C}^{n_0}\otimes\mathbb{C}^m$ with

$$S(t)=(\chi_k t)_{1\le k\le m}=(\chi_k t_j)_{1\le j\le n_0;1\le k\le m},\text{if } t=(t_j)_{1\le j\le n_0}\in L^2(\mathbb{T}^*)\otimes\mathbb{C}^{n_0}.$$

The adjoint of S, $S^*:L^2(\mathbb{T}^*)\otimes\mathbb{C}^{n_0}\otimes\mathbb{C}^m\longrightarrow L^2(\mathbb{T}^*)\otimes\mathbb{C}^{n_0}$ is defined by

$$S^*(t)=\left(\sum_{1\le k\le m}\chi_k t_{j,k}\right)_{1\le j\le n_0}\quad\text{for } t=(t_{j,k})_{1\le j\le n_0;1\le k\le m}\in L^2(\mathbb{T}^*)\otimes\mathbb{C}^{n_0}\otimes\mathbb{C}^m.$$

According to the equation (5.20) we have $\frac{1}{m}I\le S^*\circ S\le I$, (here I is the identity in $L^2(\mathbb{T}^*)\otimes\mathbb{C}^{n_0}$) thus S is one to one. One shows the following lemma:

Lemma 5.4. *There exists $C>1$ such that, for $t\in L^2(\mathbb{T}^*)\otimes\mathbb{C}^{n_0}$, we have*

$$\langle A^a_\omega St,St\rangle_{L^2(\mathbb{T}^*)\otimes\mathbb{C}^{n_0}\otimes\mathbb{C}^m}\le C\langle A^0_\omega t,t\rangle_{L^2(\mathbb{T}^*)\otimes\mathbb{C}^{n_0}},$$

where the operator A^a_ω acting on $L^2(\mathbb{T}^*)\otimes\mathbb{C}^{n_0}\otimes\mathbb{C}^m$ is defined by

$$A^a_\omega t=A^a_{\omega+}t+\Delta A^a_{\omega+,\omega}t=\left(A^a_{j,k}t_{j,k}+\Delta A^a_\omega t_{j,k}\right)_{1\le j\le n_0;1\le k\le m}.$$

Here
* $A^a_{j,k}$ *is the multiplication by ϖ_{θ^k} acting as a multplication on $L^2(\mathbb{T}^*)$.*
* $\Delta A^a_\omega=\sum_{\gamma\in\Gamma}\omega^+_\gamma\Pi_\gamma$.
* Π_γ *is the orthogonal projection on the vector $\theta\mapsto e^{i\gamma\theta}$ in $L^2(\mathbb{T}^*)$.*

Before showing this lemma, one will use it to prove Theorem 5.3. From Proposition 4.1 of [Kp1] we know A^a_ω and A^0_ω have an IDS which we will note respectively $N^a(E)$ and $N_{\mathcal{E}_0}(E)$. Moreover, for E a continuity point for $N_{\mathcal{E}_0}$, we have

$$N_{\mathcal{E}_0}(E)=\lim_{q\longrightarrow+\infty}\frac{1}{\Lambda_q}\left\{\text{ eigenvalues of } A^0_{\omega,q}\le\ E\right\},$$

where

$$A^0_{\omega,q} = \Pi^0_q A^0_\omega \Pi^0_q,$$

and for $t = (t_j)_{1\le j\le n_0} \in L^2(\mathbb{T}^*) \otimes \mathbb{C}^{n_0}$,

$$\Pi^0_q t = \left(\sum_{\gamma\in\Lambda_q} e^{i\gamma\theta} \int_{\mathbb{T}^*} e^{-i\gamma\theta} t_{j(\theta)} d\theta \right)_{1\le j\le n_0}$$

with $\Lambda_q = \{\gamma \in \Gamma; |\gamma|_\Gamma \le q\}$.
$N^a(E)$ is given by

$$N^a(E) = \lim_{q\longrightarrow+\infty} \frac{1}{\Lambda_q} \Big\{ \text{eigenvalues of } A^a_{\omega+,q} \le E \Big\},$$

where

$$A^a_{\omega,q} = \Pi^a_q A^a_\omega \Pi^a_q$$

and for $t = (t_j)_{1\le j\le n_0} \in L^2(\mathbb{T}^*) \otimes \mathbb{C}^{n_0}$

$$\Pi^a_q t = \left(\sum_{\gamma\in\Lambda_q} e^{i\gamma\theta} \int_{\mathbb{T}^*} e^{-i\gamma\theta} t_{j,k}(\theta) d\theta \right)_{1\le j\le n_0; 1\le k\le m}.$$

Let us compare $N_{\mathcal{E}_0}$ and N^a.

Lemma 5.5. [Kp1] *For any $\delta > 0$ and any $l \in \mathbb{N}$, there exists $C_l(\delta) > 0$ such that, for $q \ge 1$,*

$$||\Pi^a_{(1+\delta)q} \circ S \circ \Pi^0_q - S \circ \Pi^0_q||_{\mathcal{L}(L^2(\mathbb{T}^*)\otimes\mathbb{C}^{n_0}, L^2(\mathbb{T}^*)\otimes\mathbb{C}^{n_0}\otimes\mathbb{C}^m)} \le C_l(\delta) q^{-l}.$$

Let $E > E_+$ and $t \in \Pi^0_q(\ L^2(\mathbb{T}^*) \otimes \mathbb{C}^n)$ such that $\langle A^0_\omega t, t\rangle \le E||t||^2$. Lemma 5.4 says that there exists $C > 1$ such that

$$\langle A^a_\omega St, St\rangle \le C \cdot E||t||^2. \tag{5.21}$$

So by Lemma 5.5, we know that

$$\langle A^a_\omega(\Pi^a_{(1+\delta)q}S)t, (\Pi^a_{(1+\delta)q}S)t\rangle \le C \cdot E||t||^2 + 2|\langle A^a_\omega(1-\Pi^a_{(1+\delta)q})St, (\Pi^a_{(1+\delta)q})St\rangle|$$

$$+|\langle A^a_\omega(1-\Pi^a_{(1+\delta)q})St, (1-\Pi^a_{(1+\delta)q})St\rangle| \le (C\cdot E + C_\delta q^{-1}) \Big\| t \Big\|^2. \tag{5.22}$$

As $\frac{1}{m} I \le S^* \circ S \le I$, we have

$$||t||^2 \le \frac{m}{1 - C_\delta q^{-1}} ||(\Pi^a_{(1+\delta)q}S)t||^2. \tag{5.23}$$

Thus $\Pi^a_{(1+\delta)q}S$ is one to one from $\Pi^0_q\Big(L^2(\mathbb{T}^*) \otimes \mathbb{C}^{n_0}\Big)$ to $\Pi^0_q\Big(L^2(\mathbb{T}^*) \otimes \mathbb{C}^{n_0} \otimes \mathbb{C}^m\Big)$ and for q sufficiently large (dependent on δ and E), we have

$$\langle A^a_\omega(\Pi^a_{(1+\delta)q}S)t, (\Pi^a_{(1+\delta)q}S)t\rangle \le C \cdot m(1+\delta)E||(\Pi^a_{(1+\delta)q}S)t||^2. \tag{5.24}$$

Moreover, we have

$$\#\Big\{\text{eigenvalues of } A^0_{\omega,q} \le E\Big\}$$

$$= \sup \Bigg\{ \dim(\aleph), \aleph \text{ is a linear subspace of } \Pi^0_q\Big(L^2(\mathbb{T}^*) \otimes \mathbb{C}^{n_0}\Big)$$

$$\text{such that for all } t \in \aleph,\ \langle A^0_\omega t, t\rangle \le E||t||^2 \Bigg\}.$$

There is a similar characterization for the counting function of A^a_ω. Using (5.21) and (5.24) for q sufficiently large, we obtain

$$\#\Big\{\mathrm{eigenvalues of}\ \mathrm{A}^0_{\omega,\mathrm{q}}\ \leq \mathrm{E}\Big\} \leq \#\Big\{\mathrm{eigenvalues of}\ \mathrm{A}^{\mathrm{a}}_{\omega,(1+\delta)\mathrm{q}} \leq \mathrm{C}\cdot(1+\delta)\mathrm{m}\ \mathrm{E}\Big\}.$$

We divide by the volume of Λ_q then we take $q \to +\infty$. We obtain that, for ε sufficiently small such that $E_+ + \varepsilon$ is a continuity point of $N_{\mathcal{E}_0}(E)$, we have

$$N_{\mathcal{E}_0}(E_+ + \varepsilon) \leq (1+\delta)^d N^a\Big(C(1+\delta)m(E_+ + \varepsilon)\Big). \tag{5.25}$$

Using the right continuity of $N_{\mathcal{E}_0}(E)$ and its moste denumerable number of discontinuity points, we extend the estimate (5.25) to some interval of the form $[E_+, E_+ + E]$.We let δ go to 0 and we get

$$N_{\mathcal{E}_0}(E_+ + \varepsilon) \leq N^a\Big(C\cdot m\cdot(E_+ + \varepsilon)\Big).$$

To complete the proof of Theorem 5.3, we use Theorem 3.2 and the fact that N^a has Lifhits tails in E_+. See [Kp1].

$$\limsup_{\varepsilon\longrightarrow 0^+} \frac{\log|\log(N(E_+ + \varepsilon) - N(E_+))|}{\log\varepsilon} \leq -\frac{d}{2}.$$

so we obtain the Theorem 5.3. ♣

The proof of Lemma 5.4 We recall the following notation

$$\Delta A_\omega = -\nabla\frac{\varrho_{\omega^+} - \varrho_\omega}{\varrho_{\omega^+}\varrho_\omega}\nabla = -\nabla V_\omega\nabla.$$

Let $t = (t_j)_{1\leq j\leq n_0} \in L^2(\mathbb{T}^*)\otimes\mathbb{C}^{n_0}$; one computes

$$\langle \Delta A^0_\omega t, t\rangle = \Bigg\|\sum_{1\leq j\leq n_0;1\leq k\leq m} Op_{V_\omega,\partial_{x_i}v_j(\cdot,\theta)}(\chi_k t_j)\Bigg\|^2_{L^2(\mathbb{R}^d)}.$$

For any $1\leq i\leq d$, $1\leq j\leq n_0$ and $1\leq k\leq m$, we define

$$\widetilde{v}_{k,j,i}(x,\theta) = (\partial_{x_i}v_j(x,\theta^k))e^{i(\theta-\theta^k)x}.$$

We know that $\widetilde{v}_{k,j,i}(x,\theta)\in\mathcal{H}_\theta$ and $\Big\|\widetilde{v}_{k,j,i}\Big\|_{A_{\omega^+},\infty} < +\infty$. We set :

$$\delta v_{k,j,i}(x,\theta) = \frac{1}{\sqrt{\varpi_{\theta^k}(\theta)}}\Bigg(\widetilde{v}_{k,j,i}(x,\theta) - \partial_{x_i}v_j(x,\theta^k)\Bigg).$$

The function $\theta\mapsto(\partial_{x_i}v_j)(x,\theta)$ is analytic in θ for θ, in a neighborhood of θ^0, $(x,\theta)\mapsto\delta v_{k,j,i}(x,\theta)$ satisfies the assumptions of Lemma 5.1. We note that $U(x) = \sum_{\gamma\in\Gamma} u_\gamma \in L^p_{loc,unif}(\mathbb{R}^d)$. So, $Op_{V_\omega,\delta v_{k,j,i}(\cdot,\theta)}$ is bounded. For $1\leq i\leq d$, we have

$$\begin{aligned}
&\Bigg\|\sum_{1\leq j\leq n_0;1\leq k\leq m} Op_{V_\omega,\partial_{x_i}v_j(\cdot,\theta)}(\chi_k t_j)\Bigg\|^2_{L^2(\mathbb{R}^d)} \\
&\qquad\geq \frac{1}{2}\Bigg\|\sum_{1\leq j\leq n_0;1\leq k\leq m} Op_{V_\omega,\widetilde{v}_{k,j,i}(\cdot,\theta)}(\chi_k t_j)\Bigg\|^2_{L^2(\mathbb{R}^d)} \\
&\qquad\qquad - \Bigg\|\sum_{1\leq j\leq n_0;1\leq k\leq m} Op_{V_\omega,\delta v_{k,j,i}(\cdot,\theta)}(\sqrt{\varpi_k}\chi_k t_j)\Bigg\|^2_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

Here, we set $\overline{\omega}_k = \overline{\omega}_{\theta^k}$. So there exists $C > 0$, such that, uniformly in ω, we have

$$\sum_{i=1}^{d} \left|\left| \sum_{1\le j\le n_0;1\le k\le m} Op_{V_\omega,\widetilde{v}_{k,j,j}(\cdot,\theta)}(\chi_k t_j)\right|\right|^2_{L(\mathbb{R}^d)} \le C\Big(\langle \Delta A^0_\omega t, t\rangle + \sum_{1\le j\le n_0;1\le k\le m} ||\sqrt{\overline{\omega}_k}\chi_k t_j||^2\Big) \le C\langle A^0_\omega t, t\rangle.$$

This gives

$$\langle A^b_\omega St, St\rangle \le C\langle A^0_\omega t, t\rangle \tag{5.26}$$

where

- $A^b_\omega = A^a_{\omega^+} + \Delta A^b_\omega$
- $A^a_{\omega^+}$ is defined in Lemma 5.4,
- ΔA^b_ω is the operator defined by, for $t \in L^2(\mathbb{T}^*)\otimes\mathbb{C}^{n_0}\otimes\mathbb{C}^m$

$$\langle \Delta A^b_\omega t, t\rangle = \sum_{i=1}^{d} \left|\left| \sum_{1\le j\le n;1\le k\le m} Op_{V_\omega,\widetilde{v}_{k,j,i}(\cdot,\theta)}(\chi_k t_j)\right|\right|^2_{L^2(\mathbb{R}^d)}.$$

We can see ΔA^b_ω as an operator matrix with entries that are the operators with the following kernels

$$v^b_{\omega,jj',kk'}(\theta,\theta') = \int_{\mathbb{R}^d} V_\omega(x)\nabla v_j(x,\theta^k)\overline{\nabla v_{j'}(x,\theta^{k'})}e^{i((\theta-\theta^k)-(\theta'-\theta^{k'}))x}dx.$$

We define

$$V_\omega = \sum_{\gamma\in\Gamma}(\omega^+ - \omega_\gamma)\frac{u_\gamma}{D_\omega(x)} \text{ and } v^b_{\omega,jj',kk'}(\theta,\theta') = \sum_{\gamma\in\Gamma}\omega^+_\gamma V^b_\gamma,$$

where $D_\omega(x) = \varrho_{\omega^+}(x)(1+\sum_{\gamma\in\Gamma}\omega_\gamma u_\gamma)$ and V^b_γ is a matrix whose coefficients are operators with the following kernels

$$\int_{\mathbb{R}^d}\frac{u_\gamma}{D_\omega(x)}\nabla v_j(x,\theta^k)\overline{\nabla v_{j'}(x,\theta^{k'})}e^{i((\theta-\theta^k)-(\theta'-\theta^{k'}))x}dx.$$

Thus we obtain

$$\langle \Delta A^b_\omega t, t\rangle = \sum_{\gamma\in\Gamma}\omega^+_\gamma\langle V^b_\gamma t, t\rangle.$$

For $t \in L^2(\mathbb{T}^*)\otimes\mathbb{C}^{n_0}\otimes\mathbb{C}^m$, we have

$$\begin{aligned} C\langle V^b_\gamma t, t\rangle &\ge \int_{\mathbb{R}^d} u_\gamma(x)\left|\sum_{1\le j\le n_0;1\le k\le n}\nabla v_j(x,\theta^k)\int_{\mathbb{T}^*}e^{i(\theta-\theta^k)x}t_{j,k}(\theta)d\theta\right|^2_{\mathbb{C}^d}dx \\ &= \int_{\mathbb{R}^d} u(x)\left|\sum_{1\le j\le n_0;1\le k\le n}\nabla v_j(x,\theta^k)\int_{\mathbb{T}^*}e^{i(\theta-\theta^k)x}e^{i\gamma\theta}t_{j,k}(\theta)d\theta\right|^2_{\mathbb{C}^d}dx. \end{aligned}$$

We define

$$e^{i(\theta-\theta^k)x} = \left(\frac{e^{i(\theta-\theta^k)x}-1}{\sqrt{\omega_k}}\right)\cdot\sqrt{\omega_k}+1$$

and

$$g_k(x,\theta) = \frac{e^{i(\theta-\theta^k)x}-1}{\sqrt{\overline{\omega}_k(\theta)}}.$$

As θ^k is the single zero of $\overline{\omega}_k(\theta)$ and that it is not degenerated, there exists $C > 0$, such that for $x \in \mathbb{R}^d, \theta\in\mathbb{T}^*$ and $1 \le k \le m$, we have

$$|g_k(x,\theta)| \le C(1+|x|). \tag{5.27}$$

Now we develop $\langle V_\gamma^b t, t\rangle$. This gives

$$C\langle V_\gamma^b t,t\rangle \geq \int_{\mathbb{R}^d} u(x)\left|\sum_{1\leq j\leq n_0;1\leq k\leq m}\nabla v_j(x,\theta^k)\int_{\mathbb{T}^*} e^{-i\theta\gamma}t_{j,k}(\theta)d\theta\right|^2_{\mathbb{C}^d} dx \tag{5.28}$$

$$+2Re\Bigg[\int_{\mathbb{R}^d} u(x)\left(\sum_{1\leq j\leq n_0;1\leq k\leq m}\nabla v_j(x,\theta^k)\int_{\mathbb{T}^*} g_k(x,\theta)e^{-i\theta\gamma}\sqrt{\varpi_k(\theta)}t_{j,k}(\theta)d\theta\right)_{\mathbb{C}^d}\cdot$$
$$\overline{\left(\sum_{1\leq j\leq n_0;1\leq k\leq m}\nabla v_j(x,\theta^k)\int_{\mathbb{T}^*} e^{-i\theta\gamma}t_{j,k}(\theta)d\theta\right)_{\mathbb{C}^d}}\,dx\Bigg]$$

$$+\int_{\mathbb{R}^d} u(x)\left|\sum_{1\leq j\leq n_0;1\leq k\leq m}\nabla v_j(x,\theta^k)\int_{\mathbb{T}^*} g_k(x,\theta)e^{-i\theta\gamma}\sqrt{\varpi_k(\theta)}t_{j,k}(\theta)d\theta\right|^2_{\mathbb{C}^d} dx.$$

We define

$$A_\gamma = \int_{\mathbb{R}^d} u(x)\left|\sum_{1\leq j\leq n_0;1\leq k\leq m}\nabla v_j(x,\theta^k)\int_{\mathbb{T}^*} e^{-i\theta\gamma}t_{j,k}(\theta)d\theta\right|^2_{\mathbb{C}^d} dx$$

and

$$B_\gamma = \int_{\mathbb{R}^d} u(x)\left|\sum_{1\leq j\leq n_0;1\leq k\leq m}\nabla v_j(x,\theta^k)\int_{\mathbb{T}^*} g_k(x,\theta)e^{-i\theta\gamma}\sqrt{\varpi_k(\theta)}t_{j,k}(\theta)d\theta\right|^2_{\mathbb{C}^d} dx.$$

From (5.28), using of the Cauchy-Schwartz inequality, we find that:

$$\langle V_\gamma^b t,t\rangle \geq (\sqrt{A_\gamma}-\sqrt{B_\gamma})^2 \geq (1-\alpha)A_\gamma + (1-\frac{1}{\alpha})B_\gamma,$$

for any $0<\alpha<1$. If we take $\alpha=\frac{1}{2}$, we obtain that there exists $C>0$ such that

$$A_\gamma \leq 2B_\gamma + 2C\langle V_\gamma^b t,t\rangle. \tag{5.29}$$

When we develop the sum in A_γ, we obtain

$$A_\gamma = \sum_{1\leq j,j'\leq n_0;1\leq kk'\leq m} v_{jk,j'k'}\hat{t}_{j,k}(\gamma)\overline{\hat{t}_{j',k'}(\gamma)}$$

where

$*\ \hat{t}_{j,k}(\gamma) = \int_{\mathbb{T}^*} e^{-i\theta\gamma}t_{j,k}(\theta)d\theta,$

$$*\ \ v_{jk,j'k'} = \int_{\mathbb{R}^d} u_\gamma \nabla v_j(x,\theta^k)\overline{\nabla v_{j'}(x,\theta^{k'})}dx$$
$$= \sum_{i=1}^d \int_{\mathbb{R}^d} u_\gamma \partial_{x_i} v_j(x,\theta^k)\overline{\partial_{x_i} v_{j'}(x,\theta^{k'})}dx.$$

By [Naj] it is prooved tha the matrix $V=(v_{jkj'k'})_{1\leq j,j'\leq n_0;1\leq k,k'\leq m}$ is defined positive. So we obtain that there exists $C>0$ such that we have

$$\sum_{1\leq j\leq n_0;1\leq k\leq m}\left|\int_{\mathbb{T}^*} e^{-i\theta\gamma}t_{j,k}(\theta)d\theta\right|^2 \leq CA_\gamma. \tag{5.30}$$

We multiplie by $\omega_\gamma^+ = \omega^+ - \omega_\gamma$ the two members of (5.30) then we sum on γ Γ,

$$\sum_{\gamma\in\Gamma}\omega_\gamma^+ \sum_{1\leq j\leq n;1\leq k\leq m}\left|\int_{\mathbb{T}^*} e^{-i\theta\gamma}t_{j,k}(\theta)d\theta\right|^2 \leq C\left(\sum_{\gamma\in\Gamma}\omega_\gamma^+ A_\gamma\right).$$

By taking (5.29) into account we obtain that

$$\sum_{\gamma\in\Gamma}\omega_\gamma^+\sum_{1\le j\le n;1\le k\le m}\left|\int_{\mathbb{T}^*}e^{-i\theta\gamma}t_{j,k}(\theta)d\theta\right|^2\le C\Bigg(\sum_{\gamma\in\Gamma}\omega_\gamma^+B_\gamma+\sum_{\gamma\in\Gamma}\omega^+\langle V_\gamma^b t,t\rangle\Bigg).$$

From assumption **(H.2)**, we have

$$\int_{\mathbb{R}^d}u(x)(1+|x|)^2dx<\infty. \tag{5.31}$$

So,

$$\sum_{\gamma\in\Gamma}\omega_\gamma^+\sum_{1\le j\le n_0;1\le k\le m}\left|\int_{\mathbb{T}^*}e^{-i\theta\gamma}t_{j,k}(\theta)d\theta\right|^2$$
$$\le C\Big(\sum_{1\le j\le n_0;1\le k\le m}\int_{\mathbb{T}^*}\overline{\omega}_k(\theta)|t_{j,k}(\theta)|^2d\theta+\langle\Delta A_\omega^b t,t\rangle\Big).$$

Using this last inequality and the definition of A_ω^a and ΔA_ω^a one finds that there exists $C>0$ such that, $\forall t\in L^2(\mathbb{T}^*)\otimes\mathbb{C}^{n_0}$, we have

$$\langle\Delta A_\omega^a St,St\rangle\le C\Big(\langle A_{\omega^+}^a St,St\rangle+\langle\Delta A_\omega^b St,St\rangle\Big). \tag{5.32}$$

As $\Delta A_\omega^b\ge0$, we know that

$$\langle A_{\omega^+}^a St,St\rangle\le\Big(\langle A_{\omega^+}^a St,St\rangle+\langle\Delta A_\omega^b St,St\rangle\Big). \tag{5.33}$$

From (5.26),(5.32) and (5.33), we deduce that $\forall t\in L^2(\mathbb{T}^*)\otimes\mathbb{C}^{n_0}$

$$\langle A_\omega^a St,St\rangle\le C\cdot\langle A_\omega^b St,St\rangle\le C\langle A_\omega^0 t,t\rangle.$$

This ends the proof of Lemma 5.4. ♣

5.2. **The lower bound.** We set
- $\kappa=1/2$ if n is degenerated E_+.
- $\kappa=1/3$ if n is nondgenerated in the neighborhood of E_+.
- $s=1$ if n is non degenerated in the neighborhood of E_+.
- $s=\frac{2}{(\nu-d)}$ if n is degenerated in the neighborhood of E$_+$.

Theorem 5.6. *Let A_ω, the operator defined by (1.1). We asume that* **(H.1)**,**(H.2)**, **(H.3)** *and* **(H.4)** *are satisfied; then, we have*

$$\liminf_{\varepsilon\to0^+}\frac{\log\Big|\log\Big(N(E_++\varepsilon)-N(E_+)\Big)\Big|}{\log\varepsilon}\ge-s\frac{d}{2}. \tag{5.34}$$

The proof of Theorem 5.6 : By assumption, there is a spectral gap below E_+ of length at least $\delta'>0$. Thus, for $\varepsilon<\delta'$ we have

$$N(E_++\varepsilon)-N(E_+)=N(E_++\varepsilon)-N(E_+-\varepsilon).$$

To prove Theorem 5.6, we will lower bound $N(E_++\varepsilon)-N(E_+-\varepsilon)$. To do this, for N large, we will show that A_{ω,Λ_N} (A_{ω,Λ_N} is A_ω restricted to Λ_N with Dirichlet boundray conditions) has a large number of eigenvalues in the interval $[E_+-\varepsilon,E_++\varepsilon]$ with large a probability. To do this we will construct a family of approximate eigenfunction associated to approximate eigenvalues of A_{ω,Λ_N} in $[E_+-\varepsilon,E_++\varepsilon]$. These functions will be constructed from an eigenfunction of $A_{\omega+}$ associated with E_+. Locating this eigenfunction in θ and imposing to ω_γ^+ to be small for γ in some well choosen box, one obtains an approximate eigenfunction of A_{ω,Λ_N}. Locating the eigenfunction in x in several disjoined places, one obtains several eigenfunctions two by two orthogonal.
In order to simplify the notations, assume in what follows that $\theta^0=0$ is a point

where $E_1(\theta)$ takes the value E_+ in nondegenerate, (respe. degenerate) way if n is nondegenerated (resp. degenerate). There exists $C > 0$ such as,
• if n is nondegenerated, we set $f(\cdot,\theta) = v_1(\cdot,\theta)$, ($V_1$ is the vector constructed in Lemma 3.1)one has

$$||(A_{\omega^+}(\theta) - E_+)f(\cdot,\theta)||_{L^2(C_0)} \leq C|\theta|^2. \tag{5.35}$$

• if n is degenerated, in the same way as in the Lemma 3.1, one can construct functions $(v_j)_{1\leq j\leq q}$ which are analytical in θ in a neighborhood of θ^0 and which generate the same vector space as $(w_j)_{1\leq j\leq q}$. From [Kp1] we know that there exits $(f_j(\theta))_{1\leq j\leq q}$ such that for $\theta = (\theta_1, \theta')$ the function $\vec{f}(\cdot,\theta) = \sum_{1\leq j\leq q} f_j(\theta)v_j(\cdot,\theta)$ verifies

$$|\theta_1| \leq ||\vec{f}(\cdot,\theta)||_{L^2(C_0)} = ||f(\theta)||_{\mathbb{C}^q}$$

and

$$||(A_{\omega^+}(\theta) - E_+)\vec{f}(\cdot,\theta)||_{L^2(C_0)} \leq C(|\theta_1|\cdot|\theta'|^2 + |\theta'|^3 + |\theta|^4). \tag{5.36}$$

Then we set

$$f(\cdot,\theta) = \frac{\vec{f}(\cdot,\theta)}{|\theta_1|}. \tag{5.37}$$

Let $0 < \xi < 1$ a small constant. Let $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi \geq 0$, the support of $\chi \subset [\frac{\xi}{2}, \xi]$ and $\int_{[\frac{\xi}{2},\xi]} \chi(t)^2 dt = 2$.
For $\varepsilon > 0$, we define

$$\mathcal{W}_\varepsilon(\theta) = \varepsilon^{-(d-1+2\kappa)/4}\chi(\varepsilon^{-\kappa}\theta_1)\prod_{j=2}^{d}\chi(\varepsilon^{-\frac{1}{2}}\theta_j) \in L^2(\mathbb{T}^*)$$

and

$$\mathcal{W}_\varepsilon^f(\cdot,\theta) = \mathcal{W}_\varepsilon(\theta)\cdot f(\cdot,\theta).$$

Now let us estimate $||(A_{\omega^+} - E_+)\mathcal{W}_\varepsilon^f||^2$; we have

$$||(A_{\omega^+} - E_+)\mathcal{W}_\varepsilon^f||^2 = \frac{1}{\mathrm{vol}(\mathbb{T}^*)}\int_{\mathbb{T}^*} ||(A_{\omega^+}(\theta) - E_+)f(\cdot,\theta)||^2_{L^2(C_0)}|\mathcal{W}_\varepsilon(\theta)|^2 d\theta.$$

• If n is nondegenerated, from the equation (5.35), we find

$$\begin{aligned} ||(A_{\omega^+} - E_+)\mathcal{W}_\varepsilon^f||^2_{\mathcal{H}} &\leq C^2\int_{\mathbb{T}^*}|\theta|^4|\mathcal{W}_\varepsilon(\theta)|^2 d\theta \\ &\leq C^2\varepsilon^2\int_{[\frac{\xi}{2},\xi]^d}|\theta|^4\prod_{j=1}^{d}\chi^2(\theta_j)d\theta \\ &\leq \frac{\varepsilon^2}{8}, \text{ if } \xi \text{ is small enough.} \end{aligned} \tag{5.38}$$

• If n is degenerated, we use the equation (5.36) to get that,

$$\begin{aligned} ||(A_{\omega^+} - E_+)\mathcal{W}_\varepsilon^f||^2_{\mathcal{H}} &\leq C^2\int_{\mathbb{T}^*}(|\theta'|^2 + |\theta'|^3/|\theta_1| + |\theta|^4/|\theta_1|)^2|\mathcal{W}_\varepsilon(\theta)|^2 d\theta \\ &\leq C^2\varepsilon^2\int_{[\frac{\xi}{2},\xi]^d}(|\theta'|^2 + \varepsilon^{1/6}|\theta'|^3/|\theta_1| + |\theta|^4/|\theta_1|)^2\prod_{j=1}^{d}\chi^2(\theta_j)d\theta \\ &\leq \frac{\varepsilon^2}{8}, \text{ if } \xi \text{ is small enough.} \end{aligned} \tag{5.39}$$

For $\beta \in \Gamma$, we define

$$\mathcal{W}^f_{\varepsilon,\beta}(\cdot,\theta) = e^{-i\beta\cdot\theta}\mathcal{W}^f_{\varepsilon}(\cdot,\theta) \text{ and } \mathcal{W}^f_{\alpha,\varepsilon,\beta,\zeta}(\cdot,\theta) = e^{-i\beta\cdot\theta}(\Pi_{\Lambda_\alpha(\zeta)}\mathcal{W}^f_{\varepsilon})(\cdot,\theta),$$

where $\Lambda_\alpha(\zeta)$ is the cube defined by

$$\Lambda_\alpha(\zeta) = \{\gamma \in \Gamma;\ \text{for } 1 \le j \le d;\ |\gamma_j| \le \zeta^{-(\frac{1}{2}+\alpha)}\}$$

and $\Pi_{\Lambda_\alpha(\zeta)}$ is the orthogonal projection on $\Lambda_\alpha(\zeta)$.
We set

$$\mathcal{U}^f_{\varepsilon,\beta}(x) = \int_{\mathbb{T}^*} \mathcal{W}^f_{\varepsilon,\beta}(x,\theta)d\theta \text{ and } \mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}(x) = \int_{\mathbb{T}^*} \mathcal{W}^f_{\alpha,\varepsilon,\beta,\zeta}(x,\theta)d\theta.$$

For N large and well choosen β and $(\omega_\gamma)_{\gamma\in\Gamma}$, $\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}(x)$ will be an approximate eigenfunction of A_{ω,Λ_N} associated with an approximate eigenvalue in the interval $[E_+ - \varepsilon, E_+ + \varepsilon]$.
We show initially that $||\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}||_{L^2(\mathbb{R}^d)} > C > 0$. Note that

$$\begin{aligned}(\mathrm{vol}(\mathbb{T}^*))||\mathcal{U}^f_{\varepsilon,\beta}||^2_{L^2(\mathbb{R}^d)} &= ||\mathcal{W}^f_{\varepsilon,\beta}||^2_{\mathcal{H}} \\ &= \int_{\mathbb{T}^*} ||f(\cdot,\theta)||^2_{L^2(C_0)}|\mathcal{W}_\varepsilon(\theta)|^2 d\theta \ge 2^d.\end{aligned}$$

As in [Kp1] we remark that $\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}$ and $\mathcal{U}^f_{\varepsilon,\beta}$ are close to each others. More precisely, for any $n \in \mathbb{N}$ and $\beta \in \Gamma$, there exists $C_n > 0$ such that

(5.40)
$$(\mathrm{vol}(\mathbb{T}^*)) \cdot ||\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta} - \mathcal{U}^f_{\varepsilon,\beta}||_{L^2(\mathbb{R})} = ||\mathcal{W}^f_{\alpha,\varepsilon,\beta,\zeta} - \mathcal{W}^f_{\varepsilon,\beta}||_{\mathcal{H}} \le C_n \varepsilon^{-n\kappa}\zeta^{n(\frac{1}{2}+\alpha)}.$$

So, if $\zeta = \varepsilon^s$ (s was defined in the beginning of this section), for ε sufficiently small, one has

$$||\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}||_{L^2(\mathbb{R}^d)} \ge 1.$$

Now we have to look to the conditions under which we have

$$\left|\left|\Big(A_\omega - E_+\Big)\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}(x)\right|\right|^2 \le \varepsilon^2.$$

We have

$$\begin{aligned}(5.41)\quad \left|\left|\Big(A_{\omega,\Lambda_N} - E_+\Big)\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}(x)\right|\right|^2 &\le \left|\left|\Big(A_\omega - E_+\Big)\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}(x)\right|\right|^2 \\ &\le 2\left|\left|\Big(A_{\omega^+} - E_+\Big)\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}(x)\right|\right|^2 + 2\left|\left|\Delta A_\omega \mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}(x)\right|\right|^2.\end{aligned}$$

Equations (5.38) and (5.39) give the bound of the first member of (5.41), it just remains to us to controle the second term of (5.41). We have

(5.42)
$$\left|\left|\Big(\Delta A_\omega\Big)\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}(x)\right|\right|^2 \le 2\left|\left|\Big(\nabla V_\omega\Big)\cdot\Big(\nabla\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}(x)\Big)\right|\right|^2 + 2\left|\left|V_\omega \Delta\mathcal{U}^f_{\alpha,\varepsilon,\beta,\zeta}(x)\right|\right|^2.$$

Let $\omega^+_\gamma = \omega^+ - \omega_\gamma$. To estimate (5.42), one needs the following lemmas

Lemma 5.7. *There exists $K > 0$, such that*

$$(5.43)\quad \left|\left|\Big(\nabla V_\omega\Big)\cdot\Big(\nabla\mathcal{U}^f_{\alpha,\varepsilon,\beta,\varepsilon^s}(x)\Big)\right|\right|^2 \le \varepsilon^4 + K\Big(\varepsilon^{s\alpha(\nu-d)}\cdot\varepsilon + \sup_{\gamma\in\beta+2\Lambda_\alpha(\varepsilon^s)} \omega^+_\gamma\Big)^2.$$

Lemma 5.8. *There exists $K > 0$, such that*

$$(5.44)\quad \left|\left|V_\omega \Delta\mathcal{U}^f_{\alpha,\varepsilon,\beta,\varepsilon^s}(x)\right|\right|^2 \le \varepsilon^4 + K\Big(\varepsilon^{s\alpha(\nu-d)}\cdot\varepsilon + \sup_{\gamma\in\beta+2\Lambda_\alpha(\varepsilon^s)} \omega^+_\gamma\Big)^2.$$

Before showing these two lemmas let us use them to finish the proof of Theorem 5.6.
Combining (5.43) and (5.44) and taking 5.42) into account we find that there exists $K>0$ such as

$$\Big|\Big|(\Delta A_\omega)\mathcal{U}^f_{\alpha,\varepsilon,\beta,\varepsilon^s}(x)\Big|\Big|^2 \leq \varepsilon^3 + K\Big(\varepsilon^{s\alpha(\nu-d)}\cdot\varepsilon + \sup_{\gamma\in\beta+2\Lambda_\alpha(\varepsilon^s)}\omega^+_\gamma\Big)^2. \tag{5.45}$$

By (5.38),(5.39) and (5.45), it follows that:

$$\Big|\Big|\Big(A-E_+\Big)\mathcal{U}^f_{\alpha,\varepsilon,\beta,\varepsilon^s}\Big|\Big|^2 \leq \frac{\varepsilon^2}{2} + K\Big(\varepsilon^{s\alpha(\nu-d)}\cdot\varepsilon + \sup_{\gamma\in\beta+2\Lambda_\alpha(\varepsilon^s)}\omega^+_\gamma\Big)^2. \tag{5.46}$$

Now, for N large, we divide Λ_N into $N(\varepsilon)$ disjoined cubes of size $2\Lambda_\alpha(\varepsilon^s)$. For $\alpha<\frac{1}{2}$, there exists $C>0$ such as $N(\varepsilon)$ satisfies

$$N(\varepsilon)\simeq\frac{(2N)^d}{\varepsilon^{-sd(\frac{1}{2}+\alpha)}}\geq\frac{(N\varepsilon^s)^d}{C}. \tag{5.47}$$

We have:

$\cup_{j=1}^{N(\varepsilon)}(\beta_j+\Lambda_\alpha(\varepsilon^s))\subset\Lambda_N$ and for $j\neq j'(\beta_j+2\Lambda_\alpha(\varepsilon^s))\cap(\beta_{j'}+2\Lambda_\alpha(\varepsilon^s))=\emptyset$.

This implies that for $j\neq j'$, $\mathcal{U}^f_{\alpha,\varepsilon,\beta_j,\varepsilon^s}$ and $\mathcal{U}^f_{\alpha,\varepsilon,\beta_{j'},\varepsilon^s}$ are orthogonal.
We denote the counting function of the eigenvalues of A_{ω,Λ_N} below E by $\Theta_{\Lambda_N}(E)$, then

$$\mathbb{E}\Big(\Theta_{\Lambda_N}(E+\varepsilon)-\Theta_{\Lambda_N}(E-\varepsilon)\Big)=\mathbb{E}\Big(\#\Big\{\text{eigenvalues of }\Pi_N A_\omega\Pi_N\text{ in }[E_+-\varepsilon,E_++\varepsilon]\Big\}\Big)$$

(5.48)

$$\geq\mathbb{E}\Big(\#\Big\{1\leq j\leq N(\varepsilon);||(A-E_+)\mathcal{U}^f_{\alpha,\varepsilon,\beta_j,\varepsilon^s}||_{L^2(\mathbb{R}^d)}\leq\varepsilon\Big\}\Big)\geq\mathbb{E}\Big(\sum_{j=1}^{N(\varepsilon)}B_j(\omega)\Big),$$

where

$$B_j(\omega)=\begin{cases}1 & \text{if } K\left(\varepsilon^{s\alpha(\nu-d)}\cdot\varepsilon+\sup_{\gamma\in\beta+2\Lambda_\alpha(\varepsilon^s)}\omega^+_\gamma\right)^2\leq\frac{\varepsilon^2}{2}.\\ 0 & \text{if not.}\end{cases}$$

The $(B_j)_{1\leq j\leq N(\varepsilon)}$ are independants, identically distributed, Bernoulli random variables. So equations (5.48) and (5.47), imply that there exists $C>0$, such that one has

$$\begin{aligned}&N_{\Lambda_N}(E+\varepsilon)-N_{\Lambda_N}(E-\varepsilon)=\\ &\qquad\frac{1}{((2N+1))^d}\mathbb{E}\Big(\#\Big\{\text{eigenvalues of }\Pi_N A_\omega\Pi_N\text{ in }[E_+-\varepsilon,E_++\varepsilon]\Big\}\Big)\\ &\qquad\qquad\geq\frac{N(\varepsilon)}{(2N+1)^d}\mathbb{P}(B_1=1)\geq\frac{1}{C}\varepsilon^{sd}\mathbb{P}(B_1=1).\end{aligned}$$

We let N go to the infinity then for $\varepsilon>0$ small, we obtain

$$N(E_++\varepsilon)-N(E_+-\varepsilon)\geq\frac{1}{C}\varepsilon^{sd}\mathbb{P}(B_1=1). \tag{5.49}$$

It just remains to estimate $\mathbb{P}(B_1=1)$. If for $1\leq j\leq N(\varepsilon)$, and any $\gamma\in\beta_j+\Lambda_\alpha(\varepsilon)$, one has $\omega^+_\gamma\leq\dfrac{\varepsilon}{2K}$, then for ε rather small

$$K\Big(\varepsilon^{s\alpha(\nu-d)}\cdot\varepsilon+\sup_{\gamma\in\beta+2\Lambda_\alpha(\varepsilon^s)}\omega^+_\gamma\Big)^2\leq\varepsilon^2\cdot K\Big(\varepsilon^{s\alpha(\nu-d)}+\frac{1}{2K}\Big)^2\leq\frac{\varepsilon^2}{2}.$$

As the random variables are independent identically distributed, one has the estimate

$$\mathbb{P}(B_j = 1) \geq \Big(\mathbb{P}(\omega_0^+ \leq \frac{\varepsilon}{2K})\Big)^{\#\Lambda_\alpha(\varepsilon^s)}.$$

Hence, taking the double logarithme of (5.49), using assumption **(H.3)** and the fact that $\#\Lambda_\alpha(\varepsilon^s) = \varepsilon^{-sd(\frac{1}{2}+\alpha)}$ we get that

$$\lim_{\varepsilon\to 0^+} \frac{\log\Big|\log\Big(N(E_+ + \varepsilon) - N(E_+)\Big)\Big|}{\log\varepsilon} \geq -s\frac{d}{2} - sd\alpha. \tag{5.50}$$

The equation (5.50) is true for any $\alpha > 0$, by choosing α small we end the proof of Theorem 5.6. ♣

The proof of Lemma 5.7:

$$\Big|\Big(\nabla V_\omega\Big)\cdot\Big(\nabla \mathcal{U}^f_{\alpha,\varepsilon,\beta,\varepsilon^s}(x)\Big)\Big|\Big|^2 \leq 2^d \sum_{i=1}^d \Big|\Big|(\partial_{x_i} V_\omega)(\partial_{x_i}\mathcal{U}^f_{\alpha,\varepsilon,\beta,\varepsilon^s})(x)\Big|\Big|^2. \tag{5.51}$$

We recall that

$$\omega_\gamma^+ = \omega^+ - \omega_\gamma \text{ and } V_\omega = \frac{\varrho_{\omega^+} - \varrho_\omega}{\varrho_{\omega^+}\cdot\varrho_\omega} = \frac{1}{D_\omega(x)}\sum_{\gamma\in\Gamma}\omega_\gamma^+ u_\gamma,$$

here $D_\omega(x) = \varrho_{\omega^+}(x)\cdot\Big(1 + \sum_{\gamma\in\Gamma}\omega_\gamma u_\gamma\Big)$. We compute

$$\partial_{x_i} V_\omega = \frac{1}{D_\omega(x)}\sum_{\gamma\in\Gamma}\omega_\gamma^+\partial_{x_i}u_\gamma - \frac{1}{D^2_\omega(x)}\sum_{\gamma\in\Gamma}(\omega^+ - \omega_\gamma)u_\gamma\partial_{x_i}D_\omega.$$

So,

$$\begin{aligned}\Big|\Big|(\partial_{x_i}V_\omega)(\partial_{x_i}\mathcal{U}^f_{\alpha,\varepsilon,\beta,\varepsilon^s})(x)\Big|\Big|^2 &\leq 2\Big|\Big|\frac{1}{D_\omega(x)}\sum_{\gamma\in\Gamma}\omega_\gamma^+\partial_{x_i}u_\gamma(\mathcal{U}^{\partial_{x_i}f}_{\alpha,\varepsilon,\beta,\varepsilon^s})(x)\Big|\Big|^2 \\ &+ 2\Big|\Big|\Big(\frac{\partial_{x_i}D_\omega(x)}{D^2_\omega(x)}\Big)\sum_{\gamma\in\Gamma}\omega_\gamma^+ u_\gamma(\mathcal{U}^{\partial_{x_i}f}_{\alpha,\varepsilon,\beta,\varepsilon^s})(x)\Big|\Big|^2.\end{aligned} \tag{5.52}$$

We interest to the behavior of the quantity in the right-hand side of the equation (5.52). Using the fact that $1/C < ||D_\omega(x)||_\infty (C > 0)$, we obtain that

$$\Big|\Big|\frac{1}{D_\omega(x)}\sum_{\gamma\in\Gamma}\omega_\gamma^+\partial_{x_i}u_\gamma(\mathcal{U}^{\partial_{x_i}f}_{\alpha,\varepsilon,\beta,\varepsilon^s})(x)\Big|\Big|^2 \leq \varepsilon^5 + \int_{\mathbb{R}^d}\Big(\sum_{\gamma\in\Gamma}\omega_\gamma^+\partial_{x_i}u_\gamma\Big)^2\Big|\mathcal{U}^{\partial_{x_i}f}_{\varepsilon,\beta}(x)\Big|^2 dx. \tag{5.53}$$

We used an estimate like (5.40) to replace $\mathcal{U}^{\partial_{x_i}f}_{\alpha,\varepsilon,\beta,\varepsilon^s}$ with $\mathcal{U}^{\partial_{x_i}f}_{\varepsilon,\beta}$.

Let $S_{\beta,\varepsilon} = \int_{\mathbb{R}^d}\Big(\sum_{\gamma\in\Gamma}\omega_\gamma^+\partial_{x_i}u_\gamma\Big)^2\Big|\mathcal{U}^{\partial_{x_i}f}_{\varepsilon,\beta}(x)\Big|^2 dx$. Using assupmtion **(H.2)**, we find

$$S_{\beta,\varepsilon} \leq \sum_{\eta\in\Gamma}\Big(\sum_{\gamma\in\Gamma}(1 + |\eta-\gamma|)^{-\nu}\omega_\gamma^+\Big)^2\cdot\int_{C_0} g_+^2(x)\Big|\mathcal{U}^{\partial_{x_i}f}_{\varepsilon,\eta-\beta}(x)\Big|^2 dx. \tag{5.54}$$

There exists $B > 0$ such that

$$\sum_{\eta\notin\Lambda_\alpha(\varepsilon^s)}(1 + |\eta|)^{-\nu} \leq B\cdot\varepsilon^{\alpha s(\nu-d)}\cdot\varepsilon^{\frac{s}{2}(\nu-d)} \leq B\cdot\varepsilon^{\alpha s(\nu-d)}\cdot\varepsilon.$$

Using the fact that the random variables $(\omega_\gamma)_{\gamma\in\Gamma}$ are bounded, one finds that there exists $B'>0$ such that

$$S_{\beta,\varepsilon}\le\sum_{\eta\in\Gamma}\Big(B'\varepsilon^{s\alpha(\nu-d)}\cdot\varepsilon+\sum_{\gamma\in\eta+\Lambda_\alpha(\varepsilon^s)}(1+|\eta-\gamma|)^{-\nu}\omega_\gamma^+\Big)^2\cdot\int_{C_0}g_+^2(x)\Big|\mathcal{U}_{\varepsilon,\eta-\beta}^{\partial_{x_i}f}(x)\Big|^2dx.$$

One splites the sum on $\eta\in\Gamma$ in two parts according to whether η belongs to $\beta+\Lambda_\alpha(\varepsilon^s)$ or not.

• For the sum on $\eta\notin\beta+\Lambda_\alpha(\varepsilon^s)$, there exists $C>0$ such that

$$\begin{aligned}(5.55)\quad&\sum_{\eta\notin\beta+\Lambda_\alpha(\varepsilon^s)}\Big(C\varepsilon^{s\alpha(\nu-d)}\cdot\varepsilon+\sum_{\gamma\in\eta+\Lambda_\alpha(\varepsilon^s)}(1+|\eta-\gamma|)^{-\nu}\omega_\gamma^+\Big)^2\cdot\Big(\int_{C_0}g_+^2(x)\Big|\mathcal{U}_{\varepsilon,\eta-\beta}^{\partial_{x_i}f}(x)\Big|^2dx\Big)\\&\le C\sum_{\eta\notin\Lambda_\alpha(\varepsilon)}\int_{C_0}g_+^2(x)\Big|\mathcal{U}_{\varepsilon,\eta}^{\partial_{x_i}f}(x)\Big|^2dx.\end{aligned}$$

Using the non stationary phase and following the same computation donne in [Kp1] one proves that, for n entire, there exists $C_n>0$ such that

$$(5.56)\qquad\sum_{\eta\notin\Lambda_\alpha(\varepsilon^s)}\Big(\int_{C_0}g_+^2(x)\Big|\mathcal{U}_{\varepsilon,\eta}^{\partial_{x_i}f}(x)\Big|^2dx\Big)\le C_n\varepsilon^{\alpha sn}.$$

• For the sum on η in $\beta+\Lambda_\alpha(\varepsilon^s)$, one obtains

$$\sum_{\eta\in\beta+\Lambda_\alpha(\varepsilon^s)}\Big(C\varepsilon^{s\alpha(\nu-d)}\cdot\varepsilon+\sum_{\gamma\in\Lambda_\alpha(\varepsilon^s)}(1+|\gamma|)^{-\nu}\omega_{\gamma+\eta}^+\Big)^2\cdot\int_{C_0}g_+^2(x)\Big|\mathcal{U}_{\varepsilon,\eta}^{\partial_{x_i}f}(x)\Big|^2dx$$

$$\begin{aligned}(5.57)\quad&\le\Big(\varepsilon^{s\alpha(\nu-d)}\cdot\varepsilon+\Big[\sum_{\gamma\in\Lambda_\alpha(\varepsilon^s)}(1+|\gamma|)^{-\nu}\Big]\sup_{\gamma\in\beta+2\Lambda_\alpha(\varepsilon^s)}\omega_\gamma^+\Big)^2\\&\cdot\sum_{\eta\in\Lambda_\alpha(\varepsilon^s)}\int_{C_0}g_+^2(x)\Big|\mathcal{U}_{\varepsilon,\eta}^{\partial_{x_i}f}(x)\Big|^2dx.\end{aligned}$$

Using the fact that $(\partial_{x_i}f)(x,\theta)$ verifies assumptions of Lemma 5.1, (see Remark 5.2) and $g_+(x)\in L^p(C_0)$, we see that there exists $K>0$ such that

$$\begin{aligned}(5.58)\quad&\sum_{\eta\in\Lambda_\alpha(\varepsilon^s)}\int_{C_0}g_+^2(x)\Big|\mathcal{U}_{\varepsilon,\eta}^{\partial_{x_i}f}(x)\Big|^2dx\cdot\\&\sum_{\eta\in\Lambda_\alpha(\varepsilon^s)}\int_{C_0}g_+^2(x)\Big|\int_{\mathbb{T}^*}e^{i\eta\cdot\theta}\mathcal{W}_\varepsilon(\theta)(\partial_{x_i}f)(x,\theta)d\theta\Big|^2dx\le K.\end{aligned}$$

So we use (5.53), (5.54), (5.55), (5.56), (5.57) and (5.58), to obtain that

$$\begin{aligned}&\Big|\Big|\frac{1}{D_\omega(x)}\sum_{\gamma\in\Gamma}(\omega^+-\omega_\gamma)(\partial_{x_i}u_\gamma)(\mathcal{U}_{\varepsilon,\beta}^{\partial_{x_i}f})(x)\Big|\Big|^2\le\\&\varepsilon^5+K\Big(\varepsilon^{s\alpha(\nu-d)}\cdot\varepsilon+\sum_{\gamma\in\Lambda_\alpha(\varepsilon^s)}(1+|\gamma|)^{-\nu}\sup_{\gamma\in\beta+2\Lambda_\alpha(\varepsilon^s)}\omega_\gamma^+\Big)^2.\end{aligned}$$

We can use the same idea to prove that there exits $K>0$, such that

$$\Big|\Big|\Big(\frac{\partial_{x_i} D_\omega(x)}{D^2_\omega(x)}\Big)\sum_{\gamma\in\Gamma}(\omega^+-\omega_\gamma)u_\gamma\mathcal{U}^{\partial_{x_i}f}_{\varepsilon,\beta}\Big|\Big|^2$$
$$\leq \varepsilon^5 + K\Big(\varepsilon^{s\alpha(\nu-d)}\cdot\varepsilon + \sum_{\gamma\in\Lambda_\alpha(\varepsilon^s)}(1+|\gamma|)^{-\nu}\sup_{\gamma\in\beta+2\Lambda_\alpha(\varepsilon^s)}\omega^+_\gamma\Big)^2.$$

By taking laste estimates and the equations (5.51) into account we obtain Lemma 5.7.

The proof of Lemma 5.8 The proof of Lemma 5.8 is similar to that of Lemma 5.8. Indeed by noticing that $(\partial^2_{x_i} f)(x,\theta)$ verifies the same propreties which are useful as $(\partial_{x_i} f)(x,\theta)$ and following the same step as for the proof of (5.43), we prove (5.44).

6. The proof of Theorem 2.3

One of the techniques used to prove localization is the multi-scales analysis. This analysis makes possible to obtain information on the operator in the hole space starting from information on the operator restricted to cubes of finished size. We briefly recall the technique of multi-scales analysis.

6.1. **Multi-scales analysis.** We introduce some notations. Let Λ be a cube of $\mathbb{R}^d$. We note $A_{\omega,\Lambda}$ the restriction of A_ω on Λ with Dirichlet boundary condition. $A_{\omega,\Lambda}$ is a positive operator with compact resolvent $R_\Lambda(z)=(A_{\omega,\Lambda}-z)^{-1}$ (see [Re, Si4]).

For $k\in\mathbb{N}^*$, $\Lambda=\Lambda_k(x)$ indicates the cube of center $x\in\Gamma$ and with side length $2k$. We write $A_{x,k}$ to note $A_{\omega,\Lambda_k(x)}$ and $R_{x,k}(z)$ to note $R_{\Lambda_k(x)}(z)$. The norm in $\mathcal{L}(L^2(\Lambda_k(x)))$ is noted by $\|\cdot\|_{x,k}$. The multiplication operator by φ (where $\varphi\in L^\infty(\Lambda)$) and $L^2(\Lambda)$ will be simply indicated by φ. In what follows we take $k\in 3\mathbb{N}$.

Let

$$\Upsilon_k(x)=\Big\{y\in\Gamma;\ ||y-x||=\frac{k}{3}-1\Big\}.$$

We set

$$\Gamma_{x,k}=\sum_{y\in\Upsilon_k(x)}\chi_y.$$

where χ_y is the characteristic function of the unit cube centered in y. Let us note that $|\Upsilon_k(x)|\leq cdk^{d-1}$.

Definition Let $m>0, E>0, x\in\Gamma$. We said that $\Lambda_k(x)$ is (m,E)-regular if $E\notin\sigma(A_{x,k})$ and

$$||\Gamma_{x,k}R_{x,k}(E)\chi_x||_{x,k}\leq e^{-m\frac{k}{2}}. \tag{6.59}$$

The multi-scales analysis is based on two fundamental assumptions, namely:

(P1) (k_0,m_0,p) (initial estimate) For $E_0\in[E_+,E_++\varepsilon_0], p>d$ and $m_0>0$, there exists $k_0\in\mathbb{N}$ such that

$$\mathbb{P}\Big\{\Lambda_{k_0}(0)\text{ is }(m_0,E_0)-\text{regular}\Big\}\geq 1-\frac{1}{k_0^p}.$$

(P2) (Wegner estimate) There exists $\eta_0>0, C_W>0$ such that for any E in $[E_+,E_++\eta_0/2]$, for any $\eta<\eta_0/2$ we have

$$\mathbb{P}\Big\{\omega|d(\sigma(A_{0,k}),E)\leq\eta\Big\}\leq C_W\eta|\Lambda_k|^2.$$

The multi-scales analysis is an induction argument over increasing length scales k_j. They are defined recursively by $k_{j+1}=[k_j^\zeta]_3$ where $\zeta\in]1,2[$ and $[t]_3$ is the greatest multiple of 3 which is smaller then t. For each scale k_j one shows than $R_{0,k_j}(z)$ decrease exponentially with probability tending to 1 if j tends to the infinity. Such operation consists in proving that if we have **(P2)** then

$$\textbf{(P1)}\ (k_j, m_j, p) \Rightarrow \textbf{(P1)}\ (k_{j+1}, m_{j+1}, p'). \tag{6.60}$$

The decrease rate m_{j+1} and the scale k_{j+1} verifies : for any $\xi > 0$ there exists c_1, c_2, c_3 such that

$$m_{j+1} \geq m_j\Big(1 - \frac{4k_j}{k_{j+1}}\Big) - \frac{c_1}{k_j} - c_2\frac{\log k_{j+1}}{k_{j+1}}, \tag{6.61}$$

$$k_{j+1}^{p'} \leq c_3\Big(\frac{k_{j+1}}{k_j}\Big)^{2d} k_j^{2p} + \frac{k_{j+1}^{-\xi}}{2}. \tag{6.62}$$

The idea of the proof of (6.60) is to divide a large cube of side k_{j+1} into small cubes of side k_j and variable centers x and then to develop $R_{0,k_{j+1}}(z)$ according to $R_{x,k_j}(z)$. Such operation is based on Simon-Lieb inequality (SLI). (See Lemma 26 of [Fig, Kle]).
To make this development successfully one needs:
1) a succession of regular cubes $\Lambda_{k_j}(x)$ which join the sides of $\Lambda_{k_{j+1}}(0)$. It is shown that this is the case with a very large probability.
2) a fundamental estimate of $R_{0,k_{j+1}}(E)$ for each $E \in [E_+, E_+ + \varepsilon_0]$, with a sufficiently large probability. This is a consequence of the Wegner estimate (see assumption **(P2)**).
We lead thus to a decease estimate of the resolvent $R_{0,k_{j+1}}(z)$ with a rate slightly smaller compared to that of R_{0,k_j}, but with a higher probability.
All the proofs of localization (except in the discrete case [Aiz,Mol].) for the multi-dimensional case use the method of the multi-scales analysis.

6.2. **Localization for the acoustic operator.** We prove that from Theorem 2.1, we are able to verifie the necessary assumptions to start the multi-scales analysis. Indeed Damanik and Stollmann in [Dam, Sto] (see references in this article), prove that one can prove Theorem 2.3 by a simple verification of **(P1)** and **(P2)**. Indeed an iteration of construction done in section 6.1 and a probability argument leads to the estimates claimed in Theorem 2.3. For more details, see [Dam,Sto].
The assumption **(P2)** being satisfied by our operator (see remark 30 of [Fig, Kle] and [Sto]), **(P1)** remains to be proved.
This was already done in [Ki,Sto,St1], [Dam, Sto] for Schrödinger and in [Fig,Kle] and [Sto] for the model (1.1) under the assumption of a special mode of disorder. More precisely their approach requires a sufficiently fast decrease of the probability mesure $\mathbb{P}_{\omega_0}$ to the approach of ω^+ and ω^-. Indeed, they assumed that there exists $\tau > \frac{d}{2}$: such that $\forall \varepsilon > 0$,

$$\mathbb{P}_{\omega_0}([\omega^-, \omega^- + \varepsilon]) \leq \varepsilon^\tau, \text{ and } \mathbb{P}_{\omega_0}([\omega^+ - \varepsilon, \omega^+]) \leq \varepsilon^\tau.$$

According to whether one considers the upper or lower edge of the gap.
In what follows, to check **(P1)** one will adapt a new approach which uses the existence of Lifshitz tails of the IDS at the edges of the gaps.
We consider the following assumption
(P' 1) $\mathbb{P}\Big\{d(\sigma(A_{0,k_0}), [E_+, E_+ + \varepsilon_0]) < \frac{1}{2}k_0^{-\alpha}\Big\} \leq \frac{1}{k_0^p}$, with $\alpha \in]0,1[$.
In the appendix we prove the following result:

Lemma 6.1. **(P'1)** *implies* **(P1)** *for certain* $m_0 \geq Ck_0^{-2}$.

From previous arguments we claim that Theorem 2.3 is equivalent to:

Proposition 6.2. *Under the assumptions of Theorem 2.3, the assumption* **(P'1)** *is satisfiede.*

6.3. **The proof of Proposition 6.2.** To verify **(P'1)**, we will compare the number of states of A_ω and of $A_{0,k}$ in an interval of energy which contains E_+. A comparison technique between $N(A_\omega, E)$ and $N(A_{0,k}, E)$ is needed. At the bottom of the spectrum such technique is used with the Chebichev inequality by F. Klopp in [Kp4] and by Kirsch, Stollman and Stolz in [Ki, Sto, St1]. In the medium of the spectrum one finds the result of I.Veselić in [Ves]. Theses results are given in the Schrödinger case. We will adapt these techniques to our model.

Proposition 6.3. *For all $n \in \mathbb{N}$ and $\alpha \in]0,1[$ there exists $k_1 \in \mathbb{N}$ such that if $k \geq k_1$ we have*

$$\mathbb{E}\big(N_{\omega,k}(E_+ + k^{-\alpha}) - N_{\omega,k}(E_+)\big) \leq k^{-n}. \tag{6.63}$$

Proof For the proof we need the following lemma :

Lemma 6.4. *there exists $c_1 > 1$ such that for any $\varphi \in C_0^\infty(\mathbb{R}^d)$ for $n \in \mathbb{N}$ and $k \in \mathbb{N}^*$ we have*

$$\Big|\mathbb{E}(\langle\varphi, dN_{\omega,k}\rangle) - \langle\varphi, dN\rangle\Big| \leq \Big(\frac{c_1}{k}\Big)^n \sup_{x\in\mathbb{R}, 0\leq j\leq n+4}\Big|\frac{d^j\varphi}{dx^j}(x)\Big|. \tag{6.64}$$

The proof of Lemma 6.4 is similar to that of Lemma 4.4 by using the property of the analytic extensions.
Let φ a Gevrey class function, of exponent $\nu > 1$ (see [Hor]) such that φ is supported in $[-1,2]$, $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ on $[0,1]$. For $\varepsilon > 0$, we define

$$\varphi_{E_+,\varepsilon}(\cdot) = \varphi\Big(\frac{\cdot - E_+}{\varepsilon}\Big).$$

From Lemma 6.4 and properties of Gevrey class functions, we deduce that there exists $c_1 > 1$ such that for any $k \geq 1$, $n \geq 1$ and $0 < \varepsilon < 1$ we have

$$\Big|\mathbb{E}\Big(\langle\varphi_{E_+,\varepsilon}, dN_{\omega,k}\rangle - \langle\varphi_{E_+,\varepsilon}, dN\rangle\Big)\Big| \leq \Big(\varepsilon^{-1}n^\nu\Big)^4\Big(\frac{c_1}{k\varepsilon}\Big)^n n^{n\nu}.$$

By optimizing the last quantity in n, we obtain the existence of $c > 0$ such that for $k \geq 1$ and $0 < \varepsilon < 1$ we have

$$\Big|\mathbb{E}(\langle\varphi_{E_+,\varepsilon}), dN_{\omega,k}\rangle - \langle\varphi_{E_+,\varepsilon}, dN\rangle\Big| \leq \Big(k + \varepsilon^{-1}\Big)^4 e^{(-k\varepsilon/c)^{1/\nu} + c(k\varepsilon/c)^{-1/\nu}}.$$

This implies that

$$\mathbb{E}\Big(N_{\omega,k}(E_+ + \varepsilon) - N_{\omega,k}(E_+)\Big) \leq$$
$$N(E_+ + 2\varepsilon) - N(E - \varepsilon) + \Big(k + \varepsilon^{-1}\Big)^4 e^{(-k\varepsilon/c)^{1/\nu} + c(k\varepsilon/c)^{-1/\nu}}.$$

Using Lifshitz tails, for ε small we find

$$\mathbb{E}\Big(N_{\omega,k}(E_+ + \varepsilon) - N_{\omega,k}(E_+)\Big) \leq e^{-\varepsilon^{-d/3}} + \Big(k + \varepsilon^{-1}\Big)^4 e^{(-k\varepsilon/c)^{1/\nu} + c(k\varepsilon/c)^{-1/\nu}}.$$

If we chose $0 < \varepsilon < 1$ such that $\varepsilon = k^{-\alpha}$,for k large and $\alpha \in]0,1[$, we obtain

$$\mathbb{E}\Big(N_{\omega,k}(E_+ + k^{-\alpha}) - N_{\omega,k}(E_+)\Big) \leq e^{-k^{\alpha d/3}} + \Big(k + k^\alpha\Big)^4 e^{-(k/c)^{(1-\alpha)/\nu} + (k/c)^{-(1-\alpha)/\nu}}.$$

This implies that for any $n \in \mathbb{N}$, there exists $k_1 > 0$ such that if $k \geq k_1$, we have

$$\mathbb{E}\Big(N_{\omega,k}(E_+ + k^{-\alpha}) - N_{\omega,k}(E_+)\Big) \leq k^{-n}. \tag{6.65}$$

So the proof of Proposition 6.3 is ended. ♣
To estimate the probability to find an eigenvalue of $A_{\omega,k}(\theta^0)$ in the interval $[E_+, E_+ +$

$\varepsilon]$. Here $A_{\omega,k}(\theta^0) = A_{\omega,k}|_{\Lambda_k}^{\theta^0}$ is $A_{\omega,k}$ restricted to Λ_k with θ^0-periodiques conditions. More precisely we have

Lemma 6.5. [Ves]*Let* $\theta^0 \in \mathbb{T}_k^*$. *For any* $\varepsilon > 0$ *we have*

$$\mathbb{P}\Big(\{\omega|\sigma(A_{\omega,k}(\theta^0)) \cap [E_+, E_+ + \varepsilon] \neq \emptyset\}\Big) \leq \frac{(2\pi)^d}{\mathrm{vol}(\mathbb{T}_k^*)}\mathbb{E}\Big(N_{\omega,k}(E_+ + \varepsilon + ck^{-1}) - N_{\omega,k}(E_+)\Big). \tag{6.66}$$

For $\alpha < 1$, (α given by Proposition 6.3) and $0 < \varepsilon \leq \varepsilon_1$ there exists $k_2 \in \mathbb{N}$ such that for $k > k_2$ we have $\varepsilon + ck^{-1} \leq 2k^{-\alpha}$ where k_2 depends on α and c. Using the equation (6.65) we obtain that for $k > k_2$;

$$\mathbb{E}\Big(N_{\omega,k}(E_+ + 2k^{-\alpha}) - N_{\omega,k}(E_+)\Big) \leq Ck^{-n}.$$

Lemma 6.5 thus gives :

$$\mathbb{P}\Big(\{\omega|\sigma(A_{\omega,k}(\theta^0)) \cap [E_+, E_+ + k^{-\alpha}] \neq \emptyset\}\Big) \leq ck^{-n+d}.$$

Let $p > d$. Let n large given in the Proposition 6.3 such that $n > p + d$. There exists $k_3 \in \mathbb{N}$ such that if $k \geq k_3$ we have $ck^{-n+d} \leq k^{-p}$. This implies that

$$\mathbb{P}\Big(\{\omega|\sigma(A_{\omega,k_3}(\theta^0)) \cap [E_+, E_+ + k_3^{-\alpha}] \neq \emptyset\}\Big) \leq k_3^{-p}. \tag{6.67}$$

As we need an estimate like (6.67) for $A_\omega|_{\Lambda_k}^{\theta^0}$, notice that there exists $k_0 \in \mathbb{N}$ such that $(2)^{-1/\alpha}k_3 < k_0 < k_3$, ($\Lambda_{k_0} \subset \Lambda_{k_3}$, k_0 depending on the support of u, defining the potential) and verify

$$A_{\omega,k_3}|_{\Lambda_{k_0}}^{\theta^0} = A_\omega|_{\Lambda_{k_0}}^{\theta^0}.$$

However

$$\mathbb{P}\Big(\{\omega|\sigma(A_{\omega,k_3}|_{\Lambda_{k_0}}^{\theta^0}) \cap [E_+, E_+ + k_3^{-\alpha}] \neq \emptyset\}\Big) \leq \mathbb{P}\Big(\{\omega|\sigma(A_{\omega,k_3}(\theta^0)) \cap [E_+, E_+ + k_3^{-\alpha}] \neq \emptyset\}\Big)$$

and for any $p > 0$ we have $k_3^{-p} \leq k_0^{-p}$ so

$$\mathbb{P}\Big(\{\omega|\sigma(A_\omega|_{\Lambda_{k_0}}^{\theta^0}) \cap [E_+, E_+ + k_3^{-\alpha}] \neq \emptyset\}\Big) \leq k_0^{-p}. \tag{6.68}$$

This implies **(P' 1)** for $\varepsilon_0 = k_3^{-\alpha} - \frac{1}{2}k_0^{-\alpha} > 0$.♣

7. Appendix

The proof of Lemma 4.4 The proof consists in using Lemma 4.3 and the Combes-Thomas argument (see [Fig,Kle]) to obtain the exponential decreasing of the resolvante. To estimate the right-hand side of the equation (4.10), we write that

$$\Big|\Big|\mathrm{tr}\Big[\chi_{C_0}\Big((\lambda_0 + A_{\omega,k})^{-q}(z - A_{\omega,k})^{-1} - (\lambda_0 + A_\omega)^{-q}(z - A_\omega)^{-1}\Big)\Big]\Big|\Big|_{\mathrm{tr}} \leq A + B$$

with

$$A = \Big|\Big|\mathrm{tr}\Big(\chi_{C_0}\big[(z - A_{\omega,k})^{-1} - (z - A_\omega)^{-1}\big](\lambda_0 + A_\omega)^{-q}\chi_{C_0}\Big)\Big|\Big|$$

and

$$B = \Big|\Big|\mathrm{tr}\Big(\chi_{C_0}(z - A_{\omega,k})^{-1}\big[(\lambda_0 + A_{\omega,k})^{-q} - (\lambda_0 + A_\omega)^{-q}\big]\chi_{C_0}\Big)\Big|\Big|.$$

We control separately A and B. For A, we obtain

$$A \leq \Big|\Big|\mathrm{tr}\Big(\chi_{C_0}(z - A_{\omega,k})^{-1}(A_\omega - A_{\omega,k})(z - A_\omega)^{-1}(\lambda_0 + A_\omega)^{-q}\chi_{C_0}\Big)\Big|\Big|$$

$$\leq \Big|\Big|\chi_{C_0}(z-A_{\omega,k})^{-1}\Big(\nabla\frac{1}{\varrho_\omega(x)\varrho_{\omega,k}(x)}\Big(\sum_{\gamma\in\Gamma}\omega_\gamma u_\gamma - \sum_{\gamma\in C_k\cap\Gamma}\omega_\gamma\sum_{\beta\in(2k+1)\Gamma}u_{\gamma+\beta}\Big)\nabla\Big)\cdot(z-A_\omega)^{-1}(\lambda_0+A_\omega)^{-q}\chi_{C_0}\Big|\Big|_{\mathrm{tr}}.$$

As

$$\sum_{\gamma\in\Gamma}\omega_\gamma u_\gamma - \sum_{\gamma\in C_k\cap\Gamma}\omega_\gamma\sum_{\beta\in(2k+1)\Gamma}u_{\gamma+\beta} = \sum_{\gamma\in\Gamma,|\gamma|>k}(\omega_{\widetilde{\gamma}}-\omega_\gamma)u_\gamma$$

where $\widetilde{\gamma}=\gamma \pmod{(2k+1)\Gamma}$, we obtain that

$$A\leq \Big|\Big|\chi_{C_0}(z-A_{\omega,k})^{-1}\Big(\nabla\frac{1}{\varrho_\omega(x)\varrho_{\omega,k}(x)}\Big(\sum_{|\gamma|>k,\gamma\in\Gamma}(\omega_{\widetilde{\gamma}}-\omega_\gamma)u_\gamma\Big)\nabla\Big)\cdot (z-A_\omega)^{-1}(\lambda_0+A_\omega)^{-q}\chi_{C_0}\Big|\Big|_{\mathrm{tr}}.$$

However, the random variables $(\omega_\gamma)_{\gamma\in\Gamma}$ are bounded, therefore there exists $M>0$ such that we have

$$\begin{aligned}A\leq M\sum_{\gamma\in\Gamma,|\gamma|>k}&\Big|\Big|\chi_{C_0}(z-A_{\omega,k})^{-1}\Big(\nabla\frac{1}{\varrho_\omega(x)\varrho_{\omega,k}(x)}u_\gamma\nabla\Big)(z-A_\omega)^{-1}(\lambda_0+A_\omega)^{-q}\chi_{C_0}\Big|\Big|_{\mathrm{tr}}\\
&\sum_{\beta\in\Gamma,\gamma\in\Gamma,|\gamma|>k}\Big|\Big|\chi_{C_0}(z-A_{\omega,k})^{-1}\Big(\nabla\frac{1}{\varrho_\omega(x)\varrho_{\omega,k}(x)}u_\gamma\nabla\Big)(z-A_\omega)^{-1}\chi_{\beta+C_0}\Big|\Big|_{\mathcal{L}(L^2(\mathbb{R}^d))}\cdot\Big|\Big|\chi_{\beta+C_0}(\lambda_0+A_\omega)^{-q}\chi_{C_0}\Big|\Big|_{\mathrm{tr}}.\end{aligned}$$

Using of the Combes-Thomas argument (see [Fig, Kle]), we find that there exits $K>0$ such that

$$\Big|\Big|\chi_{C_0}(z-A_{\omega,k})^{-1}\Big(\nabla\frac{1}{\varrho_\omega(x)\varrho_{\omega,k}(x)}u_\gamma\nabla\Big)(z-A_\omega)^{-1}\chi_{\alpha+C_0}\Big|\Big|_{\mathcal{L}(L^2(\mathbb{R}^d))}\leq\frac{K}{|Im(z)|^2}e^{-m_z|\beta|_\Gamma/K}\Big|\Big|\tau_{\beta+\gamma}u\Big|\Big|_{L^p(C_0)}$$

thus, from Lemma 4.3 we obtain

$$A\leq K\sum_{\beta,\gamma\in\Gamma,|\gamma|>k}\frac{e^{-m_z|\beta|_\Gamma/K}}{|Im(z)|^2}||\tau_{\beta+\gamma}u||_{L^p(C_0)}.$$

We can proceed in the same manner to estimate B. Finely, we obtain

$$\Big|\mathbb{E}\Big(\langle\varphi,dN_{\omega,k}\rangle\Big)-\langle\varphi,dN\rangle\Big|\leq\mathbb{E}\Big(\frac{K}{\pi}\int_{\mathbb{R}^2}\Big|\frac{\partial\widetilde{\varphi}}{\partial\overline{z}}(z)\Big|\cdot K\sum_{\alpha,\gamma\in\Gamma,|\gamma|>k}\frac{e^{-m_z|\beta|_\Gamma/K}}{|Im(z)|^2}||\tau_{\beta+\gamma}u||_{L^p(C_0)}dxdy\Big).$$

This ends the proof Lemma 4.4. ♣

The proof of Lemma 6.1 The proof is based on the Combes-Thomas argument which allows one to infer the following inequality

$$||\chi_y R_{x,k}(E)\chi_x||_{x,k}\leq\frac{c}{d(E,\sigma(A_{x,k}))}\cdot e^{-c'd(E,\sigma(A_{x,k}))\|x-y\|}.$$

We have

$$\begin{aligned}||\Gamma_{x,k}R_{x,k}(E)\chi_x||_{x,k} &\leq \sum_{y\in\Upsilon(x)} \|\chi_y R_{x,k}(E)\chi_x\|_{x,k} \\ &\leq cdk^{d-1}\frac{1}{d(E,\sigma(A_{x,k}))}\cdot e^{-c'd(E,\sigma(A_{x,k}))(\frac{k}{3}-1)}.\end{aligned}$$

We remark

$$\mathbb{P}\Big\{\Lambda_{k_0}(0) \text{ is } (m_0,E_0)-\text{regular}\Big\} \geq 1-\frac{1}{k_0^p}$$

$$\Leftrightarrow \mathbb{P}\Big\{\Lambda_{k_0}(0) \text{ is not } (m_0,E_0)-\text{regular}\Big\} \leq \frac{1}{k_0^p}$$

and that

$$\mathbb{P}\Big\{\Lambda_{k_0}(0) \text{ is not } (m_0,E_0)-\text{regular}\Big\} \leq$$

$$\mathbb{P}\Big\{cdk_0^{d-1}\frac{1}{d(E,\sigma(A_{0,k_0}))}\cdot e^{-c'd(E,\sigma(A_{0,k_0}))(\frac{k_0}{3}-1)} > e^{-m_0\frac{k_0}{2}}\Big\}.$$

however

$$\mathbb{P}\Big\{d\Big(\sigma(A_{0,k_0}\Big),[E_+,E_++\varepsilon_0]) < \frac{1}{2}k_0^{-\alpha}\Big\} \leq \frac{1}{k_0^p} \Leftrightarrow$$

$$\mathbb{P}\Big\{cdk_0^{d-1}\frac{1}{d(E,\sigma(A_{0,k_0}))}\cdot e^{-c'd(E,\sigma(A_{0,k_0}))(\frac{k_0}{3}-1)} > cdk_0^{(d-1)+\alpha}e^{-\frac{k_0^{-\alpha}}{2}(\frac{k_0}{3}-1)}\Big\} \leq \frac{1}{k_0^p}.$$

Let $m_0 \geq ck_0^{-2}$ such that $e^{-m_0\frac{k_0}{2}} > cdk_0^{(d-1)+\alpha}e^{-\frac{k_0^{-\alpha}}{2}(\frac{k_0}{3}-1)}$. Thus we obtain that if **(P'1)** is satisfied we have

$$\mathbb{P}\Big\{cdk_0^{d-1}\frac{1}{d(E,\sigma(A_{0,k_0}))}\cdot e^{-c'd(E,\sigma(A_{0,k_0}))(\frac{k_0}{2}-1)} > e^{-m_0\frac{k_0}{2}}\Big\} \leq \frac{1}{k_0^p}.$$

This ends the proof of Lemma 6.1. ♣

References

[Aiz,Mol] M. Aizenman and Molchanov *Localization at large disorder and at extreme energies: an elementary derivation.* Commun. Math. Phys. (1993) **157** p 245-278.

[Ca,La] R. Carmona J. Lacroix *Spectral Theory of Random Schrödinger Operators*Birkhäuser Boston .Basel.Berlin

[Co,Hi] T.M. Combes and P.D. Hislop *Localization for some continous random Hamiltonians in d-dimensions.* J. Funt. Anal.**124** (1994) p 149-180.

[Dam,Sto] D. Damanik and P. Stollmann *Multi-scale Analysis implyes Strong Dynamical Localization.* preprint.

[Dr,Kl] V. Dreifus and A. Klein *A new proof of localization in the Anderson teight binding model* . Commun. Math. Phys. **124** 1989 p 285.

[Fig,Kle] A. Figotin and A. Klein *Localization of Clasical Waves I : Acoustic Waves.* Commu. Math. Phy. 1996 **180** p 439-48.

[Fig,Kuc] A.Figotin and P.Kuchment *Band-gap structure of spectra of periodic dielectric and acoustic media* .II Two-diensional photnic crystals : Siam.J.Math. (1996) **56**, N6 p1561-1620.

[Hor] L. Hörmander *The Analysis of Linear Partial Differential Operators.* Springer Verlag Heidelberg, 1983

[Kat] T. Kato *Perturbation Theory for linear operators* Berlin New York Springer-Verlag. 1976.

[Kir1] W. Kirsh *Random Schrödinger operators* A Cours Lecture Notes In Phy (1989) **345** Springer-Verlag, Berlin p 264-370.

[Kir,Mar1] W. Kirsch. F. Martinelli *On the density of states of Scrodinger operators witn random potential.* J.Phy A:Math Gen 1982 p 2139-2156.

[Kir,Mar2] W. Kirsch and F.Martinelli *Large Deviation and Lifshitz Singularity of The integrated Density of States of Random Hamiltonians* Commun. Math.Phy. (1983) **89** p 27-40.

[Kir,Mar3] W. Kirsch. F. Martinelli *On the the Spectrum Of Schrödinger operators With a Random Potential.* Commu. Math. Phys (1982) **85** p 329-350

[Ki,Sto,St] W. Kirsch, P. Stollmann and G. Stolz *Localization for random perturbation of periodic Schrodinger operators.* Random Oper. Stochastic Equations (1998) **6** p 241-268.

[Koz] S.M. Kozlov*Averging of random strucures.* Soviet Math. Dokl. (1978) **19** N 4 p 188-20

[Kp1] F. Klopp *Internal Lifshits Tails For Random Perturbations Of Periodic Schrödinger Operators* Duke Mathematical Journal (1999) **98** N2 p 335-396.

[Kp2] F.Klopp *Band edge behavior of the integrated density of states of random Jacobi matrices in dimension 1.*Jour. Stt.Phys.

[Kp3] F. Klopp *A; Asymptotic Expansion For The Densty Of States Of A Random Schrödinger operators With Bernoulli Disorder* Random Oprators and Stochastic Equations **3** p 315-331.

[Kp4] F. Klopp, *Localization for some continus Random Schrödinger operators* Commu. Math. Phy (1995) **167** p 553-569.

[Kuc] P. Kuchment spectrum structure and quantum states of disordred condensend systems. Sov. Phys. Usp (1993) **7** P 549-568.

[Lif] I. Lifshitz *Structure Of the energy specrum of impurities imurity bands in disordred solid solutions* Soviet Phy. JETP (1963) **17** p 1159-1170.

[Na] S.Nakao. *On the spectral distribution of the Schrödinger operators with random potontial.*Japan.J.Math (1977) **3** p111-139.

[Naj] H.Najar. *Asymptotique de Lifshitz pour les opérateurs acoustiques aléatoires* Thesis From the universtity of Paris 13 presented in november 2000.

[Pas] L.A. Pastur *Behavir of somme Wiener integrals as $t \to +\infty$ and the density of states of Schrödinger equation with random potentiels.* Tlo.Mat Fiz (Russian) (1977) **32** p 88-95.

[Pas,Fig] L. Pastur and A. Figotin: *Spectra of Random and Almost-Periodic Operators* Springer-Verlag.

[Re,Si4] M. Reed and B.Simon *Methods of Modern Mathemathical vol IV : Analysis of Operators* Academic Press, New York, 1978.

[Sim1] B.Simon *Lifshitz tails for the random model.* Journal of statistical physics 1985 **38** p 65-76.

[sim2] B. Simon *Schrödinger semi groups.* Bull. Am. Math. Soc. (1982) **7** p 447-526.

[Sj] J. Sjöstrand *Microlocal analysis for the periodic magnnetic Schrödinger equation and related questions.* Lecture Notes in Math 1495, Springer-Verlag, Berlin (1991) 237-332.

[Sto] P. Stollmann *Localization for acoustic waves in random perturbation of periodic media.* Israel J. Math. **107** 1998 p 125-139.

[Ves] I. Vesilić *Localization for Random perturbation of periodics Schrödinger operators With Regular Floquet eigenvalues.* preprint.