On the Hamiltonian Hopf bifurcations in the $3D$ Hénon–Heiles family

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Abstract

An axially symmetric perturbed isotropic harmonic oscillator undergoes several bifurcations as the parameter λ adjusting the relative strength of the two terms in the cubic potential is varied. We show that three of these bifurcations are Hamiltonian Hopf bifurcations. To this end we analyse an appropriately chosen normal form. It turns out that the linear behaviour is not that of a typical Hamiltonian Hopf bifurcation as the eigenvalues completely vanish at the bifurcation. However, the nonlinear structure is that of a Hamiltonian Hopf bifurcation. The result is obtained by formulating geometric criteria involving the normalized Hamiltonian and the reduced phase space.

1 Introduction

Resonant equilibria of Hamiltonian systems serve as models in many fields of physics, ranging from nonlinear optics to molecular dynamics. Specifically, perturbations of the

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isotropic harmonic oscillator, in 1:1:1–resonance, are used to describe galactic dynamics, cf. [23, 24]. In [11] an additional axial symmetry was imposed and the so–called *Hénon–Heiles family* in three degrees of freedom has been studied. This family is defined by

$$H(x,y) \;=\; \frac{1}{2}(y \mid y) + \frac{1}{2}(x \mid x) \;+\; \varepsilon \,(\frac{x_3^3}{3} + \lambda(x_1^2 + x_2^2)x_3) \;\;, \tag{1}$$

where $(..|..)$ denotes the standard inner product and λ is an external parameter the perturbing force field depends upon. One may think of ε as being introduced by a scaling $x \mapsto \varepsilon x$, $p \mapsto \varepsilon p$, $H \mapsto \varepsilon^2 H$, whence (1) describes the dynamics close to the origin. Note that we put the frequency of the resonant oscillator equal to 1 ; this can always be achieved by rescaling time.

The $2D$ Hénon–Heiles family, which is embedded in (1) as *e.g.* $x_2 = y_2 = 0$, has been systematically analysed, cf. [2, 15, 21, 4, 3, 19, 13]. There are only five values of λ where the first nontrivial normal form is not structurally stable. At the value $\lambda = -1$, originally studied by Hénon and Heiles [14], a higher order normal form is necessary to correctly resolve the $\mathbb{Z}_3$–equivariant dynamics. This leads to a whole sequence of bifurcations as λ is varied near -1 , see [3]. The degeneracy of the normal form for $\lambda = +1$, on the other hand, persists through all orders and is due to the integrability of the $2D$ Hénon–Heiles family for this value of λ , cf. [1, 7, 11]. Other values of λ where (1), with $x_2 = y_2 = 0$, defines an integrable system are $\lambda = 0$ and $\lambda = \frac{1}{6}$. While the normal form is structurally stable at $\lambda = \frac{1}{6}$, a degenerate bifurcation (detailed in [13]) takes place as λ passes through 0 . Finally, there are two Hamiltonian pitchfork bifurcations when λ varies through $\frac{1}{2}$ and through $\frac{5}{2}$, see [2, 15, 21, 4, 3, 19].

The $3D$ Hénon–Heiles family being axially symmetric, one may reduce immediately to two degrees of freedom and a normal form defines again an integrable approximation. The exceptional values $\lambda = -1, 0, \frac{1}{2}, 1, \frac{5}{2}$ are inherited from the $2D$ Hénon–Heiles family. At $\lambda = 6$ a family of periodic orbits with normal behaviour in 2:1–resonance emanates from the central equilibrium. For all other values of λ one has again a structurally stable normal form, which allows to describe the dynamical properties of the original system (1) as well. See [11] for a detailed exposition of these results. Again a higher order normal form is needed when λ is close to -1 . While such a study over a range of λ–values has not been done yet, a higher order normal form (and the system (1) as well) has been analysed in [22, 8, 9] at the value $\lambda = -1$ itself. Integrability–induced degeneracies are again encountered at $\lambda = 0$ and $\lambda = 1$, while the integrability of (1) at $\lambda = \frac{1}{6}$ does not destabilize the normal form. It is conjectured in [11] that Hamiltonian Hopf bifurcations take place as λ passes through $\frac{1}{2}$ and through $\frac{5}{2}$.

To understand the reasoning behind this conjecture, let us review the reduction of the normal form from three to one degree of freedom, cf. [6, 11]. The imposed axial symmetry makes the third component S of the angular momentum an integral of motion. Fixing $S = \sigma$ and dividing out the axial S^1–action reduces to two degrees of freedom. For $\sigma \neq 0$ the reduced phase space is diffeomorphic to $\mathbb{C}^2$, but for $\sigma = 0$ it is only homeomorphic to $\mathbb{C}^2$, having a conical singularity at $\{0\}\times\mathbb{C}\subseteq\mathbb{C}\times\mathbb{C}$. This singularity reflects that the axial S^1–action is not free, but leaves the whole third subsystem $x_1 = y_1 = x_2 = y_2 = 0$ fixed. In a second step the *oscillator symmetry*, introduced by means of normalization with respect to the unperturbed isotropic harmonic oscillator, is reduced. Fixing $H_0 = \eta$ and dividing out this S^1–action yields the twice reduced phase space $V_{\sigma,\eta}$.

Every point of $V_{\sigma,\eta}$ represents a whole S^1–family (generated by rotation about the vertical axis) of ellipses centered at the origin. This renders the nature of the occurring singularities very intuitive. If the ellipse in question is an equatorial circle then the whole S^1–family consists of just this circle, which has S^1 as isotropy subgroup of the combined $(S^1 \times S^1)$–action. Correspondingly, $V_{\sigma,\eta}$ shrinks to a point as $\sigma \to \pm\eta$. Furthermore $V_{\sigma,\eta} = \emptyset$ for $|\sigma| > \eta$. When $|\sigma| < \eta$ the twice reduced phase space $V_{\sigma,\eta}$ is a topological sphere with a conical singularity. This singularity reflects that an equatorial ellipse is mapped to itself by a π–rotation, *i.e.* it has $\mathbb{Z}_2$ as isotropy subgroup of $S^1 \times S^1$. Finally there is a second conical singularity for $\sigma = 0$, that is inherited from the first reduction. Here the ellipse degenerates to a vertical line, having S^1 as isotropy subgroup of $S^1 \times S^1$. See [6] for more details on how the various singularities stratify the four–dimensional semi–algebraic set $\bigcup\limits_{\sigma,\eta} V_{\sigma,\eta}$.

The bifurcations of the normal form of (1) at $\lambda = -1, \frac{1}{2}, 1, \frac{5}{2}$ are related to this second singularity of $V_{0,\eta}$. For these λ–values both the twice reduced phase space and the energy level set enter the singularity with the same slope. When λ passes through $\frac{1}{2}$ or $\frac{5}{2}$ the set of singular values of the energy–momentum mapping displays the swallowtail surface typical of a Hamiltonian Hopf bifurcation. The aim of the present paper is to prove that this bifurcation indeed takes place and to unveil its peculiarities. The cases $\lambda = \pm 1$ are more degenerate as the energy level set through the singularity touches $V_{0,\eta}$ along a whole line. For $\lambda = -1$ further normalization allows to break this degeneracy. In the case $\lambda = +1$ this degeneracy is due to the integrability of (1) and persists for all (higher order) normal forms.

This paper is organized as follows. In the next section we detail the normal form of (1) and how to reduce it by means of invariants. Section 3 contains the linear analysis of the Hamiltonian Hopf bifurcations at $\lambda = -1, \frac{1}{2}, 1, \frac{5}{2}$ of the first nontrivial normal form. In Section 4 we derive a geometric criterion for determining the nonlinear structure of a Hamiltonian Hopf bifurcation. This is applied in Section 5 to the cases $\lambda = \frac{1}{2}, \frac{5}{2}$ where the first nontrivial normal form is decisive. A higher order normal form is used in Section 6 to analyse the nonlinear structure of the Hamiltonian Hopf bifurcation near $\lambda = -1$. In the final section we put our results in context and discuss open questions.

2 Preliminaries on the 1:1:1–resonance

We expect the normal form of (1) to undergo a Hamiltonian Hopf bifurcation as λ passes through $\frac{1}{2}$ or through $\frac{5}{2}$. The pertinent transversality conditions involve the integral of motion S ; therefore we refrain from reducing the axial symmetry. Instead, we follow the approach of [22] and (first) reduce the S^1–action generated by the unperturbed isotropic harmonic oscillator.

We consider an isotropic harmonic oscillator perturbed by a cubic potential. On $\mathbb{R}^6 = T^*\mathbb{R}^3$ with the standard symplectic form the system has the Hamiltonian (1). The potential is chosen to be the most general cubic expression such that the system is axially symmetric. More precisely, the system is invariant under the axial S^1–action

$$\begin{array}{rccc} \rho : & S^1 \times T^*\mathbb{R}^3 & \longrightarrow & T^*\mathbb{R}^3 \\ & (\nu,(x,y)) & \mapsto & (\exp_\nu x, \exp_\nu y) \end{array}$$

where

$$\exp_\nu \; = \; \exp \begin{pmatrix} 0 & -\nu & 0 \\ \nu & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \; = \; \begin{pmatrix} \cos\nu & -\sin\nu & 0 \\ \sin\nu & \cos\nu & 0 \\ 0 & 0 & 1 \end{pmatrix} \; .$$

The action corresponds to spatial rotation about the third axis and the corresponding integral is the third component of the angular momentum, *i.e.*

$$S \; = \; x_1 y_2 - x_2 y_1 \; .$$

The system (1) has been normalized twice in [11], yielding a fourth order polynomial $\bar{H}_\lambda$ in the x_i and y_i . This normal form Hamiltonian commutes not only with S , but furthermore with the quadratic part $H_0 = \frac{1}{2}(x_1^2+y_1^2)+\frac{1}{2}(x_2^2+y_2^2)+\frac{1}{2}(x_3^2+y_3^2)$ of H . Therefore, it can be written in terms of the invariants corresponding to the action of H_0 . These invariants can be easily derived using complex conjugate co–ordinates, see [12, 22, 10]. The H_0–invariants are

$$\begin{array}{lll} \pi_1 = x_1^2 + y_1^2 \; , & \pi_2 = x_2^2 + y_2^2 \; , & \pi_3 = x_3^2 + y_3^2 \; , \\ \pi_4 = x_1 x_2 + y_1 y_2 \; , & \pi_5 = x_1 x_3 + y_1 y_3 \; , & \pi_6 = x_2 x_3 + y_2 y_3 \; , \\ \pi_7 = x_1 y_2 - y_1 x_2 \; , & \pi_8 = x_1 y_3 - y_1 x_3 \; , & \pi_9 = x_2 y_3 - y_2 x_3 \; . \end{array}$$

By construction the normal form of H with respect to H_0 can be expressed in these invariants. Note that the normalization in [11] is performed in such a way that the resulting normal form is still invariant under the S^1–action generated by $S = \pi_7$. The normal form reads

$$\begin{aligned} \bar{H}^\lambda \; = \; & \frac{1}{2}(\pi_1+\pi_2+\pi_3) \; + \; \frac{\varepsilon^2}{48}\Big(-5\pi_3^2 \; + \; 2\lambda(\pi_5^2+\pi_6^2-(\pi_8^2+\pi_9^2) - 6(\pi_1+\pi_2)\pi_3) \\ & - \; \lambda^2(12(\pi_5^2+\pi_6^2-(\pi_8^2+\pi_9^2)) + 5(\pi_1+\pi_2)^2 + 8(\pi_1+\pi_2)\pi_3 + 4\pi_7^2)\Big) \; . \end{aligned}$$

By writing the normal form as a function of $\pi_1, \ldots, \pi_9$ we have effectively divided out the S^1–action generated by H_0 . To complete the reduction to two degrees of freedom we put $H_0 = \eta$. This is a regular reduction, whence the reduced phase space is a (smooth) symplectic manifold. According to a theorem of Moser [20] this space is isomorphic to $\mathbb{CP}^2$. As a subspace of $\mathbb{R}^9$ with co–ordinates $\pi_1, \ldots, \pi_9$, it is determined by the relations

$$\begin{array}{ll}
\pi_1\pi_2 = \pi_4^2 + \pi_7^2 \ , & \pi_1\pi_6 = \pi_4\pi_5 + \pi_7\pi_8 \ , \\
\pi_1\pi_3 = \pi_5^2 + \pi_8^2 \ , & \pi_4\pi_6 = \pi_2\pi_5 + \pi_7\pi_9 \ , \\
\pi_2\pi_3 = \pi_6^2 + \pi_9^2 \ , & \pi_3\pi_4 = \pi_5\pi_6 + \pi_8\pi_9 \ , \\
\pi_1 + \pi_2 + \pi_3 = 2\eta \ . &
\end{array} \tag{2}$$

To further reduce from $\mathbb{CP}^2$ to $V_{\sigma,\eta}$ one would have to fix $S = \sigma$ and divide out the S^1–action generated by $S = \pi_7$. This can again be done by expressing everything in invariants (now of the axial S^1–action), and since the five invariants generating the algebra of $(S^1 \times S^1)$–invariant functions play a prominent rôle in the sequel, let us write them down here :

$$H_0 \ , \quad S \ , \quad \pi_5^2 + \pi_6^2 - (\pi_8^2 + \pi_9^2) \ , \quad \pi_5\pi_8 + \pi_6\pi_9 \ , \quad \pi_1 + \pi_2 \ .$$

However, note that we continue our study on the (once) reduced phase space $\mathbb{CP}^2$.

3 Linearizing at the normal mode

From the analysis in [11] it is clear that the bifurcation takes place at $\pi_1 + \pi_2 = y_1^2 + y_2^2 + x_1^2 + x_2^2 = 0$. Thus we are considering a bifurcation from the normal mode determined by $x_1 = x_2 = y_1 = y_2 = 0$ and $y_3^2 + x_3^2 = 2\eta$. At the reduced phase space this mode corresponds to the point $\pi^\eta = (\pi_1, \ldots, \pi_9) = (0, 0, 2\eta, 0, 0, 0, 0, 0, 0)$. Consequently, in order to study the bifurcation a chart is to be constructed for the reduced phase space at this point. Note that there is a strong analogy with the approach followed in the analysis of the Lagrange top in [5].

Recall the normalized Hamiltonian

$$\begin{aligned}
\bar{H}^\lambda \ = \ H_0 \ + \ \frac{\varepsilon^2}{48}\big(&-20H_0^2 \ - \ 4\lambda^2\pi_7^2 \ + \ 2\lambda(1-6\lambda)(\pi_5^2 + \pi_6^2 - (\pi_8^2 + \pi_9^2)) \\
&+ \ 4(5 - 6\lambda - 4\lambda^2)H_0(\pi_1 + \pi_2) \ - \ (5 - 12\lambda - 3\lambda^2)(\pi_1 + \pi_2)^2\big) \ .
\end{aligned} \tag{3}$$

Note that on the reduced phase space $H_0 = \eta$ is actually constant.

In order to obtain the Poisson structure or symplectic structure on the reduced phase space the Poisson brackets for the invariants π_i have to be computed. This is done in Table 1 which gives us the structure matrix Π for the Poisson structure on $\mathbb{R}^9$. This Poisson structure has a natural restriction to the reduced phase space $\mathbb{CP}^2$. On this submanifold the structure matrix is non–degenerate and thus defines a symplectic structure.

Table 1: The bracket relations $\{\pi_i, \pi_j\}$ defining the structure matrix Π .

$\{\ ,\ \}$	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9
π_1	0	0	0	$2\pi_7$	$2\pi_8$	0	$-2\pi_4$	$-2\pi_5$	0
π_2	0	0	0	$-2\pi_7$	0	$2\pi_9$	$2\pi_4$	0	$-2\pi_6$
π_3	0	0	0	0	$-2\pi_8$	$-2\pi_9$	0	$2\pi_5$	$2\pi_6$
π_4	$-2\pi_7$	$2\pi_7$	0	0	π_9	π_8	$\pi_1 - \pi_2$	$-\pi_6$	$-\pi_5$
π_5	$-2\pi_8$	0	$2\pi_8$	$-\pi_9$	0	π_7	$-\pi_6$	$\pi_1 - \pi_3$	π_4
π_6	0	$-2\pi_9$	$2\pi_9$	$-\pi_8$	$-\pi_7$	0	π_5	π_4	$\pi_2 - \pi_3$
π_7	$2\pi_4$	$-2\pi_4$	0	$\pi_2 - \pi_1$	π_6	$-\pi_5$	0	π_9	$-\pi_8$
π_8	$2\pi_5$	0	$-2\pi_5$	π_6	$\pi_3 - \pi_1$	$-\pi_4$	$-\pi_9$	0	π_7
π_9	0	$2\pi_6$	$-2\pi_6$	π_5	$-\pi_4$	$\pi_3 - \pi_2$	π_8	$-\pi_7$	0

Linearizing the Hamiltonian vector field $X_{\bar{H}^\lambda}$ at π^η gives

$$Y_\eta \;=\; DX_{\bar{H}^\lambda}\Big|_{T_{\pi^\eta}(\mathbb{CP}^2)} \;=\; \frac{\varepsilon^2\eta}{48}\begin{pmatrix} 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta \\ \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \end{pmatrix}$$

with $\alpha = 40(1-\lambda-2\lambda^2)$ and $\beta = -8(5-7\lambda+2\lambda^2)$. Here $T_{\pi^\eta}(\mathbb{CP}^2)$ is the tangent space to $\mathbb{CP}^2$ at π^η with co–ordinates $(0,0,0,0,\pi_5,\pi_6,0,\pi_8,\pi_9)$, whence $T_{\pi^\eta}(\mathbb{CP}^2)$ can be identified with $\mathbb{R}^4$.

The eigenvalues of Y_η are

$$\pm\frac{\varepsilon^2\eta}{6}\sqrt{20\lambda^4 - 60\lambda^3 + 5\lambda^2 + 60\lambda - 25} \ .$$

The eigenvalues are all equal to zero at the values $\lambda = -1, \frac{1}{2}, 1, \frac{5}{2}$. In the intervals $]-1, \frac{1}{2}[$ and $]1, \frac{5}{2}[$ the eigenvalues are purely imaginary. At the exceptional values the linear vector field becomes

$$Y_\eta|_{\lambda=-1} \;=\; -\frac{7}{3}\varepsilon^2\eta\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \quad Y_\eta|_{\lambda=\frac{1}{2}} \;=\; -\frac{1}{3}\varepsilon^2\eta\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \tag{4}$$

$$Y_\eta|_{\lambda=1} \;=\; -\frac{5}{3}\varepsilon^2\eta\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} , \quad Y_\eta|_{\lambda=\frac{5}{2}} \;=\; -\frac{35}{3}\varepsilon^2\eta\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} . \tag{5}$$

In all cases the linearized vector field is nilpotent. Let us compare this with a standard Hamiltonian Hopf bifurcation.

As the bifurcation parameter of a standard Hamiltonian Hopf bifurcation is varied, two pairs of imaginary eigenvalues meet at a double pair $\pm\Omega\mathrm{i}$ of imaginary eigenvalues and turn into a complex quartet $\pm\Re\pm\mathrm{i}\Im$. Correspondingly, the linearized vector field at the bifurcation reads $\Omega\cdot S + N$ with a semi–simple matrix S and a nilpotent matrix N . As long as $\Omega\neq 0$ one may normalize with respect to S and thus push an S^1–symmetry through the Taylor series of the Hamiltonian H . To establish that indeed a Hamiltonian Hopf bifurcation takes place at $\lambda=\lambda_*$ one then has to check that the (normalized) energy–momentum mapping $\mathcal{EM}=(H,S)$ starts as

$$\mathcal{EM} \;=\; (\,\Omega\cdot S + N + (\lambda-\lambda_*)M\,,\,S\,) \;+\; (\,a_1\cdot M^2\,,\,\ldots\,) \;+\;\ldots \tag{6}$$

with $a_1\neq 0$, where M is the *adjoint* of N , *i.e.* N, M and $P=\{N,M\}$ span a Lie algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. See [16, 18] for more details.

In the present situation the coefficient Ω of S in (3) vanishes. On the other hand, (3) already commutes with S because of the axial symmetry imposed on (1). Therefore, it suffices to check (6), with $\Omega=0$, to establish that a Hamiltonian Hopf bifurcation takes place.

4 A geometric criterion

We now briefly review the treatment of a standard Hamiltonian Hopf bifurcation given in [16, 17] and derive a geometric criterion that can be used to verify the necessary transversality conditions. Let us start from a family H^ν of Hamiltonians that already is invariant under the S^1–action generated by S . Here S is again the semi–simple part of the quadratic terms of H^0 , *i.e.* at the parameter value $\nu=0$ where the bifurcation takes place. In case the Hamiltonian system under consideration is not already S^1–symmetric, this may be arranged by passing to a suitable normal form. Denoting by N the nilpotent part, by M its adjoint and $P=\{N,M\}$, the S^1–symmetry allows to express H^ν as

$$H^\nu \;=\; \Omega(\nu)\cdot S \;+\; N \;+\; a_0(\nu)\,M \;+\; H_4^\nu(S,N,M,P) \;+\;\ldots$$

with $a_0(0)=0$. After a (further) normalization with respect to the nilpotent part N the higher order terms become independent of N, P and we have

$$H_4^\nu \;=\; H_4^\nu(S,M) \;=\; a_1(\nu)M^2 \;+\; a_2(\nu)SM \;+\; a_3(\nu)S^2 \;\;.$$

For a (standard) Hamiltonian Hopf bifurcation to take place one needs in fact three transversality conditions to hold true.

(HH1) The coefficient $\Omega(0)$ of S at the bifurcation value $\nu=0$ has to be nonzero. We have already discussed that this condition can be relaxed if the S^1–symmetry generated by S is externally imposed.

(HH2) The derivative $a_0'(0)$ of the coefficient of M at the bifurcation value $\nu = 0$ should not vanish. This allows in fact to reparametrise and work with $a_0(\nu) \equiv \nu$.

(HH3) The coefficient $a_1(0)$ of M^2 at the bifurcation value $\nu = 0$ has to be different from zero. Its sign distinguishes between the supercritical and the subcritical case.

We claim that the conditions (HH2&3) may be checked after reduction to one degree of freedom.

By expressing everything in the invariants S, N, M, P we have effectively reduced the S^1–symmetry. To obtain the one–degree–of–freedom systems we fix $S = \sigma$. In the standard Hamiltonian Hopf bifurcation the reduced phase space is then given by

$$U_\sigma \;=\; \{\, (N, M, P) \in \mathbb{R}^3 \;\mid\; R_\sigma(N, M, P) = 0\,,\; N \geq 0\,,\; M \geq 0 \,\}$$

where

$$R_\sigma(N, M, P) \;=\; \frac{1}{2}P^2 - 2NM + \frac{\sigma^2}{2} \;.$$

Note that U_σ is (half of) a hyperboloid for $\sigma \neq 0$ and a cone when $\sigma = 0$, with $(N, M, P) = (0, 0, 0)$ as singular point. The Poisson bracket reads

$$\{f, g\} \;=\; (\, \nabla f \times \nabla g \mid \nabla R_\sigma \,)$$

and the reduced Hamiltonian becomes

$$\mathcal{H}^{\nu,\sigma} \;=\; N \;+\; \nu M \;+\; a_1(\nu) M^2 \;+\; a_2(\nu)\sigma M \;+\; \ldots$$

where we have omitted constant terms.

The (relative) equilibria occur where a level set $\{\mathcal{H}^{\nu,\sigma} = h\}$ touches the reduced phase space U_σ . As U_σ is a surface of revolution and $\mathcal{H}^{\nu,\sigma}$ does not depend on P , this can only happen on $\{P = 0\}$. For $\sigma \neq 0$ one would have to study the relative position of the two curves

$$\begin{aligned} N \;&=\; \frac{\sigma^2}{4} \cdot \frac{1}{M} \\ N \;&=\; h \;-\; \nu M \;-\; a_1(\nu) M^2 \;-\; a_2(\nu)\sigma M \;+\; \ldots \end{aligned}$$

while for $\sigma = 0$ we can concentrate on how

$$N \;=\; h \;-\; \nu M \;-\; a_1(\nu) M^2 \;+\; \ldots \tag{7}$$

lies with respect to the N– and M–axes.

The level sets through the singularity have energy $h = 0$. For $\nu = 0$ the coefficient of M vanishes, so (7) enters the singularity tangent to the M–axis. Since 'passing through'

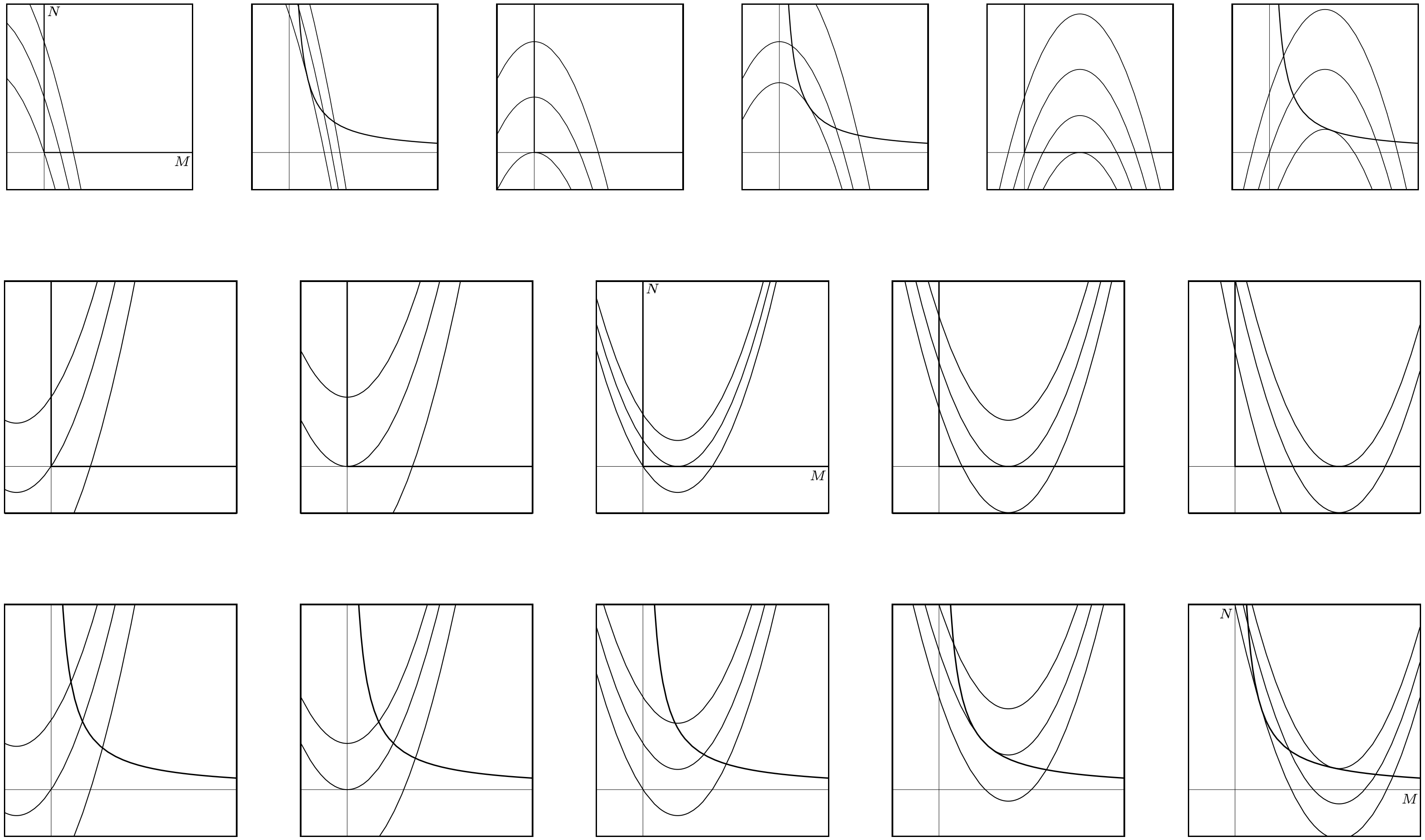

Figure 4.1: Relative positions of three energy level sets $\{\mathcal{H}^{\nu,\sigma} = h\}$ and the reduced phase space U_σ within the co–ordinate plane $P = 0$ during a supercritical and during a subcritical Hamiltonian Hopf bifurcation.

the singularity already means 'touching it' (*i.e.* leads to an equilibrium), this is a further degeneracy. The equilibrium at the origin bifurcates and changes from elliptic (where (7) remains outside the cone) to hyperbolic (where (7) enters the cone through the singularity). The condition (HH3) ensures that (7) has quadratic contact with the M–axis. There are two cases, see *figure 4.1.*

I. The coefficient $a_1(0)$ is positive. The curve (7) remains at the bifurcation value outside the cone, *i.e.* it is tangent to the line $N = 0$ from below. We call the Hamiltonian Hopf bifurcation supercritical. In this case an elliptic periodic orbit, more precisely a whole family of elliptic periodic orbits, detaches from the origin.

II. The coefficient $a_1(0)$ is negative. The curve (7) lies inside the cone, *i.e.* it is tangent to the line $N = 0$ from above. We call the bifurcation subcritical. In this case a hyperbolic periodic orbit, more precisely a swallowtail of hyperbolic and elliptic periodic orbits, shrinks into the origin. There is a subordinate (periodic) centre–saddle bifurcation where the elliptic and the hyperbolic periodic orbits meet and vanish in a one–parameter family of parabolic periodic orbits.

To give a concise formulation of the criteria we have just proven we need some preparations.

Definition 4.1 *We say that a family H^ν undergoes a linear Hamiltonian Hopf bifurcation at $\nu = \nu_0$ if the quadratic part has a nontrivial Jordan–Chevalley decomposition $H_2^{\nu_0} = \Omega(\nu_0) \cdot S + N$ (with semi–simple S and nilpotent N) and either $\Omega(\nu_0) \neq 0$ or the family H^ν is invariant under the S^1–action generated by S .*

Replacing, if necessary, H^ν by a normal form with respect to S , we may reduce to one degree of freedom. The bifurcating equilibrium gets reduced to a conic singularity τ_0 of the reduced phase space U_0 at $\sigma = 0$, reflecting that 'there is locally a copy of $\mathfrak{sl}(2,\mathbb{R})$ within the Poisson algebra': locally around τ_0 we may embed U_0 in $\mathbb{R}^3 \cong \mathfrak{sl}(2,\mathbb{R})$. Denoting the reduced Hamiltonian by $\mathcal{H}^{\nu,\sigma}$, the level set $L^{\nu_0} = \{\, \tau \in \mathbb{R}^3 \mid \mathcal{H}^{\nu_0,0}(\tau) = \mathcal{H}^{\nu_0,0}(\tau_0) \,\}$ enters the singularity with the same slope as U_0 at some direction of the cone. This direction identifies the (adjoint M of the) nilpotent part N of the linearization. Thus, the only information that cannot be read off from the reduced one–degree–of–freedom system is the coefficient $\Omega(\nu_0)$ of S . In particular, the bifurcation value ν_0 is determined by L^{ν_0} being tangent to the cone U_0 .

Lemma 4.2 *The family H^ν satisfies condition (HH2) at ν_0 if and only if L^ν enters the cone as ν passes through ν_0 with a nonzero rate of change of the angle between L^ν and U_0 .*

This rate of change would vanish if *e.g.* $a_0(\nu) = \nu^3$.

Lemma 4.3 *The family H^ν satisfies condition (HH3) at ν_0 if and only if L^{ν_0} has second order contact with U_0 in τ_0 .*

If this order of contact is of higher order, the bifurcation is degenerate, cf. [18].

Proposition 4.4 *A family H^ν that has a linear Hamiltonian Hopf bifurcation undergoes a Hamiltonian Hopf bifurcation (of co–dimension one) if and only if, locally at τ_0 , the family L^ν of level sets is in general position with respect to the reduced phase space U_0 . In case L^{ν_0} remains outside the cone, the bifurcation is supercritical. If $L^{\nu_0} \cap U_0$ contains other points near τ_0 a subcritical Hamiltonian Hopf bifurcation takes place.*

The bifurcation is robust with respect to S^1–symmetric perturbations, and even with respect to all Hamiltonian perturbations if $\Omega(\nu_0) \neq 0$, cf. [16, 17].

5 Transversality conditions, fourth order analysis

Now we apply the criterion derived in the previous section to the normal form (3). This first nontrivial normal form is not unanimously named in the literature. It is called the 'normal form of order two' in [11] since (1) has been normalized twice. Following [3] one would call it the 'fourth order normal form' since, in the original variables $x_1, \ldots, y_3$, it is a 4th–degree polynomial.

We carry out our analysis on the twice reduced phase space, *i.e.* in one degree of freedom. The reduction from $\mathbb{CP}^2$ to $V_{\sigma,\eta}$ is performed by fixing the values η of $H_0 = \frac{1}{2}(\pi_1 + \pi_2 + \pi_3)$ and σ of $S = \pi_7$ and expressing everything in the invariants

$$\begin{aligned} \operatorname{Re}\tau_1 &= (\pi_5^2 + \pi_6^2) - (\pi_8^2 + \pi_9^2) \\ \operatorname{Im}\tau_1 &= -2(\pi_8\pi_5 + \pi_9\pi_6) \\ \tau_2 &= \pi_1 + \pi_2 \ . \end{aligned}$$

We follow [6, 11] and rescale these variables according to

$$\tau_1 \mapsto 4\eta^2 \tau_1 \ , \quad \tau_2 \mapsto 2\eta \, \tau_2 \tag{8}$$

whence the *distinguished parameters* σ and η only enter through the quotient $\mu = \frac{\sigma}{\eta}$. The twice reduced phase space becomes in this way

$$V_\mu \;=\; \left\{ \tau = (\tau_1, \tau_2) \in \mathbb{C} \times \mathbb{R} \;\middle|\; R_\mu(\tau) = 0 \, , \; |\mu| \leq \tau_2 \leq 1 \right\} \ ,$$

with

$$R_\mu(\tau) \;=\; (\operatorname{Re}\tau_1)^2 + (\operatorname{Im}\tau_1)^2 \;-\; (1-\tau_2)^2(\tau_2^2 - \mu^2)$$

and has the Poisson bracket

$$\{ f , g \} \quad = \quad (\nabla f \times \nabla g \mid \nabla R_\mu) \ .$$

Furthermore the (twice) reduced Hamiltonian is obtained from (3). Modulo constant terms this gives

$$\mathcal{H}^\lambda(\tau) \quad = \quad 2\lambda(1-6\lambda)\mathrm{Re}\tau_1 \ + \ 2(5-6\lambda-4\lambda^2)\tau_2 \ - \ (5-12\lambda-3\lambda^2)\tau_2^2 \ . \tag{9}$$

The (linear) Hamiltonian Hopf bifurcations at $\lambda = -1, \frac{1}{2}, 1, \frac{5}{2}$ are related to the conical singularity of V_0 at $\tau = 0$; locally around this point the Poisson algebra contains a copy of $\mathfrak{sl}(2,\mathbb{R})$. From (4) we know that for $\lambda = -1$ and $\lambda = \frac{1}{2}$ the local rôle of N is played by $\pi_8^2 + \pi_9^2 \sim \tau_2 - \mathrm{Re}\,\tau_1$, with adjoint $\pi_5^2 + \pi_6^2 \sim \tau_2 + \mathrm{Re}\,\tau_1$ and $P = \{N, M\}$ corresponding to $\pi_5\pi_8 + \pi_6\pi_9 = -\frac{1}{2}\mathrm{Im}\,\tau_1$. In order to study the transversality conditions we consider the tangency of V_0 and $\mathcal{H}^\lambda$ in the $(\mathrm{Im}\,\tau_1, \tau_2, \mathrm{Re}\,\tau_1)$–space, where we may restrict to the plane $\mathrm{Im}\,\tau_1 = 0$. When $\lambda = -1, \frac{1}{2}$ the parabola

$$\{\mathcal{H}^\lambda(\tau) = h\} \cap \{\mathrm{Im}\,\tau_1 = 0\} \quad : \quad \mathrm{Re}\tau_1 \quad = \quad p_\lambda(\tau_2) \tag{10}$$

enters the origin with the same slope $+1$ as the upper arc of the contour

$$V_{\mu=0} \cap \{\mathrm{Im}\,\tau_1 = 0\} \quad : \quad \mathrm{Re}\tau_1 \quad = \quad \pm q(\tau_2) \ . \tag{11}$$

The two right hand sides are given by

$$p_\lambda(\tau_2) \quad = \quad \frac{5-12\lambda-3\lambda^2}{2\lambda(1-6\lambda)}\tau_2^2 \ - \ \frac{5-6\lambda-4\lambda^2}{\lambda(1-6\lambda)}\tau_2 \ + \ \frac{h}{2\lambda(1-6\lambda)}$$

and

$$q(\tau_2) \quad = \quad \tau_2(1-\tau_2) \ ,$$

respectively. The transversality condition (HH2) is met since

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}p'_\lambda(0) \quad = \quad \frac{5(1-12\lambda+8\lambda^2)}{\lambda^2(1-6\lambda)^2} \tag{12}$$

does not vanish at $\lambda = -1, \frac{1}{2}$, indeed

$$\left.\frac{\mathrm{d}}{\mathrm{d}\lambda}p'_\lambda(0)\right|_{\lambda=-1} \quad = \quad \frac{15}{7} \quad \text{and} \quad \left.\frac{\mathrm{d}}{\mathrm{d}\lambda}p'_\lambda(0)\right|_{\lambda=\frac{1}{2}} \quad = \quad -15 \ .$$

For the transversality condition (HH3) we have to check that

$$u_\lambda(\tau_2) \quad = \quad p_\lambda(\tau_2) \ - \ q(\tau_2)$$

has a nonzero second derivative at $\tau_2 = 0$. Since

$$u''_\lambda(\tau_2) \;\equiv\; \frac{5 - 12\lambda - 3\lambda^2}{\lambda(1-6\lambda)} \;+\; 2$$

we obtain

$$u''_{-1}(0) \;=\; 0 \quad \text{and} \quad u''_{\frac{1}{2}}(0) \;=\; \frac{15}{4} \;>\; 0 \;.$$

Thus, we find at $\lambda = -1$ a degenerate Hamiltonian Hopf bifurcation in the normal form (3) of (1). In Section 6 we will see that a higher order normal form of (1) breaks this degeneracy.

Theorem 5.1 *At* $\lambda = \frac{1}{2}$ *the 3D Hénon–Heiles family* (1) *undergoes a supercritical Hamiltonian Hopf bifurcation, provided that* ε *is sufficiently small.*

Proof : At $\lambda = \frac{1}{2}$ the parabola (10) through the origin stays outside the contour (11), see *figure 5.1(a)*, in particular outside the local cone. This establishes the above result for the normal form (3), implying that (1) has to undergo a supercritical Hamiltonian Hopf bifurcation at some λ_* close to $\frac{1}{2}$.

In three degrees of freedom the singularity $\tau = 0$ corresponds to vertical oscillations about the origin, and as λ passes through λ_* a family of invariant normally elliptic 2–tori detaches from the origin. For $\sigma = 0$ these 2–tori are in (internal) 1:0–resonance, *i.e.* they consist of periodic orbits. As shown in [11], a family of non–vertical rectilinear periodic orbits bifurcates off from the vertical oscillations as λ passes through the value $\frac{1}{2}$. Thus, we have indeed $\lambda_* = \frac{1}{2}$; **q.e.d.**

We now turn to $\lambda = 1, \frac{5}{2}$. From (5) we know that the local rôles of N and M are interchanged here, *i.e.* the parabola (10) enters the origin with the same slope -1 as the lower arc of the contour (11). The transversality condition (HH2) is again met since the evaluation of (12) yields

$$\left.\frac{\mathrm{d}}{\mathrm{d}\lambda} p'_\lambda(0)\right|_{\lambda=1} \;=\; -\frac{5}{3} \quad \text{and} \quad \left.\frac{\mathrm{d}}{\mathrm{d}\lambda} p'_\lambda(0)\right|_{\lambda=\frac{5}{2}} \;=\; \frac{3}{35} \;.$$

For the transversality condition (HH3) we now have to check the second derivative of

$$l_\lambda(\tau_2) \;=\; p_\lambda(\tau_2) \;+\; q(\tau_2)$$

at $\tau_2 = 0$ which is

$$l''_\lambda(0) \;=\; \frac{5 - 12\lambda - 3\lambda^2}{\lambda(1-6\lambda)} \;-\; 2 \;.$$

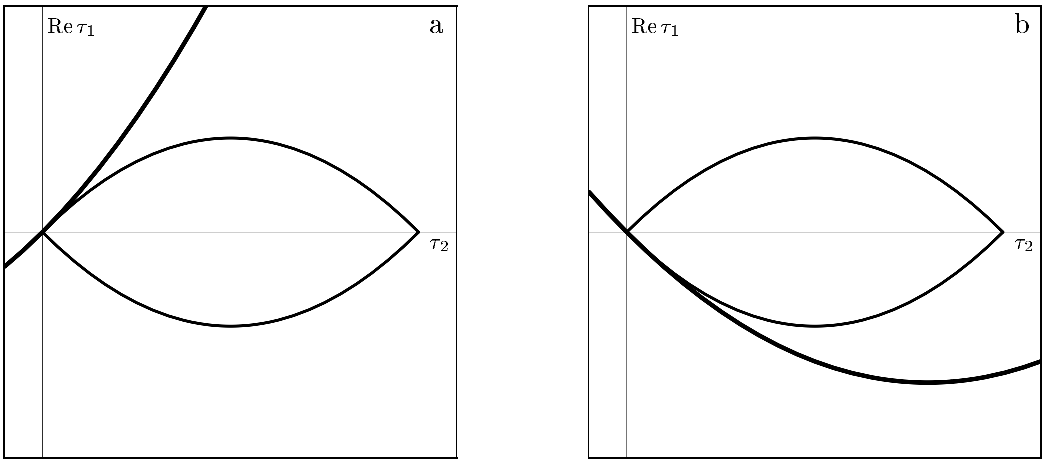

Figure 5.1: Relative position within $\operatorname{Im}\tau_1 = 0$ of the energy level (10) to $h = 0$ and the contour (11) of the twice reduced phase space V_0 at $\lambda = \frac{1}{2}$ and at $\lambda = \frac{5}{2}$.

Thus, we obtain

$$l''_{-1}(0) \;=\; 0 \quad \text{and} \quad l''_{\frac{5}{2}}(0) \;=\; -\frac{3}{4}$$

whence the bifurcation at $\lambda = -1$ is degenerate. This degeneracy is structural and persists through all higher order normal forms of (1), see [11]. It is due to the fact that for this value of λ the $3D$ Hénon–Heiles family (1) is integrable.

At $\lambda = \frac{5}{2}$ the parabola (10) through the origin stays outside the contour (11), see *figure 5.1(b)*, in particular outside the local cone. This proves that (3) undergoes a supercritical Hamiltonian Hopf bifurcation, whence (1) has to undergo a supercritical Hamiltonian Hopf bifurcation at some λ^* close to $\frac{5}{2}$. The results in [3], repeated in the next section, suggest that in fact $\lambda^* = \frac{5}{2}$.

6 Transversality conditions, sixth order analysis

One further normalization does not alter the normal form (3) since all normalized terms of odd degree vanish. The second nontrivial normal form is thus obtained through normalizing four times and contains monomials up to degree six. Again we rescale according to (8), omit the constant terms

$$\eta \;-\; \frac{\varepsilon^2\eta^2}{12}(5+\lambda^2\mu^2) \;-\; \frac{\varepsilon^4\eta^3}{432}\big(235+\lambda^2(43-372\lambda+369\lambda^2)\mu^2\big) \tag{13}$$

and divide the Hamiltonian by $\frac{\varepsilon^2\eta^2}{12}$. The normal form then reads

$$\mathcal{K}_\mu^{\lambda,\delta}(\tau) \;=\; \mathcal{H}^\lambda(\tau) \;+\; \frac{\varepsilon^2\eta}{36} K_\mu^\lambda(\tau) \tag{14}$$

with

$$\begin{aligned} K_\mu^\lambda(\tau) \;=\;& 2\lambda(41-42\lambda-656\lambda^2-48\lambda^3)\,\mathrm{Re}\,\tau_1 \\ &-\; 2\lambda(1-\lambda)(41-61\lambda-606\lambda^2)\,\tau_2\,\mathrm{Re}\,\tau_1 \\ &+\; (705-720\lambda-251\lambda^2-972\lambda^3-172\lambda^4)\,\tau_2 \\ &-\; (705-1440\lambda+189\lambda^2-1764\lambda^3+900\lambda^4)\,\tau_2^2 \\ &+\; 5(1-\lambda)(47-97\lambda-9\lambda^2-201\lambda^3)\,\tau_2^3 \\ &-\; \lambda^2(43-372\lambda+369\lambda^2)\,\mu^2\,\tau_2 \end{aligned}$$

and $\mathcal{H}^\lambda(\tau)$ given in (9). Note that the normal form now does depend on $\mu = \frac{\sigma}{\eta}$, and η also enters through the small parameter $\delta = \varepsilon^2\eta$.

Adding higher order terms in the normal form will at all orders of the once reduced system introduce new terms, even the linear system will change. That is we have to repeat the complete analysis of both Sections 3 and 5. Since σ only enters squared in (13) the coefficient of the semi–simple part still vanishes at the bifurcation. Everything else can be decided in one degree of freedom.

To obtain the intersection of the energy level set $\{\mathcal{K}_{\mu=0}^{\lambda,\delta}(\tau) = h\}$ with the co–ordinate plane $\{\mathrm{Im}\,\tau_1 = 0\}$ the parabola p_λ in (10) has to be replaced by the function

$$f_{\lambda,\delta}(\tau_2) \;=\; \frac{g_{\lambda,\delta}(\tau_2)}{d_{\lambda,\delta}(\tau_2)}$$

with

$$\begin{aligned} g_{\lambda,\delta}(\tau_2) \;=\;& h \;-\; 2(5-6\lambda-4\lambda^2)\tau_2 \;+\; (5-12\lambda-3\lambda^2)\tau_2^2 \\ &-\; \tfrac{\delta}{36}[\,(705-720\lambda-251\lambda^2-972\lambda^3-172\lambda^4)\,\tau_2 \\ &\qquad -(705-1440\lambda+189\lambda^2-1764\lambda^3+900\lambda^4)\,\tau_2^2 \\ &\qquad +5(1-\lambda)(47-97\lambda-9\lambda^2-201\lambda^3)\,\tau_2^3\,] \end{aligned}$$

and

$$\begin{aligned} d_{\lambda,\delta}(\tau_2) \;=\;& 2\lambda(1-6\lambda) \;+\; \tfrac{\delta}{36}[\,2\lambda(41-42\lambda-656\lambda^2-48\lambda^3) \\ &\qquad -\; 2\lambda(1-\lambda)(41-61\lambda-606\lambda^2)\,\tau_2\,] \quad . \end{aligned}$$

For $\delta = 0$ we recover $p_\lambda = f_{\lambda,0}$. As before we consider the functions

$$l_{\lambda,\delta} \;=\; f_{\lambda,\delta}+q \quad \text{and} \quad u_{\lambda,\delta} \;=\; f_{\lambda,\delta}-q \quad .$$

Putting

$$l'_{\lambda,\delta}(0) = 0$$

yields an equation in λ , the solutions of which are the parameter values where $f_{\lambda,\delta}$ enters the cone with the same slope -1 as the lower arc of the contour (11). Here the linearization of the 2–degrees–of–freedom system on $\mathbb{CP}^2$ is nilpotent. The sole real solutions of this equation are $\lambda = 1$ and $\lambda = \frac{5}{2}$. The former is no surprise, we already know from [11] that $f_{1,\delta} = -q$ (and that the same remains true for all higher order normal forms). In particular $l''_{1,\delta}(0) = 0$.

Theorem 6.1 *At* $\lambda = \frac{5}{2}$ *the normal form* (13)–(14) *of* (1) *undergoes a supercritical Hamiltonian Hopf bifurcation, provided that* δ *is sufficiently small.*

Proof : Above we established that a linear Hamiltonian Hopf bifurcation takes place. The transversality conditions (HH2&3) are already fulfilled by the normal form (9), and replacing p_λ by $f_{\lambda,\delta}$ only introduces additional terms of order δ , leading to

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} f'_{\lambda,\delta}(0)\bigg|_{\lambda=\frac{5}{2}} \;=\; \frac{2}{105}\left(\frac{18-\delta}{4+39\,\delta}\right) \quad \text{and} \quad l''_{\frac{5}{2},\delta}(0) \;=\; -\frac{1}{6}\left(\frac{18+23\,\delta}{4+39\,\delta}\right) \quad ,$$

whence the transversality conditions (HH2&3) are met by the normal form (13)–(14) as well **;** **q.e.d.**

We now turn to tangencies in the cone along the upper arc of (11), which happen when λ satisfies

$$u'_{\lambda,\delta}(0) = 0 \quad . \tag{15}$$

From Theorem 5.1 we already know that $\lambda = \frac{1}{2}$ is a solution of (15) since the supercritical Hamiltonian Hopf bifurcation of (1) imposes that same bifurcation in all normal forms. We are interested in the second solution

$$\lambda \;=\; \lambda(\delta) \tag{16}$$

of (15), which is an unwieldy expression that satisfies $\lim\limits_{\delta\to 0}\lambda(\delta) = -1$. Thus, the higher order terms in the normal form are moving the linear Hamiltonian Hopf bifurcation slightly away from $\lambda = -1$.

Again the transversality condition (HH2) is met simply because this is already true for the normal form (9). For (HH3) we have to study $u''_{\lambda(\delta),\delta}(0)$, which is a rather complicated (though algorithmically computable) expression. Since we are only interested in its sign we may use the approximation

$$u''_{\lambda(\delta),\delta}(0) \;=\; u''_{-1,\delta}(0) \;+\; u''_{\lambda(\delta),0}(0) \;+\; \mathcal{O}(\delta^2) \quad .$$

Indeed, it is straightforward to compute $u''_{-1,\delta}(0) = -24\delta + \mathcal{O}(\delta^2)$ and $u''_{\lambda(\delta),0} = -\frac{20}{7}\frac{\mathrm{d}\lambda}{\mathrm{d}\delta}\delta + \mathcal{O}(\delta^2)$. The numerical estimate $\frac{\mathrm{d}\lambda}{\mathrm{d}\delta}(0) \approx -2.8$ then allows to conclude that

$$u''_{\lambda(\delta),\delta}(0) \;\approx\; -16\,\delta \;+\; \mathcal{O}(\delta^2)$$

is negative for $\delta > 0$ sufficiently small. Let ζ denote the action conjugate to the angle along the bifurcating periodic orbit of (1).

Theorem 6.2 *At $\lambda_* = \lambda_*(\varepsilon, \zeta)$ close to -1 the 3D Hénon–Heiles family (1) undergoes a subcritical Hamiltonian Hopf bifurcation, provided that ε and ζ are sufficiently small.*

Proof : The energy level of the normal form (14) has locally a nontrivial intersection with the reduced phase space V_0 , whence the Hamiltonian Hopf bifurcation at $\lambda = \lambda(\delta)$ in the normal form is subcritical. This implies that (1) undergoes a subcritical Hamiltonian Hopf bifurcation as well, at some value λ_* which is $\mathcal{O}(\delta^2)$–close to $\lambda(\delta)$. However, the parameter η in $\delta = \varepsilon^2 \eta$ is the value of H_0 and thus a parameter only for the normalized system. Since the action ζ is $\mathcal{O}(\varepsilon^2)$–close to η we obtain

$$\lambda_*(\varepsilon, \zeta) \quad = \quad \lambda(\varepsilon^2 \eta) \ + \ \mathcal{O}(\varepsilon^4, \zeta^2)$$

for the value λ_* where the bifurcation takes place, in particular $\lim\limits_{\varepsilon \to 0} \lambda_*(\varepsilon, \zeta) = -1$; **q.e.d.**

The external parameters ε and λ determine the form of the perturbing cubic potential in (1). The action ζ depends upon the initial conditions and is therefore an internal parameter, distinguished with respect to ε, λ . Rather than fixing ε and ζ to vary λ through $\lambda_*(\varepsilon, \zeta)$, it is much more natural to choose ε, λ and to look for a Hamiltonian Hopf bifurcation within the system (1) singled out by that choice. The action ζ then has to play the rôle of the unfolding parameter λ , which is possible since $\frac{\mathrm{d}\lambda_*}{\mathrm{d}\zeta} = \varepsilon^2 \frac{\mathrm{d}\lambda}{\mathrm{d}\delta} + \mathcal{O}(\varepsilon^4)$ does not vanish.

The situation is similar to the Lagrange top. The swallowtail surface of the Hamiltonian Hopf bifurcation is visible in the image of the energy–momentum mapping because the action ζ features as the bifurcation parameter.

Corollary 6.3 *Choose $\varepsilon > 0$ sufficiently small and then choose $\lambda < -1$ sufficiently close to -1 . Then the vertical oscillations of the system (1) undergo a subcritical Hamiltonian Hopf bifurcation as the action passes through a certain value $\zeta_* = \zeta_*(\varepsilon, \lambda)$.*

The supercritical Hamiltonian Hopf bifurcation at $\lambda = \frac{1}{2}$ cannot be re–interpreted to occur with fixed ε, λ within a single system (1). Indeed, all vertical oscillations are elliptic when $\lambda < \frac{1}{2}$ and hyperbolic when $\lambda > \frac{1}{2}$.

Here the bifurcation can be considered as a topological free product of an S^1 with a supercritical Hamiltonian Hopf bifurcation of equilibria. One has at one side of the critical value two families of 2–tori emanating from the circle $\pi_3 = 2\zeta$. When λ passes through the critical value these two families form one smooth family of 2–tori that detaches from this circle.

In the neighbourhood of the critical circle one finds, at least in the integrable approximation, a phase space filled with $\mathbb{T}^3$, *i.e.* a 3–torus bundle. For $\lambda \in]\frac{1}{2}, 1[\cup]\frac{5}{2}, \infty[\cup]-\infty, -1[$ monodromy can be obtained by considering the phase shift in the basic frequencies on the 3–tori when following a closed curve around the line of critical circles in the image of the energy–momentum mapping. It is obvious that only two of the three frequencies will play a rôle, the η frequency will play no rôle. Consequently one finds the monodromy matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

as already mentioned in [11]. In particular, the vanishing coefficient in front of the semi–simple part has no influence on the monodromy generated through the Hamiltonian Hopf bifurcation.

7 Conclusions

We have shown that the presence of Hamiltonian Hopf bifurcations can be detected from a careful geometric analysis involving the normalized Hamiltonian and the reduced phase space. Actually the finally obtained one–degree–of–freedom system is sufficient to derive this information. Candidate points are cone–like singularities of reduced phase spaces with isotropy S^1 . The parameters should allow to bring the Hamiltonian in general position.

In this way we obtained a complete treatment of the Hamiltonian Hopf bifurcations at $\lambda = \frac{1}{2}$ and near $\lambda = -1$, while it remains an open question whether the third Hamiltonian Hopf bifurcation in the $3D$ Hénon–Heiles family (1) occurs *at* $\lambda = \frac{5}{2}$ or *near* $\lambda = \frac{5}{2}$. Adding higher order terms in the normal form of the full system changes the parameter dependence at all orders of the normal form. In particular, it is conceivable that a high order term in the normal form, or even a flat term beyond any normalization, may push the bifurcation value λ^* away from $\frac{5}{2}$.

However, we do can rule out that some higher order term $\eta^k \sigma$ introduces a nonzero coefficient Ω of the semi–simple part of the linearization about the vertical oscillations. Indeed, the $3D$ Hénon–Heiles family (1) is not only $SO(2)$–symmetric with respect to axial rotations, but in fact $O(2)$–symmetric because of reversibility. Thus, σ can only enter squared, with no dynamical consequences at $\sigma = 0$. The remaining question concerning λ^* does no longer prevent us from reducing the axial symmetry and in fact boils down to whether the (periodic) Hamiltonian pitchfork bifurcation in the $2D$ Hénon–Heiles family occurs at $\lambda = \frac{5}{2}$ or near $\lambda = \frac{5}{2}$, *i.e.* where the Floquet multiplier assumes the value $+1$.

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