One non-relativistic particle coupled to a photon field

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Abstract

We investigate the ground state energy of a charged particle coupled to a photon field. First, we regard the self-energy of a "free" electron, which we describe by the Pauli-Fierz Hamiltonian. We show that, in the case of small values of the coupling constant α, the leading order term is represented by $8\pi\alpha(\Lambda - \ln[1+\Lambda])$.

Secondly, we treat the self-energy of a charged boson and provide a different proof for recovering the next to leading order term in α, which has already been obtained in [HVV].

Thirdly, we estimate from above the binding energy of a charged boson in the field of a nucleus. The first order radiative correction turns out to behave like $\ln[1+\Lambda]$ for large values of the ultraviolet-cutoff parameter Λ.

1 Introduction

In recent times great effort was made to start a program with the goal to put Quantum Electrodynamics (QED) on a firm mathematical footing. Bach, Fröhlich, and Sigal were the first who proved (in [BFS1, BFS2, BFS3]), at least for small values of various parameters,

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that atoms and molecules have a ground state in the presence of a quantized radiation field. In the one particle case Griesemer, Lieb and Loss in [GLL] succeeded in removing the restrictions on the parameters. We refer to [GLL] for an extensive list of references concerning this subject.

In the present paper we are especially interested in the behavior of one charged particle coupled to a photon field. We consider the "free" case as well as the presence of a nucleus. We are interested in the *ground state energy* of such a particle when the coupling constant α is small and the ultraviolet cutoff parameter Λ is fixed.

Without radiation field the self-energy of an electron is simply given by the bottom of the spectrum of $-\Delta$, which is equal to 0. In the case of coupling to a photon field the situation changes dramatically and the self-energy is a complicated function depending on α and Λ. The fact that evaluating this quantity is a highly non-trivial problem was pointed out in [LL] by Lieb and Loss, who, in contrary to us, considered the case of fixed α and large values of the cutoff parameter Λ.

The first operator we look at is the Pauli-Fierz operator

$$T^{\mathrm{PF}} = (p + \sqrt{\alpha}A(x))^2 + \sqrt{\alpha}\vec{\sigma} \cdot B(x) + H_f, \tag{1}$$

describing the self-energy of an electron.

The unit of energy is $mc^2/2$, where m is the electron mass, the unit length is $l_c = 2\hbar/mc$, twice the Compton wavelength of the particle, and $\alpha = e^2/\hbar c$ is the *fine structure constant*, where e is the charge of the electron. The physical value of α is $\sim 1/137$. In the present paper α plays the role of a small, dimensionless number. Our results hold for *sufficiently small values* of α. $\vec{\sigma}$ is the vector of Pauli-matrices $(\sigma_1, \sigma_2, \sigma_3)$. The operator $p = -i\nabla$, while A is the magnetic vector potential. The magnetic field is $B = \mathrm{curl}\ A$. The unit of $A(x)^2$ is $mc^2/2l_c$. We fix the gauge $\mathrm{div}A = 0$. The units are chosen as in [GLL].

The underlying Hilbert space is

$$\mathcal{H}^{\mathrm{PF}} = \mathcal{L}^2(\mathbb{R}^3; \mathbb{C}^2) \otimes \mathcal{F}$$

where $\mathcal{F}$ is the Fock space for the photon field.

The vector potential is

$$A(x) = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \frac{\chi(|k|)}{|k|^{1/2}} [\varepsilon_\lambda a_\lambda(k) e^{ikx} + \overline{\varepsilon_\lambda} a_\lambda^*(k) e^{-ikx}] dk = D(x) + D^*(x),$$

and the corresponding magnetic field reads

$$B(x) = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \frac{\chi(|k|)}{|k|^{1/2}} [(k \times i\varepsilon_\lambda) a_\lambda(k) e^{ikx} + \overline{(k \times i\varepsilon_\lambda)} a_\lambda^*(k) e^{-ikx}] dk = E(x) + E^*(x),$$

where the operators a_λ and a_λ^* satisfy the usual commutation relations

$$[a_\nu(k), a_\lambda^*(q)] = \delta(k-q)\delta_{\lambda,\nu}, \quad [a_\lambda(k), a_\nu(q)] = 0, \quad \text{etc.}$$

The vectors $\varepsilon_\lambda(k)$ are any two possible orthonormal polarization vectors perpendicular to k.

The function $\chi(|k|)$ describes the ultraviolet cutoff on the wavenumbers k. We choose χ to be the characteristic function $\Theta(\Lambda - |k|)$.

The photon field energy H_f is given by

$$H_f = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} |k| a_\lambda^*(k) a_\lambda(k) dk.$$

First, we estimate the ground state energy

$${\Sigma_0}^{\mathrm{PF}} = \inf \text{ spec } T^{\mathrm{PF}}.$$

We evaluate the leading order term in α. Recall that $\alpha\langle 0|A^2|0\rangle = 4\pi\alpha\Lambda^2$, where $|0\rangle$ is the vacuum in $\mathcal{F}$, describes the vacuum fluctuation of the field A. Thus, this term is somehow ab initio included in the self-energy. Therefore it may turn out as a slight surprise that $4\pi\alpha\Lambda^2$ is not the leading order term of ${\Sigma_0}^{\mathrm{PF}}$, which relies on the fact that the magnetic field $\sqrt{\alpha}\vec{\sigma}\cdot B$ reproduces the same number but with a negative sign. We show that the leading order term is given by $8\pi\alpha[\Lambda - \ln(1+\Lambda)]$, which is the content of our first theorem.

THEOREM 1. *In the limit of small* α

$$\inf \text{ spec } T^{\mathrm{PF}} = {\Sigma_0}^{\mathrm{PF}} = 8\pi\alpha[\Lambda - \ln(1+\Lambda)] + O(\alpha^{3/2}\Lambda^2).$$

Moreover, it turns out in the proof that *one photon* is enough to recover the leading order term in α.

Next, we omit the $\sqrt{\alpha}\vec{\sigma}\cdot B$-term and regard the operator

$$T = (p + \sqrt{\alpha}A(x))^2 + H_f,$$

which now acts on

$$\mathcal{H} = \mathcal{L}^2(\mathbb{R}^3) \otimes \mathcal{F}.$$

This model describes the interaction of a charged boson with a radiation field. It was already shown in [HVV] that the self-energy of one particle is given by

$$\inf \text{ spec } T = \Sigma_0 = 4\pi\alpha\Lambda^2 - \alpha^2\mathcal{E}_0 + o(\alpha^2),$$

in the case of small α, with $\mathcal{E}_0 \sim \Lambda^2$. Here, we state this theorem again and provide more precise error estimates by using different methods in the proof of the lower bound. The main difference lies in the fact that in the present paper we work in momentum space.

THEOREM 2. *Let*

$$\mathcal{E}_0 = 2\int \frac{(\hat{\vec{G}}(p)\cdot\hat{\vec{G}}(q))^2}{[p+q]^2+|p|+|q|}dpdq = 2\sum_{\mu,\nu=1,2}\int \frac{(\hat{\vec{G}}_\mu(p)\cdot\hat{\vec{G}}_\nu(q))^2}{[p+q]^2+|p|+|q|}dpdq, \tag{2}$$

where $\hat{\vec{G}}_\lambda(p) = \varepsilon_\lambda(p)\chi(|p|)/|p|^{-1/2}$. *Then, in the limit of small* α,

$$\Sigma_0 = \inf \text{ spec } T = 4\pi\alpha\Lambda^2 - \alpha^2\mathcal{E}_0 + O(\alpha^{5/2}\Lambda^{7/2}) + O(\alpha^{9/4}\Lambda^{15/4}). \tag{3}$$

Concerning Σ_0, it turns out in the proof that only *two* photons are needed to recover the coefficient $\mathcal{E}_0$.

Remark 3. In Theorem 2 the main error term is of order $\alpha^{9/4}\Lambda^{15/4}$. If one takes a close look at the proof one sees that this error term stems from the fact that we did not succeed in rigorously proving the inequality

$$\int \frac{[\hat{\vec{G}}(p)\cdot\hat{\vec{G}}(q)]^2}{[l+p+q]^2+|p|+|q|+|k|}dpdq \leq \int \frac{[\hat{\vec{G}}(p)\cdot\hat{\vec{G}}(q)]^2}{[p+q]^2+|p|+|q|}dpdq \qquad \forall l\in\mathbb{R}^3, \tag{4}$$

although its validity seems to be evident by geometrical considerations as well as numerical computations. Assuming (4) our proof implies an error of order $\alpha^{5/2}\Lambda^{7/2}$, but with a slight modification of this proof, which is sketched in the appendix, this error term can even be improved to $O(\alpha^{5/2}\Lambda^{5/2})$ (See Theorem 7 in the appendix).

Finally, we look at one charged boson in the field of a Coulomb potential with charge Z. If we couple the Schrödinger operator

$$-\Delta - \frac{Z}{|x|} \tag{5}$$

to a photon field by means of the Hamiltonian

$$H_Z = (p+\sqrt{\alpha}A(x))^2 + H_f - \frac{Z}{|x|},$$

then Griesemer, Lieb and Loss proved in [GLL] that a ground state exists and they show that

$$E = \inf \text{ spec } H_Z \leq -e_0 + \Sigma_0, \tag{6}$$

where $-e_0 = -\frac{Z^2}{4}$ is the ground state energy of (5). In Section 5 we will argue that at least for small α, (6) holds with strict inequality. This problem is strongly related to [HVV] where we considered a negative short-ranged potential $V(|x|)$ such that $-\Delta + V(|x|)$ has only essential spectrum and proved that coupling to a photon field can produce a bound state.

THEOREM 4. *Let $e_0 = \frac{Z^2}{4}$. Then for α small enough*

$$E - \Sigma_0 \leq -e_0 - 4\alpha e_0 \mathcal{E}[\Lambda, e_0] + O(\alpha^2), \tag{7}$$

where

$$\mathcal{E}[\Lambda, e_0] = \sum_{\lambda=1,2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\pi^2 (l \cdot \vec{G}_\lambda(p, e_0))^2}{8(1+l^2)([p+l]^2 + |p|/e_0^{1/2})} \left[1 - \frac{1}{[p+l]^2 + |p|/e_0^{1/2}} \right] dl dp,$$

with

$$\vec{G}_\lambda(p, e_0) = \frac{\varepsilon_\lambda(p)}{|p|^{1/2}} \Theta(\Lambda/e_0^{1/2} - |p|).$$

Remark 5. It is not difficult to see that for large Λ

$$\mathcal{E}[\Lambda, e_0] \sim 8\pi \ln[1 + \Lambda],$$

which is independent of e_0.

Note, that to obtain an upper bound only one photon is necessary. We strongly conjecture that this is also the case for obtaining an equality in (7) at least for the first order term in α. We also believe that such a term behaves like $\ln[1 + \Lambda]$ for large values of Λ. Nevertheless, one realizes from the proof that if we let Λ go to infinity, α at the same time has to turn to 0, in order that $\alpha\Lambda$ remains a small number.

2 NOTATIONS

A general vector state $\Psi \in \mathcal{H}$ we denote as

$$\Psi = \{\psi_0(x), \psi_1(x, x-\xi), \psi_2(x, x-\xi_1, x-\xi_2), ..., \psi_n(x, x-\xi_1, ..., x-\xi_n), ...\}.$$

For the photon variables we use relative coordinates $\eta_i = x - \xi_i$, which is reasonable since the self-energy of the particle should only depend on the distance of the photons relative to the particle.

For our purposes we denote

$$\vec{G}(x-\xi) = \sum_{\lambda=1,2} \int e^{-ik(\xi - x)} \varepsilon_\lambda \frac{\chi(|k|)}{|k|^{1/2}} dk = \sum_{\lambda=1,2} \vec{G}_\lambda(x-\xi)$$

and

$$\vec{H}(x-\xi) = \sum_{\lambda=1,2} \int e^{-ik(\xi - x)} (k \times i\varepsilon_\lambda) \frac{\chi(|k|)}{|k|^{1/2}} dk = \sum_{\lambda=1,2} \vec{H}_\lambda(x-\xi).$$

Applying D^* to a vacuum state $\psi_0(x)$ we obtain a one-photon state which, written in momentum space, reads

$$[D^*(x)\psi_0]_1(x,k,\lambda) = e^{-ikx}\varepsilon_\lambda(k)\frac{\chi(|k|)}{|k|^{1/2}}\psi_0(x)$$

and using Fourier transform $\int ..e^{-ik\xi}dk$ we get

$$[D^*(x)\psi_0]_1(x,\xi,\lambda) = \overline{\vec{G}_\lambda(x-\xi)}\psi_0(x).$$

Similarly the two-photon expression is given by

$$[D^*(x)D^*(x)\psi_0]_2(x,\xi_1,\xi_2,\lambda_1,\lambda_2) = \overline{\vec{G}_{\lambda_1}(x-\xi_1)\cdot\vec{G}_{\lambda_2}(x-\xi_2)}\psi_0(x).$$

Analogically one derives for the n-photon case

$$[D^*(x)D^*(x)\psi_n]_{n+2}(x,\eta_1,...,\eta_{n+2};\lambda_1,...,\lambda_{n+2}) = \frac{1}{\sqrt{(n+1)(n+2)}}\times$$
$$\times\sum_{i<j}\overline{\vec{G}_{\lambda_i}(\eta_i)\cdot\vec{G}_{\lambda_j}(\eta_j)}\psi_n(x,\eta_1,..,\hat{\eta}_i,\hat{\eta}_j,..,\eta_{n+2};\lambda_1,..,\hat{\lambda}_i,\hat{\lambda}_j,\lambda_{n+2}),$$

where $\hat{\cdot}$ indicates that the variable is omitted. To get the analogical expressions for E^* one just has to replace $\vec{G}$ by $\vec{H}$. In the following, in order not to overburden the paper with too many indices, we suppress the dependence on the spin variables λ_i. So we think of $\vec{G}$ as the sum $\sum_{\lambda=1,2}\vec{G}_\lambda$ and take as a definition

$$[\vec{G}\cdot\vec{G}]^2 \equiv \sum_{\lambda,\mu=1,2}[\vec{G}_\lambda\cdot\vec{G}_\mu]^2$$

as well as

$$[\vec{H}]^2 \equiv \sum_{\lambda=1,2}[\overline{\vec{H}_\lambda}\cdot\vec{H}_\lambda].$$

3 Proof of Theorem 1

3.1 Upper bound for ${\Sigma_0}^{\mathrm{PF}}$

We define a sequence of trial wave functions $\Psi_n \in \mathcal{H}$ by

$$\Psi_n = \{f_n(x)\cdot\uparrow, \sqrt{\alpha}f_n(x)h(x-\xi), 0, 0...\},$$

where

$$h(\eta) = -\mathcal{F}^{-1}\left[\frac{\vec{\sigma}\uparrow\cdot\hat{\vec{H}}(p)}{p^2+|p|}\right] \in \mathcal{L}^2(\mathbb{R}^3;\mathbb{C}^2),$$

f_n being a sequence of real-valued functions in $H^1(\mathbb{R}^3)$, with $\|f_n\| = 1$ and $\|\nabla f_n\| \to 0$ and $\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ denotes the spin-up vector. We compute (notice, $DD\Psi_n = 0$)

$$\begin{aligned}(\Psi_n, T^{\mathrm{PF}}\Psi_n) &= 4\pi\alpha\Lambda^2\|\Psi_n\|^2 + \|\nabla f_n\|^2 + \alpha\|\nabla f_n\|^2\|h\|^2 + \alpha^2\|Dh\|^2 + \\ &+\alpha\left[\|\nabla h\|^2 + (h, H_f h) + 2\Re\int f_n(x)^2\langle\uparrow, \vec{\sigma}\cdot E(x)h(x-\xi)\rangle_{\mathbb{C}^2}\, dx d(x-\xi)\right],\end{aligned} \tag{8}$$

where we used that since $f_n(x)$ is real $\int f_n(x)\nabla f_n(x) = 0$. Collecting the terms in (8) that are of order α we get

$$\|\nabla h\|^2 + (h, H_f h) + 2\Re\int f_n(x)^2\langle\uparrow, \vec{\sigma}\cdot Eh(\eta)\rangle_{\mathbb{C}^2}\, dx d\eta,$$

which reads in momentum space

$$(h(p), [p^2 + |p|]h(p)) + 2\Re\int\langle\uparrow, \vec{\sigma}\cdot\hat{\vec{H}}h(p)\rangle_{\mathbb{C}^2}\, dp, \tag{9}$$

where $h(p) = -\frac{\vec{\sigma}\uparrow\cdot\hat{\vec{H}}(p)}{p^2+|p|}$. A straightforward calculation shows

$$\begin{aligned}(9) &= -\int\frac{(\hat{\vec{H}}(p))^2}{p^2+|p|} = -\sum_{\lambda=1,2}\int\frac{(\hat{\vec{H}}_\lambda)^2}{p^2+|p|} = \\ &= -8\pi\int_0^\Lambda\frac{p^3}{p^2+p}\, dp = -4\pi[\Lambda^2 - 2[\Lambda - \ln(1+\Lambda)]].\end{aligned}$$

Inserting into (8) and using $\|\nabla f_n\| \to 0$ yields

$$\Sigma_0{}^{\mathrm{PF}} \le \lim_{n\to\infty}(\Psi_n, T^{\mathrm{PF}}\Psi_n) = 8\pi\alpha[\Lambda - \ln(1+\Lambda)] + O(\alpha^2).$$

3.2 Lower bound on $\Sigma_0{}^{\mathrm{PF}}$

In order to derive a lower bound on $\Sigma_0{}^{\mathrm{PF}}$, we take a Ψ, with $\|\Psi\|^2 = 1$, in the form domain of T^{PF}, close to Σ_0. From the previous section we know that we can find a Ψ, such that

$$(\Psi, T^{\mathrm{PF}}\Psi) \le 8\pi\alpha\Lambda.$$

We fix now such a Ψ, and first ask for an a-priori estimate on the kinetic energy of Ψ. Since $A(x)^2 \ge 0$ and, due to our choice of gauge, $[p, A] = 0$,

$$(\Psi, T^{\mathrm{PF}}\Psi) \ge \|p\Psi\|^2 + (\Psi, H_f\Psi) + 2\sqrt{\alpha}(p\Psi, A\Psi) + \sqrt{\alpha}(\Psi, \vec{\sigma}\cdot B\Psi). \tag{10}$$

Recall that $A = D + D^*$. Therefore, by Schwarz inequality

$$2\sqrt{\alpha}(p\Psi, A\Psi) = 4\sqrt{\alpha}\Re(p\Psi, D\Psi) \le 4\sqrt{\alpha 8\pi\Lambda}\|p\Psi\|^2 + 4\sqrt{\alpha/8\pi\Lambda}\|D\Psi\|^2,$$

which by use of [GLL, Lemma A.4]

$$D^*D \leq 8\pi\Lambda H_f \tag{11}$$

implies

$$2\sqrt{\alpha}(p\Psi, A\Psi) \leq 4\sqrt{\alpha 8\pi\Lambda}\left(\|p\Psi\|^2 + (\Psi, H_f\Psi)\right).$$

Inserting into (10) we infer

$$8\pi\alpha\Lambda \geq (\Psi, T^{\mathrm{PF}}\Psi) \geq (1 - 4\sqrt{8\pi\alpha\Lambda})\left(\|p\Psi\|^2 + (\Psi, H_f\Psi)\right).$$

For $\alpha\Lambda$ small enough we get the a-priori estimate

$$\|p\Psi\|^2 + (\Psi, H_f\Psi) \leq C\alpha\Lambda$$

for some constant C. Consequently for this Ψ we obtain by (11)

$$2\sqrt{\alpha}|(p\Psi, A\Psi)| \leq 4\sqrt{\alpha}\|p\Psi\|\|D\Psi\| \leq 4C\alpha^{3/2}\Lambda^{3/2}.$$

Moreover, by

$$[D, D^*] = 4\pi\Lambda^2, \tag{12}$$

we obtain

$$A^2 = (D + D^*)^2 = 4\pi\Lambda^2 + 2D^*D + 2\Re DD.$$

By means of Schwarz inequality, (11) and (12) we additionally get

$$\begin{aligned} 2\alpha|(\Psi, DD\Psi)| &\leq \alpha^{1/2}\|D\Psi\|^2 + \alpha^{3/2}\|D^*\Psi\|^2 \\ &\leq 20\pi C\alpha^{3/2}\Lambda^2. \end{aligned}$$

Hence, summing up above inequalities,

$$\begin{aligned} &(\Psi, T^{\mathrm{PF}}\Psi) = \\ &= \|p\Psi\|^2 + (\Psi, H_f\Psi) + 2\sqrt{\alpha}(p\Psi, A\Psi) + \alpha(\Psi, A^2\Psi) + \sqrt{\alpha}(\Psi, \vec{\sigma}\cdot B\Psi) \\ &\geq 4\pi\alpha\Lambda^2 + \|p\Psi\|^2 + (\Psi, H_f\Psi) + \sqrt{\alpha}(\Psi, \vec{\sigma}\cdot B\Psi) - O(\alpha^{3/2}\Lambda^2) - O(\alpha^{3/2}\Lambda^{3/2}). \end{aligned} \tag{13}$$

Recall that a general state Ψ can be written as

$$\Psi = \{\psi_0(x), \psi_1(x, x-\xi), \psi_2(x, x-\xi_1, x-\xi_2), ..., \psi_n(x, x-\xi_1, ..., x-\xi_n), ...\}.$$

and that

$$[E^*(x)\psi_n]_{n+1}(x,\eta_1,...,\eta_{n+1}) = \frac{1}{\sqrt{n+1}}\sum_{i=1}^{n+1}\vec{H}(\eta_i)\psi_n(x,\eta_1,..,\hat{\eta}_i,..,\eta_{n+1}),$$

where $\eta_i = x - \xi_i$ and $\hat{\eta}_i$ indicates that this variable is omitted. In that case, the Fourier transform reads

$$\mathcal{F}[E^*(x)\psi_n](l,k_1,..,k_{n+1}) = \frac{1}{\sqrt{n+1}}\mathcal{S}[\hat{\vec{H}}(p)\psi_n(l,k_1,..,k_n)],$$

where $\mathcal{S}$ means symmetrization with respect to (k,p), with

$$k = (k_1,...,k_n) \quad \text{and} \quad p = k_{n+1};$$

namely,

$$\mathcal{S}[\hat{\vec{H}}(p)\psi_n(l,k_1,..,k_n)] = \sum_{i=1}^{n+1}\hat{\vec{H}}(k_i)\psi_n(l,k_1,..,\hat{k}_i,..,k_{n+1}).$$

For simplicity we use the same notation for ψ_n in configuration as well as in momentum space. Furthermore, let us denote

$$\mathcal{E}[\psi_n,\psi_{n+1}] = \|p\psi_{n+1}\|^2 + (\psi_{n+1}, H_f\psi_{n+1}) + 2\sqrt{\alpha}\Re(\vec{\sigma}\cdot E^*\psi_n,\psi_{n+1}).$$

Thus, inserting into (13) leads to

$$(\Psi, T^{\mathrm{PF}}\Psi) \geq 4\pi\alpha\Lambda^2\|\Psi\|^2 + \sum_{n=0}^{\infty}\mathcal{E}[\psi_n,\psi_{n+1}] - O(\alpha^{3/2}\Lambda^2).$$

From now on we look at $\mathcal{E}[\psi_n,\psi_{n+1}]$ in the momentum space, i.e.

$$\mathcal{E}[\psi_n,\psi_{n+1}] = (\psi_{n+1},([l+\textstyle\sum_{i=1}^n k_i + p]^2 + \sum_{i=1}^n |k_i| + |p|)\psi_{n+1}) + 2\sqrt{\alpha}\Re(\vec{\sigma}\cdot\mathcal{F}[E^*\psi_n],\psi_{n+1}).$$

Our goal is to estimate $\mathcal{E}[\psi_n,\psi_{n+1}]$ from below, by keeping ψ_n fixed and varying over ψ_{n+1}.

The respective minimum is achieved by

$$\bar{\psi}_{n+1}(l,k,p) = -\sqrt{\alpha}\frac{\vec{\sigma}\cdot\mathcal{F}[E^*\psi_n](l,k,p)}{[l+\sum_{i=1}^n k_i + p]^2 + \sum_{i=1}^n |k_i| + |p|},$$

which can be seen by variational principle or simply by Schwarz inequality. Indeed,

$$\|f\|^2 + 2\Re(f,g) \geq -\|g\|^2,$$

where

$$f = ([l + \sum_{i=1}^n k_i + p]^2 + \sum_{i=1}^n |k_i| + |p|)^{1/2}\psi_{n+1}$$

and

$$g = \sqrt{\alpha}\frac{\vec{\sigma}\cdot\mathcal{F}[E^*\psi_n](l,k,p)}{([l + \sum_{i=1}^n k_i + p]^2 + \sum_{i=1}^n |k_i| + |p|)^{1/2}}.$$

Furthermore,

$$\begin{aligned}
&\mathcal{E}[\psi_n, \bar{\psi}_{n+1}] = -\|g\|^2 = \\
&= \left(\frac{-\alpha\vec{\sigma}\cdot\mathcal{F}[E^*\psi_n]}{([l + \sum_{i=1}^n k_i + p]^2 + \sum_{i=1}^n |k_i| + |p|)^{1/2}}, \frac{\vec{\sigma}\cdot\mathcal{F}[E^*\psi_n]}{([l + \sum_{i=1}^n k_i + p]^2 + \sum_{i=1}^n |k_i| + |p|)^{1/2}}\right) \\
&= \left(\frac{-\alpha\sum_{i=1}^3 \sigma_i\mathcal{F}[E_i^*\psi_n]}{([l + \sum_{i=1}^n k_i + p]^2 + \sum_{i=1}^n |k_i| + |p|)^{1/2}}, \frac{\sum_{j=1}^3 \sigma_j\mathcal{F}[E_j^*\psi_n]}{([l + \sum_{i=1}^n k_i + p]^2 + \sum_{i=1}^n |k_i| + |p|)^{1/2}}\right).
\end{aligned}$$

The well know anti-commutation relations of the Pauli matrices

$$\sigma_i\sigma_j + \sigma_j\sigma_i = 2I_{\mathbb{C}^2}\delta_{i,j}$$

imply

$$\begin{aligned}
&\mathcal{E}[\psi_n, \bar{\psi}_{n+1}] = \qquad (14)\\
&= \left(\frac{-\alpha\sum_{i=1}^3 \mathcal{F}[E_i^*\psi_n]}{([l + \sum_{i=1}^n k_i + p]^2 + \sum_{i=1}^n |k_i| + |p|)^{1/2}}, \frac{\mathcal{F}[E_i^*\psi_n]}{([l + \sum_{i=1}^n k_i + p]^2 + \sum_{i=1}^n |k_i| + |p|)^{1/2}}\right).
\end{aligned}$$

Using

$$\mathcal{F}[E_i^*(x)\psi_n](l,k_1,\dots,k_{n+1}) = \frac{1}{\sqrt{n+1}}\mathcal{S}[H_i(p)\psi_n(l,k_1,\dots,k_n)],$$

one realizes that (14) consists of two different terms $I+II$. The first one is given by the $n+1$ diagonal terms $H_i(k_m)H_i(k_m)$ in (14) and the second one by the $(n+1)n$ off-diagonal terms $H_i(k_n)H_i(k_m)$. First, due to the symmetry of ψ_n and the normalization factor $1/(n+1)$,

$$I = -\alpha\int\frac{(\hat{\vec{H}}(p))^2|\psi_n(l,k)|^2}{[l + \sum_{i=1}^n k_i + p]^2 + \sum_{i=1}^n |k_i| + |p|}dldkdp,$$

which is the main part, and is independent of the photons energy. The relation $(\hat{\vec{H}}(p))^2 = 2\chi(|p|)|p|$, where the factor 2 comes from the two photon polarizations $\lambda = 1,2$, implies

$$\begin{aligned}
I &= -2\alpha\int\frac{\chi(|p|)|p||\psi_n(l,k)|^2}{[l + \sum_{i=1}^n k_i + p]^2 + \sum_{i=1}^n |k_i| + |p|}dldkdp \\
&\geq -2\alpha\int\frac{\chi(|p|)|p||\psi_n(l,k)|^2}{[l + \sum_{i=1}^n k_i + p]^2 + |p|}dldkdp.
\end{aligned}$$

Next, we fix $l' = l + \sum_{i=1}^n k_i$ and apply the inequality

$$\int \frac{\chi(|p|)|p|}{[l'+p]^2+|p|}dp \leq \int \frac{\chi(|p|)|p|}{p^2+|p|}dp \quad \forall l' \in \mathbb{R}^3,$$

which can be verified by explicit computations. Indeed, the function

$$f(l') = \int_0^\Lambda \int_0^\pi \frac{\chi(|p|)|p|}{l'^2+|l'||p|\cos\theta+p^2+|p|} p^2 sin\theta \, d\theta d|p|$$

turns out to be monotone decreasing in $|l'|$.

Therefore, we arrive at

$$I \geq -2\alpha\|\psi_n\|^2 \int_{\mathbb{R}^3} \frac{\chi(|p|)|p|}{p^2+|p|}dp = -4\pi\alpha\|\psi_n\|^2[\Lambda^2 - 2[\Lambda - \ln(1+\Lambda)]].$$

Secondly, we have

$$II = -\alpha n \sum_{i=1}^3 \int \frac{\overline{H_i(q)\psi_n(l,q,k)}\psi_n(l,p,k)H_i(p)}{[l+q+p+\sum_{i=2}^n k_i]^2+|q|+|p|+\sum_{i=2}^n|k_i|} dldqdpdk$$

with $k = (k_2, ..., k_n)$. In contrary to the usual definition we denote the one-particle density corresponding to ψ_n as

$$\rho_{\psi_n}(p) = \int |\psi_n(l,p,k)|^2 dldk$$

which is normalized to one.

By using Schwarz' inequality twice we obtain

$$\begin{aligned} II &\geq -\alpha 2n \sum_{i=1}^3 \int \frac{|H_i(p)||H_i(q)||\psi_n(l,p,k)||\psi_n(l,q,k)|}{|p|+|q|} dldqdpdk \\ &\geq -\alpha 2n \sum_{i=1}^3 \int \frac{|H_i(p)||H_i(q)|}{(|p|+|q|)\sqrt{|p||q|}} \sqrt{\rho_{\psi_n}(p)|p|}\sqrt{\rho_{\psi_n}(q)|q|} dqdp \\ &\geq -\alpha 6n \int \rho_{\psi_n}(p)|p|dp \left[\int\int \frac{\chi(|p|)\chi(|q|)}{(|p|+|q|)^2} dpdq\right]^{1/2}. \end{aligned}$$

By a scaling argument, one easily realizes that the second term in the last line is of order Λ^2. Since $n\int \rho_{\psi_n}(p)|p|dp = (\psi_n, H_f\psi_n)$ we arrive at

$$II \geq -O(\alpha\Lambda^2)\pi\Lambda^2(\psi_n, H_f\psi_n).$$

Altogether

$$(\Psi, T^{\mathrm{PF}}\Psi) \geq \sum_n \|\psi_n\|^2 8\pi\alpha[\Lambda - \ln(1+\Lambda)] - O(\alpha^{3/2}\Lambda^2) - O(\alpha^2\Lambda^3),$$

since

$$\alpha\Lambda^2 \sum_n (\psi_n, H_f\psi_n) = \alpha\Lambda^2(\Psi, H_f\Psi) = O(\alpha^2\Lambda^3).$$

With $\|\Psi\|^2 = \sum_n \|\psi_n\|^2 = 1$ the proof of Theorem 1 is finished.

4 Proof of Theorem 2

4.1 Upper bound for Σ_0

We proceed similarly to [HVV]. We define the sequence of trial wave functions

$$\Psi_n = \{f_n(x), 0, \alpha f_n(x)\varphi(x-\xi_1, x-\xi_2), 0, 0, \dots\},$$

where we assume $\|f_n\| = 1$, $\|\nabla f_n\| \to 0$ and

$$\varphi(\eta_1, \eta_2) = -\sqrt{2}\mathcal{F}^{-1}\left[\frac{\hat{\vec{G}}(p)\cdot\hat{\vec{G}}(q)}{[p+q]^2 + |p| + |q|}\right]. \tag{15}$$

Then,

$$(\Psi_n, T\Psi_n) = \|\nabla f_n\|^2 + \alpha^2\|\nabla\varphi\|^2 + \alpha^2(\varphi, H_f\varphi) + \alpha^2\|\nabla f_n\|^2\|\varphi\|^2 + \alpha(\Psi_n, A^2\Psi_n).$$

Recall that

$$\alpha(\Psi_n, A^2\Psi_n) = 4\pi\alpha\Lambda^2 + 2\alpha(\Psi_n, D^*D\Psi_n) + 2\alpha^2\Re(f_n, DDf_n\varphi).$$

By [GLL, Lemma A.4] we know

$$2\alpha(\Psi_n, D^*D\Psi_n) \le 18\pi\alpha^3\Lambda(\varphi, H_f\varphi),$$

whereas

$$\begin{aligned} 2\Re(f_n, DDf_n\varphi) &= -2\sqrt{2}\|f_n\|^2\int \vec{G}(\eta_1)\cdot\vec{G}(\eta_2)\varphi(\eta_1,\eta_2)d\eta_1 d\eta_2 \\ &= -4\int\frac{(\hat{\vec{G}}(p)\cdot\hat{\vec{G}}(q))^2}{[p+q]^2+|p|+|q|}dpdq = -2\mathcal{E}_0. \end{aligned}$$

Since

$$\|\nabla\varphi\|^2 + (\varphi, H_f\varphi) = \mathcal{E}_0$$

we arrive at

$$\Sigma_0 \le \lim_{n\to\infty}(\Psi_n, T\Psi_n) = 4\pi\alpha\Lambda^2 - \alpha^2\mathcal{E}_0 + O(\alpha^3\Lambda).$$

4.2 Lower bound on Σ_0.

We start this section with an a-priori estimate on Σ_0.

LEMMA 6. *Let $\alpha\Lambda$ be small enough. Then for some $C > 0$*

$$T \geq 4\pi\alpha\Lambda^2 - \tfrac{1}{2}C\alpha^2\Lambda^3 + \tfrac{1}{2}p^2 + \tfrac{1}{2}H_f \tag{16}$$

in the sense of quadratic forms.

Proof. For Ψ in the form domain of T we have

$$(\Psi, T\Psi) = \|p\Psi\|^2 + 2\sqrt{\alpha}(p\Psi, A(x)\Psi) + \alpha\|A(x)\Psi\|^2 + (\Psi, H_f\Psi).$$

Notice that $A(x) = D(x) + D^*(x)$ and $[p, A(x)] = 0$ imply

$$2\sqrt{\alpha}(p\Psi, A(x)\Psi) = 4\sqrt{\alpha}\Re(p\Psi, D\Psi) \leq \sqrt{\alpha\Lambda}\|p\Psi\|^2 + 16\sqrt{\alpha/\Lambda}\|D\Psi\|^2. \tag{17}$$

Moreover,

$$\alpha\|A\Psi\|^2 = 4\pi\alpha\Lambda^2\|\Psi\|^2 + 2\alpha(\Psi, D^*D\Psi) + 2\alpha\Re(\Psi, DD\Psi).$$

By Schwarz inequality

$$2\alpha\Re DD \leq C^{-1}D^*D + C\alpha^2 DD^* \quad \text{for all } C > 0. \tag{18}$$

The inequality (18) together with (11) and (12) implies

$$\begin{aligned}(\Psi, T\Psi) \;&\geq\; (4\pi\alpha\Lambda^2 - 4\pi C\alpha^2\Lambda^2)\|\Psi\|^2 + (\tfrac{1}{2} - \sqrt{\alpha\Lambda})\|p\Psi\|^2 \\ &\quad + \left(\tfrac{1}{2} - 8\pi[16\sqrt{\alpha\Lambda} + C^{-1}\Lambda + \alpha^2 C\Lambda]\right)(\Psi, H_f\Psi) + \tfrac{1}{2}\{\|p\Psi\|^2 + (\Psi, H_f\Psi)\}.\end{aligned}$$

With C large enough and $\alpha\Lambda$ sufficiently small we arrive at the statement of the Lemma. $\square$

We now take a $\Psi \in \mathcal{H}$ close to Σ_0, $(\Psi, T\Psi) \leq 4\pi\alpha\Lambda^2$. Equation (16) immediately yields

$$(\Psi, H_f\Psi) + \|p\Psi\|^2 \leq C\alpha^2\Lambda^3. \tag{19}$$

Therefore, we infer by (11) and (19)

$$\sqrt{\alpha}(\Psi, pA\Psi) = \sqrt{\alpha}2\Re(p\Psi, D\Psi) \leq \sqrt{\alpha}\|p\Psi\|\Lambda^{1/2}(\Psi, H_f\Psi)^{1/2} \leq O(\alpha^{5/2}\Lambda^{7/2}).$$

Consequently,

$$(\Psi, T\Psi) \geq 4\pi\alpha\Lambda^2\|\Psi\|^2 + \sum_{n=0}^{\infty}\mathcal{E}[\psi_n, \psi_{n+2}] + 2\alpha(\Psi, D^*D\Psi) - O(\alpha^{5/2}\Lambda^{7/2}),$$

where

$$\mathcal{E}[\psi_n, \psi_{n+2}] = \|\nabla_x \psi_{n+2}\|^2 + \sum_{i=1}^{n+2} \| |\nabla_{\xi_i}|^{1/2} \psi_{n+2}\|^2 + 2\alpha \Re \int \overline{D^* D^* \psi_n} \psi_{n+2}.$$

Using $x - \xi_i = \eta_i$ we can write

$$[D^*(x)\psi_n]_{n+1} = \frac{1}{\sqrt{n+1}} \sum_{i=1}^{n+1} \overline{\vec{G}(\eta_i)} \psi_n(x, \eta_1, \dots, \hat{\eta}_i, \dots, \eta_{n+1})$$

and

$$[D^*(x)D^*(x)\psi_n]_{n+2} = \frac{1}{\sqrt{(n+1)(n+2)}} \sum_{j=1}^{n+2} \sum_{i=1}^{n+1} \overline{\vec{G}(\eta_j)\vec{G}(\eta_i)} \psi_n(x, \eta_1, \dots, \hat{\eta}_j, \hat{\eta}_i, \dots, \eta_{n+2}),$$

where $\hat{\eta}_i$ indicates that η_i is omitted. In the following we will work with the momentum representation

$$\mathcal{F}\left[D^*(x)D^*(x)\psi_n\right]_{n+2}(l, k, p, q) = \frac{1}{\sqrt{(n+1)(n+2)}} \mathcal{S}[\hat{\vec{G}}(p) \cdot \hat{\vec{G}}(q) \psi_n(l, k)],$$

where $\mathcal{S}$ means symmetrization with respect to (k, p, q), while $k = (k_1, ..., k_n)$, $p = k_{n+1}$, $q = k_{n+2}$, i.e.

$$\mathcal{S}[\hat{\vec{G}}(p) \cdot \hat{\vec{G}}(q) \psi_n(l, k)] = \sum_{j=1}^{n+2} \sum_{i=1}^{n+1} \hat{\vec{G}}(k_j) \cdot \hat{\vec{G}}(k_i) \psi_n(l, k_1, \dots, \hat{k}_j, \hat{k}_i, ..., k_{n+2}). \tag{20}$$

Therefore we can write

$$\begin{aligned} \mathcal{E}[\psi_n, \psi_{n+2}] &= (\psi_{n+2}, [l + p + q + \textstyle\sum_{i=1}^n k_i]^2 \psi_{n+2}) + (\psi_{n+2}, [|p| + |q| + \textstyle\sum_{i=1}^n |k_i|] \psi_{n+2}) \\ &\quad + \frac{2\alpha}{\sqrt{(n+1)(n+2)}} \Re \int \overline{\mathcal{S}[\hat{\vec{G}}(p) \cdot \hat{\vec{G}}(q) \psi_n(l, k)]} \psi_{n+2}(l, k, p, q). \end{aligned} \tag{21}$$

Now we take a fixed but arbitrary photon number n and regard the term

$$\mathcal{E}[\psi_n, \psi_{n+2}] + 2\alpha(D\psi_{n+2}, D\psi_{n+2}). \tag{22}$$

First of all, it is necessary to separate the low energy–or "soft"–photons from the high energy photons. This can be done by an additional cut-off function

$$\chi_\Lambda(k) = \Theta(\Lambda - \sum_{i=1}^n |k_i|),$$

which decomposes the photon momentum space into a part, where $\sum_{i=1}^{n}|k_i| \leq \Lambda$ and into its complement, represented by $\overline{\chi_\Lambda}(k) = 1 - \Theta(\Lambda - \sum_{i=1}^{n}|k_i|)$. Respectively, we denote

$$\psi_n(l,k)\chi_\Lambda(k) = \widetilde{\psi_n}(l,k), \quad \psi_n(l,k)\overline{\chi_\Lambda}(k) = \psi_n(l,k)\overline{\chi}.$$

Therefore, we rewrite (22) as (we think of the terms being given in momentum space, e.g. $D^*\psi_n\overline{\chi} = \frac{1}{\sqrt{n+1}}\sum_{i=1}^{n+1}\hat{\vec{G}}(k_i)\psi_n(l,k_1,\ldots,\hat{k}_i,\ldots,k_{n+1})\overline{\chi})$

$$(22) = \mathcal{E}[\widetilde{\psi_n},\psi_{n+2}] + 2\alpha[\Re(D^*\psi_n\overline{\chi}, D\psi_{n+2}) + (D\psi_{n+2}, D\psi_{n+2})].$$

Using first Schwarz' inequality and afterwards (11) and (12) we derive

$$\begin{aligned}2\alpha[\Re(D^*\psi_n\overline{\chi}, D\psi_{n+2}) + (D\psi_{n+2}, D\psi_{n+2})] &\geq -\frac{\alpha}{2}(D^*\psi_n\overline{\chi}, D^*\psi_n\overline{\chi}) \geq \\ &\geq -\frac{\alpha}{2}[\Lambda^2\|\psi_n\overline{\chi}\|^2 + \Lambda(\psi_n\overline{\chi}, H_f\psi_n\overline{\chi})].\end{aligned} \tag{23}$$

Due to the definition of $\chi_\Lambda(k)$ and its complement we conclude

$$(\psi_n\overline{\chi}, H_f\psi_n\overline{\chi}) \equiv \sum_{i=1}^{n}\int(\psi_n(l,k)\overline{\chi}(k))^2|k_i| \geq \Lambda\|\psi_n\overline{\chi}\|^2$$

and consequently

$$\|\psi_n\overline{\chi}\|^2 \leq \frac{1}{\Lambda}(\psi_n\overline{\chi}, H_f\psi_n\overline{\chi}).$$

Thus, (23) is bounded below

$$(23) \geq -\alpha\Lambda(\psi_n\overline{\chi}, H_f\psi_n\overline{\chi}).$$

Next, we attack the term $\mathcal{E}[\widetilde{\psi_n},\psi_{n+2}]$ where we keep $\widetilde{\psi_n}$ fixed and minimize with respect to ψ_{n+2}. Standard computations yield the minimum

$$\bar{\psi}_{n+2}(l,k,p,q) = \frac{-\alpha}{\sqrt{(n+1)(n+2)}}\mathcal{S}[\hat{\vec{G}}(p)\cdot\hat{\vec{G}}(q)\widetilde{\psi_n}(l,k)].$$

Plugging this into (21) we arrive at

$$\mathcal{E}[\widetilde{\psi_n},\bar{\psi}_{n+2}] = \frac{-\alpha^2}{(n+1)(n+2)}\int\frac{\overline{\mathcal{S}[\hat{\vec{G}}(p)\cdot\hat{\vec{G}}(q)\widetilde{\psi_n}(l,k)]}\mathcal{S}[\hat{\vec{G}}(p)\cdot\hat{\vec{G}}(q)\widetilde{\psi_n}(l,k)]}{[l+p+q+\sum_{i=1}^{n}k_i]^2 + |p| + |q| + \sum_{i=1}^{n}|k_i|}. \tag{24}$$

By means of (20) and the symmetry of $\widetilde{\psi_n}$ we see that (24) consists of three different terms $I + II + III$.

First,

$$I=-2\alpha^2\int\frac{|\widetilde{\psi_n}(l,k)|^2[\hat{\vec{G}}(p)\cdot\hat{\vec{G}}(q)]^2}{[l+p+q+\sum_{i=1}^n k_i]^2+|p|+|q|+\sum_{i=1}^n|k_i|},$$

which is the main part and will be bounded independently of the photon energy of ψ_n.

Now, we keep l and k fixed. Obviously,

$$\int\frac{[\hat{\vec{G}}(p)\cdot\hat{\vec{G}}(q)]^2}{[l+p+q+\sum_{i=1}^n k_i]^2+|p|+|q|+\sum_{i=1}^n|k_i|}dpdq\le\int\frac{[\hat{\vec{G}}(p)\cdot\hat{\vec{G}}(q)]^2}{[l+p+q+\sum_{i=1}^n k_i]^2+|p|+|q|}dpdq.$$

By the symmetry of the functions $\hat{\vec{G}}(p)\cdot\hat{\vec{G}}(q)$, $[p+q]^2$ and $|p|+|q|$

$$f(l')=\int\frac{[\hat{\vec{G}}(p)\cdot\hat{\vec{G}}(q)]^2}{[l'+p+q]^2+|p|+|q|}dpdq\le\int\frac{[\hat{\vec{G}}(p)\cdot\hat{\vec{G}}(q)]^2}{[p+q]^2+|p|+|q|}dpdq, \tag{25}$$

obviously holds at least for small $|l'|$ which can be seen by differentiation with respect to l', since $f'(0)=0$ and $f(t\vec{j})$ is monotone decreasing as function of $t>0$ at least for small t, where $\vec{j}\in\mathbb{R}^3$ is kept fix. Of course there is strong evidence that $f(0)$ is not only a local but a global maximum, which is confirmed by numerical calculations. Equation (25) would immediately imply

$$I\ge-2\alpha^2\|\widetilde{\psi_n}\|^2\int\frac{[\hat{\vec{G}}(p)\cdot\hat{\vec{G}}(q)]^2}{[p+q]^2+|p|+|q|}dpdq. \tag{26}$$

Nevertheless, since we are not able to prove it analytically, we use the very rough estimate

$$|f'(l')|\le|l'|\int\frac{[\hat{\vec{G}}(p)\cdot\hat{\vec{G}}(q)]^2}{[|p|+|q|]^2}dpdq+\int\frac{[\hat{\vec{G}}(p)\cdot\hat{\vec{G}}(q)]^2}{[|p|+|q|]}dpdq\le|l'|\Lambda^2+\Lambda^3,$$

and decompose $\widetilde{\psi_n}$ into a part $\widetilde{\psi_n}^i$, where

$$\forall(l,k),\quad|l+\textstyle\sum_{i=1}^n k_i|\le(\alpha^2\Lambda^3)^{1/4},$$

and the outer part $\widetilde{\psi_n}^o$, where

$$\forall(l,k),\quad|l+\textstyle\sum_{i=1}^n k_i|\ge(\alpha^2\Lambda^3)^{1/4}.$$

By means of these definitions

$$\alpha^2\int\frac{|\widetilde{\psi_n}^i(l,k)|^2[\hat{\vec{G}}(p)\cdot\hat{\vec{G}}(q)]^2}{[l+p+q+\sum_{i=1}^n k_i]^2+|p|+|q|}\le$$

$$\le\alpha^2\|\widetilde{\psi_n}^i\|^2\int\frac{[\hat{\vec{G}}(p)\cdot\hat{\vec{G}}(q)]^2}{[p+q]^2+|p|+|q|}dpdq+\|\widetilde{\psi_n}^i\|^2O(\alpha^{9/4}\Lambda^{15/4}),$$

and

$$\alpha^2 \int \frac{|\widetilde{\psi_n}^o(l,k)|^2[\hat{\vec{G}}(p)\cdot\hat{\vec{G}}(q)]^2}{[l+p+q+\sum_{i=1}^n k_i]^2+|p|+|q|} \leq$$

$$\leq \alpha^2\|\widetilde{\psi_n}^o\|^2 \int \frac{[\hat{\vec{G}}(p)\cdot\hat{\vec{G}}(q)]^2}{|p|+|q|}dpdq \leq (\widetilde{\psi_n}^o, H_f\widetilde{\psi_n}^o)O(\alpha\Lambda^{3/2})$$

since

$$(\widetilde{\psi_n}^o, [l+\textstyle\sum_{i=1}^n k_i]^2\widetilde{\psi_n}^o) \geq (\alpha^2\Lambda^3)^{1/2}\|\widetilde{\psi_n}^o\|^2.$$

Consequently,

$$I \geq -2\alpha^2\|\widetilde{\psi_n}\|^2\int\frac{[\hat{\vec{G}}(p)\cdot\hat{\vec{G}}(q)]^2}{[p+q]^2+|p|+|q|}dpdq - \|\widetilde{\psi_n}\|^2O(\alpha^{9/4}\Lambda^{15/4}) - (\widetilde{\psi_n}, H_f\widetilde{\psi_n})O(\alpha\Lambda^{3/2}).$$

The terms II and III can be seen as perturbation terms, depending only on the photon energy, which is of order α^2 for Ψ close to Σ_0. That is why we will not care about constants and only estimate in terms of α and Λ.

Secondly, we have

$$II = -\alpha^2 n\int\frac{(\hat{\vec{G}}(p)\cdot\hat{\vec{G}}(q))(\hat{\vec{G}}(p)\cdot\hat{\vec{G}}(q'))\overline{\widetilde{\psi_n}(l,q,k)}\widetilde{\psi_n}(l,q',k)}{[l+p+q+q'+\sum_{i=2}^n k_i]^2+|p|+|q|+|q'|+\sum_{i=2}^n|k_i|}dldpdqdq'dk,$$

where $k=(k_2,...,k_n)$.

With

$$\rho_{\widetilde{\psi_n}}(q) = \int|\widetilde{\psi_n}(l,q,k)|^2 dldk,$$

we obtain

$$\begin{aligned} II &\geq -\alpha^2 n\int\frac{|\hat{\vec{G}}(p)|^2}{|p|}|\hat{\vec{G}}(q)|\sqrt{\rho_{\widetilde{\psi_n}}(q)\rho_{\widetilde{\psi_n}}(q')}|\hat{\vec{G}}(q')|dpdqdq' \\ &\geq -\alpha^2\Lambda n\left[\int\frac{|\hat{\vec{G}}(q)|}{|q|^{1/2}}\sqrt{\rho_{\widetilde{\psi_n}}(q)|q|}dq\right]^2 \\ &\geq -\alpha^2\Lambda n\int\rho_{\widetilde{\psi_n}}(q)|q|dq\int\frac{|\hat{\vec{G}}(q)|^2}{|q|}dq \geq -(\alpha\Lambda)^2(\widetilde{\psi_n}, H_f\widetilde{\psi_n}). \end{aligned}$$

Thereby we used

$$\int\hat{\vec{G}}(p)^2/|p|dp = 8\pi\Lambda$$

and Schwarz inequality.

Thirdly, there remains the part

$$III = -\alpha^2 n^2 \int \frac{\hat{\vec{G}}(p)\cdot\hat{\vec{G}}(q)\hat{\vec{G}}(p')\cdot\hat{\vec{G}}(q')\overline{\widetilde{\psi_n}(l,p,q,k)}\widetilde{\psi_n}(l,p',q',k)}{[l+p+p'+q+q'+\sum_{i=3}^n k_i]^2+|p|+|p'|+|q|+|q'|+\sum_{i=3}^n|k_i|} dldpdp'dqdq'dk$$

with $k=(k_3,...,k_n)$. We denote the two particle density as

$$\rho_{\widetilde{\psi_n}}(p,q) = \int |\widetilde{\psi_n}(l,p,q,k)|^2 dldk.$$

By applying Schwarz' inequality we get

$$\begin{aligned} III &\geq -\alpha^2 n^2 \int \frac{|\hat{\vec{G}}(p)||\hat{\vec{G}}(q)||\hat{\vec{G}}(p')||\hat{\vec{G}}(q')|}{|p|+|q|+|p'|+|q'|}\sqrt{\rho_{\widetilde{\psi_n}}(p,q)}\sqrt{\rho_{\widetilde{\psi_n}}(p',q')}dpdqdp'dq' \\ &\geq -\alpha^2 n^2 \int \rho_{\widetilde{\psi_n}}(p,q)|p||q|dpdq\left[\int \frac{|\hat{\vec{G}}(p)|^2|\hat{\vec{G}}(q)|^2|\hat{\vec{G}}(p')|^2|\hat{\vec{G}}(q')|^2}{|p||q||p'||q|(|p|+|q|+|p'|+|q'|)^2}dpdqdp'dq'\right]^{1/2}. \end{aligned}$$

Since the second term in the last line is proportional to Λ we arrive at

$$III \geq -\alpha^2\Lambda n^2 \int \rho_{\widetilde{\psi_n}}(p,q)|p||q|dpdq.$$

Now, we recall the definition of $\chi_\Lambda(k)$ and obtain

$$n^2\int \rho_{\widetilde{\psi_n}}(p,q)|p||q|dpdq = n\int |\widetilde{\psi_n}(p,k_2,...k_n)|^2|p|\sum_{i=2}^n|k_i|dpdk_2..dk_n \leq \Lambda(\widetilde{\psi_n},H_f\widetilde{\psi_n}).$$

Collecting above terms and summing over all n we get

$$\begin{aligned} \sum_{n=0}^\infty \mathcal{E}[\psi_n,\psi_{n+2}] + 2\alpha(\Psi,D^*D\Psi) &\geq -\alpha^2\sum_{n=0}^\infty \|\psi_n\|^2 2\int \frac{[\hat{\vec{G}}(p)\cdot\hat{\vec{G}}(q)]^2}{[p+q]^2+|p|+|q|}dpdq \\ &\quad -(\alpha\Lambda)^2\sum_{n=0}^\infty(\widetilde{\psi_n},H_f\widetilde{\psi_n}) - \alpha\Lambda\sum_{n=0}^\infty(\psi_n\overline{\chi},H_f\psi_n\overline{\chi}) \\ &\quad -\left\{\sum_{n=0}^\infty\|\psi_n\|^2\alpha^{9/4}\Lambda^{15/4} + \sum_{n=0}^\infty(\widetilde{\psi_n},H_f\widetilde{\psi_n})\alpha\Lambda^{3/2}\right\}. \end{aligned}$$

According to our choice of Ψ (see (19))

$$\begin{aligned} \sum_{n=0}^\infty \mathcal{E}[\psi_n,\psi_{n+2}] + 2\alpha(\Psi,D^*D\Psi) &\geq -\alpha^2\|\Psi\|^2 2\int \frac{[\hat{\vec{G}}(p)\cdot\hat{\vec{G}}(q)]^2}{[p+q]^2+|p|+|q|}dpdq \\ &\quad -(\alpha\Lambda)^4\Lambda - (\alpha\Lambda)^3\Lambda - \left[\|\Psi\|^2\alpha^{9/4}\Lambda^{15/4} + (\alpha\Lambda)^3\Lambda^{3/2}\right]. \end{aligned}$$

Altogether ($\|\Psi\| = 1$)

$$(\Psi, T\Psi) \geq \alpha\Lambda^2 - \alpha^2 2 \int \frac{[\hat{\vec{G}}(p) \cdot \hat{\vec{G}}(q)]^2}{[p+q]^2 + |p| + |q|} dpdq - O(\alpha^{5/2}\Lambda^{7/2}) - [O(\alpha^{9/4}\Lambda^{15/4})]. \qquad (27)$$

With the bracket [...] we indicate that (27) would hold without this term if we use inequality (25).

5 Proof of Theorem 4

Proof of Theorem 4. An upper bound to $E = \inf \operatorname{spec} H_Z$ can simply be derived by a trial function $\Psi \in \mathcal{H}$. In order to get a good upper bound we optimize the energy over functions $\Psi = \{\phi(x), \psi(x, x-\xi), 0, 0, ...\}$, where $\phi(x)$ is the ground state of $-\Delta - Z/|x|$, i.e.

$$(-\Delta - Z/|x|)\phi(x) = -e_o\phi(x).$$

It is well known that

$$\phi(x) = \frac{1}{\sqrt{\pi}} a^{3/2} e^{-|x|a},$$

with $a = e_0^{1/2}$.

Our choice of the comparison wave function contains our strong conjecture that only one photon is necessary to obtain the energy up to order α, for small values of α.

We compute

$$\begin{aligned}(\Psi, H_Z\Psi) &= (\phi, [-\Delta - \frac{Z}{|x|}]\phi) + \|(\nabla_x + \nabla_\eta)\psi(x,\eta)\|^2 + (\psi, H_f\psi) \\ &\quad + 4\sqrt{\alpha}\Re(p\phi, D\psi) + 4\pi\alpha\Lambda^2\|\Psi\|^2 + \alpha(\psi, D^*D\psi) - (\psi, \frac{Z}{|x|}\psi). \qquad (28)\end{aligned}$$

Next we omit $(\psi, \frac{Z}{|x|}\psi)$, which is obviously positive, and minimize the expression (28) over ψ. This leads to the requirement of minimizing

$$\mathcal{E}[\psi] = (\psi, [[l+p]^2 + |p|]\psi) + 4\sqrt{\alpha}\Re \int \overline{\hat{\phi}(l)} l \cdot \vec{G}(p)\psi(l,p) dldp \qquad (29)$$

where we denote $\hat{\phi}(l) = \mathcal{F}[\phi(x)]$ and $\mathcal{F}[\psi(x,\eta)] = \psi(l,k)$. We again work in momentum space. By analogical arguments as in the previous sections we can see that the minimizer of (29) is given by

$$\bar{\psi}(l,p) = \frac{2\sqrt{\alpha}\hat{\phi}(l)(l \cdot \hat{\vec{G}}(p))}{[l+p]^2 + |p|}$$

and the corresponding minimum reads

$$\mathcal{E}[\bar{\psi}] = -4\alpha \int \frac{\hat{\phi}(l)^2 (l \cdot \vec{G}(p))^2}{[l+p]^2 + |p|} dldp.$$

Therefore, with $\Psi = \{\phi, \bar{\psi}, 0, 0...\}$,

$$\begin{aligned} E \leq (\Psi, H_Z \Psi)/(\Psi, \Psi) &= 4\pi\alpha\Lambda^2 - e_0 + 4\alpha e_0 \int \frac{\hat{\phi}(l)^2 (l \cdot \hat{\vec{G}}(p))^2}{[[l+p]^2 + |p|]^2} dldp \\ &\quad -4\alpha \int \frac{\hat{\phi}(l)^2 (l \cdot \hat{\vec{G}}(p))^2}{[l+p]^2 + |p|} dldp + O(\alpha^2) \end{aligned} \tag{30}$$

where the first integral on the right-hand side comes from the normalization factor $e_0/(1 + \|\bar{\psi}\|^2)$. We explicitly compute

$$\hat{\phi}(l)^2 = \frac{8a^5}{\pi^2 (a^2 + l^2)^4} \equiv \hat{\phi}_a(l)^2.$$

Next, we introduce the scaling

$$l \to la, \;\; p \to pa.$$

Then the terms on the r.h.s. of (30) can be written as

$$\int \frac{\hat{\phi}_a(l)^2 (l \cdot \hat{\vec{G}}(p))^2}{[l+p]^2 + |p|} dldp = a^2 \int \frac{\hat{\phi}_1(l)^2 (l \cdot \hat{\vec{G}}(p,a))^2}{[l+p]^2 + |p|/a} dldp,$$

and

$$\int \frac{\hat{\phi}_a(l)^2 (l \cdot \hat{\vec{G}}(p))^2}{[[l+p]^2 + |p|]^2} dldp = \int \frac{\hat{\phi}_1(l)^2 (l \cdot \hat{\vec{G}}(p,a))^2}{[[l+p]^2 + |p|/a]^2} dldp,$$

with $\hat{\vec{G}}_\lambda(p,a) = \varepsilon_\lambda(p)|p|^{-1/2}\Theta(\Lambda/a - |p|)$.

Since $\Sigma_0 = 4\pi\alpha\Lambda^2 + O(\alpha^2)$ and $e_0^2 = a$

$$E - \Sigma_0 \leq -e_0 - 4\alpha e_0 \int \frac{\hat{\phi}_1(l)^2 (l \cdot \hat{\vec{G}}(p,a))^2}{[l+p]^2 + |p|/a} \left[1 - \frac{1}{[l+p]^2 + |p|/a}\right] dldp + O(\alpha^2).$$

Obviously the second term in the integral converges as Λ goes to infinity. Therefore we see that for large Λ

$$\mathcal{E}[\Lambda, a] \sim 2 \cdot 4\pi \|\hat{\phi}_1 l\|^2 \int_0^{\Lambda/a} \frac{p}{p^2 + p/a} dp = 8\pi \ln(1 + \Lambda),$$

where the factor 2 stems from the two photon polarizations. $\square$

A Appendix

THEOREM 7. *Assume (4) to be valid. Then for $\alpha\Lambda$ small*

$$\Sigma_0 = 4\pi\alpha\Lambda^2 - \alpha^2\mathcal{E}_0 + O(\alpha^{5/2}\Lambda^{5/2}). \tag{31}$$

Proof. It suffices to improve our a-priori estimate on Σ_0.

LEMMA 8. *Let Ψ be such that $(\Psi, T\Psi) \leq 4\pi\alpha\Lambda^2$. Then for some constant C*

$$\|p\Psi\|^2 + (\Psi, H_f\Psi) \leq C\alpha^2\Lambda^2.$$

Proof. Using the methods of the previous sections we can estimate

$$(\Psi, T\Psi) \geq (\tfrac{1}{2} - 4\sqrt{8\pi\alpha\Lambda})[\|p\Psi\|^2 + (\Psi, H_f\Psi)] + \alpha\|A\Psi\|^2 + \tfrac{1}{2}[\|p\Psi\|^2 + (\Psi, H_f\Psi)].$$

Mimicking the proof of Theorem 2 and thereby using (4) we can bound from below

$$\begin{aligned} &(\tfrac{1}{2} - 4\sqrt{8\pi\alpha\Lambda})[\|p\Psi\|^2 + (\Psi, H_f\Psi)] + \alpha\|A\Psi\|^2 \geq \\ \geq \quad & 4\pi\alpha\Lambda^2 - \alpha^2(\tfrac{1}{2} - 4\sqrt{8\pi\alpha\Lambda})^{-1}\mathcal{E}_0 - (\alpha\Lambda)(\tfrac{1}{2} - 4\sqrt{8\pi\alpha\Lambda})^{-1}(\Psi, H_f\Psi), \end{aligned}$$

which implies the Lemma using the fact that $\mathcal{E}_0 \sim \Lambda^2$. □

Repeating the proof of Theorem 2 yields (31). □

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