# The Periodic Multidimensional Schrodinger Operator, Part 2, Asymptotic Formulae for Bloch Functions and Fermi Surfaces

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**Abstract**

In this paper we obtain asymptotic formulas for Bloch function of periodic Schrodinger operator $-\Delta+q(x)$, of arbitrary dimension, when corresponding quasimomentum lies near a diffraction hyperplane, but far from the intersection of the two diffraction hyperplanes. Moreover, we write asymptotic formulas for Bloch function and Fermi survaces, when corresponding quasimomentum lies far from the diffraction hyperplanes, obtained in my previous papers in improved and enlarged form.

## 1 Introduction

In this paper we consider the Bloch function of the Schrodinger operator

$L(q)=-\Delta+q(x),\ x\in R^d,\ d\geq 2$, with a periodic (relative to a lattice $\Omega$) potential $q(x)\in W_2^l(R^d/\Omega)$, where $l\geq 6(3^d(d+1)^2)+d$. Let $L_t(q)$ be the operator generated in the fundamental domain $F\equiv R^d/\Omega$ of the lattice $\Omega$ by the expression $L(q)$ and the quasiperiodic conditions: $u(x+\omega)=e^{i\langle t,\omega\rangle}u(x)$, for all $\omega\in\Omega$, where $t\in F^*\equiv R^d/\Gamma$ and $\Gamma$ is the lattice dual to $\Omega$. Bloch function $\Psi_{1,t}(x),\ \Psi_{2,t}(x),...$, are the eigenfunctions of the operator $L_t(q)$ corresponding to the eigenvalues $\Lambda_1(t)\leq\Lambda_2(t)\leq ...$ of $L_t(q)$. In case $q(x)=0$

these eigenvalues and eigenfunctions are $\mid \gamma + t \mid^2$ and $e^{i(\gamma+t,x)}$ respectively, for all $\gamma \in \Gamma$. We consider the Bloch function $\Psi_{n,t}(x)$ corresponding to the eigenvalues $\Lambda_n(t)$ of the order of $\rho^2$ ( written as $\Lambda_n(t) \sim \rho^2$, which means that $c_1\rho^2 <\mid \Lambda_n(t) \mid< c_2\rho^2$, where by $c_i$ for $i = 1, 2, ...$ we denote the positive independent of $\rho$ constants, whose exact values are inessential) for big paramater $\rho$. For this we write the potential $q(x) \in W_2^l$ in the form

$$q(x) = \sum_{\gamma_1 \in \Gamma(\rho^\alpha)} q_{\gamma_1} e^{i(\gamma_1,x)} + O(\rho^{-p\alpha}), \tag{1}$$

where $\alpha = \frac{1}{q}$, $q = 4(3^d(d + 1)) + 1$, $\Gamma(\rho^\alpha) = \{\gamma \in \Gamma : 0 <\mid \gamma \mid< \rho^\alpha)\}$ and $p = l - d$ (see Part 1 ,i.e., [10]). Define $\alpha_k = 3^k\alpha$, for $k = 1, 2, ..., d$ and introduse the sets $V_{\gamma_1}(\rho^{\alpha_1}) \equiv \{x \in R^d :\mid\mid x \mid^2 - \mid x + \gamma_1 \mid^2\mid< \rho^{\alpha_1}\}$,

$E_1(\rho^{\alpha_1}, p) \equiv \bigcup_{\gamma_1\in\Gamma(p\rho^\alpha)} V_{\gamma_1}(\rho^{\alpha_1})$, $U(\rho^{\alpha_1}, p) \equiv R^d\backslash E(\rho^{\alpha_1}, p)$,

$E_k(\rho^{\alpha_k}, p) \equiv \bigcup_{\gamma_1,\gamma_2,...,\gamma_k\in\Gamma(p\rho^\alpha)}(\cap_{i=1}^k V_{\gamma_i}(\rho^{\alpha_k}))$, where the intersection $\cap_{i=1}^k V_{\gamma_i}$ in definition of $E_k$ is taken by $\gamma_1, \gamma_2, ..., \gamma_k$ which are linearly independent and the length of $\gamma_i$ is not more than the length of other vectors in $\Gamma \cap \gamma_i R$. The domain $U(\rho^{\alpha_1}, p)$ is said to be a non-resonance domain and the eigenvalue $\mid \gamma + t \mid^2$ is called a non-resonance eigenvalue if $\gamma + t \in U(\rho^{\alpha_1}, p)$, i.e., if $\gamma + t$ is far from the diffruction hyperplanes $\{x \in R^d :\mid x \mid^2=\mid x + \gamma_1 \mid^2\}$ for all $\gamma_1 \in \Gamma(p\rho^\alpha)$. The domains $V_{\gamma_1}(\rho^{\alpha_1})$ is called resonance domain and the eigenvalue $\mid \gamma + t \mid^2$ is called a resonance eigenvalue if $\gamma + t \in V_{\gamma_1}(\rho^{\alpha_1})$. For the first time in papers [1-3] in the nonresonanse domain ( in [2] for $d = 3$ and in [1,3] for arbitrary dimension $d \geq 2$) we constructed the set $B$ with the following properties:

Pr.1. If $\gamma + t \in B$ then there exists a unique eigenvalue $\Lambda_{N(\gamma+t)}(t) \equiv \Lambda(\gamma + t)$ of operator $L_t(q)$ satisfying $\Lambda(\gamma + t) = F(\gamma + t) + o(\varepsilon_1)$ ( see (14)), where $F(\gamma+t)$, which is called the known part of $\Lambda(\gamma+t)$, explicitly expressed by $q_\gamma$ and eigenvalues of $L_t(0)$. This is a simple eigenvalue of $L_t(q)$. Therefore we call the set $B$ a simple set.

Pr.2. The eigenfunction $\Psi_{N(\gamma+t)}(x) \equiv \Psi_{\gamma+t}(x)$ is close to $e^{i\langle\gamma+t,x\rangle}$, namely

$$\Psi_{N(\gamma+t)}(x) = e^{i\langle\gamma+t,x\rangle} + O(\rho^{-\alpha_1}), \tag{2}$$

(see Theorem 3 of paper [2] and Theorem 1 of this paper).

Pr.3. The set $B$ contains a part $I_\rho^{'}(q(x))$ of the translated isoenergetic surfaces $\{\gamma+t : \Lambda(\gamma+t) = \rho^2\}$ of $L_t(q)$ corresponding $\rho^2$ (see Section4) whose measure is asymptotically equal to the measure of the translated isoenergetic surfaces $B(\rho) \equiv \{x \in R^d :| x |= \rho\}$ of $L(0)$ (see Theorem 6). The nonemptyness of $I_\rho^{'}$ proves the validity of Bethe-Zommerfeld conjecture.

Construction of the set $B$ consists of two steps.

Step 1. We prove that all big eigenvalues of the operator $L_t(q)$, lie in the $\varepsilon_1$ neighborhood of a numbers $A(\gamma + t)$, where $\gamma \in \Gamma$, and $A(\gamma + t)$ is the known part of the eigenvalues. In fact for the construction of this set we looked for motion (purtibation ) of all eigenvalues of $L_t(0)$. They are very dense ( there are in averige $\rho^{d-2}$ eigenvalues in interval $(\rho^2, \rho^2 + 1)$).

Step 2. By eliminating the set of quasimomenta $\gamma + t$, for which the known parts $F(\gamma + t)$ of $\Lambda(\gamma + t)$ is situated from the known part $A(\gamma^{'} + t)$, $\gamma^{'} \neq \gamma$ of other eigenvalues at a distance less than $2\varepsilon_1$ we constructed the set $B$ such that if $\gamma + t \in B$ then the following condition (called simplicity condition for eigenvalue $\Lambda(\gamma + t) \equiv \Lambda_{N(\gamma+t)}(t)$ ) satisfies

$$| F(\gamma + t) - A(\gamma^{'} + t) |\geq 2\varepsilon_1, \ \forall\gamma^{'} \neq \gamma. \tag{3}$$

It follows from (3) that the eigenvalue $\Lambda(\gamma+t)$ can not coincide with other eigenvalues. It easily implies that ( see Theorem 1) the Fourier coefficients $b(N(\gamma + t), \gamma^{'}) = (\Psi_{N(\gamma+t)}(x), e^{i(\gamma^{'}+t,x)})$, for $\gamma^{'} \neq \gamma$, of any eigenfunction $\Psi_{N(\gamma+t)}(x)$ corresponding to the eigenvalue $\Lambda_{N(\gamma+t)}(t)$ are so small that

$$\sum_{\gamma^{'}\in\Gamma\backslash\gamma} | b(N, \gamma^{'}) |^2= O(\rho^{-2\alpha_1}) \tag{4}$$

which equivalent to (2). The main result of papers [1-3] were the construction of the simple set $B$ in neighborhood of the appoximated isoenergetic surface $S_\rho = \{\gamma + t \in U(\rho^{\alpha_1}, p) : F(\gamma + t) = \rho^2\}$ ( if $d = 2, 3$ then $F(\gamma + t) =| \gamma + t |^2$) with the properties Pr.1.-Pr.3., though in those papers we emphasized the Bethe-Zommerfeld conjecture. Note that for this conjecture and for Pr.1, Pr.3. it is enough to prove that the right side of (4) is less than $\frac{1}{4}$ ( we prove it in [1-3] and as noted in Theorem 3 of [2] the proof of this fact does not differ from the proof of (4)). From (2) by simple reduction we got (see [5]) the asymptotic formula ( see Theorem 1 of this paper)

$$\Psi_{\gamma+t}(x) = e^{i(\gamma+t,x)} + \Phi_{k-1}(x) + O(\rho^{-k\alpha_1}), k = 1, 2, ... . \tag{5}$$

In papers [6,7] and [8,9] by different method asymptotic formulas for Bloch Functions of 2 and 3 dimensional operator $L(q)$ were obtained.

In this paper we construct simple sets in the resonance domain and obtain asymptotic formulas of arbitrary order for Bloch functions of the $d$-dimensional periodic Schrodinger operator $L(q)$ for arbitrary dimension $d$, when corresponding quasimomentum lies in these simple sets , that is, near a diffruction plane. Note that we construct the simple sets in the nonresonance domain in such a form that it contains a big part of the translated isoenergetic surfaces of $L(q)$, but in the resonance domain in such a form that it can be easily used for constructive determination ( in Part 3) a family of spectral invariants by given Floquet spectrum. Then ( in Part 4) we give an algorithm to reconstruct the potential $q(x)$ by these spectral invariants. The nonresonance case, i.e., the case when the quasimomentum lies far from the diffruction planes was considered in [1-3,5]. But in those papers these results are written briefly. The enlarged variant is written in [4] which can not be used as reference. Here we write the nonresonance case in improved, enlarged form and such a form that can be easily used for the resonance case. Moreover it helps when reading the resonance case.

In this paper for the different types of the measures of the subset $A$ of $R^d$ we use the same notation $\mu(A)$, because it will be clear which measure we mean. By $\mid A \mid$ we denote the number of elements of the set $A$ and use the following obvious fact. If $a \sim \rho$ then the number of elements of the set $\{\gamma + t : \gamma \in \Gamma\}$ satisfying $\mid\mid \gamma + t \mid - a \mid < 1$ is less than $c_\Gamma \rho^{d-1}$, where $c_\Gamma$ is a constant depending on $\Gamma$. Therefore the number of eigenvalues of $L_t(q)$ lying in $(a^2 - \rho, a^2 + \rho)$ is less than $c_\Gamma \rho^{d-1}$.

# 2 Asymptotic Formulas for Bloch Functions in the Nonresonance Domain

We use the following statements of [10] ( see Theorem 1, Remark 2 of [10]).

Stat.1. For every $\gamma + t \in U(\rho^{\alpha_1}, p)$ of the order of $\rho$ there exists an eigenvalue $\Lambda_N$, denoted by $\Lambda(\gamma + t)$ or $\Lambda_{N(\gamma+t)}(t)$, of $L_t$ satisfying

$$\Lambda_N \equiv \Lambda(\gamma + t) = \mid \gamma + t \mid^2 + F_{k-1}(\gamma + t) + O(\rho^{-6k\alpha}) \tag{6}$$

for $k = 1, 2, ..., k_1 \equiv [\frac{1}{6}(p - \frac{1}{2}q(d-1))]$, where $F_s(x)$ for $x = (x_1, x_2, ..., x_d) \in U(\rho^{\alpha_1}, p)$ and $\mid x \mid \sim \rho$ satisfies the relations

$$F_0 = 0, F_s(x) = O(\rho^{-2\alpha_1}),\ \frac{\partial F_s(x)}{\partial x_i} = O(\rho^{-2\alpha_1 + \alpha}), \forall s \geq 1, \forall i \tag{7}$$

Stat.2. Suppose $|\gamma^{'}+t|\sim\rho$ and for some normalized eigenfunction $\Psi_N(x)$ corresponding to $\Lambda_N(t)$ the inequalities

$$|\Lambda_N-|\gamma^{'}+t|^2|<\frac{1}{2}\rho^{\alpha_1},\quad |b(N,\gamma^{'})|\equiv|(\Psi_N(x),e^{i(\gamma^{'}+t,x)})|>c_4\rho^{-c\alpha} \tag{8}$$

hold. If $\gamma^{'}+t\in U(\rho^{\alpha_1},p)$ then $\Lambda_N(t)$ satisfies the formulas

$$\Lambda_N=|\gamma^{'}+t|^2+F_{k-1}(\gamma^{'}+t)+O(\rho^{-6k\alpha}),\forall k=1,2,...,k_2\equiv[\frac{1}{6}(p-c)]. \tag{9}$$

If $\gamma^{'}+t\in(\cap_{i=1}^{k}V_{\gamma_i}(\rho^{\alpha_k}))\backslash E_{k+1}$, then there is $j$ such that

$$\Lambda_N=\lambda_j(\gamma^{'}+t)+O(\rho^{-(p-c-\frac{d}{4}3^d)\alpha}), \tag{10}$$

where $\lambda_1(\gamma^{'}+t)\leq\lambda_2(\gamma^{'}+t)\leq...\leq\lambda_{b_k}(\gamma^{'}+t)$ are the eigenvalues of the matrix $C(\gamma'+t,\gamma_1,\gamma_2,...,\gamma_k)$ and $b_k=O(\rho^{d\alpha+\frac{k}{2}\alpha_{k+1}})$. Moreover we have

$$|r_i(x)-r_i(x^{'})|\leq 2\rho^{\frac{1}{2}\alpha_d}|x-x^{'}|,\forall i=1,2,...,b_k, \tag{11}$$

where $r_i(x)\equiv\lambda_i(x)-|x|^2$ . Every eigenvalue of $L_t(q)$ satisfies either (9) or (10). Besides we use the well-known formula

$$(\Lambda_N-|\gamma+t|^2)b(N,\gamma)=(\Psi_{N,t}(x)q(x),e^{i\langle\gamma+t,x\rangle})=$$
$$\sum_{\gamma_1\in\Gamma(\rho^\alpha)}q_{\gamma_1}b(N,\gamma-\gamma_1)+O(\rho^{-p\alpha}), \tag{12}$$

( see (7), (8) of [10]) and the obvious relations

$$\sum_{\gamma\in\Gamma(\rho^\alpha)}|q_\gamma|=O(1),\quad ||\gamma+t|^2-|\gamma+\gamma^{'}+t|^2|>\rho^{\alpha_1}, \tag{13}$$
$$|\Lambda_N-|\gamma+\gamma^{'}+t|^2|>\frac{1}{2}\rho^{\alpha_1}.$$

for $\gamma+t\in U(\rho^{\alpha_1},p)$, $\Lambda_N=\Lambda(\gamma+t)$ and $\gamma^{'}\in\Gamma(p\rho^\alpha)$ , which follow from (6) and the definition of $U(\rho^{\alpha_1},p)$. Defining $\varepsilon_1=\rho^{-d-2\alpha}$, taking $c=p-dq-3^dd$ in (8), (9), (10) and $k=k_2$ in (6), (9), and using
$6k_2\alpha>d+2\alpha$, $(p-c-\frac{d}{4}3^d)\alpha>d+2\alpha$ we obtain

$$\begin{aligned}\Lambda_N-F(\gamma+t)&=o(\varepsilon_1)\text{ if }\Lambda_N=\Lambda(\gamma+t),\\ \Lambda_N-F(\gamma^{'}+t)&=o(\varepsilon_1)\text{ if (8) holds and }\gamma^{'}+t\in U(\rho^{\alpha_1},p),\\ \Lambda_N-\lambda_j(\gamma^{'}+t)&=o(\varepsilon_1)\text{ if (8) holds and }\gamma^{'}+t\in E_k\backslash E_{k+1},\end{aligned} \tag{14}$$

where $F(\gamma+t)=|\gamma+t|^2+F_{k_2-1}(\gamma+t)$.

In this section we construct the set $B\subset U(\rho^{\alpha_1},p)$ with the properties 1-3 ( see introduction) and obtain the asymptotic formulae for Bloch function $\Psi_{\gamma+t}(x)$ when $\gamma+t\in B$ . As noted in the introduction (see Step 1 and Step 2) for construction of $B$ we use the simplicity condition for the eigenvalue $\Lambda(\gamma+t)=\Lambda_{N(\gamma+t)}(t)$. Since the known parts of the other eigenvalues are $F(\gamma^{'}+t)$ and $\lambda_j(\gamma'+t)$ in this case the simplicity conditions are

$$|F(\gamma+t)-F(\gamma^{'}+t)|\geq 2\varepsilon_1,\ \forall\gamma^{'}\in K\backslash\{\gamma\},\ \gamma^{'}+t\in U(\rho^{\alpha_1},p), \tag{15}$$

$$|F(\gamma+t)-\lambda_j(\gamma^{'}+t)|\geq 2\varepsilon_1,\forall\gamma^{'}\in K,\gamma^{'}+t\in E_k\backslash E_{k+1},j=1,2,...,b_k \tag{16}$$

where $K$ is the set of $\gamma^{'}\in\Gamma$ satisfying $|F(\gamma+t)-|\gamma^{'}+t|^2|<\frac{1}{3}\rho^{\alpha_1}$. Thus $B$ is the set of $\gamma+t\in U(\rho^{\alpha_1},p)$ satisfying the simplicity conditions (15), (16). The existence and properties of the set $B$ will be analized in Section 4. These simplicity conditions imply the asymptotic formulas for Bloch functions.

**Theorem 1** *If $\gamma+t\in B$ and $|\gamma+t|\sim\rho$ then there exists a unique eigenvalue $\Lambda_N=\Lambda(\gamma+t)$ satisfying (6). This is a simple eigenvalue and the corresponding eigenfunction $\Psi_{\gamma+t}(x)\equiv\Psi_{N(\gamma+t)}(x)$ satisfies formulae (5) for $k=1,2,...,n=[\frac{1}{3}(c-\frac{1}{2}q(d-1))]$, where $c=p-dq-3^dd$, $\Phi_0(x)=0$, $\Phi_1(x)=\sum_{\gamma_1\in\Gamma(\rho^\alpha)}\dfrac{q_{\gamma_1}e^{i(\gamma+t+\gamma_1,x)}}{(|\gamma+t|^2-|\gamma+\gamma_1+t|^2)}$, and $\Phi_{k-1}(x)$ is a linear combination of $e^{i(\gamma+t+\gamma^{'},x)}$ for $\gamma^{'}\in\Gamma((k-1)\rho^\alpha)\cup\{0\}$ with explicitly expressed coefficients (24), (25).*

**Proof** By Stat.1. there exists an eigenvalue $\Lambda_N$ satisfying (6). Let $\Psi_N$ be any normalized eigenfunction corresponding to $\Lambda_N$. First we prove that $\Psi_N$ satisfies formula (2), that is , formula (5) for $k=1$, which is equivalent to $b(N,\gamma)=1+O(\rho^{-\alpha_1})$. Since the normalized eigenfunction is defined up to constant of modulas 1 we can suppose that $\arg b(N,\gamma)=0$. Hence it suffices to prove (4). Using (6), definition of $K$ , and then (12), we get

$$\begin{aligned}\frac{1}{4}\rho^{\alpha_1} &< |\Lambda_N-|\gamma^{'}+t|^2|<\frac{1}{2}\rho^{\alpha_1},\forall\gamma^{'}\notin K, \\ \sum_{\gamma^{'}\notin K} &| \ b(N,\gamma^{'})|^2\leq\sup|q(x)|^2\,O(\rho^{-2\alpha_1})=O(\rho^{-2\alpha_1}).\end{aligned} \tag{17}$$

Now we prove that conditions (15), (16) imply

$$| b(N,\gamma^{'}) |\leq c_4\rho^{-c\alpha}, \forall\gamma^{'}\in K\backslash\{\gamma\}, \sum_{\gamma^{'}\in K\backslash\{\gamma\}} | b(N,\gamma^{'}) |^2 = O(\rho^{-2n\alpha_1}). \quad (18)$$

If for $\gamma^{'}+t\in U(\rho^{\alpha_1},p)$ and $\gamma^{'}\in K\backslash\{\gamma\}$ the inequality in (18) is not true then by the second inequality of (17) the inequalities in (8), and hence the second relation in (14) holds. This and first relation in (14) contradict (15). Similarly if for $\gamma^{'}+t\in(E_k\backslash E_{k+1})$ and $\gamma^{'}\in K$ the inequality in (18) does not hold then the third relation in (14) holds. This and first relation in (14) contradict (16). Hence the inequality in (18) holds. Since $| K |= O(\rho^{q(d-1)\alpha})$ the inequality in (18) implies the equality in (18). This together with the last relation in (17) shows that (4) holds for any normalized eigenfunction $\Psi_N$ corresponding to any eigenvalue $\Lambda_N$ satisfying (6). If there exist two different eigenvalues or multiple eigenvalue satisfying (6) then there exist two orthogonal normalized eigenfunction satisfying (4) which is imposible. Thus (5) for $k=1$ is proved. To prove formula (5) for arbitrary $k\leq n$ we prove the following equivalent relations

$$\sum_{\gamma^{'}\in\Gamma^c(k-1)} | b(N,\gamma+\gamma^{'}) |^2 = O(\rho^{-2k\alpha_1}), \quad (19)$$

$$\Psi_N = b(N,\gamma)e^{i(\gamma+t,x)} + \sum_{\gamma^{'}\in\Gamma((k-1)\rho^{\alpha})} b(N,\gamma+\gamma^{'})e^{i(\gamma+t+\gamma^{'},x)} + H_k(x), \quad (20)$$

where $\Gamma^c(m)\equiv\Gamma\backslash(\Gamma(m\rho^{\alpha})\cup\{0\})$ and $\| H_k(x) \|= O(\rho^{-k\alpha_1})$. The case $k=1$ is proved accoding to (4). Assume that (19) is true for $k=m$ . Then by (20) for $k=m$, and by (1) we have $\Psi_N(x)(q(x)) = H(x)+O(\rho^{-m\alpha_1})$, where $H(x)$ is linear conbination of $e^{i(\gamma+t+\gamma^{'},x)}$ for $\gamma^{'}\in\Gamma(m\rho^{\alpha})\cup\{0\}$. Hence $(H(x),e^{i(\gamma+t+\gamma^{'},x)})=0$ for $\gamma^{'}\in\Gamma^c(m)$. Therefore using (12) and (17), we get

$$\sum_{\gamma^{'}} | b(N,\gamma+\gamma^{'}) |^2 = \sum_{\gamma^{'}} \left| \frac{(O(\rho^{-m\alpha_1}), e^{i(\gamma+t+\gamma^{'},x)})}{\Lambda_N - | \gamma+\gamma^{'}+t |^2} \right|^2 = O(\rho^{-2(m+1)\alpha_1}), \quad (21)$$

where sum is taken under conditions $\gamma^{'}\in\Gamma^c(m)$ and $\gamma+\gamma^{'}\notin K$. This together with (18) gives (19) for $k=m+1$. Thus formula (20) is also proved. Here

$b(N,\gamma)$ and $b(N,\gamma+\gamma^{'})$ for $\gamma^{'}\in\Gamma((n-1)\rho^{\alpha})$ can be calculated as follows. In (12) replase $\gamma$ by $\gamma+\gamma^{'}$. Apply this $n$ times and every times isolate the terms with maltiplicant $b(N,\gamma)$, that is, apply (12) for $b(N,\gamma+\gamma^{'})$, where $\gamma^{'}\in\Gamma((n-1)\rho^{\alpha})$, and hence $\gamma^{'}\neq 0$. Then apply (12) for $b(N,\gamma+\gamma^{'}-\sum_{i=1}^{j}\gamma_i)$ when $\gamma^{'}-\sum_{i=1}^{j}\gamma_i\neq 0$ where $\gamma_i\in\Gamma(\rho^{\alpha})$, $j=1,2,...,n-1$. Then using the equality in (13) and the relations

$\mid\Lambda_N-\mid\gamma+t+\gamma^{'}-\sum_{i=1}^{j}\gamma_i\mid^2\mid>\frac{1}{2}\rho^{\alpha_1}$ ( see the second inequality in (13) and use the fact that $\gamma^{'}-\sum_{i=1}^{j}\gamma_i\in\Gamma(p\rho^{\alpha})$ since $p>3n$),

$\Lambda_N=P(\gamma+t)+O(\rho^{-n\alpha_1})$, where $P(\gamma+t)=\mid\gamma+t\mid^2+F_{k_1-1}(\gamma+t)$

( see (6) and use $6k_1\geq 3n$), we obtain

$$b(N,\gamma+\gamma^{'})=\sum_{k=1}^{n-1}A_k(\gamma^{'})b(N,\gamma)+O(\rho^{-n\alpha_1}), \tag{22}$$

where

$A_1(\gamma^{'})\equiv\dfrac{q_{\gamma^{'}}}{P(\gamma+t)-\mid\gamma+\gamma^{'}+t\mid^2}=\dfrac{q_{\gamma^{'}}}{\mid\gamma+t\mid^2-\mid\gamma+\gamma^{'}+t\mid^2}+O(\rho^{-3\alpha_1})$

( see (7)),

$A_k(\gamma^{'})=\sum_{\gamma_1,...,\gamma_{k-1}}\dfrac{q_{\gamma_1}q_{\gamma_2}...q_{\gamma_{k-1}}q_{\gamma^{'}-\gamma_1-\gamma_2-...-\gamma_{k-1}}}{\prod_{j=0}^{k-1}(P(\gamma+t)-\mid\gamma+t+\gamma^{'}-\sum_{i=1}^{j}\gamma_i\mid^2)}=O(\rho^{-k\alpha_1})$,

$$\sum_{\gamma^{*}\in\Gamma((n-1)\rho^{\alpha})}\mid A_1(\gamma^{*})\mid^2=O(\rho^{-2\alpha_1}),\quad\sum_{\gamma^{*}\in\Gamma((n-1)\rho^{\alpha})}\mid A_k(\gamma^{*})\mid=O(\rho^{-k\alpha_1}), \tag{23}$$

for $k>1$. Now from (20) for $k=n$ and (22), we obtain

$$\Psi_N=b(N,\gamma)e^{i(\gamma+t,x)}+\sum_{\gamma^{*}\in\Gamma((n-1)\rho^{\alpha})}\sum_{k=1}^{n-1}(A_k(\gamma^{*})b(N,\gamma)+O(\rho^{-n\alpha_1}))e^{i(\gamma+t+\gamma^{*},x)})+H_n$$

Since $\parallel\Psi_N\parallel=1,\arg b(N,\gamma)=0$, the functions $e^{i(\gamma+t,x)}$, $H_n(x)$, $e^{i(\gamma+t+\gamma^{*},x)}$, $(\gamma^{*}\in\Gamma((n-1)\rho^{\alpha}))$ are orthogonal and $\parallel H_n\parallel=O(\rho^{-n\alpha_1})$ we have

$1=\mid b(N,\gamma)\mid^2+\sum_{k=1}^{n-1}(\sum_{\gamma^{*}\in\Gamma((n-1)\rho^{\alpha})}\mid A_k(\gamma^{*})b(N,\gamma)\mid^2+O(\rho^{-n\alpha_1}))$,

$$b(N,\gamma)=(1+\sum_{k=1}^{n-1}(\sum_{\gamma^{*}\in\Gamma((n-1)\rho^{\alpha})}\mid A_k(\gamma^{*})\mid^2))^{-\frac{1}{2}}+O(\rho^{-n\alpha_1})). \tag{24}$$

(See the second equality in (23).) Now from (22) we obtain

$$b(N,\gamma+\gamma^{'})=(\sum_{k=1}^{n-1}A_k(\gamma^{'}))(1+\sum_{k=1}^{n-1}\sum_{\gamma^*}\mid A_k(\gamma^*)\mid^2)^{-\frac{1}{2}}+O(\rho^{-n\alpha_1}) \qquad (25)$$

Consider the case $n=2$. By (24), (23), (25) we have $b(N,\gamma)=1+O(\rho^{-2\alpha_1})$,

$$b(N,\gamma+\gamma^{'})=A_1(\gamma^{'})+O(\rho^{-2\alpha_1})=\frac{q_{\gamma^{'}}}{\mid\gamma+t\mid^2-\mid\gamma+\gamma^{'}+t\mid^2}+O(\rho^{-2\alpha_1})$$

for all $\gamma^{'}\in\Gamma(\rho^{\alpha})$. These and (20) for $k=2$ imply the formula for $\Phi_1\diamondsuit$

# 3 Asymptotic Formulas for Bloch Functions in the Single Resonance Domains.

In this section we use the Stat.1, Stat.2 of section 2, and the following statements of [10] about the detail analysis of eigenvalues corresponding to single resonance domain $V_{\delta}^{'}(\rho^{\alpha_1})\equiv V_{\delta}(\rho^{\alpha_1})\backslash E_2$, for $\delta\in\Gamma(p\rho^{\alpha})$.

Stat.1. Let $\Omega_\delta$ be the sublattice $\{d\in\Omega:(d,\delta)=0\}$ of $\Omega$ in the hyperplane $H_\delta=\{x\in R^d:(x,\delta)=0\}$, $\Gamma_\delta$ be the dual lattice of $\Omega_\delta$ and $F_\delta=H_\delta/\Gamma_\delta$. If $t\in F^*$, $\gamma\in\Gamma$, $\gamma+t\in V_\delta(\rho^{\alpha_1})$, $\mid\gamma\mid\sim\rho$ then we have

$$t = a+\tau+\mid\delta\mid^{-2}(t,\delta)\delta,\ \ \gamma+t=\beta+\tau+(j+v)\delta, \qquad (26)$$

$$\mid j\delta\mid< r_1\equiv\mid 2\delta\mid^{-1}\rho^{\alpha_1}+\mid 2\delta\mid, \qquad (27)$$

where $a\in\Gamma_\delta$, $\beta\in\Gamma_\delta$, $\tau\in F_\delta$, $j\in Z$, $v\equiv v(\beta)\equiv v(\beta,t)$ is the fractional part of $\mid\delta\mid^{-2}(t,\delta)-(2\pi)^{-1}(\beta-a,\delta^*)$, and $\delta^*$ is the element of $\Omega$ satisfying $(\delta^*,\delta)=2\pi$, ( see (40), (44) and Lemma 2 of [10]). Moreover we have $\mu(F_\delta)>c_{14}\rho^{-d\alpha}$, $d_\delta\equiv\sup_{\tau,\tau^{'}\in F_\delta}\mid\tau-\tau^{'}\mid=O(1)$. (See Remark 2 of [10]).

Stat.2. The eigenvalues and eigenfunctions of the Schrodinger operator $L_t(q^\delta(x))$ with directional potential $q^\delta(x)=\sum q_{n\delta}e^{in(\delta,x)}=Q(s)$ are $\lambda_{j,\beta}(v,\tau)=\mu_j(v(\beta,t))+\mid\beta+\tau\mid^2$, $\Phi_{j,\beta}(x)=e^{i(\beta+\tau,x)}\varphi_{j,v(\beta,t)}(s)$, for $j\in Z$, $\beta\in\Gamma_\delta$, where $s=(\delta,x)$, $\mu_j(v)$ and $\varphi_{j,v}(s)$ are the eigenvalues and eigenfunctions of the operator $T_v(Q(s))$ generated by the boundary value problem

$-\mid\delta\mid^2 y''(s)+Q(s)y(s)=\mu y(s)$, $y(s+2\pi)=e^{i2\pi v}y(s)$.

We use the following relations for $\mu_j(v)$ and $\varphi_{j,v}(s)$ :

$$| \quad \mu_j(v)- | (j+v)\delta |^2 | \leq \sup | Q(s) |, \forall j \in Z \quad (28)$$

$$| \quad \mu_j(v) - \mu_{j'}(v) | > \frac{2}{\ln \rho}, \forall j' \neq j, \forall v \in A(\varepsilon(\rho)) (29)$$

$$(\varphi_{j',v'}(s), e^{i(m+v')s}) = O(| j'\delta |^{-l-1}), \forall | j' | > 2 | m |, \quad (30)$$

$$\sum_m | (\varphi_{j',v'}(s), e^{i(m+v')s}) | = O(1) \quad (31)$$

where $v' \in [0,1)$, $A(\varepsilon(\rho)) \equiv (\varepsilon(\rho), \frac{1}{2} - \varepsilon(\rho)) \cup (\frac{1}{2} + \varepsilon(\rho), 1 - \varepsilon(\rho))$, $\varepsilon(\rho) \to 0$ as $\rho \to \infty$. See (41), (50), (66) and Lemma 3 of [10].

Stat.3. For every eigenvalue $\lambda_{j,\beta}(v,\tau) \sim \rho^2$ of $L_t(q^\delta(x))$ with $v \in A(\varepsilon(\rho))$ $\beta + \tau + (j+v)\delta \in V'_\delta(\rho_1)$, there exists an eigenvalue $\Lambda_N \equiv \Lambda_N(\lambda_{j,\beta})$ of $L_t$ satisfying the formulas ( see Theorem 9, and (91) of [10]).

$$\Lambda_N = \lambda_{j,\beta}(v,\tau) + E_{k-1}(\lambda_{j,\beta}(v,\tau)) + O(\rho^{-k\alpha_2}(\ln \rho)^{2k}), \quad (32)$$

for $k = 1, 2, ..., k_3 \equiv [\frac{1}{9}(p - \frac{1}{2}q(d-1)]$, where $E_0 = 0$, $\tau = (\tau_1, \tau_2, ..., \tau_{d-1})$,

$$E_{k-1} = O(\rho^{-\alpha_2} \ln \rho), \ \frac{\partial(E_{k-1}(\lambda_{j,\beta}(v,\tau)))}{\partial \tau_i} = O(\rho^{-2\alpha_2+\alpha} \ln \rho), \ \forall i. \quad (33)$$

If $0 <| \beta' - \beta |< (p-1)\rho^\alpha$, $| j'\delta |< \frac{1}{2}\rho^{\frac{1}{2}\alpha_2}$ then

$$| \Lambda_N - \lambda_{j',\beta'} | > \frac{1}{2}\rho^{\alpha_2}, | \Lambda_N - \lambda_{j',\beta} | > (\ln \rho)^{-1}, \forall j' \neq j, \ \forall v \in A(\varepsilon(\rho)) \quad (34)$$

(see (75), (76) of [10] and (32), (33) of this paper.)

Stat.4. If $| j'\delta |< r$, where $r > r_1$, ( see (27) ) and $| \beta' | \sim \rho$ then

$$\begin{aligned} &(q(x) - Q(s))\Phi_{j',\beta'}(x) \quad (35) \\ = &\sum_{(j_1,\beta_1) \in Q(\rho^\alpha, 9r)} A(j', \beta', j' + j_1, \beta' + \beta_1)\Phi_{j'+j_1,\beta'+\beta_1}(x) + O(\rho^{-p\alpha}), \end{aligned}$$

$$(\Lambda_N - \lambda_{j',\beta'})b(N, j', \beta') = (\Psi_N(x)(q(x) - Q(s)), \Phi_{j',\beta'}(x))$$

$$\sum_{(j_1,\beta_1) \in Q(\rho^\alpha, 9r)} A(j', \beta', j' + j_1, \beta' + \beta_1)b(N, j' + j_1, \beta' + \beta_1) + O(\rho^{-p\alpha}), \quad (36)$$

where $Q(\rho^{\alpha},9r)=\{(j,\beta):\mid j\delta\mid<9r,0<\mid\beta\mid<\rho^{\alpha}\}$, $b(N,j^{'},\beta^{'})=(\Psi_{N},\Phi_{j^{'},\beta^{'}})$

$$\sum_{(j_1,\beta_1)\in Q(\rho^{\alpha},9r)}\mid A(j^{'},\beta^{'},j^{'}+j_{1,}\beta^{'}+\beta_1)\mid<c_5. \tag{37}$$

Here $c_5$ does not depend on $(j^{'},\beta^{'})$ ( see (68) and Lemma 5 of [10]).

In this section we construct an asymptotically full part $B_\delta$ of $V_{\delta}^{'}(\rho^{\alpha_1})$ such that if $\gamma+t=\beta+\tau+(j+v)\delta\in B_\delta$ then the eigenvalue $\Lambda_N(\lambda_{j,\beta}(v,\tau))$ satisfying (32) is a simple eigenvalue and the corresponding eigenfunction $\Psi_N(x)$ is close to $\Phi_{j,\beta}(x)$. As noted in the introduction (see Step 1 and Step 2) and section 2 for the construction of $B_\delta$ we use the simplicity conditions for the eigenvalue $\Lambda_N(\lambda_{j,\beta})$. The inequality (29) and formula (32) imply that $\Lambda_N(\lambda_{j,\beta})$ can not coincide with the eigenvalue $\Lambda_{N^{'}}$ which is close to $\lambda_{j^{'},\beta}$ for $j^{'}\neq j$. In order that $\Lambda_N(\lambda_{j,\beta})$ can not coincide with any other eigenvalue we use the following two simplicity conditions

$$\mid\ E(\lambda_{j,\beta}(v,\tau))-F(\gamma^{'}+t)\mid\geq 2\varepsilon_1,\forall\gamma^{'}\in M_1 \tag{38}$$

$$\mid\ E(\lambda_{j,\beta}(v,\tau))-\lambda_i(\gamma^{'}+t)\mid\geq 2\varepsilon_1,\forall\gamma^{'}\in M_2,\forall i=1,2,...,b_k \tag{39}$$

where $E(\lambda_{j,\beta}(v,\tau))=\lambda_{j,\beta}(v,\tau)+E_{k_3-1}(\lambda_{j,\beta}(v,\tau))$, $k_3\alpha_2>d+2\alpha$, $M_1$ is the set of $\gamma^{'}\in\Gamma$ satisfying $\gamma^{'}+t\in U(\rho^{\alpha_1},p),\gamma^{'}\in M$, $M_2$ is the set of $\gamma^{'}\in\Gamma$ satisfying $\gamma^{'}+t\notin U(\rho^{\alpha_1},p),\gamma^{'}\in M$ and having orthogonal decomposition (see (26)) $\gamma^{'}+t=\beta^{'}+\tau+(j^{'}+v(\beta^{'}))\delta$, with $\beta^{'}\neq\beta$, and $M$ is the set of $\gamma^{'}\in\Gamma$ satisfying $\mid\ E(\lambda_{j,\beta}(v,\tau))-\mid\gamma^{'}+t\mid^2\mid<\frac{1}{3}\rho^{\alpha_1}$. The set $B_\delta$ is the set of $\gamma+t=\beta+\tau+(j+v(\beta,t))\delta\in V_{\delta}^{'}(\rho^{\alpha_1})$ for $\beta\in\Gamma_\delta$, $\tau\in F_\delta$, $j\in Z$ $v(\beta,t)\in A(\varepsilon(\rho))$ satisfying (38), (39). The existence and properties of the sets $B_\delta$, will be analized in the next section. Recall that in section 2 the simplicity conditions (15), (16) implied the asymptotic formula (2). Similarly here the simplicity conditions (38), (39) imply the asymptotic formula for Block function in the resonance domain $V_{\delta}^{'}(\rho^{\alpha_1})$. For this we use

**Lemma 2** *Let $\Lambda_N\equiv\Lambda_N(\lambda_{j,\beta}(v,\tau))$ be an eigenvalue of $L_t$ satisfying (32), where $\beta+\tau+(j+v)\delta=\gamma+t\in V_{\delta}^{'}(\rho^{\alpha_1})$. If for $\gamma^{'}+t=\beta^{'}+\tau+(j^{'}+v(\beta^{'}))\delta$ at least one of the following conditions:*

*Cond.1: $\gamma^{'}\in M,\beta^{'}\neq\beta$. Cond.2: $\mid\beta-\beta^{'}\mid>(p-1)\rho^{\alpha}$.*

*Cond.3: $\mid\beta-\beta^{'}\mid\leq(p-1)\rho^{\alpha}$, $\mid j^{'}\delta\mid\geq h\equiv 10^{-p}\rho^{\frac{1}{2}\alpha_2}$ hold, then*

$$\mid b(N,\gamma^{'})\mid\leq c_4\rho^{-c\alpha}. \tag{40}$$

**Proof** Repeating the proof of the inequality in(18) and instead of the simplicity conditions (15), (16) and the set $K$ using simplicity conditions (38), (39), and the set $M$ we obtain the proof of (40) under Cond.1. Suppose Cond.2. holds. Then for any vector $\gamma_i = \beta_i + a_i\delta \in \Gamma(\rho^\alpha)$, where $\beta_i \in \Gamma_\delta, a_i \in R$ , $1 \leq i \leq n \leq (p-1)$, we have

$$| \quad \beta_i |< \rho^\alpha, | a_i\delta |< \rho^\alpha, \beta^{'} - \sum_{i=1}^{n} \beta_i \neq \beta \tag{41}$$

$$\gamma^{'} + t - \sum_{i=1}^{n} \gamma_i \quad = \quad \beta^{'} - \sum_{i=1}^{n} \beta_i + \tau + (j^{'} + v(\beta^{'}) - \sum_{i=1}^{n} a_i)\delta. \tag{42}$$

( See (26)) If $\gamma^{'} - \sum_{i=1}^{n} \gamma_i \in M$, then $b(N, \gamma^{'} - \sum_{i=1}^{n} \gamma_i) = O(\rho^{-c\alpha})$ becouse the Cond.1 holds ( see (41)). If $\gamma^{'} \notin M$ then applying (12) $(p-1)$-times and each time instead of $b(N, \gamma^{'} - \sum_{i=1}^{j} \gamma_i)$ for $\gamma^{'} - \sum_{i=1}^{j} \gamma_i \in M$ writing $O(\rho^{-c\alpha})$ and using the equality in (13) we obtain

$$b(N, \gamma^{'}) = \sum_{\gamma_1, \gamma_2, \ldots, \gamma_{p-1}} \frac{q_{\gamma_1} q_{\gamma_2} \ldots q_{\gamma_{p-1}} b(N, \gamma^{'} - \sum_{i=1}^{p-1} \gamma_i)}{\prod_{j=0}^{p-2} (\Lambda_N - | \gamma^{'} - \sum_{i=1}^{j} \gamma_i + t |^2)} + O(\rho^{-c\alpha}), \tag{43}$$

where the sum is taken under the conditions $\gamma^{'} - \sum_{i=1}^{j} \gamma_i \notin M$, for $j \leq p-1$. These conditions, the definition of $M$, and (32) imply that

$| \Lambda_N - | \gamma^{'} - \sum_{i=1}^{j} \gamma_i + t |^2 > \frac{1}{4}\rho^{\alpha_1}$. Therefore (43) and the equality in (13) give (40). Now assume that Cond.3 holds. First we prove that if

$\gamma^{'} - \sum_{i=1}^{s} \gamma_i \in M$, where $s = 0, 1, ..., n$ and $n$, $\gamma_i$ is defined in the case of Cond.2, then $\beta^{'} - \sum_{i=1}^{s} \beta_i \neq \beta$. Assume the converse, i.e., $\beta^{'} - \sum_{i=1}^{n} \beta_i = \beta$. Then the inequalities $| a_i\delta |< \rho^\alpha$, (see (41)) $| j^{'}\delta |\geq h$ and (42) give

$| \gamma^{'} + t - \sum_{i=1}^{s} \gamma_i |^2 \geq | \beta + \tau |^2 + \frac{1}{2}h^2$. This,

$E(\lambda_{j,\beta}(v, \tau)) =| \beta + \tau |^2 + \mu_j(v) + O(\rho^{-\alpha_2} \ln \rho)$ ( see (32), (33)) and

$\mu_j(v) =| (j + v)\delta |^2 + O(1) = O(\rho^{2\alpha_1})$ ( see (28), (27)) imply

$$E(\lambda_{j,\beta}(v, \tau)) =| \beta + \tau |^2 + O(\rho^{2\alpha_1}), \tag{44}$$

$| E(\lambda_{j,\beta}(v, \tau)) - | \gamma^{'} + t - \sum_{i=1}^{n} \gamma_i |^2 |> \rho^{\alpha_1}$ which contradicts $\gamma^{'} - \sum_{i=1}^{n} \gamma_i \in M$. Using the proved fact for $s = 0$ which means that if $\gamma^{'} \in M$ then $\beta^{'} \neq \beta$ and taking into account that (40) holds under Cond.1 we get the proof of (40) under Cond.3 and $\gamma^{'} \in M$ . Consider the case $\gamma^{'} \notin M$. To prove (40) in this case we repeat the prove of the case of Cond.2, that is, use (43) and etc.$\diamondsuit$

**Theorem 3** *If $\gamma+t=\beta+\tau+(j+v(\beta))\delta\in B_\delta$, $|\gamma|\sim\rho$ then there exists a unique eigenvalue $\Lambda_N$ satisfying (32). This is a simple eigenvalue and the corresponding eigenfunction $\Psi_N$ satisfies the asymptotic formula*

$$\Psi_N=\Phi_{j,\beta}(x)+O(\rho^{-\alpha_2}\ln\rho). \tag{45}$$

**Proof** The proof is similar to that of Theorem1. First let us analýze the very similar part , that is, the repeating part of it. By Stat.3. of this section there exists the eigenvalue $\Lambda_N$ satisfying (32). Let $\Psi_N$ be any normalized eigenfunction corresponding to $\Lambda_N$. Arguing as in the proof of theorem 1 we see that under the assumption $\arg b(N,j,\beta)=0$ we need to show that

$$\sum_{(j',\beta')\in K_0}|b(N,j',\beta')|^2=O(\rho^{-2\alpha_2}(\ln\rho)^2), \tag{46}$$

where $K_0=\{(j',\beta'):j'\in Z,\beta'\in\Gamma_\delta,(j',\beta')\neq(j,\beta)\}$. To prove (46) we devide the set $K_0$ into subsets: $K_1^c$, $K_1\cap S(p-1)$, $K_1\cap S^c(p-1)$, where

$K_1^c=K_0\setminus K_1, S^c(n)=K_0\setminus S(n), K_1=\{(j',\beta')\in K_0:|\Lambda_N-\lambda_{j',\beta'}|<h^2\}$,

$S(n)=\{(j',\beta')\in K_0:|\beta-\beta'|\leq n\rho^\alpha, |j'\delta|<10^nh\}$. (For $h$ see lemma 2). If $(j',\beta')\in K_1^c$ then using (36) and the definition of $K_1^c$ we have

$$\sum_{(j',\beta')\in K_1^c}|b(N,j',\beta')|^2=\sup|q-q^\delta|^2\,O(\rho^{-2\alpha_2})=O(\rho^{-2\alpha_2}). \tag{47}$$

To consider the set $K_1\cap S(p-1)$ we first prove that

$$K_1\cap S(n)=K_1\cap\{(j',\beta):j'\in Z\}\subset\{(j',\beta):|j'\delta|<2h\}, \tag{48}$$

for $n=1,2,...,p-1$. Take any element $(j',\beta)$ from $K_1\cap\{(j',\beta):j'\in Z\}$. By definition of $K_1$ and by (32) (44) we have

$|O(\rho^{2\alpha_1})-|(j'+v')\delta|^2|<2h^2, |j'\delta|<2h$, since $h^2\sim\rho^{\alpha_2}$ and $\alpha_2=3\alpha_1$. Hence the inclusion in (48) is proved and $K_1\cap\{(j',\beta):j'\in Z\}\subset K_1\cap S(n)$. If the inclusion $K_1\cap S(n)\subset K_1\cap\{(j',\beta):j'\in Z\}$ does not hold then there is an element $(j',\beta')$ of $K_1\cap S(n)$ such that

$0<|\beta-\beta'|\leq n\rho^\alpha\leq(p-1)\rho^\alpha$, $|j'\delta|<10^nh<\frac{1}{2}\rho^{\frac{1}{2}\alpha_2}$. Hence by first inequality in (34) we have , $|\Lambda_N-\lambda_{j',\beta'}|>\frac{1}{2}\rho^{\alpha_2}>h^2$ which contradicts $(j',\beta')\in K_1$. So (48) is proved. Now using it we prove

$$\sum_{(j',\beta')\in K_1\cap S(p-1)}|b(N,j',\beta')|^2\leq\sum_{j'\neq j,|j'\delta|<2h}|b(N,j',\beta)|^2=O(\rho^{-2\alpha_2}(\ln\rho)^2). \tag{49}$$

For estimation of $b(N,j^{'},\beta)$, for $\mid j^{'}\delta \mid< 2h$, we use (36) as follows. In (36) replase $\beta^{'}$ and $r$ by $\beta$ and $2h$. Then the sum in (36) is taken under condition $(j_1,\beta_1)\in Q(\rho^{\alpha},18h)$, that is, $\mid j_1\delta \mid< 18h$ , $\mid \beta_1 \mid< \rho^{\alpha}$. Hence in right side of (36) the multiplicant $b(N,j^{'}+j_1,\beta+\beta_1)$ takes part, where

$(j^{'}+j_1,\beta+\beta_1)\in D(\beta)\equiv\{(j,\beta+\beta_1):\mid j\delta \mid< \frac{1}{2}\rho^{\frac{1}{2}\alpha_2}, 0<\mid \beta_1 \mid<\rho^{\alpha}\}$, becouse $20h<\frac{1}{2}\rho^{\frac{1}{2}\alpha_2}$. Put $\mid b(N,j_0,\beta+\beta_0)\mid=\max\mid b(N,j,\beta+\beta_1)\mid$, where max is taken under condition $(j,\beta+\beta_1)\in D(\beta)$. By definition of $D(\beta)$ and by (34) we have $\mid \Lambda_N-\lambda_{j_0,\beta+\beta_0}\mid>\frac{1}{2}\rho^{\alpha_2}$. This together with (36) gives $\mid b(N,j_0,\beta+\beta_0)\mid=O(\rho^{-\alpha_2})$. Using this, (36) for $r=2h$, and (37), we get $\mid b(N,j^{'},\beta)\mid\leq c_6\mid b(N,j_0,\beta+\beta_0)\mid\frac{1}{|\mu_j-\mu_{j^{'}}|}=\frac{1}{|\mu_j-\mu_{j^{'}}|}O(\rho^{-\alpha_2})$

for $j^{'}\neq j$, $\mid j^{'}\delta \mid< 2h$. Hence by (28), (29) we have

$$\sum_{j^{'}\neq j,|j^{'}\delta|<2h}\mid b(N,j^{'},\beta)\mid^2=O(\rho^{-2\alpha_2})\sum_{j^{'}\neq j}\mid \mu_j-\mu_{j^{'}}\mid^{-2}=O(\rho^{-2\alpha_2}(\ln\rho)^2). \tag{50}$$

Thus (49) is proved. It remains to consider $K_1\cap S^c(p-1)$. We prove that

$$b(N,j^{'},\beta^{'})\mid=O(\rho^{-\frac{c}{9}\alpha_2})=O(\rho^{-c\alpha}),\forall(j^{'},\beta^{'})\in K_1\cap S^c(p-1) \tag{51}$$

For this using a decomposition of $\varphi_{j^{'},v(\beta^{'})}(s)$ by $\{e^{i(m+v(\beta^{'}))s}:m\in Z\}$ we get

$$b(N,j^{'},\beta^{'})=\sum_m(\varphi_{j^{'},v(\beta^{'})}(s),e^{i(m+v(\beta^{'}))s})(\Psi_N(x),e^{i(\beta^{'}+\tau+(m+v(\beta^{'}))\delta,x)}). \tag{52}$$

If $\mid \beta-\beta^{'}\mid>(p-1)\rho^{\alpha}$ then Lemma 2 ( see Cond. 2) and (31), (52) give the proof (51). So we need to consider the case $\mid \beta-\beta^{'}\mid\leq(p-1)\rho^{\alpha}$. Then by definition of $S^c(p-1)$ we have $\mid j^{'}\delta\mid\geq 10^{p_1-1}h$. Divite the sum in (52) into two part $\sum_1$ and $\sum_2$, where the summutions are taken under conditions $\mid m\delta\mid\geq h$ and $\mid m\delta \mid<h$ respectively. In first case by (31) and Lemma 2 ( see Cond.3 ) we have $\sum_1=O(\rho^{-c\alpha})$. In second case the inequality $\mid j^{'}\mid>2\mid m\mid$ holds. Therefore using (30), taking into account that $\mid j^{'}\delta\mid\sim\rho^{\alpha_2}$ and the number of summand in $\sum_2$ is less than $\rho^{\alpha_2}$, we get $\sum_2=O(\rho^{-c\alpha})$. The estimations for $\sum_1$, $\sum_2$ give (51). Now using $\mid K_1\mid=O(\rho^{(d-1)q\alpha})$, we get

$$\sum_{(j^{'},\beta^{'})\in K_1\cap S^c(p-1)}\mid b(N,j^{'},\beta^{'})\mid^2=O(\rho^{-(2c-(d-1)q)\alpha}), \tag{53}$$

This, (47), (49) give the proof of (46) and (45). The theorem is proved$\diamondsuit$

Now using Theorem 3, we prove the more precise asymptotic formulas.

**Theorem 4** *The eigenfunction $\Psi_N$ defined in Theorem 3, besides formula (45), satisfies the following asymptotic formulas*

$$\Psi_N = \Phi_{j,\beta}(x) + \Phi_{k-1}(x) + O(\rho^{-k\alpha_2}\ln\rho) \tag{54}$$

*for $k = 1, 2, ..., n_1 \equiv [\frac{1}{9}(c - \frac{1}{2}q(d-1)]$, where $c = p - dq - 3^d d$, $\Phi_0(x) = 0$, $\Phi_{k-1}(x)$ is linear conbination of $\Phi_{j,\beta}(x)$ and $\Phi_{j',\beta'}(x)$ for $(j',\beta') \in S(k-1)$ with explicitly expressed by $q(x), \lambda_{j',\beta'}, \Phi_{j',\beta'}(x)$ coefficients.*

**Proof** By Theorem 2 formula (54) for $k = 1$ is proved. To prove it for arbitrary $k$ ( $k \le n_1$) we prove the following equivalent formulas

$$\sum_{(j',\beta')\in S^c(k-1)} \mid b(N,j',\beta') \mid^2 = O(\rho^{-2k\alpha_2}(\ln\rho)^2), \tag{55}$$

$$\Psi_N = \sum_{(j',\beta')\in S(k-1)\cup(j,\beta)} b(N,j',\beta')\Psi_{j',\beta'} + O(\rho^{-k\alpha_2}\ln\rho). \tag{56}$$

First consider $S^c(k-1)\cap K_1$. It follows from $S(k-1)\cap K_1 = S(p-1)\cap K_1$ (see (48)) and $S(k-1) \subset S(p-1)$ for $0 < k < p$ ( see definition of $S(k-1)$) that $(S(p-1))\backslash S(k-1)) \cap K_1 = \emptyset$ , and hence
$S^c(k-1) = S^c(p-1)\cup\ (S(p-1)\backslash S(k-1))$,
$S^c(k-1)\cap K_1 = S^c(p-1)\cap K_1$. Therefore by (53) the equality (55), when sum is taken over $(j',\beta') \in S^c(k-1)\cap K_1$ holds. Thus we need to prove

$$\sum_{(j',\beta')\in S^c(k-1)\cap K_1^c} \mid b(N,j',\beta') \mid^2 = O(\rho^{-2k\alpha_2}(\ln\rho)^2). \tag{57}$$

We prove this by induction. By formula (35) and (45) we have
$\Psi_N(x)(q(x) - Q(s)) = H(x) + O(\rho^{-\alpha_2}\ln\rho)$, where $H(x)$ is linear conbination of $\Phi_{j,\beta}(x)$ and $\Phi_{j',\beta'}(x)$ for $(j',\beta') \in S(1)$, since $\mid j\delta \mid < r_1 < h$ (see (27)). Hence $H(x)$ orthogonal to $\Phi_{j',\beta'}(x)$ for $(j',\beta') \in S^c(1)$. So by (36),

$$\begin{aligned}\sum_{(j',\beta')\in S^c(1)\cap K_1^c} \mid b(N,j',\beta') \mid^2 &= \sum \mid \frac{(O(\rho^{-\alpha_2}\ln\rho), \Phi_{j',\beta'})}{\Lambda_N - \lambda_{j',\beta'}} \mid^2 \\ &= O(\rho^{-4\alpha_2}(\ln\rho)^2)\end{aligned} \tag{58}$$

Hence (57) for $k = 2$ is proved. Assume that this is true for $k = m$. Then (56), for $k = m$ holds. This and (35) for $r = 10^{m-1}h$ give

$\Psi_N(x)(q(x)-Q(s)) = H(x) + O(\rho^{-m\alpha_2}\ln\rho)$, where $H(x)$ is a linear combination of $\Phi_{j,\beta}(x)$ and $\Psi_{j',\beta'}(x)$ for $(j',\beta') \in S(m)$. So $H(x)$ is orthogonal to $\Psi_{j',\beta'}(x)$ for $(j',\beta') \in S^c(m)$. Using this and repeating the proof of (57) for $k=2$ we obtain the proof of (57) for $k=m+1$. Thus (55), (56) are proved. Here $b(N,j,\beta)$ and $b(N,j',\beta')$ for $(j',\beta') \in S(k-1)$ can be calculated in the same way as we found $b(N,\gamma)$ and $b(N,\gamma+\gamma')$ for $\gamma' \in \Gamma((n-1)\rho^{\alpha})$ in Theorem 1. Naimly we apply the formula (36) $2k+2$ times and each time isolate the terms with multiplicant $b(N,j,\beta)$. Then in the obtained expression instead of $\Lambda_N$ writing the right side of (32) for $k=k_3$ we write $b(N,j',\beta')$ in term of $b(N,j,\beta)$ up to $O(\rho^{-k\alpha_2})$. Substituting the obtained formula for $b(N,j',\beta')$ into (56) and taking into account that $\Psi_N$ is a normalised eigenfunction with $\arg b(N,j,\beta)=0$ we find $b(N,j,\beta)$ and then $b(N,j',\beta')\diamondsuit$

# 4 Simple Sets and Isoenergetic surfaces

In this section we consider the simple sets in the nonresonance and resonance domains. Moreover we construct a "big" part of isoenergetic surfaces corresponding to $\rho^2$ for big $\rho$. The isoenergetic surfaces of $L(q)$ corresponding to $\rho^2$ is the set $I_\rho(q(x)) = \{t \in F^* : \exists N, \Lambda_N(t) = \rho^2\}$. In the case $q(x)=0$ isoenergetic surface is $I_\rho(0) = \{t \in F^* : \exists\gamma \in \Gamma, \mid \gamma+t \mid^2 = \rho^2\}$. The sphere $B(\rho) = \{x \in R^d : \mid x \mid^2 = \rho^2\} = \{\gamma + t : t \in F^*, \gamma \in \Gamma, \mid \gamma+t \mid^2 = \rho^2\}$ is the translation of $I_\rho(0)$ by the vectors $\gamma \in \Gamma$ to the sphere $B(\rho)$. We call this the translated isoenergetic surfaces of $L(0)$ corresponding to $\rho^2$. Similarly, we call the sets $\{t \in F^* : \Lambda(\gamma+t) = \rho^2\} \subset I_\rho(q(x))$ and $\{\gamma+t : \Lambda(\gamma+t) = \rho^2\}$, where $\Lambda(\gamma+t) = \Lambda_{N(\gamma+t)}(t)$ is the unique eigenvalue ( see Theorem 1) satisfying (6), the parts of isoenergetic surfaces and translated isoenergetic surfaces of $L(q)$ corresponding to $\rho^2$. In this section we construct the subsets $I_\rho^{''} \subset I_\rho$ and $I_\rho^{'}$ of these parts of isoenergetic surfaces and translated isoenergetic surfaces respectively and prove that the measures of these subsets are asymptotically equal to the measure of isoenergetic surfaces of $L(0)$ (see (65), Theorem 6(c)). In other word we construct a big part (in some sense) of isoenergetic surfaces $I_\rho(q(x))$ of $L(q)$. As we see below the set $I_\rho^{''}$ is a translation of $I_\rho^{'}$ by vectors $\gamma \in \Gamma$ to $F^*$ and the set $I_\rho^{'}$ lies in $\varepsilon$ nieghborhood of the surface $S_\rho = \{x \in U(\rho^{\alpha_1}, p) : F(x) = \rho^2\}$. Due to (6), (7) ( replace $\gamma+t$ by $x$ in $F(\gamma+t)$ ) it is natural to call $S_\rho$ the approximated isoenergetic surfaces in the nonresonance domain. Here we construct a part of the simple set $B$ in nieghborhood of $S_\rho$ that containes $I_\rho^{'}$. For this we consider the surface $S_\rho$.

As we noted in introduction and section 2 in order that the nonresonance eigenvalue $\Lambda(\gamma+t)$ can not coincide with any other nonresonance eigenvalue $\Lambda(\gamma+t+b)$ we choose the $\gamma+t$ such that $\mid F(\gamma+t)-F(\gamma+t+b)\mid > 2\varepsilon_1$ for $\gamma+t+b\in U(\rho^{\alpha_1},p)$ and $b\in\Gamma\backslash\{0\}$ (see (15)). Therefore we eliminate

$$P_b=\{x: x, x+b\in U(\rho^{\alpha_1},p), \mid F(x)-F(x+b)\mid < 3\varepsilon_1\} \tag{59}$$

for $b\in\Gamma\backslash\{0\}$ from $S_\rho$. Denote the remaining part of $S_\rho$ by $S_\rho^{'}$. Since the part $I_\rho^{'}$ of the translated isoenergetic surfaces lies in the $\varepsilon$ neighbourhood of the approximated isoenergetic surfaces $S_\rho^{'}$ we consider $U_\varepsilon(S_\rho^{'})=\cup_{a\in S_\rho^{'}}U_\varepsilon(a)\}$, where $\varepsilon=\frac{\varepsilon_1}{7\rho}$, $U_\varepsilon(a)=\{x\in R^d :\mid x-a\mid<\varepsilon\}$. In this set the first simplicity condition (15) holds (see Lemma 5(a)). Denote by

$Tr(E)=\{\gamma+x\in U_\varepsilon(S_\rho^{'}):\gamma\in\Gamma, x\in E\}$ and

$Tr_{F^\star}(E)\equiv\{\gamma+x\in F^\star:\gamma\in\Gamma, x\in E\}$ the translations of $E\subset R^d$ into $U_\varepsilon(S_\rho^{'})$ and $F^\star$ respectively. In order that the second simplicity condition (16) also holds, we discart from $U_\varepsilon(S_\rho^{'})$ the translation $Tr(A(\rho))$ of $A(\rho)$, where $A(\rho)=\cup_{k=1}^{d-1}(\cup_{\gamma_1,\gamma_2,...,\gamma_k\in\Gamma(p\rho^\alpha)}(\cup_{i=1}^{b_k(\gamma_1,\gamma_2,...,\gamma_k)}A_{k,i}(\gamma_1,\gamma_2,...,\gamma_k)))$, $A_{k,i}(\gamma_1,...,\gamma_k)=\{x\in(\cap_{i=1}^{k}V_{\gamma_i}(\rho^{\alpha_k})\backslash E_{k+1})\cap K_\rho:\lambda_i(x)\in(\rho^2-3\varepsilon_1,\rho^2+3\varepsilon_1)\}$, and $K_\rho=\{x\in R^d :\mid\mid x\mid^2-\rho^2\mid<\rho^{\alpha_1}\}$. As a result we construct the part $U_\varepsilon(S_\rho^{'})\backslash Tr(A(\rho))$ of the simple set $B$ (see Theorem 6(a)) which contains the set $I_\rho^{'}$ (see Theorem 6(c)). For this we need the following lemma.

**Lemma 5** *(a) If $x\in U_\varepsilon(S_\rho^{'})$ and $x+b\in U(\rho^{\alpha_1},p)$, where $b\in\Gamma$, then*

$$\mid F(x)-F(x+b)\mid > 2\varepsilon_1 \tag{60}$$

*Hence for $\gamma+t\in U_\varepsilon(S_\rho^{'})$ the first simplicity condition (15) holds.*

*(b) If $x\in U_\varepsilon(S_\rho^{'})$ then $x+b\notin U_\varepsilon(S_\rho^{'})$ for all $b\in\Gamma$ .*

*(c)If $E\subset R^d$ is bounded set then $\mu(Tr(E))\le\mu(E)$.*

*(d) If $E\subset U_\varepsilon(S_\rho^{'})$ then $\mu(Tr_{F^\star}(E))=\mu(E)$*

**Proof** $(a)$ If $x\in U_\varepsilon(S_\rho^{'})$ then there is a point $a$ in $S_\rho^{'}$ such that $x\in U_\varepsilon(a)$. Since $S_\rho^{'}$ does not contain the set $P_b$ ( see (59)) we have

$$\mid F(a)-F(a+b)\mid\ge 3\varepsilon_1 \tag{61}$$

On the other hand using (7) and the relations

$\mid x \mid < \rho + 1$, $\mid x - a \mid < \varepsilon$, $\mid x + b - a - b \mid < \varepsilon$ we obtain

$$\mid F(x) - F(a) \mid < 3\rho\varepsilon, \mid F(x+b) - F(a+b) \mid < 3\rho\varepsilon. \tag{62}$$

These inequalities together with (61) give (60), since $6\rho\varepsilon < \varepsilon_1$.

$(b)$ This is a simple consequence of $(a)$. Indeed if $x$ and $x+b$ lie in $U_\varepsilon(S_\rho^{'})$ then there are points $a$ and $c$ in $S_\rho^{'}$ such that $x \in U_\varepsilon(a)$ and $x + b \in U_\varepsilon(c)$. Repeating the proof of (62) we get $\mid F(c) - F(x+b) \mid < 3\rho\varepsilon$. This, the first ineguality in (62), and the relations $F(a) = \rho^2, F(c) = \rho^2$ (see the definition of $S_\rho$) give $\mid F(x) - F(x+b) \mid < \varepsilon_1$ which contradicts (60).

$(c)$ Clearly for any bounded set $E$ there are only finite number of the vectors $\gamma_1, \gamma_2, ..., \gamma_s$ such that $E(k) \equiv (E + \gamma_k) \cap U_\varepsilon(S_\rho^{'}) \neq \emptyset, k = 1, 2, ..., s$ and $Tr(E)$ is the union of the sets $E(k)$. For $E(k) - \gamma_k$ we have the relations $\mu(E(k) - \gamma_k) = \mu(E(k)), E(k) - \gamma_k \subset E$. Moreover by $(b)$

$(E(k) - \gamma_k) \cap (E(j) - \gamma_j) = \emptyset$ for $k \neq j$. Therefore $(c)$ is true.

$(d)$ Now let $E \subset U_\varepsilon(S_\rho^{'})$. Then by $(b)$ the set $E$ can by devided into finite number of the pairwise disjoint sets $E_1, E_2, ..., E_n$ such that there are the vectors $\gamma_1, \gamma_2, ..., \gamma_n$ satisfying $(E_k + \gamma_k) \subset F^\star, (E_k + \gamma_k) \cap (E_j + \gamma_j) \neq \emptyset$,for $k \neq j$ and $k, j = 1, 2, ..., n$. Using $\mu(E_k + \gamma_k) = \mu(E_k)$ we get the proof of $(d)$, becouse $Tr_{F^\star}(E)$ and $E$ are union of pairwise disjoint sets $E_k + \gamma_k$ and $E_k$ for $k = 1, 2, ..., n$ respectively$\diamondsuit$

**Theorem 6** $(a)$ *The set $U_\varepsilon(S_\rho^{'}) \backslash Tr(A(\rho))$ is a subset of $B$. Hence if $\gamma + t$ lies in this subset then Theorem 1 holds. For every connected open subset $E$ of $U_\varepsilon(S_\rho^{'}) \backslash Tr(A(\rho)$ there exists a unique index $N$ such that*

*$\Lambda(\gamma + t) = \Lambda_N(t)$ for all $\gamma + t \in E$. Moreover*

$$\frac{\partial}{\partial t_j}\Lambda(\gamma + t) = \frac{\partial}{\partial t_j} \mid \gamma + t \mid^2 + O(\rho^{1-2\alpha_1}), \forall j = 1, 2, ..., d. \tag{63}$$

$(b)$ *For the part $V_\rho \equiv S_\rho^{'} \backslash U_\varepsilon(Tr(A(\rho)))$ of the approximated isoenergetic surface the following holds*

$$\mu(V_\rho) = (1 + O(\rho^{-\alpha}))\mu(B(\rho)). \tag{64}$$

*Moreover $U_\varepsilon(V_\rho)$ lies in the subset $U_\varepsilon(S_\rho^{'}) \backslash Tr(A(\rho))$ of the simple set $B$.*

$(c)$ *The Isoenergetic surface $I(\rho)$ contains a set $I_\rho^{''}$ which consists of the smooth surfaces and has the measure*

$$\mu(I_\rho^{''}) = \mu(I_\rho^{'}) = (1 + O(\rho^{-\alpha}))\mu(B(\rho)), \tag{65}$$

*where $I_\rho^{'}$ is a part of the translated isoenergetic surfaces of $L(q)$ that is contained in the subset $U_\varepsilon(S_\rho^{'})\backslash Tr(A(\rho))$ of the simple set $B$. In particular the number $\rho^2$ for $\rho \gg 1$ lies in the spectrum of $L(q)$, that is, the number of the gaps in the spectrum of $L(q)$ is finite.*

**Proof** $(a)$ Suppose $\gamma + t \equiv x \in U_\varepsilon(S_\rho^{'})\backslash Tr(A(\rho))$, that is , there is $a \in S_\rho^{'}$ such that $x \in U_\varepsilon(a)$. Then, by lemma 5(a), (15) holds. Now we prove that (16) holds too. The first inequality in (62) and the relation $F(a) = \rho^2$ give

$$F(\gamma+t) \in (\rho^2 - 3\rho\varepsilon, \rho^2 + 3\rho\varepsilon) \subset (\rho^2 - \varepsilon_1, \rho^2 + \varepsilon_1), \forall \gamma + t \in U_\varepsilon(S_\rho^{'}) \quad (66)$$

On the other hand $\gamma + t \notin Tr(A(\rho))$. It means that for any $\gamma^{'} \in \Gamma$, we have $\gamma^{'} + t \notin A(\rho)$. If $\gamma^{'} \in K$ and $\gamma^{'} + t \in E_k\backslash E_{k+1}$ then by definition of $K$ ( see 16) the inequality $\mid F(\gamma+t) - \mid \gamma^{'} + t \mid^2 \mid < \frac{1}{3}\rho^{\alpha_1}$ holds. This and (66) imply that $\gamma^{'} + t \in (E_k\backslash E_{k+1}) \cap K_\rho$. Sinse $\gamma^{'} + t \notin A(\rho)$ we have $\lambda_i(\gamma + t) \notin (\rho^2 - 3\varepsilon_1, \rho^2 + 3\varepsilon_1)$ for $\gamma^{'} \in K$ and $\gamma^{'} + t \in E_k\backslash E_{k+1}$. Therefore (16) follows from (66). Thus $U_\varepsilon(S_\rho^{'})\backslash Tr(A(\rho)) \subset B$.

Now let $E$ be a connected open subset of $U_\varepsilon(S_\rho^{'})\backslash Tr(A(\rho) \subset B$. By Theorem 1 for $a \in E \subset U_\varepsilon(S_\rho^{'})\backslash Tr(A(\rho)$ the eigenvalue $\Lambda(a)$ is simple and the corresponding eigenfunction $\Psi_a(x)$ satisfies $\mid (\Psi_a(x), e^{i(a,x)}) \mid^2 > \frac{1}{2}$. So for $y \in E$ there exists a unique $N(y)$ such that $\Lambda(y) = \Lambda_{N(y)}(y), \Psi_y(x) = \Psi_{N(y),y}(x)$,

$$\mid (\Psi_{N(y),y}(x), e^{i(y,x)}) \mid^2 > \frac{1}{2}. \quad (67)$$

On the other hand for fixed $N$ the functions $\Lambda_N(t)$ and $(\Psi_{N,t}(x), e^{i(t,x)})$ are continuous in neighborhood of $t$ if $\Lambda_N(t)$ is a simple eigenvalue. Therefore for $a \in E$ there exists a neighborhood $U(a) \subset E$ of $a$ such that $\mid (\Psi_{N(a),y}(x), e^{i(y,x)}) \mid^2 > \frac{1}{2}$, for every $y \in U(a)$. This and (67) yield that $N(y) = N(a)$ for $y \in U(a)$, since there is a unique integer $N$ satisfying (67). Hence we proved the Proposition 1 which is $\forall a \in E, \exists U(a) \subset E : N(y) = N(a), \forall y \in U(a)$. Now let $a_1$ and $a_2$ be two points of $E$ and $C \subset E$ be the arc which joins these points. (Note that the open connected subset of $R^d$ is arcwise connected). Let $U(y_1), U(y_2), ..., U(y_k)$ be finite subcover of the open cover $\cup_{a \in C} U(a)$ of the compact $C$, where $U(a)$ is a neighborhood of $a$ satisfying Proposition 1. By Proposition 1 $N(y) = N(y_i) = N_i$ for $y \in U(y_i)$. Clearly if $U(y_i) \cap U(y_j) \neq \emptyset$ then $N_i = N_j$ since for $a \in U(y_i) \cap U(y_j)$ there is a unique $N(a)$. Thus $N_1 = N_2 = ... = N_k$ and $N(a_1) = N(a_2)$.

To calculate the partial derivatives of the function $\Lambda(\gamma+t)=\Lambda_N(t)$ we write the operator $L_t$ in the form $-\triangle-(2it,\nabla)+(t,t)$. Then, clearly

$$\begin{aligned}\frac{\partial}{\partial t_j}\Lambda_N(t) &= 2t_j(\Phi_{N,t}(x),\Phi_{N,t}(x))-2i(\frac{\partial}{\partial x_j}\Phi_{N,t}(x),\Phi_{N,t}(x)), &(68)\\ \Phi_{N,t}(x) &= \sum_{\gamma^{'}\in\Gamma} b(N,\gamma^{'})e^{i(\gamma^{'},x)} &(69)\end{aligned}$$

where $\Phi_{N,t}(x)=e^{-i(t,x)}\Psi_{N,t}(x)$. If $\mid\gamma^{'}\mid\geq 2\rho$ then using

$\Lambda_N\equiv\Lambda(\gamma+t)=\rho^2+O(\rho^{-\alpha})$, ( see (6) and (66)),

$\mid\Lambda_N-\mid\gamma^{'}-\gamma_1-\gamma_2-...-\gamma_k+t\mid^2\mid>\mid\gamma^{'}\mid^2$, for $k=0,1,...,p$, where $\mid\gamma_1\mid<\frac{1}{4p}\mid\gamma^{'}\mid$, and iterating (12) $p$ times by using the decomposition $q(x)=\sum_{\mid\gamma_1\mid<\frac{1}{4p}\mid\gamma^{'}\mid}q_{\gamma_1}e^{i(\gamma_1,x)}+O(\mid\gamma^{'}\mid^{-p})$ we get

$$\begin{aligned}b(N,\gamma^{'}) &= \sum_{\gamma_1,\gamma_2,...}\frac{q_{\gamma_1}q_{\gamma_2}...q_{\gamma_p}b(N,\gamma^{'}-\sum_{i=1}^{p}\gamma_i)}{\prod_{j=0}^{p-1}(\Lambda_N-\mid\gamma^{'}-\sum_{i=1}^{j}\gamma_i+t\mid^2)}+O(\mid\gamma^{'}\mid^{-p}), &(70)\\ b(N,\gamma^{'}) &= O(\mid\gamma^{'}\mid^{-p}),\forall\mid\gamma^{'}\mid\geq 2\rho &(71)\end{aligned}$$

By (71) the series in (69) can be differentiated term by term. Hence

$$i(\frac{\partial}{\partial x_j}\Phi_{N,t},\Phi_{N,t})=\sum_{\gamma^{'}\in\Gamma}\gamma^{'}(j)\mid b(N,\gamma^{'})\mid^2=\gamma(j)\mid b(N,\gamma)\mid^2+\Sigma_1+\Sigma_2, \quad (72)$$

where $\Sigma_1=\sum_{\mid\gamma^{'}\mid\geq 2\rho}\gamma^{'}(j)\mid b(N,\gamma^{'})\mid^2$, $\Sigma_2=\sum_{\mid\gamma^{'}\mid<2\rho,\gamma^{'}\neq\gamma}\gamma^{'}(j)\mid b(N,\gamma^{'})\mid^2$. By (2), (4), $\gamma(j)\mid b(N,\gamma)\mid^2=\gamma(j)(1+O(\rho^{-2\alpha_1})$, $\sum_2=O(\rho^{-2\alpha_1+1})$ and by (71), $\sum_1=O(\rho^{-2\alpha_1})$. Therefore (68) and (72) give (63).

$(b)$ First we note that if $a\in V_\rho$ then $U_\varepsilon(a)\in U_\varepsilon(S_\rho^{'})\backslash Tr(A(\rho))$, because the relation $a\in S_\rho^{'}$ implies that $U_\varepsilon(a)\in U_\varepsilon(S_\rho^{'})$ and the relation $a\notin U_\varepsilon(Tr(A(\rho)))$ implies that $U_\varepsilon(a)\cap Tr(A(\rho))=\emptyset$. To prove (64) first we estimate the measure of $S_\rho, S_\rho^{'}, U_{2\varepsilon}(A(\rho))$, namely we prove

$$\begin{aligned}\mu(S_\rho) &= (1+O(\rho^{-\alpha}))\mu(B(\rho)), &(73)\\ \mu(S_\rho^{'}) &= (1+O(\rho^{-\alpha}))\mu(B(\rho)), &(74)\\ \mu(U_{2\varepsilon}(A(\rho))) &= O(\rho^{-\alpha})\mu(B(\rho))\varepsilon. &(75)\end{aligned}$$

( See below, Estimations 1, 2, 3). The estimation (64) of the measure of set $V_\rho$ is done in Estimation 4 by using Estimations 1, 2, 3.

$(c)$ In Estimation 5 we prove the formula (65). The Theorem is proved $\diamondsuit$

In Estimations 1-5 we use the notations: $G(+i,a)=\{x\in G, x_i>a\}$, $G(-i,a)=\{x\in G, x_i<-a\}$, where $x=(x_1,x_2,...,x_d), a>0$. It is not hard to verify that for any subset $G$ of $U_\varepsilon(S_\rho^{'})\cup U_{2\varepsilon}(A(\rho))$ , hence for all considered sets $G$ in these estimations, and for any $x\in G$ the followings hold

$$\rho-1<\mid x\mid<\rho+1, G\subset(\cup_{i=1}^{d}(G(+i,\rho d^{-1})\cup G(-i,\rho d^{-1})) \tag{76}$$

Indeed if $x\in S_\rho^{'}$ then $F(x)=\rho^2$ and by (7) we have $\mid x\mid=\rho+O(\rho^{-1-2\alpha_1})$. Hence the inequalities in (76) hold for $x\in U_\varepsilon(S_\rho^{'})$. If $x\in A(\rho)$ then by definition of $A(\rho)$ we have $x\in K_\rho$, and hence $\mid x\mid=\rho+O(\rho^{-1+\alpha_1})$. Therefore the inequalities in (76) hold for $x\in U_{2\varepsilon}(A(\rho))$ too. The inclution in (76) follows from these inequalities.

If $G\subset S_\rho$ then for $x\in G(+k,\rho^{-\alpha})$ by (7) we have $\frac{\partial F(x)}{\partial x_k}>0,$ . Therefore to calculate the measure of $G(+k,a)$ for $a\geq\rho^{-\alpha}$ we use the formula

$$\mu(G(+k,a))=\int\limits_{\Pr_k(G(+k,a))}(\frac{\partial F}{\partial x_k})^{-1}\mid grad(F)\mid dx_1...dx_{k-1}dx_{k+1}...dx_d, \tag{77}$$

where $\Pr_k(G)\equiv\{(x_1,x_2,...,x_{k-1},x_{k+1},x_{k+2},...,x_d):x\in G\}$ is the projection of $G$ on the hyperplane $x_k=0$. Instead of $\Pr_k(G)$ we write $\Pr(G)$ if $k$ is unambiguous. If $D$ is $m-$dimensional subset of $R^m$ then we use

$$\mu(D)=\int\limits_{\Pr_k(D)}\mu(D(x_1,...x_{k-1},x_{k+1},...,x_m))dx_1...dx_{k-1}dx_{k+1}...dx_m, \tag{78}$$

where $D(x_1,...x_{k-1},x_{k+1},...,x_m)=\{x_k:(x_1,x_2,...,x_m)\in D\}$.

ESTIMATION 1. Here we prove (73) by using (77). During this estimation the set $S_\rho$ is redenoted by $G$. If $x\in G$ then $x\notin V_b(\rho^{\alpha_1}))$, for all $b\in\Gamma(p\rho^\alpha)$. Since the rotation does not change the measure we choose the coordinate axis so that the direction of some $b\in\Gamma(p\rho^\alpha)$ coinsides with the direction of $(1,0,0,...,0)$, that is, $b=(b_1,0,0,...,0), b_1>0$. Then the relation $x\notin V_b(\rho^{\alpha_1}))$ implies that $\mid x_1\mid>a$, where $a=(\rho^{\alpha_1}-b_1^2)(2b_1)^{-1}>\rho^\alpha$ because $\mid b\mid<p\rho^\alpha, \alpha_1=3\alpha$. Therefore $G=G(+1,a)\cup G(-1,a)$. Now we estimate $\mu(G(+1,a))$ by using (77) and

$$\frac{\partial F}{\partial x_1}>\rho^\alpha,\ (\frac{\partial F}{\partial x_1})^{-1}\mid grad(F)\mid=\frac{\mid x\mid}{x_1}+O(\rho^{-2\alpha}), \tag{79}$$

$$\Pr(G(+1,a))\subset C(\rho), \Pr(G(+1,a))\supset\Pr(C^{'}(\rho)), \tag{80}$$

where $x \in G(+1,a)$, $C(\rho) = \{(x_2,...,x_d) : x_2^2 + x_3^2 + ... + x_q^2 \le \rho^2\}$,

$C^{'}(\rho) = \{x \in B(\rho) : x_1 \ge 2a\} \cap U(2\rho^{\alpha_1}, p)$. Here (79) follows from (7) If $x = (x_2,...,x_d) \in \Pr(G(+1,a))$ then there is $x_1$ such that $x_1 > a > \rho^{\alpha}$,$x \in G$,

$$x_1^2 + x_2^2 + x_3^2 + ... + x_d^2 = \rho^2 + O(\rho^{-2\alpha_1}) \tag{81}$$

(see (7)) which implies first inclusion in (80). Therefore using (79), (77) and doing the same for $G(-1,a)$, we get

$$\mu(G(+1,a)) = \int\limits_{\Pr(G(+1,a))} \frac{|\, x\,|}{x_1} dx_2 dx_3 ... dx_d + O(\rho^{-\alpha})\mu(B(\rho)), \tag{82}$$

$$\mu(G) < (1 + c_7\rho^{-\alpha})\mu(B(\rho)). \tag{83}$$

Now we prove the second inclusion in (80). If $(x_2,...,x_d) \in \Pr C^{'}(\rho)$ then by definition of $C^{'}(\rho)$ there exists $x_1$ such that

$$x_1 > 2a > 2\rho^{\alpha},\ x_1^2 + x_2^2 + ... + x_d^2 = \rho^2, |\sum_{i \ge 1}(2x_i b_i - b_i^2)\,| \ge 2\rho^{\alpha_1} \tag{84}$$

for all $(b_1, b_2, ..., b_d) \in \Gamma(p\rho^{\alpha})$. Therefore for $h = \rho^{-\alpha}$ we have

$(x_1 + h)^2 + x_2^2 + ... + x_q^2 > \rho^2 + \rho^{-\alpha}, (x_1 - h)^2 + x_2^2 + ... + x_q^2 < \rho^2 - \rho^{-\alpha}$. This and (7) give $F(x_1 + h, x_2, ..., x_d) > \rho^2$, $F(x_1 - h, x_2, ..., x_d) < \rho^2$. Since $F$ is a continuous function there is $y_1 \in (x_1 - h, x_1 + h)$ such that (see (84))

$$y_1 > a, F(y_1, x_2, ..., x_d) = \rho^2, |\ 2y_1 b_1 - b_1^2 + \sum_{i \ge 2}(2x_i b_i - b_i^2)\,| > \rho^{\alpha_1}, \tag{85}$$

because the expression under the absolute value in (85) differ from the expression under the absolute value in (84) by $2(y_1 - x_1)b_1$, whose absolute value is less than $\rho^{\alpha_1}$. The relations in (85) means that $(x_2,...,x_d) \in \Pr G(+1,a)$. Hence the second inclusion in (80) is proved. Using this, (82), and doing the same for $G(-1,a)$ we obtain $\mu(G) > \mu(C^{'}(\rho)) + c_8\rho^{-\alpha}\mu(B(\rho))$. This inequality implies $\mu(G) > (1 - c_9\rho^{-\alpha})\mu(B(\rho))$ since $\mu(C^{'}(\rho)) = (1 + O(\rho^{-\alpha}))\mu(B(\rho))$ (see (32) of [10]). This together with (83), gives (73).

ESTIMATION 2 Here we prove (74). For this we estimate the measure of the set $S_\rho \cap P_b$ ( see (59)) by using (77). During this estimation the set $S_\rho \cap P_b$ is redenoted by $G$. We choose the coordinate axis so that the direction of $b$ coincides with the direction of $(1,0,0,...,0)$, i.e., $b=(b_1,0,0,...,0)$ and $b_1>0$.Clearly if $(x_1,x_2,...,x_d)\in G$ then ( see definition of $S_\rho$ and $F(x)$)

$$\begin{aligned} x_1^2+x_2^2+...+x_d^2+F_{k_2-1}(x) &= \rho^2, & (86)\\ (x_1-b_1)^2+x_2^2+x_3^2+...+x_d^2+F_{k_2-1}(x-b) &= \rho^2+h & (87)\\ (2x_1-b_1)b_1 &= O(\rho^{-2\alpha_1}), & (88)\end{aligned}$$

where $h\in(-3\varepsilon_1,3\varepsilon_1)$ and (88) follows from (86), (87), (7). By (76), (88),

$$\mid b_1\mid<2\rho+3,\ x_1=\frac{b_1}{2}+O(\rho^{-2\alpha_1}b_1^{-1}),\mid x_1^2-(\frac{b_1}{2})^2\mid=O(\rho^{-2\alpha_1}) \tag{89}$$

Conside two cases. Case 1: $b\in\Gamma_1\equiv\{b\in\Gamma:\mid\rho^2-(\frac{b_1}{2})^2\mid<3d\rho^{-2\alpha}\}$. In this case from (89), (86), and (7), we obtain

$$x_1^2=\rho^2+O(\rho^{-2\alpha}),\mid x_1\mid=\rho+O(\rho^{-2\alpha-1}),x_2^2+x_3^2+...+x_d^2=O(\rho^{-2\alpha}). \tag{90}$$

Therefore $G\subset G(+1,a)\cup G(-1,a)$, where $a=\rho-\rho^{-1}$. Using (77), the obvious relation $\mu(\Pr_1(G(+1,a))=O(\rho^{-(d-1)\alpha})$ (see (90)) and the fact that the expression under the integral (77) for $k=1$ is equal to $1+O(\rho^{-\alpha})$ (see (79) and (90)) and doing the same for $G(-1,a)$ we get $\mu(G)=O(\rho^{-(d-1)\alpha})$. Since $\mid\Gamma_1\mid=O(\rho^{d-1})$ we have

$$\mu(\cup_{b\in\Gamma_1}(S_\rho\cap P_b)=O(\rho^{-(d-1)\alpha+d-1})=O(\rho^{-\alpha})\mu(B(\rho)). \tag{91}$$

Case 2: $b\notin\Gamma_1$, i.e., $\mid\rho^2-(\frac{b_1}{2})^2\mid\geq3d\rho^{-2\alpha}$. By (89), (86), (7) we have

$$\mid x_1^2-\rho^2\mid>2d\rho^{-2\alpha},x_2^2+x_3^2+...+x_d^2>d\rho^{-2\alpha},\mid x_k\mid>\rho^{-\alpha}, \tag{92}$$

for some $k=2,3,...,d$. Therefore $G\subset\cup_{k\geq2}(G(+k,\rho^{-\alpha})\cup G(-k,\rho^{-\alpha}))$. We calculate $\mu(G(+d,\rho^{-\alpha}))$ by (77). Redenote by $D$ the set $\Pr_d G(+d,\rho^{-\alpha})$. If $x\in G(+d,\rho^{-\alpha})$ then by (86) and (7) the under integral expression in (77) for $k=d$ is $O(\rho^{1+\alpha})$. Therefore the first equality in

$$\mu(D)=O(\varepsilon_1\mid b\mid^{-1}\rho^{d-2}),\ \mu(G(+d,\rho^{-\alpha}))=O(\rho^{d-1+\alpha}\varepsilon_1\mid b\mid^{-1}) \tag{93}$$

implies the second equality in (93). To prove the first equality of (93) we use (78) for $m=d-1$, and the relations $\mu(\Pr_1 D)=O(\rho^{d-2})$,

$$\mu(D(x_2,x_3,...,x_{d-1}))<6\varepsilon_1\mid b\mid^{-1},\ \forall(x_2,x_3,...,x_{d-1})\in\Pr D. \tag{94}$$

First relation follows from the fact that $\mid x \mid< \rho+1$ for $x\in G(+d,\rho^{-\alpha})$ (see (76)). So we need to prove (94). If $x_1\in D(x_2,x_3,...,x_{d-1})$ then (86) and (87) holds. Subtructing (86) from (87), we get

$$2x_1b_1-(b_1)^2+F_{k_2-1}(x-b)-F_{k_2-1}(x)=h \tag{95}$$

where $x_2,x_3,...,x_{d-1}$ are fixed . Hence we have two equations (86) and (95) with respect two unknown $x_1$ and $x_d$. Using (7), the implicit function theorem, and the inequalities $\mid x_d \mid> \rho^{-\alpha}$, $\alpha_1>2\alpha$ from (86), we obtain

$$x_d=f(x_1),\frac{df}{dx_1}=\frac{2x_1+O(\rho^{-2\alpha_1+\alpha})}{2x_d+O(\rho^{-2\alpha_1+\alpha})}=\frac{x_1}{x_d}+O(\rho^{-\alpha_1}) \tag{96}$$

Substituting this in (95), we get

$$2x_1b_1-b_1^2+F_{k_2-1}(x_1-b_1,x_2,...,x_{d-1},f(x_1))-F_{k_2-1}(x_1,...,x_{d-1},f)=h \tag{97}$$

Using (7), (96), the first equality in (89) and $\mid x_d \mid> \rho^{-\alpha}$ we see that the absolute value of the derivative (w.r.t. $x_1$) of the rigth side of (97) satisfies

$\mid 2b_1+O(\rho^{-2\alpha_1+\alpha})\frac{df}{dx_1}\mid=\mid 2b_1+O(\rho^{-2\alpha_1+\alpha})\frac{x_1}{x_d}+O(\rho^{-3\alpha_1+\alpha})\mid> b_1$. Therefore from (97) by implicit function theorem, we get $\mid \frac{dx_1}{dh}\mid< \frac{1}{|b|}$. This inequality and relation $h\in(-3\varepsilon_1,3\varepsilon_1)$ imply (94). Thus (93) is proved. In the same way we get the same estimation for $G(+k,\rho^{-\alpha})$ and $G(-k,\rho^{-\alpha})$ for $k\geq 2$. Hence $\mu(S_\rho\cap P_b)=O(\rho^{d-1+\alpha}\varepsilon_1\mid b\mid^{-1})$, for $b\notin\Gamma_1$. Since $\mid b\mid< 2\rho+3$ ( see (89)) and $\varepsilon_1=\rho^{-d-2\alpha}$, we infer $\mu(\cup_{b\notin\Gamma_1}(S_\rho\cap P_b))=O(\rho^{2d-1+\alpha}\varepsilon_1)=O(\rho^{-\alpha})\mu(B(\rho))$. This, (91) and (73) give the proof of (74).

ESTIMATION 3. Here we prove (75). Let $G\equiv U_{2\varepsilon}(A_{k,j}(\gamma_1,\gamma_2,...,\gamma_k))$, where $\gamma_1,\gamma_2,...,\gamma_k\in\Gamma(p\rho^\alpha)$. We turn the coordinate axis so that

$Span\{\gamma_1,\gamma_2,...,\gamma_k\}=\{x=(x_1,x_2,...,x_k,0,0,...,0)\}$. Then by (35) of [10] we have $x_n=O(\rho^{\alpha_k+(k-1)\alpha})$, for $n\leq k$, $x\in G$. This, (76), and

$\alpha_k+(k-1)\alpha<1$ give $G\subset(\cup_{i>k}(G(+i,\rho d^{-1})\cup G(-i,\rho d^{-1}))$,

$\mu(\Pr_i(G(+i,\rho d^{-1})))=O(\rho^{k(\alpha_k+(k-1)\alpha)+(d-1-k)})$, for $i>k$. Now using this and (78) for $m=d$ we prove

$$\mu(G(+i,\rho d^{-1}))=O(\varepsilon\rho^{k(\alpha_k+(k-1)\alpha)+(d-1-k)}),\forall i>k. \tag{98}$$

For this we redenote by $D$ the set $G(+i,\rho d^{-1})$ and prove

$$\mu((D(x_1,x_2,...x_{i-1},x_{i+1},...x_d))\leq(42d^2+4)\varepsilon,\forall i>k, \tag{99}$$

for all $(x_1, x_2, ...x_{i-1}, x_{i+1}, ...x_d) \in \Pr_i(D)$. To prove (99) it is sufficient to show that if both $x = (x_1, x_2, ..., x_i, ...x_d)$ and $x' = (x_1, x_2, ..., x_i', ..., x_d)$ are in $D$ then $\mid x_i - x_i' \mid \leq (42d^2 + 4)\varepsilon$. Assume the converse. Then $\mid x_i - x_i' \mid > (42d^2 + 4)\varepsilon$. We may assume that $\rho d^{-1} < x_i < x_i'$ ( see definition of $D$). Since $x$ and $x'$ lie in the $2\varepsilon$ neighborhood of $A_{k,j}$ there exist points $a$ and $a'$ in $A_{k,j}$ such that $\mid x - a \mid < 2\varepsilon$ and $\mid x' - a' \mid < 2\varepsilon$. Then we have the inequalities

$\rho d^{-1} - 2\varepsilon < a_i < a_i'$, $a_i' - a_i > 42d^2\varepsilon$, $(a_i')^2 - (a_i)^2 > 2(\rho d^{-1} - 2\varepsilon)(a_i' - a_i)$,

$\mid\mid a_s \mid - \mid a_s' \mid\mid < 4\varepsilon$, $\mid\mid a_s \mid^2 - \mid a_s' \mid^2\mid < 12\rho\varepsilon$ for $s \neq i$, and hence

$\sum_{s \neq i} \mid\mid a_s \mid^2 - \mid a_s' \mid^2\mid < 12d\rho\varepsilon < \frac{2}{7}\rho d^{-1}(a_i - a_i')$,

$\mid\mid a \mid^2 - \mid a' \mid^2\mid > \frac{3}{2}\rho d^{-1} \mid a_i' - a_i \mid$ . On the other hand by (11),

$\mid r_j(a) - r_j(a') \mid < \rho^{\frac{1}{2}\alpha_d} \mid a - a' \mid < \frac{1}{2}\rho d^{-1} \mid a_i' - a_i \mid$ . Therefore

$\mid \lambda_j(a) - \lambda_j(a') \mid \geq \mid a \mid^2 - \mid a' \mid^2\mid - \mid r_j(a) - r_j(a') \mid > \rho d^{-1} \mid a_i' - a_i \mid >$ $42d\rho\varepsilon > 6\varepsilon_1$ which controdicts the fact that both $\lambda_j(a)$ and $\lambda_j(a')$ lie in $(\rho^2 - 3\varepsilon_1, \rho^2 + 3\varepsilon_1)$, becouse $a$ and $a'$ lie in $A_{k,j}$ ( see definition $A_{k,j}$).

Thus (99), hense (98) is proved. In the same way we get the same formula for $G(-i, \frac{\rho}{d})$. So $\mu(U_{2\varepsilon}(A_{k,j}(\gamma_1, \gamma_2, ..., \gamma_k))) = O(\varepsilon\rho^{k(\alpha_k + (k-1)\alpha) + d - 1 - k})$, where $j = 1, 2, ..., b_k(\gamma_1, \gamma_2, ..., \gamma_k)$, and $\gamma_1, \gamma_2, ..., \gamma_k \in \Gamma(p\rho^\alpha)$. From this using that $b_k = O(\rho^{d\alpha + \frac{k}{2}\alpha_{k+1}})$ ( see Stat.2 of section 2) and the number of the vectors $(\gamma_1, \gamma_2, ..., \gamma_k)$ for $\gamma_1, \gamma_2, ..., \gamma_k \in \Gamma(p\rho^\alpha)$ is $O(\rho^{dk\alpha})$, we obtain (75), since

$d\alpha + \frac{k}{2}\alpha_{k+1} + dk\alpha + k(\alpha_k + (k-1)\alpha) + d - 1 - k \leq d - 1 - \alpha$ for $k \geq 1$.

ESTIMATION 4. Here we prove (64). During this estimation we denote by $G$ the set $S_\rho' \cap U_\varepsilon(Tr(A(\rho))$. Since $V_\rho \equiv S_\rho' \backslash G$, by (74) and (76), it is enough to prove $\mu(G(+i, \rho d^{-1})) = O(\rho^{-\alpha})\mu(B(\rho))$, for $i = 1, 2, ..., d$. (The same estimation for $G(-i, \rho d^{-1})$ can be proved in the same way.) We prove this by using (77). By (7) for $x \in G(+i, \rho d^{-1})$ the under integral expression in (77) for $k = i, a = \rho d^{-1}$ is less than $d + 1$. Therefore it is sufficient to prove

$$\mu(\Pr(G(+i, \rho d^{-1})) = O(\rho^{-\alpha})\mu(B(\rho)) \tag{100}$$

Clearly, if $(x_1, x_2, ...x_{i-1}, x_{i+1}, ...x_d) \in \Pr_i(G(+i, \rho d^{-1}))$ then

$\mu(U_\varepsilon(G)(x_1, x_2, ...x_{i-1}, x_{i+1}, ...x_d)) \geq 2\varepsilon$ and by (78),

$$\mu(U_\varepsilon(G)) \geq 2\varepsilon\mu(\Pr(G(+i, \rho d^{-1})). \tag{101}$$

Hense to prove (100) we need to estimate $\mu(U_\varepsilon(G))$. For this we prove that

$$U_\varepsilon(G) \subset U_\varepsilon(S_\rho'), U_\varepsilon(G) \subset U_{2\varepsilon}(Tr(A(\rho))), U_\varepsilon(G) \subset Tr(U_{2\varepsilon}(A(\rho))). \tag{102}$$

The first and second inclusions follow from $G \subset S_\rho'$ and $G \subset U_\varepsilon(Tr(A(\rho)))$ respectively (see definition of $G$ ). Now we prove the third inclusion in (102).

If $x \in U_\varepsilon(G)$ then by the second inclusion of (102) there exists $b$ such that $b \in Tr(A(\rho))$, $\mid x-b \mid < 2\varepsilon$. Then by the definition of $Tr(A(\rho))$ there are $\gamma \in \Gamma$ and $c \in A(\rho)$ such that $b = \gamma + c$. Therefore $\mid x-\gamma-c \mid = \mid x-b \mid < 2\varepsilon$, $x-\gamma \in U_{2\varepsilon}(c) \subset U_{2\varepsilon}(A(\rho))$. This together with $x \in U_\varepsilon(G) \subset U_\varepsilon(S_\rho^{'})$ (see the first inclusion of (102)) give $x \in Tr(U_{2\varepsilon}(A(\rho)))$ ,i.e., the third inclusion in (102) is proved. The third inclusion, Lemma 5 (c) and (75) imply that $\mu(U_\varepsilon(G)) = O(\rho^{-\alpha})\mu(B(\rho))\varepsilon$. This and (101) give (100)$\diamondsuit$

ESTIMATION 5 Here we prove (65). Divide the set $V_\rho \equiv V$ into pairwise disjoint subsets $V^{'}(\pm 1, \rho d^{-1}) \equiv V(\pm 1, \rho d^{-1})$,

$V^{'}(\pm i, \rho d^{-1}) \equiv V(\pm i, \rho d^{-1}) \backslash (\cup_{j=1}^{i-1}(V(\pm j, \rho d^{-1})))$, for $i = 2, 3, ..., d$. If $a \in V^{'}(+i, \rho d^{-1})$ then by Theorem 6(b), $U_\varepsilon(a) \in U_\varepsilon(S_\rho^{'}) \backslash Tr(A(\rho))$. Hence by Theorem 7($a$) there exists $N$ such that $\Lambda(y) = \Lambda_N(y)$ for $y \in U_\varepsilon(a)$ and derivative of $\Lambda(y)$ with respect to $i$-th coordinate is positive for $y \in U_\varepsilon(a)$. Therefore $\Lambda(y)$ is a continuous and increasing function in $[a-\varepsilon e_i, a+\varepsilon e_i] \subset U_\varepsilon(a)$, where $e_1 = (1, 0, 0, ..., 0)$, $e_2 = (0, 1, 0, ..., 0)$, .... Since $F(a) = \rho^2$ for $a \in V_\rho \subset S_\rho$ it follows from (7) and (14) that $F(a-\varepsilon e_i) < \rho^2 - c_{10}\varepsilon_1$, $F(a+\varepsilon e_i) > \rho^2 + c_{10}\varepsilon_1$ and $\Lambda(a-\varepsilon e_i) < \rho^2$, $\Lambda(a+\varepsilon e_i) > \rho^2$. Thus there exists a unique point $y(a) \in [a-\varepsilon e_i, a+\varepsilon e_i]$ such that $\Lambda(y(a,i)) = \rho^2$. Doing the same for $V^{'}(-i, \rho d^{-1})$ we define $I_\rho^{''} = \cup_{i=1}^{d}(I_\rho^{''}(+i) \cup I_\rho^{''}(-i))$, where $I_\rho^{''}(\pm i) = \{y(a,i) : a \in V^{'}(\pm i, \rho d^{-1})\})$. To estimate the measure of $I_\rho^{''}$ we compare the measure of $V^{'}(\pm i, \rho d^{-1})$ with the measure of $I_\rho^{''}(\pm i)$ by using (77) and the obvious relations

$$\Pr(V^{'}(\pm i, \rho d^{-1})) = \Pr(I_\rho^{''}(\pm i)) = O(\rho^{d-1}) \tag{103}$$

$$(\frac{\partial F}{\partial x_i})^{-1} \mid grad(F) \mid - (\frac{\partial \Lambda}{\partial x_i})^{-1} \mid grad(\Lambda) \mid = O(\rho^{-2\alpha_1}) \tag{104}$$

Here (103) follows from inequality in (76), since $I_\rho^{''} \subset U_\varepsilon(V_\rho) \subset U_\varepsilon(S_\rho^{'})$. Formulas (7), (63) give (104). Clearly using (103), (104), and (77) we get $\mu(V^{'}(\pm i, \rho d^{-1})) - \mu(I_\rho^{''}(\pm i)) = O(\rho^{d-1-2\alpha_1})$. On the other hand if

$y \equiv (y_1, y_2, ..., y_d) \in I_\rho^{''}(+i) \cap I_\rho^{''}(+j)$ for $i < j$ then there are $a \in V^{'}(+i, \rho d^{-1})$ and $a^{'} \in V^{'}(+j, \rho d^{-1})$ such that $y = y(a,i) = y(a^{'}, j)$ and $y \in [a-\varepsilon e_i, a+\varepsilon e_i]$, $y \in [a^{'}-\varepsilon e_j, a^{'}+\varepsilon e_j]$. These inclusions imply that $\rho d^{-1} - \varepsilon \le y_i \le \rho d^{-1}$. Therefore $\mu(\Pr_j(I_\rho^{''}(+i) \cap I_\rho^{''}(+j))) = O(\varepsilon\rho^{d-2})$. This argument, (77) and (63) imply that the measure of any pairwise intersection of the sets $I_\rho^{''}(+i)$, $I_\rho^{''}(-i)$ for $i = 1, 2, ..., d$ is equal to $O(\varepsilon\rho^{d-2})$. Thus $\mu(I_\rho^{''}) = \mu(V_\rho) + O(\rho^{-2\alpha_1})\mu(B(\rho))$. This and (64) yeild (65) for $I_\rho^{''}$. Now we define $I_\rho^{'}$ as follows. If $\gamma + t \in I_\rho^{''}$ then $\Lambda(\gamma+t) = \rho^2$, where $\Lambda(\gamma+t) =$

$\Lambda_{N(\gamma+t)}(t) =\mid \gamma + t \mid^2 + O(\rho^{-2\alpha_1})$ is a unique eigenvalue ( see Theorem 1, and (7)) satisfying (6). Clearly there are only the finite vectors $\gamma_1, \gamma_2, ..., \gamma_s \in \Gamma$ satisfying $\Lambda(\gamma_k + t) = \rho^2$. Hence $I^{''}_\rho$ is the union of pairwise disjoint subsets $I^{''}_{\rho,k} \equiv \{\gamma_k + t \in I^{''}_\rho : \Lambda(\gamma_k + t) = \rho^2\}$ for $k = 1, 2, ...s$. The translation $I^{'}_{\rho,k} = I^{''}_{\rho,k} - \gamma_k \equiv \{t \in F^* : \gamma_k + t \in I^{''}_{\rho,k}\}$ of $I^{''}_{\rho,k}$ is a part of the isoenergetic surfaces $I_\rho$ of $L(q(x))$. Put $I^{'}_\rho = \cup_{k=1}^s I^{'}_{\rho,k}$. If $t \in I^{'}_{\rho,k} \cap I^{'}_{\rho,m}$ for $k \neq m$ then $\gamma_k + t \in I^{''}_\rho \subset U_\varepsilon(S_\rho)$ and $\gamma_m + t \in U_\varepsilon(S_\rho)$ which contradict to Lemma 5(b). Hence $I^{'}_\rho$ is union of the pairwise disjoint subsets $I^{'}_{\rho,k}$ for $k = 1, 2, ...n$. Thus $\mu(I^{'}_\rho) = \sum_k \mu(I^{'}_{\rho,k}) = \sum_k \mu(I^{''}_{\rho,k}) = \mu(I^{''}_\rho) = (1 + O(\rho^{-\alpha}))\mu(B(\rho))\diamondsuit$

Now we consider the simple set $B_\delta$ in the resonance set $V_\delta(\rho^{\alpha_1})$. By (26) every vectors $w$ of $R^d$ has decomposition $w \equiv \beta + \tau + (j+v)\delta$, where $\beta \in H_\delta$, $\tau \in H_\delta/\Gamma_\delta \equiv F_\delta$, $j \in Z$, $v \in [0, 1)$. Hence the space $R^d$ is union of pairwise disjoint sets $P(\beta, j) \equiv \{\beta + \tau + (j + v)\delta : \tau \in F_\delta, v \in [0, 1)\}$ for $\beta \in H_\delta$, $j \in Z$. To prove that $B_\delta$ has an asymptotically full measure in $V_\delta(\rho^{\alpha_1})$, that is, the ratio of $\mu(B_\delta \cap (R(2\rho)\backslash R(\rho)))$ to $\mu(V_\delta(\rho^{\alpha_1}) \cap (R(2\rho)\backslash R(\rho)))$, where $R(\rho) = \{x \in R^d :\mid x \mid< \rho\}$, tends to 1 as $\rho \to \infty$ we construct sets $R_1(\rho)$, $S_1(\rho)$, $R_2(\rho)$, $S_2(\rho)$ of indices and a subset $P^{'}(\beta, j)$ of $P(\beta, j)$ such that

$$\begin{aligned} \lim_{\rho\to\infty} \frac{\mid S_i(\rho) \mid}{\mid R_i(\rho) \mid} &= 1,\ \forall i = 1, 2;\ \lim_{\beta\to\infty} \frac{\mu(P^{'}(\beta, j))}{\mu(P(\beta, j))} = 1, \qquad (105)\\ B_\delta \cap (R(2\rho)\backslash R(\rho)) &\supset \cup_{j\in S_1,\beta\in S_2} P^{'}(\beta, j), \qquad (106)\\ V_\delta(\rho^{\alpha_1}) \cap (R(2\rho)\backslash R(\rho)) &\subset \cup_{j\in R_1,\beta\in R_2} P(\beta, j). \qquad (107)\end{aligned}$$

These sets are defined as follows

$R_1(\rho) = \{j \in Z :\mid j \mid< \frac{\rho^{\alpha_1}}{2|\delta|^2} + \frac{3}{2}\}$, $S_1(\rho) = \{j \in Z :\mid j \mid< \frac{\rho^{\alpha_1}}{2|\delta|^2} - \frac{3}{2}\}$,

$R_2(\rho) = \{\beta \in \Gamma_\delta : \beta \in R_\delta(2\rho + d_\delta + 1)\backslash R_\delta(\rho - d_\delta - 1))\}$,

$S_2(\rho) = \{\beta \in \Gamma_\delta : \beta \in (R_\delta(2\rho - d_\delta - 1)\backslash R_\delta(\rho + d_\delta + 1))\backslash(\bigcup_{b\in\Gamma_\delta(\rho^{\alpha_d})} V^\delta_b(\rho^{\frac{1}{2}}))\}$,

where $R_\delta(\rho) = \{x \in H_\delta :\mid x \mid< \rho\}$, $\Gamma_\delta(\rho^{\alpha_d}) = \{b \in \Gamma_\delta :\mid b \mid< \rho^{\alpha_d}\}$,

$V^\delta_b(\rho^{\frac{1}{2}}) = \{x \in H_\delta :\mid\mid x + b \mid^2 - \mid x \mid^2\mid< \rho^{\frac{1}{2}}\}$, and for $d_\delta$ see Stat.1, of Section 3. To define $P^{'}(\beta, j)$ we introduse the sets

$A^{'}(\beta, b, \rho) = \{v \in [0, 1) : \exists j \in Z, \mid 2(\beta, b) + \mid b \mid^2 + \mid (j+v)\delta \mid^2\mid< 4d_\delta\rho^{\alpha_d}\}$,

$A^{'}(\beta, \rho) = \bigcup_{b\in\Gamma_\delta(\rho^{\alpha_d})} A^{'}(\beta, b, \rho)$, $S_3(\beta, \rho) = A(\varepsilon(\rho))\backslash A^{'}(\beta, \rho)$. For $j \in S_1$, $\beta \in S_2$ $v \in S_3(\beta, \rho)$ put $S_4(\beta, j, v, \rho) = \{\tau \in F_\delta : \beta + \tau + (j + v)\delta \in B_\delta\}$,

$P^{'}(\beta, j) = \{\beta + \tau + (j + v)\delta : v \in S_3(\beta, \rho), \tau \in S_4(\beta, j, v, \rho)\}$.

**Lemma 7** *Let* $w \equiv \beta + \tau + (j + v)\delta$. *Then*

(*a*) $w \in V_\delta(\rho^{\alpha_1}) \Rightarrow j \in R_1$. $j \in S_1 \Rightarrow w \in V_\delta(\rho^{\alpha_1})$.

$(b)$ $w \in R(2\rho)\backslash R(\rho)) \Rightarrow \beta \in R_2$. $\beta \in S_2 \Rightarrow w \in R(2\rho)\backslash R(\rho))$.
$j \in S_1, \beta \in S_2 \Rightarrow w \in R(2\rho)\backslash R(\rho)) \cap V_\delta^{'}(\rho^{\alpha_1})$.
$(c)$ *The relations (106), (107) and the first equality in (105) hold.*

**Proof** $(a)$ From definition of $V_\delta(\rho^{\alpha_1}), R_1, S_1$ using that $(\beta + \tau, \delta) = 0$, and $\mid v \mid < 1$, by direct calculation we get the both implications.

$(b)$ The inclution $w \in R(2\rho)\backslash R(\rho))$, that is, the inequalities $\rho^2 < \mid \beta + \tau \mid^2 + \mid (j+v)\delta \mid^2 < 4\rho^2$ and the relations $\mid (j+v)\delta \mid = O(\rho^{\alpha_1})$, $\mid \tau \mid \leq d_\delta = O(1)$ ( see Stat.1 of Section 3) imply that $\beta \in R_2$. By the similar calculations we obtain the second implication. This with the second implication of $(a)$ imply that if $j \in S_1, \beta \in S_2$ then $w \in R(2\rho)\backslash R(\rho)) \cap V_\delta(\rho^{\alpha_1})$. Therefore to prove the third implication it is enough to show that $w \notin V_a(\rho^{\alpha_1})$ for $a \in \Gamma(p\rho^\alpha)\backslash \delta R$. Using $a = a_1 + a_2\delta$ ( see (26)), where $a_1 \in \Gamma_\delta$ and $\mid a_1 \mid < p\rho^\alpha, \mid a_2\delta \mid < p\rho^\alpha$, we obtain $\mid w + a \mid^2 - \mid w \mid^2 = d_1 + d_2$, where $d_1 = \mid \beta + a_1 \mid^2 - \mid \beta \mid^2$, $d_2 = \mid (j + a_2 + v)\delta \mid^2 - \mid (j+v)\delta \mid^2 + 2(a_1, \tau)$. The requirements on $j, a_1, a_2$ imply that $d_2 = O(\rho^{2\alpha_1})$. On the other hand the condition $\beta \in S_2$ gives $\beta \notin V_a^\delta(\rho^{\frac{1}{2}})$, i.e., $\mid d_1 \mid \geq \rho^{\frac{1}{2}}$. Since $2\alpha_1 < \frac{1}{2}$ we have $\mid\mid w + a \mid^2 - \mid w \mid^2 \mid > \frac{1}{2}\rho^{\frac{1}{2}}$, $w \notin V_a(\rho^{\alpha_1})$.

$(c)$ The (107) follows from first implication of $(a)$ and first implication of $(b)$. If $w \equiv \beta + \tau + (j+v)\delta$ belongs to the right-hand side of (106) then by the third implication of $(b)$, $w \in V_\delta^{'}(\rho^{\alpha_1}) \cap (R(2\rho)\backslash R(\rho))$ and by the definitions of $P^{'}(\beta, j)$, $S_3(\beta, \rho)$, $S_4(\beta, j, v)$ we have $w \in B_\delta$. It remains to prove that the first equality in (105) holds. From the definitions of $R_1, S_1$ and inequalities $\mid \delta \mid < \rho^\alpha, \alpha_1 > 2\alpha$ we obtain that the first equality in (105) for $i = 1$ holds. If $\beta \in R_2$ then $\beta + F_\delta \subset R(2\rho + 2d_\delta + 1)\backslash R(\rho - 2d_\delta - 1)$. This implies that, $\mid R_2 \mid < (\mu(F_\delta))^{-1}\mu(R(2\rho + 2d_\delta + 1)\backslash R(\rho - 2d_\delta - 1))$, becouse the translations $\beta + F_\delta$ of $F_\delta$ for $\beta \in \Gamma_\delta$, are pairwise disjoint sets having measure $\mu(F_\delta)$. Suppose $\beta + \tau \in D(\rho) \equiv (R_\delta(2\rho - 1)\backslash R_\delta(\rho+1))\backslash(\bigcup_{b \in \Gamma_\delta(\rho^{\alpha_d})} V_b^\delta(2\rho^{\frac{1}{2}}))$. Then $2\rho - 1 < \mid \beta + \tau \mid < \rho + 1$, $\mid\mid \beta + \tau + b \mid^2 - \mid \beta + \tau \mid^2 \mid \geq 2\rho^{\frac{1}{2}}$, and $\beta \in S_2$. Therefore the sets $\beta + F_\delta$ for $\beta \in S_2$ is cover of $D(\rho)$. Thus $\mid S_2 \mid \geq (\mu(F_\delta))^{-1}\mu(D(\rho)$. This, the estimation for $\mid R_2 \mid$, and the obvious relation $(\mu(D(\rho))(\mu(R(2\rho + 2d_\delta + 1)\backslash R(\rho - 2d_\delta - 1)))^{-1} \to 1$ as $\rho \to \infty$ for $d\alpha_d < \frac{1}{2}$ ( see Remark 1 of [10]) give the first equality in (105) for $i = 2\diamondsuit$

**Theorem 8** *The simple set $B_\delta$ has an asymptoticully full measure in the resonance set $V_\delta(\rho^{\alpha_1})$, in the sence that* $\lim_{\rho\to\infty} \frac{\mu(B_\delta \cap (R(2\rho)\backslash R(\rho)))}{\mu(V_\delta(\rho^{\alpha_1}) \cap (R(2\rho)\backslash R(\rho)))} = 1$.

**Proof** The proof of the Theorem follows from (105), (106) and (107). By Lemma $7(c)$ we need to prove the last equality in (105). Since the translation

$P(\beta, j) - \beta - j\delta$ of $P(\beta, j)$ and the translation $P^{'}(\beta, j) - \beta - j\delta$ of $P^{'}(\beta, j)$ are $\{\tau + v\delta : v \in [0, 1), \tau \in F_\delta\}$ and $\{\tau + v\delta : v \in S_3(\beta, \rho), \tau \in S_4(\beta, j, v, \rho)\}$ respectively, it is sufficient to prove

$$\lim_{\rho\to\infty} \mu(S_3(\beta, \rho)) = 1, \ \mu(S_4(\beta, j, v, \rho)) = \mu(F_\delta)(1 + O(\rho^{-\alpha})), \tag{108}$$

where $j \in S_1, \beta \in S_2, v \in S_3(\beta, \rho)$, and $O(\rho^{-\alpha})$ does not depend on $v$. To prove the first equality in (108) it is enough to show that

$$\mu(A^{'}(\beta, \rho)) = O(\rho^{-\alpha}), \tag{109}$$

becouse $\mu(A(\varepsilon(\rho)) \to 1$ as $\rho \to \infty$ (see Stat.2 of Section 3). Since

$\mid \Gamma_\delta(\rho^{\alpha_d}) \mid = O(\rho^{(d-1)\alpha_d})$ we need to prove that $\mu(A^{'}(\beta, b, \rho)) = O(\rho^{-d\alpha_d})$. In other word we prove that

$$\mu\{s \in R :\mid f(s) \mid < 4d_\delta\rho^{\alpha_d}\} = O(\rho^{-d\alpha_d}), \tag{110}$$

where $f(s) = 2(\beta, b)+ \mid b \mid^2 + s^2 \mid \delta \mid^2$ . The condition $\beta \in S_2$ ( see definition of $P^{'}(\beta, j)$) yields $\mid 2(\beta, b)+ \mid b \mid^2 \mid \geq \rho^{\frac{1}{2}}$, for $\mid b \mid < \rho^{\alpha_d}$. This, $\mid f(s) \mid < 4d_\delta\rho^{\alpha_d}$, and $\alpha_d < \frac{1}{2}$ imply that $s^2 \mid \delta \mid^2 > \frac{1}{2}\rho^{\frac{1}{2}}$. Therefore $\mid f^{'}(s) \mid > \mid \delta \mid \rho^{\frac{1}{4}}$ and (110) holds, since $1 > 4(d+1)\alpha_d$. Thus (109), and hence the first equality in (108) is proved.

Now for $j \in S_1$, $\beta \in S_2$ and $v \in S_3(\beta, \rho)$ we consider the set $S_4(\beta, j, v, \rho)$, that is, consider the conditions (38), (39). For this we use the decompositions (see (26)) $\gamma + t = \beta + \tau + (j + v)\delta$, $\gamma^{'} + t = \beta^{'} + \tau + (j^{'} + v(\beta^{'}, t))\delta$,

$\lambda_{j,\beta}(v, \tau) = \mu_j(v)+ \mid \beta + \tau \mid^2$, $\lambda_i(\gamma^{'} + t) =\mid \gamma^{'} + t \mid^2 + r_i(\gamma^{'} + t)$. Denoting by $b$ the $\beta^{'} - \beta$ we write the decomposition of $\gamma^{'} + t$ in the form

$\gamma^{'} + t = \beta + b + \tau + (j^{'} + v(\beta + b, t))\delta$. Then to every $\gamma^{'} \in \Gamma$ there corresponds $b = b(\gamma^{'}) \in \Gamma_\delta$. For $\gamma^{'} \in M_1$ denote by $B^1(\beta, b(\gamma^{'}), j, v)$ the set of $\tau$ not satisfying (38), and for $\gamma^{'} \in M_2$ denote by $B^2(\beta, b(\gamma^{'}), j, v)$ the set of $\tau$ not satisfying (39), where $M_i$, for $i = 1, 2$ is defined in (38), (39). Clearly if $\tau \in F_\delta\backslash(\cup_{s=1,2}(\cup_{\gamma^{'}\in M_s}(B^s(\beta, b(\gamma^{'}), j, v))$ then the inequalities (38), (39) hold, that is, $\tau \in S_4(\beta, j, v, \rho)$. Therefore using $\mu(F_\delta) > c_{14}\rho^{-d\alpha}$ ( see Stat.1 of Section 3) and proving that

$$\mu(\cup_{\gamma^{'}\in M_s} B^s(\beta, b(\gamma^{'}), j, v)) = O(\rho^{-(d+1)\alpha}), \ \forall s = 1, 2, \tag{111}$$

we get the proof of the second equality in (108). So we prove (111). Using the above notations it is not hard to verify that if $\tau \in B^s(\beta, b(\gamma^{'}), j, v)$ then

$$\mid 2(\beta, b)+ \mid b \mid^2 + \mid (j^{'} + v(\beta + b))\delta \mid^2 + 2(b, \tau) - \mu_j(v) + h_s(\gamma^{'} + t) \mid < 2\varepsilon_1 \tag{112}$$

where $h_1 = F_{k_2-1} - E_{k_3-1}$, $h_2 = r_i - E_{k_3-1}$. First we prove that $B^s(\beta, b(\gamma^{'}), j, v)$ for $b(\gamma^{'}) \in \Gamma_\delta(\rho^{\alpha_d})$, $\gamma^{'} \in M_s$ is the empty set, that is, (112) for $s = 1, 2$ and $b \equiv b(\gamma^{'}) \in \Gamma_\delta(\rho^{\alpha_d})$ does not hold. The assumption $v \in S_3(\beta, \rho)$ implies that $v \notin A^{'}(\beta, \rho)$. This means that the absolute value of the sum

$2(\beta, b) + \mid b \mid^2 + \mid (j^{'} + v(\beta + b))\delta \mid^2$ of the first three summand in the right hand side of (112) is not less than $4d_\delta\rho^{\alpha_d}$. Therefore it is sufficient to prove that the absolute value of sum of 4-th, 5-th, and 6-th summands $2(b, \tau)$, $\mu_j(v)$, and $h_s(\gamma^{'} + t)$ is less than $3d_\delta\rho^{\alpha_d}$. The relations $b \in \Gamma_\delta(\rho^{\alpha_d})$, $\tau \in F_\delta$ imply that $\mid 2(b, \tau) \mid < 2d_\delta\rho^{\alpha_d}$. The inclusion $j \in S_1$ and (28) imply that $\mu_j(v) = O(\rho^{2\alpha_1})$. By (7) and (33), $h_i = O(\rho^{\alpha_1})$. Now we prove that $r_i = O(\rho^{\alpha_1})$ which implies $\mid h_2 \mid = O(\rho^{\alpha_1})$ (see (33)), and hence ends the proof of the emptyness of $B^s(\beta, b, j, v)$ for $s = 1, 2$ and $b \in \Gamma(\rho^{\alpha_d})$. The inclusion $\tau \in B^2(\beta, b(\gamma^{'}), j, v)$, means that (39) does not holds , that is, $\mid E(\lambda_{j,\beta}(v, \tau)) - \lambda_i(\gamma^{'} + t) \mid < 2\varepsilon_1$. On the other hand $\gamma^{'} \in M_2$, implies that $\gamma^{'} \in M$ ( see the definitions of $M_2$, and $M$) and hence $\mid E(\lambda_{j,\beta}(v, \tau)) - \mid \gamma^{'} + t \mid^2 \mid \leq \frac{1}{3}\rho^{\alpha_1}$ The last two inequalities imply that $\lambda_i(\gamma^{'} + t) - \mid \gamma^{'} + t \mid^2$, that is $r_i(\gamma^{'} + t)$ is equal to $O(\rho^{\alpha_1})$. Thus for $\mid b \mid < \rho^{\alpha_d}$ the sets $B^1(\beta, b, j, v)$ and $B^{2}(\beta, b, j, v)$ are empty.

To estimate the measure of the set $B^s(\beta, b(\gamma^{'}), j, v)$ for all $\gamma^{'} \in M_s$ and $\mid b(\gamma^{'}) \mid \geq \rho^{\alpha_d}$ we choose the coordinate axis so that the direction of $b$ coincides with the direction of $(1, 0, 0, ..., 0)$, i.e., $b = (b_1, 0, 0, ..., 0), b_1 > 0$ and the direction of $\delta$ coincides with the direction of $(0, 0, ..., 0, 1)$. Then $H_\delta$ and $B^s(\beta, b, j, v)$ can be considered as $R^{d-1}$ and as subset of $F_\delta \subset R^{d-1}$ respectively. We estimate the measure of $B^s(\beta, b, j, v)$ by using (78) for $D = B^s(\beta, b, j, v), m = d - 1, k = 1$. For this we prove that

$$\mu((B^s(\beta, b, j, v))(\tau_2, \tau_3, ..., \tau_{d-1})) < 4\varepsilon_1 \mid b \mid^{-1}, \tag{113}$$

for all fixed $(\tau_2, \tau_3, ..., \tau_{d-1})$. Assume the converse. Then there are two points $\tau = (\tau_1, \tau_2, \tau_3, ..., \tau_{d-1}) \in F_\delta$ and $\tau^{'} = (\tau_1^{'}, \tau_2, \tau_3, ..., \tau_{d-1}) \in F_\delta$ of $B^s(\beta, b, j, v)$, such that $\mid \tau_1 - \tau_1^{'} \mid \geq 4\varepsilon_1 \mid b \mid^{-1}$. Since (112) holds for $\tau^{'}$ and $\tau$ we have

$$\mid 2b_1(\tau_1 - \tau_1^{'}) + g_s(\tau) - g_s(\tau^{'}) \mid < 4\varepsilon_1, \tag{114}$$

where $g_s(\tau) = h_s(\beta^{'} + \tau + (j^{'} + v(\beta + b))\delta)$. By (7), (33), (11), and $\mid b \mid \geq \rho^{\alpha_d}$,

$$\mid \quad g_1(\tau) - g_1(\tau^{'}) \mid < \rho^{-\alpha_1} \mid \tau_1 - \tau_1^{'} \mid < b_1 \mid \tau_1 - \tau_1^{'} \mid, \tag{115}$$

$$\mid \quad g_2(\tau) - g_2(\tau^{'}) \mid < 3\rho^{\frac{1}{2}\alpha_d} \mid \tau_1 - \tau_1^{'} \mid < b_1 \mid \tau_1 - \tau_1^{'} \mid . \tag{116}$$

The inequality $b_1 \mid \tau_1 - \tau_1^{'} \mid \geq 4\varepsilon_1$ together with (115) and (116) controdicts (114) for $s = 1$ and for $s = 2$ respectively. Hence (113) is proved. Since

$B^s(\beta, b, j, v) \subset F_\delta$ and $d_\delta = O(1)$ ( see Stat. 1, of Section 2), we have $\mu(\Pr B^s(\beta, b, j, v)) = O(1)$. Therefore formula (78), the inequalities (113) and $\mid b \mid \geq \rho^{\alpha_d}$ yield $\mu((B^s(\beta, b(\gamma^{'}), j, v) = O(\varepsilon_1 \mid b(\gamma^{'}) \mid^{-1}) = O(\rho^{-\alpha_d}\varepsilon_1)$ for $\gamma^{'} \in M_s \subset M$ and $s = 1, 2$. This implies (111) because $\mid M \mid = O(\rho^{d-1})$ and $O(\rho^{d-1-\alpha_d}\varepsilon_1) = O(\rho^{-(d+1)\alpha})$, where $\varepsilon_1 = \rho^{-d-2\alpha}$ ( see Section 2)$\diamondsuit$

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