A CLASS OF MATRIX-VALUED SCHRÖDINGER OPERATORS WITH PRESCRIBED FINITE-BAND SPECTRA

FRITZ GESZTESY AND LEV A. SAKHNOVICH

ABSTRACT. We construct a class of matrix-valued Schrödinger operators with prescribed finite-band spectra of maximum spectral multiplicity. The corresponding matrix potentials are shown to be stationary solutions of the KdV hierarchy. The methods employed in this paper rely on matrix-valued Herglotz functions, Weyl–Titchmarsh theory, pencils of matrices, and basic inverse spectral theory for matrix-valued Schrödinger operators.

1. INTRODUCTION

While basic aspects of inverse spectral theory for matrix-valued Schrödinger operators were established some time ago, finer properties such as isospectral sets (manifolds) of potentials, for instance, in the periodic or algebro-geometric finite-band cases, are still in their infancy. This paper intends to make a modest contribution to this circle of ideas. More precisely, given a closed set $\Sigma \subset \mathbb{R}$ of the type

$$\Sigma = \left\{ \bigcup_{j=0}^{n-1} [E_{2j}, E_{2j+1}] \right\} \cup [E_{2n}, \infty), \tag{1.1}$$

where

$$\{E_\ell\}_{0 \leq \ell \leq 2n} \subseteq \mathbb{R}, \ n \in \mathbb{N}, \ \text{with } E_\ell < E_{\ell+1}, \ 0 \leq \ell \leq 2n-1, \tag{1.2}$$

we construct $m \times m$ matrix-valued Schrödinger operators $H_\Sigma = -d^2/dx^2 \mathcal{I}_m + \mathcal{Q}_\Sigma$ in $L^2(\mathbb{R})^{m \times m}$ ($\mathcal{I}_m$ the identity matrix in $\mathbb{C}^{m \times m}$, $m \in \mathbb{N}$) with prescribed spectrum Σ,

$$\operatorname{spec}(H_\Sigma) = \Sigma, \tag{1.3}$$

of uniform spectral multiplicity $2m$. The constructed matrix potentials $\mathcal{Q}_\Sigma$ will turn out to be reflectionless in the sense discussed in [23], [40], and [56], that is, the half-line Weyl–Titchmarsh matrices $\mathcal{M}_{\pm,\Sigma}(z,x)$ associated with H_Σ, the half-lines $[x, \pm\infty)$, and a Dirichlet boundary condition at $x \in \mathbb{R}$, satisfy

$$\lim_{\varepsilon \downarrow 0} \mathcal{M}_{+,\Sigma}(\lambda + i\varepsilon, x) = \lim_{\varepsilon \downarrow 0} \mathcal{M}_{-,\Sigma}(\lambda - i\varepsilon, x), \tag{1.4}$$
$$\lambda \in \bigcup_{j=0}^{n-1} (E_{2j}, E_{2j+1}) \cup (E_{2n}, \infty), \ x \in \mathbb{R}.$$

Date: December 14, 2001.

Key words and phrases. Matrix-valued Schrödinger operators, finite-band spectra, Weyl–Titchmarsh matrices.

Especially, $\mathcal{M}_{+,\Sigma}(\cdot,x)$ is the analytic continuation of $\mathcal{M}_{-,\Sigma}(\cdot,x)$ through the set Σ, and vice versa. In other words, $\mathcal{M}_{+,\Sigma}(\cdot,x)$ and $\mathcal{M}_{-,\Sigma}(\cdot,x)$ are the two branches of an analytic matrix-valued function $\mathcal{M}_\Sigma(\cdot,x)$ on the two-sheeted Riemann surface of $\big(\prod_{\ell=0}^{2n}(z-E_\ell)\big)^{1/2}$. These facts imply a purely absolutely continuous spectrum Σ of the associated Schrödinger operator H_Σ of uniform (maximal) multiplicity $2m$.

Before we turn to a brief description of the contents of each section, it seems appropriate to mention some of the pertinent results and especially, the most recent activities in connection with (inverse) spectral theory of matrix-valued Schrödinger operators. The basic Weyl–Titchmarsh theory of singular Hamiltonian systems and their basic spectral theory were developed by Hinton and Shaw, Kogan and Rofe-Beketov, Orlov, and others (see, e.g., [43], [45]–[49], [54], [55], [57], [58], [81], [91], [92], [94], [97], [101, Ch. 9], and the references therein). Various aspects of direct spectral theory, including investigations of the nature of the spectrum involved, (regularized) trace formulas, uniqueness theorems, etc., appeared in [10], [12]–[15], [20], [23], [24], [35], [37], [39], [60], [82], [89], [90]. General asymptotic expansions of Weyl–Titchmarsh matrices as the (complex) spectral parameter tends to infinity under optimal regularity assumptions on the coefficients are of relatively recent origin and can be found in [20], [21] (see also [95], [107]). The inverse scattering formalism for Schrödinger operators has been studied by a variety of authors and we refer, for instance, to [1], [2], [3], [7], [75], [77], [78], [108]. General inverse spectral theory, the existence of transformation operators, etc., are discussed in [68], [69], [85], [94], [97], [98], [100], [101], and the references therein. Inverse monodromy problems have recently been discussed in [4], [5], [6], [15], [68], [69], [96], [101], and the literature cited therein. More specific inverse spectral problems, such as compactness of the isospectral set of periodic Schrödinger operators [13], special isospectral matrix-valued Schrödinger operators, and Borg-type uniqueness theorems (for periodic coefficients as well as eigenvalue problems on compact intervals) were recently studied in [17], [18], [19], [21], [23], [26], [51], [52], [68], [69], [101], [102], [103], [104]. Moreover, direct spectral theory in the particular case of periodic Schrödinger operators (i.e., Floquet theory and the like) has been studied in [12], [14], [23], [25], [26], [34], [53], [59], [60], [86], [102], [103], [110]–[112], with many more pertinent references to be found therein. Apart from Floquet theoretic applications in connection with Schrödinger operators already briefly touched upon, we also need to mention applications to random Schrödinger operators associated with strips as discussed, for instance, in [16], [54], [56], and especially to nonabelian completely integrable systems. Since the literature associated with the latter topic is of enormous proportions, we can only refer to a few pertinent publications, such as, [7], [8], [11], [27], [30], [70], [71], [75], [79], [80], [83], [93], [95]–[97], [99]. The interested reader will find a wealth of additional material in these references.

Section 2 summarizes basic results in Weyl–Titchmarsh theory and some elements of inverse spectral theory for matrix-valued Schrödinger operators. Polynomial pencils of matrices are briefly reviewed in Section 3. In Section 4 we present our principal new result, the construction of $m \times m$ matrix-valued Schrödinger operators H_Σ with spectrum Σ (cf. (1.1)) and uniform maximal spectral multiplicity $2m$. In our final Section 5 we prove that $\mathcal{Q}_\Sigma$ satisfies a stationary KdV equation (in fact, we explicitly identify the first equation in the stationary KdV hierarchy satisfied by $\mathcal{Q}_\Sigma$) and derive matrix-valued trace formulas for $\mathcal{Q}_\Sigma$ and higher-order KdV invariants.

2. Basic Facts on Weyl–Titchmarsh Theory

In this section we briefly recall basic elements of the Weyl–Titchmarsh theory for matrix-valued Schrödinger operators. Throughout this paper all matrices will be considered over the field of complex numbers $\mathbb{C}$, and the corresponding linear space of $k \times \ell$ matrices will be denoted by $\mathbb{C}^{k\times\ell}$, $k, \ell \in \mathbb{N}$. Moreover, $\mathcal{I}_k$ denotes the identity matrix in $\mathbb{C}^{k\times k}$, $\mathcal{M}^*$ the adjoint (i.e., complex conjugate transpose), $\mathcal{M}^t$ the transpose of a matrix $\mathcal{M}$, $\operatorname{diag}(m_1, \dots, m_k) \in \mathbb{C}^{k\times k}$ a diagonal $k \times k$ matrix, and $\mathrm{AC}_{\mathrm{loc}}(\mathbb{R})$ denotes the set of locally absolutely continuous functions on $\mathbb{R}$. The spectrum, point spectrum (the set of eigenvalues), essential spectrum, absolutely continuous spectrum, and singularly continuous spectrum of a self-adjoint linear operator T in a separable complex Hilbert space are denoted by $\operatorname{spec}(T)$, $\operatorname{spec}_{\mathrm{p}}(T)$, $\operatorname{spec}_{\mathrm{ess}}(T)$, $\operatorname{spec}_{\mathrm{ac}}(T)$, $\operatorname{spec}_{\mathrm{sc}}(T)$, respectively.

The basic assumption for this section will be the following.

Hypothesis 2.1.
(i) Fix $m \in \mathbb{N}$, suppose $\mathcal{Q} = \mathcal{Q}^* \in L^1_{\mathrm{loc}}(\mathbb{R})^{m\times m}$ and introduce the differential expression

$$\mathcal{L} = -\mathcal{I}_m \frac{d^2}{dx^2} + \mathcal{Q}, \quad x \in \mathbb{R}. \tag{2.1}$$

(ii) Suppose $\mathcal{L}$ is in the limit point case at $\pm\infty$.

Given Hypothesis 2.1 (i) we consider the matrix-valued Schrödinger equation

$$-\psi''(z,x) + \mathcal{Q}(x)\psi(z,x) = z\psi(z,x) \text{ for a.e. } x \in \mathbb{R}, \tag{2.2}$$

where $z \in \mathbb{C}$ plays the role of a spectral parameter and ψ is assumed to satisfy

$$\psi(z,\cdot), \psi'(z,\cdot) \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R})^{m\times m}. \tag{2.3}$$

Throughout this paper, x-derivatives are abbreviated by a prime $'$.

Let $\Psi(z,x,x_0)$ be a $2m \times 2m$ normalized fundamental system of solutions of (2.2) at some $x_0 \in \mathbb{R}$ which we partition as

$$\Psi(z,x,x_0) = \begin{pmatrix} \theta(z,x,x_0) & \phi(z,x,x_0) \\ \theta'(z,x,x_0) & \phi'(z,x,x_0) \end{pmatrix}. \tag{2.4}$$

Here $'$ denotes d/dx, $\theta(z,x,x_0)$ and $\phi(z,x,x_0)$ are $m \times m$ matrices, entire with respect to $z \in \mathbb{C}$, and normalized according to

$$\Psi(z,x_0,x_0) = \mathcal{I}_{2m}, \tag{2.5}$$

that is,

$$\theta(z,x_0,x_0) = \phi'(z,x_0,x_0) = \mathcal{I}_m, \quad \phi(z,x_0,x_0) = \theta'(z,x_0,x_0) = 0. \tag{2.6}$$

In this context, we briefly recall a set of formulas needed later in Section 4. Introducing

$$\mathcal{J} = \begin{pmatrix} 0 & -\mathcal{I}_m \\ \mathcal{I}_m & 0 \end{pmatrix}, \tag{2.7}$$

one infers

$$\Psi(\bar z,x,x_0)^* \mathcal{J} \Psi(z,x,x_0) = \mathcal{J}, \tag{2.8}$$

which implies $\mathcal{J}\Psi(z,x,x_0)(\Psi(\bar z,x,x_0)\mathcal{J})^* = \mathcal{I}_{2m}$ and hence

$$\Psi(z,x,x_0)\mathcal{J}\Psi(\bar z,x,x_0)^* = \mathcal{J}. \tag{2.9}$$

Writing out (2.8) and (2.9) explicitly yields

$$\theta'(\bar{z},x,x_0)^*\theta(z,x,x_0)-\theta(\bar{z},x,x_0)^*\theta'(z,x,x_0)=0, \tag{2.10}$$

$$\phi'(\bar{z},x,x_0)^*\phi(z,x,x_0)-\phi(\bar{z},x,x_0)^*\phi'(z,x,x_0)=0, \tag{2.11}$$

$$\phi'(\bar{z},x,x_0)^*\theta(z,x,x_0)-\phi(\bar{z},x,x_0)^*\theta'(z,x,x_0)=\mathcal{I}_m, \tag{2.12}$$

$$\theta(\bar{z},x,x_0)^*\phi'(z,x,x_0)-\theta'(\bar{z},x,x_0)^*\phi(z,x,x_0)=\mathcal{I}_m, \tag{2.13}$$

and

$$\phi(z,x,x_0)\theta(\bar{z},x,x_0)^*-\theta(z,x,x_0)\phi(\bar{z},x,x_0)^*=0, \tag{2.14}$$

$$\phi'(z,x,x_0)\theta'(\bar{z},x,x_0)^*-\theta'(z,x,x_0)\phi'(\bar{z},x,x_0)^*=0, \tag{2.15}$$

$$\phi'(z,x,x_0)\theta(\bar{z},x,x_0)^*-\theta'(z,x,x_0)\phi(\bar{z},x,x_0)^*=\mathcal{I}_m, \tag{2.16}$$

$$\theta(z,x,x_0,\alpha)\phi'(\bar{z},x,x_0)^*-\phi(z,x,x_0,\alpha)\theta'(\bar{z},x,x_0)^*=\mathcal{I}_m. \tag{2.17}$$

Next, assuming $-\infty\le a<b\le\infty$, we consider the spaces

$$N(z,\pm\infty)=\{\phi\in L^2((c,\pm\infty))^m \mid -\phi''+\mathcal{Q}\phi=z\phi \text{ a.e. on } (c,\pm\infty)\}, \tag{2.18}$$

for some $c\in\mathbb{R}$ and $z\in\mathbb{C}$. (Here $(\phi,\psi)_{\mathbb{C}^n}=\sum_{j=1}^n\overline{\phi}_j\psi_j$ denotes the standard scalar product in $\mathbb{C}^n$, abbreviating $\chi\in\mathbb{C}^n$ by $\chi=(\chi_1,\cdots,\chi_n)^t$.) Both dimensions of the spaces in (2.18), $\dim_{\mathbb{C}}(N(z,\infty))$ and $\dim_{\mathbb{C}}(N(z,-\infty))$, are constant for $z\in\mathbb{C}_\pm=\{\zeta\in\mathbb{C}\mid\pm\mathrm{Im}(\zeta)>0\}$ (see, e.g., [55]). One then recalls that $\mathcal{L}$ in (2.1) is in the limit point case at $\pm\infty$ whenever

$$\dim_{\mathbb{C}}(N(z,\pm\infty))=m \text{ for all } z\in\mathbb{C}\backslash\mathbb{R}. \tag{2.19}$$

Since the potential $\mathcal{Q}_\Sigma$ to be constructed in Section 4 will automatically lead to the limit point case at $\pm\infty$, we decided to limit our considerations mainly to this situation. In this context we note the well-known fact that if $\mathcal{L}$ in (2.1) is in the limit point case at $\pm\infty$, then the $m\times m$ Weyl–Titchmarsh matrices associated with $\mathcal{L}$, the half-lines $[x,\pm\infty)$, and a Dirichlet boundary condition at x, are given by

$$\mathcal{M}_\pm(z,x_0)=\Psi'_\pm(z,x,x_0)\Psi_\pm(z,x,x_0)^{-1}\big|_{x=x_0},\quad z\in\mathbb{C}\backslash\mathbb{R}, \tag{2.20}$$

where $\Psi_\pm$ satisfy $(\mathcal{L}-z\mathcal{I}_m)\Psi_\pm(z,\cdot,x_0)=0$ and

$$\Psi_\pm(z,\cdot,x_0)\in L^2([x_0,\pm\infty))^{m\times m}. \tag{2.21}$$

The actual normalization of $\Psi_\pm(z,\cdot,x_0)$ is clearly irrelevant and hence $\Psi_\pm(z,\cdot,x_0)$ can be replaced by $\Psi_\pm(z,\cdot,x_0)C$, where C is any nonsingular $m\times m$ matrix.

For later reference we summarize the principal results on $\mathcal{M}_\pm(z,x_0)$ in the next theorem. First we recall the following definition.

Definition 2.2. *A map $\mathcal{M}\colon\mathbb{C}_+\to\mathbb{C}^{n\times n}$, $n\in\mathbb{N}$, extended to $\mathbb{C}_-$ by $\mathcal{M}(\bar{z})=\mathcal{M}(z)^*$ for all $z\in\mathbb{C}_+$, is called an $n\times n$ Herglotz matrix*[1] *if it is analytic on $\mathbb{C}_+$ and $\mathrm{Im}(\mathcal{M}(z))\ge 0$ for all $z\in\mathbb{C}_+$.*

Here we denote $\mathrm{Im}(\mathcal{M})=(\mathcal{M}-\mathcal{M}^*)/2i$ and $\mathrm{Re}(\mathcal{M})=(\mathcal{M}+\mathcal{M}^*)/2$.

In the scalar context $n=1$, the condition $\mathrm{Im}(\mathcal{M}(z))\ge 0$ in Definition 2.2 can be replaced by $\mathrm{Im}(m(z))>0$ for the corresponding scalar counterpart $m(z)$.

[1]There appears to be considerable confusion in the literature since Nevanlinna, Pick, Nevanlinna–Pick matrix, in addition to Herglotz matrix, are also in use. In part these discrepancies can be traced back to the use of the upper half-plane $\mathbb{C}_+$ versus the open unit disk D, and in some cases the geographical location of the author in question determines the preferred notation. Following a tradition in mathematical physics, we adopt the notion of Herglotz functions in this paper.

Theorem 2.3 ([21], [40], [45], [46], [49], [56]). *Assume Hypothesis* 2.1 *and suppose that* $z \in \mathbb{C}\backslash\mathbb{R}$, $x_0 \in \mathbb{R}$. *Then*
(*i*) $\pm\mathcal{M}_\pm(z,x_0)$ *is an* $m\times m$ *matrix-valued Herglotz function of maximal rank. In particular,*

$$\operatorname{Im}(\pm\mathcal{M}_\pm(z,x_0)) > 0, \quad z\in\mathbb{C}_+, \tag{2.22}$$

$$\mathcal{M}_\pm(\overline{z},x_0) = \mathcal{M}_\pm(z,x_0)^*, \tag{2.23}$$

$$\operatorname{rank}(\mathcal{M}_\pm(z,x_0)) = m, \tag{2.24}$$

$$\lim_{\varepsilon\downarrow 0}\mathcal{M}_\pm(\lambda+i\varepsilon,x_0) \text{ exists for a.e. } \lambda\in\mathbb{R}. \tag{2.25}$$

Local singularities of $\pm\mathcal{M}_\pm(z,x_0)$ *and* $\mp\mathcal{M}_\pm(z,x_0)^{-1}$ *are necessarily real and at most of first order in the sense that*

$$\mp\lim_{\epsilon\downarrow 0}(i\epsilon\,\mathcal{M}_\pm(\lambda+i\epsilon,x_0)) \geq 0, \quad \lambda\in\mathbb{R}, \tag{2.26}$$

$$\pm\lim_{\epsilon\downarrow 0} i\epsilon\mathcal{M}_\pm(\lambda+i\epsilon,x_0)^{-1} \geq 0, \quad \lambda\in\mathbb{R}. \tag{2.27}$$

(*ii*) $\pm\mathcal{M}_\pm(z,x_0)$ *admit the representations*

$$\pm\mathcal{M}_\pm(z,x_0) = \operatorname{Re}(\pm\mathcal{M}_\pm(\pm i,x_0)) + \int_{\mathbb{R}} d\Omega_\pm(\lambda,x_0)\left((\lambda-z)^{-1} - \lambda(1+\lambda^2)^{-1}\right), \tag{2.28}$$

where

$$\int_{\mathbb{R}} \|d\Omega_\pm(\lambda,x_0)\|_{\mathbb{C}^{m\times m}}\,(1+\lambda^2)^{-1} < \infty \tag{2.29}$$

and

$$\Omega_\pm((\lambda,\mu],x_0) = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_{\lambda+\delta}^{\mu+\delta} d\nu\,\operatorname{Im}(\pm\mathcal{M}_\pm(\nu+i\varepsilon,x_0)). \tag{2.30}$$

(*iii*) *Define the* $2m\times m$ *matrices*

$$\begin{aligned}\Psi_\pm(z,x,x_0) &= \begin{pmatrix}\psi_\pm(z,x,x_0)\\ \psi'_\pm(z,x,x_0)\end{pmatrix}\\ &= \begin{pmatrix}\theta(z,x,x_0) & \phi(z,x,x_0)\\ \theta'(z,x,x_0) & \phi'(z,x,x_0)\end{pmatrix}\begin{pmatrix}\mathcal{I}_m\\ \mathcal{M}_\pm(z,x_0)\end{pmatrix},\end{aligned} \tag{2.31}$$

then

$$\operatorname{Im}(\mathcal{M}_\pm(z,x_0)) = \operatorname{Im}(z)\int_{x_0}^{\pm\infty} dx\,\psi_\pm(z,x,x_0)^*\psi_\pm(z,x,x_0). \tag{2.32}$$

(*iv*) *Denote by* $C_\varepsilon\subset\mathbb{C}_+$ *the open sector with vertex at zero, symmetry axis along the positive imaginary axis, and opening angle* ε, *with* $0<\varepsilon<\pi/2$. *Then*

$$\mathcal{M}_\pm(z,x_0) \underset{\substack{|z|\to\infty\\ z\in C_\varepsilon}}{=} \pm z^{1/2}\mathcal{I}_m + o(1). \tag{2.33}$$

Necessary and sufficient conditions for $\mathcal{M}_\pm(z,x_0)$ to be the half-line $m\times m$ Weyl–Titchmarsh matrix associated with a Schrödinger operator on $[x_0,\pm\infty)$ in terms of the corresponding measures $\Omega_\pm(\cdot,x_0)$ in the Herglotz representation (2.28) of $\mathcal{M}_\pm(z,x_0)$ can be derived using the matrix-valued extension of the classical inverse spectral theory approach due to Gelfand and Levitan [33], as worked out by Rofe-Beketov [85]. The following result describes sufficient conditions for a monotonically

nondecreasing matrix function to be the matrix spectral function of a half-line Schrödinger operator. It extends well-known results in the scalar case $m = 1$ (cf. [65, Sects. 2.5, 2.9], [66], [76, Sect. 26.5], [106]).

Theorem 2.4 ([85]). *Let Ω_+ be a monotonically nondecreasing $m \times m$ matrix-valued function on $\mathbb{R}$ satisfying the following two conditions.*
(i) Whenever $f \in C([x_0, \infty))^{m\times 1}$ with compact support contained in $[x_0, \infty)$ and

$$\int_{\mathbb{R}} F(\lambda)^* d\Omega_+(\lambda)\, F(\lambda) = 0, \text{ then } f = 0 \text{ a.e.}, \tag{2.34}$$

where

$$F(\lambda) = \lim_{R\uparrow\infty} \int_{x_0}^{R} dx\, \frac{\sin(\lambda^{1/2}(x - x_0))}{\lambda^{1/2}} f(x), \quad \lambda \in \mathbb{R}. \tag{2.35}$$

(ii) Define

$$\widetilde{\Omega}_+(\lambda) = \begin{cases} \Omega_+(\lambda) - \frac{2}{3\pi}\lambda^{3/2}, & \lambda \geq 0 \\ \Omega_+(\lambda), & \lambda < 0 \end{cases} \tag{2.36}$$

and assume the limit

$$\lim_{R\uparrow\infty} \int_{-\infty}^{R} d\widetilde{\Omega}_+(\lambda)\, \frac{\sin(\lambda^{1/2}(x - x_0))}{\lambda^{1/2}} = \Phi(x) \tag{2.37}$$

exists and $\Phi \in L^\infty([x_0, R])^{m\times m}$ for all $R > x_0$. Moreover, suppose that for some $r \in \mathbb{N}_0$, $\Phi^{(r+1)} \in L^1([x_0, R])^{m\times m}$ for all $R > x_0$, and $\Phi(x_0) = 0$.
Then Ω_+ is the matrix spectral function of a self-adjoint Schrödinger operator H_+ in $L^2([x_0, \infty))^m$ associated with the $m \times m$ matrix-valued differential expression $\mathcal{L}_+ = -d^2/dx^2 \mathcal{I}_m + \mathcal{Q}$, $x > x_0$, with a Dirichlet boundary condition at x_0, a self-adjoint boundary condition at ∞ (if necessary), and a self-adjoint potential matrix $\mathcal{Q}$ with $\mathcal{Q}^{(r)} \in L^1([x_0, R])^{m\times m}$ for all $R > x_0$.

Next, assuming Hypothesis 2.1, we introduce the self-adjoint Schrödinger operator H in $L^2(\mathbb{R})^m$ by

$$\begin{aligned} &H = -\mathcal{I}_m \frac{d^2}{dx^2} + \mathcal{Q}, \\ &\operatorname{dom}(H) = \{g \in L^2(\mathbb{R})^m \mid g, g' \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R})^m;\, (-g'' + \mathcal{Q}g) \in L^2(\mathbb{R})^m\}. \end{aligned} \tag{2.38}$$

The resolvent of H then reads

$$((H - z)^{-1} f)(x) = \int_{\mathbb{R}} dx'\, \mathcal{G}(z, x, x') f(x'), \quad z \in \mathbb{C}\backslash\mathbb{R},\ f \in L^2(\mathbb{R})^m, \tag{2.39}$$

with the Green's matrix $\mathcal{G}(z, x, x')$ of H given by

$$\begin{aligned} \mathcal{G}(z, x, x') = \psi_\mp(z, x, x_0)[\mathcal{M}_-(z, x_0) - \mathcal{M}_+(z, x_0)]^{-1} \psi_\pm(\overline{z}, x', x_0)^*, \\ x \lesseqgtr x',\ z \in \mathbb{C}\backslash\mathbb{R}. \end{aligned} \tag{2.40}$$

Introducing

$$\mathcal{N}_\pm(z, x_0) = \mathcal{M}_-(z, x_0) \pm \mathcal{M}_+(z, x_0), \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{2.41}$$

the $2m \times 2m$ Weyl–Titchmarsh function $\mathcal{M}(z,x_0)$ associated with H on $\mathbb{R}$ is then given by

$$\begin{aligned}\mathcal{M}(z,x_0) &= \big(\mathcal{M}_{j,j'}(z,x_0)\big)_{j,j'=1,2} \\ &= \begin{pmatrix} \mathcal{M}_\pm(z,x_0)\mathcal{N}_-(z,x_0)^{-1}\mathcal{M}_\mp(z,x_0) & \mathcal{N}_-(z,x_0)^{-1}\mathcal{N}_+(z,x_0)/2 \\ \mathcal{N}_+(z,x_0)\mathcal{N}_-(z,x_0)^{-1}/2 & \mathcal{N}_-(z,x_0)^{-1} \end{pmatrix}, \\ &\qquad\qquad z \in \mathbb{C}\backslash\mathbb{R}. \end{aligned} \tag{2.42}$$

The basic results on $\mathcal{M}(z,x_0)$ then read as follows.

Theorem 2.5 ([40], [45], [46], [49], [56]). *Assume Hypothesis* 2.1 *and suppose that* $z \in \mathbb{C}\backslash\mathbb{R}$, $x_0 \in \mathbb{R}$. *Then,*
(i) $\mathcal{M}(z,x_0)$ *is a matrix-valued Herglotz function of rank* $2m$ *with representation*

$$\mathcal{M}(z,x_0) = \mathrm{Re}(\mathcal{M}(i,x_0)) + \int_{\mathbb{R}} d\Omega(\lambda,x_0)\, \big((\lambda - z)^{-1} - \lambda(1+\lambda^2)^{-1}\big), \tag{2.43}$$

where

$$\int_{\mathbb{R}} \|d\Omega(\lambda,x_0)\|_{\mathbb{C}^{2m\times 2m}}\, (1+\lambda^2)^{-1} < \infty \tag{2.44}$$

and

$$\Omega((\lambda,\mu],x_0) = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0} \frac{1}{\pi}\int_{\lambda+\delta}^{\mu+\delta} d\nu\, \mathrm{Im}(\mathcal{M}(\nu+i\varepsilon,x_0)). \tag{2.45}$$

(ii) $z \in \mathbb{C}\backslash\mathrm{spec}(H)$ *if and only if* $\mathcal{M}(z,x_0)$ *is holomorphic near* z.

Here $\mathrm{spec}(T)$ denotes the spectrum of T. Later on we will denote by $\mathrm{spec}_{\mathrm{ac}}(T)$ the absolutely continuous spectrum of T.

Finally, we state the following characterization of $\mathcal{M}(z,x_0)$ to be used later on. In the scalar context $m = 1$ this has been used by Rofe-Beketov [87], [88] (see also [65, Sect. 7.3]).

Theorem 2.6 ([87], [88]). *Assume Hypothesis* 2.1, *suppose that* $z \in \mathbb{C}\backslash\mathbb{R}$, $x_0 \in \mathbb{R}$, *and let* $\ell, r \in \mathbb{N}_0$. *Then the following assertions are equivalent.*
(i) $\mathcal{M}(z,x_0)$ *is the* $2m\times 2m$ *Weyl–Titchmarsh matrix associated with a Schrödinger operator* H *in* $L^2(\mathbb{R})^m$ *of the type* (2.38) *with an* $m \times m$ *matrix-valued potential* $\mathcal{Q} \in L^1_{\mathrm{loc}}(\mathbb{R})$ *and* $\mathcal{Q} \in C^\ell((-\infty,x_0))$ *and* $\mathcal{Q} \in C^r((x_0,\infty))$.
(ii) $\mathcal{M}(z,x_0)$ *is of the type* (2.42) *with* $\mathcal{M}_\pm(z,x_0)$ *being half-line* $m \times m$ *Weyl–Titchmarsh matrices on* $[x_0,\pm\infty)$ *corresponding to a Dirichlet boundary condition at* x_0 *and a self-adjoint boundary condition at* $-\infty$ *and/or* ∞ *(if any) which are associated with an* $m \times m$ *matrix-valued potential* $\mathcal{Q}$ *satisfying* $\mathcal{Q} \in C^\ell((-\infty,x_0))$ *and* $\mathcal{Q} \in C^r((x_0,\infty))$, *respectively.*
If (i) or (ii) holds, then the $2m \times 2m$ *matrix-valued spectral measure* $\Omega(\cdot,x_0)$ *associated with* $\mathcal{M}(z,x_0)$ *is determined by* (2.42) *and* (2.45).

Next, we consider variations of the reference point $x_0 \in \mathbb{R}$. In analogy to (2.20), we note that in the case where the Schrödinger differential expression $\mathcal{L}$ is in the limit point case at $\pm\infty$,

$$\mathcal{M}_\pm(z,x) = \Psi'_{\pm,x}(z,x,x_0)\Psi_\pm(z,x,x_0)^{-1}, \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{2.46}$$

represents the corresponding half-line Weyl–Titchmarsh matrix on $[x,\pm\infty)$, $x \in \mathbb{R}$, with $\Psi_\pm(z,\cdot,x_0)$ defined in (2.31). Again the actual normalization of $\Psi_\pm$ is, of course, irrelevant. Since $\Psi_\pm$ satisfies the second-order linear $m \times m$ matrix-valued

differential equation (2.2), $\mathcal{M}_\pm$ in (2.46) satisfies the matrix-valued Riccati-type equation (independently of any limit point assumptions at $\pm\infty$)

$$\mathcal{M}'_\pm(z,x) + \mathcal{M}_\pm(z,x)^2 = \mathcal{Q}(x) - z\mathcal{I}_m, \quad x \in \mathbb{R},\ z \in \mathbb{C}\backslash\mathbb{R}. \tag{2.47}$$

The asymptotic high-energy behavior of $\mathcal{M}_\pm(z,x)$ as $|z| \to \infty$ has recently been determined in [21] under minimal smoothness conditions on $\mathcal{Q}$ and without assuming that $\mathcal{L}$ is in the limit point case at $\pm\infty$. Here we recall just a special case of the asymptotic expansion proved in [21] which is most suited for our discussion in Section 4. We denote by $C_\varepsilon \subset \mathbb{C}_+$ the open sector with vertex at zero, symmetry axis along the positive imaginary axis, and opening angle ε, with $0 < \varepsilon < \pi/2$.

Theorem 2.7 ([21]). *Fix $x_0 \in \mathbb{R}$ and let $x \geq x_0$. In addition to Hypothesis* 2.1 *suppose that $\mathcal{Q} \in C^\infty([x_0, \pm\infty))^{m\times m}$ and that $\mathcal{L}$ is in the limit point case at $\pm\infty$. Let $\mathcal{M}_\pm(z,x)$, $x \geq x_0$, be defined as in* (2.46). *Then, as $|z| \to \infty$ in C_ε, $\mathcal{M}_\pm(z,x)$ has an asymptotic expansion of the form* ($\operatorname{Im}(z^{1/2}) > 0$, $z \in \mathbb{C}_+$)

$$\mathcal{M}_\pm(z,x) \underset{\substack{|z|\to\infty \\ z\in C_\varepsilon}}{=} \pm i\mathcal{I}_m z^{1/2} + \sum_{k=1}^{N} \mathcal{M}_{\pm,k}(x) z^{-k/2} + o(|z|^{-N/2}), \quad N \in \mathbb{N}. \tag{2.48}$$

The expansion (2.48) *is uniform with respect to* $\arg(z)$ *for $|z| \to \infty$ in C_ε and uniform in x as long as x varies in compact subsets of $[x_0, \infty)$. The expansion coefficients $\mathcal{M}_{\pm,k}(x)$ can be recursively computed from*

$$\begin{aligned} \mathcal{M}_{\pm,1}(x) &= \mp\frac{i}{2}\mathcal{Q}(x), \quad \mathcal{M}_{\pm,2}(x) = \frac{1}{4}\mathcal{Q}'(x), \\ \mathcal{M}_{\pm,k+1}(x) &= \pm\frac{i}{2}\bigg(\mathcal{M}'_{\pm,k}(x) + \sum_{\ell=1}^{k-1} \mathcal{M}_{\pm,\ell}(x)\mathcal{M}_{\pm,k-\ell}(x)\bigg), \quad k \geq 2. \end{aligned} \tag{2.49}$$

The asymptotic expansion (2.48) *can be differentiated to any order with respect to x.*

Remark 2.8.
(*i*) *Due to the recursion relation* (2.49), *the coefficients $\mathcal{M}_{\pm,k}$ are universal polynomials in $\mathcal{Q}$ and its x-derivatives (i.e., differential polynomials in $\mathcal{Q}$). That the asymptotic expansion* (2.48) *can be differentiated to arbitrary order in x follows from repeated use of the Riccati-type equation* (2.47).
(*ii*) *In the case where $\mathcal{Q}$ and its x-derivatives are in $L^1(\mathbb{R})^{m\times m}$, or in the case where $\mathcal{Q}$ is periodic and hence Floquet theory applies, the proof of the existence of an asymptotic expansion of the type* (2.48) *follows in a routine manner by iterating appropriate Volterra-type integral equations. The general case, however, is intricate as is evident from the treatment in* [21].

Finally, in addition to (2.40), one infers for the $2m \times 2m$ Weyl–Titchmarsh function $\mathcal{M}(z,x)$ associated with H on $\mathbb{R}$ in connection with arbitrary half-lines

$[x, \pm\infty)$, $x \in \mathbb{R}$,

$$\mathcal{M}(z,x) = \big(\mathcal{M}_{j,j'}(z,x)\big)_{j,j'=1,2}, \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{2.50}$$

$$\begin{aligned}\mathcal{M}_{1,1}(z,x) &= \mathcal{M}_{\pm}(z,x)[\mathcal{M}_{-}(z,x) - \mathcal{M}_{+}(z,x)]^{-1}\mathcal{M}_{\mp}(z,x)\\ &= \psi_{+}'(z,x,x_0)[\mathcal{M}_{-}(z,x_0) - \mathcal{M}_{+}(z,x_0)]^{-1}\psi_{-}'(\overline{z},x,x_0)^*, \end{aligned}\tag{2.51}$$

$$\begin{aligned}\mathcal{M}_{1,2}(z,x) &= 2^{-1}[\mathcal{M}_{-}(z,x) - \mathcal{M}_{+}(z,x)]^{-1}[\mathcal{M}_{-}(z,x) + \mathcal{M}_{+}(z,x)]\\ &= \psi_{+}(z,x,x_0)[\mathcal{M}_{-}(z,x_0) - \mathcal{M}_{+}(z,x_0)]^{-1}\psi_{-}'(\overline{z},x,x_0)^*, \end{aligned}\tag{2.52}$$

$$\begin{aligned}\mathcal{M}_{2,1}(z,x) &= 2^{-1}[\mathcal{M}_{-}(z,x) + \mathcal{M}_{+}(z,x)][\mathcal{M}_{-}(z,x) - \mathcal{M}_{+}(z,x)]^{-1}\\ &= \psi_{+}'(z,x,x_0)[\mathcal{M}_{-}(z,x_0) - \mathcal{M}_{+}(z,x_0)]^{-1}\psi_{-}(\overline{z},x,x_0)^*, \end{aligned}\tag{2.53}$$

$$\begin{aligned}\mathcal{M}_{2,2}(z,x) &= [\mathcal{M}_{-}(z,x) - \mathcal{M}_{+}(z,x)]^{-1}\\ &= \psi_{+}(z,x,x_0)[\mathcal{M}_{-}(z,x_0) - \mathcal{M}_{+}(z,x_0)]^{-1}\psi_{-}(\overline{z},x,x_0)^*. \end{aligned}\tag{2.54}$$

Introducing the convenient abbreviation,

$$\mathcal{M}(z,x) = \begin{pmatrix} \mathfrak{h}(z,x) & -\mathfrak{g}_2(z,x) \\ -\mathfrak{g}_1(z,x) & \mathfrak{g}(z,x) \end{pmatrix}, \quad z \in \mathbb{C}\backslash\mathbb{R}, \ x \in \mathbb{R}, \tag{2.55}$$

one then verifies from (2.50)–(2.55) and from $\mathcal{M}(\overline{z},x)^* = \mathcal{M}(z,x)$, $\mathcal{M}_{\pm}(\overline{z},x)^* = \mathcal{M}_{\pm}(z,x)$ that

$$\mathfrak{g}(\overline{z},x)^* = \mathfrak{g}(z,x), \quad \mathfrak{g}_2(\overline{z},x)^* = \mathfrak{g}_1(z,x), \quad \mathfrak{h}(\overline{z},x)^* = \mathfrak{h}(z,x), \tag{2.56}$$

$$\mathfrak{g}(z,x)\mathfrak{g}_1(z,x) = \mathfrak{g}_2(z,x)\mathfrak{g}(z,x), \tag{2.57}$$

$$\mathfrak{h}(z,x)\mathfrak{g}_2(z,x) = \mathfrak{g}_1(z,x)\mathfrak{h}(z,x), \tag{2.58}$$

$$\mathfrak{g}(z,x) = [\mathcal{M}_{-}(z,x) - \mathcal{M}_{+}(z,x)]^{-1}, \tag{2.59}$$

$$\mathfrak{g}(z,x)\mathfrak{h}(z,x) - \mathfrak{g}_2(z,x)^2 = -(1/4)\mathcal{I}_m, \tag{2.60}$$

$$\mathfrak{h}(z,x)\mathfrak{g}(z,x) - \mathfrak{g}_1(z,x)^2 = -(1/4)\mathcal{I}_m, \tag{2.61}$$

$$\mathcal{M}_{\pm}(z,x) = \mp(1/2)\mathfrak{g}(z,x)^{-1} - \mathfrak{g}(z,x)^{-1}\mathfrak{g}_2(z,x) \tag{2.62}$$

$$\mp (1/2)\mathfrak{g}(z,x)^{-1} - \mathfrak{g}_1(z,x)\mathfrak{g}(z,x)^{-1}. \tag{2.63}$$

Moreover, the Riccati-type equations (2.47) imply the following results needed in Section 4.

Lemma 2.9. *Let $z \in \mathbb{C}\backslash\mathbb{R}$ and define $\mathcal{M}_{\pm}$ by* (2.46) *so that $\mathcal{M}_{\pm}$ satisfy the Riccati-type equation* (2.47). *Then, for a.e. $x \in \mathbb{R}$,*

$$\mathfrak{g}' = -(\mathfrak{g}_1 + \mathfrak{g}_2), \tag{2.64}$$

$$\mathfrak{g}_1' = -(\mathcal{Q} - z\mathcal{I}_m)\mathfrak{g} - \mathfrak{h} \tag{2.65}$$

$$= (-\mathfrak{g}'' + \mathfrak{g}\mathcal{Q} - \mathcal{Q}\mathfrak{g})/2, \tag{2.66}$$

$$\mathfrak{g}_2' = -\mathfrak{g}(\mathcal{Q} - z\mathcal{I}_m) - \mathfrak{h} \tag{2.67}$$

$$= (-\mathfrak{g}'' + \mathcal{Q}\mathfrak{g} - \mathfrak{g}\mathcal{Q})/2, \tag{2.68}$$

$$\mathfrak{h}' = -\mathfrak{g}_1(\mathcal{Q} - z\mathcal{I}_m) - (\mathcal{Q} - z\mathcal{I}_m)\mathfrak{g}_2, \tag{2.69}$$

$$\mathfrak{h} = [\mathfrak{g}'' - \mathfrak{g}(\mathcal{Q} - z\mathcal{I}_m) - (\mathcal{Q} - z\mathcal{I}_m)\mathfrak{g}]/2 \tag{2.70}$$

if $\mathcal{Q} \in L^1_{\rm loc}(\mathbb{R})^{m\times m}$, *and*

$$\mathfrak{g}_1'' = -2(\mathcal{Q} - z\mathcal{I}_m)\mathfrak{g}' - \mathcal{Q}'\mathfrak{g} + \mathfrak{g}_1\mathcal{Q} - \mathcal{Q}\mathfrak{g}_1, \tag{2.71}$$

$$\mathfrak{g}_2'' = -2\mathfrak{g}'(\mathcal{Q} - z\mathcal{I}_m) - \mathfrak{g}\mathcal{Q}' + \mathcal{Q}\mathfrak{g}_2 - \mathfrak{g}_2\mathcal{Q} \tag{2.72}$$

if in addition $\mathcal{Q}' \in L^1_{\rm loc}(\mathbb{R})^{m\times m}$.

Proof. (2.47) rewritten in terms of $\mathfrak{g}$, $\mathfrak{g}_1$, $\mathfrak{g}_2$ yields

$$\begin{aligned}&\pm (1/2)\mathfrak{g}^{-1}\mathfrak{g}'\mathfrak{g}^{-1} + \mathfrak{g}^{-1}\mathfrak{g}'\mathfrak{g}^{-1}\mathfrak{g}_2 - \mathfrak{g}^{-1}\mathfrak{g}_2' + (1/4)\mathfrak{g}^{-2} + \mathfrak{g}^{-1}\mathfrak{g}_2\mathfrak{g}^{-1}\mathfrak{g}_2\\ &\pm (1/2)\mathfrak{g}^{-2}\mathfrak{g}_2 \pm (1/2)\mathfrak{g}^{-1}\mathfrak{g}_2\mathfrak{g}^{-1} = \mathcal{Q} - z\mathcal{I}_m.\end{aligned} \tag{2.73}$$

Taking the difference of the two equations in (2.73) yields (2.64). Adding the two equations in (2.73) and using (2.57), (2.60), (2.61), and (2.64) yields (2.65) and (2.67). Combining (2.65), (2.67), and (2.64) implies (2.70). Inserting (2.70) into (2.65) and (2.67) yields (2.66) and (2.68). (2.69) follows from differentiating $\mathfrak{h} = \mathfrak{g}_1^2\mathfrak{g}^{-1} - (1/4)\mathfrak{g}^{-1}$, inserting $\mathfrak{g}'$ and $\mathfrak{g}_1'$ from (2.64) and (2.65), and making repeated use of the identities (2.57), (2.61). Finally, (2.71) (resp. (2.72)) follows from differentiating (2.65) (resp. (2.67)) inserting (2.69) for $\mathfrak{h}'$. Alternatively, (2.67)–(2.72) follow directly from (2.65)–(2.71) using (2.56). □

3. Polynomial Pencils of Matrices in a Nutshell

Since self-adjoint polynomial pencils of matrices play a role in our principal section 4, we briefly review some of the corresponding definitions and basic results, mainly following the monograph of Markus [72] and papers by Markus and Matsaev [73], [74]. While all results below are discussed for operator pencils by Markus and Matsaev, we will only quote them in the matrix context, for simplicity.

Given $m \in \mathbb{N}$, we denote by

$$\mathcal{A}(z) = \sum_{k=0}^{n} \mathcal{A}_k z^k, \quad \mathcal{A}_k \in \mathbb{C}^{m\times m},\ 1 \le k \le n,\ z \in \mathbb{C}, \tag{3.1}$$

a polynomial pencil of $m \times m$ matrices (in short, a pencil) in the following. $\mathcal{A}$ is called of degree $n \in \mathbb{N}_0$ if $\mathcal{A}_n \neq 0$ and monic if $\mathcal{A}_n = \mathcal{I}_m$.

Definition 3.1. *Let* $\mathcal{A}$ *be a pencil of the type* (3.1).
(*i*) *The spectrum of* $\mathcal{A}$, *denoted by* $\operatorname{spec}(\mathcal{A})$, *is defined by*

$$\operatorname{spec}(\mathcal{A}) = \{z \in \mathbb{C} \,|\, \mathcal{A}(z) \text{ is not invertible}\}. \tag{3.2}$$

$z_0 \in \mathbb{C}$ *is called an eigenvalue of* $\mathcal{A}$ *if* $\mathcal{A}(z_0)f_0 = 0$ *has a solution* $f_0 \in \mathbb{C}^m \setminus \{0\}$.
(*ii*) *A monic pencil* $\mathcal{C}$ *is called a* (*right*) *divisor of* $\mathcal{A}$ *if*

$$\mathcal{A}(z) = \mathcal{B}(z)\mathcal{C}(z), \quad z \in \mathbb{C} \tag{3.3}$$

for some pencil $\mathcal{B}$. *If in addition* $\operatorname{spec}(\mathcal{B}) \cap \operatorname{spec}(\mathcal{C}) = \emptyset$, *then* $\mathcal{C}$ *is called a* (*right*) *spectral divisor of* $\mathcal{A}$.
(*iii*) $\mathcal{Z} \in \mathbb{C}^{m\times m}$ *is called a* (*matrix*) *root of the pencil* $\mathcal{A}$ *if*

$$\mathcal{A}(\mathcal{Z}) = 0, \tag{3.4}$$

where $\mathcal{A}(\mathcal{Z})$ *is defined as*

$$\mathcal{A}(\mathcal{Z}) = \sum_{k=0}^{n} \mathcal{A}_k \mathcal{Z}^k. \tag{3.5}$$

$\mathcal{Z} \in \mathbb{C}^{m\times m}$ is called a (matrix) spectral root of the pencil $\mathcal{A}$ if $(z\mathcal{I}_m - \mathcal{Z})$ is a spectral divisor of $\mathcal{A}$.
(iv) The pencil $\mathcal{A}$ is called self-adjoint if $\mathcal{A}_k = \mathcal{A}_k^*$ for all $1 \le k \le n$ (i.e., $\mathcal{A}(\overline{z})^* = \mathcal{A}(z)$ for all $z \in \mathbb{C}$).
(v) A self-adjoint pencil $\mathcal{A}$ is called weakly hyperbolic if $\mathcal{A}_n > 0$ and for all $f \in \mathbb{C}^m\backslash\{0\}$, the roots of the polynomial $(f, \mathcal{A}(\cdot)f)_{\mathbb{C}^m}$ are real. If in addition all these zeros are distinct, the pencil $\mathcal{A}$ is called hyperbolic.
(vi) Let $\mathcal{A}$ be a weakly hyperbolic pencil and denote by $\{p_j(\mathcal{A}, f)\}_{1\le j\le n}$,

$$p_j(\mathcal{A}, f) \le p_{j+1}(\mathcal{A}, f),\ 1 \le j \le n-1,\ f \in \mathbb{C}^m\backslash\{0\}, \tag{3.6}$$

the roots of the polynomial $(f, \mathcal{A}(\cdot)f)_{\mathbb{C}^m}$ ordered in magnitude. The range of the roots $p_j(\mathcal{A}, f)$, $f \in \mathbb{C}^m\backslash\{0\}$ is denoted by $\Delta_j(\mathcal{A})$ and called the jth root zone of $\mathcal{A}$.
(vii) A hyperbolic pencil $\mathcal{A}$ is called strongly hyperbolic if $\overline{\Delta_j(\mathcal{A})}$ and $\overline{\Delta_k(\mathcal{A})}$ are mutually disjoint for $j \ne k$, $1 \le j, k \le n$.

Moreover, the Vandermonde matrix corresponding to a collection $\{\mathcal{Z}_1, \dots, \mathcal{Z}_n\} \subset \mathbb{C}^{m\times m}$ is defined by

$$\mathcal{V}(\mathcal{Z}_1, \dots, \mathcal{Z}_n) = \begin{pmatrix} \mathcal{I}_m & \mathcal{I}_m & \dots & \mathcal{I}_m \\ \mathcal{Z}_1 & \mathcal{Z}_2 & \dots & \mathcal{Z}_n \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{Z}_1^{n-1} & \mathcal{Z}_2^{n-1} & \dots & \mathcal{Z}_n^{n-1} \end{pmatrix}. \tag{3.7}$$

Theorem 3.2 (Markus [72], Sect. 29)**.** *Let $\mathcal{A}$ be a pencil of the type* (3.1).
(i) $(z\mathcal{I}_m - \mathcal{Z})$ is a divisor of $\mathcal{A}$ if and only if $\mathcal{A}(\mathcal{Z}) = 0$. (This justifies the notation introduced in the last part of Definition 3.1 *(ii).)*
(ii) Let $\mathcal{A}$ be a monic pencil of degree n and $\mathcal{Z}_1, \dots, \mathcal{Z}_n$ spectral roots of $\mathcal{A}$. Then the following assertions are equivalent:
(α) $\mathcal{V}(\mathcal{Z}_1, \dots, \mathcal{Z}_n)$ is invertible.
(β) $\operatorname{spec}(\mathcal{Z}_j) \cap \operatorname{spec}(\mathcal{Z}_k) = \emptyset$, $j \ne k$, $1 \le j, k \le n$.
(γ) $\operatorname{spec}(\mathcal{A}) = \bigcup_{j=1}^n \operatorname{spec}(\mathcal{Z}_j)$.
(iii) If $\mathcal{A}$ is a self-adjoint pencil, then $\operatorname{spec}(\mathcal{A})$ is symmetric with respect to $\mathbb{R}$.

Theorem 3.3 (Markus [72], Sect. 31)**.** *Let $\mathcal{A}$ be a pencil of the type* (3.1).
(i) If $\mathcal{A}$ is a weakly hyperbolic pencil, then $\operatorname{spec}(\mathcal{A}) \subseteq \mathbb{R}$.
(ii) The root zones $\Delta_j(\mathcal{A})$, $1 \le j \le n$, of a weakly hyperbolic pencil $\mathcal{A}$ are intervals (possibly degenerating to a point).
(iii) If $\mathcal{A}$ is a weakly hyperbolic pencil of order n with root zones $\{\Delta_j(\mathcal{A})\}_{1\le j\le n}$ and $\Delta_k(\mathcal{A}) = \{\lambda_k\}$ for some $k \in \{1, \dots, n\}$, then $\mathcal{A}(z) = (z - \lambda_k)\mathcal{B}(z)$, where $\mathcal{B}$ is a weakly hyperbolic pencil of order $n-1$, with root zones

$$\Delta_1(\mathcal{A}), \dots, \Delta_{k-1}(\mathcal{A}), \Delta_{k+1}(\mathcal{A}), \dots, \Delta_n(\mathcal{A}). \tag{3.8}$$

(iv) If $\mathcal{A}$ is a weakly hyperbolic pencil then $\Delta_j(\mathcal{A}) \cap \Delta_{j+1}(\mathcal{A})$, $1 \le j \le n-1$, consists of at most one point.
(v) If $\mathcal{A}$ is a hyperbolic pencil, then $\Delta_j(\mathcal{A}) \cap \Delta_k(\mathcal{A}) = \emptyset$, $j \ne k$, $1 \le j, k \le n$. Thus, a hyperbolic pencil is strongly hyperbolic if and only if

$$\overline{\Delta_j(\mathcal{A})} \cap \overline{\Delta_{j+1}(\mathcal{A})} = \emptyset, \quad 1 \le j \le n-1. \tag{3.9}$$

(vi) Suppose $\mathcal{A}$ is a self-adjoint pencil of degree n, $\mathcal{A}_n > 0$, and $\mathcal{A}(\lambda) \ne 0$ for all $\lambda \in \mathbb{R}$. Then $\mathcal{A}$ is a weakly hyperbolic pencil if and only if there exist numbers $\gamma_1 < \gamma_2 < \dots < \gamma_{n-1}$ such that $(-1)^j \mathcal{A}(\gamma_j) \ge 0$, $1 \le j \le n-1$.

(*vii*) *Suppose $\mathcal{A}$ is a self-adjoint pencil of degree n with $\mathcal{A}_n > 0$. Then $\mathcal{A}$ is a strongly hyperbolic pencil if and only if there exist numbers $\gamma_1 < \gamma_2 < \cdots < \gamma_{n-1}$ such that $(-1)^j \mathcal{A}(\gamma_j) > 0$, $1 \leq j \leq n-1$.*

Theorem 3.4 (Markus [72], Sect. 31)**.** *Let $\mathcal{A}$ be a pencil of the type* (3.1).
(*i*) *Suppose $\mathcal{A}$ is a weakly hyperbolic pencil and $\overline{\Delta_{j_0-1}(\mathcal{A})} \cap \overline{\Delta_{j_0}(\mathcal{A})} = \overline{\Delta_{j_0}(\mathcal{A})} \cap \overline{\Delta_{j_0+1}(\mathcal{A})} = \emptyset$ for some $j_0 \in \{1, \dots, n\}$. Then $\mathcal{A}$ has a spectral root $\mathcal{Z}_{j_0}$ such that $\operatorname{spec}(\mathcal{A}) \cap \overline{\Delta_{j_0}(\mathcal{A})} = \operatorname{spec}(\mathcal{Z}_{j_0})$ and $\mathcal{Z}_{j_0}$ is similar to a self-adjoint matrix.*
(*ii*) *A strongly hyperbolic pencil $\mathcal{A}$ has n spectral roots $\{\mathcal{Z}_j\}_{1 \leq j \leq n}$ such that $\operatorname{spec}(\mathcal{Z}_j) = \operatorname{spec}(\mathcal{A}) \cap \overline{\Delta_j(\mathcal{A})}$ and each $\mathcal{Z}_j$, $1 \leq j \leq n$, is similar to a self-adjoint matrix.*

Theorem 3.5 (Markus [72], Sect. 31, Markus and Matsaev [73], [74])**.** *Let $\mathcal{A}$ be a pencil of the type* (3.1).
(*i*) *A weakly hyperbolic monic pencil $\mathcal{A}$ of degree n is decomposable as*

$$\mathcal{A}(z) = (z\mathcal{I}_m - \mathcal{Y}_n)(z\mathcal{I}_m - \mathcal{Y}_{n-1}) \cdots (z\mathcal{I}_m - \mathcal{Y}_1), \tag{3.10}$$

with $\operatorname{spec}(\mathcal{Y}_j) \subset \overline{\Delta_j(\mathcal{A})}$, $1 \leq j \leq n$.
(*ii*) *Let $\mathcal{A}$ be a strongly hyperbolic monic pencil of degree n. Then $\mathcal{A}$ is decomposable as*

$$\mathcal{A}(z) = (z\mathcal{I}_m - \mathcal{Y}_n)(z\mathcal{I}_m - \mathcal{Y}_{n-1}) \cdots (z\mathcal{I}_m - \mathcal{Y}_1), \tag{3.11}$$

with $\operatorname{spec}(\mathcal{Y}_j) \subset \overline{\Delta_j(\mathcal{A})}$, $1 \leq j \leq n$. Moreover, each $\mathcal{Y}_j$ is similar to a spectral root $\mathcal{Z}_j$ of $\mathcal{A}$ and hence,

$$\operatorname{spec}(\mathcal{Y}_j) = \operatorname{spec}(\mathcal{Z}_j) = \operatorname{spec}(\mathcal{A}) \cap \overline{\Delta_j(\mathcal{A})}, \quad 1 \leq j \leq n. \tag{3.12}$$

4. A Class of Matrix-Valued Schrödinger Operators with Prescribed Finite-Band Spectra

This section is devoted to the construction of a class of matrix-valued Schrödinger operators with a prescribed finite-band spectrum of uniform maximum multiplicity, the principal result of this paper.

To begin our analysis we start with a useful result on (scalar) Herglotz functions. Even though the result is probably well-known to experts, we provide an elementary proof for completeness.

Let

$$\{E_\ell\}_{0 \leq \ell \leq 2n} \subseteq \mathbb{R}, \ n \in \mathbb{N}, \ \text{with } E_\ell < E_{\ell+1}, \ 0 \leq \ell \leq 2n-1, \tag{4.1}$$

and introduce the polynomial

$$R_{2n+1}(z) = \prod_{\ell=0}^{2n} (z - E_\ell), \quad z \in \mathbb{C}. \tag{4.2}$$

Moreover, we define the square root of R_{2n+1} by

$$R_{2n+1}(\lambda)^{1/2} = \lim_{\varepsilon \downarrow 0} R_{2n+1}(\lambda + i\varepsilon)^{1/2}, \quad \lambda \in \mathbb{R}, \tag{4.3}$$

and

$$R_{2n+1}(\lambda)^{1/2} = |R_{2n+1}(\lambda)^{1/2}| \begin{cases} (-1)^n i & \text{for } \lambda \in (-\infty, E_0), \\ (-1)^{n+j} i & \text{for } \lambda \in (E_{2j-1}, E_{2j}),\ j = 1, \dots, n, \\ (-1)^{n+j} & \text{for } \lambda \in (E_{2j}, E_{2j+1}),\ j = 0, \dots, n-1, \\ 1 & \text{for } \lambda \in (E_{2n}, \infty), \end{cases} \quad \lambda \in \mathbb{R} \tag{4.4}$$

and analytically continue $R_{2n+1}^{1/2}$ from $\mathbb{R}$ to all of $\mathbb{C}\backslash\Sigma$, where Σ is defined by

$$\Sigma = \left\{ \bigcup_{j=0}^{n-1} [E_{2j}, E_{2j+1}] \right\} \cup [E_{2n}, \infty). \tag{4.5}$$

In this context we also mention the useful formula

$$\overline{R_{2n+1}(\overline{z})^{1/2}} = -R_{2n+1}(z)^{1/2}, \quad z \in \mathbb{C}_+ . \tag{4.6}$$

Theorem 4.1. *Let $z \in \mathbb{C}\backslash\Sigma$ and $n \in \mathbb{N}$. Define $R_{2n+1}^{1/2}$ as in (4.1)–(4.4) followed by an analytic continuation to $\mathbb{C}\backslash\Sigma$. Moreover let F_n and H_{n+1} be two monic polynomials of degree n and $n+1$, respectively. Then*

$$\frac{iF_n(z)}{R_{2n+1}(z)^{1/2}} \tag{4.7}$$

is a Herglotz function if and only if all zeros of F_n are real and there is precisely one zero in each of the intervals $[E_{2j-1}, E_{2j}]$, $1 \leq j \leq n$. Moreover, if $iR_{2n+1}^{-1/2} F_n$ is a Herglotz function, then it can be represented in the form

$$\frac{iF_n(z)}{R_{2n+1}(z)^{1/2}} = \frac{1}{\pi} \int_\Sigma \frac{F_n(\lambda) d\lambda}{R_{2n+1}(\lambda)^{1/2}} \frac{1}{\lambda - z}, \quad z \in \mathbb{C}\backslash\Sigma. \tag{4.8}$$

Similarly,

$$\frac{iH_{n+1}(z)}{R_{2n+1}(z)^{1/2}} \tag{4.9}$$

is a Herglotz function if and only if all zeros of H_{n+1} are real and there is precisely one zero in each of the intervals $(-\infty, E_0]$ and $[E_{2j-1}, E_{2j}]$, $1 \leq j \leq n$. Moreover, if $iR_{2n+1}^{-1/2} H_{n+1}$ is a Herglotz function, then it can be represented in the form

$$\begin{aligned} \frac{iH_{n+1}(z)}{R_{2n+1}(z)^{1/2}} &= \operatorname{Re}\left(\frac{iH_{n+1}(i)}{R_{2n+1}(i)^{1/2}} \right) \\ &\quad + \frac{1}{\pi} \int_\Sigma \frac{H_{n+1}(\lambda) d\lambda}{R_{2n+1}(\lambda)^{1/2}} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right), \quad z \in \mathbb{C}\backslash\Sigma. \end{aligned} \tag{4.10}$$

Proof. We start with the case of $F_n(z)/R_{2n+1}(z)^{1/2}$ in (4.7). Consider a closed counterclockwise oriented contour $\Gamma_{R,\varepsilon}$ which consists of the semicircle $C_\varepsilon = \{z \in \mathbb{C} \,|\, z = E_0 + \varepsilon \exp(i\alpha),\ -\pi/2 \leq \alpha \leq \pi/2\}$ centered at E_0, the straight line $L_+ = \{z \in \mathbb{C}_+ \,|\, z = x + i\varepsilon,\ E_0 \leq x \leq R\}$, the following part of the circle of radius $(R^2 + \varepsilon^2)^{1/2}$ centered at E_0, $C_R = \{z \in \mathbb{C} \,|\, z = E_0 + (R^2 + \varepsilon^2)^{1/2} \exp(i\beta),\ \arctan(\varepsilon/R) \leq \beta \leq 2\pi - \arctan(\varepsilon/R)\}$, and the straight line $L_- = \{z \in \mathbb{C}_- \,|\, z = x - i\varepsilon,\ E_0 \leq x \leq R\}$.

Then, for $\varepsilon > 0$ small enough and $R > 0$ sufficiently large, one infers

$$\begin{aligned} \frac{iF_n(z)}{R_{2n+1}(z)^{1/2}} &= \frac{1}{2\pi i} \oint_{\Gamma_{R,\varepsilon}} \frac{1}{\zeta - z} \frac{iF_n(\zeta)}{R_{2n+1}(\zeta)^{1/2}} d\zeta \\ &\underset{\varepsilon \downarrow 0, R \uparrow \infty}{=} \frac{1}{\pi} \int_\Sigma \frac{1}{\lambda - z} \frac{F_n(\lambda) d\lambda}{R_{2n+1}(\lambda)^{1/2}}. \end{aligned} \tag{4.11}$$

Here we used (4.4) to compute the contributions of the contour integral along $[E_0, R]$ in the limit $\varepsilon \downarrow 0$ and note that the integral over C_R tends to zero as $R \uparrow 0$ since

$$\frac{F_n(\zeta)}{R_{2n+1}(\zeta)^{1/2}} \underset{\zeta \to \infty}{=} O\big(|\zeta|^{-1/2}\big). \tag{4.12}$$

Next, utilizing the fact that F_n is monic and using (4.4) again, one infers that $F_n(\lambda) d\lambda / R_{2n+1}(\lambda)^{1/2}$ represents a positive measure supported on Σ if and only if F_n has precisely one zero in each of the intervals $[E_{2j-1}, E_{2j}]$, $1 \leq j \leq n$. In other words,

$$\frac{F_n(\lambda)}{R_{2n+1}(\lambda)^{1/2}} \geq 0 \text{ on } \Sigma \tag{4.13}$$

if and only if F_n has precisely one zero in each of the intervals $[E_{2j-1}, E_{2j}]$, $1 \leq j \leq n$. The Herglotz representation theorem, Theorem 2.3, then finishes the proof of (4.8). The proof of (4.10) follows along similar lines taking into account the additional residues at $\pm i$ inside $\Gamma_{R,\varepsilon}$ which are responsible for the real part on the right-hand side of (4.10). □

Theorem 4.1 can be improved by invoking ideas developed in the Appendix of [61] (cf. also [105]). We will pursue this further in [9].

Corollary 4.2. *Let $z \in \mathbb{C} \backslash \Sigma$ and $m, n \in \mathbb{N}$. Define $R_{2n+1}^{1/2}$ as in (4.1)–(4.4) followed by an analytic continuation to $\mathbb{C} \backslash \Sigma$. Moreover let $\mathcal{F}_n$ and $\mathcal{H}_{n+1}$ be two monic $m \times m$ matrix pencils of degree n and $n+1$, respectively. Then $(i/2) R_{2n+1}^{-1/2} \mathcal{F}_n$ is a Herglotz matrix if and only if the root zones $\Delta_j(\mathcal{F}_n)$ of $\mathcal{F}_n$ satisfy*

$$\Delta_j(\mathcal{F}_n) \subseteq [E_{2j-1}, E_{2j}], \quad 1 \leq j \leq n. \tag{4.14}$$

Analogously, $(i/2) R_{2n+1}^{-1/2} \mathcal{H}_{n+1}$ is a Herglotz matrix if and only if the root zones $\Delta_j(\mathcal{H}_{n+1})$ of $\mathcal{H}_{n+1}$ satisfy

$$\Delta_0(\mathcal{H}_{n+1}) \subset (-\infty, E_0], \quad \Delta_j(\mathcal{H}_{n+1}) \subseteq [E_{2j-1}, E_{2j}], \quad 1 \leq j \leq n. \tag{4.15}$$

If (4.14) (resp., (4.15)) holds, then $\mathcal{F}_n$ (resp., $\mathcal{H}_{n+1}$) is a strongly hyperbolic pencil.

Proof. We recall that $(i/2) R_{2n+1}^{-1/2} \mathcal{F}_n$ is an $m \times m$ Herglotz matrix if and only if $(f, (i/2) R_{2n+1}^{-1/2} \mathcal{F}_n f)_{\mathbb{C}^m}$ is a Herglotz function for all $f \in \mathbb{C}^m \backslash \{0\}$. Thus, it suffices to apply Theorem 4.1, identifying F_n and $(f, \mathcal{F}_n f)_{\mathbb{C}^m}$, to arrive at (4.14). The same argument applied to $\mathcal{H}_{n+1}$ yields (4.15). □

Next, we define the following $2m \times 2m$ matrix $\mathcal{M}_\Sigma(z, x_0)$ which will turn out to be the underlying Weyl–Titchmarsh matrix associated with a class of $m \times m$ matrix-valued Schrödinger operators with prescribed finite-band spectra. We introduce,

for fixed $x_0 \in \mathbb{R}$,

$$\begin{aligned} \mathcal{M}_\Sigma(z,x_0) &= \big(\mathcal{M}_{\Sigma,p,q}(z,x_0)\big)_{1\leq p,q\leq 2} \\ &= \frac{i}{2R_{2n+1}(z)^{1/2}} \begin{pmatrix} \mathcal{H}_{n+1,\Sigma}(z,x_0) & -\mathcal{G}_{2,n-1,\Sigma}(z,x_0) \\ -\mathcal{G}_{1,n-1,\Sigma}(z,x_0) & \mathcal{F}_{n,\Sigma}(z,x_0) \end{pmatrix}, \quad z\in\mathbb{C}\backslash\Sigma. \end{aligned} \tag{4.16}$$

Here $R_{2n+1}(z)^{1/2}$ is defined as in (4.1)–(4.4) followed by analytic continuation into $\mathbb{C}\backslash\Sigma$ and the polynomial matrix pencils $\mathcal{F}_{n,\Sigma}$, $\mathcal{G}_{1,n-1,\Sigma}$, $\mathcal{G}_{2,n-1,\Sigma}$, and $\mathcal{H}_{n+1,\Sigma}$ are introduced as follows:

(*i*) $\mathcal{F}_{n,\Sigma}(\cdot,x_0)$ is an $m\times m$ monic matrix pencil of degree n, that is, $\mathcal{F}_{n,\Sigma}(\cdot,x_0)$ is of the type

$$\mathcal{F}_{n,\Sigma}(z,x_0) = \sum_{\ell=0}^n \mathcal{F}_{n-\ell,\Sigma}(x_0) z^\ell, \quad \mathcal{F}_{0,\Sigma}(x_0) = \mathcal{I}_m, \ z\in\mathbb{C} \tag{4.17}$$

and

$$\frac{i}{2R_{2n+1}^{1/2}} \mathcal{F}_{n,\Sigma}(\cdot,x_0) \text{ is assumed to be an } m\times m \text{ Herglotz matrix.} \tag{4.18}$$

Hence $\mathcal{F}_{n,\Sigma}(\cdot,x_0)$ is a self-adjoint (in fact, strongly hyperbolic) pencil,

$$\mathcal{F}_{n,\Sigma}(\overline{z},x_0)^* = \mathcal{F}_{n,\Sigma}(z,x_0), \quad z\in\mathbb{C} \tag{4.19}$$

and $(i/2)R_{2n+1}^{-1/2}\mathcal{F}_{n,\Sigma}$ and $2iR_{2n+1}^{1/2}\mathcal{F}_{n,\Sigma}^{-1}$ admit the Herglotz representations

$$\frac{i}{2R_{2n+1}(z)^{1/2}}\mathcal{F}_{n,\Sigma}(z,x_0) = \frac{1}{2\pi}\int_\Sigma \frac{d\lambda}{R_{2n+1}(\lambda)^{1/2}} \mathcal{F}_{n,\Sigma}(\lambda,x_0)\frac{1}{\lambda-z}, \quad z\in\mathbb{C}\backslash\Sigma, \tag{4.20}$$

$$\begin{aligned} &iR_{2n+1}(z)^{1/2}\mathcal{F}_{n,\Sigma}(z,x_0)^{-1} \\ &= \frac{1}{\pi}\int_\Sigma d\lambda\, R_{2n+1}(\lambda)^{1/2}\mathcal{F}_{n,\Sigma}(\lambda,x_0)^{-1}\left(\frac{1}{\lambda-z}-\frac{\lambda}{1+\lambda^2}\right) \\ &\quad + \Gamma_{\Sigma,0}(x_0) - \sum_{k=1}^N (z-\mu_k(x_0))^{-1}\Gamma_{\Sigma,k}(x_0), \\ &\qquad z\in\mathbb{C}\backslash\{\Sigma\cup\{\mu_k(x_0)\}_{1\leq k\leq N}\}, \end{aligned} \tag{4.21}$$

where

$$\begin{aligned} &\Gamma_{\Sigma,0}(x_0) = \Gamma_{\Sigma,0}(x_0)^* \in \mathbb{C}^{m\times m}, \quad 0\leq \Gamma_{\Sigma,k}(x_0)\in\mathbb{C}^{m\times m}, \ 1\leq k\leq N, \\ &\sum_{k=1}^N \operatorname{rank}(\Gamma_{\Sigma,k}(x_0)) \leq mn, \quad \mu_k(x_0)\in\bigcup_{j=1}^n [E_{2j-1},E_{2j}], \quad 1\leq k\leq N. \end{aligned} \tag{4.22}$$

In fact, there are precisely m numbers $\mu_k(x_0)$ in $[E_{2j-1},E_{2j}]$ for each $1\leq j\leq n$, counting multiplicity (they are the points z where $\mathcal{F}_{n,\Sigma}(z,x_0)$ is not invertible).

(ii) Given these facts we now define

$$\mathcal{G}_{1,n-1,\Sigma}(z,x_0)=\bigg(\sum_{k=1}^{N}\frac{\varepsilon_k(x_0)}{z-\mu_k(x_0)}\Gamma_{\Sigma,k}(x_0)\bigg)\mathcal{F}_{n,\Sigma}(z,x_0), \tag{4.23}$$

$$\mathcal{G}_{2,n-1,\Sigma}(z,x_0)=\mathcal{F}_{n,\Sigma}(z,x_0)\bigg(\sum_{k=1}^{N}\frac{\varepsilon_k(x_0)}{z-\mu_k(x_0)}\Gamma_{\Sigma,k}(x_0)\bigg), \tag{4.24}$$

$$\varepsilon_k(x_0)\in\{1,-1\},\ 1\le k\le N,\quad z\in\mathbb{C}\backslash\{\mu_k(x_0)\}_{1\le k\le N}, \tag{4.25}$$

and

$$\begin{aligned}\mathcal{H}_{n+1,\Sigma}(z,x_0)&=R_{2n+1}(z)\mathcal{F}_{n,\Sigma}(z,x_0)^{-1} \\ &+\bigg(\sum_{k=1}^{N}\frac{\varepsilon_k(x_0)}{z-\mu_k(x_0)}\Gamma_{\Sigma,k}(x_0)\bigg)\mathcal{F}_{n,\Sigma}(z,x_0)\bigg(\sum_{\ell=1}^{N}\frac{\varepsilon_\ell(x_0)}{z-\mu_\ell(x_0)}\Gamma_{\Sigma,\ell}(x_0)\bigg), \\ &\qquad z\in\mathbb{C}\backslash\{\mu_k(x_0)\}_{1\le k\le N}.\end{aligned} \tag{4.26}$$

Lemma 4.3. *Let $z\in\mathbb{C}\backslash\{\mu_k(x_0)\}_{1\le k\le N}$. $\mathcal{G}_{p,n-1,\Sigma}(\cdot,x_0)$, $p=1,2$, are $m\times m$ polynomial matrix pencils of equal degree at most $n-1$ and $\mathcal{H}_{n+1,\Sigma}(\cdot,x_0)$ is a self-adjoint $m\times m$ monic matrix pencil of degree $n+1$. Moreover, the following identities hold.*

$$\mathcal{G}_{2,n-1,\Sigma}(\overline{z},x_0)^*=\mathcal{G}_{1,n-1,\Sigma}(z,x_0), \tag{4.27}$$

$$\mathcal{F}_{n,\Sigma}(z,x_0)\mathcal{G}_{1,n-1,\Sigma}(z,x_0)=\mathcal{G}_{2,n-1,\Sigma}(z,x_0)\mathcal{F}_{n,\Sigma}(z,x_0), \tag{4.28}$$

$$\mathcal{H}_{n+1,\Sigma}(z,x_0)\mathcal{G}_{2,n-1,\Sigma}(z,x_0)=\mathcal{G}_{1,n-1,\Sigma}(z,x_0)\mathcal{H}_{n+1,\Sigma}(z,x_0), \tag{4.29}$$

$$\mathcal{F}_{n,\Sigma}(z,x_0)\mathcal{H}_{n+1,\Sigma}(z,x_0)-\mathcal{G}_{2,n-1,\Sigma}(z,x_0)^2=R_{2n+1}(z)\mathcal{I}_m, \tag{4.30}$$

$$\mathcal{H}_{n+1,\Sigma}(z,x_0)\mathcal{F}_{n,\Sigma}(z,x_0)-\mathcal{G}_{1,n-1,\Sigma}(z,x_0)^2=R_{2n+1}(z)\mathcal{I}_m. \tag{4.31}$$

Proof. The identities (4.28)–(4.31) are obvious from (4.23)–(4.26). Similarly, (4.27) is clear from (4.23)–(4.25) and (4.19). By (4.21), one infers

$$\lim_{z\to\mu_k(x_0)}\Gamma_{\Sigma,k}(x_0)\mathcal{F}_{n,\Sigma}(z,x_0)=0=\lim_{z\to\mu_k(x_0)}\mathcal{F}_{n,\Sigma}(z,x_0)\Gamma_{\Sigma,k}(x_0) \tag{4.32}$$

and hence $\mathcal{G}_{p,n-1,\Sigma}(\cdot,x_0)$, $p=1,2$, are polynomial matrix pencils of degree at most $n-1$. By (4.27),

$$\deg(\mathcal{G}_{1,n-1,\Sigma}(\cdot,x_0))=\deg(\mathcal{G}_{2,n-1,\Sigma}(\cdot,x_0))\le n-1. \tag{4.33}$$

Next, using (4.21) again, one notes that

$$iR_{2n+1}(\mu_k(x_0))^{1/2}\Gamma_{\Sigma,k}(x_0)=-\Gamma_{\Sigma,k}(x_0)[(d/dz)\mathcal{F}_{n,\Sigma}(\mu_k(x_0))]\Gamma_{\Sigma,k}(x_0),\quad 1\le k\le N \tag{4.34}$$

and thus, combining (4.26) and (4.34),

$$\begin{aligned}&\operatorname{res}_{z=\mu_k(x_0)}\mathcal{H}_{n+1,\Sigma}(z,x_0) \\ &=iR_{2n+1}(\mu_k(x_0))^{1/2}\Gamma_{\Sigma,k}(x_0)+\Gamma_{\Sigma,k}(x_0)[(d/dz)\mathcal{F}_{n,\Sigma}(\mu_k(x_0))]\Gamma_{\Sigma,k}(x_0)=0, \\ &\qquad 1\le k\le N.\end{aligned} \tag{4.35}$$

Hence, $\mathcal{H}_{n+1,\Sigma}$ is indeed a polynomial matrix pencil of degree $n+1$. □

In fact, $\mathcal{H}_{n+1,\Sigma}$ is a strongly hyperbolic pencil as shown in Theorem 4.8.

In the following it will be convenient to use the following set of assumptions.

Hypothesis 4.4. *Let $m, n \in \mathbb{N}$. Define R_{2n+1} as in (4.1), (4.2) and $R_{2n+1}^{1/2}$ as in (4.3), (4.4) followed by an analytic continuation to $\mathbb{C}\backslash\Sigma$, with Σ introduced in (4.5). Moreover, let the polynomial $m\times m$ matrix pencils $\mathcal{F}_{n,\Sigma}(\cdot,x_0)$, $\mathcal{G}_{1,n-1,\Sigma}(\cdot,x_0)$, $\mathcal{G}_{2,n-1,\Sigma}(\cdot,x_0)$, and $\mathcal{H}_{n+1,\Sigma}(\cdot,x_0)$ be defined as in (4.17), (4.18), (4.23)–(4.26).*

Next, we introduce

$$\begin{aligned}
&\mathcal{M}_{\pm,\Sigma}(z,x_0)\\
&= \pm iR_{2n+1}(z)^{1/2}\mathcal{F}_{n,\Sigma}(z,x_0)^{-1} - \mathcal{G}_{1,n-1,\Sigma}(z,x_0)\mathcal{F}_{n,\Sigma}(z,x_0)^{-1} &&\text{(4.36a)}\\
&= \pm iR_{2n+1}(z)^{1/2}\mathcal{F}_{n,\Sigma}(z,x_0)^{-1} - \mathcal{F}_{n,\Sigma}(z,x_0)^{-1}\mathcal{G}_{2,n-1,\Sigma}(z,x_0), &&\text{(4.36b)}\\
&\qquad\qquad z\in\mathbb{C}\backslash\{\Sigma\cup\{\mu_k(x_0)\}_{1\leq k\leq N}\}
\end{aligned}$$

and

$$\mathcal{N}_{\pm,\Sigma}(z,x_0) = \mathcal{M}_{-,\Sigma}(z,x_0) \pm \mathcal{M}_{+,\Sigma}(z,x_0), \quad z\in\mathbb{C}\backslash\{\Sigma\cup\{\mu_k(x_0)\}_{1\leq k\leq N}\}. \tag{4.37}$$

We also introduce the open interior Σ^o of Σ defined by $\Sigma^o = \bigcup_{j=0}^{n-1}(E_{2j},E_{2j+1})\cup(E_{2n},\infty)$. Then one verifies the following fundamental facts.

Theorem 4.5. *Assume Hypothesis 4.4 and let $z\in\mathbb{C}\backslash\{\Sigma\cup\{\mu_k(x_0)\}_{1\leq k\leq N}\}$. Moreover, introduce the $2m\times 2m$ matrix $\mathcal{M}_\Sigma(\cdot,x_0)$ as in (4.16), (4.37) and the $m\times m$ matrices $\mathcal{M}_{\pm,\Sigma}(\cdot,x_0)$ as in (4.36). Then,*
(i) $\pm\mathcal{M}_{\pm,\Sigma}(\cdot,x_0)$ are $m\times m$ Herglotz matrices with representations

$$\begin{aligned}
\pm\mathcal{M}_{\pm,\Sigma}(z,x_0) = \frac{1}{\pi}\int_\Sigma d\lambda\, R_{2n+1}(\lambda)^{1/2}\mathcal{F}_{n,\Sigma}(\lambda,x_0)^{-1}\left(\frac{1}{\lambda-z}-\frac{\lambda}{1+\lambda^2}\right)\\
+\Gamma_{\Sigma,0}(x_0) - \sum_{k=1}^N \frac{1\pm\varepsilon_k(x_0)}{z-\mu_k(x_0)}\Gamma_{\Sigma,k}(x_0), \qquad\qquad\text{(4.38)}\\
z\in\mathbb{C}\backslash\{\Sigma\cup\{\mu_k(x_0)\}_{1\leq k\leq N}\}.
\end{aligned}$$

Moreover, $\mathcal{M}_{\pm,\Sigma}(\cdot,x_0)$ are the half-line $\mathcal{M}$-matrices associated with self-adjoint Schrödinger operators $H^D_{\pm,x_0,\Sigma}$ in $L^2([x_0,\pm\infty))^m$, with a Dirichlet boundary condition at the point x_0 and an $m\times m$ matrix-valued potential $\mathcal{Q}_\Sigma$ satisfying

$$\mathcal{Q}_\Sigma = \mathcal{Q}_\Sigma^* \in L^1_{\rm loc}(\mathbb{R})^{m\times m}\cap C^\infty(\mathbb{R}\backslash\{x_0\})^{m\times m}, \tag{4.39}$$

given by

$$\begin{aligned}
&H^D_{\pm,x_0,\Sigma} = -\mathcal{I}_m\frac{d^2}{dx^2}+\mathcal{Q}_\Sigma, &&\text{(4.40)}\\
&\operatorname{dom}(H^D_{\pm,x_0,\Sigma}) = \{g\in L^2((x_0,\pm\infty))^m \,|\, g,g'\in AC([x_0,c])^m \text{ for all } c\gtrless x_0;\\
&\qquad\qquad \lim_{\varepsilon\downarrow 0} g(x_0\pm\varepsilon) = 0;\ (-g''+\mathcal{Q}_\Sigma g)\in L^2((x_0,\pm\infty))^m\}.
\end{aligned}$$

(ii) The differential expression $\mathcal{L}_\Sigma = -\mathcal{I}_m\frac{d^2}{dx^2}+\mathcal{Q}_\Sigma$ is in the limit point case at $\pm\infty$.
(iii) The matrix $\mathcal{M}_\Sigma(\cdot,x_0)$, defined by

$$\begin{aligned}
\mathcal{M}_\Sigma(z,x_0) &= \big(\mathcal{M}_{\Sigma,p,q}(z,x_0)\big)_{1\leq p,q\leq 2} &&\text{(4.41)}\\
&= \begin{pmatrix} \mathcal{M}_{\pm,\Sigma}(z,x_0)\mathcal{N}_{-,\Sigma}(z,x_0)^{-1}\mathcal{M}_{\mp,\Sigma}(z,x_0) & \mathcal{N}_{-,\Sigma}(z,x_0)^{-1}\mathcal{N}_{+,\Sigma}(z,x_0)/2\\ \mathcal{N}_{+,\Sigma}(z,x_0)\mathcal{N}_{-,\Sigma}(z,x_0)^{-1}/2 & \mathcal{N}_{-,\Sigma}(z,x_0)^{-1}\end{pmatrix},
\end{aligned}$$

is a $2m \times 2m$ Herglotz matrix admitting a representation of the type (2.43)*, with measure $\Omega_\Sigma(\cdot, x)$ given by*

$$d\Omega_\Sigma(\lambda, x_0) = \begin{cases} \frac{1}{2\pi R_{2n+1}(\lambda)^{1/2}} \begin{pmatrix} \mathcal{H}_{n+1,\Sigma}(\lambda,x_0) & -\mathcal{G}_{2,n-1,\Sigma}(\lambda,x_0) \\ -\mathcal{G}_{1,n-1,\Sigma}(\lambda,x_0) & \mathcal{F}_{n,\Sigma}(\lambda,x_0) \end{pmatrix} d\lambda, & \lambda \in \Sigma^o, \\ 0, & \lambda \in \mathbb{R}\backslash\Sigma. \end{cases} \tag{4.42}$$

In addition, $\mathcal{M}_\Sigma(\cdot, x_0)$ is the Weyl–Titchmarsh $\mathcal{M}$-matrix associated with the self-adjoint Schrödinger operator H_Σ in $L^2(\mathbb{R})^m$ defined by

$$\begin{aligned} &H_\Sigma = -\mathcal{I}_m \frac{d^2}{dx^2} + \mathcal{Q}_\Sigma, \\ &\operatorname{dom}(H_\Sigma) = \{g \in L^2(\mathbb{R})^m \mid g, g' \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R})^m;\, (-g'' + \mathcal{Q}_\Sigma g) \in L^2(\mathbb{R})^m\}. \end{aligned} \tag{4.43}$$

(iv) H_Σ has purely absolutely continuous spectrum Σ,

$$\operatorname{spec}(H_\Sigma) = \operatorname{spec}_{ac}(H_\Sigma) = \Sigma, \quad \operatorname{spec}_{\mathrm{p}}(H_{\Sigma_n}) = \operatorname{spec}_{\mathrm{sc}}(H_{\Sigma_n}) = \emptyset, \tag{4.44}$$

with $\operatorname{spec}(H_{\Sigma_n})$ of uniform spectral multiplicity $2m$.

Proof. The representations (4.38) for $\mathcal{M}_{\pm,\Sigma}$ immediately follow from combining (4.21), (4.23), and (4.36a). These representations also prove that $\pm\mathcal{M}_{\pm,\Sigma}(\cdot, x_0)$ are $m \times m$ Herglotz matrices (cf. Theorem 2.3). Combining (4.38) with Theorem 2.4, taking into account Lemma 8.3.2 in [65], then yields the properties stated for $\mathcal{Q}_\Sigma$. That $\mathcal{L}_\Sigma = -\mathcal{I}_m \frac{d^2}{dx^2} + \mathcal{Q}_\Sigma$ is in the limit point case at $\pm\infty$ can be proved in analogy to Wienholtz's proof [109] of a result originally due to Povzner [84], reproduced as Theorem 35 in [41, p. 58]. The corresponding details will be presented in [22]. Equation (4.41) follows from (4.16), (4.23)–(4.26), and (4.36). Relation (4.44) follows from the explicit formula (4.42) of the spectral measure. In particular, the support property $\operatorname{supp}(\Omega_\Sigma) = \Sigma$ of the measure Ω_Σ in (4.42) proves $\operatorname{spec}(H_\Sigma) = \Sigma$, etc. The uniform maximum spectral multiplicity $2m$ then follows from the fact that $\operatorname{rank}(d\Omega_\Sigma/d\lambda) = 2m$ on the interior Σ^o of Σ. □

At this point we cannot yet infer continuity of $\mathcal{Q}_\Sigma$ at the boundary point x_0. We will subsequently return to this issue in Theorem 4.8.

In the following we will apply these facts to our concrete class of matrix-valued Schrödinger operators discussed in Theorem 4.5. In order to find the corresponding Weyl–Titchmarsh matrices $\mathcal{M}_{\pm,\Sigma}(z, x)$, we need some preparations. We denote by $\psi_{\pm,\Sigma}(z, x, x_0)$ the Weyl solutions (2.31) associated with $\mathcal{Q}_\Sigma$, that is,

$$\psi_{\pm,\Sigma}(z, x, x_0) = \theta_\Sigma(z, x, x_0) + \phi_\Sigma(z, x, x_0)\mathcal{M}_{\pm,\Sigma}(z, x_0), \quad z \in \mathbb{C}\backslash\Sigma, \tag{4.45}$$

where, in obvious notation, $\theta_\Sigma(z, x, x_0)$, $\phi_\Sigma(z, x, x_0)$ denote the fundamental system (2.4) corresponding to $\mathcal{Q}_\Sigma$. Then straightforward computations of the right-hand sides of (2.46)–(2.49) (taking into account (2.16), (2.17), (4.28), and (4.31)) yield

$$\begin{aligned} \mathcal{M}_\Sigma(z, x) &= \big(\mathcal{M}_{\Sigma,p,q}(z, x)\big)_{1\le p,q\le 2} \\ &= \frac{i}{2R_{2n+1}(z)^{1/2}} \begin{pmatrix} \mathcal{H}_{n+1,\Sigma}(z, x) & -\mathcal{G}_{2,n-1,\Sigma}(z, x) \\ -\mathcal{G}_{1,n-1,\Sigma}(z, x) & \mathcal{F}_{n,\Sigma}(z, x) \end{pmatrix}, \quad z \in \mathbb{C}\backslash\Sigma, \end{aligned} \tag{4.46}$$

where we abbreviated

$$\begin{aligned}
\mathcal{F}_{n,\Sigma}(z,x) &= \theta_\Sigma(z,x,x_0)\mathcal{F}_{n,\Sigma}(z,x_0)\theta_\Sigma(\overline{z},x,x_0)^* \\
&\quad + \phi_\Sigma(z,x,x_0)\mathcal{H}_{n+1,\Sigma}(z,x_0)\phi_\Sigma(\overline{z},x,x_0)^* \\
&\quad - \phi_\Sigma(z,x,x_0)\mathcal{G}_{1,n-1,\Sigma}(z,x_0)\theta_\Sigma(\overline{z},x,x_0)^* \\
&\quad - \theta_\Sigma(z,x,x_0)\mathcal{G}_{2,n-1,\Sigma}(z,x_0)\phi_\Sigma(\overline{z},x,x_0)^*, \tag{4.47} \\
\mathcal{G}_{1,n-1,\Sigma}(z,x) &= -\theta_\Sigma'(z,x,x_0)\mathcal{F}_{n,\Sigma}(z,x_0)\theta_\Sigma(\overline{z},x,x_0)^* \\
&\quad - \phi_\Sigma'(z,x,x_0)\mathcal{H}_{n+1,\Sigma}(z,x_0)\phi_\Sigma(\overline{z},x,x_0)^* \\
&\quad + \phi_\Sigma'(z,x,x_0)\mathcal{G}_{1,n-1,\Sigma}(z,x_0)\theta_\Sigma(\overline{z},x,x_0)^* \\
&\quad + \theta_\Sigma'(z,x,x_0)\mathcal{G}_{2,n-1,\Sigma}(z,x_0)\phi_\Sigma(\overline{z},x,x_0)^*, \tag{4.48} \\
\mathcal{G}_{2,n-1,\Sigma}(z,x) &= -\theta_\Sigma(z,x,x_0)\mathcal{F}_{n,\Sigma}(z,x_0)\theta_\Sigma'(\overline{z},x,x_0)^* \\
&\quad - \phi_\Sigma(z,x,x_0)\mathcal{H}_{n+1,\Sigma}(z,x_0)\phi_\Sigma'(\overline{z},x,x_0)^* \\
&\quad + \phi_\Sigma(z,x,x_0)\mathcal{G}_{1,n-1,\Sigma}(z,x_0)\theta_\Sigma'(\overline{z},x,x_0)^* \\
&\quad + \theta_\Sigma(z,x,x_0)\mathcal{G}_{2,n-1,\Sigma}(z,x_0)\phi_\Sigma'(\overline{z},x,x_0)^*, \tag{4.49} \\
\mathcal{H}_{n+1,\Sigma}(z,x) &= \theta_\Sigma'(z,x,x_0)\mathcal{F}_{n,\Sigma}(z,x_0)\theta_\Sigma'(\overline{z},x,x_0)^* \\
&\quad + \phi_\Sigma'(z,x,x_0)\mathcal{H}_{n+1,\Sigma}(z,x_0)\phi_\Sigma'(\overline{z},x,x_0)^* \\
&\quad - \phi_\Sigma'(z,x,x_0)\mathcal{G}_{1,n-1,\Sigma}(z,x_0)\theta_\Sigma'(\overline{z},x,x_0)^* \\
&\quad - \theta_\Sigma'(z,x,x_0)\mathcal{G}_{2,n-1,\Sigma}(z,x_0)\phi_\Sigma'(\overline{z},x,x_0)^*. \tag{4.50}
\end{aligned}$$

Considerations of this type can be found in [65, Sect. 8.2] in the special scalar case $m = 1$ and in the matrix context $m \in \mathbb{N}$ in [101, Sect. 9.4].

Differentiating (4.47)–(4.50) with respect to x (taking into account that $\theta'' = (\mathcal{Q}_\Sigma - z\mathcal{I}_m)\theta$, $\phi'' = (\mathcal{Q}_\Sigma - z)\phi$) then yields (4.51)–(4.58) below. Alternatively, these results directly follow from Lemma 2.9 identifying $\mathfrak{g}$ and $(i/2)R_{2n+1}^{-1/2}\mathcal{F}_n$, $\mathfrak{g}_p$ and $(i/2)R_{2n+1}^{-1/2}\mathcal{G}_{p,n-1}$, $p = 1, 2$, and $\mathfrak{h}$ and $(i/2)R_{2n+1}^{-1/2}\mathcal{H}_{n+1}$, respectively.

Lemma 4.6. *Assume Hypothesis* 4.4 *and let* $(z,x) \in \mathbb{C} \times \mathbb{R}$. *Then*

$$\begin{aligned}
\mathcal{F}_{n,\Sigma}' &= -(\mathcal{G}_{1,n-1,\Sigma} + \mathcal{G}_{2,n-1,\Sigma}), && (4.51) \\
\mathcal{G}_{1,n-1,\Sigma}' &= -(Q_\Sigma - z\mathcal{I}_m)\mathcal{F}_{n,\Sigma} - \mathcal{H}_{n+1,\Sigma} && (4.52) \\
&= (-\mathcal{F}_{n,\Sigma}'' + \mathcal{F}_{n,\Sigma}\mathcal{Q}_\Sigma - \mathcal{Q}_\Sigma\mathcal{F}_{n,\Sigma})/2, && (4.53) \\
\mathcal{G}_{1,n-1,\Sigma}'' &= -2(\mathcal{Q}_\Sigma - z\mathcal{I}_m)\mathcal{F}_{n,\Sigma}' - \mathcal{Q}_\Sigma'\mathcal{F}_{n,\Sigma} + \mathcal{G}_{1,n-1,\Sigma}\mathcal{Q}_\Sigma - \mathcal{Q}_\Sigma\mathcal{G}_{1,n-1,\Sigma}, && (4.54) \\
\mathcal{G}_{2,n-1,\Sigma}' &= -\mathcal{F}_{n,\Sigma}(Q_\Sigma - z\mathcal{I}_m) - \mathcal{H}_{n+1,\Sigma} && (4.55) \\
&= (-\mathcal{F}_{n,\Sigma}'' + \mathcal{Q}_\Sigma\mathcal{F}_{n,\Sigma} - \mathcal{F}_{n,\Sigma}\mathcal{Q}_\Sigma)/2, && (4.56) \\
\mathcal{G}_{2,n-1,\Sigma}'' &= -2\mathcal{F}_{n,\Sigma}'(\mathcal{Q}_\Sigma - z\mathcal{I}_m) - \mathcal{F}_{n,\Sigma}\mathcal{Q}_\Sigma' + \mathcal{Q}_\Sigma\mathcal{G}_{2,n-1,\Sigma} - \mathcal{G}_{2,n-1,\Sigma}\mathcal{Q}_\Sigma, && (4.57) \\
\mathcal{H}_{n+1,\Sigma}' &= -\mathcal{G}_{1,n-1,\Sigma}(Q_\Sigma - z\mathcal{I}_m) - (Q_\Sigma - z\mathcal{I}_m)\mathcal{G}_{2,n-1,\Sigma}, && (4.58) \\
\mathcal{H}_{n+1,\Sigma} &= [\mathcal{F}_{n,\Sigma}'' - \mathcal{F}_{n,\Sigma}(\mathcal{Q}_\Sigma - z\mathcal{I}_m) - (\mathcal{Q}_\Sigma - z\mathcal{I}_m)\mathcal{F}_{n,\Sigma}]/2. && (4.59)
\end{aligned}$$

In particular, one also verifies the following facts from (2.56)–(2.63).

Lemma 4.7. *Assume Hypothesis* 4.4 *and let* $(z,x) \in \mathbb{C} \times \mathbb{R}$. *Then*

$$\mathcal{F}_{n,\Sigma}(\overline{z},x)^* = \mathcal{F}_{n,\Sigma}(z,x), \quad \mathcal{H}_{n+1,\Sigma}(\overline{z},x)^* = \mathcal{H}_{n+1,\Sigma}(z,x),$$
$$\mathcal{G}_{2,n-1,\Sigma}(\overline{z},x)^* = \mathcal{G}_{1,n-1,\Sigma}(z,x), \tag{4.60}$$
$$\mathcal{F}_{n,\Sigma}(z,x)\mathcal{G}_{1,n-1,\Sigma}(z,x) = \mathcal{G}_{2,n-1,\Sigma}(z,x)\mathcal{F}_{n,\Sigma}(z,x), \tag{4.61}$$
$$\mathcal{H}_{n+1,\Sigma}(z,x)\mathcal{G}_{2,n-1,\Sigma}(z,x) = \mathcal{G}_{1,n-1,\Sigma}(z,x)\mathcal{H}_{n+1,\Sigma}(z,x), \tag{4.62}$$
$$\mathcal{H}_{n+1,\Sigma}(z,x)\mathcal{F}_{n,\Sigma}(z,x) - \mathcal{G}_{1,n-1,\Sigma}(z,x)^2 = R_{2n+1}(z)\mathcal{I}_m, \tag{4.63}$$
$$\mathcal{F}_{n,\Sigma}(z,x)\mathcal{H}_{n+1,\Sigma}(z,x) - \mathcal{G}_{2,n-1,\Sigma}(z,x)^2 = R_{2n+1}(z)\mathcal{I}_m. \tag{4.64}$$

Proof. (4.60) is clear from (4.27), the fact that $\mathcal{F}_{n,\Sigma}(\cdot,x_0)$ and $\mathcal{H}_{n+1,\Sigma}(\cdot,x_0)$ are self-adjoint $m \times m$ matrix pencils, (4.48), and (4.49). Similarly, (4.61)–(4.64) follow from elementary (but somewhat tedious) calculations directly from (4.47)–(4.50), invoking (2.10)–(2.17) and (4.28)–(4.31) repeatedly. □

Combining (2.50)–(2.54) and (4.46) then yields

$$\mathcal{M}_{\pm,\Sigma}(z,x)$$
$$= \pm i R_{2n+1}(z)^{1/2}\mathcal{F}_{n,\Sigma}(z,x)^{-1} - \mathcal{G}_{1,n-1,\Sigma}(z,x)\mathcal{F}_{n,\Sigma}(z,x)^{-1} \tag{4.65a}$$
$$= \pm i R_{2n+1}(z)^{1/2}\mathcal{F}_{n,\Sigma}(z,x)^{-1} - \mathcal{F}_{n,\Sigma}(z,x)^{-1}\mathcal{G}_{2,n-1,\Sigma}(z,x), \tag{4.65b}$$
$$z \in \mathbb{C}\backslash\mathbb{R}.$$

One observes that for each $x \in \mathbb{R}$, $\mathcal{M}_{+,\Sigma}(\cdot,x)$ is the analytic continuation of $\mathcal{M}_{-,\Sigma}(\cdot,x)$ through the set Σ, and vice versa,

$$\lim_{\varepsilon \downarrow 0} \mathcal{M}_{+,\Sigma}(\lambda + i\varepsilon, x) = \lim_{\varepsilon \downarrow 0} \mathcal{M}_{-,\Sigma}(\lambda - i\varepsilon, x), \tag{4.66}$$
$$\lambda \in \bigcup_{j=0}^{n-1} (E_{2j}, E_{2j+1}) \cup (E_{2n}, \infty), \ x \in \mathbb{R}.$$

In other words, for each $x \in \mathbb{R}$, $\mathcal{M}_{+,\Sigma}(\cdot,x)$ and $\mathcal{M}_{-,\Sigma}(\cdot,x)$ are the two branches of an analytic matrix-valued function $\mathcal{M}_\Sigma(\cdot,x)$ on the two-sheeted Riemann surface of $R_{2n+1}^{1/2}$. Thus, the corresponding potential $\mathcal{Q}_\Sigma$ is *reflectionless* in the sense discussed in [23], [40], and [56].

Thus, one obtains the following results.

Theorem 4.8. *Assume Hypothesis* 4.4 *and let* $z \in \mathbb{C}\backslash\mathbb{R}$ *and* $x \in \mathbb{R}$. *Then*
(*i*) $\mathcal{M}_{\pm,\Sigma}(z,\cdot)$ *in* (4.65) *satisfy the matrix-valued Riccati-type equation*

$$\mathcal{M}'_{\pm,\Sigma}(z,x) + \mathcal{M}_{\pm,\Sigma}(z,x)^2 = \mathcal{Q}_\Sigma(x) - z\mathcal{I}_m, \quad x \in \mathbb{R}, \ z \in \mathbb{C}\backslash\mathbb{R}. \tag{4.67}$$

Moreover, $\mathcal{M}_{\pm,\Sigma}(z,x)$ *in* (4.65) *are the* $m \times m$ *Weyl–Titchmarsh matrices associated with* $H^D_{\pm,x,\Sigma}$ *on the half-lines* $[x, \pm\infty)$ *and thus for each* $x \in \mathbb{R}$, $\mathcal{M}_\Sigma(z,x)$ *in* (4.46) *is a* $2m \times 2m$ *Weyl–Titchmarsh matrix associated with* H_Σ *on* $\mathbb{R}$. *In particular,* $\mathcal{M}_\Sigma(\cdot,x_0)$ *is a* $2m \times 2m$ *Herglotz matrix of* H_Σ *admitting a representation of the type* (2.43), *with measure* $\Omega_\Sigma(\cdot,x)$ *given by*

$$d\Omega_\Sigma(\lambda,x) = \begin{cases} \frac{1}{2\pi R_{2n+1}(\lambda)^{1/2}} \begin{pmatrix} \mathcal{H}_{n+1,\Sigma}(\lambda,x) & -\mathcal{G}_{2,n-1,\Sigma}(\lambda,x) \\ -\mathcal{G}_{1,n-1,\Sigma}(\lambda,x) & \mathcal{F}_{n,\Sigma}(\lambda,x) \end{pmatrix} d\lambda, & \lambda \in \Sigma^o, \\ 0, & \lambda \in \mathbb{R}\backslash\Sigma. \end{cases} \tag{4.68}$$

(*ii*) $\mathcal{F}_{n,\Sigma}(\cdot,x)$ *and* $\mathcal{H}_{n+1,\Sigma}(\cdot,x)$ *are strongly hyperbolic* (*and hence self-adjoint*) $m \times m$ *monic matrix pencils of degree* n *and* $n+1$, *respectively, and* $\mathcal{G}_{p,n-1,\Sigma}(\cdot,x)$,

$p = 1, 2$, are $m \times m$ matrix pencils of degree $n - 1$.
(iii) $\mathcal{Q}_\Sigma \in C^\infty(\mathbb{R})^{m\times m}$.

Proof. (4.67) is clear from Lemma 4.6 and (4.65). Since $\mathcal{Q}_\Sigma \in C^\infty(\mathbb{R})^{m\times m}$, the initial value problems

$$\mathcal{M}'_\pm(z,x) + \mathcal{M}_\pm(z,x)^2 = \mathcal{Q}_\Sigma(x) - z\mathcal{I}_m, \quad x \in \mathbb{R},\ z \in \mathbb{C}\backslash\mathbb{R}, \tag{4.69a}$$

$$\mathcal{M}_\pm(z,x_0) = \mathcal{M}_{\pm,\Sigma}(z,x_0), \tag{4.69b}$$

with $\mathcal{M}_{\pm,\Sigma}(z,x_0)$ given by (4.36), has a unique solution. Since at $x = x_0$ this solution coincides with the Weyl–Titchmarsh M-matrix $\mathcal{M}_{\pm,\Sigma}(z,x)|_{x=x_0}$ in (4.65) (using the initial condition (2.5) in (4.47)–(4.50)), $\mathcal{M}_{\pm,\Sigma}(z,x)$ represents the Weyl–Titchmarsh matrices associated with $H^D_{\pm,\Sigma}$ on the half-lines $[x, \pm\infty)$. This proves part (i). By the known leading asymptotic behavior (2.33) of $\mathcal{M}_{\pm,\Sigma}(\cdot,x)$ (valid for each $x_0 \in \mathbb{R}$, see [21]) and that of the diagonal Green's matrix $\mathcal{G}_\Sigma(\cdot,x,x) = \mathcal{M}_{\Sigma,2,2}(\cdot,x)$ of H_Σ as $|z| \to \infty$, $\mathcal{F}_{n,\Sigma}(\cdot,x)$ and $\mathcal{H}_{n+1,\Sigma}(\cdot,x)$ are monic matrix pencils of degree n and $n+1$, respectively, and $\mathcal{G}_{p,n-1,\Sigma}(\cdot,x)$, $p = 1,2$, are $m \times m$ matrix pencils of degree $n-1$. Since the diagonal blocks of each Herglotz matrix are also Herglotz matrices, one concludes that $(i/2)R_{2n+1}^{-1/2}\mathcal{F}_{n,\Sigma}$ and $(i/2)R_{2n+1}^{-1/2}\mathcal{H}_{n+1,\Sigma}$ are Herglotz matrices. By Corollary 4.2 this then proves that $\mathcal{F}_{n,\Sigma}$ and $\mathcal{H}_{n+1,\Sigma}$ are strongly hyperbolic pencils and hence item (ii) holds. As in Theorem 4.5, $\mathcal{Q}_\Sigma \in C^\infty((-\infty,x)\cup(x,\infty))^{m\times m}$. Since $x \in \mathbb{R}$ is arbitrary, this proves (iii). □

It should be emphasized that the construction of $\mathcal{Q}_\Sigma$ along the lines of Section 4 in the scalar case $m = 1$ is due to Levitan [62] (see also [63], [64], [65, Ch. 8], [67]).

5. Trace Formulas and Connections with the Stationary Matrix KdV Hierarchy

In this section we introduce the stationary matrix Korteweg–de Vries (KdV) hierarchy and show that the class of finite-band potentials $\mathcal{Q}_\Sigma$ constructed in Section 4 satisfies some (and hence infinitely many) equations of the stationary KdV equations. We also introduce trace formulas for KdV invariants.

In order to extend the recursive approach constructing KdV Lax pairs in the scalar (Abelian) context to the present matrix-valued (non-Abelian) setting, we focus on an efficient approach introduced by Dubrovin [28] (in the scalar case $m = 1$). Recalling (2.50)–(2.63) and Lemma 2.9, we state the following matrix-version of Dubrovin's generating function approach to higher-order Lax pairs.

We start by introducing the following hypothesis.

Hypothesis 5.1. *Fix $m \in \mathbb{N}$, suppose $\mathcal{Q} = \mathcal{Q}^* \in C^\infty(\mathbb{R})^{m\times m}$ and introduce the differential expression*

$$\mathcal{L} = -\mathcal{I}_m \frac{d^2}{dx^2} + \mathcal{Q}, \quad x \in \mathbb{R}. \tag{5.1}$$

Suppose $\mathcal{L}$ is in the limit point case at $\pm\infty$ and introduce the corresponding self-adjoint operator H in $L^2(\mathbb{R})^m$ by

$$H = -\mathcal{I}_m \frac{d^2}{dx^2} + \mathcal{Q}, \tag{5.2}$$
$$\operatorname{dom}(H) = \{g \in L^2(\mathbb{R})^m \mid g, g' \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R})^m;\ (-g'' + \mathcal{Q}g) \in L^2(\mathbb{R})^m\}.$$

Given Hypothesis 5.1, we introduce the generating operator P_z by

$$P_z = \Big(\mathfrak{g}(z,\cdot)\frac{d}{dx} + \mathfrak{g}_2(z,\cdot)\Big)(\mathcal{L} - z\mathcal{I}_m)^{-1}, \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{5.3}$$
$$\operatorname{dom}(P_z) = \bigcup_{E\in\mathbb{C}\backslash\{\mathbb{R}\cup\{z\}\}} \ker(\mathcal{L} - E\mathcal{I}_m),$$

where $\ker(\mathcal{L} - E\mathcal{I}_m)$ denotes the algebraic nullspace of $\mathcal{L} - E\mathcal{I}_m$ (as opposed to the functional analytic nullspace in $L^2(\mathbb{R})^m$) and $(\mathcal{L} - z\mathcal{I}_m)^{-1}$ acts in the obvious manner by

$$(\mathcal{L} - z\mathcal{I}_m)^{-1}\psi = (E - z)^{-1}\psi, \quad \psi \in \ker(\mathcal{L} - E\mathcal{I}_m). \tag{5.4}$$

The precise operator theoretic properties of P_z will be irrelevant in the following.

Lemma 5.2. *Assume Hypothesis* 5.1 *and let* $z \in \mathbb{C}\backslash\mathbb{R}$, $x \in \mathbb{R}$. *Then,*

$$\Big[\Big(\mathfrak{g}(z,\cdot)\frac{d}{dx} + \mathfrak{g}_2(z,\cdot)\Big)(\mathcal{L} - z\mathcal{I}_m)^{-1}, \mathcal{L}\Big]\Big|_{\bigcup_{E\in\mathbb{C}\backslash\{\mathbb{R}\cup\{z\}\}}\ker(\mathcal{L}-E\mathcal{I}_m)}$$
$$= -2\mathfrak{g}'(z,\cdot)\Big|_{\bigcup_{E\in\mathbb{C}\backslash\{\mathbb{R}\cup\{z\}\}}\ker(\mathcal{L}-E\mathcal{I}_m)}. \tag{5.5}$$

Proof. Fix $\psi \in \ker(\mathcal{L} - E\mathcal{I}_m)$ for some $E \in \mathbb{C}\backslash\{\mathbb{R}\cup\{z\}\}$. Then one computes

$$\begin{aligned}
&\Big[\Big(\mathfrak{g}(z,\cdot)\frac{d}{dx} + \mathfrak{g}_2(z,\cdot)\Big)(\mathcal{L} - z\mathcal{I}_m)^{-1}, \mathcal{L}\Big]\psi \\
&= (E - z)^{-1}\{[2\mathfrak{g}_2' + \mathfrak{g}'' + \mathfrak{g}\mathcal{Q} - \mathcal{Q}\mathfrak{g}]\psi' \\
&\quad [\mathfrak{g}_2'' + 2\mathfrak{g}'(\mathcal{Q} - z\mathcal{I}_m) + \mathfrak{g}\mathcal{Q}' + \mathfrak{g}_2\mathcal{Q} - \mathcal{Q}\mathfrak{g}_2 + 2(z - E)\mathfrak{g}']\psi\} \\
&= -2\mathfrak{g}'\psi,
\end{aligned} \tag{5.6}$$

using (2.68) and (2.72). ∎

A variant of Dubrovin's idea of a generating operator for KdV Lax pairs was also used by Olmedilla, Martínez Alonso, and Guil [79]. Their approach, however, focuses on formal operator expansions and formal pseudo-differential operators. For a different approach we refer to [27, Ch. 15], [32].

Next, we recall that

$$\mathfrak{g}(z,x) = [\mathcal{M}_-(z,x) - \mathcal{M}_+(z,x)]^{-1}, \tag{5.7}$$
$$\mathfrak{g}_1(z,x) = \frac{1}{2}[\mathcal{M}_-(z,x) + \mathcal{M}_+(z,x)][\mathcal{M}_-(z,x) - \mathcal{M}_+(z,x)]^{-1}, \tag{5.8}$$
$$\mathfrak{g}_2(z,x) = \frac{1}{2}[\mathcal{M}_-(z,x) - \mathcal{M}_+(z,x)]^{-1}[\mathcal{M}_-(z,x) + \mathcal{M}_+(z,x)], \tag{5.9}$$
$$\mathfrak{h}(z,x) = \mathcal{M}_\pm(z,x)[\mathcal{M}_-(z,x) - \mathcal{M}_+(z,x)]^{-1}\mathcal{M}_\mp(z,x), \tag{5.10}$$

and note that by Theorem 2.7 the right-hand sides of (5.7) and (5.9) admit asymptotic expansions in cones avoiding the spectrum of H. In particular, one thus

obtains the asymptotic expansions

$$\mathfrak{g}(z,x) \underset{\substack{|z|\to\infty\\ z\in C_\varepsilon}}{=} \frac{i}{2z^{1/2}} \sum_{k=0}^{\infty} \widehat{\mathfrak{R}}_k(x) z^{-k}, \quad \widehat{\mathfrak{R}}_0(x) = \mathcal{I}_m, \tag{5.11}$$

$$\mathfrak{g}_p(z,x) \underset{\substack{|z|\to\infty\\ z\in C_\varepsilon}}{=} \frac{i}{2z^{1/2}} \sum_{k=0}^{\infty} \widehat{\mathfrak{G}}_{p,k}(x) z^{-k}, \quad \widehat{\mathfrak{G}}_{p,0}(x) = 0, \quad p = 1,2, \tag{5.12}$$

$$\mathfrak{h}(z,x) \underset{\substack{|z|\to\infty\\ z\in C_\varepsilon}}{=} \frac{iz^{1/2}}{2} \sum_{k=0}^{\infty} \widehat{\mathfrak{H}}_k(x) z^{-k}, \quad \widehat{\mathfrak{H}}_0(x) = \mathcal{I}_m, \tag{5.13}$$

for some coeficients $\widehat{\mathfrak{R}}_k$, $\widehat{\mathfrak{G}}_{p,k}$, $p = 1,2$, and $\widehat{\mathfrak{H}}_k$, which are universal differential polynomials in $\mathcal{Q}$ by Remark 2.8 (i). Explicitly, one obtains

$$\widehat{\mathfrak{R}}_0 = \mathcal{I}_m, \quad \widehat{\mathfrak{R}}_1 = \tfrac{1}{2}\mathcal{Q}, \quad \widehat{\mathfrak{R}}_2 = -\tfrac{1}{8}\mathcal{Q}'' + \tfrac{3}{8}\mathcal{Q}^2, \tag{5.14}$$

$$\widehat{\mathfrak{G}}_{1,0} = -\tfrac{1}{4}\mathcal{Q}', \quad \widehat{\mathfrak{G}}_{1,1} = \tfrac{1}{16}\mathcal{Q}''' - \tfrac{1}{8}(\mathcal{Q}^2)' - \tfrac{1}{8}\mathcal{Q}'\mathcal{Q}, \tag{5.15}$$

$$\widehat{\mathfrak{G}}_{2,0} = -\tfrac{1}{4}\mathcal{Q}', \quad \widehat{\mathfrak{G}}_{2,1} = \tfrac{1}{16}\mathcal{Q}''' - \tfrac{1}{8}(\mathcal{Q}^2)' - \tfrac{1}{8}\mathcal{Q}\mathcal{Q}', \tag{5.16}$$

$$\widehat{\mathfrak{H}}_0 = \mathcal{I}_m, \quad \widehat{\mathfrak{H}}_1 = -\tfrac{1}{2}\mathcal{Q}, \quad \widehat{\mathfrak{H}}_2 = \tfrac{1}{8}\mathcal{Q}'' - \tfrac{1}{8}\mathcal{Q}^2, \tag{5.17}$$

etc.

Motivated by Lemma 5.2, we now introduce the $m\times m$ matrix-valued differential expressions $\widehat{\mathcal{P}}_{2k+1}$ by

$$\widehat{\mathcal{P}}_{2k+1} = \sum_{\ell=0}^{k} \Big(\widehat{\mathfrak{R}}_\ell \frac{d}{dx} + \widehat{\mathfrak{G}}_{2,\ell}\Big) \mathcal{L}^{k-\ell}, \quad k \in \mathbb{N}_0. \tag{5.18}$$

In analogy to Lemma 5.2 one then obtains the following result.

Lemma 5.3. *Assume Hypothesis* 5.1 *and let* $z \in \mathbb{C}\backslash\mathbb{R}$, $x \in \mathbb{R}$. *Then,*

$$\big[\widehat{\mathcal{P}}_{2k+1}, \mathcal{L}\big] = 2\widehat{\mathfrak{R}}'_{k+1}, \quad k \in \mathbb{N}_0. \tag{5.19}$$

Proof. Assuming $\psi \in \ker(\mathcal{L} - z\mathcal{I}_m)$ one computes

$$\begin{aligned}
\big[\widehat{\mathcal{P}}_{2k+1}, \mathcal{L}\big]\psi &= \sum_{\ell=0}^{k} z^{k-\ell} \big(2\widehat{\mathfrak{G}}'_{2,\ell} + \widehat{\mathfrak{R}}''_\ell + \widehat{\mathfrak{R}}_\ell \mathcal{Q} - \mathcal{Q}\widehat{\mathfrak{R}}_\ell\big)\psi' \\
&+ \sum_{\ell=0}^{k} z^{k-\ell} \big(\widehat{\mathfrak{G}}''_{2,\ell} + 2\widehat{\mathfrak{R}}'_\ell(\mathcal{Q} - z\mathcal{I}_m) + \widehat{\mathfrak{R}}_\ell \mathcal{Q}' + \widehat{\mathfrak{G}}_{2,\ell}\mathcal{Q} - \mathcal{Q}\widehat{\mathfrak{G}}_{2,\ell}\big)\psi \\
&= \sum_{\ell=0}^{k} z^{k-\ell} \big(2\widehat{\mathfrak{G}}'_{2,\ell} + \widehat{\mathfrak{R}}''_\ell + \widehat{\mathfrak{R}}_\ell \mathcal{Q} - \mathcal{Q}\widehat{\mathfrak{R}}_\ell\big)\psi' - 2\widehat{\mathfrak{R}}'_0\psi \\
&+ \sum_{\ell=0}^{k} z^{k-\ell} \big(\widehat{\mathfrak{G}}''_{2,\ell} - 2\widehat{\mathfrak{R}}'_{\ell+1} + 2\widehat{\mathfrak{R}}'_\ell \mathcal{Q} + \widehat{\mathfrak{R}}_\ell \mathcal{Q}' + \widehat{\mathfrak{G}}_{2,\ell}\mathcal{Q} - \mathcal{Q}\widehat{\mathfrak{G}}_{2,\ell}\big)\psi + 2\widehat{\mathfrak{R}}'_{k+1}\psi \\
&= 2\widehat{\mathfrak{R}}'_{k+1}\psi.
\end{aligned} \tag{5.20}$$

Here we used $\widehat{\mathfrak{R}}'_0 = 0$ and

$$2\widehat{\mathfrak{G}}'_{2,\ell} + \widehat{\mathfrak{R}}''_\ell + \widehat{\mathfrak{R}}_\ell \mathcal{Q} - \mathcal{Q}\widehat{\mathfrak{R}}_\ell = 0, \quad \ell \in \mathbb{N}_0, \tag{5.21}$$

$$\widehat{\mathfrak{G}}''_{2,\ell} - 2\widehat{\mathfrak{R}}'_{\ell+1} + 2\widehat{\mathfrak{R}}'_\ell \mathcal{Q} + \widehat{\mathfrak{R}}_\ell \mathcal{Q}' + \widehat{\mathfrak{G}}_{2,\ell}\mathcal{Q} - \mathcal{Q}\widehat{\mathfrak{G}}_{2,\ell} = 0, \quad \ell \in \mathbb{N}_0, \tag{5.22}$$

which follow from inserting the asymptotic expansion (5.11) and (5.12) into (2.68) and (2.72) (which is permitted by Theorem 2.7). Relation (5.20) implies (5.19) since $\widehat{\mathcal{P}}_{2k+1}$ and $\mathcal{L}$ are $m \times m$ matrix-valued differential expressions of finite-order while $\bigcup_{z\in\mathbb{C}} \ker(\mathcal{L} - z\mathcal{I}_m)$ is an infinite-dimensional space of $C^\infty(\mathbb{R})^m$-functions. □

Introducing

$$\mathcal{P}_{2k+1} = \sum_{\ell=0}^{k} c_{k-\ell} \widehat{\mathcal{P}}_{2\ell+1}, \quad k \in \mathbb{N}_0, \tag{5.23}$$

where

$$\{c_\ell\}_{\ell=1,\dots,k} \subset \mathbb{C}, \quad c_0 = 1 \tag{5.24}$$

denotes a set of constants, the pairs $(\mathcal{P}_{2k+1}, \mathcal{L})$, $k \in \mathbb{N}_0$, by definition, represent the Lax pairs of the (matrix-valued) KdV hierarchy. More precisely, varying $\{c_\ell\}_{\ell\in\mathbb{N}} \subset \mathbb{C}$, the set of evolution equations,

$$\frac{d}{dt}\mathcal{L} - [\mathcal{P}_{2k+1}, \mathcal{L}] = 0, \quad k \in \mathbb{N}_0, \tag{5.25}$$

or equivalently, the set of equations,

$$\mathrm{KdV}_k(\mathcal{Q}) = \mathcal{Q}_t - 2\sum_{\ell=0}^{k} c_{k-\ell} \widehat{\mathfrak{R}}'_{\ell+1}(\mathcal{Q}, \dots) = 0, \quad k \in \mathbb{N}_0, \tag{5.26}$$

represents the (matrix-valued) KdV hierarchy of evolution equations. The corresponding stationary KdV hierarchy, characterized by

$$\mathcal{Q}_t = 0, \text{ or equivalently, by } [\mathcal{P}_{2k+1}, \mathcal{L}] = 0, \quad k \in \mathbb{N}_0, \tag{5.27}$$

is then given by

$$\text{s-KdV}_k(\mathcal{Q}) = -2\sum_{\ell=0}^{k} c_{k-\ell} \widehat{\mathfrak{R}}'_{\ell+1}(\mathcal{Q}, \dots) = 0, \quad k \in \mathbb{N}_0. \tag{5.28}$$

Remark 5.4. *By Remark* 2.8, *each* $\widehat{\mathfrak{R}}_\ell$ *is a universal polynomial in* $\mathcal{Q}$ *and its* x*-derivatives and occasionally we slightly abuse notation and indicate this by writing* $\widehat{\mathfrak{R}}_\ell(\mathcal{Q}, \dots)$ *for* $\widehat{\mathfrak{R}}_\ell(x)$, $\widehat{\mathfrak{R}}'_{\ell+1}(\mathcal{Q}, \dots)$ *for* $\widehat{\mathfrak{R}}'_{\ell+1}(x)$, *etc. Explicit formulas for the differential polynomials* $\widehat{\mathfrak{R}}_\ell(\mathcal{Q}, \dots)$ *were derived for instance, in* [8] *and* [83].

In order to make the connection with the finite-band formalism of Section 4 we now recall

$$\frac{i}{2R_{2n+1}(z)^{1/2}}\mathcal{F}_{n,\Sigma}(z,x) = [\mathcal{M}_{-,\Sigma}(z,x) - \mathcal{M}_{+,\Sigma}(z,x)]^{-1}, \tag{5.29}$$

$$\begin{aligned}&\frac{i}{2R_{2n+1}(z)^{1/2}}\mathcal{G}_{1,n-1,\Sigma}(z,x)\\ &\quad = \frac{1}{2}[\mathcal{M}_{-,\Sigma}(z,x) + \mathcal{M}_{+,\Sigma}(z,x)][\mathcal{M}_{-,\Sigma}(z,x) - \mathcal{M}_{+,\Sigma}(z,x)]^{-1},\end{aligned} \tag{5.30}$$

$$\begin{aligned}&\frac{i}{2R_{2n+1}(z)^{1/2}}\mathcal{G}_{2,n-1,\Sigma}(z,x)\\ &\quad = \frac{1}{2}[\mathcal{M}_{-,\Sigma}(z,x) - \mathcal{M}_{+,\Sigma}(z,x)]^{-1}[\mathcal{M}_{-,\Sigma}(z,x) + \mathcal{M}_{+,\Sigma}(z,x)],\end{aligned} \tag{5.31}$$

$$\begin{aligned}&\frac{i}{2R_{2n+1}(z)^{1/2}}\mathcal{H}_{n+1,\Sigma}(z,x)\\ &\quad = \mathcal{M}_{\pm,\Sigma}(z,x)[\mathcal{M}_{-,\Sigma}(z,x) - \mathcal{M}_{+,\Sigma}(z,x)]^{-1}\mathcal{M}_{\mp,\Sigma}(z,x),\end{aligned} \tag{5.32}$$

and note that (5.29)–(5.32) admit expansions convergent in a neighborhood of infinity. In particular,

$$\frac{1}{R_{2n+1}(z)^{1/2}}\mathcal{F}_{n,\Sigma}(z,x) \underset{\substack{|z|\to\infty\\ z\in C_\varepsilon}}{=} \frac{1}{z^{1/2}}\sum_{k=0}^{\infty}\widehat{\mathfrak{R}}_{k,\Sigma}(x)z^{-k}, \quad \widehat{\mathfrak{R}}_{0,\Sigma}(x) = \mathcal{I}_m, \tag{5.33}$$

$$\frac{1}{R_{2n+1}(z)^{1/2}}\mathcal{G}_{p,n-1,\Sigma}(z,x) \underset{\substack{|z|\to\infty\\ z\in C_\varepsilon}}{=} \frac{1}{z^{1/2}}\sum_{k=0}^{\infty}\widehat{\mathfrak{G}}_{p,k,\Sigma}(x)z^{-k}, \quad \widehat{\mathfrak{G}}_{p,0,\Sigma}(x) = 0,\ p=1,2, \tag{5.34}$$

$$\frac{1}{R_{2n+1}(z)^{1/2}}\mathcal{H}_{n+1,\Sigma}(z,x) \underset{\substack{|z|\to\infty\\ z\in C_\varepsilon}}{=} \frac{1}{z^{1/2}}\sum_{k=0}^{\infty}\widehat{\mathfrak{H}}_{k,\Sigma}(x)z^{-k}, \quad \widehat{\mathfrak{H}}_{0,\Sigma}(x) = \mathcal{I}_m, \tag{5.35}$$

for $|z|$ sufficiently large. Here the coefficients $\widehat{\mathfrak{R}}_{k,\Sigma}$ and $\widehat{\mathfrak{G}}_{2,k,\Sigma}$ are the universal differential polynomials $\widehat{\mathfrak{R}}_k = \widehat{\mathfrak{R}}_k(\mathcal{Q}_\Sigma,\dots)$ and $\widehat{\mathfrak{G}}_{2,k} = \widehat{\mathfrak{G}}_{2,k}(\mathcal{Q}_\Sigma,\dots)$ in (5.11) and (5.12) (with $\mathcal{Q}$ replaced by $\mathcal{Q}_\Sigma$). We also recall

$$\mathcal{F}_{n,\Sigma}(z,x) = \sum_{\ell=0}^{n}\mathcal{F}_{n-\ell,\Sigma}(x)z^\ell, \quad \mathcal{F}_{0,\Sigma}(x) = \mathcal{I}_m, \tag{5.36}$$

$$\mathcal{G}_{p,n-1,\Sigma}(z,x) = \sum_{\ell=0}^{n-1}\mathcal{G}_{p,n-1-\ell,\Sigma}(x)z^\ell, \quad p=1,2, \tag{5.37}$$

$$\mathcal{H}_{n+1,\Sigma}(z,x) = \sum_{\ell=0}^{n}\mathcal{H}_{n+1-\ell,\Sigma}(x)z^\ell, \quad \mathcal{H}_{0,\Sigma}(x) = \mathcal{I}_m. \tag{5.38}$$

Since we seek the connection between the set of coefficients $\widehat{\mathfrak{R}}_{k,\Sigma}$, $\widehat{\mathfrak{G}}_{p,k,\Sigma}$, $\widehat{\mathfrak{H}}_{k,\Sigma}$ and $\mathcal{F}_{k,\Sigma}$, $\mathcal{G}_{p,k,\Sigma}$, $\mathcal{H}_{k,\Sigma}$, we next consider the following elementary expansions. Let

$$\eta\in\mathbb{C} \ \text{ such that } \ |\eta| < \min\{|E_0|^{-1},\dots,|E_{2n}|^{-1}\}. \tag{5.39}$$

Then

$$\bigg(\prod_{\ell=0}^{2n}\big(1-E_\ell\eta\big)\bigg)^{-1/2}=\sum_{k=0}^{\infty}\hat{c}_k\big(\underline{E}\big)\eta^k, \tag{5.40}$$

where

$$\begin{aligned}&\hat{c}_0\big(\underline{E}\big)=1,\\ &\hat{c}_k\big(\underline{E}\big)=\sum_{\substack{j_0,\dots,j_{2n}=0\\ j_0+\cdots+j_{2n}=k}}^{k}\frac{(2j_0)!\cdots(2j_{2n})!}{2^{2k}(j_0!)^2\cdots(j_{2n}!)^2}E_0^{j_0}\cdots E_{2n}^{j_{2n}},\quad k\in\mathbb{N}.\end{aligned} \tag{5.41}$$

The first few coefficients explicitly read

$$\begin{aligned}&\hat{c}_0\big(\underline{E}\big)=1,\ \hat{c}_1\big(\underline{E}\big)=\frac{1}{2}\sum_{\ell=0}^{2n}E_\ell,\\ &\hat{c}_2\big(\underline{E}\big)=\frac{1}{4}\sum_{\substack{\ell_1,\ell_2=0\\ \ell_1<\ell_2}}^{2n}E_{\ell_1}E_{\ell_2}+\frac{3}{8}\sum_{\ell=0}^{2n}E_\ell^2,\quad\text{etc.}\end{aligned} \tag{5.42}$$

Similarly, one has

$$\bigg(\prod_{\ell=0}^{2n}\big(1-E_\ell\eta\big)\bigg)^{1/2}=\sum_{k=0}^{\infty}c_k\big(\underline{E}\big)\eta^k, \tag{5.43}$$

where

$$\begin{aligned}&c_0\big(\underline{E}\big)=1,\\ &c_k\big(\underline{E}\big)=-\sum_{\substack{j_0,\dots,j_{2n}=0\\ j_0+\cdots+j_{2n}=k}}^{k}\frac{(2j_0)!\cdots(2j_{2n})!}{2^{2k}(j_0!)^2\cdots(j_{2n}!)^2(2j_0-1)\cdots(2j_{2n}-1)}E_0^{j_0}\cdots E_{2n}^{j_{2n}},\\ &\qquad\qquad k\in\mathbb{N}.\end{aligned} \tag{5.44}$$

The first few coefficients explicitly are given by

$$\begin{aligned}&c_0\big(\underline{E}\big)=1,\quad c_1\big(\underline{E}\big)=-\frac{1}{2}\sum_{\ell=0}^{2n}E_\ell,\\ &c_2\big(\underline{E}\big)=\frac{1}{4}\sum_{\substack{\ell_1,\ell_2=0\\ \ell_1<\ell_2}}^{2n}E_{\ell_1}E_{\ell_2}-\frac{1}{8}\sum_{\ell=0}^{2n}E_\ell^2,\quad\text{etc.}\end{aligned} \tag{5.45}$$

Lemma 5.5. *Assume Hypothesis* 5.1 *and let* $x \in \mathbb{R}$. *Then,*

$$\mathcal{F}_{\ell,\Sigma}(x) = \sum_{k=0}^{\ell} c_{\ell-k}(\underline{E}) \widehat{\mathfrak{R}}_{k,\Sigma}(x), \quad \ell = 0, \dots, n, \tag{5.46}$$

$$\widehat{\mathfrak{R}}_{\ell,\Sigma}(x) = \sum_{k=0}^{\ell} \hat{c}_{\ell-k}(\underline{E}) \mathcal{F}_{k,\Sigma}(x), \quad \ell = 0, \dots, n, \tag{5.47}$$

$$\mathcal{G}_{p,\ell,\Sigma}(x) = \sum_{k=0}^{\ell} c_{\ell-k}(\underline{E}) \widehat{\mathfrak{G}}_{p,k+1,\Sigma}(x), \quad \ell = 0, \dots, n-1, \ p = 1, 2, \tag{5.48}$$

$$\widehat{\mathfrak{G}}_{p,\ell+1,\Sigma}(x) = \sum_{k=0}^{\ell} \hat{c}_{\ell-k}(\underline{E}) \mathcal{G}_{p,k,\Sigma}(x), \quad \ell = 0, \dots, n-1, \ p = 1, 2, \tag{5.49}$$

$$\mathcal{H}_{\ell,\Sigma}(x) = \sum_{k=0}^{\ell} c_{\ell-k}(\underline{E}) \widehat{\mathfrak{H}}_{k,\Sigma}(x), \quad \ell = 0, \dots, n+1, \tag{5.50}$$

$$\widehat{\mathfrak{H}}_{\ell,\Sigma}(x) = \sum_{k=0}^{\ell} \hat{c}_{\ell-k}(\underline{E}) \mathcal{H}_{k,\Sigma}(x), \quad \ell = 0, \dots, n+1. \tag{5.51}$$

Proof. It suffices to prove (5.46) and (5.47) and so we omit the analogous proofs of (5.48)–(5.51). Since for $|z|$ sufficiently large,

$$\begin{aligned} z^{-n} \mathcal{F}_{n,\Sigma}(z,x) &= \sum_{\ell=0}^{n} \mathcal{F}_{\ell,\Sigma}(x) z^{-\ell} \\ &= z^{-n-(1/2)} R_{2n+1}(z)^{1/2} \sum_{\ell=0}^{\infty} \widehat{\mathfrak{R}}_{\ell,\Sigma}(x) z^{-\ell} \\ &= \sum_{k=0}^{\infty} c_k(\underline{E}) z^{-k} \sum_{\ell=0}^{\infty} \widehat{\mathfrak{R}}_{\ell,\Sigma}(x) z^{-\ell} \\ &= \sum_{\ell=0}^{\infty} \bigg(\sum_{k=0}^{\ell} c_{\ell-k}(\underline{E}) \widehat{\mathfrak{R}}_{k,\Sigma}(x) \bigg) z^{-\ell} \end{aligned} \tag{5.52}$$

and hence (5.46). Equation (5.47) is then clear from (5.46) and

$$\sum_{\ell=0}^{k} \hat{c}_{k-\ell}(\underline{E}) c_{\ell}(\underline{E}) = \delta_{k,0}, \quad k \in \mathbb{N}_0. \tag{5.53}$$

The latter follows from multiplying (5.40) and (5.43), comparing coefficients of η^k. □

Given these preliminaries we can now state the principal result of this section.

Theorem 5.6. *The self-adjoint finite-band potential* $\mathcal{Q}_\Sigma \in C^\infty(\mathbb{R})^{m \times m}$, *discussed in Theorems* 4.5 *and* 4.8, *is a stationary KdV solution satisfying*

$$\text{s-KdV}_n(\mathcal{Q}_\Sigma) = -2 \sum_{\ell=0}^{n} c_{n-\ell}(\underline{E}) \widehat{\mathfrak{R}}'_{\ell+1}(\mathcal{Q}_\Sigma, \dots) = 0, \tag{5.54}$$

with $c_\ell(\underline{E})$ *defined in* (5.44) *and* $\widehat{\mathfrak{R}}_{\ell+1}$ *the universal differential polynomials* (*with respect to* $\mathcal{Q}$) *in* (5.11).

Proof. Introducing the $m \times m$ matrix-valued differential expression $\mathcal{P}_{2k+1,\Sigma}$ by

$$\mathcal{P}_{2n+1,\Sigma} = \sum_{\ell=0}^{n} c_{n-\ell}(\underline{E}) \widehat{\mathcal{P}}_{2\ell+1} = \sum_{\ell=0}^{n} \Big(\mathcal{F}_{n-\ell,\Sigma}(\cdot) \frac{d}{dx} + \mathcal{G}_{2,n-1-\ell,\Sigma}(\cdot) \Big) \mathcal{L}^{\ell} \tag{5.55}$$

(cf. (5.46), (5.48), and (5.23)), one computes for $\psi \in \ker(\mathcal{L} - z\mathcal{I}_m)$, $z \in \mathbb{C}$,

$$\mathcal{P}_{2n+1,\Sigma}\psi = \mathcal{F}_{n,\Sigma}(z,x)\psi' + \mathcal{G}_{2,n-1,\Sigma}(z,x)\psi, \tag{5.56}$$

and hence,

$$\begin{aligned} [\mathcal{P}_{2n+1,\Sigma}, \mathcal{L}]\psi &= [2\mathcal{G}_{2,n-1,\Sigma} + \mathcal{F}''_{n,\Sigma}\mathcal{F}_{n,\Sigma}\mathcal{Q}_\Sigma - \mathcal{Q}_\Sigma\mathcal{F}_{n,\Sigma}]\psi' \\ &+ [\mathcal{G}''_{2,n-1,\Sigma} + 2\mathcal{F}'_{n,\Sigma}(\mathcal{Q}_\Sigma - z\mathcal{I}_m) + \mathcal{F}_{n,\Sigma}\mathcal{Q}'_\Sigma + \mathcal{G}_{2,n-1,\Sigma}\mathcal{Q}_\Sigma - \mathcal{Q}_\Sigma\mathcal{G}_{2,n-1,\Sigma}]\psi = 0 \end{aligned} \tag{5.57}$$

by (4.56) and (4.57). Since $z \in \mathbb{C}$ is arbitrary, this implies

$$[\mathcal{P}_{2n+1,\Sigma}, \mathcal{L}] = 0, \tag{5.58}$$

completing the proof by (5.27), (5.28). □

Next, we turn to a discussion of trace formulas for the finite-band potential $\mathcal{Q}_\Sigma$ in terms of (matrix) roots of $\mathcal{F}_{n,\Sigma}$ and $\mathcal{H}_{n+1,\Sigma}$.

Theorem 5.7. *Let $(z,x) \in \mathbb{C} \times \mathbb{R}$ and assume $\mathcal{Q}_\Sigma$ to be the self-adjoint finite-band potential discussed in Theorems* 4.5 *and* 4.8. *In addition, let the monic self-adjoint matrix pencils $\mathcal{F}_{n,\Sigma}$ and $\mathcal{H}_{n+1,\Sigma}$ be given by* (4.47) *and* (4.50). *Then $\mathcal{F}_{n,\Sigma}(\cdot,x)$ and $\mathcal{H}_{n+1,\Sigma}(\cdot,x)$ are strongly hyperbolic pencils and hence admit the factorizations*

$$\mathcal{F}_{n,\Sigma}(z,x) = (z\mathcal{I}_m - \mathcal{U}_n(x))(z\mathcal{I}_m - \mathcal{U}_{n-1}(x)) \cdots (z\mathcal{I}_m - \mathcal{U}_1(x)), \tag{5.59}$$

$$\mathcal{H}_{n+1,\Sigma}(z,x) = (z\mathcal{I}_m - \mathcal{V}_n(x))(z\mathcal{I}_m - \mathcal{V}_{n-1}(x)) \cdots (z\mathcal{I}_m - \mathcal{V}_0(x)), \tag{5.60}$$

with

$$\operatorname{spec}(\mathcal{V}_0(x)) = \operatorname{spec}(\mathcal{H}_{n+1,\Sigma}) \cap \overline{\Delta_0(\mathcal{H}_{n+1,\Sigma})} \subset (-\infty, E_0], \tag{5.61}$$

$$\operatorname{spec}(\mathcal{U}_j(x)) = \operatorname{spec}(\mathcal{F}_{n,\Sigma}) \cap \overline{\Delta_j(\mathcal{F}_{n,\Sigma})} \subseteq [E_{2j-1}, E_{2j}], \quad 1 \leq j \leq n, \tag{5.62}$$

$$\operatorname{spec}(\mathcal{V}_j(x)) = \operatorname{spec}(\mathcal{H}_{n+1,\Sigma}) \cap \overline{\Delta_0(\mathcal{H}_{n+1,\Sigma})} \subseteq [E_{2j-1}, E_{2j}], \quad 1 \leq j \leq n. \tag{5.63}$$

Moreover, one obtains the sequence of trace formulas

$$\sum_{\substack{j_1,j_2,\dots,j_k=0 \\ j_1<j_2<\cdots<j_k}}^{n} \mathcal{U}_{j_k}(x) \cdots \mathcal{U}_{j_2}(x)\mathcal{U}_{j_1}(x) = \sum_{\ell=0}^{k} c_{k-\ell}(\underline{E}) \widehat{\mathfrak{R}}_{\ell,\Sigma}(x), \quad 1 \leq k \leq n, \tag{5.64}$$

$$0 = \sum_{\ell=0}^{k} c_{k-\ell}(\underline{E}) \widehat{\mathfrak{R}}_{\ell,\Sigma}(x), \quad k \geq n+1, \tag{5.65}$$

$$\sum_{\substack{j_0,j_1,\dots,j_k=0 \\ j_0<j_1<\cdots<j_k}}^{n} \mathcal{V}_{j_k}(x) \cdots \mathcal{V}_{j_1}(x)\mathcal{V}_{j_0}(x) = \sum_{\ell=0}^{k} c_{k-\ell}(\underline{E}) \widehat{\mathfrak{H}}_{\ell,\Sigma}(x), \quad 1 \leq k \leq n, \tag{5.66}$$

$$0 = \sum_{\ell=0}^{k} c_{k-\ell}(\underline{E}) \widehat{\mathfrak{H}}_{\ell,\Sigma}(x), \quad k \geq n+1. \tag{5.67}$$

In particular, in the special case $k = 1$, $\mathcal{Q}_\Sigma$ satisfies the trace formulas

$$\mathcal{Q}_\Sigma(x) = \bigg(\sum_{\ell=0}^{2n} E_\ell\bigg)\mathcal{I}_m - 2\sum_{j=1}^{n} \mathcal{U}_j(x), \tag{5.68}$$

$$= -\bigg(\sum_{\ell=0}^{2n} E_\ell\bigg)\mathcal{I}_m + 2\sum_{k=0}^{n} \mathcal{V}_k(x). \tag{5.69}$$

In addition, one obtains

$$\mathcal{Q}_\Sigma^{(r)} \in C^\infty(\mathbb{R})^{m\times m} \cap L^\infty(\mathbb{R})^{m\times m} \text{ for all } r \in \mathbb{N}_0. \tag{5.70}$$

Proof. By (5.29) and (5.32), $i/R_{2n+1}^{1/2}\mathcal{F}_{n,\Sigma}(\cdot,x)$ and $i/R_{2n+1}^{-1/2}\mathcal{H}_{n+1,\Sigma}(\cdot,x)$ are Herglotz matrices and hence (4.14) and (4.15) apply. In particular, $\mathcal{F}_{n,\Sigma}(\cdot,x)$ and $\mathcal{H}_{n+1,\Sigma}(\cdot,x)$ are strongly hyperbolic pencils. Since both are monic, the factorizations (5.59) and (5.60), as well as (5.61)–(5.63), hold by Theorem 3.5 (ii). By (5.33) one infers

$$\begin{aligned} z^{-n}\mathcal{F}_{n,\Sigma}(z,x) &= \sum_{k=0}^{n}\bigg(\sum_{\substack{j_1,j_2,\dots,j_k=0\\ j_1<j_2<\cdots<j_k}}^{n} \mathcal{U}_{j_k}(x)\cdots\mathcal{U}_{j_2}(x)\mathcal{U}_{j_1}(x)\bigg)z^{-k} \\ &\underset{\substack{|z|\to\infty\\ z\in C_\varepsilon}}{=} z^{-n-(1/2)}R_{2n+1}(z)^{1/2}\sum_{k=0}^{\infty}\widehat{\mathfrak{R}}_{k,\Sigma}(x)z^{-k} \\ &\underset{\substack{|z|\to\infty\\ z\in C_\varepsilon}}{=} \bigg(\prod_{\ell=0}^{2n}\big(1-(E_\ell/z)\big)\bigg)^{1/2}\sum_{k=0}^{\infty}\widehat{\mathfrak{R}}_{k,\Sigma}(x)z^{-k} \\ &\underset{\substack{|z|\to\infty\\ z\in C_\varepsilon}}{=} \sum_{k=0}^{\infty}\bigg(\sum_{\ell=0}^{k} c_\ell(\underline{E})\widehat{\mathfrak{R}}_{k-\ell,\Sigma}(x)\bigg)z^{-k}. \end{aligned} \tag{5.71}$$

Comparing coefficients z^{-k}, $k \in \mathbb{N}$, then yields (5.64) and (5.65). In the special case $k = 1$ one infers

$$-\sum_{j=1}^{n}\mathcal{U}_j(x) = -\frac{1}{2}\bigg(\sum_{\ell=0}^{2n}E_\ell\bigg)\mathcal{I}_m + \widehat{\mathfrak{R}}_{1,\Sigma}(x) \tag{5.72}$$

and since $\widehat{\mathfrak{R}}_{1,\Sigma} = \mathcal{Q}_\Sigma/2$ by (5.14), the trace formula (5.68) for $\mathcal{Q}_\Sigma$ results. (5.66), (5.67), and (5.69) are proved analogously. By (5.62), $\mathcal{U}_j \in L^\infty(\mathbb{R})^{m\times m}$, $1 \le j \le n$ and hence

$$\widehat{\mathfrak{R}}_k \in C^\infty(\mathbb{R})^{m\times m} \cap L^\infty(\mathbb{R})^{m\times m}, \quad k \in \mathbb{N}_0 \tag{5.73}$$

(since $\mathcal{Q}_\Sigma \in C^\infty(\mathbb{R})^{m\times m}$ by Theorem 4.8 (iii)). An analysis of the recursion relation (2.49) for $\mathcal{M}_{\pm,k}$ combined with (5.7), (5.11) then proves that $\widehat{\mathfrak{R}}_k$ is of the form

$$\widehat{\mathfrak{R}}_k = d_k\mathcal{Q}^{(2k-2)} + \mathcal{R}_k(\mathcal{Q}^{(2k-4)},\dots), \quad k \ge 2, \tag{5.74}$$

with $d_k \in \mathbb{R}$ appropriate constants and $\mathcal{R}_k$ abbreviating a differential polynomial in $\mathcal{Q}$ which contains $\mathcal{Q}^{(2k-4)}$ as the highest derivative of $\mathcal{Q}$. Hence one infers (5.70). □

The factorizations (5.59), (5.60), eigenvalue distributions (5.61)–(5.63), and trace formulas (5.64)–(5.69) are extensions of well-known formulas in the scalar case $m = 1$ (see, e.g., [28], [31], [36], [38], [50], [67]).

Finally, the property $\mathcal{Q}_\Sigma \in C^\infty(\mathbb{R})^{m\times m}$ in Theorem 4.8 (iii) can be improved upon by using the system (4.51), (4.52), (4.55), and (4.58). In fact, writing

$$\mathcal{F}_{n,\Sigma}(z,x) = \sum_{\ell=0}^{n} \mathcal{F}_{n-\ell,\Sigma}(x) z^\ell, \quad \mathcal{F}_{0,\Sigma}(x) = \mathcal{I}_m, \tag{5.75}$$

$$\mathcal{G}_{p,n-1,\Sigma}(z,x) = \sum_{\ell=0}^{n-1} \mathcal{G}_{p,n-1-\ell,\Sigma}(x) z^\ell, \quad p = 1,2, \tag{5.76}$$

$$\mathcal{H}_{n+1,\Sigma}(z,x) = \sum_{\ell=0}^{n+1} \mathcal{H}_{n+1-\ell,\Sigma}(x) z^\ell, \quad \mathcal{H}_{0,\Sigma}(x) = \mathcal{I}_m, \tag{5.77}$$

one obtains the following result.

Lemma 5.8. *Assume Hypothesis* 4.4 *and let* $(z,x) \in \mathbb{C}\times\mathbb{R}$. *Then the coefficients in* (5.75)–(5.77) *satisfy the autonomous nonlinear first-order system*

$$\mathcal{F}'_{\ell,\Sigma} = -(\mathcal{G}_{1,\ell-1,\Sigma} + \mathcal{G}_{2,\ell-1,\Sigma}), \quad 1 \le \ell \le n, \tag{5.78}$$

$$\mathcal{G}'_{1,\ell,\Sigma} = -(\mathcal{F}_{1,\Sigma} - \mathcal{H}_{1,\Sigma})\mathcal{F}_{\ell+1,\Sigma} + \mathcal{F}_{\ell+2,\Sigma} - \mathcal{H}_{\ell+2,\Sigma}, \quad 0 \le \ell \le n-1, \tag{5.79}$$

$$\mathcal{G}'_{2,\ell,\Sigma} = -\mathcal{F}_{\ell+1,\Sigma}(\mathcal{F}_{1,\Sigma} - \mathcal{H}_{1,\Sigma}) + \mathcal{F}_{\ell+2,\Sigma} - \mathcal{H}_{\ell+2,\Sigma}, \quad 0 \le \ell \le n-1, \tag{5.80}$$

$$\mathcal{H}'_{\ell,\Sigma} = \mathcal{G}_{1,\ell-1,\Sigma} + \mathcal{G}_{2,\ell-1,\Sigma} - \mathcal{G}_{1,\ell-2,\Sigma}(\mathcal{F}_{1,\Sigma} - \mathcal{H}_{1,\Sigma}) - (\mathcal{F}_{1,\Sigma} - \mathcal{H}_{1,\Sigma})\mathcal{G}_{2,\ell-2,\Sigma}, \quad 1 \le \ell \le n+1, \tag{5.81}$$

$$\mathcal{F}_{n+1,\Sigma} = 0, \quad \mathcal{G}_{p,n,\Sigma} = \mathcal{G}_{p,-1,\Sigma} = 0, \quad p = 1,2. \tag{5.82}$$

Moreover, $\mathcal{F}_{\ell,\Sigma}$, $0 \le \ell \le n$, $\mathcal{G}_{p,\ell,\Sigma}$, $0 \le \ell \le n-1$, $p = 1,2$, $\mathcal{H}_{\ell,\Sigma}$, $0 \le \ell \le n+1$, *and hence* $\mathcal{Q}_\Sigma$, *are all analytic in an open neighborhood containing the real axis.*

Proof. Inserting (5.75)–(5.77) into (4.51), (4.52), (4.55), and (4.58) yields

$$\mathcal{F}'_{\ell,\Sigma} = -(\mathcal{G}_{1,\ell-1,\Sigma} + \mathcal{G}_{2,\ell-1,\Sigma}), \quad 1 \le \ell \le n, \tag{5.83}$$

$$\mathcal{G}'_{1,\ell,\Sigma} = -\mathcal{Q}_\Sigma\mathcal{F}_{\ell+1,\Sigma} + \mathcal{F}_{\ell+2,\Sigma} - \mathcal{H}_{\ell+2,\Sigma}, \quad 0 \le \ell \le n-1, \tag{5.84}$$

$$\mathcal{G}'_{2,\ell,\Sigma} = -\mathcal{F}_{\ell+1,\Sigma}\mathcal{Q}_\Sigma + \mathcal{F}_{\ell+2,\Sigma} - \mathcal{H}_{\ell+2,\Sigma}, \quad 0 \le \ell \le n-1, \tag{5.85}$$

$$\mathcal{H}'_{\ell,\Sigma} = \mathcal{G}_{1,\ell-1,\Sigma} + \mathcal{G}_{2,\ell-1,\Sigma} - (\mathcal{G}_{1,\ell-2,\Sigma}\mathcal{Q}_\Sigma + \mathcal{Q}_\Sigma\mathcal{G}_{2,\ell-2,\Sigma}), \quad 1 \le \ell \le n+1, \tag{5.86}$$

$$\mathcal{F}_{n+1,\Sigma} = 0, \quad \mathcal{G}_{p,n,\Sigma} = \mathcal{G}_{p,-1,\Sigma} = 0, \quad p = 1,2, \tag{5.87}$$

$$\mathcal{Q}_\Sigma = \mathcal{F}_{1,\Sigma} - \mathcal{H}_{1,\Sigma}. \tag{5.88}$$

Insertion of (5.88) into (5.84)–(5.86) yields the autonomous nonlinear first-order system (5.78)–(5.82). Given the initial conditions

$$\begin{aligned} &\mathcal{F}_{\ell,\Sigma}(x_0), \quad 1 \le \ell \le n, \\ &\mathcal{G}_{p,\ell,\Sigma}(x_0), \quad p = 1,2, \ 0 \le \ell \le n-1, \\ &\mathcal{H}_{\ell,\Sigma}(x_0), \quad 1 \le \ell \le n+1, \end{aligned} \tag{5.89}$$

determined by (5.75)–(5.77) and the half-line Weyl–Titchmarsh matrices $\mathcal{M}_\pm(z,x_0)$ in (4.36), the maximal interval of existence of the solution of the autonomous system (5.78)–(5.82), (5.89) is all of $[0,\infty)$ and $(-\infty,0]$, and hence all of $\mathbb{R}$ (cf. [42, p. 18]), applying (5.70). Thus, one recovers the $C^\infty(\mathbb{R})^{m\times m}$-property $\mathcal{F}_\ell$, $\mathcal{G}_{p,\ell}$, $p = 1,2$,

$\mathcal{H}_\ell$, and hence that of $\mathcal{Q}_\Sigma$. Moreover, since the Picard iterations are convergent in sufficiently small circles in $\mathbb{C}$ centered around each $x \in \mathbb{R}$ (cf. [44, Sect. 2.3]), the unique solution obtained by these Picard iterations is analytic in each of these circles. □

We conclude with a remark that puts the construction of $\mathcal{Q}_\Sigma$ in Section 4 into proper perspective.

Remark 5.9. *The simplest examples of potentials $\mathcal{Q}_\Sigma$ described in Theorem* 4.8 *are of the type*

$$\mathcal{Q}_\Sigma(x) = \operatorname{diag}(q_{1,\Sigma}(x), \dots, q_{m,\Sigma}(x)), \tag{5.90}$$

where $q_{j,\Sigma}$, $1 \leq j \leq m$, are isospectral algebro-geometric finite-band potentials associated with scalar Schrödinger operators in $L^2(\mathbb{R})$ with spectrum Σ. The next simplest and closely related set of such examples for $\mathcal{Q}_\Sigma$ then will be of the type

$$\mathcal{Q}_\Sigma(x) = \mathcal{U} \operatorname{diag}(q_{1,\Sigma}(x), \dots, q_{m,\Sigma}(x)) \mathcal{U}^{-1}, \tag{5.91}$$

where $\mathcal{U} \in \mathbb{C}^{m\times m}$ is a unitary $m \times m$ matrix independent of $x \in \mathbb{R}$. At first sight one might think that perhaps all matrix potentials $\mathcal{Q}_\Sigma$ are of the form (5.91). *That this is certainly not the case will be argued next. Indeed, assuming that* (5.91) *holds for some unitary $m \times m$ matrix $\mathcal{U}$ independent of x, one infers that*

$$\mathcal{Q}_\Sigma^{(r)}(x) = \mathcal{U} \operatorname{diag}(q_{1,\Sigma}^{(r)}(x), \dots, q_{m,\Sigma}^{(r)}(x)) \mathcal{U}^{-1} \text{ for all } r \in \mathbb{N}_0. \tag{5.92}$$

Since by Remark 2.8 *(i) and Lemma* 5.5 *all coefficients of $\mathcal{F}_{n,\Sigma}$, $\mathcal{G}_{p,n-1,\Sigma}$, $p = 1, 2$, and $\mathcal{H}_{n+1,\Sigma}$ are differential polynomials with respect to $\mathcal{Q}_\Sigma$,* (5.92) *implies*

$$\mathcal{U}\mathcal{F}_{n,\Sigma}(z,x)\mathcal{U}^{-1} = \operatorname{diag}(F_{n,\Sigma,1}(z,x), \dots, F_{n,\Sigma,m}(z,x)), \tag{5.93}$$

$$\mathcal{U}\mathcal{G}_{p,n-1,\Sigma}(z,x)\mathcal{U}^{-1} = \operatorname{diag}(G_{p,n-1,\Sigma,1}(z,x), \dots, G_{p,n-1,\Sigma,m}(z,x)), \quad p = 1, 2, \tag{5.94}$$

$$\mathcal{U}\mathcal{H}_{n+1,\Sigma}(z,x)\mathcal{U}^{-1} = \operatorname{diag}(H_{n+1,\Sigma,1}(z,x), \dots, H_{n+1,\Sigma,m}(z,x)), \tag{5.95}$$

$$\mathcal{U}\mathcal{M}_{\pm,\Sigma}(z,x)\mathcal{U}^{-1} = \operatorname{diag}(m_{\pm,\Sigma,1}(z,x), \dots, m_{\pm,\Sigma,m}(z,x)). \tag{5.96}$$

Consequently, one obtains for all $z, z' \in \mathbb{C}_+$, $x, x' \in \mathbb{R}$,

$$[\mathcal{Q}_\Sigma^{(r)}(x), \mathcal{Q}_\Sigma^{(r)}(x')] = 0, \quad r \in \mathbb{N}_0, \tag{5.97}$$

$$[\mathcal{F}_{n,\Sigma}(z,x), \mathcal{F}_{n,\Sigma}(z',x')] = 0, \tag{5.98}$$

$$[\mathcal{G}_{p,n-1,\Sigma}(z,x), \mathcal{G}_{p,n-1,\Sigma}(z',x')] = 0, \quad p = 1, 2, \tag{5.99}$$

$$[\mathcal{H}_{n+1,\Sigma}(z,x), \mathcal{H}_{n+1,\Sigma}(z',x')] = 0, \tag{5.100}$$

$$[\mathcal{M}_{\pm,\Sigma}(z,x), \mathcal{M}_{\pm,\Sigma}(z',x')] = 0. \tag{5.101}$$

In particular,

$$[\mathcal{F}_{n,\Sigma}(z,x_0), \mathcal{F}_{n,\Sigma}(z',x_0)] = 0 \text{ for all } z, z' \in \mathbb{C}_+. \tag{5.102}$$

Thus, whenever

$$[\mathcal{F}_{n,\Sigma}(z,x_0), \mathcal{F}_{n,\Sigma}(z',x_0)] \neq 0 \text{ for some } z, z' \in \mathbb{C}_+ \tag{5.103}$$

(which can easily be arranged for $n \geq 2$), (5.91) *cannot hold for a unitary $m \times m$ matrix $\mathcal{U}$ independent of x.*

Additional results, including extensions of Borg's and Hochstadt's theorems in the special cases $n = 0, 1$, respectively, will appear in [9].

Acknowledgements. We thank Robert Carlson, Konstantin Makarov, Mark Malamud, Fedor Rofe-Beketov, and Barry Simon for many helpful discussions on the material presented in this paper. In particular, we are grateful to Konstantin Makarov for a critical reading of this manuscript.

References

[1] Z. S. Agranovich and V.A. Marchenko, *The Inverse Problem of Scattering Theory*, Gordon and Breach, New York, 1963.

[2] D. Alpay and I. Gohberg, *Inverse spectral problem for differential operators with rational scattering matrix functions*, J. Diff. Eq. **118**, 1–19 (1995).

[3] D. Alpay and I. Gohberg, *Inverse problem for Sturm-Liouville operators with rational reflection coefficient*, Integral Equations Operator Theory **30**, 317–325 (1998).

[4] D. Z. Arov and H. Dym, *J-inner matrix functions, interpolation and inverse problems for canonical systems,I: foundations*, Integral Equations Operator Theory **29**, 373–454 (1997).

[5] D. Z. Arov and H. Dym, *J-inner matrix functions, interpolation and inverse problems for canonical systems,II: the inverse monodromy problem*, Integral Equations Operator Theory **36**, 11–70 (2000).

[6] D. Z. Arov and H. Dym, *J-inner matrix functions, interpolation and inverse problems for canonical systems,III: more on the inverse monodromy problem*, Integral Equations Operator Theory **36**, 127–181 (2000).

[7] N. Asano and Y. Kato, *Algebraic and Spectral Methods for Nonlinear Wave Equations*, Longman, New York, 1990.

[8] I. Avramidi and R. Schimming, *A new explicit expression for the Korteweg–de Vries hierarchy*, Math. Nachr. **219**, 45–64 (2000).

[9] E. D. Belokolos, F. Gesztesy, K. A. Makarov, and L. A. Sakhnovich, *Matrix-valued generalizations of the theorems of Borg and Hochstadt*, preprint, 2001.

[10] R. Benguria and M. Loss, *A simple proof of a theorem of Laptev and Weidl*, Math. Res. Lett. **7**, 195–203 (2000).

[11] F. Calogero and A. Degasperis, *Nonlinear evolution equations solvable by the inverse spectral transfor. II*, Nuovo Cim. **39B**, 1–54 (1977).

[12] R. Carlson, *Large eigenvalues and trace formulas for matrix Sturm-Liouville problems*, SIAM J. Math. Anal. **30**, 949–962 (1999).

[13] R. Carlson, *Compactness of Floquet isospectral sets for the matrix Hill's equation*, Proc. Amer. Math. Soc. **128**, 2933–2941 (2000).

[14] R. Carlson, *Eigenvalue estimates and trace formulas for the matrix Hill's equation*, J. Diff. Eq. **167**, 211–244 (2000).

[15] R. Carlson, *An inverse problem for the matrix Schrödinger equation*, preprint, 2001.

[16] R. Carmona and J. Lacroix, *Spectral Theory of Random Schrödinger operators*, Birkhäuser, Boston, MA, 1990.

[17] H.-H. Chern, *On the construction of isospectral vectorial Sturm-Liouville differential equations*, preprint, 1998.

[18] H.-H. Chern, C. K. Law, and H.-J. Wang, *Extension of Ambarzumyan's theorem to the general boundary conditions*, preprint, 2000.

[19] H.-H. Chern and C-L. Shen, *On the n-dimensional Ambarzumyan's theorem*, Inverse Problems **13**, 15–18 (1997).

[20] S. L. Clark, *Asymptotic behavior of the Titchmarsh–Weyl coefficient for a coupled second order system*, in "*Ordinary and Delay Differential Equations*", J. Wiener and J. K. Hale (eds.), Longman, New York, 1992, pp. 24–28.

[21] S. Clark and F. Gesztesy, *Weyl–Titchmarsh M-function asymptotics for matrix-valued Schrödinger operators*, Proc. London Math. Soc. **82**, 701–724 (2001).

[22] S. Clark and F. Gesztesy, in preparation.

[23] S. Clark, F. Gesztesy, H. Holden, and B. M. Levitan, *Borg-type theorems for matrix-valued Schrödinger and Dirac operators*, J. Diff. Eqs. **167**, 181–210 (2000).

[24] S. Clark and D. Hinton, *A Liapunov inequality for linear Hamiltonian systems*, Math. Inequ. Appl. **1**, 201–209 (1998).
[25] V. I. Derguzov, *The spectrum of Hamilton's operator with periodic coefficients*, Vestnik Leningrad Univ. Math. **12**, 280–285 (1980).
[26] B. Després, *The Borg theorem for the vectorial Hill's equation*, Inverse Probl. **11**, 97–121 (1995).
[27] L. A. Dickey, *Soliton Equations and Hamiltonian Systems*, World Scientific, Singapore, 1991.
[28] B. A. Dubrovin, *Periodic problems for the Korteweg-de Vries equation in the class of finite-gap potentials*, Funct. Anal. Appl. **9**, 215–223 (1975).
[29] B. A. Dubrovin, *Completely integrable Hamiltonian Systems associated with Matrix operators and Abelian varieties*, Funct. Anal. Appl. **11**, 265–277 (1977).
[30] B. A. Dubrovin, *Matrix finite-zone operators*, Revs. Sci. Tech. **23**, 20–50 (1983).
[31] B. A. Dubrovin, V. B. Matveev, and S. P. Novikov, *Non-linear equations of Korteweg-de Vries type, finite-zone linear operators, and Abelian varieties*, Russian Math. Surv. **31:1**, 59–146 (1976).
[32] I. M. Gel'fand and L. A. Dikii, *The resolvent and Hamiltonian systems*, Funct. Anal. Appl. **11**, 93–105 (1977).
[33] I. M. Gelfand and B. M. Levitan, *On the determination of a differential equation from its spectral function*, Izv. Akad. Nauk SSR. Ser. Mat. **15**, 309–360 (1951) (Russian); English transl. in Amer. Math. Soc. Transl. Ser. 2 **1**, 253–304 (1955).
[34] I. M. Gelfand and V. B. Lidskij, *On the structure of the regions of stability of linear canonical systems of differential equations with periodic coefficients*, in "Izrail M. Gelfand, Collected Papers", Vol. I, S. G. Gindikin, V. W. Guillemin, A. A. Kirillov, B. Kostant, S. Sternberg (eds.), Springer, Berlin, 1987, pp. 466–504. (Amer. Math. Soc. Transl. (2) **8**, 143–181 (1958).)
[35] F. Gesztesy and H. Holden, *On trace formulas for Schrödinger-type operators*, in Multiparticle Quantum Scattering with Applications to Nuclear, Atomic and Molecular Physics, D. G. Truhlar and B. Simon (eds.), Springer, New York, 1997, pp. 121–145.
[36] F. Gesztesy, H. Holden, B. Simon, and Z. Zhao, *Higher order trace relations for Schrödinger operators*, Revs. Math. Phys. **7**, 893–922 (1995).
[37] F. Gesztesy, A. Kiselev, and K. A. Makarov, *Uniqueness Results for Matrix-Valued Schrödinger, Jacobi, and Dirac-Type Operators*, Math. Nachr., to appear.
[38] F. Gesztesy, R. Ratnaseelan, and G. Teschl, *The KdV hierarchy and associated trace formulas*, in proceedings of the *International Conference on Applications of Operator Theory*, I. Gohberg, P. Lancaster, P. N. Shivakumar (eds.), Operator Theory: Advances and Applications, Vol. 87, Birkhäuser, 1996, pp. 125–163.
[39] F. Gesztesy and B. Simon, *On local Borg-Marchenko uniqueness results*, Commun. Math. Phys. **211**, 273–287 (2000).
[40] F. Gesztesy and E. Tsekanovskii, *On matrix-valued Herglotz functions*, Math. Nachr. **218**, 61–138 (2000).
[41] I. M. Glazman, *Direct Methods of Qualitative Spectral Analysis of Singular Differential Operators*, Israel Program for Scientific Translations, Jerusalem, 1965.
[42] J. K. Hale, *Ordinary Differential Equations*, 2nd ed., Krieger, Malabar, Fl, 1980.
[43] E. Hille, *Lectures on Ordinary Differential Equations*, Addison-Wesley, Reading, 1969.
[44] E. Hille, *Ordinary Differential Equations in the Complex Domain*, Dover, New York, 1997.
[45] D. B. Hinton and J. K. Shaw, *On Titchmarsh–Weyl $M(\lambda)$-functions for linear Hamiltonian systems*, J. Diff. Eqs. **40**, 316–342 (1981).
[46] D. B. Hinton and J. K. Shaw, *On the spectrum of a singular Hamiltonian system*, Quaest. Math. **5**, 29–81 (1982).
[47] D. B. Hinton and J. K. Shaw, *Hamiltonian systems of limit point or limit circle type with both endpoints singular*, J. Diff. Eqs. **50**, 444–464 (1983).
[48] D. B. Hinton and J. K. Shaw, *On boundary value problems for Hamiltonian systems with two singular points*, SIAM J. Math. Anal. **15**, 272–286 (1984).
[49] D. B. Hinton and J. K. Shaw, *On the spectrum of a singular Hamiltonian system, II*, Quaest. Math. **10**, 1–48 (1986).
[50] A. R. Its and V. B. Matveev, *Schrödinger operators with finite-gap spectrum and N-soliton solutions of the Korteweg-de Vries equation*, Theoret. Math. Phys. **23**, 343–355 (1975).
[51] M. Jodeit and B. M. Levitan, *Isospectral vector-valued Sturm-Liouville problems*, Lett. Math. Phys. **43**, 117–122 (1998).

[52] M. Jodeit and B. M. Levitan, *The isospectrality problem for some vector-valued Sturm-Liouville boundary problems*, Russ. J. Math. Phys **6**, 375–393 (1999).
[53] R. A. Johnson, *m-Functions and Floquet exponents for linear differential systems*, Ann. Mat. Pura Appl., Ser. 4, **147**, 211–248 (1987).
[54] R. Johnson, S. Novo, and R. Obaya, *Ergodic properties and Weyl M-functions for random linear Hamiltonian systems*, Proc. Roy. Soc. Edinburgh **130A**, 1045–1079 (2000).
[55] V. I. Kogan and F. S. Rofe-Beketov, *On square-integrable solutions of symmetric systems of differential equations of arbitrary order*, Proc. Roy. Soc. Edinburgh **74A**, 1–40 (1974).
[56] S. Kotani and B. Simon, *Stochastic Schrödinger operators and Jacobi matrices on the strip*, Commun. Math. Phys. **119**, 403–429 (1988).
[57] A. M. Krall, $M(\lambda)$ *theory for singular Hamiltonian systems with one singular point*, SIAM J. Math. Anal. **20**, 664–700 (1989).
[58] A. M. Krall, $M(\lambda)$ *theory for singular Hamiltonian systems with two singular points*, SIAM J. Math. Anal. **20**, 701–715 (1989).
[59] M. G. Krein, *Foundations of the theory of λ-zones of stability of a canonical system of linear differential equations with periodic coefficients*, Amer. Math. Soc. Transl. (2) **120**, 1–70 (1983).
[60] M. G. Krein, *On tests for stable boundedness of solutions of periodic canonical systems*, Amer. Math. Soc. Transl. (2) **120**, 71–110 (1983).
[61] M. G. Krein and A. A. Nudelman, *The Markov Moment Problem and Extremal Problems*, Amer. Math. Soc., Providence, 1977.
[62] B. M. Levitan, *On the solvability of the inverse Sturm-Liouville problem on the entire line*, Sov. Math. Dokl. **18**, 597–600 (1977).
[63] B. M. Levitan, *On the solvability of the inverse Sturm–Liouville problem on the whole axis in the case of an infinite number of lacunas in the spectrum*, Sov. Math. Dokl. **18**, 964–967 (1977).
[64] B. M. Levitan, *An inverse problem for the Sturm–Liouville operator in the case of finitezone and infinite-zone potentials*, Trans. Moscow Math. Soc. 1984, No. 1, 1–34.
[65] B.M. Levitan, *Inverse Sturm-Liouville Problems*, VNU Science Press, Utrecht, 1987.
[66] B. M. Levitan and M. G. Gasymov, *Determination of a differential equation by two of its spectra*, Russ. Math. Surv. **19:2**, 1–63 (1964).
[67] B. M. Levitan and A. V. Savin, *The inverse problem on the half-line for finite zone potentials*, Moscow Univ. Math. Bull. **43**, No. 1, 27–34 (1988).
[68] M. M. Malamud, *Similarity of Volterra operators and related questions of the theory of differential equations of fractional order*, Trans. Moscow Math. Soc. **55**, 57–122 (1994).
[69] M. M. Malamud, *Uniqueness questions in inverse problems for systems of ordinary differential equations on a finite interval*, Trans. Moscow Math. Soc. **60**, 173–224 (1999).
[70] Yu. I. Manin, *Matrix solitons and bundles over curves with singularities*, Funct. Anal. Appl. **12**, 286–295 (1978).
[71] V. A. Marchenko, *Nonlinear Equations and Operator Algebras*, Reidel, Dordrecht, 1988.
[72] A. S. Markus, *Introduction to the Spectral Theory of Polynomial Operator Pencils*, Translations of Mathematical Monographs, Vol. 71, Amer. Math. Soc., Providence, RI, 1988.
[73] A. S. Markus and V. I. Matsaev, *Factorization of a weakly hyperbolic bundle*, Funct. Anal. Appl. **10**, 69–71 (1976).
[74] A. S. Markus and V. I. Matsaev, *On the spectral factorization of holomorphic operator-functions*, Sel. Math. Sov. **4**, 325–354 (1985).
[75] L. Martínez Alonso and E. Olmedilla, *Trace identities in the inverse scattering transform method associated with matrix Schrödinger operators*, J. Math. Phys. **23**, 2116–2121 (1982).
[76] M. A. Naimark, *Linear Differential Operators, Part II*, F. Ungar, New York, 1968.
[77] R. G. Newton and R. Jost, *The construction of potentials from the S-matrix for systems of differential equations*, Nuovo Cim. **1**, 590–622 (1955).
[78] E. Olmedilla, *Inverse scattering transform for general matrix Schrödinger operators and the related symplectic structure*, Inverse Problems **1**, 219–236 (1985).
[79] E. Olmedilla, L. Martínez Alonso, and F. Guil, *Infinite-dimensional Hamiltonian systems associated with matrix Schrödinger operators*, Nuovo Cim. **61 B**, 49–61 (1981).
[80] P. J. Olver and V. V. Sokolov, *Integrable evolution equations on assciative algebras*, Commun. Math. Phys. **193**, 245–268 (1998).

[81] S. A. Orlov, *Nested matrix disks analytically depending on a parameter, and theorems on the invariance of ranks of radii of limiting disks*, Math. USSR Izv. **10**, 565–613 (1976).

[82] V. G. Papanicolaou, *Trace formulas and the behavior of large eigenvalues*, SIAM J. Math. Anal. **26**, 218–237 (1995).

[83] I. Polterovich, *Heat kernel asymptotics for Laplace type operators and matrix KdV hierarchy*, preprint, 2000.

[84] A. Ya. Povzner, *The expansion of arbitrary functions in eigenfunctions of the operator* $-\Delta u + cu$, Mat. Sbornik **32**, 109–156 (1953) (Russian.) English translation in Amer. Math. Soc. Transl. (2) **60**, 1–49 (1967).

[85] F. S. Rofe-Beketov, *Expansions in eigenfunctions of infinite systems of differential equations in the non-self-adjoint and self-adjoint cases*, Mat. Sb. **51**, 293–342 (1960). (Russian.)

[86] F. S. Rofe-Beketov, *The spectrum of non-selfadjoint differential operators with periodic coefficients*, Sov. Math. Dokl. **4**, 1563–1566 (1963).

[87] F. S. Rofe-Beketov, *The spectral matrix and the inverse Sturm-Liouville problem on the axis* $(-\infty, \infty)$, Teor. Funktsiĭ Funkts. Analiz Prilozh. **4**, 189–197 (1967). (Russian.)

[88] F. S. Rofe-Beketov, *The inverse Sturm-Liouville problem for the spectral matrix on the whole axis and associated problems*, in *Integral Equations and Inverse Problems*, V. Petkov and R. Lazarov Eds.), Longman, New York, 1991, p. 234–238.

[89] F. S. Rofe-Beketov and A. M. Hol'kin, *On the connection between spectral and oscillation properties of the Sturm-Liouville matrix problem*, Math. USSR Sbornik **31**, 365–378 (1977).

[90] F. S. Rofe-Beketov and A. M. Kholkin, *Spectral Analysis of Differential Operators. Connection between Spectral and Oscillatory Properties*, Mariupol, 2001.

[91] A. L. Sakhnovich, *Asymptotic behavior of spectral functions of an S-node*, Sov. Math. (Iz. VUZ) **32:9**, 92–105 (1988).

[92] A. L. Sakhnovich, *Spectral functions of a canonical system of order* $2n$, Math. USSR Sbornik **71**, 355–369 (1992).

[93] A. Sakhnovich, *Iterated Backlund-Darboux transform for canonical systems*, J. Funct. Anal. **144**, 359–370 (1997).

[94] A. Sakhnovich, *Canonical systems and transfer matrix-functions*, Proc. Amer. Math. Soc. **125**, 1451–1455 (1997).

[95] L. A. Sakhnovich, *Evolution of spectral data and nonlinear equations*, Ukrain. Math. J. **40**, 459–461 (1988).

[96] L. A. Sakhnovich, *Inverse problems for equations systems*, in *Matrix and Operator Valued Functions: The Vladimir Petrovich Potapov Memorial Volume*, I. Gohberg and L. A. Sakhnovich (eds.), Operator Theory: Advances and Applications, Vol. 72, Birkhäuser, Basel, 1994, pp. 202–211.

[97] L. A. Sakhnovich, *Method of operator identities and problems of analysis*, St. Petersburg Math. J. **5**, 1–69 (1994).

[98] L. A. Sakhnovich, *Spectral problem on half-axis*, Methods Funct. Anal. Topology **2**, 128–140 (1996).

[99] L. A. Sakhnovich, *Interpolation Theory and its Applications*, Kluwer, Dordrecht, 1997.

[100] L. A. Sakhnovich, *Spectral analysis of a class of canonical differential systems*, St. Petersburg Math. J. **10**, 147–158 (1999).

[101] L. A. Sakhnovich, *Spectral Theory of Canonical Differential Systems. Method of Operator Identities*, Operator Theory: Advances and Applications, Vol. 107, Birkhäuser, Basel, 1999.

[102] C.-L. Shen, *Some eigenvalue problems for the vectorial Hill's equation*, Inverse Probl. **16**, 749–783 (2000).

[103] C.-L. Shen, *On some inverse spectral problems for vectorial Sturm-Liouville equations*, Inverse Probl., to appear.

[104] C.-L. Shen and C.-T. Shieh, *Two inverse eigenvalue problems for vectorial Sturm-Liouville equations*, Inverse Probl. **14**, 1331–1343 (1998).

[105] M. Sodin and P. Yuditskii, *Almost periodic Sturm-Liouville operators with Cantor homogeneous spectrum*, Comment. Math. Helvetici **70**, 639–658 (1995).

[106] C. Thurlow, *A generalisation of the inverse spectral theorem of Levitan and Gasymov*, Proc. Roy. Soc. Edinburgh **84A**, 185–196 (1979).

[107] I. Trooshin, *Asymptotics for the spectral and Weyl functions of the operator-valued Sturm-Liouville problem*, in "*Inverse Problems and Related Topics*", G. Nakamura, S. Saitoh,

J. K. Seo, and M. Yamamoto (eds.), Chapman & Hall/CRC, Res. Notes Math. **419**, Boca Raton, FL, 2000, pp. 189–208.

[108] M. Wadati and T. Kamijo, *On the extension of inverse scattering method*, Progr. Theoret. Phys. **52**, 397–414 (1974).

[109] E. Wienholtz, *Halbbeschränkte partielle Differentialoperatoren zweiter Ordnung vom elliptischen Typus*, Math. Ann., **135**, 50–80 (1958).

[110] V. A. Yakubovich, *Nonoscillation of linear periodic Hamiltonian equations, and related topics*, St. Petersburg Math. J. **3**, 1165–1188 (1992).

[111] V. A. Yakubovich and V. M. Starzhinskii, *Linear Differential Equations with Periodic Coefficients 1*, Wiley, New York, 1975.

[112] V. A. Yakubovich and V. M. Starzhinskii, *Linear Differential Equations with Periodic Coefficients 2*, Wiley, New York, 1975.

Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
E-mail address: fritz@math.missouri.edu
URL: http://www.math.missouri.edu/people/fgesztesy.html

735 Crawford Ave., Brooklyn, NY 11223, USA
E-mail address: Lev.Sakhnovich@verizon.net