Semi-Classical Asymptotics in Magnetic Bloch Bands

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Abstract This article gives a simple construction of wave packets localized near semi-classical trajectories for an electron subject to external electric and magnetic fields. We assume that the magnetic and electric potentials are slowly varying perturbations of the potential of a constant magnetic field and a periodic lattice potential, respectively.

0. Introduction

When one ignores the interactions between electrons in a crystal, one is led to consider the one-body Hamiltonian governing the motion of a single electron through a lattice of ions in the presence of external electric and magnetic fields. If the external fields are constant or slowly varying on the scale of the lattice spacing, simplified "semi-classical" theories for the electron dynamics have been used since the 1930's. In [8] a multiple-scale method was used to construct wave packets obeying the semi-classical dynamics in the presence of weak, constant magnetic fields. In the present article we take a constant magnetic field and the electric field of the lattice of ions as the unperturbed fields, and consider slowly varying perturbations of them. This means that the electric and magnetic potetentials have the form

$$\mathcal{V} = V_0(x) + V(\epsilon x, \epsilon t) \text{ and } \mathcal{A} = \frac{\omega \times x}{2} + A(\epsilon x, \epsilon t)$$

respectively, where V_0 is periodic. We are interested in the asymptotic solutions of the Schrödinger equation with these potentials as $\epsilon \to 0$. Note that this implies that the *fields* resulting from the perturbation terms are both slowing varying and relatively small. Since we assume that A and V are smooth functions, the multi-scale approach leads to asymptotics to all orders in ϵ over any fixed time interval $[0, T]$. The wave packets that we construct are localized in space-time near the semi-classical trajectories, but their supports are large relative to the lattice spacing.

We think that this method gives an easy derivation of rigorous semi-classical asymptotics.

1. The Unperturbed Hamiltonian

Semi-classical dynamics involve the spectral band functions for the unperturbed Hamiltonian. In this section, we will define the magnetic Bloch bands, and state our assumptions on the unperturbed Hamiltonian. We first give a brief review of the magnetic translation operators [19].

The Hamiltonian for an electron in a crystal lattice Γ_0 of $\mathbf{R}^3$ in the presence of a constant magnetic field $\omega = (\omega_1, \omega_2, \omega_3)$ is given by

$$H_0 = \frac{1}{2m}(-ih\frac{\partial}{\partial x} + e\frac{\omega \times x}{2})^2 + V_0(x),$$

where $V_0 \in C^\infty(\mathbf{R}^n; \mathbf{R})$ is periodic with respect to Γ_0. Here m and e are the electron mass and charge respectively. To simplify notation, we will use the system of units in which $h = 2m = e = 1$ from now on.

In general, H_0 will not commute with ordinary translations, but it will commute with magnetic translations T^ω_α, $\alpha \in \Gamma_0$, given by

$$T^\omega_\alpha u(x) = e^{i\langle \frac{\omega\times x}{2}, \alpha\rangle} u(x-\alpha).$$

The T^ω_α satisfy the commutation relations:

$$T^\omega_\alpha T^\omega_\beta = e^{-\frac{i}{2}\langle \omega, \alpha\times\beta\rangle} T^\omega_{\alpha+\beta} = e^{-i\langle \omega, \alpha\times\beta\rangle} T^\omega_\beta T^\omega_\alpha. \tag{1}$$

Throughout this work, we assume that

$$\langle \omega, \Gamma_0 \times \Gamma_0\rangle \subset 2\pi\mathbf{Q}.$$

Then we can find a sublattice Γ of Γ_0 such that

$$\langle \omega, \Gamma \times \Gamma\rangle \subset 2\pi\mathbf{Z}. \tag{2}$$

In the following, we denote by $\Gamma^* := \{\gamma^* \in \mathbf{R}^n; \gamma\cdot\gamma^* \in 2\pi\mathbf{Z}, \forall\gamma \in \Gamma\}$ the dual lattice of Γ. A fundamental domain of Γ is denoted by M, the one of Γ^* by M^*.

For $\alpha \in \Gamma$, we set

$$\mathcal{T}^\omega_\alpha = e^{-\frac{i}{2}\theta(\alpha)} T^\omega_\alpha,$$

where $\theta(\alpha) = \omega_1\alpha_2\alpha_3 + \omega_3\alpha_1\alpha_2 - \omega_2\alpha_1\alpha_3$. A simple computation and (2) show that

$$\frac{1}{2}\Big[\theta(\alpha) + \theta(\beta) - \theta(\alpha+\beta) - \langle\omega, \alpha\times\beta\rangle\Big] = \omega_2\alpha_1\beta_3 - \omega_3\alpha_1\beta_2 - \omega_1\alpha_2\beta_3 \in 2\pi\mathbf{Z},$$

which together with (1) gives

$$\mathcal{T}_\alpha^\omega \mathcal{T}_\beta^\omega = \mathcal{T}_{\alpha+\beta}^\omega, \ \ \forall \gamma, \beta \in \Gamma. \tag{3}$$

We now introduce the magnetic Bloch-Floquet transformation:

$$Uu(x,k) = \sum_{\alpha \in \Gamma} e^{i(x-\alpha)\cdot k} \mathcal{T}_\alpha^\omega u(x), \ \ x, k \in \mathbf{R}^n.$$

By (3), we get

$$\mathcal{T}_\alpha^\omega Uu(x,k) = Uu(x,k),$$

$$Uu(x, k+\alpha^*) = e^{ix\cdot\alpha^*} Uu(x,k),$$

from which we prove easily that U extends to a unitary operator

$$L^2(\mathbf{R}^n) \to \int_{M^*}^{\oplus} \mathcal{H}_\omega dk = L^2(M^*; \mathcal{H}_\omega),$$

where $\mathcal{H}_\omega = \{u \in L^2_{\mathrm{loc}}(\mathbf{R}^n); \mathcal{T}_\gamma^\omega u = u, \ \forall \gamma \in \Gamma\}$. Moreover, one has

$$UH_0U^{-1} = \int_{M^*}^{\oplus} H_0(k) dk, \tag{4}$$

where by definition the last expression is the selfadjoint operator with domain $L^2(M^*; \mathcal{H}_\omega^2)$, given by

$$H_0(k)u(x) = \Big[(-i\frac{\partial}{\partial x} + \frac{\omega \times x}{2} + k)^2 + V_0(x)\Big] u(x), \tag{5}$$

for all $k \in M^*$ and all $u \in \mathcal{H}_\omega^2 := \{u \in \mathcal{H}_\omega; \partial^\alpha u \in \mathcal{H}_\omega, \, |\alpha| \le 2\}$.

Since the resolvent of $H_0(k)$ is compact, $H_0(k)$ has complete set of (normalized) eigenfunctions $\Phi_l(\cdot, k) \in \mathcal{H}_\omega^2$, $l \in \mathbf{N}$, called magnetic Bloch states. The corresponding eigenvalues accumulate at infinity and we enumerate them according to their multiplicities,

$$E_1(k) \le E_2(k) \le \ldots.$$

Since $e^{-ix\gamma^*} H_0(k) e^{ix\gamma^*} = H_0(\gamma^* + k)$, $E_l(k)$ is periodic with respect to Γ^*. Ordinary perturbation theory shows that $E_j(k)$ are continuous functions in k for every fixed j. The closed intervals $\Lambda_j := E_j(M^*)$ are called magnetic Bloch bands. Combining this with (4), we get

$$\sigma(H_0) = \cup_{l=1}^\infty \Lambda_l, \ \ \Lambda_l = E_l(M^*).$$

We make the following crucial hypothesis on the spectrum of the unperturbed magnetic Schrödinger operator

(H) *There exists l such that $E_l(k)$ is simple for all $k \in M^*$, i.e.*

$$E_{l-1}(k) < E_l(k) < E_{l+1}(k),\ \forall k \in M^*.$$

2. The perturbed Hamiltonian

As stated in the Introduction, we consider the time-dependent Schrödinger equation with perturbed magnetic and electric potentials:

$$-i\frac{\partial u}{\partial t} = \left[\left(-i\frac{\partial}{\partial x} + \frac{\omega \times x}{2} + A(\epsilon x, \epsilon t)\right)^2 + V_0(x) + V(\epsilon x, \epsilon t)\right] u. \tag{6}$$

With the change of variables

$$s = \epsilon t \text{ (adiabatic scale) and } y = \epsilon x \text{(long spatial scale)},$$

equation (6) becomes

$$-i\epsilon\frac{\partial u}{\partial s} = \left[\left(-i\epsilon\frac{\partial}{\partial y} + \frac{\omega \times y}{2\epsilon} + A(y, s)\right)^2 + V_0(\frac{y}{\epsilon}) + V(y, s)\right] u \tag{7}$$

When $\epsilon \to 0$ there are two spatial scales in the equation (7): the first one of the order of the linear dimensions γ of the periodicity cell and the second of order γ/ϵ on which the perturbations of the potentials vary appreciably. This makes possible to use the method of two scale expansions ($x = y/\epsilon$ and y are regarded as independent variables), (c.f. Buslaev [4], Guillot-Ralston-Trubowitz [8], Gérard-Martinez-Sjöstrand [7], Horn [10], Ralston [18] and Dimassi-Sjöstrand [5]). More precisely, one can consider the following equation in the independent variables x and y:

$$-i\epsilon\frac{\partial v}{\partial s} = \left[\left(-i\epsilon\frac{\partial}{\partial y} - i\frac{\partial}{\partial x} + \frac{\omega \times x}{2} + A(y, s)\right)^2 + V_0(x) + V(y, s)\right] v. \tag{8}$$

Note that, if in the solution $v(x, y, s, \epsilon)$ of (8) we let $x = y/\epsilon$ then it becomes a solution of (7). In the variable x, $v(x, y, s, \epsilon)$ is required to be periodic.

We look for local solution of the form

$$v(x,y,s,\epsilon)=e^{i\phi(y,s)/\epsilon}m(x,y,s,\epsilon), \tag{9}$$

where

$$m(x,y,s,\epsilon)=m_0(x,y,s)+\epsilon m_1(x,y,s)+\cdots. \tag{10}$$

Inserting (9) in (8), we get

$$\begin{aligned}
&\frac{\partial\phi}{\partial s}m-i\epsilon\frac{\partial m}{\partial s} && (11)\\
&=\left[\left(-i\epsilon\frac{\partial}{\partial y}-i\frac{\partial}{\partial x}+A_0(x)+k(y,s)\right)^2+V_0(x)+V(y,s)\right]m\\
&=\left[\left(-i\frac{\partial}{\partial x}+A_0(x)+k(y,s)\right)^2+V_0(x)+V(y,s)\right]m\\
&-i\epsilon\left[2\left(-i\frac{\partial}{\partial x}+A_0(x)+k(y,s)\right)\cdot\frac{\partial}{\partial y}+\frac{\partial}{\partial y}\cdot k(y,s)\right]m\\
&\qquad-\epsilon^2\Delta_y m,
\end{aligned}$$

where $k(y,s)$ is given by:

$$k(y,s)=A(y,s)+\frac{\partial\phi}{\partial y}(y,s). \tag{12}$$

Substituting (10) into the last expression in (11) and requiring that each term in the resulting asymptotic expansion vanish, we get the sequence of equations:

$$\frac{\partial\phi}{\partial s}m_0=\left[H_0\left(k(y,s)\right)+V(y,s)\right]m_0, \tag{13}$$

$$Km_0=\left[H_0\left(k(y,s)\right)+V(y,s)-\frac{\partial\phi}{\partial s}(y,s)\right]m_1, \tag{T_1}$$

$$Km_{j-1}+\Delta_y m_{j-2}=\left[H_0\left(k(y,s)\right)+V(y,s)-\frac{\partial\phi}{\partial s}(y,s)\right]m_j,\ j=2,3,... \tag{T_j}$$

Here

$$K=i\Big[\frac{\partial H_0}{\partial k}(k(y,s))\cdot\frac{\partial}{\partial y}+\frac{\partial}{\partial y}\cdot k(y,s)-\frac{\partial}{\partial s}\Big]$$

3. Eikonal equation and semi-classical dynamics

From now on we assume that (H) holds. We let Φ_l be the normalized eigenfunction corresponding to $E_l(k)$:

$$H_0(k)\Phi_l = E_l(k)\Phi_l, \tag{14}$$

$$\langle \Phi_l, \Phi_l \rangle = 1. \tag{15}$$

Since l is fixed we shall suppress the subscript l in $E_l(k)$ and Φ_l.

Equation (13) tells us that for all (y,s), $m_0(x,y,s)$ is an eigenfunction of $H_0(k(y,s))$ with eigenvalue $\frac{\partial \phi}{\partial s}(y,s) - V(y,s)$. Hence, under assumption H, we can satisfy (13) by choosing

$$\frac{\partial \phi}{\partial s}(y,s) = E\left(A(y,s) + \frac{\partial \phi}{\partial y}(y,s)\right) + V(y,s), \text{(eikonal equation)}, \tag{16}$$

and setting

$$m_0(x,y,s) = f_0(y,s)\Phi(x,k(y,s)). \tag{17}$$

Since (16) is derived from the Hamiltonian

$$h(y,s,\eta,\sigma) = \sigma - E\left(A(y,s)+\eta\right) - V(y,s),$$

we see that the equations of motion of the magnetic Bloch electron are:

$$\dot{y} = -\frac{\partial E}{\partial k}(A(y,s)+\eta), \quad \dot{s} = 1 \tag{18}$$

and

$$\dot{\eta} = -\sum_{i=1}^{3} \dot{y}_i \frac{\partial A_i}{\partial y}(y,s) + \frac{\partial V}{\partial y}(y,s), \;\; \dot{\sigma} = -\dot{y}\cdot\frac{\partial A}{\partial s}(y,s) + \frac{\partial V}{\partial s}(y,s). \tag{19}$$

It is convenient to rewrite the above equations in the new set of cordinates $(y,s,k,\sigma) = (y,s,\eta + A(y),\sigma)$. It follows from (18) and (19) that

$$\dot{y} = -\frac{\partial E}{\partial k}(k), \quad \dot{s} = 1 \tag{20}$$

$$\begin{aligned}\dot{k} &= \dot{\eta} + \sum_{i=1}^{3} \frac{\partial A}{\partial y_i}(y,s)\dot{y}_i + \frac{\partial A}{\partial s}(y,s)\dot{s} \\ &= \sum_{i=1}^{3}\Big(\frac{\partial A}{\partial y_i}(y,s) - \frac{\partial A_i}{\partial y}(y,s)\Big)\dot{y}_i + \frac{\partial A}{\partial s}(y,s) + \frac{\partial V}{\partial y}(y,s) \\ &= -\dot{y}\times B(y,s) + \frac{\partial A}{\partial s}(y,s) + \frac{\partial V}{\partial y}(y,s).\end{aligned} \tag{21}$$

Here $B = \nabla_y \times A(y,s)$ is the magnetic field corresponding to $A(y,s)$.

In (20) we see the formula for the velocity of a wave packet in terms of the gradient of the function $E(k)$ in momentum space, and in (21) the statement that the momentum k of a packet is governed by the classical equation of motion in terms of the additional external magnetic and electric field $(A(y,s), V(y,s))$.

4. Transport equations

In the following we will denote $H_0(k)$, $\Phi(x,k)$ and $E(k)$ simply by H_0, Φ and E respectively.

By the Fredholm alternative, equation (T_1) has a solution m_1 if and only if the right hand side of (T_1) is orthogonal to

$$\ker\left[H_0\left(k(y,s)\right) + V(y,s) - \frac{\partial\phi}{\partial s}\right].$$

In view of (17), this is equivalent to

$$0 = \langle\Phi, \frac{\partial m_0}{\partial s} - \frac{\partial H_0}{\partial k}\cdot\frac{\partial m_0}{\partial y} - \frac{\partial}{\partial y}\cdot k(y,s)m_0\rangle. \tag{22}$$

Since $E(k)$ is simple eigenvalue, we may assume that $\Phi(\cdot,k)$ is analytic in k. Thus, taking the gradient with respect to k in

$$\left[H_0 - E\right]\frac{\partial\Phi}{\partial k} = \left[\frac{\partial E}{\partial k} - \frac{\partial H_0}{\partial k}\right]\Phi, \tag{23}$$

and taking the inner product with Φ we get

$$\frac{\partial E}{\partial k} = \langle\Phi, \frac{\partial H_0}{\partial k}\Phi\rangle. \tag{24}$$

Here we have used the fact that $\langle\Phi(\cdot,k), \Phi(\cdot,k)\rangle = 1$.

Substituting (17) into the right hand side of (22) and using (24), we get the transport equation for f_0

$$0 = \frac{\partial f_0}{\partial s} - \frac{\partial E}{\partial k}\cdot\frac{\partial f_0}{\partial y} + h(y,s)f_0, \tag{25}$$

where h is the function given by

$$h(y,s) = \langle\Phi, \frac{\partial\Phi}{\partial s}\rangle - \langle\Phi, \frac{\partial H_0}{\partial k}\cdot\frac{\partial\Phi}{\partial y}\rangle - \frac{\partial}{\partial y}\cdot k(y,s). \tag{26}$$

In order to study the transport properties of magnetic bloch electron, we must compute the real and imaginary parts of $h(y,s)$.

Differentiating (24) with respect to y, and noting that $\frac{\partial H_0}{\partial k_j}$ is self-adjoint,

$$\frac{\partial}{\partial y}\cdot\frac{\partial E}{\partial k}(k(y,s)) = \langle\frac{\partial H_0}{\partial k}\cdot\frac{\partial}{\partial y}\Phi,\Phi\rangle + \langle\Phi,\frac{\partial H_0}{\partial k}\cdot\frac{\partial}{\partial y}\Phi\rangle + 2\frac{\partial}{\partial y}\cdot k(y,s)$$

$$= 2\mathrm{Re}\langle\Phi,\frac{\partial H_0}{\partial k}\cdot\frac{\partial}{\partial y}\Phi\rangle + 2\frac{\partial}{\partial y}\cdot k(y,s),$$

and using that $\langle\Phi,\frac{\partial\Phi}{\partial s}\rangle\in i\mathbf{R}$, (which follows from $\langle\Phi,\Phi\rangle = 1$), we obtain from (26)

$$\mathrm{Re}\ h(y,s) = -\frac{1}{2}\frac{\partial}{\partial y}\cdot\frac{\partial E}{\partial k}(k(y,s)). \tag{27}$$

In the following, we are interested in computing the imaginary part of $h(y,s)$. We use the Einstein summation convention (like indices are summed from 1 to 3).

Taking the imaginary part of (26) and using (23) as well as $\langle\Phi,\frac{\partial\Phi}{\partial s}\rangle\in i\mathbf{R}$, we obtain

$$\begin{aligned}\mathrm{Im}\ h &= \mathrm{Im}\{\langle\Phi,-\frac{\partial H_0}{\partial k_j}\frac{\partial\Phi}{\partial y_j}\rangle\} + \ \mathrm{Im}\ \{\langle\Phi,\frac{\partial\Phi}{\partial s}\rangle\} \\ &= \mathrm{Im}\{\langle-\frac{\partial E}{\partial k_j}\Phi,\frac{\partial\Phi}{\partial y_j}\rangle + \langle(H_0-E)\frac{\partial\Phi}{\partial k_j},\frac{\partial\Phi}{\partial y_j}\rangle\} + \mathrm{Im}\{\langle\Phi,\frac{\partial\Phi}{\partial s}\rangle\} \\ &= -\frac{i}{2}\langle(H_0-E)\frac{\partial\Phi}{\partial k_j},\left(\frac{\partial k_l}{\partial y_j}-\frac{\partial k_j}{\partial y_l}\right)\frac{\partial\Phi}{\partial k_l}\rangle + i\langle\frac{\partial E}{\partial k_j}\Phi,\frac{\partial\Phi}{\partial y_j}\rangle - i\langle\Phi,\frac{\partial\Phi}{\partial s}\rangle\end{aligned} \tag{28}$$

We further notice that

$$(\frac{\partial k_l}{\partial y_j}-\frac{\partial k_j}{\partial y_l})\frac{\partial\Phi}{\partial k_l} = (\frac{\partial A_l}{\partial y_j}-\frac{\partial A_j}{\partial y_l})\frac{\partial\Phi}{\partial k_l}.$$

Thus, letting M be the anti-symmetric matrix corresponding to the magnetic field $B = \nabla_y\times A(y,s)$

$$M = \begin{pmatrix} 0 & B_3 & -B_2 \\ -B_3 & 0 & B_1 \\ B_2 & -B_1 & 0\end{pmatrix},$$

(28) becomes

$$\mathrm{Im}\ h = -\frac{i}{2}\langle(H_0-E)\frac{\partial\Phi}{\partial k_j},(M\frac{\partial\Phi}{\partial k})_j\rangle + i\langle\frac{\partial E}{\partial k_j}\Phi,\frac{\partial\Phi}{\partial y_j}\rangle - i\langle\Phi,\frac{\partial\Phi}{\partial s}\rangle.$$

Let $s \to (y,s)$ be the solution of the Hamilton equations (18) and (19). Along (y,s) the above differential equation takes the form

$$\begin{aligned}\text{Im } h &= -\frac{i}{2}\langle (H_0 - E)\frac{\partial \Phi}{\partial k_j}, (M\frac{\partial \Phi}{\partial k})_j\rangle - i\langle \Phi, \dot{y}\frac{\partial \Phi}{\partial y_j}\rangle - i\langle \Phi, \frac{\partial \Phi}{\partial s}\rangle \\ &= -\frac{i}{2}\langle (H_0 - E)\frac{\partial \Phi}{\partial k_j}, (M\frac{\partial \Phi}{\partial k})_j\rangle - i\langle \Phi, \dot{\Phi}\rangle.\end{aligned}$$

This, together with (25) and (27), yields

$$\begin{aligned}&\frac{\partial f_0}{\partial s} - \frac{\partial E}{\partial k}\cdot\frac{\partial f_0}{\partial y} - \frac{1}{2}(\frac{\partial}{\partial y}\cdot\frac{\partial E}{\partial k})f_0 \\ &+ \frac{1}{2}\langle (H_0 - E)\frac{\partial \Phi}{\partial k_j}, (M\frac{\partial \Phi}{\partial k})_j\rangle f_0 + \langle \Phi, \dot{\Phi}\rangle f_0 = 0.\end{aligned}$$

Setting

$$L = \text{ Im }\left\{(\langle (H_0 - E)\frac{\partial \Phi}{\partial k_2}, \frac{\partial \Phi}{\partial k_3}\rangle, \langle (H_0 - E)\frac{\partial \Phi}{\partial k_3}, \frac{\partial \Phi}{\partial k_1}\rangle, \langle (H_0 - E)\frac{\partial \Phi}{\partial k_1}, \frac{\partial \Phi}{\partial k_2}\rangle)\right\}$$

we then get

$$\frac{\partial f_0}{\partial s} - \frac{\partial E}{\partial k}\cdot\frac{\partial f_0}{\partial y} - \frac{1}{2}(\frac{\partial}{\partial y}\cdot\frac{\partial E}{\partial k})f_0 + (iL\cdot B + \langle \Phi, \dot{\Phi}\rangle)f_0 = 0. \tag{29}$$

Note that L is an angular momentum which satisfies the usual commutation relations.

The transport equation (29) implies that if we begin with a localized packet, i.e. if we chose $f_0(y,0)$ with a small support, the packet will move along the trajectory predicted by the semi-classical theory given by the equation (19) and (20). The coefficient

$$\frac{1}{2}\frac{\partial}{\partial y}\cdot\frac{\partial E}{\partial k}$$

insures that $\int_{\mathbf{R}^3}|f_0(y,s)|^2 dy$ is independent on s, and thus

$$\int_{\mathbf{R}^3}|u(y,s,\epsilon)|^2 dy = \int_{\mathbf{R}^3}|u(y,0,\epsilon)|^2 dy + \mathcal{O}(\epsilon).$$

The term $iL\cdot B + \langle \Phi, \dot{\Phi}\rangle$, which is derived from the term of order ϵ in the equation (11) plays exactly the role in these constructions that the subprincipal symbol plays in the construction of asymptotic solutions to $P(y, \epsilon D_y, \epsilon)u = 0$, (Duistermaat [5]). In physics literature the term $L.B$ is well known, see Wilkinson [20] and also Bellissard [2], Helffer-Sjöstrand [9].

When $A(y,s) = \frac{\omega_1 \times y}{2}$, it is called the Rammal-Wilkinson term. The term $\langle \Phi, \dot{\Phi} \rangle$ gives rise to the Berry phase progression [3] in the solution.

Remark

Formula (29) is not consistent with formula (6) in [8]. A calculation error in passing from formula (A.3) to (A.4) in the Appendix to [8] led to the omission of part of the Berry phase term in formula (6) in [8].

We now derive the transport equation for m_1. As for T_1, T_2 can be solved for m_2 if and only if

$$\langle \Phi, Km_1 + \Delta_y m_0 \rangle = 0. \tag{30}$$

On the other hand, since $x \to m_1(x, \cdot, \cdot)$ is required to be in $\mathcal{H}_\omega$, we can write

$$m_1(x,y,s) = f_1(y,s)\Phi(x, k(y,s)) + m_1^\perp(x,y,s),$$

with

$$\langle \Phi(\cdot, k(y,s)), m_1^\perp(x,y,s) \rangle = 0.$$

Inserting the above equation in (30), we get

$$-i\langle \Phi, K(f_1\Phi) \rangle = -i\langle \Phi, Km_1^\perp + \Delta_y m_0 \rangle. \tag{31}$$

Clearly, the L. H. S of (31) equals the one of (22) with f_1 instead of f_0. Thus, (29) yeilds

$$\begin{aligned} \frac{\partial f_1}{\partial s} - \frac{\partial E}{\partial k} \cdot \frac{\partial f_1}{\partial y} - \frac{1}{2}(\frac{\partial}{\partial y} \cdot \frac{\partial E}{\partial k}) f_1 + (iL \cdot B + \langle \Phi, \dot{\Phi} \rangle) f_1 & \\ = -i\langle \Phi, Km_1^\perp + \Delta_y m_0 \rangle & \end{aligned} \tag{32}$$

Since $m_1^\perp$ is determined by T_1, (32) reduces to an inhomogeneous version of the transport equation (29). This gives m_1. In the same way we compute all the terms m_j.

We can now construct a wave packet for (7) by solving the eikonal and transport equations along the solutions of (19) and (20) with initial condition $y = y_0, \eta = \eta_0$ at $s = 0$ and taking a linear superposition over (y_0, η_0).

Using the Gaussian beams ansatz, (see Arnaud [1] and Ralston [17]), and choosing the initiale phase condition so that $\mathrm{Im}\{\phi(y,0)\} \geq 0$, $\mathrm{Im}\{\phi(y_0,0)\} = 0$, $\mathrm{Im}\{\phi_{yy}\}(y_0,0) > 0$, $\phi_y(y_0,0) = \eta_0$, we can easily solve the eikonal and transport equations along the solutions of (19) and (20).

Remark

In [13] and [14], Ming-Che Chang and Qian Niu consider the same unperturbed Hamiltonian that is used in this article, and introduce perturbations of the electric and magnetic fields. They find a set of equations for the semi-classical dynamics in which the velocity ($\dot{y}$ in (20)) has an extra term involving the Berry curvature. We do not know what assumptions on the perturbations will lead to these equations. Our assumptions lead to (20) and (21).

References

[1] J. A. Arnaud, Hamiltonian theory of beam mode propagation. *Progress in Optics XI. Ed. E. Wolf. North Holland, 249–304 (1973).*

[2] J. Bellissard, C^* algebra in solid state Physics (2D electrons in a uniform magnetic field), *Talk given at at the Warwick Conference on operator algebras (July 1987).* .

[3] B. Simon, Phys. Rev. Lett. **51**, 2167-2170(1983).

[4] V. S. Buslaev, Semiclassical approximation for equations with periodic coefficients. *Russ. Math. Surveys.* **42 (6)**, 97–125 (1987).

[5] M. Dimassi, J. Sjöstrand, *Spectral Asymptotics in the Semi-Classical Limit.* London Math. Soc. Lecture note Ser., 268 Cambridge Univ. Press, Cambridge, 1999.

[6] J. J. Duistermaat, Oscillatory integrals, Lagrange immersions and unfolding of singularities. *Commun. Pure Appl. Math.* **27**, *207–281 (1974).*

[7] C. Gérard, A. Martinez, J. Sjöstrand, *A Mathematical Approach to the effective Hamiltonian in perturbed periodic Problems.* Comm. Math. Phys., **142**, 217–244 (1991).

[8] J.-C. Guillot, J. Ralston, E. Trubowitz, Semi-classical methods in solid state physics. *Comm. Math. Phys.,* **116**, 401–15 (1988).

[9] B. Helffer, J. Sjöstrand, Analyse semi-classique pour l'équation de Harper. *Bull. de la SMF,* **116 (4)**, mémoire no 34 (1988).

[10] W. Horn, Semi-classical constructions in solid state physics, *Commun. in P. D. E.* **(16)**, *255–290 (1991).*

[11] J. Keller, S. Rubinow, Asymptotic solution of eigenvalue problems. *Ann. Phys.* **9**, *24–75 (1960).*

[12] M. Kohmoto, Berry's phase of Bloch electrons in electromagnetic fields. *Journal of the phys. Soc. of Japan* **(62)**, *659–663, (1993)*

[13] V.P. Maslov, M. V. Fedoriuk, *Semiclassical approximation in quantum mechanics.* Dordrecht. D . Reidel (1981).

[14] Ming-Che Chang, Qian Niu, Berry phase, hyperorbits, and the Hofstadter spectrum *Phys. Review Letters,* **(75)** *1348–1351. (1996)*

[15] Ming-Che Chang, Qian Niu, Berry phase, hyperorbits, and the Hofstadter spectrum: Semiclassical dynamics in Magnetic Bloch bands, *Phys. Review B,* **(53)** *7010–7022. (1996)*

[16] R. Peierls, Zur Theorie des diamagnetimus von Leitungselektronen. *Z. Phys.* **80**, *763–791 (1933).*

[17] J. Ralston, On the Construction of quasimodes associated with stable periodic orbits. *Commun. math. phys.* **(51)**, *219–242 (1976).*

[18] J. Ralston Magnetic Breakdown, *Astérisque* **(210)**, *263–282 (1992).*

[19] J. Zak, Dynamics of electrons in solids in external fields. *Phys. Rev.* **168**, *686–695 (1968).*