C*–ALGEBRAS AND TOPOLOGICAL DYNAMICS IN SPECTRAL ANALYSIS

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Abstract. We present some developments and applications of a recent algebraic approach to certain problems in spectral theory. Our principal topic is to determine in a natural way subsets of the essential spectrum of certain self-adjoint operators, by extending [22]. Following [3], we also point out localization results for energies situated in the complements of these subsets.

1. Introduction

In the last years, a new approach to some important problems in the spectral analysis of self-adjoint operators emerged. It is based on a formalism relying on C*-algebras techniques and it was developed by Vladimir Georgescu and his collaborators. Meant first to give a unified and conceptually clear framework for the spectral analysis of the N-body problem, it came out to be well-suited to some other systems with rich internal structure, such that anisotropic Schrödinger operators and field theoretical models. As relevant references I quote [1], [2], [4], [5], [6], [7], [8], [9], [10], [14], [15], [16] and [18]. The theory is certainly still at an incipient stage, but I do not feel insecure to assert that new ranges of application will be found and that for many interesting problems it will give definitive answers.

The present text is an extended version of the talk delivered by the author at the OAMP Conference in Constanta, July, 2001. It contains a part of the "standard material", picked up from the references above, as well as a personal point of view on this subject, by improving upon [3], [21] and [22]. In particular, I shall emphasise the role played by compactifications. I thank Werner Amrein and Radu Purice for delightful collaboration and Vladimir Georgescu for very useful discussions, for communicating me some of his works before publication and for suggesting the use of aproximate units.

An extensive presentation of the formalism and of the applications is outside my aim. Good expositions may be found in [2] and [15] and we hope that there will also be some contributions on this in the present volume. We point out only that the main idea consists in ascribing to well-specified classes of self-adjoint operators C*-algebras which reflect well their common properties. Some features of these

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operators are not sensitive to certain ideals of the C*-algebras; for example, if the difference of the resolvents of two operators belongs to the ideal of compact operators, their essential spectrum is the same. Thus, a study of the quotient of the C*-algebras by these ideals will give information on the operators.

The algebraic approach has for the moment three ranges of application: (a) the essential spectrum, (b) the Mourre estimate and (c) localization properties. The last one is discovered and developed in [3]. I would like to touch on (a) and (c). There is no place here for detailed proofs, for optimal results or for exploring many examples, so that the text will be rather condensed. But most of the justifications are standard knowledge or simple extensions of arguments comprised in the references cited above.

In the remaining part of the Introduction we present informally the two problems we want to solve and sketch solutions.

Let $\mathfrak{H}$ be a family of self-adjoint operators in the Hilbert space $\mathcal{H}$, defined by some remarkable property $\mathfrak{RP}$.

Problem 1. Decide for each element H of $\mathfrak{H}$ that a certain set $\Sigma(H, \mathfrak{RP})$ is included into the essential spectrum of H.

Problem 2. Associate to $\mathfrak{RP}$ a family $\boldsymbol{W}$ of bounded operators of norm one such that if η is a real continuous function with compact support disjoint from $\Sigma(H, \mathfrak{RP})$ and if $\varepsilon > 0$, then there is an element W of the family such that $\parallel W\eta(H) \parallel \leq \varepsilon$.

This is very vague. We shall give an example which is extremely simple and intuitive, hence available to a very simple treatment, but which shows already the main features of the topic. In $\mathcal{H} = L^2(\mathbb{R})$ we consider the Schrödinger operator $H = H_0 + V$, where H_0 is (say) the positive Laplace operator and V is the operator of multiplication with a bounded, uniformly continuous function having a limit at plus infinity: $V(x) \to c$ when $x \to +\infty$. Then it is simple to prove that $[c, \infty)$ is included in the essential spectrum of H. The behaviour of V "to the left" (which is not submitted to constraints) may introduce spectrum and even essential spectrum below c. But let $\eta : \mathbb{R} \to \mathbb{R}$ be continuous, with a compact support situated below the value c and $\boldsymbol{W}$ the family of all operators χ_a of multiplication with characteristic functions of intervals of the form $[a, \infty)$. Then one can show that for any $\varepsilon > 0$ there exists $a \in \mathbb{R}$ (positive and sufficiently large) such that $\parallel \chi_a \eta(H) \parallel \leq \varepsilon$, although $\parallel \chi_a \parallel = 1$ and $\parallel \eta(H) \parallel = \sup_{\lambda \in \sigma(H)} \mid \eta(\lambda) \mid$. There is a simple, interesting consequence in terms of (non-)propagation: we shall have $\parallel \chi_a e^{-itH} \eta(H) f \parallel \leq \varepsilon \parallel f \parallel$, uniformly in $t \in \mathbb{R}$ and $f \in \mathcal{H}$. So, at energies bellow c, even when propagation towards infinity is possible, it does not take place to the right.

Since most of the cases one would like to treat are much more intricate than the one above (see also [11]), it is nice to have a unified strategy. We shall try to describe it for the case in which the operators H act in $\mathcal{H} = L^2(X)$ and are of the form $H_0 + V$. Here $X = \mathbb{R}^n$, $H_0 = h(P)$ with $P = -i\nabla$ and $h : X \to \mathbb{R}$ continuous and diverging at infinity and $V : X \to \mathbb{R}$ is a continuous function having a certain type of anisotropic behaviour at infinity. Various extensions are possible.

An informal answer to Problems 1 and 2 in the setting above is as follows:

(A) To V we associate a C*-algebra $\mathcal{C}$ of bounded, uniformly continuous functions on X having "the same type of anisotropy as V". In fact, the results will be valid for any potential belonging to this C*-algebra.

(B) Let us assume that $\mathcal{C}$ is invariant under the translations by elements of X and that it contains the constant functions and the continuous functions that tend to zero at infinity. Then the spectrum (character space) $X_{\mathcal{C}}$ of $\mathcal{C}$ will be a compact space containing X densely (a *compactification* of X) and X will act as a group of homeomorphisms of $X_{\mathcal{C}}$. X will be left invariant and the relevant information on the essential spectrum of H comes from the "boundary at infinity" $Z_{\mathcal{C}} = X_{\mathcal{C}} \setminus X$, which is the space of a topological dynamical system with group X.

(C) Fix a closed, invariant subset $F \subset Z_{\mathcal{C}}$. Denote by $\boldsymbol{W}$ the family of sets of the form $W = \mathcal{W} \cap X$, where $\mathcal{W}$ is any element of a base of neighbourhoods of F in $X_{\mathcal{C}}$. On the other hand, V admits a continuous extension to $X_{\mathcal{C}}$; we denote the restriction to F of this extension by V^F.

(D) There is a way to construct from h and V^F an "observable" (a kind of self-adjoint operator) H^F. Let us denote its spectrum by Σ^F. Then $\Sigma^F \subset \boldsymbol{\sigma}_{\text{ess}}(H)$ and if $\text{supp}(\eta) \cap \Sigma^F = \emptyset$, then for any $\varepsilon > 0$ we have $\| \chi_W \eta(H) \| \leq \varepsilon$ for some $W \in \boldsymbol{W}$. We set χ_W for the operator of multiplication by the characteristic function of W.

This is a description of the solution we envisaged, but there is definitely some hidden machinery. We hope to be able to clarify the picture above in the main body of the text. Let us just make some extra remarks which might help the reader to meet the subsequent material with some sympathy.

(i) We emphasized the operations to perform at the level of the potentials, since at this level the interesting facts happen. The "kinetic energy" is taken into account in the following way: We construct the C*-algebra $\mathfrak{C}_{\mathcal{C}}$ generated by $\mathcal{C}$ and the C*-algebra $\mathfrak{C}_{\mathbb{C}}$ spanned by the resolvents of all operators of the form $h(P)$, with h as above. The salient feature of this C*-algebra is to contain the functional calculus of H: $\eta(H) \in \mathfrak{C}_{\mathcal{C}}$ for any suitable η. We say that H is *affiliated* to $\mathfrak{C}_{\mathcal{C}}$.

(ii) To F we associate an ideal $\mathcal{C}^F$ consisting in all the functions of $\mathcal{C}$ that vanish near F. By mixing it with the C*-algebra of the kinetic energies one gets an ideal $\mathfrak{C}_{\mathcal{C}^F}$ of $\mathfrak{C}_{\mathcal{C}}$. The observable H^F mentioned in (D) is the canonical image of the initial operator H in the quotient $\mathfrak{C}_{\mathcal{C}}/\mathfrak{C}_{\mathcal{C}^F}$, which is generated (in a subtile sense) by $\mathfrak{C}_{\mathbb{C}}$ and the C*-algebra $C(F)$ of continuous functions on F, to which V^F belongs.

(iii) Very often the observable H^F is still obscure, because F is too big. Then we look for *invariant decompositions* of F, i.e. for families $\{F_j\}_{j\in J}$ of closed, invariant sets such that $F = \cup_{j\in J} F_j$. The best is to understand the quasi-orbit structure of the action of X on F; this will lead to the most refined invariant decomposition. Then each component F_j will give an "asymptotic Hamiltonian" H^{F_j} and we shall have $\Sigma^F \equiv \boldsymbol{\sigma}(H^F) = \overline{\cup_{j\in J}\boldsymbol{\sigma}(H^{F_j})} \subset \boldsymbol{\sigma}_{\text{ess}}(H)$. The resamblence with the HVZ formula is not fortuitous. As a rule, the observables H^{F_j} will be easy to interpret as quite familiar self-adjoint operators of the form $H_j = h(P) + V_j$.

(iv) If $F = Z_{\mathcal{C}}$ we have $\mathcal{C}^F = C_0(X)$ (the ideal of all complex continuous functions on X that tend to zero at infinity), $\mathfrak{C}_{\mathcal{C}^F} = K(\mathcal{H})$ (the ideal of all compact operators in $\mathcal{H}$) and $\Sigma^F = \boldsymbol{\sigma}_{\text{ess}}(H)$. Thus, to express conveniently the essential spectrum of H one has to invariantly decompose the boundary $Z_{\mathcal{C}}$.

Sections 2 and 3 are devoted to a more precise and detailed presentation of the formalism. We indicate all the relevant constructions, but the proofs are only described, and not done. In Section 4 we study a class of examples, by improving some results of [3] and [22]. Particular potentials that we cover here are products

UU' or sums $U + U'$, where U has asymptotically vanishing oscillation and U' is minimal (this is a much more general concept that almost periodicity).

2. C*-algebras and spectral theory

Most self-adjoint operators of interest in mathematical physics are unbounded. To link them to a C*-algebra we make use of their functional calculus, which contains relevant information about the operators themselves. In particular, the spectrum and several of its subsets can be expressed by means of this functional calculus. This is fortunate, since one can transport it by an injective morphism and calculate the spectrum in some other C*-algebra which is easier to understand or presenting richer structure. This morphism may present no Hilbert space implementation and the second C*-algebra may be not naturally represented; this is the source of a lot of malleability. The fact that the spectral analysis of systems with a rich interior structure has had such a difficult start is due mainly to the reluctance people had to abandon a purely Hilbertian formalism.

In this text we shall call briefly *ideal* in a C*-algebra a closed, *-invariant, two-sided ideal. The term *morphism* will denote a morphism of C*-algebras (involution is preserved), unital when this makes sense. We set $C_0(\mathbb{R}) := \{\eta : \mathbb{R} \to \mathbb{C} \mid \eta \text{ continuous}, \eta(\lambda) \to 0 \text{ if } |\lambda| \to \infty\}$. If $\mathcal{H}$ is a Hilbert space, $B(\mathcal{H})$ will be the C*-algebra of all bounded, linear operators and $K(\mathcal{H})$ the ideal of all compact operators.

[2.1] Observables affiliated to C*-algebras.

An observable affiliated to the C-algebra* $\mathfrak{C}$ is a morphism $H : C_0(\mathbb{R}) \to \mathfrak{C}$, i.e. a functional calculus. Instead of $H(\eta)$ we shall always write $\eta(H)$. Assume that we know that H is an observable affiliated to a larger C*-algebra $\mathfrak{D}$ and that we checked that $\eta_z(H) \in \mathfrak{C}$ with $\eta_z(\lambda) = (\lambda - z)^{-1}$, for some number $z \notin \mathbb{R}$. Then H is in fact affiliated to $\mathfrak{C}$. Observables affiliated to $B(\mathcal{H})$ are generalized self-adjoint operators, i.e. self-adjoint operators with a domain which is dense in some closed subspace $\mathcal{K}$ of $\mathcal{H}$. In the present article only the case $\mathcal{K} = \mathcal{H}$ will occur. Since any C*-algebra can be represented faithfully in a Hilbert space, any observable can be realized as a generalized self-adjoint operator, but this will be useful only if the representation is natural in some convenient sense.

The following perturbative situation will often be met in the sequel: Let $\mathfrak{A}$ and $\mathfrak{B}$ be two C*-subalgebras of $B(\mathcal{H})$, H_0 a (usual) self-adjoint operator affiliated to $\mathfrak{A}$ and V an element of $\mathfrak{B}$. We consider the C*-algebra $\mathfrak{C}$ generated by $\mathfrak{A} \cup \mathfrak{B}$, the smallest one containing both $\mathfrak{A}$ and $\mathfrak{B}$. Then by a simple application of the Neumann series, it is seen that the resolvent $(H - z)^{-1}$ of $H = H_0 + V$ belongs to $\mathfrak{C}$ for $|\Im z|$ large enough, i.e. H is affiliated to $\mathfrak{C}$.

One may transport observables by morphisms: Let $\pi : \mathfrak{C} \to \mathfrak{D}$ a morphism and H an observable affiliated to $\mathfrak{C}$. Then $\pi[H] : C_0(\mathbb{R}) \to \mathfrak{D}$ given by $\eta(\pi[H]) := \pi[\eta(H)]$ is an observable affiliated to $\mathfrak{D}$, called *the image of H through π*. The most important case is obtained when $\mathfrak{D}$ is the quotient $\mathfrak{C}/\mathfrak{K}$ of $\mathfrak{C}$ by an ideal $\mathfrak{K}$ and π is the canonical morphism. The flexibility of the formalism is already apparent: even if $\mathfrak{C}$ is composed of bounded operators in a Hilbert space, the quotient $\mathfrak{C}/\mathfrak{K}$ has an abstract nature. This does not prevent it to be a central object in spectral analysis for good choices of $\mathfrak{K}$.

[2.2] The $\mathfrak{K}$-essential spectrum.

Let H be an observable affiliated to the C*-algebra $\mathfrak{C}$ and $\mathfrak{K}$ an ideal in $\mathfrak{C}$. The $\mathfrak{K}$-*essential spectrum* of H is

$$\boldsymbol{\sigma}_{\mathfrak{K}}^{\mathfrak{C}}(H) := \{\lambda \in \mathbb{R} \mid \text{ if } \eta \in C_0(\mathbb{R}) \text{ and } \eta(\lambda) \neq 0, \text{ then } \eta(H) \notin \mathfrak{K}\}.$$

When $\mathfrak{K} = \{0\}$, the lower index will no longer be written. One sees immediately that $\boldsymbol{\sigma}_{\mathfrak{K}}^{\mathfrak{C}}(H) = \boldsymbol{\sigma}^{\mathfrak{C}/\mathfrak{K}}(\pi[H])$, with $\pi : \mathfrak{C} \to \mathfrak{C}/\mathfrak{K}$ the canonical morphism. The notion is inspired by the particular cases in which $\mathfrak{C}$ is a C*-subalgebra of $B(\mathcal{H})$ and $\mathfrak{K}$ is either $\{0\}$ or $K(\mathcal{H})$: by standard spectral theory $\boldsymbol{\sigma}^{\mathfrak{C}}(H)$ is the usual spectrum of the self-adjoint operator H, while $\boldsymbol{\sigma}_{\mathfrak{C}\cap K(\mathcal{H})}^{\mathfrak{C}}(H) = \boldsymbol{\sigma}_{\text{ess}}(H)$.

It follows easily that morphisms do not increase the spectrum: if $\pi : \mathfrak{C} \to \mathfrak{D}$ then

$$\boldsymbol{\sigma}^{\mathfrak{D}}(\pi[H]) \subset \boldsymbol{\sigma}^{\mathfrak{C}}(H)$$

and one has equality if π is a monomorphism (an injective morphism).

Now let $\mathfrak{K}_1$ and $\mathfrak{K}_2$ be two ideals in $\mathfrak{C}$ such that $\mathfrak{K}_1 \subset \mathfrak{K}_2$. Since there is a (surjective) morphism : $\mathfrak{C}/\mathfrak{K}_1 \to \mathfrak{C}/\mathfrak{K}_2$, we have $\boldsymbol{\sigma}_{\mathfrak{K}_2}^{\mathfrak{C}}(H) \subset \boldsymbol{\sigma}_{\mathfrak{K}_1}^{\mathfrak{C}}(H)$. In particular, if $\mathfrak{C} \subset B(\mathcal{H})$ and $\mathfrak{K}_1 = \mathfrak{C} \cap K(\mathcal{H})$, then $\boldsymbol{\sigma}_{\mathfrak{K}_2}^{\mathfrak{C}}(H) \subset \boldsymbol{\sigma}_{\text{ess}}(H)$ and we have already got a preliminary answer to the Problem 1 from the Introduction. One has to affiliate the operator H to some C*-algebra $\mathfrak{C}$ and find in $\mathfrak{C}$ an ideal that contains $\mathfrak{C} \cap K(\mathcal{H})$. Then perform the quotient $\mathfrak{C}/\mathfrak{K}$, determine the image of H in this quotient and calculate the spectrum of this image; it will be a subset of the essential spectrum. There are two reasons to consider also ideals larger than $\mathfrak{C} \cap K(\mathcal{H})$. First, they are directly useful for solving Problem 2. Second, sometimes $\mathfrak{C}/\mathfrak{C} \cap K(\mathcal{H})$ is extremely complicated but $\mathfrak{C}/\mathfrak{K}$ is simple or has some remarkable structure.

[2.3] Approximate units.

What about Problem 2 in a general setting? Let H be a self-adjoint operator in the Hilbert space $\mathcal{H}$, affiliated to the C*-subalgebra $\mathfrak{C}$ of $B(\mathcal{H})$. Let $\mathfrak{K} \subset \mathfrak{C}$ be an ideal of $B(\mathcal{H})$. It is quite easy to see that $\eta(H) \in \mathfrak{K}$ if $\eta \in C_0(\mathbb{R})$ and $\text{supp}(\eta) \cap \boldsymbol{\sigma}_{\mathfrak{K}}^{\mathfrak{C}}(H) = \emptyset$. Choose an approximate unit $(\theta_W)_{W \in \boldsymbol{W}}$ for $\mathfrak{K}$, composed of elements of $B(\mathcal{H})$. Then, by setting $\chi_W = 1 - \theta_W$, one has $\| \chi_W S \| \to 0$ for all $S \in \mathfrak{K}$. By aplying this to $S = \eta(H)$, one gets a solution for Problem 2.

[2.4]. Sufficient families of ideals.

The solutions are not yet satisfactory and this must be considered only a first step. We show now how to further reduce the observable that is affiliated to the quotient and to express its spectrum in terms of presumably simpler observables. When calculating the quotient of a C*-algebra $\mathfrak{C}$ by an ideal $\mathfrak{K}$, it may be useful to embed $\mathfrak{K}$ into a family of larger ideals $(\mathfrak{K}_j)_{j \in J}$. We would like to choose this family in an economical way and the C*-algebras $\mathfrak{C}/\mathfrak{K}_j$ should be "small and comprehensible".

Let $\mathfrak{C}$ be a C*-algebra and $\mathfrak{K}$ an ideal of $\mathfrak{C}$. A family of ideals $(\mathfrak{K}_j)_{j \in J}$ will be called $\mathfrak{K}$-*sufficient* if (i) $\cap_{j \in J} \mathfrak{K}_j = \mathfrak{K}$ and (ii) $\mathfrak{K}_j$ is not contained in $\mathfrak{K}_k$ if $j \neq k$. When $\mathfrak{K} = \{0\}$, we say simply *sufficient* instead of $\{0\}$-sufficient.

In fact we can pass very simply from a general $\mathfrak{K}$ to $\mathfrak{K} = \{0\}$. Let $(\mathfrak{K}_j)_{j \in J}$ a $\mathfrak{K}$-sufficient family of ideals in $\mathfrak{C}$. Then $\mathfrak{J}_j := \mathfrak{K}_j/\mathfrak{K}$ may be identified with an ideal of $\mathfrak{C}/\mathfrak{K}$ and $(\mathfrak{J}_j)_{j \in J}$ is a sufficient family of ideals in $\mathfrak{C}/\mathfrak{K}$.

The basic technical result is the existence of a canonical monomorphism from the quotient into a direct product C*-algebra

$$\mathfrak{C}/\mathfrak{K} \hookrightarrow \prod_{j\in J} \mathfrak{C}/\mathfrak{K}_j$$

as soon as $(\mathfrak{K}_j)_{j\in J}$ is a $\mathfrak{K}$-sufficient family of ideals of $\mathfrak{C}$. This follows from the fact that the kernel of π_j is $\mathfrak{K}_j$. Thus the kernel of $(\pi_j)_{j\in J} : \mathfrak{C} \to \prod_{j\in J} \mathfrak{C}/\mathfrak{K}_j$ is $\mathfrak{K}$.

By using this, the invariance of the spectrum under monomorphisms and the fact that the spectrum of an observable affiliated to a direct product is the closure of the union of the spectra of its components, we get the easy but fundamental

Proposition. *Let $(\mathfrak{K}_j)_{j\in J}$ a $\mathfrak{K}$-sufficient family of ideals of the C*-algebra $\mathfrak{C}$. Let H be an observable affiliated to $\mathfrak{C}$. For any $j \in J$, let us denote by $H_j \equiv \pi_j[H]$ the image of H by the canonical surjection $\pi_j : \mathfrak{C} \to \mathfrak{C}/\mathfrak{K}_j$. Then*

$$\boldsymbol{\sigma}^{\mathfrak{C}}_{\mathfrak{K}}(H) = \overline{\bigcup_{j\in J} \boldsymbol{\sigma}^{\mathfrak{C}}_{\mathfrak{K}_j}(H)} = \overline{\bigcup_{j\in J} \boldsymbol{\sigma}^{\mathfrak{C}/\mathfrak{K}_j}(H_j)}.$$

In the sequel I shall call H_j (or some isomorphic image of it) *an asymptotic observable for H.*

[2.5] Is there really "an abstract theory"?.

Let us first sum up what we have obtained:

Theorem 1. *We place ourselves in the framework of the Proposition, but assume that $\mathfrak{C}$ is represented in the Hilbert space $\mathcal{H}$ and that $K(\mathcal{H}) \subset \mathfrak{K}$. Let $\{\theta_W\}_{W\in\boldsymbol{W}} \subset B(\mathcal{H})$ an approximate unit for $\mathfrak{K}$ and set $\chi_W := 1 - \theta_W$.*

(a) One has $\overline{\bigcup_{j\in J} \boldsymbol{\sigma}^{\mathfrak{C}/\mathfrak{K}_j}(H_j)} \subset \boldsymbol{\sigma}_{\rm ess}(H)$; if $\mathfrak{K} = K(\mathcal{H})$, equality holds.

(b) If $\eta \in C_0(\mathbb{R})$ and supp$(\eta) \cap \overline{\bigcup_{j\in J} \boldsymbol{\sigma}^{\mathfrak{C}/\mathfrak{K}_j}(H_j)} = \emptyset$, then for any $\varepsilon > 0$ there exists some $W \in \boldsymbol{W}$ such that $\| \chi_W \eta(H) \| \le \varepsilon$. In particular, one has $\| \chi_W e^{-itH}\eta(H)f \| \le \varepsilon \| f \|$, uniformly in $t \in \mathbb{R}$ and $f \in \mathcal{H}$.

It is easy to see that, once the right definitions are given, some simple standard facts in C*-algebra theory give the result. I prefer to say that this simplicity is a good thing. The development above is a systematic well-suited language and now everyone knows what to look for. To cover interesting examples there is still much effort to invest. In particular, the asymptotic observables H_j should be comprehensible. I shall try to show in the next Section that for a rather large class of situations this effort still has a unified and general setting and that the whole approach deserves to be called *a theory.*

3. Topological dynamics and spectral theory

In this Section we apply the formalism developped in Section 2 to self-adjoint operators (Hamiltonians) of the form $H = H_0 + V$ acting in the Hilbert space $\mathcal{H} = L^2(X)$, for X an abelian, locally compact group, with $H_0 = h(P)$, h and V being suitable functions. These Hamiltonians are affiliated to C*-algebras $\mathfrak{C}_{\mathcal{C}} := \langle \mathcal{C} \cdot \mathfrak{C}_{\mathbb{C}} \rangle$ generated by an algebra $\mathcal{C}$ of multiplication operators and an algebra $\mathfrak{C}_{\mathbb{C}}$

of convolution operators. The choice of $\mathcal{C}$ will reflect the anisotropy of the class of potentials we consider. In [**3.1**] we shall study the quotient $\mathcal{C}/\mathcal{C}^F$ of $\mathcal{C}$ by the ideal $\mathcal{C}^F$ associated to a closed, invariant subset F of the character space of $\mathcal{C}$. In [**3.2**], by using [**3.1**] and the concept of crossed product, we express the quotient $\mathfrak{C}_{\mathcal{C}}/\mathfrak{C}_{\mathcal{C}^F}$ in a manner that permits the application of [**2.2**] and [**2.3**]. The paragraph [**3.3**] is a concrete version of [**2.4**]; we obtain sufficient families of ideals from invariant decompositions of the set F. In [**3.4**] we formulate an explicit result on the essential spectrum and localization properties of H.

If Y is a locally compact space, we set $BC(Y)$ for the C*-algebra of all bounded, continuous functions : $Y \to \mathbb{C}$ and $BC_u(Y)$ for the C*-subalgebra consisting of uniformly continuous elements. $C_0(Y)$ is the ideal of all functions in $BC(Y)$ that tend to zero at infinity, i.e. that are arbitrarily small outside a sufficiently large compact subset of Y. If the space Y happens to be compact, all the classes above coincide with $C(Y)$, the C*-algebra of all complex continuous functions on Y.

We denote by X an abelian, locally compact group and by $X^\sharp$ its dual, which is also a locally compact group in a standard way. We shall always assume that X is not compact; this is the case of interest and without this hypothesis the discussion below becomes slightly degenerate. The group X will play the role of configuration space. The standard cases are $X = \mathbb{R}^n$ and $X = \mathbb{Z}^n$, but the fact that such a general setting is permitted pleades in favour of the method.

[3.1] X-algebras.

For any C*-subalgebra $\mathcal{C}$ of $BC(X)$, we denote by $X_{\mathcal{C}}$ *its character space* (or *spectrum*), consisting of all non-null morphisms from $\mathcal{C}$ to $\mathbb{C}$. Endowed with the simple convergence topology, $X_{\mathcal{C}}$ becomes a locally compact space, which is compact if and only if $\mathcal{C}$ is unital. By Gelfand theory, $\mathcal{C}$ is canonically isomorphic to $C_0(X_{\mathcal{C}})$; to $\varphi \in \mathcal{C}$ we assign the function $\varphi_{\mathcal{C}} : X_{\mathcal{C}} \to \mathbb{C}$, given by $\varphi_{\mathcal{C}}(\nu) = \nu(\varphi)$ for all character ν of $\mathcal{C}$. If $\mathcal{C}$ is unital and contains $C_0(X)$, the points $x \in X$ correspond injectively to characters $\nu_x(\varphi) = \varphi(x)$ and $X_{\mathcal{C}}$ is *a compactification of* X, i.e. a compact space containing a dense subset homeomorphic to X. Frequently X will be identified to its image in $X_{\mathcal{C}}$; this image is open.

All facts above are true for all locally compact spaces X; we shall exploit now the fact that X is also an abelian group. Let us denote by α the action of X on itself $\alpha_x(y) = x+y$, $x, y \in X$ as well as its natural action on functions $[\alpha_x(\varphi)](y) = \varphi[\alpha_{-x}(y)] = \varphi(y-x)$, $x, y \in X$, $\varphi \in BC(X)$. By restricting to $BC_u(X)$, this action becomes continuous. We call X-*algebra* an unital C*-subalgebra $\mathcal{C}$ of $BC_u(X)$ which is X-invariant: $\varphi \in \mathcal{C}$ and $x \in X$ imply $\alpha_x(\varphi) \in \mathcal{C}$. Then the character space $X_{\mathcal{C}}$ becomes the space of a (topological) dynamical system $(X_{\mathcal{C}}, \alpha_{\mathcal{C}}, X)$. It is always true that the ideals of $\mathcal{C}$ are in a one-to-one correspondence with the closed subsets of the compact space $X_{\mathcal{C}}$. In fact, in the isomorphic image $C(X_{\mathcal{C}})$, the single ideals are of the form $C^F(X_{\mathcal{C}}) = \{\varphi_{\mathcal{C}} \in C(X_{\mathcal{C}}) \mid \varphi_{\mathcal{C}}|_F = 0\}$ for some closed subset F of $X_{\mathcal{C}}$; we denote by $\mathcal{C}^F$ the inverse image of $C^F(X_{\mathcal{C}})$ by the Gelfand isomorphism. It follows easily that the ideal $\mathcal{C}^F$ is X-invariant if and only if F is X-invariant. We shall be mostly interested in the case in which $\mathcal{C}$ contains $C_0(X)$; then $\mathcal{C}$ will be called *an X-algebra of the first kind.* In this case, the action $\alpha_{\mathcal{C}}$ is an extension of α. By assuming that the closed set F is a subset of *the corona space* $Z_{\mathcal{C}} := X_{\mathcal{C}} \setminus X$ we get the inclusion $C_0(X) \subset \mathcal{C}^F$.

The quotient $\mathcal{C}/\mathcal{C}^F$ is naturally isomorphic to $C(X_{\mathcal{C}})/C^F(X_{\mathcal{C}})$, that is in turn isomorphic to $C(F)$. If $\mathcal{C}$ is of the first kind, the composed isomorphism can be described as follows: Let $\varphi \in \mathcal{C}$, let $\varphi_{\mathcal{C}}$ be the unique continuous extension to the compactification $X_{\mathcal{C}}$ of X and let $\varphi^F := \varphi_{\mathcal{C}}|_F$. Then the mapping $\varphi \mapsto \varphi^F$ is an epimorphism (surjective morphism) from $\mathcal{C}$ to $C(F)$ and its kernel is $\mathcal{C}^F$. If one interprets φ as "the potential", φ^F may be called *the asymptotic potential corresponding to* F. The abstract nature of F has been for quite long an obstacle for understanding anisotropic operators.

[3.2] Crossed products.

In the preceding paragraph we settled all the preparations we need at an abelian level. We are now in a position to treat the problem of the affiliation of the self-adjoint operator $H = H_0 + V = h(P) + V$, with $h : X^\sharp \to \mathbb{R}$ a continuous function diverging at infinity ($|h(p)| \to \infty$ when $x \to \infty$) and V is an element of an X-algebra $\mathcal{C}$. In fact, we choose $\mathcal{C}$ as to capture the anisotropy properties of V. It is convenient to assume that $\mathcal{C}$ is an X-algebra of the first kind, but we shall be sloppy on this point; we admit occasionally $\mathcal{C} = \mathbb{C}$ or $\mathcal{C} = C_0(X)$.

Let us denote by $\mathfrak{C}_{\mathbb{C}}$ the C*-algebra of all operators of the form $\psi(P)$, $\psi \in C_0(X^\sharp)$, where the convolution operator $\psi(P)$ is defined to be unitarily equivalent to the operator of multiplication by ψ in $L^2(X^\sharp)$, via the Fourier transformation. It is obvious that $H_0 = h(P)$ should be affiliated to $\mathfrak{C}_{\mathbb{C}}$, that is a C*-subalgebra of $B(\mathcal{H})$, $\mathcal{H} = L^2(X)$.

For some X-algebra $\mathcal{C}$, we also consider in $B(\mathcal{H})$ the C*-subalgebra $\mathfrak{C}_{\mathcal{C}} := \langle \mathcal{C} \cdot \mathfrak{C}_{\mathbb{C}} \rangle$ which is the norm closure of the set of finite sums of the form $\varphi_1\psi_1(P) + \cdots + \varphi_N\psi_N(P)$, with $\varphi_k \in \mathcal{C}$ (interpreted as an operator of multiplication in $\mathcal{H}$) and $\psi_k \in C_0(X)$. A simple application of the Neumann series and a commutation argument show that H is affiliated to $\mathfrak{C}_{\mathcal{C}}$.

We consider now an X-invariant, closed subset F of $X_{\mathcal{C}}$ and the ideal $\mathcal{C}^F$ as in **[3.1]**. We would like to rise the isomorphism $\mathcal{C}/\mathcal{C}^F \cong C(F)$ to an isomorphism of $\mathfrak{C}_{\mathcal{C}}/\mathfrak{C}_{\mathcal{C}^F}$, where we set $\mathfrak{C}_{\mathcal{C}^F} := \left\langle \mathcal{C}^F \cdot \mathfrak{C}_{\mathbb{C}} \right\rangle$. An obstacle to a direct solution is the fact that an object such as $\left\langle \mathcal{C}/\mathcal{C}^F \cdot \mathfrak{C}_{\mathbb{C}} \right\rangle$ does not make sense without a suitable reinterpretation; $\mathcal{C}/\mathcal{C}^F$ is not a C*-subalgebra of $B(\mathcal{H})$.

The reinterpretation needs the concept of crossed product; we shall just use it without explanations, refering especially to [15] for details. Since $\mathcal{C}$ is an X-algebra, then $\mathfrak{C}_{\mathcal{C}}$ is a C*-algebra isomorphic to the crossed product algebra $\mathcal{C} \rtimes X$, defined in terms of the action α of X on $\mathcal{C}$. Analogously, we have $\mathfrak{C}_{\mathcal{C}^F} \cong \mathcal{C}^F \rtimes X$. But a well-known result states that the quotient $[\mathcal{C} \rtimes X] / \left[\mathcal{C}^F \rtimes X\right]$ is isomorphic to $\left[\mathcal{C}/\mathcal{C}^F\right] \rtimes X$, hence also with $C(F) \rtimes X$ (crossed product defined in terms of the action of the group X on the C*-algebra $C(F)$; here the fact that F was assumed invariant is crucial). We point out that the notation $\mathfrak{C}_{\mathbb{C}}$ is justified by either of the formulae $\mathfrak{C}_{\mathbb{C}} = \langle \mathbb{C} \cdot \mathfrak{C}_{\mathbb{C}} \rangle$ or $\mathfrak{C}_{\mathbb{C}} = \mathbb{C} \rtimes X$.

So let us denote by H^F the image of the observable H through the epimorphism $\mathfrak{C}_{\mathcal{C}} \to C(F) \rtimes X$ with kernel $\mathfrak{C}_{\mathcal{C}^F}$, that induces the isomorphism above. By assuming that $\mathcal{C}$ is an X-algebra of the first kind and that $F \subset Z_{\mathcal{C}}$, we have $C_0(X) \subset \mathcal{C}^F$. It is known that $\langle C_0(X) \cdot \mathfrak{C}_{\mathbb{C}} \rangle = K(\mathcal{H})$. Then, by a straightforward application of **[2.2]**, we get $\boldsymbol{\sigma}(H^F) \subset \boldsymbol{\sigma}_{\mathrm{ess}}(H)$. One has equality if F coincides with the corona set $Z_{\mathcal{C}} = X_{\mathcal{C}} \setminus X$.

As we have already said at an abstract level, if the support of $\eta \in C_0(X)$ is disjoint from $\boldsymbol{\sigma}(H^F)$, then $\eta(H) \in \mathfrak{C}_{\mathcal{C}^F}$. We need an explicit approximate unit for the ideal $\mathfrak{C}_{\mathcal{C}^F}$. Let $\boldsymbol{W}$ be a filter base in X, i.e. a non-void family of non-void subsets of X such that if $W_1, W_2 \in \boldsymbol{W}$, then $W \subset W_1 \cap W_2$ for some $W \in \boldsymbol{W}$. We shall say that the filter base $\boldsymbol{W}$ *is adjacent to* F if the set $\mathrm{Cl}(\boldsymbol{W}) := \cap_{W \in \boldsymbol{W}} \overline{W}$ of all cluster points of $\boldsymbol{W}$ is contained in F. It follows easily that a continuous function on $X_{\mathcal{C}}$ that is null on F has to converge to zero along the filter base $\boldsymbol{W}$. Then, by taking into account the explicit form of $\mathfrak{C}_{\mathcal{C}^F} := \langle \mathcal{C}^F \cdot \mathfrak{C}_{\mathbb{C}} \rangle$, it is easy to see that $\{\theta_W := 1 - \chi_W\}_{W \in \boldsymbol{W}}$ is an approximate unit for this ideal; χ_W is the characteristic function of W. So, for any $\varepsilon > 0$, we shall have $\| \chi_W \eta(H) \| \leq \varepsilon$ if $W \in \boldsymbol{W}$ is "far enough". This is the localization property Problem 2 was asking for.

[3.3] Invariant decompositions.

The point that still has to be elucidated is the nature of the observable H^F. To do that, we express it by means of a suitable decomposition of the set F and by applying [**2.4**]. In [**3.4**], by making an extra assumption which is usually verified, the "components" H_j that appear are bona fide self-adjoint operators and the (closure of the) union of their spectra gives the spectrum of H^F.

Recall that the group X acts on the invariant, compact set F; the restricted action will be denoted by α^F. An *invariant decomposition* of the topological dynamical system $\mathcal{F} := (F, \alpha^F, X)$ is a family $\mathcal{D} := (F_j)_{j \in J}$ of closed, X-invariant subsets of F such that (i) $\cup_{j \in J} F_j$ is dense in F and (ii) If $j \neq k$ then F_j is not a subset of F_k.

Let $\mathcal{D} = (F_j)_{j \in J}$ be an invariant decomposition of $\mathcal{F}$. Set $i_{\mathcal{D}}(\phi) := (\phi|_{F_j})_{j \in J}$ for any $\phi \in C(F)$. Then $i_{\mathcal{D}} : C(F) \to \prod_{j \in J} C(F_j)$ is a monomorphism (rarely surjective). To see the connection with [**2.4**], the reader has only to check that $\left(\mathcal{C}^{F_j}\right)_{j \in J}$ is a $\mathcal{C}^F$-sufficient family of ideals in $\mathcal{C}$ and, by applying the Proposition, we get a monomorphism $\mathcal{C}/\mathcal{C}^F \to \prod_{j \in J} \left[\mathcal{C}/\mathcal{C}^{F_j}\right]$ which is $i_{\mathcal{D}}$ in disguise.

Note that elements $(\phi_j)_{j \in J}$ of the image of $i_{\mathcal{D}}$ satisfy $\lim_{x \to 0} \sup_j \| \alpha_x^{F_j}[\phi_j] - \phi_j \| = 0$, i.e. (by definition) they belong to *the restricted direct product* $\prod'_{j \in J} C(F_j)$. We denoted by $\alpha_x^{F_j}$ the restriction of α_x^F to F_j. The product action $\prod_{j \in J} \alpha^{F_j}$ acts continuously in $\prod'_{j \in J} C(F_j)$, so we can form the crossed product $\left[\prod'_{j \in J} C(F_j)\right] \rtimes X$, which is naturally isomorphic to a C*-subalgebra of $\prod_{j \in J} [C(F_j) \rtimes X]$. Then $i_{\mathcal{D}}$ generates a monomorphism $I_{\mathcal{D}} : C(F) \rtimes X \to \prod_{j \in J} [C(F_j) \rtimes X]$. By combining it with the isomorphism $\mathfrak{C}_{\mathcal{C}}/\mathfrak{C}_{\mathcal{C}^F} \cong C(F) \rtimes X$, pointed out in [**3.2**], one gets a canonical embedding of $\mathfrak{C}_{\mathcal{C}}/\mathfrak{C}_{\mathcal{C}^F}$ into $\prod_{j \in J} [C(F_j) \rtimes X]$ and this is the basic structure result.

Let us denote by H^{F_j} the image of the observable H^F through the j'th component of $I_{\mathcal{D}}$. Since the spectrum of an observable in a direct product is the closure of the union of the spectra of the components, we get $\boldsymbol{\sigma}(H^F) = \overline{\cup_{j \in J} \boldsymbol{\sigma}(H^{F_j})}$.

[3.4] Essential spectrum and localization.

We succeeded to explose the observable H^F into a cluster $(H^{F_j})_{j \in J}$ of smaller ones, which may be simpler. It is often useful to work with the most refined invariant decomposition $\mathcal{D}_{\max} = (F_j)_{j \in J}$, for which every component F_j is a (maximal) quasi-orbit. In this case it is impossible to further decompose H^{F_j} by the method above.

Frequently, there is a means to interpret the observable H^{F_j} as a self-adjoint operator acting in $\mathcal{H}$. Suppose that we are given monomorphisms $\rho_j : C(F_j) \to BC_u(X)$. They will generate embeddings of $C(F_j) \rtimes X$ into $BC_u(X) \rtimes X \cong \langle BC_u(X) \cdot \mathfrak{C}_{\mathbb{C}} \rangle$, that is a C*-subalgebra of $B(\mathcal{H})$. The image of the observable H^{F_j} will be now a self-adjoint operator H_j, may be not densely defined. In fact, in most of the applications, the domain of H_j happens to be dense.

We formulate the main result. It follows straightforwardly from all the discussion above, by writing carefully the explicit form of the morphisms that come into play. For the convenience of the reader we recall the

Framework. *(i) X is an abelian, locally compact group, with dual $X^\sharp$.*

(ii) $\mathcal{C}$ is an X-algebra. We denote by $X_{\mathcal{C}}$ its compact character space and by $Z_{\mathcal{C}}$ the corona set $X_{\mathcal{C}} \setminus X$.

(iii) F is a closed, X-invariant subset of $Z_{\mathcal{C}}$ and $\boldsymbol{W}$ is a filter base on X adjacent to F.

(iv) $h : X^\sharp \to \mathbb{R}$ is a continuous function, satisfying $\lim_{p\to\infty}|h(p)| = \infty$. We set $H_0 = h(P)$.

We shall call *generalized Schrödinger operator* the self-adjoint operator $H := H_0 + V$ in $\mathcal{H} = L^2(X)$, where V is the operator of multiplication with the real-valued function (also denoted by) V, which we choose in $\mathcal{C}$. This function will be fixed from now on and its continuous extension to $X_{\mathcal{C}}$ will be denoted by $V_{\mathcal{C}}$.

Theorem 2. *(a) The self-adjoint operator H is affiliated to the C*-subalgebra $\mathfrak{C}_{\mathcal{C}} := \langle \mathcal{C} \cdot \mathfrak{C}_{\mathbb{C}} \rangle$ of $B(\mathcal{H})$.*

(b) Let $\mathcal{D} = (F_j)_{j\in J}$ be an invariant decomposition of the dynamical system $\mathcal{F} = (F, \alpha^F, X)$. For any $j \in J$, let $\rho_j : C(F_j) \to BC_u(X)$ be a monomorphism. Set $V_j := \rho_j \left[V_{\mathcal{C}}|_{F_j} \right]$ and $H_j := H_0 + V_j$. Then

$$\overline{\bigcup_{j\in J} \boldsymbol{\sigma}[H_j]} \subset \boldsymbol{\sigma}_{\text{ess}}(H). \tag{1-1}$$

(c) If the support of $\eta \in C_0(X)$ is disjoint from $\overline{\bigcup_{j\in J} \boldsymbol{\sigma}[H_j]}$, then for each $\varepsilon > 0$ there exists $W \in \boldsymbol{W}$ such that $\| \chi_W \eta(H) \| < \varepsilon$. In particular, one has $\| \chi_W e^{-itH} \eta(H) f \| < \varepsilon \| f \|$ uniformly in $t \in \mathbb{R}$ and $f \in \mathcal{H}$.

Remark. Much larger classes of Hamiltonians may be treated by using more sophisticated affiliation criteria such as those in [2] or [9].

4. An example; product potentials

We shall give a single example of application of the formalism developed above. Some others are scattered in the references quoted in the Introduction, especially in [15]. The N-body problem can also be treated by compactifications and invariant decompositions, but we shall not do this here. In [22] and [3] it is shown how to treat products of dynamical systems, connected to Schrödinger operators with cartesian anisotropy.

So, we consider generalized Schrödinger operators $H = H_0 + V$ with potentials including those of the form $V = UU'$, where $U, U' : X \to \mathbb{R}$ are suitable continuous functions. In [22], we considered the case in which U is a function with oscillation vanishing at infinity and U' is an almost periodic function. Let us denote by Λ the asymptotic range of U (see below); we have shown that the essential spectrum of H is the union of the spectra of the operators $(H_0 + \lambda U')_{\lambda\in\Lambda}$. Now we shall generalize, by taking U' to be a minimal function. We also present a localization result that unifies some examples from [3]. The family of minimal functions (not forming a C*-algebra) is a very general class of functions that "oscillate regularly". It contains all the almost periodic functions, but also much larger C*-algebras that appear to be important in topological dynamics, like the algebra of distal functions and the algebra of almost automorphic functions. For reasons of space, we make only a summary of the topic. Full proofs and extensions will be presented in a future publication.

[4.1] Functions with asymptotically vanishing oscillation.

We say that a function $\varphi \in BC_u(X)$ has *asymptotically vanishing oscillation*, and we write $\varphi \in VO(X)$, if the function $\alpha_y(\varphi) - \varphi$ is of class $C_0(X)$ for all $y \in X$. $VO(X)$ is an X-algebra that contains strictly the C*-algebra $C^{\rm rad}(X)$, of all complex continuous functions on X that have radial limits at infinity, uniformly in all directions. $VO(X)$ is considerably larger than $C^{\rm rad}(X)$. A C^1-function φ in $X = \mathbb{R}^n$ with $\partial_j\varphi \in C_0(X)$ for all j, is in $VO(X)$. This includes $\varphi(x) = \phi\left[(1+|x|)^p\right]$ for $p < 1$ and ϕ, ϕ' continuous and bounded.

Let X_{VO} be the character space of $VO(X)$. By identifying X with its homeomorphic image in X_{VO}, we can express this character space as the disjoint union $X_{VO} = X \sqcup Z_{VO}$. The nice feature is that $VO(X)$ is the largest X-algebra such that all the elements of its corona space Z_{VO} are fixed points under the extension of the action of the group X; this follows straightforwardly from the definition of $VO(X)$.

We point out that, for X_1, X_2 locally compact, non-compact, abelian groups, in general $VO(X_1) \otimes VO(X_2)$ is different from $VO(X_1 \times X_2)$. Potentials of class $VO(X_1) \otimes VO(X_2)$ are a simple instance of the topic "cartesian anisotropy".

[4.2] Minimal functions.

We discuss now minimal functions, following [19]; see also [13].

Recall that a dynamical system is called *minimal* if it has no proper invariant closed subset or, equivalently, if all its orbits are dense. Let $\mathcal{B}$ be an X-algebra and $X_{\mathcal{B}}$ its character space. We say that $\mathcal{B}$ is *X-simple* if it has no proper invariant ideal.

For any function $\varphi \in BC_u(X)$, we denote by $\mathcal{B}_\varphi$ the smallest X-algebra containing φ. It is not difficult to show that the following four conditions are equivalent for an X-algebra $\mathcal{B}$: (i) $\mathcal{B}$ is X-simple, (ii) $X_{\mathcal{B}}$ is minimal, (iii) $\mathcal{B}_\varphi$ is X-simple for any $\varphi \in \mathcal{B}$, (iv) $X_{\mathcal{B}_\varphi}$ is minimal for any $\varphi \in \mathcal{B}$.

The function φ is called *minimal* if $\mathcal{B}_\varphi$ is X-simple, so that an X-algebra is X-simple if and only if it is composed only of minimal functions. Unfortunately, the set of all minimal functions does not even form a vector space, so some care is needed. Among the X-simple algebras it is interesting to consider those that are maximal, i.e. that are not contained strictly in some other X-simple algebra. It is known that any two of them are isomorphic. The intersection of all the maximal X-simple algebras is still large enough to contain strictly all the distal functions ([13],

[20]) and all the almost automorphic functions ([25]). To have an idea, remark that a function is almost periodic ([12], [17]) if and only if it is both almost automorphic and distal. On $X = \mathbb{Z}$ all the functions $\cos \circ p$, $\sin \circ p$ ($p : \mathbb{Z} \to \mathbb{R}$ polynomial) are distal, hence minimal. Most of them are not almost periodic.

We point out, for further use, a characterization of minimal functions saying that "they oscillate regularly". Let $\varphi \in BC_u(X)$, $S \subset X$ and $\varepsilon > 0$. We say that $y \in X$ is an *ε-translation element* for the function φ and the set S (and we write $y \in T_\varepsilon(\varphi, S)$) if one has $\mid \varphi(x+y) - \varphi(x) \mid < \varepsilon$ for all $x \in S$. Then φ is minimal if and only if for any $\varepsilon > 0$ and any $S \subset X$ finite set, $T_\varepsilon(\varphi, S)$ is *relatively dense*, i.e. there exists some finite subset $\tilde{S}$ of X such that $\tilde{S} + T_\varepsilon(\varphi, S) = X$. Clearly, it follows that no non-null minimal function is in $C_0(X)$. In the coarse case $S = X$, ε-translation elements are called *ϵ-(quasi-)periods* and may serve to define the concept of almost periodicity, which is a kind of global version of minimality.

By using "regular oscillation", one shows easily that if φ_1 is a minimal function on X_1 and φ_2 is a minimal function on X_2, then $\varphi_1 \otimes \varphi_2$ is minimal on the direct product of groups $X_1 \times X_2$.

[4.3] Asymptotic independence.

Our intention is to consider generalized Schrödinger operators with potentials V in the C*-algebra generated by $VO(X)$ and some X-simple algebra; one covers, for example, $V = UU'$, where U has asymptotically vanishing oscillation and U' is an arbitrary minimal function. The formalism developed in Section 3 pushes us to calculate the corona space of this C*-algebra. A series of papers of S. C. Power (also dedicated to the non-commutative case) are relevant in this setting; we quote [23] and [24] for instance.

Let $\mathcal{A}$ and $\mathcal{B}$ two X-algebras and $\mathcal{C}$ the C*-algebra generated by $\mathcal{A}$ and $\mathcal{B}$. We shall assume, for convenience, that $\mathcal{A}$ is of the first kind and that $\mathcal{B} \cap C_0(X) = \{0\}$, but our discussion can be easily adapted to other situations. We would like to decide when the character space $X_\mathcal{C}$ of the X-algebra $\mathcal{C}$ may be identified to the disjoint union $X \sqcup [Z_\mathcal{A} \times X_\mathcal{B}]$. One sees immediatly that the single non-trivial matter is "surjectivity". By particularizing a result of Power, this will be true exactly when $\mathcal{A}$ and $\mathcal{B}$ are asymptotically independent, a concept that we now develop. For a bounded, continuous function φ it is useful to consider the norm of its image $\tilde{\varphi}$ in the quotient $BC(X)/C_0(X)$; we call it *the asymptotic norm* and we write $\| \varphi \|_{\text{asy}} := \| \tilde{\varphi} \|$. Two bounded, continuous functions ψ, μ are called *asymptotically independent* if $\| \psi\mu \|_{\text{asy}} = \| \psi \|_{\text{asy}} \| \mu \|_{\text{asy}}$. And $\mathcal{A}$, $\mathcal{B}$ are called *asymptotically independent* if for any $\psi \in \mathcal{A}$ and $\mu \in \mathcal{B}$, ψ and μ are asymptotically independent. It is already obvious that this is by no means a frequent situation.

Power shows essentially that $VO(X)$ and $AP(X)$ (the algebra of all almost periodic functions) are asymptotically independent, and this is used in [22]. We sketch now very briefly a proof of the generalization of this result, with $AP(X)$ replaced by an arbitrary X-simple algebra.

We introduce first the essential range. We say that a complex number λ is in *the essential range* of the boundeded, continuous function φ and write $\lambda \in R_{\text{asy}}(\varphi)$ if for any $\varepsilon > 0$, *the ε-level set* $M_\varepsilon(\varphi, \lambda) := \{x \in X \mid |\varphi(x) - \lambda| < \varepsilon\}$ is not relatively compact. It is straightforward to verify that $R_{\text{asy}}(\varphi)$ is just the usual range of $\tilde{\varphi} \in BC(X)/C_0(X)$, which can be interpreted as a continuous function defined on the boundary at infinity (corona space) of the Stone-Cečh space associated to X. A more concrete representation is given by the formula $R_{\rm asy}(\varphi) = \cap_K \overline{\varphi(X \setminus K)}$, the intersection being taken over all the compact subsets K of X and the closures are taken in X. The concept is obviously useful, since the asymptotic norm of φ is just $\sup[R_{\rm asy}(|\varphi|)]$.

The key to asymptotic independence is the study of ε-level sets. We have already introduced relative density. A subset B of X is called *replete* if it contains compact sets of any size, i.e. for any $K \subset X$ compact, there exists $x_K \in X$ such that $x_K + K \subset B$. By simple examples one sees that the intersection of (respectively) two relatively dense sets or of two replete sets may be void. But *the intersection of a relatively dense and a replete set in never relatively compact.*

We consider now two functions: $\psi \in VO(X)$ and μ that is minimal. We can show that for any $\varepsilon > 0$, $M_\varepsilon(\psi, \| \psi \|_{\rm asy})$ is replete and $M_\varepsilon(\mu, \| \mu \|_{\rm asy})$ is relatively dense. Then $M_{2C\varepsilon}(\psi\mu, \| \psi \|_{\rm asy} \| \mu \|_{\rm asy})$, which contains the intersection of the previous two ε-level sets for a suitable constant $C < \infty$, is not relatively compact. It follows that $\| \psi \|_{\rm asy} \| \mu \|_{\rm asy}$ is contained in the asymptotic range of $\psi\mu$ and this entails the fact that ψ and μ are asymptotically independent. Thus, *$VO(X)$ and any* X-*simple algebra are asymptotically independent.*

[4.4] The essential spectrum.

Let us fix an X-simple algebra $\mathcal{B}$, set $\mathcal{A} := VO(X)$ and let $\mathcal{C}$ be the C*-algebra generated by $\mathcal{A}$ and $\mathcal{B}$. By [**4.3**], we know that $X_\mathcal{C}$ can be identified to $X \sqcup [Z_\mathcal{A} \times X_\mathcal{B}]$. By [**3.4**], in order to express the essential spectrum of generalized Schrödinger operators with potentials in $\mathcal{C}$, one has to invariantly decompose the corona set $Z_\mathcal{A} \times X_\mathcal{B}$. This is extremely simple, since all the points of $Z_\mathcal{A}$ are fixed points under X, while $X_\mathcal{B}$ is minimal. We understand completely the quasi-orbit structure, and the finest invariant decomposition is given by $\mathcal{D}_{\max} = (\{s\} \times X_\mathcal{B})_{s \in Z_\mathcal{A}}$. This leads to the structure result saying that that quotient $\mathcal{C}/C_0(X)$ can be embedded into the restricted direct product $\prod'_{s \in Z_\mathcal{A}} \mathcal{B}$. We used in fact the identification ρ_s of $C(\{s\} \times X_\mathcal{B})$ with $\mathcal{B} \subset BC_u(X)$, so that our "asymptotic potentials" are already seen as functions on X. By taking crossed products and by making simple identifications, one embeds $\mathfrak{C}_\mathcal{C} \equiv \langle \mathcal{C} \cdot \mathfrak{C}_\mathbb{C} \rangle$ into $\prod_{s \in Z_\mathcal{A}} \mathfrak{C}_\mathcal{B} \equiv \prod_{s \in Z_\mathcal{A}} \langle \mathcal{B} \cdot \mathfrak{C}_\mathbb{C} \rangle$.

A simple feature of the present example is that the quasi-orbits form a *partition* of the corona space $Z_\mathcal{C}$. This is not at all the rule and, indeed, it does not hold in the N-body problem or for cartesian anisotropy.

We calculate now the essential spectrum of generalized Schrödinger operators with potentials in $\mathcal{C}$. As usual, $H_0 = h(P)$ for some continuous function $h : X^\sharp \to \mathbb{R}$ diverging at infinity. Let us fix a real function $V \in \mathcal{C}$; it will be our potential. Its most general form is $V = \lim_{N \to \infty} \sum_{k=1}^{m_N} U_{N,k} U'_{N,k}$ (uniform limit), with $m_N \in \mathbb{N}$, $U_{N,k} \in \mathcal{A} \equiv VO(X)$ and $U'_{N,k} \in \mathcal{B}$. So, all the functions $U'_{N,k}$ must belong to the *same* X-simple algebra; one may assume for exemple that they are all distal or all almost automorphic ("almost periodic" being just a particular case). If U is an element of $\mathcal{A}$, let us denote by $\hat{U}$ the continuous extension of U to $X_\mathcal{A}$ and by $\tilde{U}$ the restriction of this extension to $Z_\mathcal{A}$. For any $s \in Z_\mathcal{A}$, we set $V_s = \lim_{N \to \infty} \sum_{k=1}^{m_N} \tilde{U}_{N,k}(s) U'_{N,k} \in \mathcal{B}$. By inspecting the explicit form of the morphisms we used, we see that the image of $H = H_0 + V$ in the s'th copy of $\mathfrak{C}_\mathcal{B}$

appearing in the direct product is $H_s := H_0 + V_s$. Thus the essential spectrum of $H = H_0 + V$ is given by

$$\boldsymbol{\sigma}_{\rm ess}(H) = \bigcup_{s \in Z_{\mathcal{A}}} \boldsymbol{\sigma}(H_s)$$

(the union is automatically closed, by [9], Th.2.10). The values taken by the functions $U_{N,k}$ "at infinity" become "coupling constants" for the "indecomposable" factors $U'_{N,k}$; this gives asymptotic Hamiltonians and the union of their spectra is the essential spectrum of H.

Let us illustrate this by two simple particular cases: $V_{(1)} = UU'$ and $V_{(2)} = U + U'$, where U has asymptotically vanishing oscillation and U' is minimal. We set $H_{(i)} = H_0 + V_{(i)}$, $i = 1, 2$. By also taking into account the convenient notion of asymptotic range of U, we get respectively $\boldsymbol{\sigma}_{\rm ess}(H_{(1)}) = \bigcup_{\lambda \in R_{\rm asy}(U)} \boldsymbol{\sigma}[H_0 + \lambda U']$ and $\boldsymbol{\sigma}_{\rm ess}(H_{(2)}) = \bigcup_{\lambda \in R_{\rm asy}(U)} \boldsymbol{\sigma}[H_0 + \lambda + U'] = \boldsymbol{\sigma}(H_0 + U') + R_{\rm asy}(U)$.

[4.5] Localization.

We are in the setting of [**4.4**]. To get localization properties by Theorem 2, one has to choose an invariant closed subset F of $Z_{\mathcal{C}}$, to calculate the spectrum of the observable H^F (the image of H in $C(F) \rtimes X$) and to determine an explicit filter base $\boldsymbol{W}$ in X which shoud be related to F. The obvious form F should have is $F_{\mathcal{A}} \times X_{\mathcal{B}}$, where $F_{\mathcal{A}}$ is some closed subset of $Z_{\mathcal{A}}$ (automatically invariant).

By the same arguments as in [**4.4**], using the invariant decomposition $(\{s\} \times X_{\mathcal{B}})_{s \in F_{\mathcal{A}}}$ of F, we get that the $\mathfrak{C}_{\mathcal{C}^F}$-essential spectrum of H is given by $\boldsymbol{\sigma}^{\mathfrak{C}_{\mathcal{C}}}_{\mathfrak{C}_{\mathcal{C}^F}}(H) = \boldsymbol{\sigma}(H^F) = \bigcup_{s \in F_{\mathcal{A}}} \boldsymbol{\sigma}[H_s]$.

To proceed, we simplify and consider only the two particular cases $V_{(1)} = UU'$ and $V_{(2)} = U + U'$. The frontier $Z_{\mathcal{A}}$ is not easy to understand, so we shall consider only closed sets $F_{\mathcal{A}} \subset Z_{\mathcal{A}}$ which are suitably related to U. We recall that $\hat{U}$ is the continuous extension of U to $X_{\mathcal{A}}$ and $\tilde{U}$ its restriction to $Z_{\mathcal{A}}$. Let G be a closed subset of $\mathbb{R}$ such that its interior G^0 meets $R_{\rm asy}(U)$. We set $F_{\mathcal{A}} = \tilde{U}^{-1}(G)$; it is a closed, non-void subset of $Z_{\mathcal{A}}$.

We introduce now a convenient filter base. For any compact subset K of X we set $W_K = U^{-1}(G) \cap K^c$. The assumption $G^0 \cap R_{\rm asy}(U) \neq \emptyset$ implies that $W_K \neq \emptyset$. Since $W_K \cap W_{K'} = W_{K \cup K'}$, $\boldsymbol{W} = \{W_K\}_K$ is a filter base in X. We calculate (closures are taken in $X_{\mathcal{C}}$):

$$\cap_K \overline{W_K} = \cap_K \overline{[U^{-1}(G) \cap K^c]} \subset \cap_K \left[\overline{U^{-1}(G)} \cap \overline{K^c}\right] =$$
$$= \overline{U^{-1}(G)} \cap Z_{\mathcal{C}} \subset \tilde{U}^{-1}(G) \times X_{\mathcal{B}} = F.$$

Thus F is adjacent to $\boldsymbol{W}$. We are now in a position to state localization results; We set $\Lambda(U, G) := R_{\rm asy}(U) \cap G$.

(1) If the support of $\eta \in C_0(X)$ is disjoint from $\bigcup_{\lambda \in \Lambda(U,G)} \boldsymbol{\sigma}[H_0 + \lambda U']$, then for any $\varepsilon > 0$ there exists a compact subset K of X, large enough, such that $\| \chi_{W_K} \eta(H_{(1)}) \| \leq \varepsilon$.

(2) If the support of $\eta \in C_0(X)$ is disjoint from $\boldsymbol{\sigma}[H_0 + U'] + \Lambda(U, G)$, then for any $\varepsilon > 0$ there exists a compact subset K of X, large enough, such that $\| \chi_{W_K} \eta(H_{(2)}) \| \leq \varepsilon$.

We leave to the reader the pleasure (or may be the pain) to find sufficiently explicit examples to which these results apply. (S)He may consider situation (2) for some periodic potential U', with an interesting band structure of the spectrum of $H_0 + U'$. Of course, non-propagation into some remote spatial region (connected to U') is interesting only if we know that propagation towards infinity actually takes place in some way. For this, one has to put into evidence situations in which H has intervals of continuous spectrum in $\boldsymbol{\sigma}_{\mathrm{ess}}(H) \setminus \boldsymbol{\sigma}(H^F)$. This is also left to the reader.

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