A note on the adiabatic theorem without gap condition

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Abstract

We simplify the proof of the adiabatic theorem of quantum mechanics without gap condition of Avron and Elgart [1] by providing an elementary solution of the "commutator equation". In addition, a minor modification of their argument allows for a more direct treatment of eigenvalue crossings. We also obtain simple, explicit conditions that yield information on the rate of convergence in the adiabatic limit.

In [1], Avron and Elgart prove an adiabatic theorem without gap condition. We will show in this note how to simplify parts of their proof by giving a considerably simpler solution of the "commutator equation" (6), which is elementary even in the case of unbounded Hamiltonians. Furthermore, a minor change in the argument allows for a simpler treatment of eigenvalue crossings.

One is interested in the solution of the initial value problem

$$i\frac{d}{ds}U_\tau(s) = \tau\, H(s)\, U_\tau(s)\,, \quad U_\tau(0) = \mathbf{1} \tag{1}$$

for $s \in [0,1]$ in the limit of large τ. Here $H(s)$ is a family of self-adjoint operators on some Hilbert space $\mathcal{H}$ with a common dense domain $\mathcal{D}$ such that $R(i,\cdot) \in C^1([0,1],\mathcal{B})$ and such that, uniformly for $s \in [0,1]$, $H(s) \geq \Lambda_1 > -\infty$ and $\|H(s)\dot{R}(i,s)\| \leq \Lambda_2 < \infty$. $R(z,s) := (H(s)-z)^{-1}$ and $(\mathcal{B}, \|\cdot\|)$ denotes the space of bounded operators on $\mathcal{H}$ equipped with the usual norm. The existence of a family of unitaries $U_\tau(s)$ such that (1) holds on $\mathcal{D}$ follows from general theory, cf. [1].

For a discussion of the history, the physical interpretation and applications of the following theorem we refer to [1]. The analogous result in the context of the space-adiabatic theorem with an application to the massless Nelson model will be presented in [3].

Theorem. *Suppose that $E(s)$ is an eigenvalue of $H(s)$ and that $P(s)$ is a family of finite rank projections such that $H(s)P(s) = E(s)P(s)$ for all $s \in [0,1]$, $P(\cdot) \in C^2([0,1],\mathcal{B})$ and such that $P(s)$ is the spectral projection of $H(s)$ on $\{E(s)\}$ for almost all $s \in [0,1]$. Then the solution $U_\tau(s)$ of (1) satisfies*

$$\sup_{s\in[0,1]} \|\,(1-P(s))\,U_\tau(s)\,P(0)\,\| \to 0 \quad \textit{as} \quad \tau \to \infty\,. \tag{2}$$

Note that if $E(s)$ crosses a different eigenvalue $E_1(s)$ at some time $s = s_0$, i.e. $E(s) = E_1(s)$ only for $s = s_0$, then the rank of the spectral projection $\mathbb{1}_{\{E(s)\}}(H(s))$

jumps and thus it is not even continuous at $s = s_0$. If it is possible to continue $1\!\!1_{\{E(s)\}}(H(s))$ through the crossing in a differentiable way, such a crossing (actually a countable number of them) is included in the above formulation of the theorem.

The following proof, as well as the original one in [1], actually yields a stronger statement than (2): Let $U_{\mathrm{A}}(s)$ be the adiabatic time evolution given as the solution of (1) with $H(s)$ replaced by $H_{\mathrm{A}}(s) := H(s) + i\tau^{-1}[\dot{P}(s), P(s)]$,[1] then

$$\sup_{s\in[0,1]} \| U_\tau(s) - U_{\mathrm{A}}(s) \| \to 0 \quad \text{as} \quad \tau \to \infty\,. \tag{3}$$

The adiabatic evolution exactly intertwines $P(0)$ and $P(s)$, i.e.

$$U_{\mathrm{A}}^*(s)\, P(s)\, U_{\mathrm{A}}(s) = P(0)\,, \tag{4}$$

and thus (3) implies (2). To see (4) for $s \in [0,1]$, note that it is clearly true for $s = 0$ and that the time derivative of the left hand side vanishes because, with (9), $\dot{P}(s) = -i\,\tau\,[H_{\mathrm{A}}(s), P(s)]$.

The first general adiabatic theorem without gap condition based on a completely different strategy of proof is due to Bornemann [2]. His result is considerably weaker since only (2) is obtained and only in the strong operator topology. However, his result was the motivation for this author to relax the "spectrality" condition from $P(s) = 1\!\!1_{\{E(s)\}}(H(s))$ for all $s \in [0,1]$, as in [1], to $P(s) = 1\!\!1_{\{E(s)\}}(H(s))$ for almost all $s \in [0,1]$, as in [2].

Proof. Using the standard Cook's type argument one expresses the difference of the unitaries in terms of the difference of the generators. One finds first for $\psi \in \mathcal{D}$ that

$$\begin{aligned}
(U_\tau(s) - U_{\mathrm{A}}(s))\,\psi &= -\,U_\tau(s) \int_0^s dt \frac{d}{dt}\Big(U_\tau^*(t) U_{\mathrm{A}}(t)\Big)\psi \\
&= -\,U_\tau(s) \int_0^s dt\, U_\tau^*(t)[\dot{P}(t), P(t)] U_{\mathrm{A}}(t)\psi
\end{aligned}$$

and thus employing density of $\mathcal{D}$ that

$$\| U_\tau(s) - U_{\mathrm{A}}(s) \| = \| \int_0^s dt\, U_\tau^*(t)[\dot{P}(t), P(t)] U_{\mathrm{A}}(t) \|\,. \tag{5}$$

If one can find bounded operators $X(s)$, $Y(s)$ satisfying

$$[\dot{P}(s), P(s)] = [H(s), X(s)] + Y(s)\,, \tag{6}$$

then the integrand in (5) can be written as a time derivative plus a remainder,

$$\begin{aligned}
U_\tau^*[\dot{P}, P]\, U_{\mathrm{A}} &= -\frac{i}{\tau}\frac{d}{ds}\Big(U_\tau^*\, X\, U_\tau U_\tau^* U_{\mathrm{A}}\Big) \\
&\quad + \frac{i}{\tau}\Big(U_\tau^*\, X[\dot{P}, P] U_{\mathrm{A}} + U_\tau^* \dot{X} U_{\mathrm{A}}\Big) + U_\tau^*\, Y\, U_{\mathrm{A}}\,,
\end{aligned} \tag{7}$$

and (7) in (5) yields the bound

$$\begin{aligned}
&\sup_{s\in[0,1]} \| U_\tau(s) - U_{\mathrm{A}}(s) \| \le \\
&\qquad \sup_{s\in[0,1]} \frac{2(1 + \|\dot{P}(s)\|)\, \|X(s)\|}{\tau} + \int_0^1 ds\, \left(\frac{\|\dot{X}(s)\|}{\tau} + \|Y(s)\| \right)\,.
\end{aligned} \tag{8}$$

[1] Using $i\tau^{-1}[\dot{P}(\cdot), P(\cdot)] \in C^1([0,1], \mathcal{B})$ it is easy to see that for τ large enough also the family $H_{\mathrm{A}}(s)$ satisfies the general assumptions on $H(s)$. Hence the existence of a unitary propagator $U_{\mathrm{A}}(s)$ satisfying (1) on $\mathcal{D}$ with $H(s)$ replaced by $H_{\mathrm{A}}(s)$ follows.

If there was a gap δ separating $E(s)$ from the rest of the spectrum of $H(s)$, then $X_{\rm gap}(s) := R(E(s),s)\dot{P}(s)P(s)+P(s)\dot{P}(s)R(E(s),s)$, $Y_{\rm gap}(s)\equiv 0$ would be a solution of (6) with $X_{\rm gap}(\cdot)\in C^1([0,1],\mathcal{B})$. This is because one obtains from differentiating $P(s)^2 = P(s)$ that

$$\dot{P}(s) = (1-P(s))\,\dot{P}(s)\,P(s) + P(s)\,\dot{P}(s)\,(1-P(s))\,, \tag{9}$$

and thus $\|R(E(s),s)\dot{P}(s)P(s)\| = \|R(E(s),s)(1-P(s))\,\dot{P}(s)\,P(s)\| \le \|\dot{P}(s)\|/\delta$. Hence, in the presence of a gap (8) gives the standard result $\sup_{s\in[0,1]}\|U_{\rm A}(s)-U_\tau(s)\| = \mathcal{O}(\tau^{-1})$.

Instead of using a cutoff function $\widetilde{g}$ to define $R(E(s),s)\,\widetilde{g}((H(s)-E(s))/\Delta)$ in absence of a gap as in [1], we shift the resolvent into the complex plane and define

$$\begin{aligned} X_\Delta(s) &:= R(E(s)-i\Delta,s)\,\dot{P}(s)\,P(s) \;+\; \text{adjoint} \\ &= X_\Delta(s)\,P(s) \;+\; \text{adjoint}\,, \\ Y_\Delta(s) &:= i\,\Delta R(E(s)-i\Delta,s)\,\dot{P}(s)\,P(s) \;-\; \text{adjoint} \\ &= Y_\Delta(s)\,P(s) \;-\; \text{adjoint}\,. \end{aligned}$$

Here "$\pm$ adjoint" means that the adjoint of the first term in the sum is added resp. subtracted. With our definition the fact that $X_\Delta(s)$ and $Y_\Delta(s)$ solve (6) is obvious and the following bounds have elementary proofs even in the case of unbounded $H(s)$.

We will show that $X_\Delta(\cdot)$, $Y_\Delta(\cdot)\in C^1([0,1],\mathcal{B})$ for $\Delta\in(0,1]$ and that for some constant $C<\infty$

$$\sup_{s\in[0,1]}\|X_\Delta(s)\| \le \frac{C}{\Delta}\,, \qquad \sup_{s\in[0,1]}\left\|\dot{X}_\Delta(s)\right\| \le \frac{C}{\Delta^2}\,, \tag{10}$$

$$\sup_{s\in[0,1],\Delta\in(0,1]}\|Y_\Delta(s)\| < \infty \tag{11}$$

and

$$\|Y_\Delta(s)\| \to 0 \quad \text{as} \quad \Delta\to 0 \tag{12}$$

for almost all $s\in[0,1]$. Then the theorem follows by inserting (10)–(12) into (8), using dominated convergence and chosing $\Delta=\Delta(\tau)$ with $\lim_{\tau\to\infty}\Delta(\tau)=0$ and $\lim_{\tau\to\infty}\tau\Delta^2(\tau)=\infty$.

It suffices to establish (10)–(12) for $X_\Delta(s)P(s)$ resp. $Y_\Delta(s)P(s)$, since the adjoints have the same norms. (11) and the first bound in (10) follow from $\|R(E(s)-i\Delta,s)\|\le 1/\Delta$ and from $P(\cdot)\in C^2([0,1],\mathcal{B})$. For the second bound in (10) observe that

$$\frac{d}{ds}R(E(s)-i\Delta,s) = -R(E(s)-i\Delta,s)(\dot{H}(s)-\dot{E}(s))R(E(s)-i\Delta,s)$$

and thus, by inserting $\mathbf{1} = R(i,s)\,(H(s)-(E(s)-i\Delta)-i+(E(s)-i\Delta))$,

$$\|\dot{R}(E(s)-i\Delta,s)\| \le \left(\frac{\|\dot{H}(s)R(i,s)\|(2\Delta+1+|E(s)|)}{\Delta^2} + \frac{|\dot{E}(s)|}{\Delta^2}\right)\,.$$

Since $E(s)=\mathrm{tr}(H(s)P(s))/\mathrm{tr}P(s)$, it follows from the differentiability of $P(s)$ and $H(s)$ that also $E(s)$ is continuously differentiable and therefore $\sup_{s\in[0,1]}(|E(s)|+|\dot{E}(s)|)<\infty$. According to the assumptions on $H(s)$ we have $\sup_{s\in[0,1]}\|\dot{H}(s)R(i,s)\| = \sup_{s\in[0,1]}\|(H(s)+i)\dot{R}(i,s)\|<\infty$.

The final step in proving the adiabatic theorem without gap is to establish (12) and we repeat the argument from [1]. Since $P(s)$ has finite rank, $\lim_{\Delta\to 0}\|Y_\Delta(s)P(s)\| = 0$

follows if one can show $\lim_{\Delta\to 0}\|Y_\Delta(s)P(s)\psi\| = 0$ for all ψ. Let $\varphi := \dot{P}(s)P(s)\psi$, then

$$\begin{aligned}\lim_{\Delta\to 0}\|Y_\Delta(s)P(s)\psi\|^2 &= \lim_{\Delta\to 0}\|i\Delta R(E(s)-i\Delta,s)\varphi\|^2 \\ &= \lim_{\Delta\to 0}\int_{\mathbb{R}}\mu_\varphi(d\lambda)\frac{\Delta^2}{(\lambda-E(s))^2+\Delta^2} = \mu_\varphi(E(s))\,,\end{aligned}$$

where μ_φ denotes the spectral measure of $H(s)$ for φ. If $P(s)$ is the spectral projection on $\{E(s)\}$, then $\mu_\varphi(E(s)) = 0$ because, according to (9), $\varphi \in \mathrm{Ran}(1-P(s))$. Hence, $\lim_{\Delta\to 0}\|Y_\Delta(s)P(s)\psi\| = 0$ for almost all $s\in[0,1]$. $\square$

Remark 1. The preceding proof shows, in particular, that information about the rate of convergence in (2) can be derived from information about the limiting behavior of $R(E(s)-i\Delta,s)\,\dot{P}(s)\,P(s)$ in the limit $\Delta\to 0$. Assume that there is a function $\eta:[0,1]\to[0,1]$ with $\eta(\Delta)\geq\Delta$ such that there is a constant $C_1<\infty$ with

$$\sup_{s\in[0,1]}\|\,R(E(s)-i\Delta,s)\,\dot{P}(s)\,P(s)\,\|\;\leq\;C_1\,\frac{\eta(\Delta)}{\Delta}\,. \tag{13}$$

Then (10) and (12) can be sharpened to

$$\sup_{s\in[0,1]}\|X_\Delta(s)\|\leq C\,\frac{\eta(\Delta)}{\Delta}\,,\qquad \sup_{s\in[0,1]}\left\|\dot{X}_\Delta(s)\right\|\leq C\,\frac{\eta(\Delta)}{\Delta^2}\,,$$

$$\sup_{s\in[0,1]}\|Y_\Delta(s)\|\leq C\,\eta(\Delta)\,,$$

which inserted into (8) yields for $\Delta(\tau):=\tau^{-\frac{1}{2}}$ and some $C<\infty$ that

$$\sup_{s\in[0,1]}\|\,U_\tau(s)-U_{\mathrm{A}}(s)\,\|\;\leq\;C\,\eta(\tau^{-\frac{1}{2}})\,. \tag{14}$$

Analogously one concludes that if in addition to (13) also

$$\sup_{s\in[0,1]}\|\,\frac{d}{ds}(R(E(s)-i\Delta,s)\,\dot{P}(s)\,P(s))\,\|\;\leq\;C_2\,\frac{\eta(\Delta)}{\Delta} \tag{15}$$

holds for some $C_2<\infty$, then one obtains

$$\sup_{s\in[0,1]}\|\,U_\tau(s)-U_{\mathrm{A}}(s)\,\|\;\leq\;C\,\eta(\tau^{-1})\,. \tag{16}$$

It was shown in [1] that (13) follows with $\eta(\Delta)=\Delta^{\frac{\alpha}{2}}$, if the spectral measures $\mu_{\varphi(s)}$ are α-Hölder continuous uniformly for $\varphi(s)\in\mathrm{Ran}P(s)$ and $s\in[0,1]$. Then, according to (14), the adiabatic limit is approached at least at the rate $\tau^{-\frac{\alpha}{4}}$. This slightly improves the rate $\tau^{-\frac{\alpha}{4+\alpha}}$, which was obtained in Corollary 1 in [1] under the same assumptions.[2]

For a nontrivial example where the conditions (13) and (15) can be checked explicitly, we refer to [3].

Remark 2. With one reservation, namely the existence of $U_{\mathrm{A}}(s)$, it is possible to replace the condition $P(\cdot)\in C^2([0,1],\mathcal{B})$ in the theorem by $P(\cdot)\in C^1([0,1],\mathcal{B})$. I owe

[2]Note that the rate $\tau^{-\frac{\alpha}{2+\alpha}}$ which is stated in Corollary 1 in [1] results from a typo in the proof: In Eq. (34) in [1] α should read $\alpha/2$, which results in the rate $\tau^{-\frac{\alpha}{4+\alpha}}$.

the following argument to Alexander Elgart. One can approximate $\dot{P}(s)$ by a family $A_\varepsilon(s)$, defined, e.g., as

$$A_\varepsilon(s) = \frac{1}{\varepsilon}\int_0^1 dt\, e^{-\frac{(t-s)^2}{\varepsilon^2}}\,\dot{P}(t)\,,$$

so that $\|\dot{P}(s)-A_\varepsilon(s)\|\to 0$ as $\varepsilon\to 0$ uniformly for $s\in[0,1]$. Since $A_\varepsilon(\cdot)\in C^1([0,1],\mathcal{B})$ with $\|\dot{A}_\varepsilon(s)\|\leq C/\varepsilon^2$, it can be used to define ε-dependent solutions of the commutator equation (6) as

$$\begin{aligned} X_{\Delta,\varepsilon}(s) &:= R(E(s)-i\Delta,s)\,A_\varepsilon(s)\,P(s)\;+\;\text{adjoint}\,,\\ Y_{\Delta,\varepsilon}(s) &:= i\,\Delta R(E(s)-i\Delta,s)\,A_\varepsilon(s)\,P(s)\;-\;\text{adjoint}\\ &\quad\;+[(\dot{P}(s)-A_\varepsilon(s)),P(s)]\,, \end{aligned}$$

where in (10) and (11) only $\sup_{s\in[0,1]}\left\|\dot{X}_{\Delta,\varepsilon}\right\|\leq\frac{C}{\Delta^2\varepsilon^2}$ changes. Instead of (12) one obtains $\lim_{\Delta,\varepsilon\to 0}\|Y_{\Delta,\varepsilon}(s)\|=0$ irrespective of the order in which the limits are taken. By inserting the new bounds into (8) and chosing, for example, $\Delta=\varepsilon=\tau^{-\frac{1}{5}}$, one obtains (3) without the need for differentiability of $\dot{P}(s)$.

A warning is however in order. If we assume only $P(\cdot)\in C^1([0,1],\mathcal{B})$ then $H_{\rm A}(s)$ does not satisfy the general assumptions guaranteeing existence of $U_{\rm A}(s)$ anymore, because $(H_{\rm A}(s)+i)^{-1}$ need not be differentiable. Hence, if the condition $P(\cdot)\in C^2([0,1],\mathcal{B})$ in the theorem is replaced by $P(\cdot)\in C^1([0,1],\mathcal{B})$, one has to assume that the unitary propagator $U_{\rm A}(s)$ with generator $H_A(s)$ exists. In the special case when $H(s)$ is a family of *bounded* operators one has $H_{\rm A}(\cdot)\in C([0,1],\mathcal{B})$ and then Dyson expansion yields the existence of the unitary propagator $U_{\rm A}(s)$ even for $P(\cdot)\in C^1([0,1],\mathcal{B})$.

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References

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